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Augmented Legendrian cobordism in J^1S^1

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Abstract. We consider Legendrian links and tangles in J^1S^1 and $J^1[0,1]$ equipped with Morse complex families over a field $\mathbb{F}$ and classify them up to Legendrian cobordism. When the coefficient field is $\mathbb{F}_2$, this provides a cobordism classification for Legendrians equipped with augmentations of the Legendrian contact homology DG-algebras. A complete set of invariants, for which arbitrary values may be obtained, is provided by the fiber cohomology, a graded monodromy matrix, and a mod 2 spin number. We apply the classification to construct augmented Legendrian surfaces in J^1M with $\dim M = 2$ realizing any prescribed monodromy representation, $\Phi : \pi_1(M, x_0) \to \mathrm{GL}(\mathbf{n}, \mathbb{F})$.

1. Introduction

In this article, we classify augmented Legendrian links in the 1-jet space of the circle, J^1S^1, up to cobordism. By an augmented Legendrian link, we mean a Legendrian link Λ equipped with the additional structure of an *augmentation*, $\varepsilon : \mathcal{A} \to \mathbb{F}_2$, of its Legendrian contact homology dg-algebra. Augmented Legendrians are natural objects from the point of view of symplectic field theory, and for some purposes, they can be treated analogously to embedded Lagrangian submanifolds. For instance, Legendrians with augmentations define objects in Fukaya categories [30, 32], and an analog of the Lagrangian intersection Floer complex can be defined for pairs of augmented Legendrians using linearized homology; cf. [8]. Moreover, Legendrians that admit augmentations are more rigid in their behavior than general Legendrian submanifolds. For example, the number of Reeb chords of an augmented Legendrian in $J^1\mathbb{R}^n$ satisfies an Arnold conjecture-type lower bound that can fail for general Legendrians; see [9, 11]. Our main results show that this heightened rigidity persists at the level of cobordisms, as there are substantially more cobordism classes for Legendrians equipped with augmentations than for those without.

Augmentations of 1-dimensional Legendrian links are known to be equivalent to (or closely related to) some other important structures in symplectic topology, such as *constructible sheaves* [32, 44] and *generating families* [19, 37]. In this article, we

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treat augmented Legendrians using the correspondence between augmentations and *Morse complex families* (abbrv. *MCFs*) which are combinatorial analogs of generating families; cf. [1, 23, 25, 26, 35, 42]. As in [35], working over $\mathbb{F}_2$, the correspondence between augmentations and MCFs induces a bijection between cobordism classes; see Section 2. With this identification understood, we will also use the terminology *augmented Legendrian* when referring to an ordered pair $(\Lambda, \mathcal{C})$ where $\Lambda \subset J^1M$ is a Legendrian submanifold and $\mathcal{C}$ is an MCF.

Remark 1.1. Throughout, we will consider MCFs with coefficients in an arbitrary field $\mathbb{F}$ which is a natural level of generality for our arguments. See Section 3.3 for a discussion of an extension of the correspondence between cobordism classes of MCFs and augmentations to coefficient fields more general than $\mathbb{F}_2$; currently, there are difficulties when $\operatorname{Char}\mathbb{F} \neq 2$ stemming from the challenge in explicitly evaluating the orientation signs of holomorphic disks in the case of Legendrian surfaces.

Roughly, a Morse complex family $\mathcal{C}$ over $\mathbb{F}$ for a Legendrian link or surface, $\Lambda \subset J^1M$, consists of a family of chain complexes (C_x, d_x) of $\mathbb{F}$-modules defined for generic $x \in M$ where the generators for C_x correspond to the sheets of $\Lambda|_x$, subject to axioms coming from the 1-parameter and 2-parameter Morse theory; cf. [22]. In general, we will consider ρ-graded Legendrians and MCFs where the grading on the complexes (C_x, d_x) is by $\mathbb{Z}/\rho$ with $\rho \geq 0$. The cases $\rho = 1$ or 2 are of some special interest due to a connection between 1- and 2-graded augmentations and the Kauffman and HOMFLY-PT polynomials; see [26]. When ρ is even, ρ-graded Legendrian cobordisms are orientable; non-orientable cobordisms may occur when ρ is odd.

