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Exotic Lagrangian tori in Grassmannians

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Abstract. We describe an iterative construction of Lagrangian tori in the complex Grassmannian $\mathrm{Gr}(k, n)$, based on the cluster algebra structure of the coordinate ring of a mirror Landau–Ginzburg model proposed by Marsh and Rietsch (2020). Each torus comes with a Laurent polynomial, and local systems controlled by the k-variables Schur polynomials at the n-th roots of unity. We use this data to give examples of monotone Lagrangian tori that are neither displaceable nor Hamiltonian isotopic to each other, and that support nonzero objects in different summands of the spectral decomposition of the Fukaya category over $\mathbb{C}$.

1. Introduction

1.1. Lagrangian tori

The construction and classification of Lagrangian submanifolds is a driving question in symplectic topology, with Lagrangian tori having a prominent role. One reason for this is the origin of the field in the Hamiltonian formulation of classical mechanics. In this context, the Arnold–Liouville theorem constrains the level sets of a completely integrable system to be Lagrangian tori; see, e.g., Duistermaat [12]. A more recent motivation is the geometric description of mirror symmetry, where Lagrangian tori arise as generic fibers of Strominger–Yau–Zaslow fibrations [44]. Lagrangian tori are also of interest in low-dimensional topology: the Luttinger surgery [27] operation was used by Auroux, Donaldson, and Katzarkov [4] to study symplectic isotopy classes of plane curves; Vidussi [48] and Fintushel and Stern [13] found connections between Seiberg–Witten invariants and Lagrangian tori. In general dimension, Lagrangian tori in the standard symplectic $\mathbb{R}^{2n}$ have been the subject of much investigation: Viterbo [49] and Buhovsky [5] constrained their Maslov class; Chekanov [9] classified those of product type [9]; Chekanov and Schlenk [8], and Auroux [3] constructed examples that are not products.

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1.2. Disk potentials

A unifying way to think about these results is to consider Lagrangian tori $L \subset \mathbb{R}^{2n} = \mathbb{C}^n$ as boundary conditions for maps $u: D^2 \to \mathbb{C}^n$ satisfying the nonlinear Cauchy–Riemann type equation $\overline{\partial}_J(u) = 0$, where J is an almost-complex structure on the target that may vary from point to point and be non-integrable. One can try to understand how J-holomorphic disks change as L is deformed through Lagrangian embeddings; many known results focus on deformations by Hamiltonian isotopies. This line of thought generalizes to the global case, when $L \subset X$ is not in a Darboux chart of the symplectic manifold X; however, J-holomorphic disks are not easy to describe for an arbitrary target X. Since the work of Floer [14] and Oh [34], the monotone case has been the focus of much investigation. A symplectic manifold (X^{2N}, ω) is monotone if $[\omega]$ and the first Chern class $c_1(X)$ are positively proportional in $H^2(X;\mathbb{R})$; a Lagrangian $L \subset X$ is monotone if the area $\omega(\beta)$ of disk classes $\beta \in H_2(X, L; \mathbb{R})$ is positively proportional to their Maslov index $\mu(\beta)$. In this setting, for generic J the moduli space $\mathcal{M}_J(L, \beta)$ of unparametrized J-holomorphic disks with boundary on L, homology class β and a boundary marked point $\bullet$ is a compact manifold of dimension $\mu(\beta) + \dim(L) - 2$. One can encode counts of J-holomorphic disks in a finite generating function called *disk potential,* and try to establish general properties of the function that may imply something about its coefficients. The disk potential of a monotone Lagrangian torus $L^N \subset X^{2N}$ is defined as

$$W_L = \sum_{\beta \in H_2(X,L;\mathbb{Z})} c_\beta(L) x^{\partial\beta} \in \mathbb{C}[x_1^{\pm}, \dots, x_N^{\pm}];$$

here the degree $c_\beta(L) = \deg(\mathrm{ev}_\bullet : \mathcal{M}_J(L, \beta) \to L) \in \mathbb{Z}$ of the evaluation map $\mathrm{ev}_\bullet$ at the marked point $\bullet \in \partial D^2$ is independent of J, and rigid disks have $\mu(\beta) = 2$; monotonicity implies that $c_\beta(L) \neq 0$ for finitely many classes β. When writing the disk potential, we implicitly assume the choice of a basis of cycles $\gamma_1, \dots, \gamma_N \in H_1(L;\mathbb{Z}) \cong \mathbb{Z}^N$, so that J-holomorphic disks with boundary of class $\partial\beta = k_1\gamma_1 + \dots + k_N\gamma_N$ contribute to the monomial $x^{\partial\beta} = x^{k_1} \dots x^{k_N}$. A Hamiltonian isotopy ϕ^t gives an isomorphism $(\phi^t)_* : H_1(L;\mathbb{Z}) \to H_1(\phi^t(L);\mathbb{Z})$, and $W_{\phi^t(L)} = W_L$ in the induced basis of cycles. It is known that the critical points of W_L obstruct Hamiltonian displaceability; see Cho and Oh [10, Proposition 7.2] for toric moment fibers, Auroux [2, Proposition 6.9], and Sheridan [43, Proposition 4.2] for a general discussion. Disk potentials have been used by Vianna [46, 47] to distinguish infinitely many monotone Lagrangian tori in complex surfaces X of Fano type; see also Pascaleff and Tonkonog [36].

1.3. A cluster construction

In this article, we construct Lagrangian tori in a class of Fano manifolds of arbitrarily large dimension: the Grassmannians $\mathrm{Gr}(k,n)$ of complex k-dimensional linear subspaces in $\mathbb{C}^n$.

Construction 1.1. Given integers $1 \le k < n$, for any Plücker sequence $\mathfrak{s}$ of type (k,n) there is a corresponding Lagrangian torus $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$, equipped with a canonical basis of cycles $\gamma_d \in H_1(L_{\mathfrak{s}};\mathbb{Z})$ labeled by Young diagrams $d \subseteq k \times (n-k)$. The torus comes with a rational function $W_{\mathfrak{s}}$ of formal variables x_d.

The Plücker sequences $\mathfrak{s}$ are based on the notion of quiver mutation from representation theory; see Section 2 for more details and Example 1.3 below. The Lagrangian tori $L_{\mathfrak{s}}$ are obtained from algebraic degenerations to (singular) toric varieties $\mathrm{Gr}(k,n) \rightsquigarrow X(\Sigma_{\mathfrak{s}})$, using a general technique for constructing completely integrable systems on complex projective manifolds studied by Harada and Kaveh [21]; the notion of toric degeneration is explained in Section 3. Such degenerations of Grassmannians have been studied by Rietsch and Williams [40] in connection with the theory of Okounkov bodies [23, 25, 35]. All Plücker sequences start from a single initial seed, and the rational functions $W_{\mathfrak{s}}$ are obtained by explicit rational changes of variable from a single initial Laurent polynomial W_0, whose variables are labeled by those Young diagrams $d \subseteq k \times (n-k)$ that are rectangles. In [6], it was proved that W_0 is in fact the disk potential of the monotone Lagrangian torus fiber of the Gelfand–Cetlin integrable system introduced by Guillemin and Sternberg [20]. The formulation of the construction as iterative procedure is particularly convenient for computational purposes. To illustrate this point, we created a random walk that generates Plücker sequences $\mathfrak{s}$ of arbitrary length, and computes the corresponding Laurent polynomials $W_{\mathfrak{s}}$ explicitly; the code is available for inspection and experiments [7].

1.4. Topology of Laurent/positivity phenomena

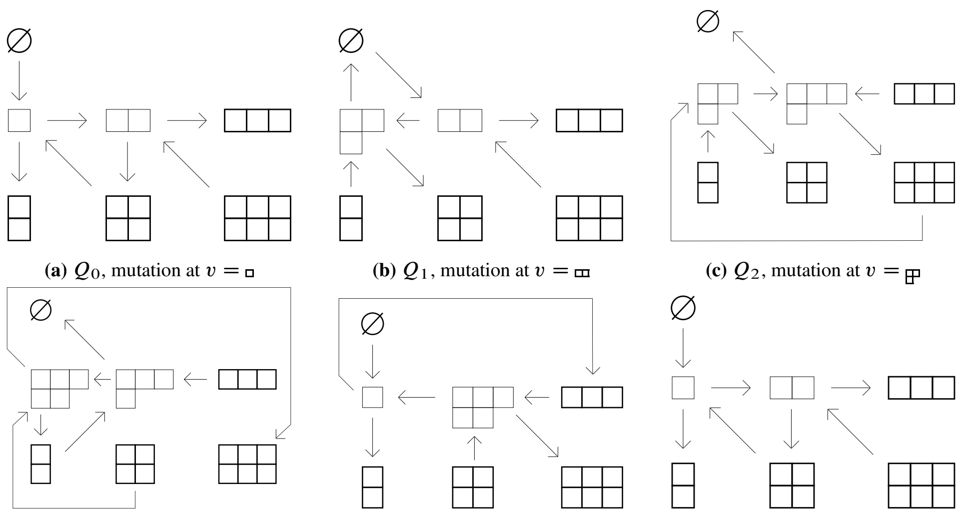
By computing a few examples of $W_{\mathfrak{s}}$, one quickly notices the following two phenomena, which are not a direct consequence of the construction:

1. the rational function $W_{\mathfrak{s}}$ is a Laurent polynomial;
2. the coefficients of each Laurent polynomial $W_{\mathfrak{s}}$ are natural numbers.

Property (1) is related to the Laurent phenomenon of cluster algebras, a notion developed by Fomin and Zelevinsky [15]. Think each x_d as a Plücker coordinate on the dual Grassmannian $\mathrm{Gr}^\vee(k,n) = \mathrm{Gr}(n-k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ in its Plücker embedding; the definition of Plücker coordinate is recalled in Definition 2.1. Each Plücker sequence $\mathfrak{s}$ singles out an open algebraic torus chart $T_{\mathfrak{s}} \subset U_{k,n} = \mathrm{Gr}^\vee(k,n) \setminus D^\vee_{FZ}$ in the complement of a particular divisor $D^\vee_{FZ}$, and the global regular functions

$\mathcal{A}_{k,n} = \mathcal{O}(U_{k,n})$ form a cluster algebra; see Scott [41]. The space $U_{k,n}$ is a smooth affine variety known as open positroid stratum, and its properties have been the focus of several works in representation theory, combinatorics, topology and mirror symmetry [24,26,37,40,42]. By a result of Marsh and Rietsch [28], one can think of each $W_{\mathfrak{s}}$ as restriction $W_{\mathfrak{s}} = W|_{T_{\mathfrak{s}}}$ of a single global regular function $W \in \mathcal{A}_{k,n}$ called *Landau–Ginzburg potential.* Property (2) is related to positivity of cluster algebras, which has been proved by Gross, Hacking, Keel, and Kontsevich [19]. Their proof consists in interpreting the coefficients of certain elements of a cluster algebra as counts of tropical curves called *broken lines.* In mirror symmetry, broken lines are expected to correspond to the J-holomorphic disks of symplectic topology, and this heuristic leads us to the following.

Conjecture 1.2 (see the more precise Conjecture 3.8). *The Laurent polynomial $W_{\mathfrak{s}}$ is an invariant of the Hamiltonian isotopy class of the Lagrangian torus $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$.*



(a) Q_0, mutation at $v = \square$ **(b)** Q_1, mutation at $v = \square\square$ **(c)** Q_2, mutation at $v = \boxplus$ **(d)** Q_3, mutation at $v = \boxplus$ **(e)** Q_4, mutation at $v = \boxplus$ **(f)** $Q_5 = Q_0$

Figure 1. A Plücker sequence of type $(2,5)$ and length five. The labeling variables x_d on the nodes are replaced by d for notational convenience.

Example 1.3. Let $k = 2$ and $n = 5$. Figure 1 represents a Plücker sequence

$$\mathfrak{s}\colon (Q_0, W_0) \to (Q_1, W_1) \to (Q_2, W_2) \to (Q_3, W_3) \to (Q_4, W_4) \to (Q_5, W_5)$$

of type $(2,5)$ and length five, where the final step and the initial one coincide. At each step (Q_i, W_i), the graph Q_i is a quiver whose nodes are labeled by Plücker coordinates x_d for some collection of Young diagrams $d \subseteq 2 \times 3$, and W_i is a Laurent polynomial of the variables x_d. A step $(Q_i, W_i) \to (Q_{i+1}, W_{i+1})$ in the sequence consists in performing a quiver mutation at a mutable node v of Q_i as described in Section 2. This procedure changes the label $l(v)$ of the node v in Q_i to a new label $l'(v)$ of the same node in Q_{i+1}. The two labels are related by the following exchange relation: $l(v)l(v')$ is a sum of two terms, obtained by taking the product of labels $l(w)$ from incoming/outgoing nodes w adjacent to v respectively. The rational function W_{i+1} is obtained from W_i by using the previous relation to replace the label $l(v)$ with $l(v')$, and becomes Laurent modulo Plücker relations, i.e., when interpreted as element of the function field $\mathrm{Frac}(\mathcal{A}_{2,5}) = \mathbb{C}(U_{2,5}) = \mathbb{C}(\mathrm{Gr}^\vee(2,5))$ of the dual Grassmannian $\mathrm{Gr}^\vee(2,5) = \mathrm{Gr}(3,5)$. The intermediate steps of Construction 1.1 produce Lagrangian tori $L_0, L_1, L_2, L_3, L_4 \subset \mathrm{Gr}(2,5)$. In this case, all the tori are monotone and the Laurent polynomials W_i for $0 \le i \le 4$ match their disk potentials W_{L_i}:

$$W_0 = x_{(1)} + \frac{x_{(1,1)}}{x_{(1)}} + \frac{x_{(2,2)}x_{\emptyset}}{x_{(1)}x_{(2)}} + \frac{x_{(3,3)}x_{\emptyset}}{x_{(2)}x_{(3)}} + \frac{x_{(2)}}{x_{(3,3)}} + \frac{x_{(2)}}{x_{(1)}} + \frac{x_{(3)}}{x_{(2)}} + \frac{x_{(2,2)}x_{\emptyset}}{x_{(1)}x_{(1,1)}} + \frac{x_{(3,3)}x_{(1)}}{x_{(2)}x_{(2,2)}},$$

