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Triangular decomposition of SL_3 skein algebras

Vijay Higgins

Abstract. We give an SL_3 analogue of the triangular decomposition of the Kauffman bracket stated skein algebras described by Lê. To any punctured bordered surface, we associate an SL_3 stated skein algebra which contains the SL_3 skein algebra of closed webs. These algebras admit natural algebra morphisms associated to the splitting of surfaces along ideal arcs. We give an explicit basis for the SL_3 stated skein algebra and show that the splitting morphisms are injective and describe their images. By splitting a surface along the edges of an ideal triangulation, we see that the SL_3 stated skein algebra of any ideal triangulable surface embeds into a tensor product of stated skein algebras of triangles. As applications, we prove that the stated skein algebras do not have zero divisors, we construct Frobenius maps at roots of unity, and we obtain a new proof that Kuperberg's web relations generate the kernel of the Reshetikhin–Turaev functor.

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1. Introduction

The representation theory of a quantum group admits a diagrammatic calculus. Ribbon diagrams in the plane depict maps between representations, and skein relations among the diagrams correspond to relations among the maps. For the case of SL_n, it has been shown that such skein theories essentially completely describe the representation theory of the quantum group $U_q(\mathfrak{sl}_n)$ (see [6, 24]). Instead of restricting our attention to diagrams in the plane, we can associate to any surface a skein module which consists of formal linear combinations of diagrams on the surface modulo local skein relations. Such a skein module admits a natural algebra structure given by superimposing one diagram on top of another. For the SL_n case, Sikora [30] has described a skein theory built from directed n-valent ribbon graphs which is particularly amenable to the algebra structure since only one color of strand is used. For the SL_2 case, this skein theory coincides with that given by the Kauffman bracket skein relations and for the SL_3 case it coincides with that given by the Kuperberg bracket skein relations.

Since the skein algebras are defined as quotients of free modules which happen to admit natural algebra structures, it is difficult to study the algebra structures explicitly. In particular, it can be difficult to construct algebra morphisms whose domains are the skein algebras. Nevertheless, the SL_2 case of the Kauffman bracket skein algebras of surfaces is relatively well studied since the diagrams are built out of curves on surfaces and various geometric and combinatorial techniques have been developed to handle them.

In [3], Bonahon and Wong defined an algebra embedding of the Kauffman bracket skein algebra into quantum Teichmüller space. This map is called the *quantum trace map* and is viewed as a quantization of the classical trace map. The quantum Teichmüller space is a certain quantum torus, an algebra whose presentation is constructed from an ideal triangulation of a surface. Inspired by the computations involved with checking that the quantum trace map is well defined, Lê developed in [26] a triangular decomposition of the Kauffman bracket skein algebra by introducing the Kauffman bracket *stated* skein algebra. The stated skein algebra is a finer version of the regular skein algebra with extra relations along the boundary allowing one to define splitting morphisms, which are algebra maps associated to the splitting of a surface along an ideal arc. The extra boundary relations are consistent with the regular skein algebra relations, so no information is lost when passing to the stated skein algebra. The splitting morphisms are injective, so no information is lost by passing to the triangular decomposition. One can define an algebra map out of a skein algebra by defining maps on stated skein algebras of triangles and then precomposing with the triangular decomposition. Using this method, Lê was able to reconstruct the quantum trace map.

Lê and Costantino further developed the theory of the Kauffman bracket stated skein algebras of surfaces in [8]. There, they highlighted the connections between the stated skein algebra, the quantized ring of coordinate functions $\mathcal{O}_q(\mathrm{SL}_2)$, and the Reshetikhin–Turaev invariant of ribbon tangles. This perspective provides hints for how to construct a stated skein algebra for Lie groups other than SL_2 .

A triangular decomposition for the algebra of functions on G-character varieties of surfaces has been developed by Korinman in [21] for quite a general class of Lie groups G. Furthermore, Korinman and Quesney [22] and Costantino and Lê [8] showed that for the SL_2 case, the triangular decomposition of the stated skein algebra fits into an exact sequence which parallels the exact sequence associated to the triangular decomposition of the character variety. One may expect that skein algebras associated to other Lie groups G have analogous triangular decompositions lining up with those of the character variety.

The goal of this paper is to give an explicit description of a stated version of the SL_3 skein algebra, analogous to the Kauffman bracket skein algebra, and to study some of its properties. The current version of this paper is adapted from an earlier version as well as from material appearing in the author's thesis [17]. We now summarize our main results about the SL_3 stated skein algebra, $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$, of a punctured bordered surface Σ.

1.1. Main results

One of the most interesting properties of our description of the SL_3 stated skein algebra by explicit skein relations is that our skein relations are confluent.

Theorem 1.1 (Theorem 5.3). *The* SL_3 *stated skein algebra* $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ *is a free module with a canonical basis which is an extension of the Sikora–Westbury canonical basis of the* SL_3 *ordinary skein algebra of closed webs* [31]. *Furthermore, any element of the stated skein algebra can be written in the basis by repeatedly applying the reduction rules labeled* (I1a)–(I4b), (B1)–(B4), (C_k), *and* (S) *which may be found in Sections* 2 *and* 5 *of this paper.*

Analogous to the Kauffman bracket stated skein algebras, our SL_3 stated skein algebras admit splitting maps associated to cutting surfaces along ideal arcs. Our basis allows us to prove important properties of the splitting maps.

Theorem 1.2 (Theorems 8.1 and 8.2). *The splitting maps are injective and their images can be explicitly described. Each splitting map fits into an exact sequence of the form described in* [8, 21, 22].

To show the injectivity of the splitting map associated to an arbitrary ideal arc, we show that it suffices to prove the injectivity of the splitting map for an ideal arc that bounds a monogon. While this case is much simpler to consider, it is still difficult to prove. We use the list of reduction rules suggested by the SL_3 skein algebra relations as in [31] and expand the list to include relations along the boundary. Using these reduction rules, we apply the diamond lemma to find an explicit basis that helps us prove the injectivity of the splitting map for an ideal arc bounding a monogon. To generalize these results to other G, we desire similar reduction rules for their skein theories or else we hope to find a replacement for the role that the basis plays in this paper.

The splitting maps allow us to study skein algebras of punctured surfaces by studying stated skein algebras of building block surfaces: the monogon $\mathfrak{M}$, the bigon $\mathfrak{B}$, and the triangle $\mathfrak{T}$. We give explicit presentations for these stated skein algebras.

Theorem 1.3 (Proposition 6.1 and Theorems 9.3 and 10.1). *We have the following identifications:*

(i) $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{M}) \cong \mathcal{R}$, *the ground ring;*

(ii) $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \cong \mathcal{O}_q(\mathrm{SL}_3)$ *as Hopf algebras;*

(iii) $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}) \cong \mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$, *the braided tensor square of* $\mathcal{O}_q(\mathrm{SL}_3)$.

In the second half of the paper, we use these tools to obtain several applications. In our first application, we use the fact that $\mathcal{O}_q(\mathrm{SL}_3)$ has no zero divisors to establish the same result for SL_3 stated skein algebras.

Theorem 1.4 (Theorem 11.1). *Suppose that Σ is a punctured bordered surface with at least one puncture and that the ground ring $\mathcal{R}$ has no zero divisors. Then $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ has no zero divisors.*

In an important construction, [4] defines the Chebyshev–Frobenius homomorphism, which produces central elements in the SL_2 skein algebra at roots of unity by threading elements of the skein algebra through Chebyshev polynomials. The works [2, 22] extend this construction to the case of SL_2 stated skein algebras. We are interested in generalizing these constructions to the case of SL_3. We make progress in this direction by adapting the arguments of [22] to establish the existence of an SL_3 analogue of the Chebyshev–Frobenius homomorphism, using the Frobenius map for the quantum group $\mathcal{O}_q(\mathrm{SL}_3)$ given in [28].

Theorem 1.5 (Theorem 12.1). *Suppose that $\mathcal{R}$ has no zero divisors and $q^{1/3}$ is a root of unity of order N coprime to* 6. *If Σ has at least one puncture, then there exists an embedding*

$$\mathcal{S}_1^{\mathrm{SL}_3}(\Sigma) \hookrightarrow Z(\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma))$$

into the center of the stated skein algebra.

In [24], Kuperberg's SL_3 web category is shown to be equivalent to a full subcategory of $U_q(\mathfrak{sl}_3)$-mod. Theorem 1.3 from this introduction provides a diagrammatic definition of the quantum group $\mathcal{O}_q(\mathrm{SL}_3)$. We can then use Theorem 1.2 to give a new, diagrammatic proof of Kuperberg's result in the context of $\mathcal{O}_q(\mathrm{SL}_3)$-comodules for any coefficient ring $\mathcal{R}$ and any choice of invertible element $q \in \mathcal{R}$.

Theorem 1.6 (Theorem 13.1). *Kuperberg's* SL_3 *web category is equivalent to a full subcategory of* $\mathcal{O}_q(\mathrm{SL}_3)$-comod, *and the Reshetikhin–Turaev functor providing the equivalence can be described explicitly in terms of a splitting map for the bigon.*

1.2. Recent related works

The study of skein algebras for higher rank lie groups has been a popular topic recently. We would like to take note of a few works that are especially related to this paper.

A construction of the SL_3 quantum trace map has been developed by works of Douglas and of Kim. In [10], Douglas proposes a definition of the quantum trace map for links while Kim extends the definition to webs and the SL_3 skein algebra in [20] by making use of the SL_3 stated skein algebra defined in the present paper.

In [13], Frohman and Sikora develop and study a reduced skein algebra, which ends up sandwiched between the ordinary skein algebra and the stated skein algebra. The reduced skein algebra is coordinatized in [11–13] and its connections to cluster algebras are further studied in [18, 20]. It turns out that the basis of the reduced stated skein algebra of Frohman-Sikora can be viewed, in an appropriate way, as a subset of the basis for the stated skein algebra described in this paper. See [11, 27] for discussion of this fact.

The study of SL_n skein algebras for $n > 3$ is continued in the paper of Lê and Sikora [27], in which they develop a theory of SL_n stated skein algebras compatible with Sikora's definition of SL_n webs [30]. Some techniques in the present paper are adaptable to the higher rank case, and Lê and Sikora have been able to establish analogues of several of our main results. The method of confluence theory does not appear to be effective for the web relations in the higher rank case, and so it remains an important problem to describe a basis of webs for SL_n skein algebras when $n > 3$.

Skein algebras have been studied by some authors from a more general perspective of skein categories using methods of factorization homology, such as in [1, 7, 15]. In particular, [15] describe a so-called internal skein algebra by way of a coend construction. In [7], an excision property for skein categories is established. The work of [16] in the SL_2 case relates stated skein algebras to internal skein algebras and properties of splitting maps to the excision property. It is believed that the methods of [16] should extend to higher rank cases as well. In contrast to the general methods of factorization homology, in this paper we are focused on studying skein algebras from the perspective of explicit skein relations.

2. The SL_3 stated skein algebra

Definition 2.1. A *punctured bordered surface* is a pair $(\Sigma', \mathcal{P})$, where Σ' is a smooth compact oriented surface, possibly with boundary, and $\mathcal{P}$ is a collection of finitely many points of Σ'. We require that each boundary component of Σ' contains at least one point of $\mathcal{P}$. We do not require Σ' to be connected. We let $\Sigma = \Sigma' \setminus \mathcal{P}$. To simplify notation, we also refer to the pair $(\Sigma', \mathcal{P})$ simply by Σ. A *boundary arc* of Σ is a connected component of $\partial\Sigma$.

For a punctured bordered surface Σ, a *web* in $\Sigma \times (0, 1)$ is an embedding of a directed ribbon graph Γ such that each interior vertex of Γ in $\mathring{\Sigma} \times (0, 1)$ is a trivalent sink or a trivalent source. We allow Γ to have univalent vertices, called *endpoints*, contained in $\partial\Sigma \times (0, 1)$ such that for each boundary arc b of Σ the vertices contained in $b \times (0, 1)$ have distinct heights. We require the web to have a vertical framing with respect to the $(0, 1)$ component and we require that strands that terminate in a univalent vertex are transverse to $\partial\Sigma$.

We consider isotopies of webs in the class of webs. In particular, our isotopies must preserve the height order of boundary points of webs for each boundary arc of Σ.

For a web Γ, a *state* is a function $s \colon \partial\Gamma \to \{-, 0, +\}$. A *stated web* is a web together with a state. We will make use of the order $- < 0 < +$ on the set $\{-, 0, +\}$. For notational purposes, it will be convenient to sometimes add states together. By identifying the state $-$ with the integer -1 and the state $+$ with the integer 1, we partially define an addition on the set $\{-, 0, +\}$ whenever the answer is contained in the set as well.

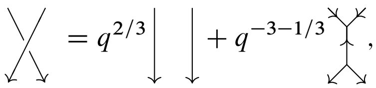
Definition 2.2. A web Γ in $\Sigma \times (0, 1)$ is in generic position if the projection $\pi \colon \Sigma \times (0, 1) \to \Sigma$ restricts to an embedding of Γ except for the possibility of transverse double points in the interior of Σ. Each web is isotopic to a web in generic position. A *stated diagram* D of a generic stated web Γ is the projection $\pi(\Gamma)$ along with the over/undercrossing information at each double point and the height orders and states of the boundary points of Γ. Web diagrams are isotopic if they are isotopic through an isotopy of the surface.

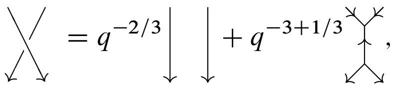
As in [26], it will be convenient for us to record the local height orders of the boundary points of a web diagram by drawing an arrow along a portion of the boundary arc of Σ.

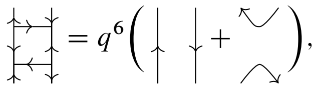
Let $\mathcal{R}$ be a unital commutative ring containing an invertible element $q^{1/3}$. The quantum integer $[n]$ denotes the Laurent polynomial defined by $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

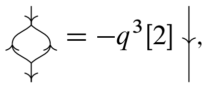
Definition 2.3. The SL_3 *stated skein algebra* $\mathcal{S}_q^{SL_3}(\Sigma)$ is the $\mathcal{R}$-module freely spanned by isotopy classes of webs in $\Sigma \times (0, 1)$ modulo the following relations.

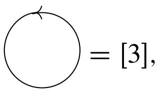
Interior relations:

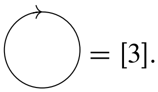

(I1a)


(I1b)

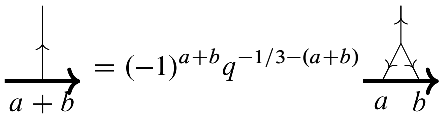

(I2)

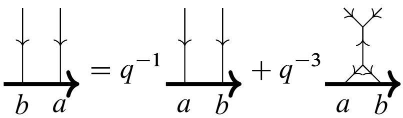

(I3)

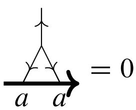

(I4a)

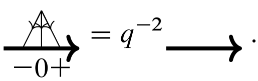

(I4b)

Boundary relations:


(for $b > a$), (B1)


(for $b > a$), (B2)


(for any $a \in \{-, 0, +\}$), (B3)

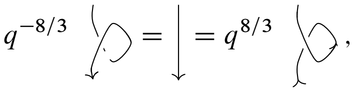

(B4)

The interior relations above hold for local diagrams contained in an embedded disk in Σ. The boundary relations hold for local diagrams in a neighborhood of a point of $\partial\Sigma$. The thicker line denotes a portion of a boundary arc while the thin lines belong to a web. The arrow along the boundary arc indicates the height order of that boundary arc. For example, in the diagram on the right side of relation (B1), the endpoint with the state b has a greater height than the endpoint with the state a.

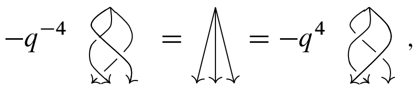
The module defined above admits a natural multiplication where the product $\Gamma_1\Gamma_2$ of two stated webs Γ_1, Γ_2 in $\Sigma \times (0,1)$ is given by isotoping Γ_1 so that it is contained in $\Sigma \times (1/2, 1)$, isotoping Γ_2 so that it is contained in $\Sigma \times (0, 1/2)$, and then taking the union of these two stated webs in $\Sigma \times (0,1)$. This gives $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ an associative, unital $\mathcal{R}$-algebra structure.

3. Consequences of the defining relations

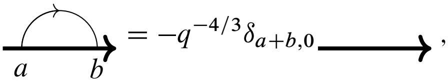
Proposition 3.1. *The following relations are consequences of the defining relations:*



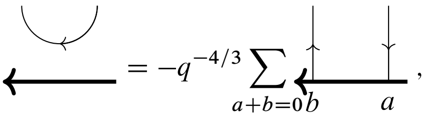
(a)



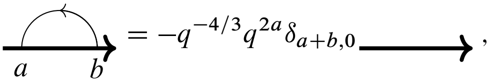
(b)



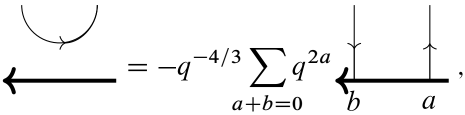
(c)



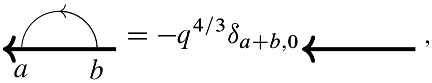
(d)



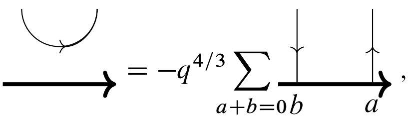
(e)



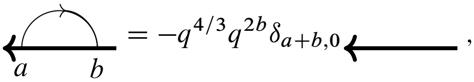
(f)



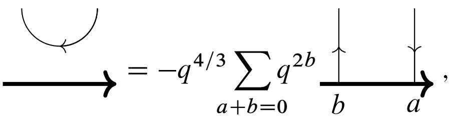
(g)



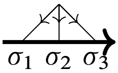
(h)



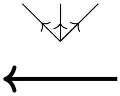
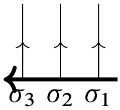
(i)



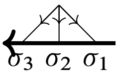
(j)



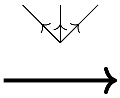
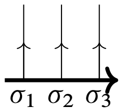
$$= \begin{cases} q^{-2}(-q)^{l(\sigma)} & \text{if } \sigma = (\sigma_1,\sigma_2,\sigma_3) \in S_3, \\ 0 & \text{if } (\sigma_1,\sigma_2,\sigma_3) \notin S_3 \end{cases} \quad \textit{(same for sinks)}, \tag{k}$$


$$= q^{-2}\sum_{\sigma\in S^3}(-q)^{l(\sigma)}$$


$$\textit{(same for sinks)}, \tag{l}$$



$$= \begin{cases} -q^{2}(-q)^{l(\sigma)} & \text{if } \sigma = (\sigma_1,\sigma_2,\sigma_3) \in S_3, \\ 0 & \text{if } (\sigma_1,\sigma_2,\sigma_3) \notin S_3 \end{cases} \quad \textit{(same for sinks)}, \tag{m}$$


$$= -q^{2}\sum_{\sigma\in S^3}(-q)^{l(\sigma)}$$


$$\textit{(same for sinks)}. \tag{n}$$

In the notation above, we consider the permutation $(-, 0, +)$ to be the identity permutation and $l(\sigma)$ denotes the length of the permutation σ.

Proof. Relations (a) and (b) follow from the defining interior relations.

The relations involving boundary orientations pointing to the right can be checked by reducing both sides according to the algorithm given by the diamond lemma described in Theorem 5.3.

The relations involving boundary orientations pointing to the left can be derived from those involving orientations pointing to the right by sliding the boundary points horizontally to reverse the height order and using the twisting relations (a) and (b). ■

4. The splitting morphism

As in [26], our stated skein algebras of punctured bordered surfaces satisfy a compatibility with the gluing and splitting of surfaces. If Σ is a punctured bordered surface and a and b are two boundary arcs of Σ, we can obtain a new punctured bordered surface $\overline{\Sigma} = \Sigma/(a = b)$ by gluing the arcs a and b together in the way compatible with the orientation of Σ. It is the reverse of this process that gives us an algebra morphism from $\mathcal{S}_q^{\mathrm{SL}_3}(\overline{\Sigma})$ to $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ associated with splitting the surface $\overline{\Sigma}$ along an ideal arc c.

Definition 4.1. If Σ is a punctured bordered surface, an *ideal arc* in Σ is a proper embedding $c\colon (0, 1) \to \mathring{\Sigma}$ such that its endpoints are (not necessarily distinct) points in the set of punctures, $\mathcal{P}$.