We now state the cobordism classification in J^1S^1. To a ρ-graded augmented Legendrian $(\Lambda, \mathcal{C}) \subset J^1S^1$ and a choice of basepoint $x_0 \in S^1$, we can associate the *fiber cohomology* at x_0, $H^*(C_{x_0}, d_{x_0})$, together with a *monodromy map*

$$\phi_{\Lambda,\mathcal{C}} : H^*(C_{x_0}, d_{x_0}) \to H^*(C_{x_0}, d_{x_0})$$

that is a degree-preserving automorphism of $H^*(C_{x_0}, d_{x_0})$. Writing $\mathbf{n} : \mathbb{Z}/\rho \to \mathbb{Z}_{\geq 0}$, $\mathbf{n}(i) = \dim H^i(C_{x_0}, d_{x_0})$ for the graded dimension of the fiber cohomology, a choice of basis for $H^*(C_{x_0}, d_{x_0})$ leads to a *graded monodromy matrix*

$$\mathbf{M}_{\Lambda,\mathcal{C}} \in \mathrm{GL}(\mathbf{n}, \mathbb{F}) := \prod_{i \in \mathbb{Z}/\rho} \mathrm{GL}(\mathbf{n}(i), \mathbb{F})$$

which is a sequence of matrices indexed by $\mathbb{Z}/\rho$ such that $\mathbf{M}_{\Lambda,\mathcal{C}}(i) \in \mathrm{GL}(n(i), \mathbb{F})$ is the matrix of

$$\phi_{\Lambda,\mathcal{C}} : H^i(C_{x_0}, d_{x_0}) \to H^i(C_{x_0}, d_{x_0}).$$

When $\operatorname{char}\mathbb{F} \neq 2$ and ρ is even, a spin invariant $\xi(\Lambda, \mathcal{C}) \in \mathbb{Z}/2$ is also defined; see Section 5.

Theorem 1.2. *Assume that either* $\operatorname{char}\mathbb{F} = 2$ *and* $\rho \neq 1$, *or* $\operatorname{char}\mathbb{F} \neq 2$ *and* ρ *is even. Then, two* ρ*-graded augmented Legendrians over* $\mathbb{F}$ *are cobordant if and only if*

- *they have the same spin invariant (if* $\operatorname{Char}\mathbb{F} \neq 2$*),*
- *their fiber cohomologies have the same graded dimension,* $\mathbf{n} : \mathbb{Z}/\rho \to \mathbb{Z}_{\geq 0}$*,*
- *their monodromy matrices are conjugate in* $\mathrm{GL}(\mathbf{n}, \mathbb{F})$.

Moreover, all possible values of these cobordism invariants arise from augmented Legendrians.

In contrast, without augmentations, two Legendrians in J^1S^1 are oriented cobordant if and only if they have the same rotation number and are homologous; cf. [4] and Section 7.

Theorem 1.2 is established as a consequence of a cobordism classification of augmented Legendrian tangles in $J^1[0,1]$. When working with (ρ-graded) Legendrians in $J^1[0,1]$, we fix the number of boundary points, n, as well as their grading, $\vec{\mu} \in (\mathbb{Z}/\rho)^n$, to be identical at the left and right boundary, and we focus on MCFs that satisfy $d_x = 0$ near the boundary. Cobordism classes of such *full augmented Legendrian* n*-tangles* form a group denoted by $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$ with the operation of side-by-side concatenation. Moreover, the monodromy matrix is well defined (not just up to conjugation).

Theorem 1.3. *There are group isomorphisms:*

- *When* $\operatorname{char}\mathbb{F} \neq 2$ *and* ρ *is even:*

$$\begin{aligned} \mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F}) &\to \mathrm{GL}(\mathbf{n},\mathbb{F}) \times \mathbb{Z}/2 \\ [(\Lambda,\mathcal{C})] &\mapsto (\mathbf{M}_{\Lambda,\mathcal{C}}, \xi(\Lambda,\mathcal{C})). \end{aligned}$$

- *When* $\operatorname{char}\mathbb{F} = 2$ *and* $(n,\rho) \neq (0,1)$*:*

$$\begin{aligned} \mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F}) &\to \mathrm{GL}(\mathbf{n},\mathbb{F}) \\ [(\Lambda,\mathcal{C})] &\mapsto \mathbf{M}_{\Lambda,\mathcal{C}}. \end{aligned}$$

In the case that $n = 0$ and $\rho = 1$, we obtain partial results. See Theorem 6.1.