$$W_1 = \frac{x_{\emptyset}x_{(2,2)}}{x_{(2,1)}} + \frac{x_{(2)}x_{(1,1)}}{x_{(2,1)}} + \frac{x_{(3,3)}x_{\emptyset}}{x_{(2)}x_{(2,1)}} + \frac{x_{(3,3)}x_{(1,1)}}{x_{(2,2)}x_{(2,1)}} + \frac{x_{(2,1)}}{x_{(2)}} + \frac{x_{(2,1)}}{x_{(1,1)}} + \frac{x_{(3,3)}x_{\emptyset}}{x_{(2)}x_{(3)}} + \frac{x_{(2)}}{x_{(3,3)}} + \frac{x_{(3)}}{x_{(2)}},$$

$$W_2 = \frac{x_{\emptyset}x_{(3,3)}x_{(1,1)}}{x_{(3,1)}x_{(2,1)}} + \frac{x_{\emptyset}}{x_{(3,1)}} + \frac{x_{(3)}x_{(1,1)}}{x_{(3,1)}} + \frac{x_{(2,1)}x_{(3)}}{x_{(3,1)}x_{(3,3)}} + \frac{x_{(3,1)}}{x_{(2,1)}} + \frac{x_{(3,1)}}{x_{(3)}} + \frac{x_{\emptyset}x_{(2,2)}}{x_{(2,1)}} + \frac{x_{(3,3)}x_{(1,1)}}{x_{(2,2)}x_{(2,1)}} + \frac{x_{(2,1)}}{x_{(1,1)}},$$

$$W_3 = \frac{x_{\emptyset}}{x_{(3,1)}} + \frac{x_{(3)}x_{(1,1)}}{x_{(3,1)}} + \frac{x_{(3,1)}}{x_{(3)}} + \frac{x_{(3,2)}x_{\emptyset}}{x_{(3,1)}} + \frac{x_{(3,2)}}{x_{(2,2)}} + \frac{x_{(1,1)}x_{(3)}}{x_{(3,2)}x_{(3,1)}} + \frac{x_{(3,3)}}{x_{(3,2)}} + \frac{x_{(2,2)}x_{(3)}}{x_{(3,2)}x_{(3,3)}} + \frac{x_{(3,1)}x_{(2,2)}}{x_{(3,2)}x_{(1,1)}},$$

$$W_4 = \frac{x_{(3,2)}}{x_{(2,2)}} + \frac{x_{(3,3)}}{x_{(3,2)}} + \frac{x_{(2,2)}x_{(3)}}{x_{(3,2)}x_{(3,3)}} + \frac{x_{(1)}}{x_{(3,2)}} + x_{(1)} + \frac{x_{\emptyset}x_{(3,2)}}{x_{(1)}x_{(3)}} + \frac{x_{\emptyset}x_{(2,2)}}{x_{(1)}x_{(1,1)}} + \frac{x_{(1,1)}}{x_{(1)}} + \frac{x_{(3)}x_{(2,2)}}{x_{(1)}x_{(3,2)}}.$$

The equality $W_{\mathfrak{s}} = W_{L_{\mathfrak{s}}}$ is an application of a general result of Nishinou, Nohara, and Ueda [30] on the behavior of disk potentials under toric degeneration, which also implies monotonicity of $L_{\mathfrak{s}}$; see Proposition 3.12. This result gives a sufficient condition for the equality $W_{\mathfrak{s}} = W_{L_{\mathfrak{s}}}$, which is the existence of a small toric resolution for the singular toric variety $X(\Sigma_{\mathfrak{s}})$; see Definition 3.11. Due to the combinatorial nature of toric varieties, for any given Plücker sequence $\mathfrak{s}$ one can check this condition in finitely many steps. In Section 4 we use this to describe a sample application in the smallest example not accessible by previous techniques.

Theorem 1.4 (see Theorem 4.16). *The Grassmannian* $\mathrm{Gr}(3,6)$ *contains at least* 6 *monotone Lagrangian tori that are non-displaceable and pairwise inequivalent under Hamiltonian isotopy.*

We call these tori *exotic,* because only one monotone torus was previously known: the Gelfand–Cetlin torus. The new examples are of the form $L_{\mathfrak{s}}$ for some Plücker sequence $\mathfrak{s}$, and are distinguished by a combination of two invariants: the number of critical points of their disk potential $W_{L_{\mathfrak{s}}}$ and the f-vector of its Newton polytope. This strategy applies without modification to arbitrary Grassmannians. If Conjecture 3.8 holds, the same arguments of Theorem 4.16 imply that the tori $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ are always nondisplaceable, and generally not Hamiltonian isotopic. Note that Conjecture 3.8 may still hold when the toric variety $X(\Sigma_{\mathfrak{s}})$ has no small toric resolution, and the result of Nishinou, Nohara, and Ueda [30] does not apply. In this case, positivity of the coefficients of $W_{\mathfrak{s}}$ suggests an enumerative interpretation in terms of counts of J-holomorphic disks with boundary on $L_{\mathfrak{s}}$. We plan to explore this in a separate work, simply pointing out here a possible interpretation in terms of low-area Floer theory in the sense of Tonkonog and Vianna [45]. For $k = 1$, one has projective spaces $\mathrm{Gr}(1,n) = \mathbb{P}^{n-1}$, and there is only one Plücker sequence $\mathfrak{s}$ of length 0; in this case $W_{\mathfrak{s}}$ is the disk potential of the Clifford torus. In particular, Construction 1.1 does not imply the existence of exotic tori in $\mathbb{P}^2$ established by Vianna [46,47]. For $k = 2$, Construction 1.1 recovers a different one studied by Nohara and Ueda [32], who introduced a collection of Lagrangian tori in $\mathrm{Gr}(2,n)$ corresponding to triangulations of an n-gon; the relation is explained in Lemma 4.4, and it implies that Conjecture 3.8 holds when $k = 2$.

1.5. Probing the spectral decomposition

Since for $k = 2$ all tori $L_{\mathfrak{s}} \subset \mathrm{Gr}(2,n)$ are monotone, it is natural to think of them as objects of the monotone Fukaya category. As described by Sheridan [43], the Fukaya category of a monotone symplectic manifold X has a spectral decomposition

$$\mathscr{F}(X) = \bigoplus_{\lambda} \mathscr{F}_{\lambda}(X).$$

The summands are A_∞-categories indexed by the eigenvalues λ of the operator $c_1\star$ of multiplication by the first Chern class acting on the small quantum cohomology. The objects of the λ-summand are monotone Lagrangians L_ξ equipped with a rank one local system ξ such that

$$m^0(L_\xi) = \sum_\beta c_\beta(L)\,\mathrm{hol}_\xi(\partial\beta) = \lambda.$$

Definition 3.7 introduces some natural local systems supported on the Lagrangian tori $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$, that are controlled by the values of k-variables Schur polynomials at certain roots of unity. These local systems generalize the ones studied in [6] for the Gelfand–Cetlin torus, that were controlled by Schur polynomials corresponding rectangular Young diagrams. When $k = 2$, we show that the corresponding objects split-generate the derived Fukaya category $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,n))$ in some cases, notably including examples where the Gelfand–Cetlin torus alone fails to do so.

Theorem 1.5 (see Theorem 4.8). *If $n = 2^t + 1$ for some $t \in \mathbb{N}^+$, then the derived Fukaya category $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,2^t+1))$ is split-generated by objects supported on a single Plücker torus.*

The Lagrangian torus in the statement is associated to a special triangulation of the n-gon, that we call *dyadic*. In fact, Section 4 contains a criterion to prove split-generation of $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,n))$ by objects supported on any number of tori $L_{\mathfrak{s}} \subset \mathrm{Gr}(2,n)$, whenever n is odd. The criterion is based on a construction of triangulations of the n-gon whose sides lengths avoid the prime numbers appearing in the factorization of n.

Theorem 1.6 (see Theorem 4.11). *Let $n > 2$ be odd, and consider its prime factorization $n = p_1^{e_1} \dots p_l^{e_l}$. If for all $1 \le i \le l$ there exists a triangulation Γ_i of $[n]$ that is p_i-avoiding, then $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,n))$ is split generated by objects supported on l Plücker tori.*

Remark 1.7. A standard consequence of split-generation is that any monotone Lagrangian supporting nonzero objects of the Fukaya category must intersect the generator.

These results seem to suggest that objects supported on the tori $L_{\mathfrak{s}}$ could split-generate $\mathbf{D}\mathcal{F}(\mathrm{Gr}(k,n))$ in general, although split-generation over $\mathbb{C}$ has a subtle relation with the location of the critical points of $W \in \mathcal{A}_{k,n}$ relative to the torus charts $T_{\mathfrak{s}} \subset U_{k,n}$. For example, split-generation over $\mathbb{C}$ fails for $\mathrm{Gr}(2,4)$, where W has two critical points in a complex codimension 2 locus of $U_{k,n}$ which is not covered by cluster charts. We plan to investigate in a separate work how the situation changes when considering bulk-deformations in the sense of Fukaya, Oh, Ohta, and Ono [17].

1.6. Mirror symmetry and abundance of Lagrangian tori

This article can be thought of as part of a broader program aimed at investigating the abundance of Lagrangian tori in Fano manifolds X with an anti-canonical divisor $D \subset X$ whose complement $U = X \setminus D$ is a cluster variety. This class includes many homogeneous varieties $X = G/P$ with $P \subset G$ parabolic subgroup of a complex linear algebraic group. The cluster variety U comes with a Langlands dual cluster variety $U^\vee$, and Gross, Hacking, Keel, and Kontsevich [19] proposed that (X, D) has a Landau–Ginzburg model $(U^\vee, W)$ in the sense of homological mirror symmetry. Here $W \in \mathcal{O}(U^\vee)$ is a regular function intrinsically defined by the cluster structure and given as a sum of theta functions, which are generating functions of discrete objects called *broken lines* in a scattering diagram. We expect that the cluster charts of $U^\vee$ will correspond to certain Lagrangian tori in $L \subset G/P$, and that the restriction of W to different cluster charts will fully determine their disk potential W_L in some cases, and in general suffice to distinguish many of their Hamiltonian isotopy classes in the spirit of Conjecture 1.2.

1.7. Algebraic and topological wall-crossing

When $W_{L_\mathfrak{s}} = W_\mathfrak{s}$, the Lagrangian tori $L_\mathfrak{s} \subset \mathrm{Gr}(k, n)$ constructed in this article have disk potentials related by algebraic wall-crossing formulas by construction. It is natural to ask if these formulas correspond to a topological wall-crossing, i.e., if the tori $L_\mathfrak{s}$ are connected by families of Lagrangian immersions that bound Maslov 0 J-holomorphic disks at some intermediate time. We do not investigate this question here, but only point out that it would be interesting to see if there is a relation between our examples and the model of Lagrangian mutation studied by Pascaleff and Tonkonog [36].

2. The iterative construction

Throughout this article, k and n are integers with $1 \leq k < n$. The symbol d denotes a Young diagram in the $k \times (n-k)$ grid, obtained by placing d_i consecutive boxes in the i-th row for all $1 \leq i \leq k$, starting from the left in each row and with $d_1 \geq d_2 \geq \cdots \geq d_k$. Chosen $0 \leq i \leq k$ and $0 \leq j \leq (n-k)$, one has a rectangular Young diagram $i \times j$, with $i \times j = \emptyset$ empty diagram if $i = 0$ or $j = 0$. A full rank $n \times (n-k)$ matrix M determines an $(n-k)$-dimensional linear subspace of $\mathbb{C}^n$ by taking its column-span. If $[M]$ is the equivalence class of M modulo column operations, write $[M] \in \mathrm{Gr}^\vee(k, n) = \mathrm{Gr}(n-k, n)$ and think of it as a point of the complex Grassmannian. Each Young diagram d has a profile path, which connects the top-right corner of the $k \times (n-k)$ grid to the bottom-left one. Labeling the steps of the path by $[n] =$

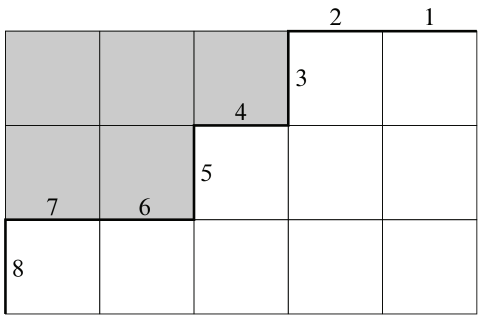


Figure 2. A Young diagram $d \subseteq 3 \times 5$ with $d^- = \{1, 2, 4, 6, 7\}$ and $d^| = \{3, 5, 8\}$.

$\{1, \ldots, n\}$, the vertical steps of d determine a set $d^| \subset [n]$ with $|d^|| = k$, while the horizontal steps determine a set $d^- \subset [n]$ with $|d^-| = n - k$; see Figure 2 for an example with $k = 3$ and $n = 8$.

Definition 2.1. If M is a full rank $n \times (n - k)$ matrix, the determinant of M at rows d^- is denoted $x_d(M)$ and called *Plücker coordinate* corresponding to d.

The Plücker coordinates define a projective embedding of $\mathrm{Gr}^\vee(k, n)$ in $\mathbb{P}^{\binom{n}{k}-1}$. If $\mathcal{I}_{k,n} \subset \mathbb{C}[x_d : d \subseteq k \times (n - k)]$ is the corresponding homogeneous ideal, each x_d is an element of the algebra $\mathcal{A}_{k,n} = \mathbb{C}[x_d : d \subseteq k \times (n - k)]/\mathcal{I}_{k,n}$ of regular functions of the affine cone over $\mathrm{Gr}^\vee(k, n)$.

2.1. Initial seed

Definition 2.2. A quiver with potential of type (k, n) is a pair (Q, W), where

1. Q is an oriented connected graph, with no edge connecting a node to itself and no oriented loops with two edges, whose nodes are labeled by Plücker coordinates $x_d \in \mathcal{A}_{k,n}$;
2. W is a Laurent polynomial in the labels of the nodes of Q;

As part of the data, the nodes of Q are partitioned in two groups, called *frozen* and *mutable*.