Let $p\colon \Sigma \to \Sigma/(a = b) =: \overline{\Sigma}$ be the projection map associated to the gluing. Then $c := p(a) = p(b)$ is an ideal arc. We will define a splitting morphism

$$\Delta_c\colon \mathcal{S}_q^{\mathrm{SL}_3}(\overline{\Sigma}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$$

by defining it on stated webs in $\overline{\Sigma} \times (0,1)$ and then checking that it is well defined on $\mathcal{S}_q^{\mathrm{SL}_3}(\overline{\Sigma})$.

For a stated web $[\Gamma, s]$ in $\overline{\Sigma} \times (0,1)$, we first isotope it so that Γ intersects $c \times (0,1)$ transversely in points of distinct heights. By defining p to act trivially on the $(0,1)$ factor, we can extend it to a map $p\colon \Sigma \times (0,1) \to \overline{\Sigma} \times (0,1)$. We then consider $p^{-1}(\Gamma)$, which is a web in $\Sigma \times (0,1)$. Except for the points of $p^{-1}(c \cap \Gamma)$, each boundary point of $p^{-1}(\Gamma)$ inherits a state from Γ.

We will say that s' is an *admissible state* for $p^{-1}(\Gamma)$ if $s'(p^{-1}(x)) = s(x)$ for all $x \in \partial\Gamma$ and if $y, z \in p^{-1}(\Gamma \cap c)$ then $s'(y) = s'(z)$.

We define the splitting morphism on a stated web $[\Gamma, s]$ in $\overline{\Sigma} \times (0,1)$ by

$$\Delta_c(\Gamma, s) = \sum_{\text{admissible } s'} [p^{-1}(\Gamma), s'].$$

Theorem 4.2. (a) *The map Δ_c described above extends linearly to a well-defined algebra morphism $\Delta_c\colon \mathcal{S}_q^{\mathrm{SL}_3}(\overline{\Sigma}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$.*

(b) *If a and b are two ideal arcs with disjoint interiors, then we have*

$$\Delta_a \circ \Delta_b = \Delta_b \circ \Delta_a.$$

As in [26], the map Δ_c is injective, but we will postpone a discussion of this fact until Section 8.

Proof. If Δ_c is well defined, then the fact that it is an algebra morphism and that it satisfies the property given in part (b) of Theorem 4.2 follows from the definition of the splitting morphism.

To check that it is well defined, we first check that the effect of passing cups, caps, vertices, and crossings past the ideal arc c commutes with the application of Δ_c. This will tell us that the splitting morphism is well defined with respect to isotopies of diagrams. Cups and caps can slide past the arc because of relations (c)–(j) from above. To slide a vertex past the arc, we can first rotate the vertex, using the fact that cups and caps can slide past the arc, until it appears as in relations (k)–(n). Since crossings can be rewritten as a linear combination of cups, caps and vertices, this allows us to pass a crossing past the arc.

If strands intersecting $c \times (0,1)$ are isotoped vertically, so as to alter their height order, then on a diagram this has the effect of a Reidemeister 2 move. Since crossings can slide past c, we can isotope the disk containing the Reidemeister 2 move on the diagram past c and then perform the move. This tells us that the splitting map is well defined on isotopy classes of webs.

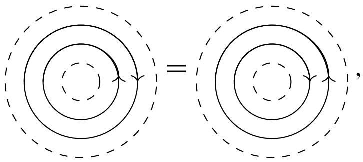
To check that the splitting morphism respects the defining relations of $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$, we observe that if c cuts through a disk or half disk appearing in one of the defining relations, we can isotope the diagram away from c first and then apply the relation. ∎

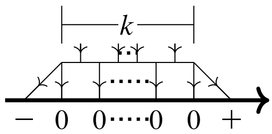
5. A basis for the stated skein algebra

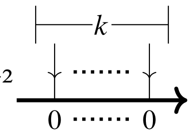
If a module is defined as a quotient of a free module by a list of relations, and if each relation can be interpreted as a reduction rule that permits the replacement of one element by a linear combination of simpler elements, then the module is a good candidate for an attempted application of the diamond lemma to produce a basis. As explained in [31], the diamond lemma can accommodate modules built out of diagrams on surfaces and it has been successful in producing bases for webs on surfaces for the cases of Kuperberg's webs of type A_1, A_2, B_2, and G_2. In [26], Lê organized the new boundary relations into reduction rules that are compatible with the reduction rules coming from the Kauffman bracket skein algebra and then applied the diamond lemma to find a basis. In this section, we will do the same for the SL_3 case.

We first summarize our goal. To apply the diamond lemma, we need to realize our skein module as a quotient of a free module by reduction rules that are terminal and locally confluent. The defining relations from Section 2 provide a starting point for a list of reduction rules. We will introduce a measure of complexity that allows us to say that the diagrams in the right side of each defining relation are simpler than the diagram on the left side. Using a reduction rule on a diagram D replaces that diagram with a linear combination of simpler diagrams. We call any linear combination of diagrams obtained by applying a sequence of reduction rules to D a *descendant* of D, and we call the diagrams appearing in the linear combination *descendant diagrams* of D. If there exists no infinite chain of descendant diagrams for D, then D can be written as a linear combination of irreducible diagrams by repeatedly applying reduction rules to the diagram and to its descendants. If no diagram admits an infinite chain of descendant diagrams, then the reduction rules are called *terminal* and this property implies that irreducible diagrams span our module. Sometimes, more than one reduction rule will apply to a diagram. If there is always a common descendant for any two ways of reducing a diagram, then the reduction rules are called *locally confluent*. If the set of reduction rules are terminal and locally confluent, then the set of irreducible diagrams forms a basis for our module, by [31, Theorem 2.3].

In anticipation of issues regarding local confluence, we need to introduce the following redundant relations:

 (S)



$= q^{3k-2}$


. (C_k)

Relation (S) allows one to switch two circles of opposite orientations whenever the two circles bound an annulus. We see from [31] that relation (S) will be necessary for our list of reduction rules to be confluent, as none of the left sides of the defining relations are applicable to the diagrams in (S) unless they happen to bound a disk. We borrow notation from [13] to say that two circles that bound an annulus on the surface and are oriented inconsistently with the boundary of the annulus form a *British highway*. For example, the two circles on the left side of the relation (S) form a British highway. The fact that we are using oriented surfaces allows us to declare the right side of (S) to be the more reduced side. The relation (S) will serve as a reduction rule that will decrease the number of British highways on any connected component that is not a torus. The torus provides an exception since parallel nontrivial circles will bound two distinct annuli. See the remark after Theorem 5.3 regarding this exception.

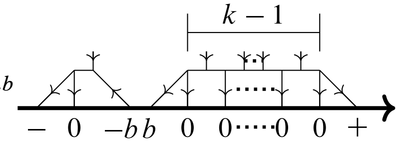
Proposition 5.1. (i) *The relations* (S) *hold in* $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ *for any annulus embedded in* Σ.

(ii) *The relations* (C_k) *hold in* $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ *for all* $k \geq 0$.

Proof. (i) (S) represents an isotopy of webs in the thickened surface $\Sigma \times (0, 1)$, so the relation holds in $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$.

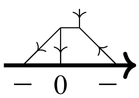
(ii) We will proceed by induction on k. (C_0) is the same as (B4), so the statement is true for $k = 0$.

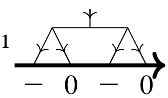
If $k > 0$ we can apply the relation (j) to the horizontal bar to the right of the top left strand to yield

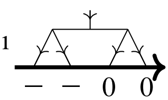
$$-q^{4/3} \sum_{b\in\{-,0,+\}} q^{-2b}$$


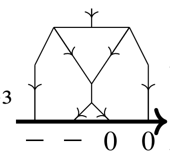
.

When $b = 0$ the right connected component of the diagram is zero by relation (B3). When $b = +$ we compute that the left portion of the diagram reduces to



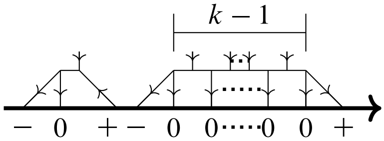
$= -q^{-1/3+1}$


$= -q^{-1/3+1}\Big(q^{-1}$


$+ q^{-3}$


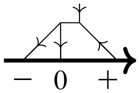
$\Big).$

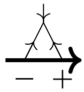
Both of the last terms reduce to 0 using (B3) after applying (I2) to the second diagram. When $b = -$ we are interested in computing

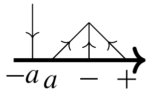
$$-q^{4/3}q^2$$ 

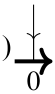
.

The right part of the diagram can be reduced by induction now while the left part of the diagram can be computed in the following manner:



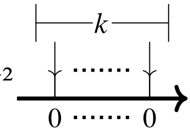
$$\overset{(B1)}{=} -q^{1/3-1}$$ 

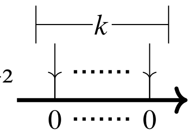
$$\overset{(h)}{=} -q^{1/3-1}(-q^{4/3}) \sum_{a\in\{-,0,+\}}$$ 

$$\overset{(k)}{=} -q^{1/3-1}(-q^{4/3})q^{-2}(-q)$$ 

.

This all reduces to

$$-q^{1/3-1}(-q^{4/3})q^{-2}(-q)(-q^{4/3}q^2)q^{3(k-1)-2}$$ 

$$= q^{3k-2}$$ 

,

which concludes the proof by induction. ∎

For the rest of this section, we will assume any boundary arcs in our diagram have an orientation that matches the one appearing in the pictures of the defining boundary relations and that this orientation dictates the height order.

A univalent endpoint of a web diagram is a *bad endpoint* if the strand attached to the endpoint is oriented out of the boundary. For example, the endpoint in the picture on the left of relation (B1) is a bad endpoint while the two endpoints on the right of the relation are good. We say that a pair of two good endpoints on the same boundary arc with states b and a are a *bad pair* if $b > a$ but the endpoint with state b is lower in the height order than the endpoint with state a. For example, the two endpoints on the left of (B2) form a bad pair, while the two endpoints in each diagram of the right side of the relation form a good pair. In the following, by the term *vertices* we mean only trivalent vertices of the web.

Definition 5.2. We define the *complexity* of a stated web diagram to be the tuple (#crossings, #bad endpoints, #bad pairs, #vertices, #connected components, #British highways) in $\mathbb{Z}^6_{\geq 0}$.

We use the lexicographic ordering on $\mathbb{Z}^6_{\geq 0}$ and note that each defining relation, each relation (C_k), and each relation (S) involve a single diagram on the left side of the equation while the right side of the equation contains only diagrams of strictly lower complexity than the one on the left side of the equation.

We say that a diagram contains a *reducible feature* if the left side of one of the relations (I1a)–(I4b), (B1)–(B4), (C_k), or (S) applies. If a diagram contains no reducible feature, we call such a diagram an *irreducible diagram*.

Theorem 5.3. *The set of isotopy classes of irreducible diagrams on Σ forms a basis for $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$.*

Remark 5.4. If Σ has a connected component that is a torus, we modify our notion of an irreducible diagram. By omitting the reduction rule (S) on any torus, the proof below can be modified to show that the remaining reduction rules will produce a basis consisting of the set of irreducible diagrams up to isotopy and circle flip moves (S) on any torus.

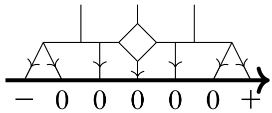
Proof. We will apply the diamond lemma in much the same setup as [26]. First, we claim that the module freely spanned by isotopy classes of web diagrams with our chosen boundary orientations modulo the defining relations along with (C_k) and (S) yields a module isomorphic to $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$. To do this, one observes that ribbon Reidemeister moves RI, RII, and RIII and the fact that a strand can pass over or under a vertex all follow from the defining interior relations, as shown in [23]. The fact that (C_k) and (S) are redundant relations completes this part of the argument.

Next, we must verify that given a diagram D, the process of iteratively applying the left sides of our relations to D and to its descendants always terminates in a linear combination of irreducible diagrams. This is guaranteed by the fact that our reduction rules involve replacing a diagram by a linear combination of diagrams of strictly lower complexity in our lexicographic ordering, as in [31, Theorem 2.2]. Thus, the set of isotopy classes of irreducible diagrams span $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$.

To show that each diagram can be uniquely written as a linear combination of irreducible diagrams, we must show the local confluence of our relations. This is the reason that we had to include the redundant relations (C_k) and (S). We must check that if more than one relation is applicable to a diagram then we can reach a common descendant regardless of which relation we choose to apply. We use the same notion of the *support* of a relation as [26]. If two relations are applicable to a diagram, but their support is disjoint, then the applications of these relations commute with each other, and thus immediately reach a common descendant.

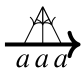
We must find local confluence for relations whose supports overlap nontrivially. If the two relations are both interior relations or (S), then we see by [31] that they are locally confluent.

There is one possible way for the support of an interior relation to intersect the support of a boundary relation: a square could be connected to the top of the relations (C_k) for some $k \geq 2$. The following diagram shows an example of an overlap of (C_4) and (I2):


.

Such a situation will terminate at 0 no matter which relation (C_k) or (I2) is applied first, as each resulting diagram will provide an opportunity to apply (B3).

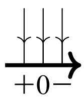
Finally, we consider the cases of overlapping supports of the defining boundary relations and the additional relations (C_k). A first easy case is an overlap of (B3) with (B3), which must be of the following form:


$(a \in \{-, 0, +\})$.

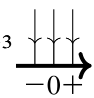
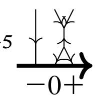
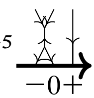
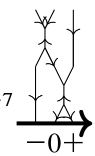
Applying (B3) to either the left triangle or the right triangle in the above diagram yields zero.

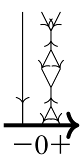
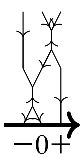
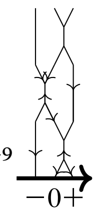
We see that the only other supports that can overlap are those of (B2) with any of (B2)–(B4), and (C_k).

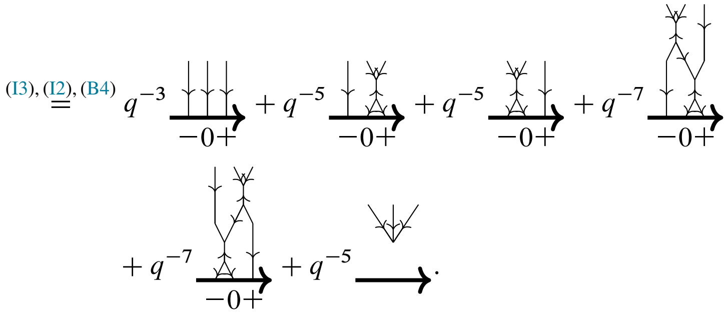
Relations (B2) *and* (B2). If (B2) overlaps with (B2), the overlap must be of the following form:


.

If we first apply (B2) to the right two endpoints, and then we continue to apply (B2) until there are no longer any bad pairs we obtain

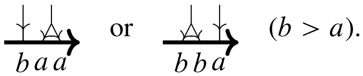
q^{-3}  $+\,2q^{-5}$  $+\,q^{-5}$  $+\,q^{-7}$ 

$+\,q^{-7}$  $+\,q^{-7}$  $+\,q^{-9}$ 

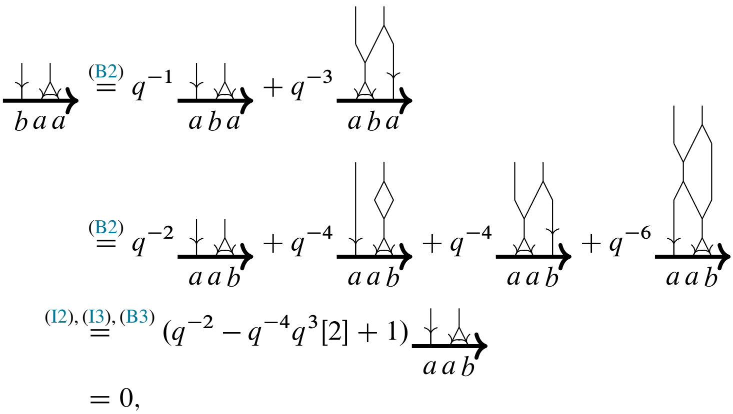


If, instead, we first apply (B2) to the left two endpoints, and then we continue to apply (B2) until there are no longer any bad pairs, we obtain the same linear combination but with the diagrams reflected in a vertical line (but with the state locations and boundary orientation remaining the same). By noting the coefficients in our last equation are symmetric with respect to this reflection, we see that we obtain the same answer in both cases.

Relations (B2) *and* (B3). If (B2) overlaps with (B3), the overlap must take one of the following forms:



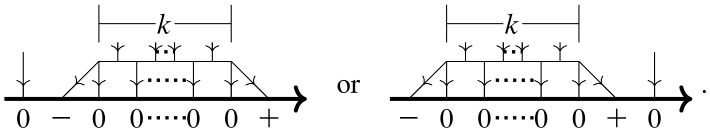
Both cases are handled symmetrically, so we will focus on the left case. If we apply (B3) first, we obtain zero. So, we must show that if we instead apply (B2) first we eventually obtain zero. We do this by computing



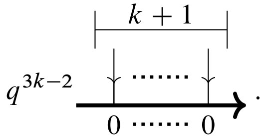
resolving this case.

Since (B4) is the same as (C_0) the last overlap we need to check is an overlap between (B2) and (C_k) for any $k \geq 0$.

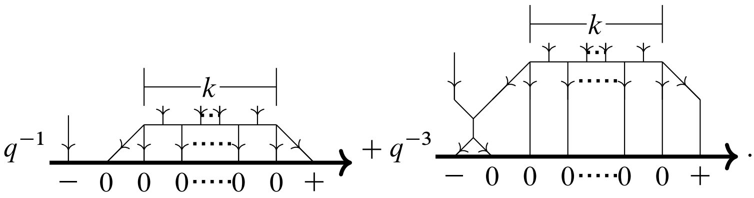
Relations (B2) *and* (C_k). There are four cases for such an overlap. Consider first the following two cases:



These two cases are handled symmetrically, so we will focus on the left case. If we apply (C_k) first, we obtain

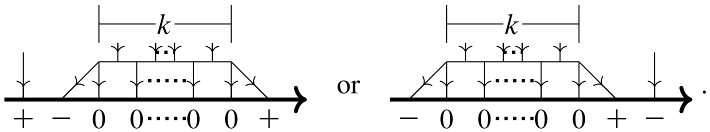


If we apply (B2) first, we obtain



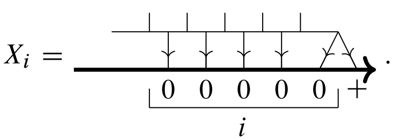
The first term in this linear combination becomes zero after applying (B3). The diagram in the second term is isotopic to the diagram appearing on the left side of (C_{k+1}). After application of (C_{k+1}) we obtain confluence in this case.

The other two possible overlaps between (B2) and (C_k) are of the following forms:



Since these two cases are handled symmetrically, we will focus on the left case.

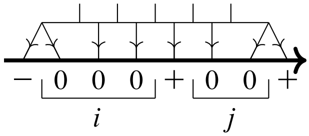
We introduce some notation to simplify this computation. We will use symbols placed next to each other to represent certain diagrams appearing next to each other. We represent the diagrams in the left case above by $\downarrow_+ \cdot C_k$. We denote by 0_i the diagram involving i parallel strands that terminate in good endpoints with states labeled 0. We also denote by X_i the diagram



By applying the relation (C_k) to $\downarrow_+ \cdot C_k$ we obtain $q^{3k-2} \downarrow_+ \cdot 0_k$. Consider the effect of using relation (B2) on $q^{3k-2} \downarrow_+ \cdot 0_k$ to get rid of bad pairs, using (B3) at each opportunity. The reduced result is of the form

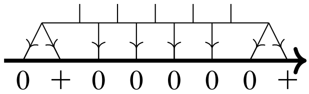
$$q^{3k-2} \sum_{l=0}^{k} q^{-l} q^{-3(k-l)} 0_l \cdot X_{k-l} = \sum_{l=0}^{k} q^{2l-2} 0_l \cdot X_{k-l}.$$

We now check that we reach the same reduced result if we instead apply (B2) first to $\downarrow_+ \cdot C_k$. We introduce another piece of notation. The diagram $A_{i,j}$ has i 0-states on the left of the $+$-state and j 0-states on the right:

$A_{i,j} =$ 

.