For an augmented Legendrian surface, $(\Sigma,\mathcal{C}) \subset J^1M$ with $\dim M = 2$; as in [42], the monodromy map construction provides a monodromy representation:

$$\Phi_{\Lambda,\mathcal{C}} : \pi_1(M, x_0) \to \mathrm{GL}(H^*(C_{x_0}, d_{x_0})).$$

As an application of the above theorems, we construct augmented Legendrian surfaces with arbitrary monodromy representations. See Section 8.

The remainder of the paper is organized as follows. Section 2 contains the definition of Morse complex families over $\mathbb{F}$ and reviews the continuation maps and

monodromy representation; previously, MCFs for surfaces have only been defined with $\mathbb{F}_2$ coefficients. Section 3 uses results from [35] to establish over $\mathbb{F}_2$ a bijection between cobordism classes of Legendrians equipped with augmentations and cobordism classes of Legendrians equipped with MCFs. We also include a discussion of extension of this bijection to general $\mathbb{F}$. Section 4 provides a set of tools for constructing and working with MCFs and in particular augmented Legendrian cobordisms including the following: a characterization of augmented Legendrian cobordisms via moves on 1-dimensional slices; the augmented D_4^- cobordism; a standard form for MCFs (the SR-form) from generalized normal rulings; and extension of MCFs over Legendrian isotopies. These methods have grown out of the literature on MCFs; e.g., M. B. Henry studies equivalence of MCFs via moves on slices in [23]; various constructions of 2-dim MCFs appear in [35,42]; for knots in $\mathbb{R}^3$, the SR-form MCFs are considered in [23,26]. As the original sources mainly restrict to $\mathbb{F}_2$ coefficients and do not consider 1-dimensional links in J^1S^1, we provide here a mostly self-contained exposition that includes fairly detailed proofs.

With these preparations in hand, Sections 5–7 turn toward establishing Theorems 1.2 and 1.3. Section 5 establishes the fiber cohomology, monodromy matrix, and spin numbers as invariants of augmented cobordism. Section 6 proves Theorem 1.3 (see Theorem 6.1). This involves introducing standard forms for augmented Legendrians up to cobordism that are certain positive permutation braids equipped with particular MCFs. It is then shown that every augmented Legendrian is cobordant to a standard form and that the standard forms are uniquely determined by their invariants. Section 7 establishes Theorem 1.2 (see Theorem 7.1) by realizing augmented Legendrians in J^1S^1 up to cobordism as closures of full n-tangles and applying Theorem 1.3. Finally, Section 8 shows how to construct augmented Legendrian surfaces with arbitrary monodromy representation and illustrates some examples in J^1T^2.

2. Morse complex families over $\mathbb{F}$

The augmented Legendrians that we consider in this paper are pairs $(\Lambda, \mathcal{C})$ consisting of a (ρ-graded) Legendrian knot or surface, $\Lambda \subset J^1M$, in a 1-jet space together with a (ρ-graded) Morse complex family (abbrv. MCF), $\mathcal{C}$. Motivated by generating families of functions and the bifurcation theory for Morse complexes in 1- and 2-parameter families, see [22], MCFs have been considered for 1-dimensional Legendrians knots in [23–25,36] (over $\mathbb{F}_2$) and [25] (over any $\mathbb{F}$) and for 2-dimensional Legendrian surfaces and cobordisms in [35,42] (over $\mathbb{F}_2$). In this section, we review Morse complex families and extend the definition to allow general coefficients in the 2-dimensional case. In Section 2.4, we discuss the monodromy representation of an MCF $\mathcal{C}$ that is a representation of $\pi_1(M)$ on the fiber cohomology of $\mathcal{C}$.

Matrix notation: Throughout the paper, we will frequently use the notation $E_{i,j}$ for a matrix whose (i, j)-entry is 1 and whose other entries are 0.