Remark 2.3. To avoid confusion, we point out that Definition 2.2 is not a special case of the notion of quiver with potential in representation theory: although Q is a quiver in the classical sense, the potential W is an element of the commutative algebra $\mathcal{A}_{k,n}$ as opposed to the non-commutative path algebra of Q.

The iterative construction we describe in this section begins with a specific quiver with potential.

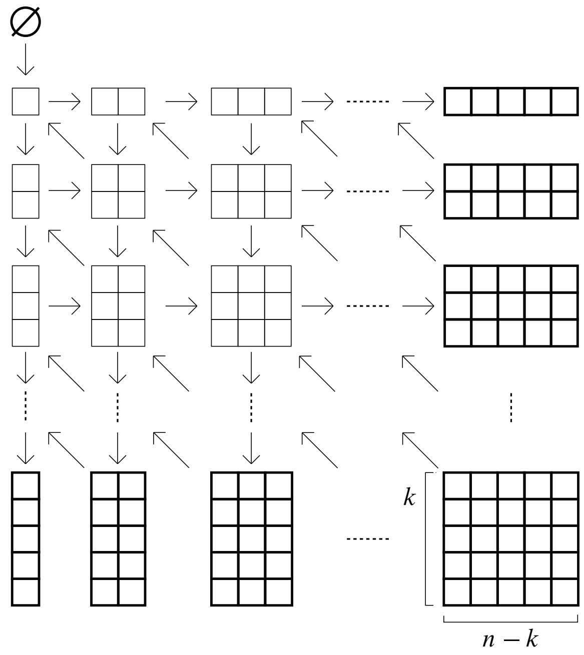


Figure 3. Initial quiver Q_0: labels x_d are indicated by d, frozen nodes in bold type.

Definition 2.4. The initial seed of type (k, n) is the quiver with potential (Q_0, W_0), where

1. Q_0 is the oriented labeled graph in Figure 3;
2. W_0 is the Laurent polynomial

$$x_{1\times 1} + \sum_{i=2}^{k}\sum_{j=1}^{n-k} \frac{x_{i\times j}x_{(i-2)\times(j-1)}}{x_{(i-1)\times(j-1)}x_{(i-1)\times j}} + \frac{x_{(k-1)\times(n-k-1)}}{x_{k\times(n-k)}} + \sum_{i=1}^{k}\sum_{j=2}^{n-k} \frac{x_{i\times j}x_{(i-1)\times(j-2)}}{x_{(i-1)\times(j-1)}x_{i\times(j-1)}}.$$

A node of Q_0 is frozen if its label is $x_{i\times j}$ with $i \times j = \emptyset$, $i = k$ or $j = n - k$; the remaining nodes are mutable.

Observe that the labels on the nodes of Q_0 are precisely the $k(n-k)+1$ variables x_d where d is a rectangular Young diagram, and n of the nodes are frozen.

2.2. Mutation step

Given a quiver with potential (Q, W) as in Definition 2.2, and fixed a mutable node v of Q, one can form a new labeled quiver Q' as follows:

1. start with $Q' = Q$, and for all length 2 paths $a \to v \to b$ with at least one mutable node among a and b, add to Q' a new edge $a \to b$;
2. modify Q' by reversing all the edges incident to v;
3. remove all oriented 2-cycles formed in Q', by deleting their arrows.

Calling $l(w)$ the label of a node w in Q, define new labels $l'(w)$ in Q' by declaring $l'(w) = l(w)$ if $w \neq v$, and

$$l'(v) = \frac{\prod_{w\to v} l(w) + \prod_{v\to w} l(w)}{l(v)}.$$

Since Q and Q' have the same nodes, the nodes of Q' inherit the property of being frozen or mutable from Q.

Definition 2.5. The *mutation of* (Q, W) *along* v is the pair (Q', W') with Q' constructed as above, and W' obtained from W by substitution

$$l(v) = \Big(\prod_{w\to v} l'(w) + \prod_{v\to w} l'(w)\Big) l'(v)^{-1}.$$

A priori, mutations of quivers with potentials as in Definition 2.2 are not necessarily quivers with potentials, since $l'(v)$ and W' are only rational functions of the Plücker coordinates x_d. The following guarantees that certain iterated mutations of the initial seed of Definition 2.4 remain quivers with potentials.

Proposition 2.6 (Scott [41, Theorem 3] and Marsh and Rietsch [28, Section 6.3]). *Given a finite sequence of mutations that starts at* (Q, W) *and ends at* (Q', W')*:*

1. *if* $(Q, W) = (Q_0, W_0)$ *is the initial seed of Definition* 2.4*, then* W' *is a Laurent polynomial in the labels of* Q'*;*
2. *if in addition each mutation of the sequence is based at some node with two incoming and two outgoing edges, then the labels of* Q' *are Plücker coordinates* x_d*.*

Proof. For the reader's convenience, we explain how the statements follow from the cited results. It suffices to prove them when the sequence of mutations consists of a single mutation, as the general case follows by applying repeatedly the same argument.

(1) Marsh and Rietsch [28, Section 6.3] (see also Rietsch and Williams [40, Proposition 9.5]) showed that the potential W_0 of the initial seed is the restriction $W_0 = W|_{T_0}$ of a regular function $W \in \mathcal{A}_{k,n}$ to an algebraic torus $T_0 \subset \mathrm{Gr}^\vee(k,n)$ defined by

$$T_0 = \{[M] \in \mathrm{Gr}^\vee(k,n)\colon l(M) \neq 0 \text{ for all } l \text{ label of } Q_0\}.$$

By Scott [41, Theorem 3] $\mathcal{A}_{k,n}$ is a cluster algebra, and the rational functions labeling the nodes of Q' are cluster variables. Just as with the labels of Q_0, one can use the labels of Q' to define an algebraic torus $T' \subset \mathrm{Gr}^\vee(k,n)$ via

$$T' = \{[M] \in \mathrm{Gr}^\vee(k,n)\colon l'(M) \neq 0 \text{ for all } l' \text{ label of } Q'\};$$

this torus is called *toric chart* in [41, Section 6]. By Definition 2.5, W' is obtained from W_0 by substitution $l(v) = (\prod_{w\to v} l'(w) + \prod_{v\to w} l'(w))l'(v)^{-1}$. This means that W' is the pull-back of W_0 along the birational map from T' to T_0 defined by the substitution formula. It is part of the statement that $\mathcal{A}_{k,n}$ is a cluster algebra that the substitution formula gives a relation

$$l(v)l'(v) - \prod_{w\to v} l'(w) - \prod_{v\to w} l'(w) \quad \in \quad \mathcal{I}_{k,n},$$

so that $W' = W|_{T'}$ is a restriction of W as well. In particular, W' is a regular function on the algebraic torus T', and hence a Laurent polynomial.

(2) If the mutation from (Q,W) to (Q',W') is based at some node v with two incoming and two outgoing edges, denote $\{v_1^+, v_2^+\}$ and $\{v_1^-, v_2^-\}$ the corresponding nodes of Q adjacent to v. The substitution formula of Definition 2.5 simplifies to $l(v) = (l'(v_1^+)l'(v_2^+) + l'(v_1^-)l'(v_2^-))l'(v)^{-1}$. By definition of mutation $l'(v_i^+) = l(v_i^+)$ and $l'(v_i^-) = l(v_i^-)$ for $i = 1, 2$. Moreover, by assumption the l labels are Plücker coordinates, meaning that $l(v_i^+) = x_{d_i^+}$ and $l(v_i^-) = x_{d_i^-}$ for $i = 1, 2$ and $l(v) = x_d$ for some Young diagrams $d, d_1^+, d_2^+, d_1^-, d_2^- \subseteq k \times (n-k)$. Scott [41, proof of Theorem 3] proves that this implies $l'(v) = x_{d'}$ for some Young diagram $d' \subseteq k \times (n-k)$ too, using combinatorial objects called *wiring arrangements.* The same phenomenon is discussed by Rietsch and Williams in [40, Lemma 5.6] in the combinatorial framework of plabic graphs; see also the proof of Proposition 3.2 for a comparison between plabic graphs and quivers. ■

Definition 2.7. A length l Plücker sequence of mutations of type (k,n), denoted $\mathfrak{s}$, is a finite sequence of pairs (Q_i, W_i) with $0 \le i \le l$ such that

1. (Q_0, W_0) is the initial seed of type (k,n) of Definition 2.4;
2. (Q_{i+1}, W_{i+1}) is obtained from (Q_i, W_i) by mutation along a mutable node with two incoming and outgoing edges, as in Definition 2.5.

If we want to suppress the length l, we denote $(Q_l, W_l) = (Q_{\mathfrak{s}}, W_{\mathfrak{s}})$ and call it the *final quiver* with potential of $\mathfrak{s}$.

2.3. Relation with polytope mutations

Later on in this article, we will be interested in how the Newton polytope $P_{\mathfrak{s}} = \mathrm{Newt}(W_{\mathfrak{s}})$ changes throughout a sequence of mutations in the sense of Definition 2.5.

Definition 2.8. A convex polytope $P \subset \mathbb{R}^N$ is Fano if the following properties hold:

1. $\dim(P) = N$;
2. $0 \in \mathbb{R}^N$ is an interior point of P;
3. the vertices $V(P)$ form a set of primitive vectors $V(P) \subset \mathbb{Z}^N$.

Proposition 2.10 proves that each $P_{\mathfrak{s}}$ is a Fano polytope. The Fano condition has the following interpretation in terms of toric geometry; see [11, 18] for some general background on toric varieties.

Lemma 2.9. *If P is a Fano polytope, then the polyhedral fan $\Sigma = \Sigma^f P$ consisting of the cones spanned by its faces is such that the associated toric variety $X(\Sigma)$ is Fano, meaning that*

1. *the anti-canonical toric Weil divisor D_Σ is $\mathbb{Q}$-Cartier;*
2. *D_Σ is ample.*

Proof. The anti-canonical toric Weil divisor is defined to be

$$D_\Sigma = \sum_{\rho \in \Sigma(1)} D_\rho,$$

where the sum is over the one-dimensional cones $\rho \in \Sigma(1)$ and D_ρ is the prime divisor corresponding to ρ. By [11, Theorem 4.2.8] D_Σ is $\mathbb{Q}$-Cartier if and only if

$$\begin{aligned} &\text{for all } \sigma \in \Sigma(N) \text{ there exists } m_\sigma \in \mathbb{Q}^N \text{ such that} \\ &\langle m_\sigma, u_\rho \rangle = -1 \quad \text{for all } \rho \in \sigma(1), \end{aligned}$$

where $\Sigma(N)$ is the set of maximal cones of Σ, $\sigma(1)$ is the set of one-dimensional cones in σ, and $u_\rho \in \mathbb{Z}^N$ is the primitive generator of ρ; if m_σ exists then it is unique, and when all $m_\sigma \in \mathbb{Z}^N$ one recorvers the stronger Cartier condition. By assumption (3) in Definition 2.8 and the fact that $\Sigma = \Sigma^f P$, one has $u_\rho \in V(P)$ for all $\rho \in \Sigma(1)$. Now, consider the polar dual polytope

$$P^\circ = \{v \in \mathbb{R}^N : \langle v, u \rangle \geq -1 \text{ for all } u \in P\};$$

since $(P^\circ)^\circ = P$ and $V(P) \subset \mathbb{Z}^N$, the polytope P° is reflexive. One equivalent formulation of reflexivity is to say that each vertex of the polar dual polytope $u_\rho \in V((P^\circ)^\circ) = V(P)$ defines a facet (or codimension one face)

$$F_{u_\rho} = P^\circ \cap \{v \in \mathbb{R}^N \colon \langle v, u_\rho\rangle = -1\}.$$

We claim that for each $\sigma \in \Sigma(N)$ one has

$$\bigcap_{\rho\in\sigma(1)} F_{u_\rho} = \{m_\sigma\},$$

where $m_\sigma \in \mathbb{Q}^N$ and it satisfies the $\mathbb{Q}$-Cartier condition. Polar duality exchanges the face fan Σ^f and the normal fan Σ^n, so that $\sigma \in \Sigma = \Sigma^f P = \Sigma^n P^\circ$; from this point of view the vector u_ρ can be thought of as inward-pointing normal to the facet $F_{u_\rho} \subset P^\circ$. By [11, Proposition 2.3.8] the intersection above describes the unique vertex $m_\sigma \in V(P^\circ)$ corresponding to the N-dimensional cone σ of the normal fan. Although in general $m_\sigma \notin \mathbb{Z}^N$, one always has $m_\sigma \in \mathbb{Q}^N$, because P° is the polar dual of a polytope P with $V(P) \subset \mathbb{Z}^N$; see [11, Exercise 2.2.1 (a)]. Finally, $\langle m_\sigma, u_\rho\rangle = -1$ for all $\rho \in \sigma(1)$ because $m_\sigma \in F_{u_\rho}$ by construction. This proves part (1), for part (2) proceed as follows. The $\mathbb{Q}$-Cartier divisor D_Σ has a support function $\phi_{D_\Sigma} : \mathbb{R}^N \to \mathbb{R}$, which is piecewise-linear on Σ and such that $\phi_{D_\Sigma}(u_\rho) = -1$ for all $\rho \in \Sigma(1)$. By [11, Theorem 6.1.7 and Lemma 6.1.13] D_Σ is ample if and only if the points $\{m_\sigma \colon \sigma \in \Sigma(N)\}$ are the vertices of the polytope

$$P_{D_\Sigma} = \{v \in \mathbb{R}^N \colon \langle v, u_\rho\rangle \geq \phi_{D_\Sigma}(u_\rho) \text{ for all } \rho \in \Sigma(1)\},$$

and moreover $m_\sigma \neq m_{\sigma'}$ for $\sigma \neq \sigma'$. This is true because $P_{D_\Sigma} = P^\circ$ and the fact that the correspondence between faces of P° and cones of its normal fan is a bijection. ■