We also note that diagrams of the form



$= 0$

are zero, as can be shown by induction on the number of zero states appearing between the two + states. The inductive hypothesis can be applied after applying (B2) once to improve the order of the states and then applying (I2) to remove the square that forms.

If we apply relation (B2) to $\downarrow_+ \cdot C_k$, then one of the resulting terms will become zero as it is of the form above. We are then left with

$$\downarrow_+ \cdot C_k = q^{-3} A_{0,k+1}.$$

Now, consider the diagram $A_{l,m}$ for some $l, m \geq 0$. We have $A_{l,0} = 0$ by relation (B3). For $m > 0$ we can apply relation (B2) followed by (I2) and, ignoring the term with the zero diagram as above, we see that

$$A_{l,m} = q^{-1} A_{l+1,m-1} + q^3 C_l \cdot X_{m-1}.$$

A repeated application of this equation yields

$$\begin{aligned} q^{-3} A_{0,k+1} &= q^{-3} q^3 \sum_{i=0}^{k} q^{-i} C_i \cdot X_{k-i} \\ &\overset{(C_i)}{=} \sum_{i=0}^{k} q^{-i} q^{3i-2} 0_i \cdot X_{k-i} \\ &= \sum_{i=0}^{k} q^{2i-2} 0_i \cdot X_{k-i}. \end{aligned}$$

Thus, we have reached local confluence in this last case. The diamond lemma now gives us the result. ■

We define the *interior skein algebra* $\mathring{\mathcal{S}}_q^{\mathrm{SL}_3}(\Sigma)$ as the module freely spanned by closed webs contained in the interior of Σ modulo the interior relations (I1a)–(I4b) only.

Corollary 5.5. There is an algebra embedding

$$\mathring{\mathcal{S}}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$$

induced by the inclusion map on diagrams.

Proof. Using the reduction rules (I1a)–(I4b) and (S), the diamond lemma applies to give a basis for $\mathring{\mathcal{S}}_q^{\mathrm{SL}_3}(\Sigma)$. This set of basis diagrams is a subset of basis diagrams of $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$, thus the inclusion induces an injective map. ■

6. Bialgebra and comodule structure associated to the bigon

The surface made by removing one point from the boundary of a closed disk is called the monogon and will be denoted $\mathfrak{M}$. The surface obtained by removing two points from the boundary of a closed disk is called the bigon and will be denoted $\mathfrak{B}$. (See Figure 1.)

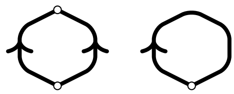


Figure 1. Bigon $\mathfrak{B}$ on the left and monogon $\mathfrak{M}$ on the right.

Proposition 6.1. $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{M}) \cong \mathcal{R}$.

Proof. We show that $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{M})$ is spanned by the empty diagram. The fact that the empty diagram is nonzero follows from the fact that it is irreducible and is thus a basis element.

Consider a web diagram W in $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{M})$. We can use relations (I1a) and (I1b) to inductively write W as a linear combination of crossingless diagrams. We can use relations (l) or (m) to get rid of vertices near the boundary. If there are strands between a vertex and the boundary we can apply relations (d) or (f) to create room for the vertex to slide over to the boundary without introducing crossings.

So, by induction we can write W as a linear combination of diagrams with no crossing and no vertex. After applying relations (I4a) and (I4b) to get rid of circles, these diagrams only have arcs connected to the single boundary arc. By applying relations (g) and (i), these diagrams become scalar multiples of the empty diagram. ∎

We recall that in [23], Kuperberg used an Euler characteristic argument to show that the module spanned by closed webs in the plane is 1-dimensional. We remark that by Proposition 6.1 along with the corollary to the construction of the basis, we obtain an alternate proof that Kuperberg's relations are enough to reduce any closed web in the plane to a scalar multiple of the empty web, and that this reduction can be performed algorithmically by iteratively applying the left sides of the interior relations. We also observe that Proposition 6.1 and the algorithm produced by the diamond lemma imply that any stated web in $\mathfrak{M}$ can be reduced to a scalar multiple of the empty diagram by iteratively applying just the left sides of the defining relations and (C_k).

We next describe the bialgebra structure of $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$. For a counit, we will construct an algebra morphism $\varepsilon\colon \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{M}) \cong \mathcal{R}$. As in [26] we will use an edge inversion map.

Definition 6.2. If b is a boundary arc of Σ with the orientation given in the defining relations of $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$, we define the inversion along b, $\mathrm{inv}_b\colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ to be the $\mathcal{R}$-module homomorphism defined on web diagrams by reversing the height order of b, switching the states to their negatives, and multiplying by scalars $C_s^{\uparrow}$ and $C_s^{\downarrow}$ for each endpoint on b. Here, we use $C_s^{\downarrow} = -q^{-4/3}$ for each good endpoint on b with any state s and we use $C_t^{\uparrow} = -q^{-4/3}q^{-2t}$ for each bad endpoint on b with a state $t \in \{-, 0, +\}$.

Proposition 6.3. *The map* inv_b *defined above is a well-defined* $\mathcal{R}$*-module automorphism.*

Proof. We must check that the map respects the defining boundary relations. To do so, we apply the map to both sides of a boundary relation and then reduce the results using the diamond lemma algorithm to see that we obtain the same answers in each case. Thus, the map is well defined. Alternatively, it is easier to use the relations in Section 3 to check that inv_b respects the relations (c), (e), (h), (j), (k), and (n). We then observe that these relations imply relations (B1)–(B4). To check that it is an automorphism, one needs to check that the obvious candidate for its inverse is well defined in the same way. ∎

We define $\varepsilon\colon \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{M})$ to be the map given by the result of inverting the right boundary arc e_r of the bigon with inv_{e_r} and then filling in the puncture. The map is well defined since it is a composition of well-defined maps. The fact that it is an algebra morphism is an easy diagrammatic observation, and can be seen in the same way as in [8].

The comultiplication $\Delta\colon \mathscr{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathscr{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \otimes \mathscr{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ is given by the splitting morphism Δ_c for an ideal arc c that travels from the bottom puncture to the top puncture. By Theorem 4.2, Δ is an algebra morphism and satisfies the coassociativity property.

To check that ε satisfies the counit property, we only need to check on generators. To find a nice set of generators, we use the method in the proof of Proposition 6.1 to see that any web in the bigon can be written as a linear combination of webs which have no crossing, no vertex, and no circle. Any trivial arc that starts and ends on the same boundary arc can be replaced by a scalar, and we are left with a linear combination of webs containing only parallel and antiparallel strands with one endpoint on each boundary arc. Thus, $\mathscr{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ has a generating set consisting of diagrams, each of which contains a single strand traveling from one boundary arc of the diagram to the other. We denote such diagrams α_{st} and β_{st} depending on the strand orientation and states. (See Figure 2.)

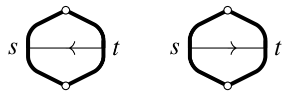
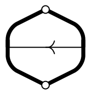
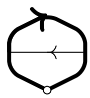


Figure 2. Generator α_{st} on the left and generator β_{st} on the right.

We use our diagrammatic definition of ε to compute that

$$\varepsilon(\alpha_{st}) = \varepsilon\Big(s$$

$$t \Big) = -q^{-4/3}q^{-2t}\Big(s$$

$$-t \Big)$$

$$\overset{\text{(i)}}{=} -q^{-4/3}q^{-2t}(-q^{4/3}q^{2t}\delta_{s-t,0}) = \delta_{st}.$$

We similarly compute that $\varepsilon(\beta_{st}) = \delta_{st}$.

By the definition of Δ, we compute that

$$\Delta(\alpha_{st}) = \sum_{l\in\{-,0,+\}} \alpha_{sl} \otimes \alpha_{lt}.$$

Similarly,

$$\Delta(\beta_{st}) = \sum_{l\in\{-,0,+\}} \beta_{sl} \otimes \beta_{lt}.$$

These equations allow us to verify that

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(\alpha_{st}) = \alpha_{st} = (\mathrm{id} \otimes \varepsilon) \circ \Delta(\alpha_{st}).$$

The same equations hold for β_{st} and we have proven the following proposition.

Proposition 6.4. *The algebra $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ has a natural bialgebra structure given by the maps Δ and ε defined above.*

The ingredients here are now the same as in [8] and so we obtain the following analogue.

Proposition 6.5. *Suppose b is a boundary arc of Σ. The map defined by splitting Σ along an ideal arc isotopic to b so as to split off a bigon $\mathfrak{B}$ whose right edge is b gives an $\mathcal{R}$-algebra homomorphism*

$$\Delta_b \colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}).$$

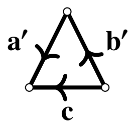
This endows $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ with a right comodule-algebra structure over $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$. Similarly, the map ${}_b\Delta$ defined by splitting off from Σ a bigon $\mathfrak{B}$ whose left edge is b gives an $\mathcal{R}$-algebra homomorphism

$${}_b\Delta \colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma).$$

This endows $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ with a left comodule-algebra structure over $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$.

7. Gluing or cutting along a triangle

Consider a punctured bordered surface Σ with two distinct boundary arcs a and b. Also consider an ideal triangle $\mathfrak{T}$, which is a disk with three points removed from its boundary. We will denote the punctured bordered surface $\Sigma \# \mathfrak{T}$ obtained by gluing Σ to $\mathfrak{T}$ along a and b. We label the edges of $\mathfrak{T}$ as in the following diagram:

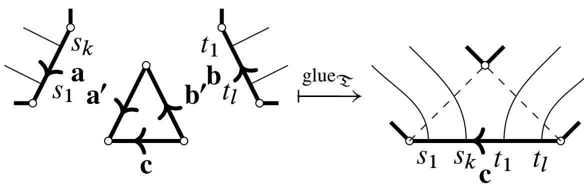


.

There is a well-defined $\mathcal{R}$-module homomorphism

$$\mathrm{glue}_{\mathfrak{T}} \colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma \# \mathfrak{T})$$

defined on diagrams by continuing the strands with endpoints on a or b until they reach c. The map is depicted in the following diagram:



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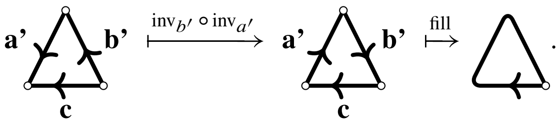
The map $\mathrm{glue}_{\mathfrak{T}}$ was introduced in [8] for the SL_2 case. In general, $\mathrm{glue}_{\mathfrak{T}}$ does not respect the algebra structure, but it gives rise to an algebra structure that is called a *self braided tensor product* in [8]. In Section 10 of this paper, we describe a special case of this structure, called the *braided tensor product*. In this section, we are interested in $\mathrm{glue}_{\mathfrak{T}}$ because it is an $\mathcal{R}$-linear isomorphism. We will show this by constructing a natural inverse.

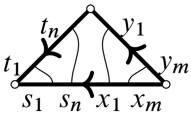
The triangle $\mathfrak{T}$ admits an analogue of the bigon's counit. We define

$$\varepsilon_{\mathfrak{T}} \colon \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{M})$$

as the map obtained by applying $\mathrm{inv}_{b'} \circ \mathrm{inv}_{a'}$ and then filling in the punctures between c and a' and between a' and b' as in the following figure:



Since $\varepsilon_{\mathfrak{T}}$ is defined as a composition of well-defined $\mathcal{R}$-linear maps, it is an $\mathcal{R}$-linear map. What makes $\varepsilon_{\mathfrak{T}}$ an analogue of ε is that if $\varepsilon_{\mathfrak{T}}$ is applied to a diagram W of the form

$W =$ 

(with any choice of strand orientations), then the result is

$$\varepsilon_{\mathfrak{T}}(W) = \Big(\prod_{i=1}^{n} \delta_{s_i,t_i}\Big)\Big(\prod_{j=1}^{m} \delta_{x_j,y_j}\Big).$$

We next define an $\mathcal{R}$-linear map

$$\mathrm{cut}_{\mathfrak{T}} \colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma \# \mathfrak{T}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma).$$

Recall the notation of the projection $p \colon \Sigma \sqcup \mathfrak{T} \to \Sigma \# \mathfrak{T}$ associated to gluing Σ to the triangle along a and b. If $a'' = p(a') = p(a)$ and $b'' = p(b') = p(b)$, we define $\mathrm{cut}_{\mathfrak{T}}$ by

$$\mathrm{cut}_{\mathfrak{T}} = (\varepsilon_{\mathfrak{T}} \otimes \mathrm{id}) \circ (\Delta_{b''} \circ \Delta_{a''}).$$

Since $(\Delta_{b''} \circ \Delta_{a''})$ cuts out a triangle, we view it as a linear map

$$\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma \# \mathfrak{T}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma),$$

so the composition above makes sense.

Proposition 7.1. *The $\mathcal{R}$-linear maps* $\mathrm{glue}_{\mathfrak{T}}$ *and* $\mathrm{cut}_{\mathfrak{T}}$ *satisfy*

$$\mathrm{cut}_{\mathfrak{T}} \circ \mathrm{glue}_{\mathfrak{T}} = \mathrm{id}_{\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)}$$

and

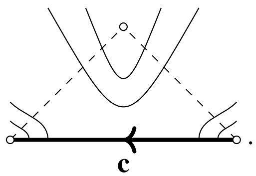
$$\mathrm{glue}_{\mathfrak{T}} \circ \mathrm{cut}_{\mathfrak{T}} = \mathrm{id}_{\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma\#\mathfrak{T})}.$$

Proof. We will check each equality on a spanning set for the skein algebra involved. For the case of $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$, we consider the spanning set consisting of all stated web diagrams. Suppose D is a stated web diagram on Σ. If we examine the diagrams that appear in the triangle cut out by $(\Delta_{b''} \circ \Delta_{a''}) \circ \mathrm{glue}_{\mathfrak{T}}(D)$, we see that they are all of the form W above. Thus, the computation for $\varepsilon_{\mathfrak{T}}(W)$ above shows that

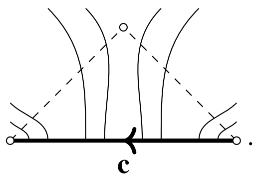
$$(\varepsilon_{\mathfrak{T}} \otimes \mathrm{id})(\Delta_{b''} \circ \Delta_{a''}) \circ \mathrm{glue}_{\mathfrak{T}}(D) = D.$$

This proves the first equality of Proposition 7.1.

For the second equality, we wish to use a smaller spanning set of $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma\#\mathfrak{T})$. Consider a stated web diagram D on $(\Sigma\#\mathfrak{T})$ and examine it in a neighborhood of $p(\mathfrak{T})$. By applying an isotopy, we can guarantee that $p(\mathfrak{T})$ contains only arcs, and that any arc that enters the triangle through one of the sides either leaves through the other side or terminates at an endpoint on c. After such an isotopy, we obtain a diagram of the following form (for some choice of strand orientations):



Using relations (f) and (j), we can break up the strands that pass through both a'' and b'' and thus write our diagram D as a linear combination of diagrams of the following form:



So, a spanning set consists of diagrams on $\Sigma\#\mathfrak{T}$ that are of the above form in a neighborhood of $p(\mathfrak{T})$. Let E be such a diagram. We see that the triangles that appear in the terms of $(\Delta_{b''} \circ \Delta_{a''})(E)$ are all of the form W above. Again, the computation of $\varepsilon_{\mathfrak{T}}(W)$ above allows us to see that

$$\mathrm{glue}_{\mathfrak{T}} \circ (\varepsilon_{\mathfrak{T}} \otimes \mathrm{id}) \circ (\Delta_{b''} \circ \Delta_{a''})(E) = E.$$

This proves the second equality of Proposition 7.1. ∎

Corollary 7.2. Suppose c is a boundary arc of a punctured bordered surface $\bar{\bar{\Sigma}}$ and that a'' and b'' are ideal arcs with disjoint interiors such that $a'' \cup b'' \cup c$ bounds an ideal triangle. Then both $\Delta_{a''}$ and $\Delta_{b''}$ are injective.

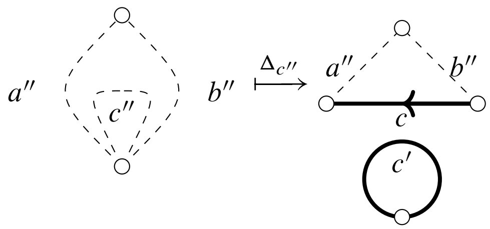
Proof. Let $\mathfrak{T}$ be the ideal triangle that is split off from $\bar{\bar{\Sigma}}$ if $\Delta_{b''} \circ \Delta_{a''}$ is applied. Then $\bar{\bar{\Sigma}} = \Sigma \# \mathfrak{T}$ for the punctured bordered surface Σ containing two distinct boundary arcs a and b resulting from the splitting maps. Proposition 7.1 tells us that $\mathrm{cut}_{\mathfrak{T}}$ is injective. By the definition of $\mathrm{cut}_{\mathfrak{T}}$, we see that $\Delta_{b''} \circ \Delta_{a''}$ is injective. Thus, $\Delta_{a''}$ is injective. By Theorem 4.2, we see that $\Delta_{b''} \circ \Delta_{a''} = \Delta_{a''} \circ \Delta_{b''}$. Thus, $\Delta_{b''}$ is injective as well. ∎

8. The triangular decomposition

We are now able to prove the following addendum to Theorem 4.2.

Theorem 8.1. *Suppose $\bar{\bar{\Sigma}}$ is a punctured bordered surface and a'' is an ideal arc on $\bar{\bar{\Sigma}}$. Then the map $\Delta_{a''}$ is injective.*

Proof. Let b'' be an ideal arc isotopic to a'' so that the ideal arcs have disjoint interiors and bound a bigon. Let c'' be an ideal arc that bounds a monogon whose ideal vertex is an endpoint of a'', and such that a'', b'', c'' have disjoint interiors and $a'' \cup b'' \cup c''$ bounds an ideal triangle. The following diagram depicts the map $\Delta_{c''}$:



.

Consider the application of $\Delta_{c''}$ to the set of basis diagrams described in Theorem 5.3. Each irreducible diagram D can be isotoped so that it does not intersect the monogon bounded by c''. This allows us to observe that $\Delta_{c''}(D)$ is an irreducible diagram on its surface as well, and that the isotopy class of D can be completely determined by the isotopy class of this irreducible representative of $\Delta_{c''}(D)$. Thus, $\Delta_{c''}$ maps a basis to a linearly independent set and we conclude that $\Delta_{c''}$ is injective.

After splitting off the monogon bounded by c'', we are left with a surface Σ that contains a boundary arc c such that $p(c) = c''$. Now, the ideal arcs a'', b'' and the boundary arc c satisfy the hypothesis of Corollary 7.2. By the corollary, $\Delta_{a''}$ is injective on the image of $\Delta_{c''}$ and thus $\Delta_{a''} \circ \Delta_{c''}$ is an injective map. The fact that these maps commute implies that $\Delta_{a''}$ is injective on $\mathcal{S}_q^{\mathrm{SL}_3}(\bar{\bar{\Sigma}})$ as well. ∎

Now, that we have determined the splitting morphisms have trivial kernels, we discuss their images.

Suppose Σ is a punctured bordered surface with distinct boundary arcs a and b. Let $\overline{\Sigma} = \Sigma/(a = b)$ and denote by c the common image of a and b under the gluing map. Recall the comodule structure maps associated to the boundary arcs a, and b. We will be interested in $\Delta_a\colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ and $\tau \circ_b \Delta\colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$, where τ only transposes the tensor factors. We are interested in the following result.

Theorem 8.2. *Let $\overline{\Sigma} = \Sigma/(a = b)$ and denote by c the common image of a and b under the gluing map. Then we have*

$$\operatorname{im}(\Delta_c) = \ker(\Delta_a - \tau \circ_b \Delta).$$

Proof. The inclusion $\operatorname{im}(\Delta_c) \subseteq \ker(\Delta_a - \tau \circ_b \Delta)$ follows by coassociativity of splitting $\overline{\Sigma}$ along c and an ideal arc isotopic to c.