2.1. Graded Legendrian submanifolds in 1-jet spaces

We assume basic familiarity with Legendrian knots and surfaces in 1-jet spaces. In particular, when M is a manifold, the 1-jet space J^1M has a standard contact structure that appears in coordinates as $dz - y\,dx$, where (x, y, z) are coordinates on $T^*M \times \mathbb{R}_z$ resulting from a local coordinate x on M. Denote by

$$\pi_{xz} : J^1M \to J^0M = M \times \mathbb{R} \quad \text{and} \quad \pi_x : J^1M \to M$$

the *front* and *base projections*. A Legendrian submanifold $\Lambda \subset J^1M$ may be conveniently presented via its front projection, $\pi_{xz}(\Lambda) \subset J^0M$, which (generically) in the 1-dimensional case has standard cusp and crossing singularities, all with different base projections. Generic singularities of 2-dimensional front projections include 1-dimensional cusp edges and crossing arcs (aka singularities of type A_2 and A_1^2), triple points (A_1^3), cusp-sheet intersections (A_2A_1), and swallowtail points (A_3). See, e.g., [4] or [42, Figure 1]. We consider compact Legendrian submanifolds, $\Lambda \subset J^1M$, that are properly embedded and such that the front projection $\pi_{xz}(\Lambda)$ appears as a product in a collar neighborhood of ∂M. When $B \subset \partial M$ is a boundary component, the *restriction* $\Lambda|_B \subset J^1B$ is defined by taking the intersection $\Lambda \cap \pi_x^{-1}(B)$ and then projecting out the y-coordinate transverse to B. We will refer to 1-dimensional Legendrians $\Lambda \subset J^1[0, 1]$ as *Legendrian tangles* and, in the case where $\Lambda|_{\{i\}}$ consists of n points for $i = 0, 1$, as *Legendrian n-tangles*.

A *Legendrian cobordism* between two compact 1-dimensional Legendrians Λ_0, $\Lambda_1 \subset J^1M$ is a compact Legendrian surface $\Sigma \subset J^1(M \times [0, 1])$ satisfying

$$\Sigma|_{M\times\{i\}} = \Lambda_{\{i\}}.$$

For any $\Lambda \subset J^1M$, there is a *trivial* or *product cobordism* that (by a slight abuse of notation) we denote as $\Lambda \times [0, 1] \subset J^1(M \times [0, 1])$; it appears in coordinates (x_1, x_2, y_1, y_2, z) with $(x_1, y_1) \in T^*M$ and $(x_2, y_2) \in T^*[0, 1]$ as $\{(x_1, x_2, y_1, 0, z) \mid (x_1, y_1, z) \in \Lambda\}$.

Let $\rho \geq 0$ be a non-negative integer. A *$\mathbb{Z}/\rho\mathbb{Z}$-valued Maslov potential* for Λ is a locally constant function

$$\mu : \Lambda \setminus \Lambda_{\text{cusp}} \to \mathbb{Z}/\rho\mathbb{Z},$$

where $\Lambda_{\text{cusp}} \subset \Lambda$ is the set of points where $\pi_{xz}|_\Lambda$ has a cusp or swallowtail singularity such that the value of μ increases by one when passing from the lower sheet to the upper sheet at a cusp edge. A *ρ-graded Legendrian submanifold* is a pair (Λ, μ) such that μ is a $\mathbb{Z}/\rho\mathbb{Z}$-valued Maslov potential for Λ.

Remark 2.1. If ρ is even and M is oriented, then any ρ-graded Legendrian $\Lambda \subset J^1M$ receives an orientation by making the requirement that the base projection $\pi_x : \Lambda \to M$ is orientation preserving (resp., reversing) on sheets of Λ where μ is even (resp., odd). In particular, when considered oriented 1-dimensional ρ-graded Legendrians in J^1M with $M = \mathbb{R}$, $[0, 1]$ or S^1, we will always use this orientation so that μ is even (resp., odd) where the orientation of the front projection is in the positive (resp., negative) x-direction.

Let $(\Lambda, \mu) \subset J^1M$ be a ρ-graded Legendrian submanifold of dimension 1 or 2, and let R be a commutative ring with identity. For each $x_0 \in M$ that is not the base projection of any singular point of $\pi_{xz}(\Lambda)$ (including crossings), we refer to the points in $\pi_x^{-1}(\{x_0\}) \cap \Lambda$ as the *sheets of* Λ *at* x_0, and we label them with descending z-coordinates as

$$S_1^{x_0}, \dots, S_n^{x_0}, \quad z(S_i^{x_0}) > z(S_{i+1}^{x_0}), \quad \text{for } 1 \le i < n = n(x_0).$$

To each such x_0, we associate a free $\mathbb{Z}/\rho\mathbb{Z}$-graded R-module:

$$C_{x_0} = \mathrm{Span}_R\{S_1^{x_0}, \dots, S_n^{x_0}\}, \quad |S_i^{x_0}| = \mu(S_i^{x_0}).$$

We will usually omit the superscript x_0 from the notation for sheets.