Akhtar, Coates, Galkin, and Kasprzyk [1] proposed a general notion of polytope mutation that should describe how the Newton polytope of a Laurent polynomial changes under the action of special birational maps of a torus. More precisely, consider a birational map of the form

$$\phi\colon (\mathbb{C}^\times)^N \dashrightarrow (\mathbb{C}^\times)^N, \quad \phi = \phi_{M_2} \circ \phi_A \circ \phi_{M_1},$$

where

$$(\mathbb{C}^\times)^N = \operatorname{Spec} \mathbb{C}[x_1^\pm, \ldots, x_N^\pm] \quad \text{and} \quad \phi_A(x_1, \ldots, x_N) = (x_1, \ldots, x_{N-1}, Ax_N)$$

for some Laurent polynomial A with $\partial_{x_N} A = 0$, and where

$$\phi_{M_i}(x_1, \ldots, x_N) = (x_1^{m_{11}} \ldots x_N^{m_{1N}}, \ldots, x_1^{m_{N1}} \ldots x_N^{m_{NN}}),$$

with $M_i = (m_{st})_{1\leq s,t\leq N} \in \mathrm{GL}(N,\mathbb{Z})$ for $i = 1,2$ are automorphisms of the torus, specified by invertible integer matrices. If f is a Laurent polynomial, one can think of it as a polynomial in the $x_N^{\pm}$ variables with coefficients C_h which are Laurent polynomials with $\partial_{x_N} C_h = 0$ and write

$$f = \sum_{-h_{\min}\leq h\leq h_{\max}} C_h(x_1,\dots,x_{N-1})x_N^h \quad \text{with } h_{\min}, h_{\max} \in \mathbb{N}.$$

Then the rational function

$$\phi_A^* f = \sum_{-h_{\min}\leq h\leq h_{\max}} \frac{C_h}{A^{-h}} x_N^h$$

is again a Laurent polynomial whenever $A^{-h}|C_t$ for all $-h_{\min} \leq h < 0$, and so is $\phi^* f = g$ because ϕ_{M_1}, ϕ_{M_2} are automorphisms. The Newton polytopes $P = \mathrm{Newt}(f)$ and $P' = \mathrm{Newt}(g)$ are convex hulls in $\mathbb{R}^N$ of the exponent vectors of the monomials in f and g respectively. The special form of ϕ_A singles out the x_N variable, and a width vector $w = (0,\dots,0,1) \in \mathbb{Z}^N$ corresponding to this choice. For heights $-h_{\min} \leq h \leq h$ one can form lattice polytopes $w_h(P) \subseteq P$ by taking the convex hull of lattice points in hyperplane sections orthogonal to the w-direction:

$$w_h(P) = \mathrm{Conv}(P \cap \{\langle\cdot, w\rangle = h\} \cap \mathbb{Z}^N).$$

In fact, $h_{\max} - h_{\min} \in \mathbb{N}$ can be thought as the width of the polytope P with respect to the w-direction, and h as a height coordinate. Calling

$$F = \mathrm{Newt}(A) \quad \text{and} \quad G_h = \mathrm{Newt}\Big(\frac{C_h}{A^{-h}}\Big) \quad \text{for all } -h_{\min} \leq h \leq h_{\max},$$

the polytope $F \subset \mathbb{R}^N$ has codimension at least one and lies at height $h = 0$. Denoting $V(P)$ the vertices of P one has

$$V(P) \cap \{\langle\cdot, w\rangle = h\} \subseteq G_h + (-h)F \subseteq w_h(P) \quad \text{for all } -h_{\min} \leq h < 0.$$

The notation $P' = \mathrm{mut}_w(P, F)$ expresses the fact that P' is a polytope mutation of P in direction w and with factor F, and the quantity $h_{\max} - h_{\min}$ is called *width* of the mutation. We now explain how mutation in the sense of Definition 2.5 is related to polytope mutations.

Proposition 2.10. *If (Q', W') is obtained from (Q, W) by mutation along a node v, then the Newton polytopes $P = \mathrm{Newt}(W)$ and $P = \mathrm{Newt}(W')$ are related by polytope mutation. In particular, $P_{\mathfrak{s}}$ and $X(\Sigma^f P_{\mathfrak{s}})$ are Fano for any Plücker sequence $\mathfrak{s}$.*

Proof. Define the complex tori

$$T = \mathrm{Spec}(\mathbb{C}[x_d^{\pm} : d \text{ label of } Q]), \quad T' = \mathrm{Spec}(\mathbb{C}[x_d^{\pm} : d \text{ label of } Q']),$$

and think of W and W' as regular functions on them. Quiver mutation along v changes the label $x_{l(v)}$ of Q into a new label $x_{l'(v)}$ in Q', and the two are related by

$$x_{l(v)}x_{l'(v)} = \prod_{w \to v} x_{l(w)} + \prod_{v \to w} x_{l(w)}.$$

Up to automorphisms of tori, one can arrange the coordinates in such a way that $x_{l(v)}$ and $x_{l'(v)}$ go last, and the common ones appear in the same order. With this choice, there is a birational transition map between the tori

$$\phi = \Big(\mathrm{id}, \Big(\prod_{w \to v} x_{l(w)} + \prod_{v \to w} x_{l(w)}\Big) x_{l(v)}^{-1}\Big).$$

The Laurent polynomial $A = \prod_{w \to v} x_{l(w)} + \prod_{v \to w} x_{l(w)}$ satisfies $\partial_{x_{l(v)}} A = 0$, and using the notation introduced in this section $\phi = \phi_A \circ \phi_{M_1}$, where ϕ_{M_1} is the automorphism of T that inverts the last coordinate. By direct inspection, one sees that the polytope $P_0 = \mathrm{Newt}(W_0)$ corresponding to the initial potential W_0 given in Definition 2.4 is Fano in the sense of Definition 2.8. It follows from [1, Proposition 2] that $P_{\mathfrak{s}}$ is Fano for every Plücker sequence $\mathfrak{s}$, and thus $X(\Sigma^f P_{\mathfrak{s}})$ is too, thanks to Lemma 2.9. ■

3. Plücker Lagrangians

In this section, Σ denotes a complete fan in $\mathbb{R}^{k(n-k)}$, and $X(\Sigma)$ its associated proper toric variety; see for example [11, 18] for background material on toric geometry. The reader familiar with symplectic manifolds and Hamiltonian torus actions can think of Σ as the normal fan $\Sigma = \Sigma^n \Delta$ of a moment polytope Δ, with the important caveat that $X(\Sigma)$ is typically singular, and not even an orbifold; in this case Δ should be thought as the closure of the open convex region obtained from the moment map of the maximal torus orbit.

We will assume that the primitive generators of the rays of Σ in the lattice $\mathbb{Z}^{k(n-k)} \subset \mathbb{R}^{k(n-k)}$ are the vertices of a convex polytope P, and alternatively think of Σ as its face fan $\Sigma = \Sigma^f P$. This condition is equivalent to $X(\Sigma)$ being Fano, and P is sometimes called a *Fano polytope*. The reader should not confuse the polytopes Δ and P: the second is always a lattice polytope, whereas the first may not be. The two polytopes are related by polar duality $\Delta = P^\circ$.

3.1. Lagrangian tori from degenerations

Definition 3.1. If $X \subset \mathbb{P}^M$ is a smooth subvariety of complex dimension N, an embedded toric degeneration $X \rightsquigarrow X(\Sigma)$ is a closed subscheme $\mathcal{X} \subset \mathbb{P}^M \times \mathbb{C}$ such that the map $p : \mathcal{X} \to \mathbb{C}$ obtained by restriction of the projection satisfies the following properties:

- $p^{-1}(\mathbb{C}^\times) \cong X \times \mathbb{C}^\times$ as schemes over $\mathbb{C}^\times$;
- $p^{-1}(0) \subset \mathbb{P}^M$ is an orbit closure for some linear torus action $(\mathbb{C}^\times)^N \curvearrowright \mathbb{P}^M$;
- $p^{-1}(0)$ is a toric variety with fan Σ.

Proposition 3.2 (Rietsch and Williams [40, Theorem 1.1]). *Every Plücker sequence* $\mathfrak{s}$ *of mutations of type* (k,n) *has an associated embedded toric degeneration* $\mathrm{Gr}(k,n) \rightsquigarrow X(\Sigma_{\mathfrak{s}})$*, where* $\Sigma_{\mathfrak{s}} = \Sigma^f P_{\mathfrak{s}}$ *is the face fan of the Newton polytope* $P_{\mathfrak{s}}$ *of the final potential* $W_{\mathfrak{s}}$*.*

Proof. For the reader's convenience, we provide details on how to specialize the result of Rietsch and Williams [40, Theorem 1.1] to recover this statement. Each step (Q_i, W_i) of the Plücker sequence $\mathfrak{s}$ corresponds to a reduced plabic graph G_i of type $\pi_{k,n}$ [40, Section 3], which is a combinatorial object encoding the quiver Q_i and the Laurent polynomial W_i simultaneously. Nodes in Q_i correspond to faces in G_i, and each arrow of Q_i is dual to an edge of G_i, with black/white nodes of the plabic graph respectively to the right/left of the arrow. The frozen nodes of Q_i correspond to boundary faces of G_i, and the mutable nodes to interior faces. Mutations at some mutable node with two incoming and outgoing arrows in Q_i correspond to a square move on the plabic graph G_i. The Plücker variables on nodes of Q_i are labeled by the Young diagrams appearing on the faces of G_i, which are induced by trips as in [40, Definition 3.5]. The Laurent polynomial W_i is a generating function counting matchings on the plabic graph G_i [40, Theorem 18.2]; see also Marsh-Scott [29] for a proof. The initial seed (Q_0, W_0) corresponds to a particular plabic graph $G_0 = G^{rec}_{k,n}$, called the *rectangle plabic graph* in [40, Section 4]. Consider the divisor $D_i \subset \mathrm{Gr}(k,n)$ cut out by the equation $x_{d_i} = 0$, with $d_i \subseteq k \times (n-k)$ one of the n frozen Young diagrams, and call $D = r_1 D_1 + \cdots + r_n D_n$ a general effective divisor with the same support. One can associate to the pair $(D, G_{\mathfrak{s}})$ a convex polytope $\Delta_{G_{\mathfrak{s}}}(D)$ known as Okounkov body [40, Section 1.2]. From now on, set $r_1 = \cdots = r_n = 1$, and call $D_{FZ} = D_1 + \cdots + D_n$ the corresponding divisor. There exists a scaling factor $r_{\mathfrak{s}} \in \mathbb{Q}^+$ such that $r_{\mathfrak{s}} \Delta_{G_{\mathfrak{s}}}(D_{FZ})$ is a normal lattice polytope [11, Definition 2.2.9]; normality is referred to as integer decomposition property in [40, Definition 17.7], and from [40, Proposition 19.4] one sees that the scaling factor mentioned there is related to ours by $r_{\mathfrak{s}} = \frac{r_{G_{\mathfrak{s}}}}{n}$. From [40, Section 17] one gets a degeneration of $\mathrm{Gr}(k,n)$ to the toric variety associated with the polytope $r_{\mathfrak{s}} \Delta_{G_{\mathfrak{s}}}(D_{FZ})$, and this is an embedded toric degeneration in the sense of Definition 3.1 with fan $\Sigma_{\mathfrak{s}} = \Sigma^n r_{\mathfrak{s}} \Delta_{G_{\mathfrak{s}}}(D_{FZ}) = \Sigma^n \Delta_{G_{\mathfrak{s}}}(D_{FZ})$, where we used that the normal fan of a polytope does not change under scaling. In [40, Theorem 1.1] and [40, Definition 10.14], an interpretation of $\Delta_{G_{\mathfrak{s}}}(r_1 D_1 + \cdots + r_n D_n)$ is given in terms of the tropicalization of $W_{\mathfrak{s}}$. Setting $r_1 = \cdots = r_n = 1$, one finds in particular that for $D_{FZ} = D_1 + \cdots + D_n$ in fact

$$\Delta_{G_{\mathfrak{s}}}(D_{FZ}) = \{v \in \mathbb{R}^{k(n-k)} \colon \langle v, u \rangle \geq -1 \text{ for every vertex } u \in P_{\mathfrak{s}}\};$$

here $P_{\mathfrak{s}}$ denotes the Newton polytope of the Laurent polynomial $W_{\mathfrak{s}}$, i.e., the convex hull of its exponents. To see this, observe that from [40, Theorem 1.1] and [40, Definitions 10.7 and 10.14] one has

$$v \in \Delta_{G_{\mathfrak{s}}}(D_{FZ}) \iff \operatorname{Trop}(W_{i|T_{\mathfrak{s}}})(v) \geq -1 \quad \text{for } i = 1, \ldots, n;$$

here each W_i is a special term of a rational function $W = W_1 + \cdots + W_n$ on $\mathrm{Gr}^\vee(k,n)$ defined in [40, Definition 10.1], and

$$T_{\mathfrak{s}} = \{[M] \in \mathrm{Gr}^\vee(k,n) \colon l(M) \neq 0 \text{ for all } l \text{ label of } Q_{\mathfrak{s}}\}$$

is a complex torus chart such that $W|_{T_{\mathfrak{s}}} = W_{\mathfrak{s}}$; compare (1) of Proposition 2.6. The symbol $\operatorname{Trop}(\cdot)$ denotes tropicalization of Laurent polynomials, which produces a piece-wise linear function defined as

$$\operatorname{Trop}\Big(\sum_u c_u x^u\Big)(v) = \min_u \langle v, u \rangle.$$

Also observe that

$$\begin{aligned} &\operatorname{Trop}(W_{i|T_{\mathfrak{s}}})(v) \geq -1 \quad \text{for } i = 1, \ldots, n \\ &\iff \min_{i=1,\ldots,n} \operatorname{Trop}(W_{i|T_{\mathfrak{s}}})(v) \geq -1 \\ &\iff \operatorname{Trop}(W|_{T_{\mathfrak{s}}})(v) \geq -1 \\ &\iff \operatorname{Trop}(W_{\mathfrak{s}})(v) \geq -1. \end{aligned}$$

Summarizing, $v \in \Delta_{G_{\mathfrak{s}}}(D_{FZ})$ is equivalent to $\langle v, u \rangle \geq -1$ for every u exponent of a monomial in $W_{\mathfrak{s}}$. By convexity, the latter condition is equivalent to asking $\langle v, u \rangle \geq -1$ only for those u that are vertices of the Newton polytope $P_{\mathfrak{s}}$ of $W_{\mathfrak{s}}$. We have thus recovered the polar dual polytope, i.e., $\Delta_{G_{\mathfrak{s}}}(D_{FZ}) = P_{\mathfrak{s}}^\circ$. Since the normal fan of a polytope equals the face fan of its polar dual and polar duality is an involution, we find that $\Sigma_{\mathfrak{s}} = \Sigma^n \Delta_{G_{\mathfrak{s}}}(D_{FZ}) = \Sigma^f P_{\mathfrak{s}}$ as in the statement. ■

Remark 3.3. The toric variety $X(\Sigma_{\mathfrak{s}})$ depends only the final step of $\mathfrak{s}$ in the following sense. Suppose that $Q_{\mathfrak{s}}$ and $Q_{\mathfrak{s}'}$ are the final quivers of two different Plücker sequences. Consider the charts

$$T_{\mathfrak{s}} = \{[M] \in \mathrm{Gr}^\vee(k,n) \colon l(M) \neq 0 \text{ for all } l \text{ label of } Q_{\mathfrak{s}}\}$$

and

$$T_{\mathfrak{s}'} = \{[M] \in \mathrm{Gr}^\vee(k,n) \colon l'(M) \neq 0 \text{ for all } l' \text{ label of } Q_{\mathfrak{s}'}\};$$

these were already considered in (1) of Proposition 2.6, were it was observed that $W_{\mathfrak{s}} = W|_{T_{\mathfrak{s}}}$ and $W_{\mathfrak{s}'} = W|_{T_{\mathfrak{s}}}$. If the quivers $Q_{\mathfrak{s}}$ and $Q'_{\mathfrak{s}}$ have equal sets of labels, then $T_{\mathfrak{s}} = T_{\mathfrak{s}'}$ and therefore the final Laurent polynomials of the two Plücker sequences are equal: $W_{\mathfrak{s}} = W_{\mathfrak{s}'}$. Since $\Sigma_{\mathfrak{s}}$ and $\Sigma_{\mathfrak{s}'}$ are the face fans of their Newton polytopes $P_{\mathfrak{s}} = P_{\mathfrak{s}'}$, it follows that $\Sigma_{\mathfrak{s}} = \Sigma_{\mathfrak{s}'}$ and thus $X(\Sigma_{\mathfrak{s}}) = X(\Sigma_{\mathfrak{s}'})$.