To prove the other inclusion, we assume that $y \in \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ satisfies

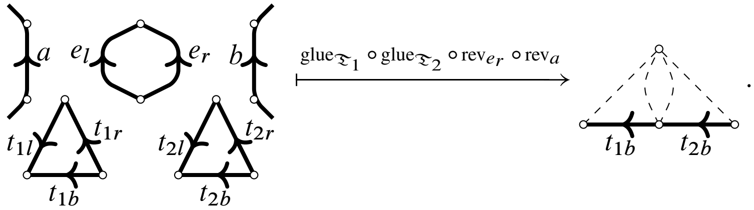
$$\Delta_a(y) = \tau \circ_b \Delta(y).$$

Our goal is to find some $x \in \mathcal{S}_q^{\mathrm{SL}_3}(\overline{\Sigma})$ such that $y = \Delta_c(x)$. The element y is represented by a linear combination of stated web diagrams on Σ. We will find a candidate for x by trying to weld the strands with endpoints on a or on b to each other. This process uses a map similar to the edge inversion maps inv before, but this time with a different choice of scalars associated to the endpoints.

For a boundary arc e with positive orientation, we define the edge reversal map rev_e to be the $\mathcal{R}$-linear automorphism of the stated skein module that reverses the height order on e, flips the states to their negatives and multiplies by the following scalars for each endpoint on e: $\overset{\downarrow}{_s}C = -q^{-4/3}q^{2s}$ for good endpoints with a state s and $\overset{\uparrow}{_s}C = -q^{-4/3}$ for bad endpoints with a state s. We can check that this map is well defined and an automorphism in the same way that we checked this for inv_e .

Let $z = \Delta_a(y) = \tau \circ_b \Delta(y)$. Denote the left boundary arc of the bigon of $\Sigma \sqcup \mathfrak{B}$ by e_l and the right arc by e_r. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be two triangles. We will use the gluing maps $\mathrm{glue}_{\mathfrak{T}}$ defined in Section 7. Denote the left, right, and bottom edges of the triangles t_{1l}, t_{2l}, t_{1r}, t_{2r}, and t_{1b}, t_{2b}, respectively. We will consider the result of reversing the arc a, reversing the arc e_r, then gluing to the triangles. To glue to $\mathfrak{T}_2$ we glue b to t_{2r} and glue e_r to t_{2l}. To glue to $\mathfrak{T}_1$, we glue e_l to t_{1r} and glue a to t_{1l}.

We can write the new element as $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$. This gluing is depicted in the following diagram:



First, we view z as $z = \tau \circ_b \Delta(y)$. Write y as a linear combination of diagrams D_i. For each i, $\tau \circ_b \Delta(D_i)$ is a linear combination of diagrams D_{ij}. Each D_{ij} has k_i endpoints on e_r, k_i endpoints on b, and the states of corresponding endpoints match. After applying rev_{e_r} to D_{ij} and then gluing to $\mathfrak{T}$, we see that there are $2k_i$ endpoints on t_{2b}, and that the endpoints which are k_i-th and $k_i + 1$-st in the height order have opposite states and opposite orientations. The scalars associated with the application of rev_{e_r} guarantee that relations (d) or (f) are applicable and allow us to reduce the number of endpoints on t_{2b}. After applying these relations k_i times for each D_i, we see that we can write $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ as a linear combination of diagrams, where no diagram has an endpoint on t_{2b}. As no reduction rule from our diamond lemma algorithm can result in an endpoint appearing on a boundary arc that previously contained no endpoints, we see that when we write $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ as a linear combination of our basis diagrams, each basis diagram that appears in the linear combination has no endpoints on t_{2b}.

Next, we view z as $z = \Delta_a(y)$. In a similar way as the last paragraph, we see that after applying rev_a and gluing to $\mathfrak{T}$, we can apply relations (d) and (f) to write $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ as a linear combination of basis diagrams such that no diagram has an endpoint on t_{1b}. By the uniqueness of this linear combination we see that we can write it as a linear combination of basis diagrams so that no diagram appearing in the linear combination has an endpoint on t_{1b} or on t_{2b}.

Now, $\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ is a linear combination of basis diagrams on $(\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2$. Consider the surface $(\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2 \setminus (t_{1b} \cup t_{2b})$. This is not a punctured bordered surface, but depending on whether the appropriate endpoint of c was a boundary puncture or was an interior puncture, this surface is either a punctured bordered surface missing an interval on its boundary or it is a punctured bordered surface missing a boundary circle. In either case, it is naturally diffeomorphic to the original punctured bordered surface $\overline{\Sigma}$ by replacing this missing boundary interval or boundary circle with a single puncture. There is a linear map defined on the submodule of $\mathcal{S}_q^{\mathrm{SL}_3}((\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2)$ spanned by basis diagrams that have no endpoints on t_{1b} or t_{2b} that takes such a basis diagram and embeds it in $(\Sigma \sqcup \mathfrak{B})\#\mathfrak{T}_1\#\mathfrak{T}_2 \setminus (t_{1b} \cup t_{2b})$. After applying this map to the linear combination

$\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a(z)$ and composing with our diffeomorphism, we obtain our candidate $x \in \mathcal{S}_q^{\mathrm{SL}_3}(\bar{\Sigma})$.

To see that x is the correct choice, we consider $\Delta_c(x)$ and then apply the same process to it as we did to y and observe that

$$\mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a \circ \Delta_a(y) = \mathrm{glue}_{\mathfrak{T}_1} \circ \mathrm{glue}_{\mathfrak{T}_2} \circ \mathrm{rev}_{e_r} \circ \mathrm{rev}_a \circ \Delta_a(\Delta_c(x)).$$

The injectivity of the maps involved allow us to conclude that $\Delta_c(x) = y$. ∎

We say that a punctured bordered surface is *ideal triangulable* if it can be obtained from a finite collection of disjoint triangles by gluing some pairs of edges together. It is known that a punctured bordered surface is ideal triangulable if it has no connected component that is one of the following surfaces: a closed surface, a sphere with fewer than three punctures, a bigon, or a monogon.

If Σ is an ideal triangulable punctured bordered surface, then the images of the glued edges are ideal arcs on Σ with disjoint interiors. These form the set of interior edges $\mathcal{E}$ for the *ideal triangulation* of Σ. Let

$$p: \bigsqcup_{i=1}^{n} \mathfrak{T}_i \to \Sigma$$

be the gluing map. If $e \in \mathcal{E}$, then its preimage $p^{-1}(e) = \{e', e''\}$ consists of two triangle edges. The composition Δ of the splitting maps Δ_e for $e \in \mathcal{E}$ gives an algebra embedding

$$\Delta: \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \bigotimes_{i=1}^{n} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i).$$

The composition ${}^{L}\Delta$ of all left comodule maps ${}_{e''}\Delta$ gives a map

$${}^{L}\Delta: \bigotimes_{i=1}^{n} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i) \to \Big(\bigotimes_{e\in\mathcal{E}} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})\Big) \otimes \Big(\bigotimes_{i=1}^{n} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i)\Big).$$

The composition Δ^R of all right comodule maps $\Delta_{e'}$ gives a map

$$\Delta^R: \bigotimes_{i=1}^{n} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i) \to \Big(\bigotimes_{i=1}^{n} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i)\Big) \otimes \Big(\bigotimes_{e\in\mathcal{E}} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})\Big).$$

Then Theorem 8.1 and Theorem 8.2 allow us to observe the following corollary.

Corollary 8.3. If Σ admits an ideal triangulation with a set of interior edges $\mathcal{E}$, then the following sequence of $\mathcal{R}$-modules is exact:

$$0 \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \xrightarrow{\Delta} \bigotimes_{i=1}^{n} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i) \xrightarrow{\Delta^R - \tau \circ {}^{L}\Delta} \Big(\bigotimes_{i=1}^{n} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i)\Big) \otimes \Big(\bigotimes_{e\in\mathcal{E}} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})\Big).$$

9. The stated skein algebra of the bigon

In [8], it was shown that the Kauffman bracket stated skein algebra of the bigon is isomorphic to $\mathcal{O}_q(\mathrm{SL}_2)$ as a Hopf algebra (with a suitable renormalization of q). They showed this by defining a bialgebra map between $\mathcal{O}_q(\mathrm{SL}_2)$ and the Kauffman bracket stated skein algebra of the bigon. The fact that this map is an isomorphism follows because it maps the canonical basis of the stated skein algebra to a well-known basis of $\mathcal{O}_q(\mathrm{SL}_2)$. There is an analogous isomorphism between our SL_3 stated skein algebra of the bigon and $\mathcal{O}_q(\mathrm{SL}_3)$. However, the proof here will require us to define maps in both directions since it is not otherwise clear that the canonical basis of the SL_3 stated skein algebra of the bigon matches up with a basis of $\mathcal{O}_q(\mathrm{SL}_3)$.

We first recall the R-matrix definition of $\mathcal{O}_q(\mathrm{SL}_3)$. Consider the free $\mathcal{R}$-module V with basis $\{x_1, x_2, x_3\}$. The standard R-matrix for SL_3 is a linear map

$$R\colon V \otimes V \to V \otimes V$$

defined by

$$R(x_i \otimes x_j) = q^{-1/3} \begin{cases} q x_i \otimes x_j & (\text{if } i = j), \\ x_j \otimes x_i & (\text{if } i > j), \\ x_j \otimes x_i + (q - q^{-1}) x_i \otimes x_j & (\text{if } i < j). \end{cases}$$

We develop some notation for the matrix entries R_{ij}^{kl} of R. We have that the entry $R(x_i \otimes x_j)$ is uniquely written as

$$R(x_i \otimes x_j) = \sum_{1 \le k,l \le 3} R_{ij}^{kl} x_k \otimes x_l.$$

We define $\mathcal{O}_q(\mathrm{SL}_3)$ as the free $\mathcal{R}$-algebra generated by elements $\{X_{ij}\}_{1 \le i,j \le 3}$ modulo the following relations:

$$\begin{cases} \sum_{1 \le k,l \le 3} R_{ij}^{kl} X_{km} X_{ln} = \sum_{1 \le k,l \le 3} R_{kl}^{mn} X_{ik} X_{jl} & (\text{for } 1 \le i, j, m, n \le 3), \\ \sum_{\sigma \in S_3} (-q)^{l(\sigma)} X_{\sigma_1 1} X_{\sigma_2 2} X_{\sigma_3 3} = 1. \end{cases}$$

Here, we consider $(\sigma_1, \sigma_2, \sigma_3) = (1, 2, 3)$ the identity permutation.

The left side of the second equation is called the *quantum determinant*, $\det_q$, of the matrix of generators $(A)_{ij} = X_{ij}$. We will also make use of notation $A[i|j]$ to mean the quantum minor of A after deleting row i and column j.

$\mathcal{O}_q(\mathrm{SL}_3)$ has a Hopf algebra structure with structure maps given by

$$\varepsilon(X_{ij}) = \delta_{ij}$$

and

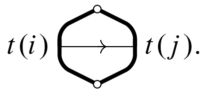
$$\Delta(X_{ij}) = \sum_{k=1}^{3} X_{ik} \otimes X_{kj}.$$

The antipode $S\colon \mathcal{O}_q(\mathrm{SL}_3) \to \mathcal{O}_q(\mathrm{SL}_3)$ is defined by

$$S(X_{ij}) = (-q)^{i-j} A[j|i].$$

For the purpose of notation to match up our stated skein algebra with the standard definition of $\mathcal{O}_q(\mathrm{SL}_3)$, we define a bijection $t\colon \{1,2,3\} \to \{-,0,+\}$ given by $t(1) = +$, $t(2) = 0$, $t(3) = -$. Since t reverses the order we have placed on the sets $\{1,2,3\}$ and $\{-,0,+\}$, we will have to take care when we apply relations (k)-(n) to diagrams.

Proposition 9.1. *There is a unique bialgebra morphism* $\phi\colon \mathcal{O}_q(\mathrm{SL}_3) \to \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ *defined by*

$\phi(X_{ij}) =$ 

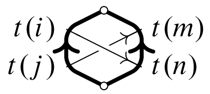
Proof. Since the elements X_{ij} generate $\mathcal{O}_q(\mathrm{SL}_3)$, the morphism will be unique if it exists. By construction, such a morphism will preserve the bialgebra structure. To prove that ϕ gives a well-defined algebra morphism, we must check that it respects the defining relations of $\mathcal{O}_q(\mathrm{SL}_3)$. We must show that the relations

$$\sum_{1\le k,l\le 3} R_{ij}^{kl}\phi(X_{km})\phi(X_{ln}) = \sum_{1\le k,l\le 3} R_{kl}^{mn}\phi(X_{ik})\phi(X_{jl})$$

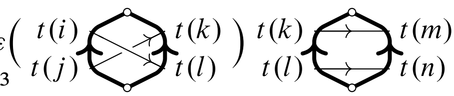
and

$$\sum_{\sigma\in S_3}(-q)^{l(\sigma)}\phi(X_{\sigma_1 1})\phi(X_{\sigma_2 2})\phi(X_{\sigma_3 3}) = 1$$

hold in $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$. For this, we recall the bialgebra structure of the bigon given in Section 6. We consider the result of applying $(\varepsilon \otimes \mathrm{id}) \circ \Delta$ to the following diagram in two different ways:

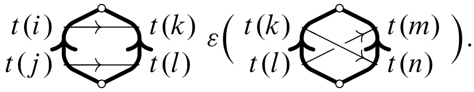


For the first way, we split the bigon along an ideal arc that stays to the right of the crossing and obtain

$\sum_{1\le k,l\le 3} \varepsilon$ 

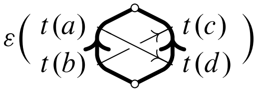
For the second way, we split the bigon along an ideal arc that stays to the left of the crossing and then apply $\mathrm{id} \otimes \varepsilon$:

$$\sum_{1 \leq k,l \leq 3}$$



The bialgebra axiom $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta$ along with the isotopy invariance of the splitting map guarantee that both answers must be the same.

We can use the defining relations to compute that



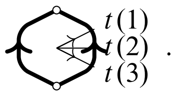
$$= R^{cd}_{ab}.$$

Equating our two answers shows that the relations

$$\sum_{1 \leq k,l \leq 3} R^{kl}_{ij} \phi(X_{km})\phi(X_{ln}) = \sum_{1 \leq k,l \leq 3} R^{mn}_{kl} \phi(X_{ik})\phi(X_{jl})$$

hold in $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$.

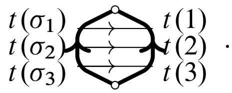
Next, we consider the following diagram:



On the one hand, we can evaluate this diagram using relation (k) from Section 3 along the right edge of the bigon. On the other hand, we could use relation (l) along the left edge of the bigon.

This gives us the relation

$$q^{-2} = q^{-2} \sum_{\sigma \in S_3} (-q)^{l(\sigma)}$$



Thus, the relation

$$\sum_{\sigma \in S_3} (-q)^{l(\sigma)} \phi(X_{\sigma_1 1})\phi(X_{\sigma_2 2})\phi(X_{\sigma_3 3}) = 1$$

holds in $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$. Thus, ϕ is well defined. ∎

To prove that ϕ is an isomorphism, we will construct an inverse function. We will define an algebra morphism $\psi \colon \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathcal{O}_q(\mathrm{SL}_3)$ by defining it on diagrams and then checking that it is well defined.

In order for ψ to be the inverse of ϕ we are forced to define it on the diagrams $\alpha_{t(i)t(j)}$ and $\beta_{t(i)t(j)}$ from Section 6 as

$$\psi(\beta_{t(i)t(j)}) = X_{ij}$$

and

$$\psi(\alpha_{t(i)t(j)}) = (-q)^{j-i} A[4-i|4-j].$$

As was noted in Section 6, the diagrams $\alpha_{t(i)t(j)}$ and $\beta_{t(i)t(j)}$ generate $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$. So, the values of ψ on these diagrams would determine ψ on $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$. However, as we do not a priori have a definition of $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ as a quotient of a free algebra by relators, it will be tricky to check that the map is well defined. Instead, we have a definition of $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ as a quotient of a free module and so we will define ψ on any diagram by giving specific directions on how to write the diagram in terms of the diagrams $\alpha_{t(i)t(j)}$ and $\beta_{t(i)t(j)}$ and then check that this process leads to a well-defined map.

Given a diagram D, we obtain $\psi(D)$ by performing the following algorithm:

- apply Δ by splitting D near the right boundary arc of $\mathfrak{B}$ so that $\Delta(D)$ is written as
$$\Delta(D) = \sum D_i \otimes E_i,$$
where the diagrams E_i each contains only parallel and antiparallel strands;
- apply $(\varepsilon \otimes \mathrm{id})$ to $\Delta(D)$ to write
$$(\varepsilon \otimes \mathrm{id})\Delta(D) = \sum \varepsilon(D_i) E_i;$$
- obtain
$$\psi(D) = \sum \varepsilon(D_i)\psi(E_i) \in \mathcal{O}_q(\mathrm{SL}_3),$$
where $\psi(E_i)$ is determined by the values of $\psi(\alpha_{t(i)t(j)})$ and $\psi(\beta_{t(i)t(j)})$ given above.

Proposition 9.2. *The map* $\psi\colon \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathcal{O}_q(\mathrm{SL}_3)$ *described above is a well-defined algebra homomorphism.*

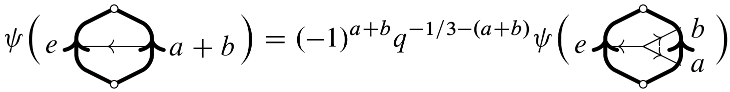
Proof. We observe that if ψ is well defined, then it does respect the natural multiplication of diagrams in $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$.

We must check that the process outlined in the bulletpoints above respects the defining relations of the stated skein algebra. We split the relations into three cases: interior relations, boundary relations along the left boundary arc of $\mathfrak{B}$, boundary relations along the right boundary arc of $\mathfrak{B}$.

Consider a relation falling under the first two cases. Such a relation only affects the diagrams D_i during the process. Since ε is well defined, application of such relations will result in identical representatives in $\mathcal{O}_q(\mathrm{SL}_3)$, and so the process respects these relations.

The case of a relation along the right boundary arc of $\mathfrak{B}$ is more difficult since it will change the diagrams E_i and will thus ultimately produce different representatives in $\mathcal{O}_q(\mathrm{SL}_3)$. It is our task to show that these representatives are equivalent. We handle each relation separately.

Relation (B1). To prove that ψ respects relation (B1) it will suffice to check that

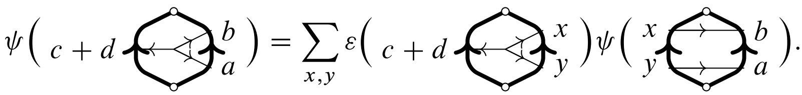


for any states $e, a, b \in \{-, 0, +\}$ with $a < b$.

Fix such e, a, b and let $i = t^{-1}(e)$ and $j = t^{-1}(a+b)$ be the corresponding integers in $\{1, 2, 3\}$. Then by the definition of ψ, the left side of our relation is $(-q)^{j-i} A[4-i|4-j]$.

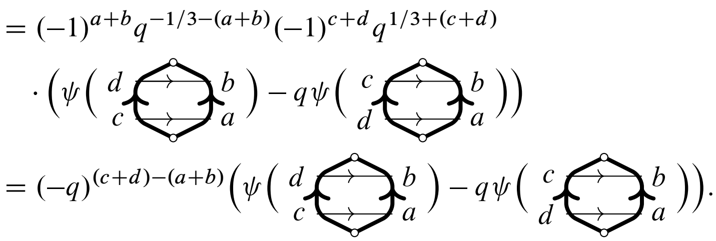
We now compute the right side of the equation. It will be convenient to let c, d be the unique states in $\{-, 0, +\}$ such that $c < d$ and $c + d = e$.

By the definition of ψ, we compute that



We will denote the values of the counit appearing in the above equation as $\varepsilon_{c+d,x,y}$. We use (B3) and (B1) to compute that $\varepsilon_{c+d,x,y} = 0$ unless $\{x, y\} = \{c, d\}$ and we use (B2) to see that $\varepsilon_{c+d,c,d} = -q\varepsilon_{c+d,d,c}$. We also use (B1) to compute that $\varepsilon_{c+d,d,c} = (-1)^{c+d} q^{1/3+(c+d)}$.

The right side of our relation becomes



We check that this formula agrees with

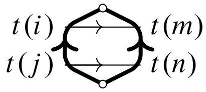
$$(-q)^{t^{-1}(a+b)-t^{-1}(c+d)}(X_{t^{-1}(d)t^{-1}(b)}X_{t^{-1}(c)t^{-1}(a)} - qX_{t^{-1}(c)t^{-1}(b)}X_{t^{-1}(d)t^{-1}(a)})$$

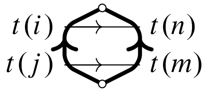
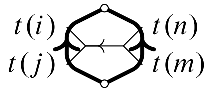
which is

$$(-q)^{(j-i)} A[4-i|4-j],$$

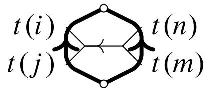
as required.

Relation (B2). To show that ψ respects relation (B2) it suffices to check that the following relation holds in $\mathcal{O}_q(\mathrm{SL}_3)$,

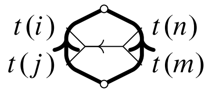
$\psi\Big($  $\Big)$

$= q^{-1}\psi\Big($  $\Big) + q^{-3}\psi\Big($  $\Big)$

for $i, j, m, n \in \{1, 2, 3\}$ such that $n < m$. So, we must show that

$\psi\Big($  $\Big) = q^3 X_{im} X_{jn} - q^2 X_{in} X_{jm}.$

From relation (I1a) and the computations of $\varepsilon(\beta_{st})$ from Section 6, we compute that

$\varepsilon\Big($  $\Big) = q^{3+1/3} R^{kl}_{ij} - q^4 \delta_{ik}\delta_{jl}.$

Thus, we must show that

$$\Big(\sum_{k,l} q^{3+1/3} R^{kl}_{ij} X_{kn} X_{lm}\Big) - q^4 X_{in} X_{jm} = q^3 X_{im} X_{jn} - q^2 X_{in} X_{jm}.$$

We apply the identity

$$\sum_{k,l} R^{kl}_{ij} X_{kn} X_{lm} = \sum_{k,l} R^{nm}_{kl} X_{ik} X_{jl}.$$

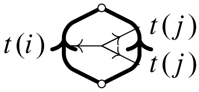
Since $n < m$, we have that $R^{nm}_{nm} = q^{-1/3}(q - q^{-1})$ and $R^{nm}_{mn} = q^{-1/3}$ are the only nonzero values of R^{nm}_{kl} as k and l vary.

The left side of our equation now becomes

$$\begin{aligned}
&\Big(\sum_{k,l} q^{3+1/3} R^{kl}_{ij} X_{kn} X_{lm}\Big) - q^4 X_{in} X_{jm} \\
&\quad = \Big(\sum_{k,l} q^{3+1/3} R^{nm}_{kl} X_{ik} X_{jl}\Big) - q^4 X_{in} X_{jm} \\
&\quad = q^3(q - q^{-1}) X_{in} X_{jm} + q^3 X_{im} X_{jn} - q^4 X_{in} X_{jm} \\
&\quad = q^3 X_{im} X_{jn} - q^2 X_{in} X_{jm},
\end{aligned}$$

as required. So, ψ respects (B2).

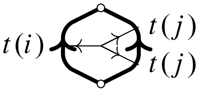
Relation (B3). To show that ψ respects (B3) we need to show that

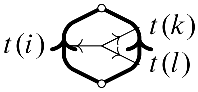
$$\psi\Bigg($$ 

$$\Bigg) = 0$$

for any $i, j \in \{1,2,3\}$.

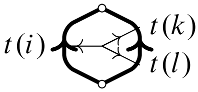
By the definition of ψ, we have

$$\psi\Bigg($$ 

$$\Bigg) = \sum_{k,l} \varepsilon\Bigg($$ 

$$\Bigg) X_{kj} X_{lj}.$$

We compute that

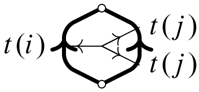
$$\varepsilon\Bigg($$ 

$$\Bigg) = 0$$

if $4 - i$ is in $\{k, l\}$ or if $k = l$.

If $l < k$ we have $\varepsilon_{ikl} = -q\varepsilon_{ilk}$. This can be computed by using relations (B2) and (I3).

Thus,

$$\psi\Bigg($$ 

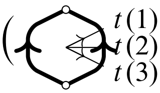
$$\Bigg) = \varepsilon_{ilk}(X_{lj} X_{kj} - q X_{kj} X_{lj})$$

for the unique suitable pair l, k for which ε_{ilk} is nonzero. The result follows from the identity

$$X_{lj} X_{kj} = q X_{kj} X_{lj}$$

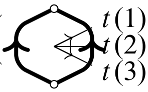
which holds in $\mathcal{O}_q(\mathrm{SL}_3)$ for $l < k$.

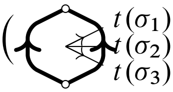
Relation (B4). To check that ψ respects relation (B4) it suffices to check

$$\psi\Bigg($$ 

$$\Bigg) = q^{-2}.$$

By the definition of ψ, we compute

$$\psi\Bigg($$ 

$$\Bigg) = \sum_{\sigma \in S_3} \varepsilon\Bigg($$ 

$$\Bigg) X_{\sigma_1 1} X_{\sigma_2 2} X_{\sigma_3 3}.$$

We see that this is equal to

$$q^{-2} \sum_{\sigma \in S_3} (-q)^{l(\sigma)} X_{\sigma_1 1} X_{\sigma_2 2} X_{\sigma_3 3} = q^{-2} \det_q = q^{-2}.$$

So, we see that ψ respects (B4) and, thus, ψ is well defined. ∎

Our previous two propositions allow us to state the following theorem.

Theorem 9.3. *We have that*

$$\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \cong \mathcal{O}_q(\mathrm{SL}_3)$$

as Hopf algebras.

Proof. In Proposition 9.1 we showed that ϕ is a well-defined map of bialgebras. To show that ϕ is an isomorphism, it suffices to show that ϕ is invertible as a map of $\mathcal{R}$-modules. We claim that ψ is its inverse.

We observe that $\psi \circ \phi(X_{ij}) = X_{ij}$ for all generators X_{ij} of $\mathcal{O}_q(\mathrm{SL}_3)$. Since ψ and ϕ are both algebra maps, this implies that

$$\psi \circ \phi = \mathrm{id}_{\mathcal{O}_q(\mathrm{SL}_3)}.$$

Similarly, $\phi \circ \psi$ agrees with $\mathrm{id}_{\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})}$ for all generating diagrams α_{st} and β_{st}. Thus,

$$\phi \circ \psi = \mathrm{id}_{\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})}.$$

Thus, $\mathcal{O}_q(\mathrm{SL}_3)$ and $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ are isomorphic as bialgebras. Since $\mathcal{O}_q(\mathrm{SL}_3)$ is a Hopf algebra, then $\mathcal{O}_q(\mathrm{SL}_3)$ and $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ are isomorphic as Hopf algebras. ∎

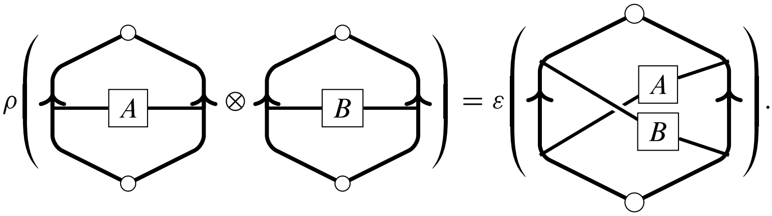
10. The stated skein algebra of the triangle

The Hopf algebra $\mathcal{O}_q(\mathrm{SL}_3)$ is equipped with a cobraiding

$$\rho\colon \mathcal{O}_q(\mathrm{SL}_3) \otimes \mathcal{O}_q(\mathrm{SL}_3) \to \mathcal{R}.$$

In [8] the cobraiding for the SL_2 case was shown to have a simple diagrammatic definition, and an analogous definition will work here as well. This cobraiding will allow us to describe the SL_3 stated skein algebra of the triangle, $\mathfrak{T}$.

We define the cobraiding $\rho\colon \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathcal{R}$ on diagrams by



In the diagrams above, the strands depict a bundle of parallel or antiparallel strands. The diagrammatic definition of the map makes it easy to see that it respects the defining relations of the stated skein algebra, so it is well defined. The argument that this satisfies the cobraiding axioms is identical to the one in [8], but we do not need to use it in this paper.

We recall that a cobraiding is determined by its values on a set of generators and so we see that the map ρ that we have defined diagrammatically satisfies

$$\rho(X_{ij} \otimes X_{kl}) = R_{ik}^{jl},$$

and thus matches up with the standard co-R-matrix.

In the situation that we have two algebras M and N which are both left comodule-algebras over $\mathcal{O}_q(\mathrm{SL}_3)$ we can endow the vector space $M \otimes N$ with a left comodule-algebra structure using the cobraiding ρ. We will denote this algebra by $M \underline{\otimes} N$ and call it the *braided tensor product* of the algebras M and N. Using Sweedler's notation, its multiplication is defined as follows:

$$(x \otimes y) \star (z \otimes t) = (x \otimes 1)\Big(\sum_{(z)(y)} \rho(z' \otimes y')(z'' \otimes y'')\Big)(1 \otimes t)$$

Equivalently, if we identify M with $M \otimes \{1\}$ and N with $\{1\} \otimes N$, then our product structure is given by

$$xy = \begin{cases} xy & \text{if } x, y \text{ both in } M \text{ or both in } N, \\ x \otimes y & \text{if } x \text{ in } M \text{ and } y \text{ in } M, \\ \sum_{(x)(y)} \rho(y' \otimes x')(y'' \otimes x'') & \text{if } x \text{ in } N \text{ and } y \text{ in } M. \end{cases}$$

Costantino and Lê showed in [8] that gluing disjoint surfaces along a triangle yields a braided tensor product of stated skein algebras for the SL_2 case. The same is true for the SL_3 case and Proposition 7.1 from this paper takes care of most of the work we need to do to show it.

Theorem 10.1. *Let Σ_1 and Σ_2 be disjoint punctured bordered surfaces. If a is a boundary arc of Σ_1 and b is a boundary arc of Σ_2, then we have an algebra isomorphism*

$$\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1) \underline{\otimes} \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_2) \cong \mathcal{S}_q^{\mathrm{SL}_3}((\Sigma_1 \sqcup \Sigma_2)\#\mathfrak{T})$$

given by the map $\mathrm{glue}_{\mathfrak{T}}$ *defined in Section 7.*

Proof. By Proposition 7.1, the map

$$\mathrm{glue}_{\mathfrak{T}} \colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1 \sqcup \Sigma_2) \to \mathcal{S}_q^{\mathrm{SL}_3}((\Sigma_1 \sqcup \Sigma_2)\#\mathfrak{T})$$

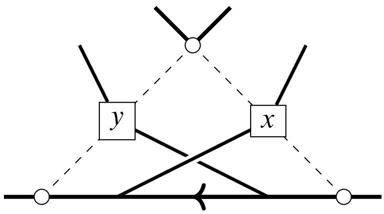
is an $\mathcal{R}$-module isomorphism. Since $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1 \sqcup \Sigma_2)$ is naturally isomorphic to $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_2)$, we see that the isomorphism claimed in Theorem 10.1 holds on the level of $\mathcal{R}$-modules. To see that it holds on the level of $\mathcal{R}$-algebras we must show that $\mathrm{glue}_{\mathfrak{T}}$ respects the algebra structure.

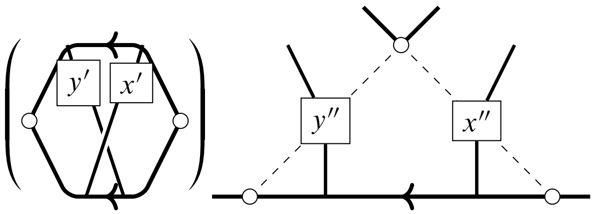
For this fact, the same diagrammatic proof in [8] works here. In each of the following cases:

- x, y are both in $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1)$,
- x, y are both in $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_2)$,
- or x is in $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1)$ while y is in $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_2)$,

it is clear that $\mathrm{glue}_{\mathfrak{T}}(x)\,\mathrm{glue}_{\mathfrak{T}}(y) = \mathrm{glue}_{\mathfrak{T}}(xy)$.

In the remaining case, we have that x is in $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_2)$ and y is in $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1)$. We diagrammatically compute that

$\mathrm{glue}_{\mathfrak{T}}(x)\,\mathrm{glue}_{\mathfrak{T}}(y) =$ 

$= \sum_{(x)(y)} \varepsilon$ 

$$= \sum_{(x)(y)} \rho(y' \otimes x')\,\mathrm{glue}_{\mathfrak{T}}(y'' \otimes x'')$$

$$= \mathrm{glue}_{\mathfrak{T}}\Big(\sum_{(x)(y)} \rho(y' \otimes x')(y'' \otimes x'')\Big)$$

$$= \mathrm{glue}_{\mathfrak{T}}(xy).$$

This shows that $\mathrm{glue}_{\mathfrak{T}}$ respects the multiplication of $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_1) \underline{\otimes} \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma_2)$ and completes our proof. ■

By applying Theorem 10.1 in the special case where Σ_1 and Σ_2 are both bigons $\mathfrak{B}$ we obtain the following corollary.

Corollary 10.2. $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}) \cong \mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$.

11. The skein algebra of a punctured surface is a domain

Suppose that our ground ring $\mathcal{R}$ is a domain, which means that if $xy = 0$ for elements $x, y \in \mathcal{R}$, then we must have that $x = 0$ or $y = 0$. Our goal in this section is to show that whenever $\mathcal{R}$ is a domain, $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ is a domain as well. We are able to prove this fact as long as Σ has at least one puncture. We state our main theorem here and then prove it in the rest of the section.

Theorem 11.1. *If $\mathcal{R}$ is a domain and Σ has at least one puncture, then $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ is a domain as well.*

We remark that this theorem also implies that the ordinary skein algebra is a domain since it embeds in the stated skein algebra.

We first prove the theorem for the cases when Σ has no ideal triangulation. A punctured bordered surface Σ is called a *small surface* if it is one of the following surfaces: a bigon, a monogon, a sphere with two punctures or a sphere with one puncture.

Proposition 11.2. *If $\mathcal{R}$ is a domain and Σ is a small surface, then $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ is a domain.*

Proof. If Σ is a monogon or a sphere with one puncture, then $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \cong \mathcal{R}$ and is a domain.

If Σ is a bigon, then one has $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \cong \mathcal{O}_q(\mathrm{SL}_3)$, which is a domain by [5, Theorem I.2.10]. The proof there is stated for $\mathcal{R} = k$, a field but their proof works for any domain $\mathcal{R}$.

Finally, if Σ is a sphere with two punctures, then by applying the splitting map associated to an ideal arc traveling from one puncture to the other, we obtain an embedding $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \hookrightarrow \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ and so our skein algebra is a domain in this case as well. ∎

If Σ is a punctured surface that is not a small surface, then Σ has an ideal triangulation and we can apply our triangular decomposition to obtain an embedding

$$\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \hookrightarrow \bigotimes_i \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i),$$

where for each triangle $\mathfrak{T}_i$, we have $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i) \cong \mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$.

So, it will suffice to show that $\bigotimes_i (\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3))$ is a domain. Since domains are not necessarily well behaved under tensor products or braided tensor products (recall that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not a domain), our result does not follow immediately from the fact that $\mathcal{O}_q(\mathrm{SL}_3)$ is a domain. However, we will still model our proof on the proof in [5] by first using properties of $\mathcal{O}_q(M_3)$ and then using a localization.

Recall that the bialgebra $\mathcal{O}_q(M_3)$ has a similar presentation as $\mathcal{O}_q(\mathrm{SL}_3)$, generated by elements X_{ij} with the only difference being that the presentation of $\mathcal{O}_q(M_3)$ does not include the relation $\det_q = 1$. The precise relations for $\mathcal{O}_q(M_3)$ are as follows:

$$X_{ij}X_{lm} = \begin{cases} q^{-1}X_{lm}X_{ij} & (i > l, j = m), \\ q^{-1}X_{lm}X_{ij} & (i = l, j > m), \\ X_{lm}X_{ij} & (i < l, j > m), \\ X_{lm}X_{ij} - (q - q^{-1})X_{im}X_{lj} & (i > l, j > m). \end{cases}$$

We define an order on our generators X_{ij} using the lexicographic ordering, meaning $X_{ij} < X_{kl}$ if $i < k$ or if both $i = k$ and $j < l$. Using the defining relations of $\mathcal{O}_q(M_3)$ as reduction rules and a standard diamond lemma argument, we have a basis for $\mathcal{O}_q(M_3)$ consisting of monomials of generators appearing in increasing order. We will soon observe that this basis of monomials is compatible with the multiplication on $\mathcal{O}_q(M_3)$ in the following sense.

Definition 11.3. A total order on a monoid $\mathcal{M}$ is compatible with its operation if it satisfies the property that for elements $m_1 \leq m_2$ and $n_1 \leq n_2$, we have $m_1 + n_1 \leq m_2 + n_2$. We will refer to such a monoid $\mathcal{M}$ as a *totally ordered monoid*.

Suppose that $\mathcal{B}$ is an $\mathcal{R}$-basis of an $\mathcal{R}$-algebra $\mathcal{A}$. If there exists an injective assignment $d\colon \mathcal{B} \to \mathcal{M}$, then d can be extended to $\mathcal{A} \setminus \{0\}$ in the following way. If $a \in \mathcal{A} \setminus \{0\}$, express $a = \sum r_i b_i$ uniquely as a finite linear combination of elements in the basis $\mathcal{B}$. Define $d(a)$ to be the maximum value of $\{d(b_i) | r_i \neq 0\}$.

We say that $\mathcal{B}$ is a *compatibly ordered basis* for $\mathcal{A}$ if it is indexed by a totally ordered monoid $\mathcal{M}$ such that the map $d\colon \mathcal{A} \to \mathcal{M}$ satisfies

$$d(b_1 b_2) = d(b_1) + d(b_2)$$

for any basis elements $b_1, b_2 \in \mathcal{B}$.

Proposition 11.4. *The monomial basis for $\mathcal{O}_q(M_3)$ described above is a compatibly ordered basis.*

Proof. To each basis monomial, we may associate a degree $d \in \mathbb{Z}_{\geq 0}^9$ by

$$d(X_{11}^{m_{11}} X_{12}^{m_{12}} \cdots X_{33}^{m_{33}}) = (m_{33}, m_{32}, \ldots, m_{11}),$$

the list of exponents of the generators, listed in reverse order. A basis element is determined uniquely by its degree, and so we have an indexing of our basis by the totally ordered monoid $\mathbb{Z}_{\geq 0}^9$.

To an arbitrary nonzero element $x \in \mathcal{O}_q(M_3)$ we can associate a degree $d(x)$ by writing x in the basis and defining $d(x)$ to be the maximum degree among all basis elements appearing with nonzero coefficients.

Suppose m_1 and m_2 are two monomial basis elements. The reduction rules imply that generators q-commute up to terms of smaller degree and so

$$d(m_1 m_2) = d(m_1) + d(m_2). \qquad \blacksquare$$

Corollary 11.5. If $\mathcal{R}$ is a domain then $\mathcal{O}_q(M_3)$ is a domain.

Proof. Suppose that x and y are nonzero elements of $\mathcal{O}_q(M_3)$. The fact that

$$d(m_1 m_2) = d(m_1) + d(m_2)$$

for all monomial basis elements m_1, m_2 implies that we also have

$$d(xy) = d(x) + d(y).$$

From this we can deduce that $d(xy) > \vec{0}$. Hence, $xy \neq 0$ and $\mathcal{O}_q(M_3)$ is a domain. $\blacksquare$

We next prove the same property holds for the braided tensor square of $\mathcal{O}_q(M_3)$.

Proposition 11.6. *If $\mathcal{R}$ is a domain then*

$$\bigotimes_i (\mathcal{O}_q(M_3) \mathbin{\underline{\otimes}} \mathcal{O}_q(M_3))$$

is a domain.

Proof. We upgrade our compatibly ordered basis of $\mathcal{O}_q(M_3)$ to a compatibly ordered basis of $\mathcal{O}_q(M_3) \mathbin{\underline{\otimes}} \mathcal{O}_q(M_3)$. We will continue to use X_{ij} to refer to the generators in the first factor and use Y_{ij} to refer to the generators in the second factor. Recall that the algebra $\mathcal{O}_q(M_3) \mathbin{\underline{\otimes}} \mathcal{O}_q(M_3)$ is isomorphic as a module to $\mathcal{O}_q(M_3) \otimes \mathcal{O}_q(M_3)$ and thus has a basis $\{m_X m_Y\}$ where m_X is a monomial of generators X_{ij} in increasing order and m_Y is a monomial of generators Y_{ij} appearing in increasing order. We claim that this basis is compatibly ordered as well.