In what follows, we endow the Grassmannian $\mathrm{Gr}(k,n)$ with the symplectic structure obtained by restriction of the Fubini-Study form on the target projective space of the Plücker embedding.

Proposition 3.4 (Harada and Kaveh [21, Theorem B]). *Every Plücker sequence $\mathfrak{s}$ of mutations of type (k,n) has an associated Lagrangian torus $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$, and it comes with a canonical basis of $H_1(L_{\mathfrak{s}};\mathbb{Z})$.*

Proof. For the reader's convenience, we provide details on how to specialize the result of Harada and Kaveh [21, Theorem B] to recover this statement. Recall from Proposition 3.2 that $\mathfrak{s}$ determines a degeneration of $\mathrm{Gr}(k,n)$ to the toric variety associated with the polytope $r_{\mathfrak{s}}\Delta_{G_{\mathfrak{s}}}(D_{FZ})$, known as Okounkov body; for a detailed description of the value semigroup underlying this Okounkov body and of why it satisfies the assumptions of [21, Theorem B], see [40, Definition 17.8 and Lemma 17.9]. From [21, Theorem B] one deduces the existence of an open set $U_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ and a smooth submersion $\mu_{\mathfrak{s}}\colon U_{\mathfrak{s}} \to \mathbb{R}^{k(n-k)}$ whose image is the interior of the polytope, and whose fibers are Lagrangian tori. Call $L_{\mathfrak{s}} = \mu_{\mathfrak{s}}^{-1}(0)$. If $p \in L_{\mathfrak{s}}$, the tangent space to the fiber at p is $T_p(L_{\mathfrak{s}}) = \ker(d_p\mu_{\mathfrak{s}})$. Therefore, the standard basis of $\mathbb{R}^{k(n-k)}$ lifts under $d_p\mu_{\mathfrak{s}}$ to a basis of $T_p(U_{\mathfrak{s}})/T_p(L_{\mathfrak{s}})$. Since the symplectic structure vanishes on $L_{\mathfrak{s}}$, the lift defines a symplectic-dual basis of $T_p(L_{\mathfrak{s}})$. Since $L_{\mathfrak{s}}$ is a torus, the vectors of this basis are tangent to natural closed loops in $L_{\mathfrak{s}}$, and their homology classes give a basis of $H_1(L_{\mathfrak{s}};\mathbb{Z})$ which is independent of the point $p \in L_{\mathfrak{s}}$. ■

Definition 3.5. The Lagrangians $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ of Proposition 3.4 are called *Plücker tori,* and the elements $\gamma_d \in H_1(L_{\mathfrak{s}};\mathbb{Z})$ of the canonical basis are called *canonical cycles.* We index the canonical cycles by Young diagrams $d \subseteq k \times (n-k)$ such that $d \neq \emptyset$ and d appears in a label x_d of the final quiver $Q_{\mathfrak{s}}$.

Remark 3.6. Note that $x_\emptyset$ is the label of a frozen node in the initial quiver Q_0. Since frozen labels do not change under mutation, $x_\emptyset$ is in fact a the label of a forzen node in any quiver $Q_{\mathfrak{s}}$.

3.2. Local systems from Schur polynomials

Given a Young diagram $d \subseteq k \times (n-k)$, the corresponding k-variables Schur polynomial is defined as

$$S_d(z_1, \ldots, z_k) = \sum_{T_d} z_1^{t_1} \ldots z_k^{t_k},$$

where the sum runs over semi-standard tableaux T_d on d. The tableaux T_d are obtained by labeling d with integers in $\{1, \ldots, k\}$, in such a way that rows are weakly increasing and columns are strictly increasing. The exponent t_i is the number of times that the integer i appears in the tableaux T_d. If I is any of the $\binom{n}{k}$ sets of roots of $z^n = (-1)^{k+1}$ with $|I| = k$, it makes sense to evaluate $S_d(I) \in \mathbb{C}$ without specifying an order on the elements of I because Schur polynomials are symmetric functions.

Definition 3.7. For each Plücker torus $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ and set I, denote ξ_I the rank one local system whose holonomy $\mathrm{hol}_{\xi_I} : H_1(L_{\mathfrak{s}}; \mathbb{Z}) \to \mathbb{C}^\times$ around the canonical cycles of Definition 3.5 is given by the formula

$$\mathrm{hol}_{\xi_I}(\gamma_d) = S_d(I) \in \mathbb{C}^\times;$$

if $S_d(I) = 0$ for some d appearing in a label x_d of the final quiver of $\mathfrak{s}$, then ξ_I is not defined.

3.3. A conjecture and some evidence

If $\mathfrak{s}$ is a Plücker sequence of type (k,n), after setting $x_\emptyset = 1$ the Laurent polynomial $W_{\mathfrak{s}}$ can be thought of as a regular function on the algebraic torus $H_1(L_{\mathfrak{s}}; \mathbb{Z}) \otimes \mathbb{C}^\times \cong (\mathbb{C}^\times)^{k(n-k)}$. Setting $x_\emptyset = 1$ corresponds to thinking $\mathcal{A}_{k,n} = \mathcal{O}(U_{k,n})$ as algebra of regular functions on $U_{k,n} = \mathrm{Gr}^\vee(k,n) \setminus D_{FZ}^\vee$ rather than on its affine cone.

The canonical cycles $\gamma_d \in H_1(L_{\mathfrak{s}}; \mathbb{Z})$ of Definition 3.5 give an isomorphism of schemes $H_1(L_{\mathfrak{s}}; \mathbb{Z}) \otimes \mathbb{C}^\times \cong (\mathbb{C}^\times)^{k(n-k)}$, where one thinks the latter torus as having coordinates x_d labeled by Young diagrams $d \subseteq k \times (n-k)$ such that $d \neq \emptyset$ and d appears in some label of the quiver $Q_{\mathfrak{s}}$. Under the identification described by Scott [41, Theorem 4], one can think that $H_1(L_{\mathfrak{s}}; \mathbb{Z}) \otimes \mathbb{C}^\times \cong T_{\mathfrak{s}} \subset \mathrm{Gr}^\vee(k,n)$ where

$$T_{\mathfrak{s}} = \{[M] \in \mathrm{Gr}^\vee(k,n) \colon x_d(M) \neq 0 \text{ for all } d \text{ label of } Q_{\mathfrak{s}}\}.$$

Conjecture 3.8. *If $\mathfrak{s}$ and $\mathfrak{s}'$ are two Plücker sequences of type (k,n), and ϕ is a Hamiltonian isotopy of* $\mathrm{Gr}(k,n)$ *such that $\phi(L_{\mathfrak{s}}) = L_{\mathfrak{s}'}$, then the induced map $\phi_* : H_1(L_{\mathfrak{s}}; \mathbb{Z}) \to H_1(L_{\mathfrak{s}'}; \mathbb{Z})$ is such that*

$$W_{\mathfrak{s}} \sim W_{\mathfrak{s}'} \circ (\phi_* \otimes \mathrm{id}_{\mathbb{C}^\times}),$$

where $\sim$ denotes equality up to automorphisms of $T_{\mathfrak{s}}$.

Remark 3.9. The reason for $\sim$ in the conjecture above is the following. Suppose that $W_{\mathfrak{s}}$ is the disk potential of $L_{\mathfrak{s}}$, i.e., $W_{\mathfrak{s}} = W_{L_{\mathfrak{s}}}$. By Hamiltonian invariance of the disk potential, if $L_{\mathfrak{s}'} = \phi(L_{\mathfrak{s}})$ then $W_{L_{\mathfrak{s}'}} = W_{L_{\mathfrak{s}}} = W_{\mathfrak{s}}$ as long as we express the disk potential of $L_{\mathfrak{s}'}$ in the basis of cycles induced by $\phi_* : H_1(L_{\mathfrak{s}};\mathbb{Z}) \to H_1(L_{\mathfrak{s}'};\mathbb{Z})$. Instead, the Laurent polynomial $W_{\mathfrak{s}'}$ expresses the disk potential of $L_{\mathfrak{s}'}$ in the canonical basis of cycles of Definition 3.5, which is a priori different from the one induced by ϕ_*.

Under some assumptions on the singularities of the toric varieties $X(\Sigma_{\mathfrak{s}})$ appearing as limits of the degenerations $\mathrm{Gr}(k,n) \rightsquigarrow X(\Sigma_{\mathfrak{s}})$, the conjecture above can be verified. We describe below how, and give some sample applications in Section 4.

Definition 3.10. If $X(\Sigma)$ is a projective toric variety, a toric resolution consists of a smooth projective toric variety $X(\widetilde{\Sigma})$ with a toric morphism $r : X(\widetilde{\Sigma}) \to X(\Sigma)$ which is a birational equivalence.

Any toric variety $X(\Sigma)$ has a toric resolution; see for example [11, Chapter 11]. Toric resolutions can be constructed by taking refinements $\widetilde{\Sigma}$ of the fan Σ, which have natural associated morphisms r. The refined fan $\widetilde{\Sigma}$ has in general more rays than Σ, and these correspond to torus invariant divisors in the exceptional locus $r^{-1}(\mathrm{Sing}\, X(\Sigma))$.

Definition 3.11. A toric resolution $r\colon X(\widetilde{\Sigma}) \to X(\Sigma)$ is small if $\widetilde{\Sigma}$ and Σ have the same rays. Being small is equivalent to $\mathrm{codim}_{\mathbb{C}}(r^{-1}(\mathrm{Sing}\, X(\Sigma))) \geq 2$; see for example [11, Proposition 11.1.10].

Proposition 3.12. *If $\mathfrak{s}$ is a Plücker sequence of type (k,n), and the toric variety $X(\Sigma_{\mathfrak{s}})$ admits a small resolution, then $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ is monotone and has disk potential $W_{\mathfrak{s}}$ with respect to the basis of canonical cycles for $H_1(L_{\mathfrak{s}};\mathbb{Z})$.*

Proof. Recall from Proposition 3.4 that there is a smooth submersion $\mu_{\mathfrak{s}}\colon U_{\mathfrak{s}} \to \mathbb{R}^{k(n-k)}$ with Lagrangian torus fibers, defined on some open set $U_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$. If $P_{\mathfrak{s}}$ is the Newton polytope of the Laurent polynomial $W_{\mathfrak{s}}$, the image of this map is the interior of the polytope described in Proposition 3.2:

$$r_{\mathfrak{s}} \Delta_{G_{\mathfrak{s}}}(D_{FZ}) = \{v \in \mathbb{R}^{k(n-k)}\colon \langle v,u\rangle \geq -r_{\mathfrak{s}} \text{ for every vertex } u \in P_{\mathfrak{s}}\}.$$

Call v a point in the interior, and $L_{\mathfrak{s}}(v) = \mu_{\mathfrak{s}}^{-1}(v)$ the corresponding Lagrangian torus fiber. Observe that $X(\Sigma_{\mathfrak{s}})$ is Fano, thanks to Proposition 2.10. The assumption that $X(\Sigma_{\mathfrak{s}})$ has a small toric resolution allows to invoke a theorem of Nishinou, Nohara, and Ueda [30, Theorem 10.1], and conclude that the disk potential of $L_{\mathfrak{s}}(v) \subset \mathrm{Gr}(k,n)$ has one monomial for each facet $\langle v,u\rangle = -r_{\mathfrak{s}}$ of $r_{\mathfrak{s}} \Delta_{G_{\mathfrak{s}}}(D_{FZ})$, with exponent $u \in H_1(L_{\mathfrak{s}}(v);\mathbb{Z})$. The subgroup of disk classes in $H_2(\mathrm{Gr}(k,n), L_{\mathfrak{s}}(v);\mathbb{Z})$ is generated by Maslov index 2 classes. This holds because $U_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ is the domain of a symplectomporhism $\psi\colon U_{\mathfrak{s}} \to X(\Sigma_{\mathfrak{s}}) \setminus D_{\Sigma_{\mathfrak{s}}}$ with the maximal torus orbit of the toric variety $X(\Sigma_{\mathfrak{s}})$ obtained by removing the standard torus invariant divisor $D_{\Sigma_{\mathfrak{s}}}$. Harada-Kaveh [21, Theorem A (1)] shows that ψ extends to a continuous map $\overline{\psi}\colon \mathrm{Gr}(k,n) \to X(\Sigma_{\mathfrak{s}})$. As explained in Nishinou, Nohara, and Ueda [30, Lemma 9.2 and Corollary 9.3], the assumption that $X(\Sigma_{\mathfrak{s}})$ has a small resolution $X(\widetilde{\Sigma_{\mathfrak{s}}})$ allows to use the map $\overline{\psi}$ to identify disk classes in $H_2(\mathrm{Gr}(k,n), L_{\mathfrak{s}}(v); \mathbb{Z})$ with classes of disks in $X(\widetilde{\Sigma_{\mathfrak{s}}})$ with boundary on the toric moment fiber over v, and these are generated by Maslov index 2 classes; see Cho and Oh [10, Theorem 5.1] for a general formula computing the Maslov index of disks with boundary in a toric moment fiber. In conclusion, $L_{\mathfrak{s}}(v)$ is monotone if and only if all Maslov index 2 classes have the same symplectic area. The symplectic area of a Maslov 2 disk with boundary u is $2\pi(\langle v, u\rangle + r_{\mathfrak{s}})$; see Cho and Oh [10, Theorem 8.1] for a proof. The choice $v = 0$ guarantees that all the areas are equal, and thus $L_{\mathfrak{s}} = L_{\mathfrak{s}}(0)$ is monotone. ■

4. Sample applications

We describe some sample applications of what seen so far to the symplectic topology of Grassmannians. These results are by no means optimal; they are meant to illustrate new phenomena, and we give some indications on how one can prove analogous statements using the same techniques.