For a basis element $m_X m_Y$, we define its degree $d(m_X m_Y) \in \mathbb{Z}^{18}_{\geq 0}$ to be the concatenation $(d(m_X), d(m_Y))$ where $d(m_X)$ was defined in the proof of Proposition 11.4 and $d(m_Y)$ is the corresponding definition using the generators Y_{ij}. We recall that in the braided tensor product $\mathcal{O}_q(M_3) \mathbin{\underline{\otimes}} \mathcal{O}_q(M_3)$ we have that

$$Y_{ij} X_{kl} = q^{-1/3} \begin{cases} q X_{kl} Y_{ij} & (i = k), \\ X_{kl} Y_{ij} + (q - q^{-1}) X_{il} Y_{kj} & (i < k), \\ X_{kl} Y_{ij} & (i > k). \end{cases}$$

We note that if $i < k$ then $X_{il} < X_{kl}$. Thus, the generators Y_{ij} and X_{kl} q-commute up to lower order terms. From this, we deduce that

$$d(m_{X_1} m_{Y_1} m_{X_2} m_{Y_2}) = d(m_{X_1} m_{Y_1}) + d(m_{X_2} m_{Y_2})$$

and consequently $\mathcal{O}_q(M_3) \mathbin{\underline{\otimes}} \mathcal{O}_q(M_3)$ is a domain.

We then use the tensor product of these bases to get a compatibly ordered basis of $\bigotimes_i (\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$ and see that it is a domain. ■

We would like to take the Ore localization of $\bigotimes_i (\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$ with respect to the multiplicative set generated by the elements $\det_{X_i}$ and $\det_{Y_i}$. This will be easy to do if we can show that these determinant elements are central. It suffices to show the following.

Proposition 11.7. *The quantum determinant elements* $\det_X$ *and* $\det_Y$ *are central elements of* $\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3)$.

Proof. We will prove that $\det_X$ is central. The argument that $\det_Y$ is central is similar.

It is well known that $\det_X$ commutes with the generators X_{ij} so we need to check that it commutes with the generators Y_{kl}. Recall from the previous proof that the commutativity relations involving Y_{kl} and X_{ij} only depend on the row indices k and i.

We must check that

$$Y_{kl}\Big(\sum_\sigma (-q)^{l(\sigma)} X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)}\Big) = \Big(\sum_\sigma (-q)^{l(\sigma)} X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)}\Big)Y_{kl}.$$

If $k = 3$ then the row index of Y is not smaller than any row indices of the generators X_{ij} and so Y_{3l} can slide past the determinant, picking up one factor of $q^{2/3}$ and two factors of $q^{-1/3}$ along the way. Thus, the relation holds if $k = 3$.

If $k = 2$, then we use the relations to slide Y_{2l} past the generators $X_{i\sigma(i)}$ to get

$$\begin{aligned} &Y_{2l}\Big(\sum_\sigma (-q)^{l(\sigma)} X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)}\Big) \\ &= \sum_\sigma (-q)^{l(\sigma)} q^{2/3} q^{-1/3} X_{1\sigma(1)}X_{2\sigma(2)}Y_{2l}X_{3\sigma(3)} \\ &= \Big(\sum_\sigma (-q)^{l(\sigma)} X_{1\sigma(1)}X_{2\sigma(2)}X_{3\sigma(3)}\Big)Y_{2l} \\ &\quad + \sum_\sigma (-q)^{l(\sigma)} (q - q^{-1}) X_{1\sigma(1)}X_{2\sigma(2)}X_{2\sigma(3)}Y_{3l} \end{aligned}$$

This last term is zero since $\sum_\sigma (-q)^{l(\sigma)} X_{1\sigma(1)}X_{2\sigma(2)}X_{2\sigma(3)}$ has a repeated row index and so is zero by properties of quantum determinants. Thus, Y_{2l} commutes with $\det_X$.

When $k = 1$ a similar computation shows that Y_{1l} commutes with $\det_X$. ■

We can then take an Ore localization of $\bigotimes_i (\mathcal{O}_q(M_3) \underline{\otimes} \mathcal{O}_q(M_3))$ to obtain the algebra

$$\bigotimes_i (\mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3)),$$

where $\mathcal{O}_q(\mathrm{GL}_3) = \mathcal{O}_q(M_3)[\det_q^{-1}]$. Since the localization of a domain is a domain (see [14, Chapter 6]), we have that $\bigotimes_i(\mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3))$ is a domain.

The proof of Theorem 11.1 then follows from the following.

Proposition 11.8. *There is an algebra embedding*

$$\bigotimes_i(\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)) \hookrightarrow \bigotimes_i(\mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3)).$$

Proof. Producing an embedding $\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3) \hookrightarrow \mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3)$ will induce the desired embedding since these algebras are free $\mathcal{R}$-modules and so the tensor product of injective maps will be an injective map.

To produce this embedding we follow the construction of an the embedding $\mathcal{O}_q(\mathrm{SL}_3) \hookrightarrow \mathcal{O}_q(\mathrm{GL}_3)$ from [5].

We show that

$$(\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3))[z_X^{\pm 1}, z_Y^{\pm 1}] \cong \mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3).$$

We will denote by X_{ij} and Y_{ij} the generators of $\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$ and by x_{ij} and y_{ij} the generators of $\mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3)$. Define

$$F\colon (\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3))[z_X^{\pm 1}, z_Y^{\pm 1}] \to \mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3)$$

on generators by

$$\begin{aligned} X_{i1} &\mapsto x_{i1}\det_x^{-1}, & X_{ij} &\mapsto x_{ij} \quad (j \neq 1), \\ Y_{i1} &\mapsto y_{i1}\det_y^{-1}, & Y_{ij} &\mapsto y_{ij} \quad (j \neq 1), \\ z_X &\mapsto \det_x, & z_Y &\mapsto \det_y . \end{aligned}$$

Since $\det_x$ and $\det_y$ are central, we can see that F respects the standard $\mathcal{O}_q(M_3)$ relations. By construction, it satisfies $F(\det_X) = 1 = F(\det_Y)$. It also satisfies the mixed relations involving Y_{ij} and X_{kl}. Thus, F is a well-defined algebra map.

We define

$$G\colon \mathcal{O}_q(\mathrm{GL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{GL}_3) \to (\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3))[z_X^{\pm 1}, z_Y^{\pm 1}]$$

on generators by

$$\begin{aligned} x_{i1} &\mapsto X_{i1}z_X, & x_{ij} &\mapsto X_{ij} \quad (j \neq 1), \\ y_{i1} &\mapsto Y_{i1}z_Y, & y_{ij} &\mapsto Y_{ij} \quad (j \neq 1), \\ \det_x^{-1} &\mapsto z_X^{-1}, & \det_y^{-1} &\mapsto z_Y^{-1}. \end{aligned}$$

G respects the relations and so is a well-defined algebra map. We can see on generators that $FG = \mathrm{id}$ and $GF = \mathrm{id}$ and so we have an isomorphism. Restricting F to $\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$ produces the desired embedding. ∎

12. The SL_3 analogue of the Frobenius homomorphism

In [4], an algebra map called the *Chebyshev–Frobenius homomorphism* was constructed, which embedded the classical skein algebra $\mathcal{S}_1^{\mathrm{SL}_2}(\Sigma)$ into the center of the skein algebra $\mathcal{S}_q^{\mathrm{SL}_2}(\Sigma)$ at a root of unity q of odd order. The map is interesting from a topological viewpoint since it has a definition in terms of threading links through Chebyshev polynomials and the fact that it is well defined follows from "miraculous cancellations" in skein theoretic computations when q is a root of unity. From an algebraic viewpoint, the map is important because it provides a source of central elements which can be used to study the representation theory of $\mathcal{S}_q^{\mathrm{SL}_2}(\Sigma)$ at roots of unity. We are interested in finding an analogous map for the case of SL_3 .

Throughout this section, we assume that $\mathcal{R}$ is a domain and $q^{1/3}$ is a root of unity of order N coprime to 6. We are interested in the relationship between $\mathcal{S}_1^{\mathrm{SL}_3}(\Sigma)$ and $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$, where the skein algebra $\mathcal{S}_1^{\mathrm{SL}_3}(\Sigma)$ is obtained from the definition of $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)$ by replacing $q^{1/3}$ by 1 in all of the defining skein relations. Our goal in this section is to prove the following.

Theorem 12.1. *Suppose that $\mathcal{R}$ is a domain and $q^{1/3}$ is a root of unity of order N coprime to* 6. *Then, for a punctured bordered surface Σ with at least one puncture per connected component, there exists an embedding*

$$F_\Sigma\colon \mathcal{S}_1^{\mathrm{SL}_3}(\Sigma) \hookrightarrow Z(\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma)).$$

The Frobenius map F_Σ will be constructed by starting with the Hopf algebra embedding $\mathcal{O}_1(\mathrm{SL}_3) \hookrightarrow \mathcal{O}_q(\mathrm{SL}_3)$ constructed in [28] and then, in some sense, extending the map to Σ. We follow the strategy of [22] from the SL_2 case.

When we say that $q^{1/3}$ has order N, we mean that $(q^{1/3})^N = 1$ and $(q^{1/3})^k \neq 1$ for $0 < k < N$. Our assumption that N is coprime to 6 guarantees that q and q^2 are also roots of unity of the same order N, which is a hypothesis used in [28].

Proposition 12.2 ([28]). *There is a Hopf algebra map*

$$F_{\mathfrak{B}}\colon \mathcal{O}_1(\mathrm{SL}_3) \to \mathcal{O}_q(\mathrm{SL}_3)$$

defined on generators by $F_{\mathfrak{B}}(X_{ij}) = (X_{ij})^N$. Furthermore, the image of $F_{\mathfrak{B}}$ is contained in the center of $\mathcal{O}_q(\mathrm{SL}_3)$.

We observe the following.

Lemma 12.3. *The Hopf algebra map $F_{\mathfrak{B}}$ is an embedding.*

Proof. The set of monomials

$$\{X_{11}^{m_{11}} X_{12}^{m_{12}} \cdots X_{33}^{m_{33}} \mid m_{11}m_{22}m_{33} = 0\}$$

forms a basis for both $\mathcal{O}_1(\mathrm{SL}_3)$ and $\mathcal{O}_q(\mathrm{SL}_3)$. Since

$$F_{\mathfrak{B}}(X_{11}^{m_{11}} X_{12}^{m_{12}} \cdots X_{33}^{m_{33}}) = X_{11}^{Nm_{11}} X_{12}^{Nm_{12}} \cdots X_{33}^{Nm_{33}},$$

we see that $F_{\mathfrak{B}}$ maps the basis of $\mathcal{O}_1(\mathrm{SL}_3)$ injectively into the basis of $\mathcal{O}_q(\mathrm{SL}_3)$. ■

We next extend this map to the case of the braided tensor square. Recall that when $q = 1$, the braided tensor square of $\mathcal{O}_1(\mathrm{SL}_3)$ is just the ordinary tensor square.

Proposition 12.4. *There is an algebra embedding*

$$F_{\mathfrak{T}} \colon \mathcal{O}_1(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_1(\mathrm{SL}_3) \hookrightarrow Z(\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3))$$

defined by $F_{\mathfrak{T}} = F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}$.

Proof. Since $\mathcal{O}_1(\mathrm{SL}_3)$ is a free $\mathcal{R}$-module, by setting $F_{\mathfrak{T}} = F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}$ we obtain an embedding of $\mathcal{R}$-modules $\mathcal{O}_1(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_1(\mathrm{SL}_3) \hookrightarrow \mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$. We need to check that this map respects the algebra structure.

Recall the notation $X_{ij} Y_{kl}$ for generators of $\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$. To see that $F_{\mathfrak{T}}$ respects the algebra structure of the braided tensor product, it will suffice to observe that the images of generators $F_{\mathfrak{T}}(Y_{kl}) = Y_{kl}^N$ commute with the generators X_{ij} in $\mathcal{O}_q(\mathrm{SL}_3) \underline{\otimes} \mathcal{O}_q(\mathrm{SL}_3)$. A symmetric argument shows the same is true for $F_{\mathfrak{T}}(X_{ij})$ and Y_{kl}.

If $k = i$, we have the relation $Y_{kl} X_{ij} = q^{2/3} X_{ij} Y_{kl}$ and so

$$Y_{kl}^N X_{ij} = (q^{2/3})^N X_{ij} Y_{kl}^N = X_{ij} Y_{kl}^N,$$

since $(q^{1/3})^N = 1$. Similarly, if $k > i$, we have the relation

$$Y_{kl} X_{ij} = q^{-1/3} X_{ij} Y_{kl}$$

and so

$$Y_{kl}^N X_{ij} = X_{ij} Y_{kl}^N$$

in this case as well.

If $k < i$, then we will use the relation

$$Y_{kl} X_{ij} = q^{-1/3} X_{ij} Y_{kl} + q^{-1/3}(q - q^{-1}) X_{kj} Y_{il}.$$

We will prove the following for $m \geq 1$ by induction:

$$Y_{kl}^m X_{ij} = (q^{-1/3})^m X_{ij} Y_{kl}^m + (q^{-1/3})^m (q - q^{-1}) \sum_{n=0}^{m-1} q^{-2n} X_{kj} Y_{il} Y_{kl}^{m-1}.$$

We are given the base case. Now, assume the inductive hypothesis. We have

$$\begin{aligned} Y_{kl}^m X_{ij} &= Y_{kl}(q^{-1/3})^{m-1} X_{ij} Y_{kl}^{m-1} \\ &\quad + (q^{-1/3})^{m-1}(q-q^{-1}) \sum_{n=0}^{m-2} q^{-2n} Y_{kl} X_{kj} Y_{il} Y_{kl}^{m-2} \\ &= (q^{-1/3})^m X_{ij} Y_{kl}^m + (q^{-1/3})^m (q-q^{-1}) X_{kj} Y_{il} Y_{kl}^{m-1} \\ &\quad + (q^{-1/3})^{m-1}(q-q^{-1}) \sum_{n=0}^{m-2} q^{-1/3} q^{-2n+2} X_{kj} Y_{il} Y_{kl}^{m-1} \\ &= (q^{-1/3})^m X_{ij} Y_{kl}^m + (q^{-1/3})^m (q-q^{-1}) \sum_{n=0}^{m-1} q^{-2n} X_{kj} Y_{il} Y_{kl}^{m-1}, \end{aligned}$$

as claimed.

When we specialize this formula to the case $m = N$ we obtain

$$Y_{kl}^N X_{ij} = X_{ij} Y_{kl}^N$$

as required since $(q^{-1/3})^N = 1$ and $(q-q^{-1}) \sum_{n=0}^{N-1} q^{-2n} = q(1-q^{-2N}) = 0$. ■

We next investigate the diagrammatic properties of our maps $F_{\mathfrak{B}}$ and $F_{\mathfrak{T}}$ when we view them as maps on the skein algebras of the bigon $\mathfrak{B}$ and triangle $\mathfrak{T}$.

Proposition 12.5. *When $\mathcal{S}_q^{\mathrm{SL}_2}(\mathfrak{B})$ is identified with $\mathcal{O}_q(\mathrm{SL}_3)$, $F_{\mathfrak{B}}$ is defined on generating strands by $F_{\mathfrak{B}}(\alpha_{t(i)t(j)}) = \alpha_{t(i)t(j)}^N$ and $F_{\mathfrak{B}}(\beta_{t(i)t(j)}) = \beta_{t(i)t(j)}^N$ for all $i, j \in \{1,2,3\}$.*

Proof. Our isomorphism $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathcal{O}_q(\mathrm{SL}_3)$ sends $\beta_{t(i)t(j)}$ to X_{ij} and so we already know that $F_{\mathfrak{B}}(\beta_{t(i)(j)}) = \beta_{t(i)t(j)}^N$.

For the strands $\alpha_{t(i)t(j)}$ we will use the antipodes S and the fact that $F_{\mathfrak{B}}$ commutes with the antipodes. For our strands $\alpha_{t(i)t(j)}$, we use the fact

$$\alpha_{t(i)t(j)} = q^{2j-2i} S(\beta_{t(\bar{j})t(\bar{\imath})}),$$

where $\bar{\imath} = 4 - i$. We then compute

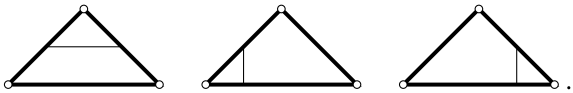
$$\begin{aligned} F_{\mathfrak{B}}(\alpha_{t(i)t(j)}) &= F_{\mathfrak{B}}(S(\beta_{t(\bar{j})t(\bar{\imath})})) = S(F_{\mathfrak{B}}(\beta_{t(\bar{j})t(\bar{\imath})})) = S((\beta_{t(\bar{j})t(\bar{\imath})})^N) \\ &= S(\beta_{t(\bar{j})t(\bar{\imath})})^N = (q^{2i-2j})^N \alpha_{t(i)t(j)}^N = \alpha_{t(i)t(j)}^N, \end{aligned}$$

as claimed. ■

Thus, even though our isomorphism $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \cong \mathcal{O}_q(\mathrm{SL}_3)$ depends on a choice of right and left boundary arcs of $\mathfrak{B}$, the definition of $F_{\mathfrak{B}}$ is invariant under this choice.

Similarly, even though $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T}) \cong \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \underline{\otimes} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ depended on a choice of bottom edge of the triangle, we aim to show that the map $F_{\mathfrak{T}}\colon \mathcal{S}_1^{\mathrm{SL}_3}(\mathfrak{T}) \to \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{T})$ is invariant under this choice.

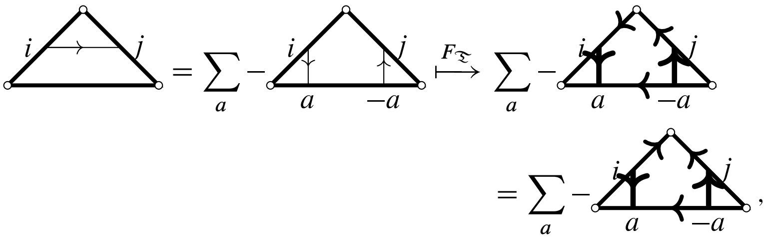
We call a stated arc in the triangle $\mathfrak{T}$ a *corner arc* if it admits a crossingless diagram and it is not homotopic to a boundary arc. The following illustrates examples of top, left, and right corner arcs:



Proposition 12.6. *The map $F_{\mathfrak{T}}$ sends a stated corner arc to its N-th power.*

Proof. If the arc is a left or right corner arc, then by the definition of $F_{\mathfrak{T}} = F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}$ and our diagrammatic interpretation of $F_{\mathfrak{B}}$, we already know that $F_{\mathfrak{T}}$ sends the arc to its N-th power. So, we just have to show the same is true for a top corner arc. We will compute this for a top corner arc with one orientation. A similar computation works for the opposite orientation.

We compute the value of $F_{\mathfrak{T}}$ on our arc by first writing it in terms of left and right corner arcs and then applying $F_{\mathfrak{T}}$.



where the thick strands denote N parallel strands. The orientation reversal of the left edge in the last equality is possible since it comes at the expense of a factor of $(q^{2/3})^{N(N-1)/2} = 1$.

We claim the last expression in our computation is the same as the N-th power of our top corner arc. To show this, we will make use of the fact that $F_{\mathfrak{B}}$ is a bialgebra map. Let e_r denote the right boundary arc of the left bigon in a disjoint union $\mathfrak{B} \sqcup \mathfrak{B}$ and denote by e_l the left boundary arc of the right bigon. Recall the maps rev_{e_r} and $\mathrm{glue}_{\mathfrak{T}}$. We have that the N-th power of our top corner arc is the same as

$$\mathrm{glue}_{\mathfrak{T}}\, \mathrm{rev}_{e_r}\, \Delta(F_{\mathfrak{B}}(\beta_{ij})) = \mathrm{glue}_{\mathfrak{T}}\, \mathrm{rev}_{e_r}(F_{\mathfrak{B}} \otimes F_{\mathfrak{B}})\Delta(\beta_{ij}),$$

which is our last expression in our computation above. This part uses the fact that rev_{e_r} multiplies each diagram by $(-1)^N = -1$, since N is odd. ∎

Now, that we have established diagrammatic interpretations of our maps $F_{\mathfrak{B}}$ and $F_{\mathfrak{T}}$, we can observe that they satisfy a compatibility with our splitting maps. Suppose that a is a boundary arc of a triangle $\mathfrak{T}$.

Lemma 12.7. *Our Frobenius maps F commute with Δ_a and ${}_a\Delta$ in the sense that*

$$(F_{\mathfrak{T}} \otimes F_{\mathfrak{B}})\Delta_a = \Delta_a F_{\mathfrak{T}}$$

and

$$(F_{\mathfrak{T}} \otimes F_{\mathfrak{B}})_a\Delta = {}_a\Delta F_{\mathfrak{T}}.$$

Proof. This follows from the fact that $\Delta \circ F_{\mathfrak{B}} = (F_{\mathfrak{B}} \otimes F_{\mathfrak{B}}) \circ \Delta$ and from an embedding $\mathfrak{B} \sqcup \mathfrak{B} \hookrightarrow \mathfrak{B} \sqcup \mathfrak{T}$. ∎

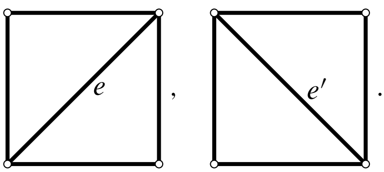
We can now construct a Frobenius map F_Σ for any ideal triangulable Σ.

Proposition 12.8. *Suppose Σ has an ideal triangulation with a set of interior edges $\mathscr{E}$. There exists an algebra embedding $F_{\Sigma,\mathscr{E}}$ of $\mathscr{S}_1^{\mathrm{SL}_3}(\Sigma)$ into the center $Z(\mathscr{S}_q^{\mathrm{SL}_3}(\Sigma))$ defined as the unique algebra map making the left square in the following diagram commute:*

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{S}_1^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta"] \arrow[d, dashed, "F_{\Sigma,\mathscr{E}}"] & \bigotimes_{i=1}^{n} \mathscr{S}_1^{\mathrm{SL}_3}(\mathfrak{T}_i) \arrow[r, "\Delta^R - \tau \circ {}^L\Delta"] \arrow[d, "\otimes_i F_{\mathfrak{T}_i}"] & \left(\bigotimes_{i=1}^{n} \mathscr{S}_1^{\mathrm{SL}_3}(\mathfrak{T}_i)\right) \otimes \left(\bigotimes_{e \in \mathscr{E}} \mathscr{S}_1^{\mathrm{SL}_3}(\mathfrak{B})\right) \arrow[d, "\otimes_i F_{\mathfrak{T}_i} \otimes_e F_{\mathfrak{B}}"] \\
0 \arrow[r] & \mathscr{S}_q^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta"] & \bigotimes_{i=1}^{n} \mathscr{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i) \arrow[r, "\Delta^R - \tau \circ {}^L\Delta"] & \left(\bigotimes_{i=1}^{n} \mathscr{S}_q^{\mathrm{SL}_3}(\mathfrak{T}_i)\right) \otimes \left(\bigotimes_{e \in \mathscr{E}} \mathscr{S}_q^{\mathrm{SL}_3}(\mathfrak{B})\right)
\end{tikzcd}$$

Proof. The horizontal rows are exact, by our triangular decomposition theorem. The right square commutes by Lemma 12.7. Thus, there exists a unique map of modules $F_{\Sigma,\mathscr{E}}$ as claimed. The map is an injective algebra map because Δ and $F_{\mathfrak{T}_i}$ are injective algebra maps. By the centrality of $F_{\mathfrak{T}_i}$ and the injectivity of Δ, we see that $F_{\Sigma,\mathscr{E}}$ is central. ∎

So, far we have defined $F_{\Sigma,\mathscr{E}}$ in terms of the ideal triangulation $\mathscr{E}$. We next aim to show that if $\mathscr{E}$ and $\mathscr{E}'$ are two ideal triangulations of Σ, then $F_{\Sigma,\mathscr{E}} = F_{\Sigma,\mathscr{E}'}$. As shown in [22] for the SL_2 case, this will follow from showing that $\Sigma = \mathfrak{Q}$ is an ideal square, then $F_{\mathfrak{Q}}$ is the same map for both triangulations of $\mathfrak{Q}$. We illustrate the two triangulations of $\mathfrak{Q}$ here. One triangulation is $\mathscr{E} = \{e\}$ and the other is $\mathscr{E}' = \{e'\}$:



The skein algebra $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{Q})$ has a nice generating set consisting of single corner arcs, single horizontal arcs, single vertical arcs, with all possible strand orientations and state labels.

Proposition 12.9. *Suppose that γ is a stated arc in our generating set for $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{Q})$. Then both $F_{\mathfrak{Q},\mathcal{E}}$ and $F_{\mathfrak{Q},\mathcal{E}'}$ send γ to γ^N. Thus, the map $F_{\mathfrak{Q}}$ is invariant under change of triangulation of $\mathfrak{Q}$.*

Proof. We must check that for the generator $\gamma \in \mathcal{S}_1^{\mathrm{SL}_3}(\mathfrak{Q})$, we have the equalities

$$\Delta_e(\gamma^N) = (F_{\mathfrak{T}} \otimes F_{\mathfrak{T}})\Delta_e$$

and

$$\Delta_{e'}(\gamma^N) = (F_{\mathfrak{T}} \otimes F_{\mathfrak{T}})\Delta_{e'}.$$

If γ can be isotoped so that it does not intersect e then the first equation is obvious. Otherwise, it can be isotoped so that it intersects e exactly once and then the first equality follows from the fact that $F_{\mathfrak{B}}$ is a bialgebra map and from an embedding $\mathfrak{B} \sqcup \mathfrak{B} \hookrightarrow \mathfrak{T} \sqcup \mathfrak{T}$. An analogous argument works for the second equality. ∎

Next, we record a compatibility of $F_{\Sigma,\mathcal{E}}$ with a partial splitting of the triangulation. Suppose a and b are two boundary arcs of a punctured bordered surface Σ and let $\overline{\Sigma} = \Sigma/(a = b)$. Then the common image of a and b on $\overline{\Sigma}$ is an ideal arc we will denote e. Suppose $\overline{\Sigma}$ has an ideal triangulation with set of interior edges $\mathcal{E}$ with $e \in \mathcal{E}$. Then Σ naturally inherits an ideal triangulation with edge set $\mathcal{E} \setminus \{e\}$. We are interested in the relationship between $F_{\overline{\Sigma},\mathcal{E}}$ and $F_{\Sigma,\mathcal{E}\setminus\{e\}}$.

Proposition 12.10. *We have that $F_{\overline{\Sigma},\mathcal{E}}$ is equal to the unique algebra map making the following diagram commute:*

$$\begin{tikzcd} \mathcal{S}_1^{\mathrm{SL}_3}(\overline{\Sigma}) \arrow[r, "\Delta_e"] \arrow[d, dashed, "F_{\overline{\Sigma},\mathcal{E}}"] & \mathcal{S}_1^{\mathrm{SL}_3}(\Sigma) \arrow[d, "F_{\Sigma,\mathcal{E}\setminus\{e\}}"] \\ \mathcal{S}_q^{\mathrm{SL}_3}(\overline{\Sigma}) \arrow[r, "\Delta_e"] & \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma). \end{tikzcd}$$

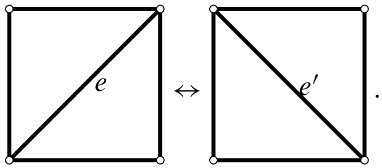
Proof. We examine the following diagram:

$$\begin{tikzcd} \mathcal{S}_1^{\mathrm{SL}_3}(\overline{\Sigma}) \arrow[r, "\Delta_e"] \arrow[d, "F_{\overline{\Sigma},\mathcal{E}}"] & \mathcal{S}_1^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta_{\mathcal{E}\setminus\{e\}}"] \arrow[d, "F_{\Sigma,\mathcal{E}\setminus\{e\}}"] & \bigotimes_i^n \mathfrak{T}_i \arrow[d, "\bigotimes_i F_{\mathfrak{T}_i}"] \\ \mathcal{S}_q^{\mathrm{SL}_3}(\overline{\Sigma}) \arrow[r, "\Delta_e"] & \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta_{\mathcal{E}\setminus\{e\}}"] & \bigotimes_i^n \mathfrak{T}_i \end{tikzcd}$$

The outer rectangle and the right square both commute by the definitions of $F_{\overline{\Sigma},\mathcal{E}}$ and $F_{\Sigma,\mathcal{E}\setminus\{e\}}$. Thus, the left square commutes. The injectivity of Δ_e and $F_{\Sigma,\mathcal{E}\setminus\{e\}}$ imply the uniqueness of $F_{\overline{\Sigma},\mathcal{E}}$. ∎

Corollary 12.11. Suppose that Σ is a punctured bordered surface with an ideal triangulation with a set of internal edges $\mathcal{E}$. The map $F_{\Sigma,\mathcal{E}}$ does not depend on the triangulation $\mathcal{E}$.

Proof. Suppose that Σ has a second ideal triangulation $\mathcal{E}'$. Then $\mathcal{E}'$ may be obtained from $\mathcal{E}$ by a finite sequences of edge flips involving an internal edge that borders two distinct faces. Thus, without loss of generality, we can assume that $\mathcal{E}$ and $\mathcal{E}'$ are identical except for a single edge flip in a square $\mathfrak{Q}$:



Let $\Delta_{\mathfrak{Q}}\colon \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \to \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma \setminus \mathfrak{Q})$ be the composition of splitting maps associated to cutting the square $\mathfrak{Q}$ out of Σ.

By Proposition 12.9, $F_{\mathfrak{Q}}$ does not depend on its triangulation. A repeated application of Proposition 12.10 implies that since both $F_{\Sigma,\mathcal{E}}$ and $F_{\Sigma,\mathcal{E}'}$ make the diagram

$$\begin{tikzcd}
\mathcal{S}_1^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta_{\mathfrak{Q}}"] \arrow[d] & \mathcal{S}_1^{\mathrm{SL}_3}(\Sigma \setminus \mathfrak{Q}) \otimes \mathcal{S}_1^{\mathrm{SL}_3}(\mathfrak{Q}) \arrow[d, "F_{\Sigma\setminus Q}\otimes F_{\mathfrak{Q}}"] \\
\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta_{\mathfrak{Q}}"] & \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma \setminus \mathfrak{Q}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{Q})
\end{tikzcd}$$

commute, we have the equality $F_{\Sigma,\mathcal{E}} = F_{\Sigma,\mathcal{E}'}$. ∎

So far, we have shown that Theorem 12.1 is true for any ideal triangulable surface Σ and that the definition of F_Σ in these cases does not depend on the triangulation. We now briefly comment on the surfaces with at least one puncture which do not admit an ideal triangulation.

Proposition 12.12. *The four punctured bordered surfaces which do not admit a triangulation admit a Frobenius map.*

Proof. The four surfaces are the monogon $\mathfrak{M}$, the bigon $\mathfrak{B}$, and the sphere with 2 or 1 punctures. If the surface Σ is the monogon or the sphere with one puncture, then $\mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \cong \mathcal{R}$, which is commutative. So, in these cases, F_Σ is determined by the fact that it sends the empty diagram to the empty diagram.

If $\Sigma = \mathfrak{B}$, we have already constructed $F_{\mathfrak{B}}$ as the map from [28] from $\mathcal{O}_1(\mathrm{SL}_3)$ to $\mathcal{O}_q(\mathrm{SL}_3)$.

If Σ is the sphere with 2 punctures, then we let c be an ideal arc connecting the two punctures and we define F_Σ to be the unique map making the following diagram commute:

$$\begin{tikzcd} \mathcal{S}_1^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta_c"] \arrow[d, dashed, "F_\Sigma"] & \mathcal{S}_1^{\mathrm{SL}_3}(\mathfrak{B}) \arrow[d, "F_{\mathfrak{B}}"] \\ \mathcal{S}_q^{\mathrm{SL}_3}(\Sigma) \arrow[r, "\Delta_c"] & \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \end{tikzcd}$$ ■

In this section, we have defined our Frobenius morphism F_Σ locally, in mostly an algebraic manner, and extended it to the whole surface. We have shown that for a triangulable surface Σ, the map F_Σ does not depend on the triangulation, and so is canonical in some sense. However, there should be a nice global definition of F_Σ that can be given without reference to a triangulation, and one which will generalize to the case of skein algebras of closed surfaces and skein modules of 3-manifolds. We would hope for a description of the image of an arbitrary web with a single connected component. For example, it is certain that a stated arc α should be sent to its N-th framed power. A knot should be threaded through an SL_3 analogue of the N-th Chebyshev polynomial analogous to the SL_2 constructions in [2, 4, 22, 25]. It is unclear what should be the image of a more complicated web, so it would be interesting to find a nice description for it. These questions are beyond the scope of the current paper but deserve to be explored in the future.

13. Kuperberg's SL$_3$ spider

The skein relations referred to as *interior relations* in this paper (relations (I1a)–(I4b) of Section 2) first appeared in [19, 23]. In the paper [23], Kuperberg noted that the link invariant computed by the skein relations is an explicit example of the SL_3 case of a more general construction of invariants of ribbon graphs arising from quantum groups due to the work of Reshetikhin and Turaev [29]. In a followup work, [24], Kuperberg showed that the collection of SL_3 webs, called the SL_3 *spider,* encodes a full subcategory of the representation category of $U_q(\mathfrak{sl}_3)$.

Roughly speaking, Kuperberg's spider can be interpreted as a category of *unstated* webs, $\mathrm{Web}^{\mathrm{SL}_3}$. An unstated web in a rectangle encodes a morphism of $U_q(\mathfrak{sl}_3)$-modules with the boundary data of the web encoding the domain and codomain of the morphism as tensor products of the standard representation of $U_q(\mathfrak{sl}_3)$ and its dual. The work of [29] describes a monoidal functor from the category $\mathrm{Web}^{\mathrm{SL}_3}$ to the category of $U_q(\mathfrak{sl}_3)$-modules. The work of Kuperberg in [24] shows that this functor is faithful and that the image of the functor is onto the full subcategory of $U_q(\mathfrak{sl}_3)$-mod generated by tensor products of the standard representation of $U_q(\mathfrak{sl}_3)$ and its dual.

Kuperberg achieved his result by using the interior skein relations to construct spanning sets of webs which enabled him to bound the dimensions of Hom-spaces in $\mathrm{Web}^{\mathrm{SL}_3}$. He then used results from Lie theory to establish the fullness of the functor and used a dimension count to establish the faithfulness of the functor. Consequently, he also proved that the spanning sets of webs were bases for the Hom-spaces in $\mathrm{Web}^{\mathrm{SL}_3}$.

Our identification of the stated skein algebra of the bigon with the quantum group $\mathcal{O}_q(\mathrm{SL}_3)$ allows us in this section to recover the result of Kuperberg in the language of $\mathcal{O}_q(\mathrm{SL}_3)$-comodules. Our web basis of *stated webs* from Section 5 is a finer tool than the web basis of *unstated webs* and it allows us to prove the fullness and faithfulness of the Reshetikhin–Turaev functor by using our results from Section 8 about the splitting map for the stated skein algebra.

Later in this section, we give precise definitions, in the context of skein algebras, of the categories $\mathrm{Web}^{\mathrm{SL}_3}$ and $\mathcal{O}_q(\mathrm{SL}_3)\text{-comod}\langle V\rangle$, and of the Reshetikhin–Turaev functor RT. The main result of the section is the following.

Theorem 13.1. *The Reshetikhin–Turaev functor* $\mathrm{RT}\colon \mathrm{Web}^{\mathrm{SL}_3} \to \mathcal{O}_q(\mathrm{SL}_3)\text{-comod}\langle V\rangle$ *is an isomorphism of braided monoidal categories.*

The theorem will follow from an interpretation of the exact sequence associated to our splitting map.

To define relevant categories and functors, it will be convenient to introduce modified versions of $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ in which we allow for one or both boundary arcs of $\mathfrak{B}$ to be designated to contain endpoints of webs *without* states and in which we do not impose any boundary skein relations along the designated boundary arcs. We can call such a boundary arc an *inactive* boundary arc. In our notation, we will use "_" on the right or left of $\mathfrak{B}$ to indicate an inactive boundary arc, which is one designated to have endpoints which are not labeled by states. For example, $\mathcal{S}_q^{\mathrm{SL}_3}(\_\mathfrak{B}\_)$ denotes the skein algebra of webs in the bigon with endpoints unlabeled by states and subject to only the interior skein relations. The notation $\mathcal{S}_q^{\mathrm{SL}_3}(\_\mathfrak{B})$ denotes the skein algebra of webs in the bigon such that any endpoints on the left boundary arc of $\mathfrak{B}$ are unlabeled by states (but endpoints on the right boundary arc are labeled by states), and which is subject to only the interior skein relations and stated skein relations along the right boundary arc. Similarly, the skein algebra $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}\_)$ denotes the skein algebra of webs in the bigon such that any endpoints on the right boundary arc of $\mathfrak{B}$ are unlabeled by states, and which is subject to only the interior skein relations and the stated skein relations along the left boundary arc.

Our theorems involving bases and splitting maps carry over to the situation of inactive boundary arcs. We use these modified versions of skein algebras to define certain categories and functors. First we observe that these new versions of our skein algebras admit a module decomposition in terms of boundary data of webs. Let $\vec{a}$

be a sequence of left and right arrows $\vec{a} = (a_1, \dots, a_k)$ for some $k \geq 0$ with each $a_i \in \{\leftarrow, \rightarrow\}$.

In this section, we will identify the state $+$ with the integer 1, the state 0 with the integer 2 and the state $-$ with the integer 3.

Definition 13.2. For an arrow sequence $\vec{a}$ we define $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ to be the submodule of $\mathcal{S}_q^{\mathrm{SL}_3}(\_\mathfrak{B})$ spanned by webs whose left boundary data, read from top to bottom, agrees with the arrow sequence $\vec{a}$. Similarly, we define $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$ to be the submodule of $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}\_)$ spanned by webs whose right boundary data, read from top to bottom, agrees with the arrow sequence $\vec{a}$. Finally, for two arrow sequences $\vec{a}, \vec{b}$ we define $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ to be the submodule of $\mathcal{S}_q^{\mathrm{SL}_3}(\_\mathfrak{B}\_)$ spanned by webs whose left boundary data agrees with $\vec{b}$ and whose right boundary data agrees with $\vec{a}$.

Proposition 13.3. *Our algebras are graded with respect to the following decompositions as $\mathcal{R}$-modules:*

(i) $\mathcal{S}_q^{\mathrm{SL}_3}(\_\mathfrak{B}) = \bigoplus_{\vec{a}} \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$, *where the direct sum is over all possible arrow sequences $\vec{a}$;*

(ii) $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}\_) = \bigoplus_{\vec{a}} \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$, *where the direct sum is over all possible arrow sequences $\vec{a}$;*

(iii) $\mathcal{S}_q^{\mathrm{SL}_3}(\_\mathfrak{B}\_) = \bigoplus_{\vec{a},\vec{b}} \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$, *where the direct sum is over all possible arrow sequences $\vec{a}, \vec{b}$.*

Proof. The proposition follows from the fact that none of our reduction rules coming from our diamond lemma algorithm will change the boundary data of a web along an inactive boundary arc, and so the algebras are graded with respect to this data. ∎

13.1. The category $\mathrm{Web}^{\mathrm{SL}_3}$

The first category we will define is Kuperberg's SL_3 web category, modified to our setting. The category $\mathrm{Web}^{\mathrm{SL}_3}$ is the monoidal $\mathcal{R}$-linear category consisting of the following data:

- an object $\vec{a}$ of $\mathrm{Web}^{\mathrm{SL}_3}$ is a sequence of arrows $\vec{a} = (a_1, a_2, \dots, a_k)$ for some $k \geq 0$ where $a_i \in \{\leftarrow, \rightarrow\}$;
- $\mathrm{Hom}(\vec{a}, \vec{b})$ is the module $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$;
- the composition of morphisms is defined on diagrams $D \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ and $E \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{c}}\mathfrak{B}_{\vec{b}})$ by horizontally gluing E on the left of D to obtain a diagram $E \circ D$ in $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{c}}\mathfrak{B}_{\vec{a}})$;
- the tensor product $\vec{a} \otimes \vec{b}$ of objects $\vec{a}$ and $\vec{b}$ is the concatenation $(\vec{a}, \vec{b})$. The tensor product of morphisms is then given by the product operation in $\mathcal{S}_q^{\mathrm{SL}_3}(\_\mathfrak{B}\_)$.