4.1. Generating the Fukaya category of Gr(2, *n*)

In this section we focus on Grassmannians of planes $\mathrm{Gr}(2,n)$. For this class of Grassmannians, Nohara and Ueda [31, 32] have used symplectic reduction techniques to construct a Catalan number C_{n-2} of integrable systems on $\mathrm{Gr}_2(n)$, each labeled by a triangulation Γ of the n-gon; the generic fibers of these systems are Lagrangian tori $L_\Gamma \subset \mathrm{Gr}_2(n)$, and the images of their Hamiltonians are lattice polytopes Δ_Γ. Explicit formulas for the disk potentials W_{L_Γ} were given as sums over edges of the triangulation Γ. We compare this construction with the case $k = 2$ of our Construction 1.1, and explore some consequences for the Fukaya category of $\mathrm{Gr}(2,n)$. When $k = 2$, the Plücker coordinates appearing as labels of a quiver $Q_{\mathfrak{s}}$ have a simple combinatorial description.

Definition 4.1. Let $n > 2$, and consider n points on S^1, labeled counter-clockwise from 1 to n. A triangulation Γ of $[n]$ is a collection of subsets $\{i, j\} \subset [n]$ with $i \neq j$, such that connecting i and j with an arc in D^2 for all $\{i, j\} \in \Gamma$ one gets a triangulation of the n-gon with vertices at $[n]$.

Remark 4.2. Every triangulation Γ of $[n]$ must contain the n sets

$$\{1,2\}, \quad \{2,3\}, \quad \ldots, \quad \{n-1,n\}, \quad \{1,n\};$$

these correspond to the edges of the n-gon; the other sets in Γ correspond to interior edges of the triangulation. Note that the n sets above are also the vertical steps $d^{|}$ of those Young diagrams $d \subseteq 2 \times (n-2)$ that label the frozen nodes in Definition 2.4 (specialized to $k = 2$).

Lemma 4.3 (Fomin and Zelevinsky [16, Proposition 12.5]; [33, Theorem 1.6]). *A collection of Young diagrams $d \subseteq 2 \times (n-2)$ labels the nodes of $Q_{\mathfrak{s}}$ for some Plücker sequence $\mathfrak{s}$ of type $(2,n)$ if and only if the set $\Gamma = \{d^{|} \subset [n]\}$ is a triangulation of $[n]$.*

Proof. A set of Plücker coordinates x_d labels the nodes of some quiver $Q_{\mathfrak{s}}$ precisely if it is a cluster in the cluster algebra structure of $\mathcal{A}_{2,n}$. By Oh, Postnikov, Speyer [33, Theorem 1.6], this is equivalent to saying that the sets $d^{|}$ of vertical steps of the corresponding Young diagrams $d \subseteq 2 \times (n-2)$ form a maximal weakly separated collection in the sense of [33, Definition 3.1]. For $k = 2$ the general notion of maximal weakly separated collection recovers the one of triangulation given in Definition 4.1. ■

Lemma 4.4 (Nohara and Ueda [31, Theorem 1.5] and [32, Theorem 1.1]). *If $k = 2$ then the Lagrangian tori $L_{\mathfrak{s}} \subset \mathrm{Gr}(2,n)$ are monotone for all $\mathfrak{s}$ and have disk potential $W_{\mathfrak{s}}$. Consequently, by Hamiltonian invariance of the disk potential, Conjecture 3.8 holds.*

Proof. In view of Proposition 3.12, it suffices to prove that for any Plücker sequence $\mathfrak{s}$ of type $(2,n)$ the toric variety $X(\Sigma_{\mathfrak{s}})$ has a small toric resolution. From Lemma 4.3, the labels of $Q_{\mathfrak{s}}$ define a triangulation $\Gamma_{\mathfrak{s}}$ of $[n]$. Nohara and Ueda [32, Theorem 1.1] describe an open embedding $\iota_{\Gamma_{\mathfrak{s}}}\colon (\mathbb{C}^{\times})^{2(n-2)} \to \mathrm{Gr}^{\vee}(k,n)$ such that $\iota_{\Gamma_{\mathfrak{s}}}^{*} W = W_{\Gamma_{\mathfrak{s}}}$, where $W \in \mathcal{A}_{k,n}$ is the Landau–Ginzburg potential defined by Marsh-Rietsch [28] and $W_{\Gamma_{\mathfrak{s}}}$ is a Laurent polynomial associated to the triangulation $\Gamma_{\mathfrak{s}}$. It was shown earlier by Nohara and Ueda [31, Proposition 7.4] that the polar dual of the Newton polytope of $W_{\Gamma_{\mathfrak{s}}}$ is a lattice polytope $\Delta_{\Gamma_{\mathfrak{s}}} = \mathrm{Newt}^{\circ}(W_{\Gamma_{\mathfrak{s}}})$ (as opposed to just a rational polytope) and that the associated toric variety $X(\Sigma^{n} \Delta_{\Gamma_{s}})$ has a small toric resolution [31, Theorem 1.5]. Since polar duality exchanges normal and face fans $X(\Sigma^{n} \Delta_{\Gamma_{\mathfrak{s}}}) = X(\Sigma^{f}\, \mathrm{Newt}(W_{\Gamma_{\mathfrak{s}}}))$. The image of the embedding $\iota_{\Gamma_{\mathfrak{s}}}$ is the cluster chart $T_{\mathfrak{s}}$ and $W_{\mathfrak{s}} = W|_{T_{\mathfrak{s}}}$, so $W_{\Gamma_{\mathfrak{s}}}$ and $W_{\mathfrak{s}}$ are Laurent polynomials related by an automorphism of the torus, therefore their Newton polytopes $\mathrm{Newt}(W_{\Gamma_{\mathfrak{s}}})$ and $P_{\mathfrak{s}}$ are equivalent under the action of $\mathrm{GL}(2(n-2), \mathbb{Z})$. We conclude that $X(\Sigma^{f}\, \mathrm{Newt}(W_{\Gamma_{\mathfrak{s}}})) \cong X(\Sigma^{f} P_{\mathfrak{s}}) = X(\Sigma_{\mathfrak{s}})$ and therefore $X(\Sigma_{\mathfrak{s}})$ has a small toric resolution too. ■

As described by Sheridan [43], the Fukaya category of a monotone symplectic manifold like the Grassmannian has a spectral decomposition

$$\mathcal{F}(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda} \mathcal{F}_\lambda(\mathrm{Gr}(k,n)).$$

The summands are A_∞-categories indexed by the eigenvalues λ of the operator $c_1\star$ of multiplication by the first Chern class acting on the small quantum cohomology. The objects of the λ-summand are monotone Lagrangians with rank one local systems L_ξ as described in Section 1. The following proposition holds for general Grassmannians.

Proposition 4.5. *For any $1 \le k < n$ and any Plücker sequence $\mathfrak{s}$ of type (k,n):*

1. *if $\mathcal{F}_\lambda(\mathrm{Gr}(k,n)) \neq 0$ then $\lambda = n(\zeta_1 + \cdots + \zeta_k)$ for some $\{\zeta_1, \dots, \zeta_k\} = I \subset \{\zeta \in \mathbb{C} \colon \zeta^n = (-1)^{k+1}\}$ with $|I| = k$;*
2. *if $X(\Sigma_\mathfrak{s})$ has a small toric resolution, then $(L_\mathfrak{s})_{\xi_I}$ is a defined and nonzero in $\mathcal{F}_\lambda(\mathrm{Gr}(k,n))$ if and only if $S_d(I) \neq 0$ for all Young diagrams d appearing as labels on the nodes of $Q_\mathfrak{s}$, and moreover $\lambda = n(\zeta_1 + \cdots + \zeta_k)$.*

Proof. (1) By [6, Proposition 1.3], $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $c_1\star$ of multiplication by the first Chern class acting on the small quantum cohomology if and only if $\lambda = n(\zeta_1 + \cdots + \zeta_k)$ for some $\{\zeta_1, \dots, \zeta_k\} = I \subset \{\zeta \in \mathbb{C} \colon \zeta^n = (-1)^{k+1}\}$ with $|I| = k$. By Auroux [3, Proposition 6.8], any monotone Lagrangian L with a rank one local system ξ having Floer cohomology $\mathrm{HF}(L_\xi, L_\xi) \neq 0$ must have $m^0(L_\xi)$ which is an eigenvalue of $c_1\star$.

(2) Think of the holonomy of a local system ξ on $L_\mathfrak{s}$ as a point on a complex torus

$$\mathrm{hol}_\xi \in \mathrm{Hom}(H_1(L_\mathfrak{s};\mathbb{Z})) \cong (\mathbb{C}^\times)^{k(n-k)}.$$

The identification depends on the choice of a basis for $H_1(L_\mathfrak{s};\mathbb{Z})$, and we use the canonical $\gamma_d \in H_1(L_\mathfrak{s};\mathbb{Z})$ of Definition 3.5. By Proposition 3.12 and the assumption of small resolution, the Lagrangian torus $L_\mathfrak{s}$ is monotone and has disk potential $W_{L_\mathfrak{s}} = W_\mathfrak{s}$. By Auroux [3, Proposition 6.9] and Sheridan [43, Proposition 4.2], one has Floer cohomology $HF((L_\mathfrak{s})_\xi, (L_\mathfrak{s})_\xi) \neq 0$ if and only if hol_ξ is a critical point of the disk potential. By Definition 3.7, the local system $\xi = \xi_I$ has $\mathrm{hol}_{\xi_I}(\gamma_d) = S_d(I)$. Rietsch [38, Lemma 4.4] proves that $S_d(I) = S_{d^T}(I^\vee)$, where $d^T \subseteq (n-k) \times k$ is the transpose Young diagram of d and $I^\vee = \{\zeta_1, \dots, \zeta_{n-k}\}$ is the set of $n-k$ distinct roots of $\zeta^n = (-1)^{n-k+1}$ obtained by looking at the roots I^c of $\zeta^n = (-1)^{k+1}$ that are not in I and declaring $I^\vee = e^{\pi i} I^c$. Consider the points $[M_{I^\vee}] \in \mathrm{Gr}^\vee(k,n)$

defined as

$$[M_{I^\vee}] = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \zeta_1 & \zeta_2 & \zeta_3 & \dots & \zeta_{n-k} \\ \zeta_1^2 & \zeta_2^2 & \zeta_3^2 & \dots & \zeta_{n-k}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_1^{n-1} & \zeta_2^{n-1} & \zeta_3^{n-1} & \dots & \zeta_{n-k}^{n-1} \end{bmatrix};$$

these are known to be the critical points of the Landau–Ginzburg potential $W \in \mathcal{A}_{k,n}$ defined by Marsh and Rietsch [28]; see [22, Theorem 1.1 and Corollary 3.12]. Observe that $S_{d^T}(I^\vee) = x_d(M_{I^\vee})/x_\emptyset(M_{I^\vee})$; this follows from the expression of Schur polynomials as determinants [6, Proposition 2.3 (1)]. After setting $x_\emptyset = 1$, one can think of $W_\mathfrak{s}$ as a regular function on the cluster chart $T_\mathfrak{s} \subset \mathrm{Gr}^\vee(k,n)$ such that $W_\mathfrak{s} = W|_{T_\mathfrak{s}}$, as explained in Proposition 2.6 (1). This means that the critical points of $W_\mathfrak{s}$ are precisely those critical points $[M_{I^\vee}] \in \mathrm{Gr}^\vee(k,n)$ of W such that $[M_{I^\vee}] \in T_\mathfrak{s}$. By definition of $T_\mathfrak{s}$ the latter condition is equivalent to $x_d(M_{I^\vee}) \neq 0$ for all Young diagrams d appearing as labels of $Q_\mathfrak{s}$, and thus ξ_I is a well-defined local system on $L_\mathfrak{s}$ such that $HF((L_\mathfrak{s})_\xi, (L_\mathfrak{s})_\xi) \neq 0$ if and only if $S_d(I) \neq 0$ for all d appearing as labels of $Q_\mathfrak{s}$. ∎

Lemma 4.6. *If n is odd, all the eigenvalues of $c_1\star$ acting on* $\mathrm{QH}(\mathrm{Gr}(2,n))$ *have algebraic multiplicity one.*