13.2. The category $\mathcal{O}_q(\mathrm{SL}_3)$-comod$\langle V\rangle$

We next give the definition of our category $\mathcal{O}_q(\mathrm{SL}_3)$-comod$\langle V\rangle$ and then give it a diagrammatic interpretation. The category $\mathcal{O}_q(\mathrm{SL}_3)$-comod$\langle V\rangle$ is the full subcategory of right $\mathcal{O}_q(\mathrm{SL}_3)$ comodules tensor-generated by the standard rank 3 comodule $V_{\rightarrow}$ and its dual $V_{\leftarrow}$.

Before giving a precise definition of our category, we fix conventions for the standard comodule $V_{\rightarrow}$ and its dual $V_{\leftarrow}$. We let $V_{\rightarrow}$ be the free $\mathcal{R}$-module with basis v_1, v_2, v_3 with coaction $V_{\rightarrow} \to V_{\rightarrow} \otimes \mathcal{O}_q(\mathrm{SL}_3)$ given by

$$v_i \mapsto \sum_{j=1}^{3} v_j \otimes X_{ji}.$$

Due to conventions associated to our definition of the stated skein relations and the splitting map, we will use a nonstandard weight basis of $V_{\leftarrow}$, meaning that our basis will not be the dual basis of our basis for $V_{\rightarrow}$. We let $V_{\leftarrow}$ be the free $\mathcal{R}$-module with basis w_1, w_2, w_3 with coaction $V_{\leftarrow} \to V_{\leftarrow} \otimes \mathcal{O}_q(\mathrm{SL}_3)$ given by

$$w_i \mapsto \sum_{j=1}^{3} w_j \otimes q^{2i-2j} S(X_{\bar{i}\bar{j}}),$$

where we use the notation $\bar{k} = 4 - k$.

Given a sequence of arrows $\vec{a} = (a_1, \ldots, a_k)$, we denote by $V_{\vec{a}}$ the tensor product

$$V_{\vec{a}} = V_{a_1} \otimes V_{a_2} \otimes \cdots \otimes V_{a_k}.$$

The category $\mathcal{O}_q(\mathrm{SL}_3)$-comod$\langle V\rangle$ consists of the following data:

- objects are the modules $V_{\vec{a}}$, which are finite tensor products of copies of $V_{\rightarrow}$ and $V_{\leftarrow}$;
- morphisms are $\mathcal{R}$-linear maps between objects which commute with the right coaction of $\mathcal{O}_q(\mathrm{SL}_3)$. We call the set of morphisms $\mathrm{Hom}_{\mathcal{O}_q(\mathrm{SL}_3)}(V_{\vec{a}}, V_{\vec{b}})$.

Recall that the splitting map

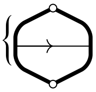
$$\Delta\colon \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B}) \to \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$$

gives $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ the structure of a right $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B})$ comodule, which is a right $\mathcal{O}_q(\mathrm{SL}_3)$ comodule structure when we use the identification $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \cong \mathcal{O}_q(\mathrm{SL}_3)$. We find the following diagrammatic interpretation of the objects of our category.

Proposition 13.4. *Given a sequence of arrows $\vec{a}$, we have that $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B}) \cong V_{\vec{a}}$ as $\mathcal{O}_q(\mathrm{SL}_3)$ comodules.*

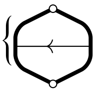
Proof. We first look at the generating cases. If $\vec{a}$ happens to be the empty sequence, then both $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ and $V_{\vec{a}}$ are isomorphic to $\mathcal{R}$ with the trivial comodule structure.

If $\vec{a} = (\rightarrow)$, then $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ has a basis

$\Bigg\{$  $i \Bigg\}_{i=1}^{3}$

and the image of this basis under the splitting map agrees with the coaction on the basis $\{v_i\}_{i=1}^3$ of $V_{\rightarrow}$.

Similarly, if $\vec{a} = (\leftarrow)$ then $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ has a basis

$\Bigg\{$  $i \Bigg\}_{i=1}^{3}$

and the image of this basis under the splitting map agrees with the coaction on the basis $\{w_i\}_{i=1}^3$ of $V_{\leftarrow}$.

If $\vec{a}$ is an arbitrary sequence of arrows, then a basis of $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ consists of a product of basis elements of $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\leftarrow}\mathfrak{B})$ and $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\rightarrow}\mathfrak{B})$. Since the splitting map is an algebra map, we have that the image of this basis under the splitting map agrees with the coaction on the standard tensor basis of $V_{\vec{a}}$. ∎

Next, we provide a diagrammatic interpretation of some of the morphisms of our category.

Proposition 13.5. *Given a diagram E in $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ and a diagram D in $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$, we obtain a diagram $E \circ D$ in $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B})$ by gluing horizontally. This gluing defines a linear map $E\colon \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B}) \to \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B})$. The linear map commutes with the coaction.*

Proof. The equation $(E \otimes \mathrm{id})\Delta(D) = \Delta(E \circ D)$ can be seen diagrammatically, so E commutes with the coaction. ∎

We now have the ingredients to define our Reshetikhin–Turaev functor.

Proposition 13.6. *We produce a functor* $\mathrm{RT}\colon \mathrm{Web}^{\mathrm{SL}_3} \to \mathcal{O}_q(\mathrm{SL}_3)\text{-comod}\langle V\rangle$ *in the following manner. On objects, we define* $\mathrm{RT}(\vec{a}) = V_{\vec{a}}$*, which we have identified with* $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$*. On morphisms,* RT *is the identity on the module* $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$*, which we have previously identified as a submodule of* $\mathrm{Hom}_{\mathcal{O}_q(\mathrm{SL}_3)}(V_{\vec{a}}, V_{\vec{b}})$.

We will eventually show that RT is an isomorphism of categories. First, we will need a diagrammatic interpretation of $\mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}})$.

13.3. The category split($\mathrm{Web}^{\mathrm{SL}_3}$)

The category split($\mathrm{Web}^{\mathrm{SL}_3}$) is the monoidal $\mathcal{R}$-linear category consisting of the following data:

- an object $\vec{a}$ of split($\mathrm{Web}^{\mathrm{SL}_3}$) is a sequence of arrows;
- $\mathrm{Hom}(\vec{a}, \vec{b})$ is the module $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$;
- the composition of morphisms is defined on diagrams

$$D_1 \otimes D_2 \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$$

  and

$$E_1 \otimes E_2 \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{c}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{b}})$$

  by gluing E_2 on the left of D_1, taking the counit, and obtaining

$$\varepsilon(E_2 \circ D_1) E_1 \otimes D_2 \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{c}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}});$$

- the tensor product $\vec{a} \otimes \vec{b}$ of objects $\vec{a}$ and $\vec{b}$ is the concatenation $(\vec{a}, \vec{b})$. The tensor product of morphisms is then given by the product operation in $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\_}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\_})$.

13.4. The category $\mathcal{R}$-comod$\langle V \rangle$

We now give the definition of the category $\mathcal{R}$-comod$\langle V \rangle$ and then give it a diagrammatic interpretation. The category $\mathcal{R}$-comod$\langle V \rangle$ is the full subcategory of right $\mathcal{R}$-comodules tensor generated by the standard rank 3 comodule $V_{\rightarrow}$ and its dual $V_{\leftarrow}$. The coaction of $\mathcal{R}$ is the trivial coaction $V_{\vec{a}} \to V_{\vec{a}} \otimes \mathcal{R}$. So, it does no harm to think of this category as the full subcategory of $\mathcal{R}$-modules tensor generated by $V_{\rightarrow}$ and $V_{\leftarrow}$.

We record the data of our category $\mathcal{R}$-comod$\langle V \rangle$.

- Objects are the modules $V_{\vec{a}}$, which are finite tensor products of copies of $V_{\rightarrow}$ and $V_{\leftarrow}$.
- Morphisms are $\mathcal{R}$-linear maps between objects which commute with the (trivial) right coaction of $\mathcal{R}$. We call the set of morphisms $\mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}})$.

Proposition 13.7. *We have the following:*

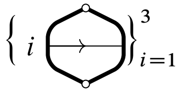
(i) $V_{\vec{b}} \cong \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B})$ *as $\mathcal{R}$-comodules;*

(ii) $(V_{\vec{a}})^* \cong \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$ *as $\mathcal{R}$-comodules, with evaluation of $E \in \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$ and $D \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ given by gluing horizontally and taking the counit to obtain $\varepsilon(E \circ D)$;*

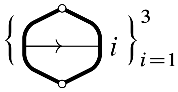
(iii) $\mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}}) \cong \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$.

Proof. We already proved (i) holds for $\mathcal{O}_q(\mathrm{SL}_3)$-comodules, so it holds for $\mathcal{R}$-comodules as well.

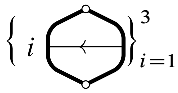
Under the pairing described in (ii), we have that the basis



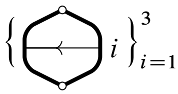
of $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\rightarrow})$ and the basis



of $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\rightarrow}\mathfrak{B})$ are dual bases. Similarly, the bases



and



are dual to each other. Thus, for an arbitrary arrow sequence $\vec{a}$, the standard basis of the tensor product $\mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$ is dual to the standard basis of the tensor product $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$.

The statement (iii) follows from the property $\mathrm{Hom}_{\mathcal{R}}(V_{\vec{a}}, V_{\vec{b}}) \cong V_{\vec{b}} \otimes (V_{\vec{a}})^*$. ∎

We now have the ingredients to prove a category isomorphism.

Proposition 13.8. *The following functor* $\mathrm{split}(\mathrm{RT})\colon \mathrm{split}(\mathrm{Web}^{\mathrm{SL}_3}) \to \mathcal{R}\text{-comod}\langle V\rangle$ *defines an isomorphism of categories:*

- *on objects,* $\mathrm{split}(\mathrm{RT})(\vec{a}) = V_{\vec{a}}$, *which is identified with* $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$;
- *on morphisms,* $\mathrm{split}(\mathrm{RT})$ *is the identity on* $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$.

13.5. Relating the categories

So, far we have discussed four $\mathcal{R}$-linear monoidal categories. We next observe that they fit into a commutative diagram of categories.

Proposition 13.9. *The following diagram of categories is commutative:*

$$\begin{tikzcd}
\mathrm{Web}^{\mathrm{SL}_3} \arrow[r, "\Delta"] \arrow[d, "\mathrm{RT}"] & \mathrm{split}(\mathrm{Web}^{\mathrm{SL}_3}) \arrow[d, "\mathrm{split(RT)}"] \\
\mathcal{O}_q^{\mathrm{SL}_3}\text{-comod}\langle V\rangle \arrow[r, "\mathrm{incl}"] & \mathcal{R}\text{-comod}\langle V\rangle
\end{tikzcd}$$

where the functor Δ *is defined as*

- *on objects,* $\Delta(\vec{a}) = \vec{a}$;
- *on morphisms,* $\Delta\colon \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}}) \to \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec{a}})$ *is the splitting map.*

Proof. We first address the functoriality of Δ. It respects the monoidal structure since Δ is an algebra map. We need to check that it respects compositions of diagrams. Suppose that $D \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ and $E \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{c}}\mathfrak{B}_{\vec{b}})$ are diagrams. We need to check that

$$\Delta(E \circ D) = \Delta(E) \circ \Delta(D).$$

Before we check this with a computation, we introduce some notation. Given an arrow sequence $\vec{b} = (b_1, \dots, b_k)$ we let $\mathrm{St}(\vec{b}) = \{1,2,3\}^k$ denote the set of sequences of states of the same length as $\vec{b}$. To verify that our equality holds, we choose to cut E very close to its right boundary so that

$$\Delta(E) = \sum_{v \in \mathrm{St}(\vec{b})} E_v \otimes {}_v E''$$

such that each diagram ${}_v E''$ consists of only parallel strands whose left endpoints are labeled with a sequence of states corresponding to the standard basis vector $v \in V_{\vec{b}}$ and each diagram E_v is the same underlying diagram as E but with its right endpoints labeled with states corresponding to v. Similarly, we choose to cut D very close to its left boundary so that

$$\Delta(D) = \sum_{w \in \mathrm{St}(\vec{b})} D'_w \otimes {}_w D$$

such that each diagram D'_w consists of only parallel strands whose right endpoints are labeled with a sequence of states corresponding to the standard basis vector $w \in V_{\vec{b}}$. This allows us to compute that

$$\begin{aligned}\Delta(E) \circ \Delta(D) &= \sum_{v,w \in \mathrm{St}(V_{\vec{b}})} \varepsilon({}_v E'' \circ D'_w) E_v \otimes {}_w D = \sum_{v,w \in \mathrm{St}(V_{\vec{b}})} \delta_{vw} E_v \otimes {}_w D \\ &= \sum_{v \in \mathrm{St}(V_{\vec{b}})} E_v \otimes {}_v D = \Delta(E \circ D),\end{aligned}$$

as required.

Next, we check that the diagram commutes. We see that it commutes for objects, so we need to check that it commutes for morphisms. We can check this on a diagram. Let $E \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}_{\vec{a}})$ be a diagram. Then $\mathrm{incl}(\mathrm{RT}(E))\colon \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B}) \to \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B})$ is defined on a diagram $D \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{a}}\mathfrak{B})$ by gluing to obtain the diagram

$$E \circ D \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec{b}}\mathfrak{B}).$$

On the other hand, $\mathrm{split}(\mathrm{RT})\Delta(E) = \sum_{(E)} E' \otimes E''$ is a morphism $\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec a}\mathfrak{B}) \to \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec b}\mathfrak{B})$ which sends a diagram $D \in \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec a}\mathfrak{B})$ to

$$\sum_{(E)} E'\varepsilon(E'' \circ D) = \sum_{(E\circ D)} (E \circ D)' \otimes \varepsilon((E \circ D)'') = E \circ D,$$

by the counit axiom. So, the diagram commutes. ∎

13.6. Proof that RT is an isomorphism

We now will observe that RT is an isomorphism on Hom-modules. The following proposition is a consequence of the identifications we have established in this section.

Proposition 13.10. *The following diagram of $\mathcal{R}$-modules commutes:*

$$\begin{tikzcd}
\mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec b}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec a}) \arrow[r, "\Delta_{\vec b \mathfrak{B}} - \Delta_{\mathfrak{B}_{\vec a}}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}"] & \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec b}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec a}) \arrow[d, "\mathrm{id}\otimes\psi\otimes\mathrm{id}"] \\
\mathrm{Hom}_{\mathcal{R}}(V_{\vec a}, V_{\vec b}) \arrow[r, "\Delta_{V_{\vec b}}\circ(-)-((-)\otimes\mathrm{id})\circ\Delta_{V_{\vec a}}"] & \mathrm{Hom}_{\mathcal{R}}(V_{\vec a}, V_{\vec b} \otimes \mathcal{O}_q(\mathrm{SL}_3))
\end{tikzcd}$$

where $\psi\colon \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \to \mathcal{O}_q(\mathrm{SL}_3)$ is our isomorphism from Proposition 9.2 and we have used identifications in the bottom row of the form $\mathrm{Hom}_{\mathcal{R}}(X, Y) = Y \otimes X^*$, *so that the vertical maps make sense.*

Corollary 13.11. The RT functor is an isomorphism of $\mathcal{R}$-linear braided monoidal categories $\mathrm{Web}^{\mathrm{SL}_3} \to \mathcal{O}_q(\mathrm{SL}_3)$-comod$\langle V\rangle$.

Proof. The functor RT is bijective on objects, so we just need to show that it induces isomorphisms on Hom-sets. For that we observe the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec b}\mathfrak{B}_{\vec a}) \arrow[r, "\Delta"] \arrow[d, "\mathrm{RT}"] & \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec b}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec a}) \arrow[r, "\Delta_{\vec b \mathfrak{B}} - \Delta_{\mathfrak{B}_{\vec a}}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}"] & \mathcal{S}_q^{\mathrm{SL}_3}({}_{\vec b}\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}) \otimes \mathcal{S}_q^{\mathrm{SL}_3}(\mathfrak{B}_{\vec a}) \arrow[d, "\mathrm{id}\otimes\psi\otimes\mathrm{id}"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_q(\mathrm{SL}_3)}(V_{\vec a}, V_{\vec b}) \arrow[r, "\mathrm{incl}"] & \mathrm{Hom}_{\mathcal{R}}(V_{\vec a}, V_{\vec b}) \arrow[r, "\Delta_{V_{\vec b}}\circ(-)-((-)\otimes\mathrm{id})\circ\Delta_{V_{\vec a}}"] & \mathrm{Hom}_{\mathcal{R}}(V_{\vec a}, V_{\vec b} \otimes \mathcal{O}_q(\mathrm{SL}_3))
\end{tikzcd}$$

The top row is exact by Theorem 8.1 and Theorem 8.2 and the bottom row is exact by the definition of a morphism of $\mathcal{O}_q(\mathrm{SL}_3)$-comod$\langle V\rangle$. The vertical maps in the middle and the right are isomorphisms. Thus, RT is an isomorphism as well, by a special case of the five lemma. ∎

Finally, we will observe that since the pairing $\langle -, - \rangle \colon \mathcal{O}_q(\mathrm{SL}_3) \otimes U_q(\mathfrak{sl}_3)$ turns right $\mathcal{O}_q(\mathrm{SL}_3)$-comodules into left $U_q(\mathfrak{sl}_3)$-modules, we obtain an embedding of categories

$$\mathrm{Web}^{\mathrm{SL}_3} \xrightarrow{\cong} \mathcal{O}_q(\mathrm{SL}_3)\text{-comod}\langle V \rangle \hookrightarrow U_q(\mathfrak{sl}_3)\text{-mod}\langle V \rangle.$$

If the pairing is nondegenerate, then the embedding will be an isomorphism. We can see this after observing the following.

Lemma 13.12. *Suppose the pairing $\langle -, - \rangle \colon \mathcal{O}_q(\mathrm{SL}_3) \otimes U_q(\mathfrak{sl}_3) \to \mathcal{R}$ is nondegenerate. If U and W are right $\mathcal{O}_q(\mathrm{SL}_3)$-comodules and $T \colon U \to W$ is an $\mathcal{R}$-linear map which commutes with the induced left $U_q(\mathfrak{sl}_3)$ action on U and W, then T commutes with the $\mathcal{O}_q(\mathrm{SL}_3)$ coaction as well.*

Proof. Fix an arbitrary $u \in U$. By assumption, we have that for any $x \in U_q(\mathfrak{sl}_3)$,

$$x.T(u) = T(x.u).$$

We expand both sides of this equation by the definitions of the actions in terms of the pairing. The left side is

$$x.T(u) = (\mathrm{id} \otimes \langle -, x \rangle)\Delta_W(T(u)).$$

The right side is

$$T(x.u) = T((\mathrm{id} \otimes \langle -, x \rangle)\Delta_U(U)) = (\mathrm{id} \otimes \langle -, x \rangle)(T \otimes \mathrm{id})\Delta_U(u).$$

These equations hold for all $x \in U_q(\mathfrak{sl}_3)$ and so by considering bases of U and W, we are able to use the fact that the pairing is nondegenerate to conclude that

$$(T \otimes \mathrm{id})\Delta_U(u) = \Delta_W T(u),$$

and T commutes with the coaction. ∎

Corollary 13.13. Whenever the pairing $\langle -, - \rangle \colon \mathcal{O}_q(\mathrm{SL}_3) \otimes U_q(\mathfrak{sl}_3) \to \mathcal{R}$ is nondegenerate, our Reshetikhin–Turaev functor gives an equivalence of braided monoidal categories $\mathrm{Web}^{\mathrm{SL}_3} \to U_q(\mathfrak{sl}_3)\text{-mod}\langle V \rangle$.

Remark 13.14. When $\mathcal{R} = \mathbb{C}$ and q is not a root of unity, then the pairing $\langle -, - \rangle$ is nondegenerate, as discussed in [32]. To work at a root of unity, one can replace $U_q(\mathfrak{sl}_3)$ with a form of Lusztig's divided powers algebra studied in [9].

Using a similar method as in the proof of Theorem 8.2, we can use the reduction rules from Theorem 5.3 to define an explicit algorithm which takes as input a morphism in split($\mathrm{Web}^{\mathrm{SL}_3}$) which commutes with the coaction and gives as output a morphism in $\mathrm{Web}^{\mathrm{SL}_3}$. The algorithm gives us a diagrammatic description of the inverse of the Reshetikhin–Turaev functor.

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Vijay Higgins
Department of Mathematics, Michigan State University, 619 Red Cedar Rd, C212 Wells Hall, East Lansing, MI 48824, USA; higgi231@msu.edu