Proof. It was explained in Proposition 4.5 (2) that the eigenvalues of $c_1\star$ acting on $\mathrm{QH}(\mathrm{Gr}(k,n))$ correspond to critical values of the Landau–Ginzburg potential W on $\mathrm{Gr}^\vee(k,n)$ defined by Marsh and Rietsch [28], and that the corresponding critical points can be explicitly described. In particular, there are $\binom{n}{k}$ critical points, and thus at most the same number of critical values. Therefore, the statement is equivalent to proving that there are precisely $\binom{n}{2}$ distinct eigenvalues. From Proposition 4.5 (1), each eigenvalue is of the form $\lambda = n(\zeta_1 + \zeta_2)$, with ζ_1 and ζ_2 distinct roots of $\zeta^n = -1$. Write $\zeta_1 = e^{\frac{\pi i}{n}a}$ and $\zeta_1 = e^{\frac{\pi i}{n}b}$ with $0 < a < b < 2n$ odd integers. The norm of one such eigenvalue is

$$|\lambda| = n\sqrt{2}\Big(1 + 2\cos\Big(\frac{\pi}{n}(b-a)\Big)\Big)^{1/2}.$$

The function $\cos(x)$ is decreasing for $0 \le x \le \pi$ and $\cos(2\pi - x) = \cos(x)$; in our case $0 \le \frac{\pi}{n}(b-a) \le \pi$ whenever $0 \le b-a \le n$. Since n is odd by assumption, by varying a and b among all odd integers with $0 < a < b < 2n$ one finds $l = (n-1)/2$ eigenvalues with $0 < |\lambda_1| < \dots < |\lambda_l|$, corresponding to $b-a$ attaining all the even integer values in the interval $[2, n-1]$. Moreover, fixed any $1 \le t \le l$, the n complex numbers $\lambda_t, (e^{\frac{2\pi}{n}i})\lambda_t, \dots, (e^{\frac{2\pi}{n}i})^{n-1}\lambda_t$ are eigenvalues of $c_1\star$ too, and they have the same norm as λ_t; see also [6, Proposition 1.12] for more on the symmetries of the spectrum of $c_1\star$. Overall, we found $n(n-1)/2 = \binom{n}{2}$ distinct eigenvalues. ∎

Lemma 4.7. *Let $d \subseteq 2 \times (n-2)$ be a Young diagram, and denote by $d^{|} = \{s, t\}$ with $1 \leq s < t \leq n$ its vertical steps. Writing an arbitrary set $I \subset \{\zeta \in \mathbb{C} \colon \zeta^n = -1\}$ with $|I| = 2$ as $I_{a,b} = \{e^{\frac{\pi}{n}ia}, e^{\frac{\pi}{n}ib}\}$ with $0 < a < b < 2n$ odd integers, then*

$$S_d(I_{a,b}) = 0 \qquad \Longleftrightarrow \qquad n \mid \frac{b-a}{2}(t-s).$$

Proof. Consider the full rank $n \times 2$ matrix

$$[M_{I_{a,b}}] = \begin{bmatrix} 1 & e^{\frac{\pi}{n}ia} & (e^{\frac{\pi}{n}ia})^2 & \dots & (e^{\frac{\pi}{n}ia})^{n-1} \\ 1 & e^{\frac{\pi}{n}ib} & (e^{\frac{\pi}{n}ib})^2 & \dots & (e^{\frac{\pi}{n}ib})^{n-1} \end{bmatrix}^T$$

As pointed out in part (2) of Proposition 4.5, one has $S_d(I_{a,b}) = 0$ if and only if $x_{d^T}(M_{I_{a,b}}) = 0$. Observe that the horizontal steps of the transpose diagram are $(d^T)^- = \{n+1-t, n+1-s\}$, so that

$$x_{d^T}(M_{I_{a,b}}) = e^{\frac{\pi}{n}ia(n-t)}e^{\frac{\pi}{n}ib(n-s)} - e^{\frac{\pi}{n}ia(n-s)}e^{\frac{\pi}{n}ib(n-t)},$$

from which $x_{d^T}(M_{I_{a,b}}) = 0$ if and only if $e^{\frac{\pi}{n}i(as+bt-at-bs)} = 1$. The last condition is verified precisely when $2n \mid (b-a)(t-s)$, and since $b-a$ is the difference of two odd integers this can be rewritten as in the statement. ∎

Theorem 4.8. *If $n = 2^t + 1$ for some $t \in \mathbb{N}^+$, the derived Fukaya category $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2, 2^t + 1))$ is split-generated by objects supported on a single Plücker torus.*

Proof. Up to replacing 2 with $n-2$, we can think of the critical points $[M_{I_{a,b}}]$ of the Landau–Ginzburg potential W on $\mathrm{Gr}^\vee(2,n)$ defined by Marsh and Rietsch [28] as being parametrized by sets $I_{a,b} = \{e^{\frac{\pi}{n}ia}, e^{\frac{\pi}{n}ib}\}$ with $0 < a < b < 2n$ odd integers; compare with Proposition 4.5 (2). We claim that there exists a Plücker sequence $\mathfrak{s}$ of type $(2,n)$ such that the corresponding cluster chart $T_{\mathfrak{s}} \subset \mathrm{Gr}^\vee(2,n)$ contains all critical points $[M_{I_{a,b}}]$. If this is true, then these will be also critical points of the Laurent polynomial $W_{\mathfrak{s}} = W|_{T_{\mathfrak{s}}}$, which is the disk potential of the monotone Lagrangian torus $L_{\mathfrak{s}} \subset \mathrm{Gr}(2,n)$ by Proposition 3.12 and Lemma 4.4. By Sheridan [43, Corollary 2.19], if the generalized eigenspace $\mathrm{QH}_\lambda(X)$ of the operator $c_1\star$ is one-dimensional, any monotone Lagrangian brane L_ξ with $HF(L_\xi, L_\xi) \neq 0$ split-generates $\mathbf{D}\mathcal{F}_\lambda(X)$. Since $n = 2^t + 1$ is odd, by Lemma 4.6 we can apply this to $X = \mathrm{Gr}(2,n)$, $L = L_{\mathfrak{s}}$ and any $\xi = \xi_{I_{a,b}}$ for all $0 < a < b < 2n$ odd integers, thus concluding that the objects $(L_{\mathfrak{s}})_{I_{a,b}}$ split-generate every summand of $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2, 2^t+1))$. The construction of the Plücker sequence $\mathfrak{s}$ mentioned in the claim above proceeds as follows.

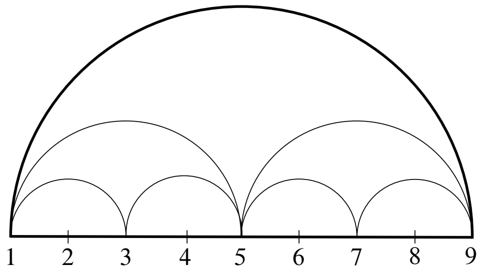


Figure 4. The dyadic triangulation of an 9-gon.

Consider the following incremental construction of a set Γ (an example with $t = 3$ is given in Figure 4):

1. start with a segment partitioned in $n - 1 = 2^t$ intervals, which are added to Γ as new edges $\{1, 2\}$, $\{2, 3\}$, ..., $\{2^t, 2^t + 1\}$;
2. partition the segment into $(n - 1)/2 = 2^{t-1}$ pairs of consecutive intervals, and add a new arc connecting the left end of the left interval to the right end of the right interval in each pair, thus adding new edges $\{1, 3\}$, $\{3, 5\}$, ...,$\{2^{t-1}, 2^{t+1}\}$ to Γ;
3. partition the segment in $(n - 1)/2^2 = 2^{t-2}$ tuples of 2^2 consecutive intervals, and add a new arc connecting the left end of the leftmost interval to the right end of the rightmost interval in each tuple, thus adding new edges $\{1, 5\}$, $\{5, 9\}$, ...,$\{2^{t+1} - 2^2, 2^{t+1}\}$ to Γ;
4. proceed as above until the initial segment is partitioned in two tuples of 2^{t-1} consecutive intervals, and add the edge $\{1, n\} = \{1, 2^t + 1\}$ to Γ, so that it becomes a triangulation of $[n]$ in the sense of Definition 4.1.

Let (Q_0, W_0) be the initial seed of Definition 2.4, and call Γ_0 the triangulation of $[n]$ corresponding to Young diagrams labeling the nodes of Q_0 as in Lemma 4.3. The triangulation Γ_0 is connected to Γ constructed above by a sequence of flips, which correspond to mutations of the quiver Q_0 at nodes with to incoming and two outgoing arrows. From Proposition 2.6, this gives a Plücker sequence of mutations of type $(2, n)$ in the sense of Definition 2.7, which ends at $(Q_{\mathfrak{s}}, W_{\mathfrak{s}})$ and such that the labels of $Q_{\mathfrak{s}}$ correspond to the triangulation $\Gamma_{\mathfrak{s}} = \Gamma$, again by Lemma 4.3. It remains to show that $[M_{I_{a,b}}] \in T_{\mathfrak{s}}$ for all odd integers a and b such that $0 < a < b < 2n$. Suppose not, then there exist some a, b and some Young diagram $d \subseteq 2 \times (n - 2)$ such that $x_d(M_{I_{a,b}}) = 0$. By Lemma 4.7, this implies that $n \mid \frac{b-a}{2}(t - s)$, where $d^{|} = \{s, t\}$ are the vertical steps of d. By construction, for any $d^{|} \in \Gamma_{\mathfrak{s}}$, if $d^{|} = \{s, t\}$ then $t - s$ is a power of 2, and since $n = 2^t + 1$ is odd by assumption we must have $n \mid \frac{b-a}{2}$. This is impossible, because $\frac{b-a}{2} < n$. ∎

Example 4.9. $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,9))$ is generated by a single Plücker torus. Note that instead the Gelfand–Cetlin torus mentioned in Section 1 does not support enough nonzero objects to generate; compare [6, Figure 2 (C)].

The arguments presented above can be generalized to prove that certain collections of Plücker tori split generate $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,n))$.

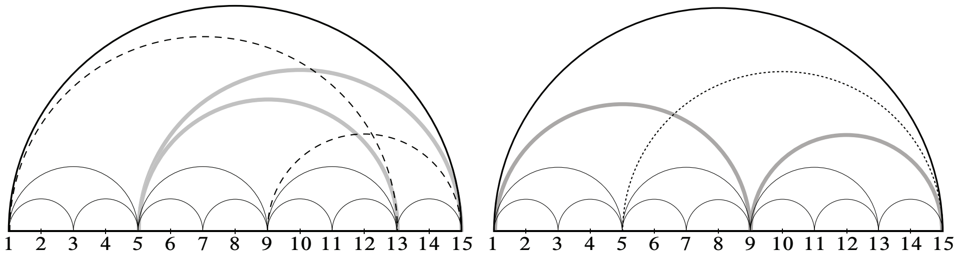
Definition 4.10. Let p be a prime number. A triangulation Γ of $[n]$ as in Definition 4.1 is called *p-avoiding* if for all $\{s,t\} \in \Gamma$ one has $p \nmid (t-s)$.

Theorem 4.11. *Let $n > 2$ be odd, and consider its prime factorization $n = p_1^{e_1} \dots p_l^{e_l}$. Assume that for all $1 \le i \le l$ there exists a triangulation Γ_i of $[n]$ that is p_i-avoiding, then $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,n))$ is split generated by objects supported on l Plücker tori.*

Proof. Recall that up to replacing 2 with $n-2$, we can think of the critical points $[M_{I_{a,b}}]$ of the Landau–Ginzburg potential W on $\mathrm{Gr}^\vee(2,n)$ defined by Marsh and Rietsch [28] as being parametrized by sets $I_{a,b} = \{e^{\frac{\pi}{n}ia}, e^{\frac{\pi}{n}ib}\}$ with $0 < a < b < 2n$ odd integers; compare with part (2) of Proposition 4.5. Denote $\mathcal{C}$ the set of all critical points of W, and for $1 \le i \le l$ define

$$\mathcal{C}_{p_i} = \{[M_{I_{a,b}}] \in \mathcal{C} \colon p_i^{e_i} \nmid \frac{b-a}{2}\}.$$

Observe that $\mathcal{C} = \mathcal{C}_{p_1} \cup \dots \cup \mathcal{C}_{p_l}$. Indeed, if $p_i^{e_i} \mid (b-a)/2$ for all $1 \le i \le l$ then $p_1^{e_1} \dots p_l^{e_l} = n \mid (b-a)/2$, against the fact that $(b-a)/2 < n$. By assumption, for each $1 \le i \le l$ there exist a triangulation Γ_i of $[n]$ that is p_i-avoiding, and arguing as in Theorem 4.8 one finds a Plücker sequence $\mathfrak{s}_i$ of type $(2,n)$ that starts with the initial seed (Q_0, W_0) and ends with $(Q_{\mathfrak{s}_i}, W_{\mathfrak{s}_i})$, and such that the labels of $Q_{\mathfrak{s}_i}$ correspond to the triangulation $\Gamma_{\mathfrak{s}_i} = \Gamma_i$ as in Lemma 4.3. Each of the l Plücker tori $L_{\mathfrak{s}_i} \subset \mathrm{Gr}(2,n)$ has an associated cluster chart $T_{\mathfrak{s}_i} \subset \mathrm{Gr}^\vee(2,n)$, and we claim that $\mathcal{C}_{p_i} \subset T_{\mathfrak{s}_i}$. Suppose not, then there exists some $[M_{I_{a,b}}] \in \mathcal{C}_{p_i}$ such that $[M_{I_{a,b}}] \notin T_{\mathfrak{s}_i}$. This means that $p_i^{e_i} \nmid \frac{b-a}{2}$ and there exists some Young diagram $d \subseteq 2 \times (n-2)$ such that $x_d(M_{I_{a,b}}) = 0$, and denoting $d^{|} = \{s,t\}$ its vertical steps $\{s,t\} \in \Gamma_{\mathfrak{s}_i}$. By Lemma 4.7, this implies that $n \mid \frac{b-a}{2}(t-s)$, and so in particular $p_i^{e_i} \mid \frac{b-a}{2}(t-s)$. Since $\Gamma_{\mathfrak{s}_i}$ is p_i-avoiding, this means that $p_i^{e_i} \mid \frac{b-a}{2}$, against the fact that $[M_{I_{a,b}}] \in \mathcal{C}_{p_i}$. As in Theorem 4.8, the assumption n odd and Lemma 4.6 guarantee, by Sheridan [43, Corollary 2.19], that any nonzero object of the Fukaya category supported on one of the l monotone Plücker tori $L_{\mathfrak{s}_1}, \dots, L_{\mathfrak{s}_l} \subset \mathrm{Gr}(2,n)$ split-generates the summand $\mathbf{D}\mathcal{F}_\lambda(\mathrm{Gr}(2,n))$ of the derived Fukaya category containing it. The objects supported on $L_{\mathfrak{s}_i}$ are obtained by endowing it with local systems $\xi_{I_{a,b}}$ as in Definition 3.7 corresponding to critical points $[M_{I_{a,b}}] \in T_{\mathfrak{s}_i}$; these are such that $HF((L_{\mathfrak{s}_i})_{\xi_{I_{a,b}}}, (L_{\mathfrak{s}_i})_{\xi_{I_{a,b}}}) \neq 0$ because the disk potential of $L_{\mathfrak{s}_i}$ is $W_{\mathfrak{s}_i} = W|_{T_{\mathfrak{s}_i}}$. ■



(a) 3-avoiding triangulation

(b) 5-avoiding triangulation

Figure 5. Two triangulations of a 15-gon.

Example 4.12. $\mathbf{D}\mathcal{F}(\mathrm{Gr}(2,15))$ is generated by two Plücker tori, whose corresponding triangulations are shown in Figure 5. To get the two triangulations, one starts by constructing a partial triangulation of the 15-gon with dyadic arcs as in Theorem 4.8 (solid arcs in Figure 5). The partial triangulation is p-avoiding for every prime $p > 2$ by construction. Since $15 = 3 \cdot 5$, by Theorem 4.11 one needs to find completions of the partial triangulation to full triangulations that are 3-avoid and 5-avoiding respectively. In Figure 5, the remaining arcs $\{i, j\}$ with $3 \mid (j - i)$ are coarsely dashed, while the one with $5 \mid (j - i)$ is finely dashed; triangulation (A) is obtained by adding two shaded arcs and is 3-avoiding, while triangulation (B) is obtained by adding two different shaded arcs and is 5-avoiding.

4.2. Exotic Lagrangian tori in Gr(3, 6)

Definition 4.13. If $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ is a Plücker Lagrangian of type (k,n), define its f-vector to be

$$\mathbf{f}(L_{\mathfrak{s}}) = (f_1, \ldots, f_{k(n-k)}) \in \mathbb{N}^{k(n-k)},$$

where f_i is the number of $(i-1)$-dimensional faces in the Newton polytope $P_{\mathfrak{s}}$ of the potential $W_{\mathfrak{s}}$.

Definition 4.14. If $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ is a Plücker Lagrangian of type (k,n), define its weight $\mathbf{wt}(L_{\mathfrak{s}}) \in \mathbb{N}$ to be the number of sets

$$I \subset \{\zeta \in \mathbb{C} \colon \zeta^n = (-1)^{k+1}\}$$

such that $|I| = k$ and $S_d(I) \neq 0$ for all Young diagrams d appearing as labels of the quiver $Q_{\mathfrak{s}}$.

Lemma 4.15. *Assume $\mathfrak{s}, \mathfrak{s}'$ are Plücker sequences of type (k,n) satisfying Conjecture 3.8. If $\mathbf{f}(L_{\mathfrak{s}}) \neq \mathbf{f}(L_{\mathfrak{s}'})$ or $\mathbf{wt}(L_{\mathfrak{s}}) \neq \mathbf{wt}(L_{\mathfrak{s}'})$, then the Lagrangian tori $L_{\mathfrak{s}}, L_{\mathfrak{s}'} \subset$ $\mathrm{Gr}(k,n)$ are not Hamiltonian isotopic.*

Proof. Suppose that there exists a Hamiltonian isotopy ϕ such that $\phi(L_{\mathfrak{s}}) = L_{\mathfrak{s}'}$. Then by assumption the induced map $\phi_*\colon H_1(L_{\mathfrak{s}};\mathbb{Z}) \to H_1(L_{\mathfrak{s}'};\mathbb{Z})$ is such that

$$W_{\mathfrak{s}} \sim W_{\mathfrak{s}'} \circ (\phi_* \otimes \mathrm{id}_{\mathbb{C}^\times}),$$

where $\sim$ denotes equality up to automorphisms of $T_{\mathfrak{s}}$. This means that the Newton polytopes $P_{\mathfrak{s}}$ and $P_{\mathfrak{s}'}$ of the Laurent polynomials $W_{\mathfrak{s}}$ and $W_{\mathfrak{s}'}$ are related by a transformation of $\mathrm{GL}(k(n-k),\mathbb{Z})$, and hence have the same f-vector, because the number of faces of any given dimension of a polytope is a unimodular invariant; this proves that $\mathbf{f}(L_{\mathfrak{s}}) = \mathbf{f}(L_{\mathfrak{s}'})$. Moreover, the Laurent polynomials $W_{\mathfrak{s}}$ and $W_{\mathfrak{s}'}$ can be thought of as regular functions on a torus $(\mathbb{C}^\times)^{k(n-k)}$, which agree up to an automorphism. Since the number of critical points of a function is invariant under automorphisms of its domain, it follows from part (2) of Proposition 4.5 that $\mathbf{wt}(L_{\mathfrak{s}}) = \mathbf{wt}(L_{\mathfrak{s}'})$. ■

Theorem 4.16. *The Grassmannian* $\mathrm{Gr}(3,6)$ *contains at least* 6 *monotone Lagrangian tori that are non-displaceable and pairwise inequivalent under Hamiltonian isotopy.*

Proof. Table 1 above contains informations about the steps of a Plücker sequence $\mathfrak{s}$ of type $(3,6)$. In each row, the reader can find the Young diagrams $d \subseteq 3\times 3$ appearing as labels of $Q_{\mathfrak{s}}$ at a given step, identified by their sets of vertical sets $\{i,j,k\} \subset [6]$. Each potential $W_{\mathfrak{s}}$ has an associated Newton polytope $P_{\mathfrak{s}}$, whose f-vector is $\mathbf{f}(L_{\mathfrak{s}})$ as in Definition 4.13. Following Definition 4.14, the weight $\mathbf{w}(L_{\mathfrak{s}})$ is computed by counting how many of the $\binom{6}{3}$ sets I of roots of $\zeta^6 = 1$ with $|I| = 3$ have the property that $S_d(I) \neq 0$ for all Young diagrams $d \subseteq 3\times 3$ that appear as labels on the nodes of the quiver $Q_{\mathfrak{s}}$. Calling $\Sigma_{\mathfrak{s}} = \Sigma^f P_{\mathfrak{s}}$ the face fan of the Newton polytope, by Proposition 3.12 the Lagrangian torus $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ is monotone and has disk potential $W_{\mathfrak{s}}$ whenever the toric variety $X(\Sigma_{\mathfrak{s}})$ has a small toric resolution in the sense of Definition 3.11. This condition can be checked algorithmically at each step, since every fan has finitely many simplicial refinements with the same rays, and every smooth refinement is in particular simplicial. For the 34 steps in Table 1, the code [7] finds small resolutions in 32 cases; the remaining 2 cases are marked gray in the table (we did not actually check all possible simplicial refinements in these cases, so small toric resolutions for them may still exist). From Lemma 4.15, we conclude that $\mathrm{Gr}(3,6)$ contains at least 6 monotone Lagrangian tori that are pairwise not Hamiltonian isotopic: rows 1, 3, 7, 8, 15, 29. Regarding nondisplaceability, it suffices to show that the 32 tori $L_{\mathfrak{s}} \subset \mathrm{Gr}(3,6)$ have Floer cohomology $HF(L_\xi, L_\xi) \neq 0$ for some local system ξ. By Auroux [3, Proposition 6.9] and Sheridan [43, Proposition 4.2], the Floer cohomology of a monotone Lagrangian torus brane L_ξ is nonzero if and only if the holonomy hol_ξ of its local system ξ is a critical point of the disk potential disk potential $W_{\mathfrak{s}}$. Therefore, it suffices to show that each of the 32 Laurent polynomials $W_{\mathfrak{s}}$

$k = 3, n = 6$			
$L_{\mathfrak{s}}$	Labels of $Q_{\mathfrak{s}}$	$\mathbf{f}(L_{\mathfrak{s}})$	$\mathbf{wt}(L_{\mathfrak{s}})$
1	123, 124, 125, 126, 156, 234, 245, 256, 345, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
2	123, 124, 125, 126, 145, 156, 234, 245, 345, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
3	123, 125, 126, 135, 145, 156, 234, 235, 345, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	6
4	123, 126, 134, 136, 146, 156, 234, 345, 346, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
5	123, 126, 156, 234, 235, 236, 245, 256, 345, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	6
6	123, 125, 126, 156, 234, 235, 245, 256, 345, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
7	123, 124, 125, 126, 134, 145, 156, 234, 345, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	18
8	123, 125, 126, 134, 135, 145, 156, 234, 345, 456	(16, 98, 322, 645, 832, 701, 378, 122, 20)	6
9	123, 124, 126, 146, 156, 234, 245, 246, 345, 456	(16, 98, 322, 645, 832, 701, 378, 122, 20)	6
10	123, 126, 156, 234, 236, 246, 256, 345, 346, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	6
11	123, 124, 126, 146, 156, 234, 246, 345, 346, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	6
12	123, 124, 126, 145, 146, 156, 234, 245, 345, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	18
13	123, 126, 146, 156, 234, 236, 246, 345, 346, 456	(16, 98, 322, 645, 832, 701, 378, 122, 20)	6
14	123, 126, 156, 234, 236, 256, 345, 346, 356, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
15	123, 126, 146, 156, 234, 236, 245, 246, 345, 456	(18, 111, 358, 700, 882, 728, 386, 123, 20)	6
16	123, 126, 156, 234, 235, 236, 256, 345, 356, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
17	123, 124, 126, 134, 145, 146, 156, 234, 345, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
18	123, 126, 134, 135, 136, 156, 234, 345, 356, 456	(16, 98, 322, 645, 832, 701, 378, 122, 20)	6
19	123, 126, 134, 136, 145, 146, 156, 234, 345, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
20	123, 126, 135, 136, 145, 156, 234, 235, 345, 456	(15, 93, 317, 661, 882, 760, 413, 132, 21)	6
21	123, 126, 136, 146, 156, 234, 236, 345, 346, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	18
22	123, 125, 126, 134, 135, 156, 234, 345, 356, 456	(18, 111, 358, 700, 882, 728, 386, 123, 20)	6
23	123, 126, 136, 156, 234, 235, 236, 345, 356, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
24	123, 126, 134, 135, 136, 145, 156, 234, 345, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	6
25	123, 124, 126, 156, 234, 245, 246, 256, 345, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	6
26	123, 126, 134, 136, 156, 234, 345, 346, 356, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	18
27	123, 126, 135, 136, 156, 234, 235, 345, 356, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	6
28	123, 124, 126, 134, 146, 156, 234, 345, 346, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
29	123, 126, 156, 234, 236, 245, 246, 256, 345, 456	(16, 98, 322, 645, 832, 701, 378, 122, 20)	18
30	123, 126, 136, 156, 234, 236, 345, 346, 356, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
31	123, 125, 126, 145, 156, 234, 235, 245, 345, 456	(14, 83, 276, 571, 766, 670, 372, 122, 20)	18
32	123, 125, 126, 135, 156, 234, 235, 345, 356, 456	(16, 98, 322, 645, 832, 701, 378, 122, 20)	6
33	123, 125, 126, 156, 234, 235, 256, 345, 356, 456	(15, 91, 302, 615, 807, 690, 376, 122, 20)	18
34	123, 124, 126, 156, 234, 246, 256, 345, 346, 456	(15, 93, 317, 661, 882, 760, 413, 132, 21)	6

Table 1

has at least one critical point. Thinking $W_{\mathfrak{s}}$ as restriction $W_{\mathfrak{s}} = W|_{T_{\mathfrak{s}}}$ of the Landau–Ginzburg potential W on $\mathrm{Gr}^{\vee}(3,6)$ defined by Marsh and Rietsch [28] to the cluster chart $T_{\mathfrak{s}} \subset \mathrm{Gr}^{\vee}(3,6)$, it suffices to show that each of the charts contains at least one critical point of W. In fact, something stronger is true: there is a critical point of W that is contained in $T_{\mathfrak{s}}$ for all $\mathfrak{s}$. As proved by Rietsch [39] (see also Karp [22]), for any $1 \leq k < n$ there is a (unique) critical point of W in the totally positive part $\mathrm{Gr}^{\vee}(k,n)_{>0} \subset \mathrm{Gr}^{\vee}(k,n)$, i.e., the locus where all Plücker coordinates are real and positive. Following the notation of Proposition 4.5 (2), this point is $[M_{I_0}] \in \mathrm{Gr}^{\vee}(k,n)$ with I_0 the set of k roots of $\zeta^n = (-1)^{k+1}$ closest to 1. Applying this to $(k,n) = (3,6)$, and recalling that $[M_{I_0}] \in T_{\mathfrak{s}}$ if and only if $x_d(M_{I_0})$ for all Young diagrams $d \subseteq 3 \times 3$ appearing as labels on the nodes of $Q_{\mathfrak{s}}$, we conclude that the total positivity of $[M_{I_0}]$ implies that it belongs to every cluster chart $T_{\mathfrak{s}}$, and this proves that all $L_{\mathfrak{s}}$ are nondisplaceable. ■

Remark 4.17. We emphasize that the arguments of Theorem 4.16 prove that any $L_{\mathfrak{s}} \subset \mathrm{Gr}(k,n)$ is nondisplaceable as long as $W_{\mathfrak{s}} = W_{L_{\mathfrak{s}}}$. This is due to the fact that the tori $L_{\mathfrak{s}}$ correspond to cluster charts $T_{\mathfrak{s}} \subset \mathrm{Gr}^\vee(k,n)$ by construction, and that W has a critical point in the intersection of all such charts.

Remark 4.18. It was shown in [6, Theorem 4.8] that the dihedral group

$$D_n = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle$$

acts on the set of critical points of $W \in \mathcal{A}_{k,n}$, and that the cluster chart $T_{\mathfrak{s}}$ is invariant under the action of the subgroup $\langle r \rangle$. Since $\mathbf{wt}(L_{\mathfrak{s}})$ is the number of critical points in $T_{\mathfrak{s}}$, the fact that $\mathbb{Z}/n\mathbb{Z}$ acts on it puts some arithmetic constraints on this number.

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