2.2. 1-dimensional Morse complex families

Assume now that $(\Lambda, \mu) \subset J^1M$ is a 1-dimensional ρ-graded Legendrian link or tangle with generic front and base projections. In particular, no two front singularities have the same base projection.

Definition 2.2. A (ρ-graded) *Morse complex family* (abbrv. MCF) over R (a commutative ring with identity) for (Λ, μ) is a tuple

$$\begin{aligned} &\mathcal{C} = (\{d_x\}, H, *, \xi) \quad \text{when char } R \neq 2, \\ \text{or} \quad &\mathcal{C} = (\{d_x\}, H, *) \quad \text{when char } R = 2, \end{aligned}$$

consisting of the following items:

(1) A collection of *homology basepoints*, $* = \{*_1, \dots, *_r\} \subset \Lambda$, assigned values $s_i \in R^*$.

(2) When char $R \neq 2$, a collection of *spin basepoints*, $\xi = \{\circ_1, \dots, \circ_s\}$, each assigned the value -1. *When* char $R = 2$*, a collection of spin basepoints is not included in the defining data for* $\mathcal{C}$.

(3) A handleslide set, H, that is a collection of formal handleslides with coefficients $b \in R$. Here, a *formal handleslide* is a triple $h = (x, u_h, l_h)$, where

$x \in M$ and $u_h, l_h \in \Lambda \setminus \Lambda_{\text{cusp}}$ are lifts of x having

$$\pi_x(u_h) = \pi_x(l_h) = x, \quad z(u_h) > z(l_h), \quad \text{and} \quad \mu(u_h) = \mu(l_h) \in \mathbb{Z}/\rho.$$

We require that all cusps, crossings, handleslides, and basepoints occur at distinct x-coordinates, and we denote by $M^{\mathcal{C}}_{\text{reg}} \subset M$ the complement of such x-coordinates.

(4) A collection of upper triangular differentials $d_x : C_x \to C_x$ of degree $+1$ mod ρ defined for each $x \in M^{\mathcal{C}}_{\text{reg}}$ that are locally constant in x, i.e., writing $A = (a_{i,j})$ for the matrix of d_x,

$$d_x S_j = \sum_i a_{i,j} S_i,$$

it is required that $A^2 = 0$; $a_{i,j} = 0$ unless $\mu(S_i) = \mu(S_j) + 1$; $a_{i,j} = 0$ when $i \geq j$; and the matrices A remain constant except when x passes a crossing, cusp, basepoint, or handleslide.

The data of $\mathcal{C}$ are subject to the following axioms. Suppose that between x_- and x_+ there is a single crossing, cusp, handleslide, or basepoint at x_0. Write $d_\pm$ and $A_\pm = (a^\pm_{i,j})$ for the differentials and their matrices at x_- and x_+.

(A1) When x_0 has a *crossing* involving strands k and $k+1$,

$$A_+ = Q_{(k\,k+1)} A_- Q_{(k\,k+1)},$$

where $Q_{(k\,k+1)}$ is the permutation matrix of the transposition $(k\ k+1)$.

(A2) When x_0 has a *left cusp* involving the sheets S^+_k and S^+_{k+1}, A_+ is obtained from A_- by inserting two new rows and columns at positions k and $k+1$ with the 2×2 block

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

appearing on the main diagonal and all other entries 0. The same requirement with the role of A_+ and A_- interchanged is made at right cusps.

(A3) When x_0 has a *handleslide* with upper (resp., lower) endpoint on S_k (resp., S_l) and coefficient b,

$$A_+ = (I + bE_{k,l})A_-(I + bE_{k,l})^{-1} = (I + bE_{k,l})A_-(I - bE_{k,l})$$

or equivalently

$$A_+(I + B) = (I + B)A_-, \quad \text{where } B = bE_{k,l},$$

where $E_{i,j}$ is the matrix with i, j-entry 1 and all other entries 0.

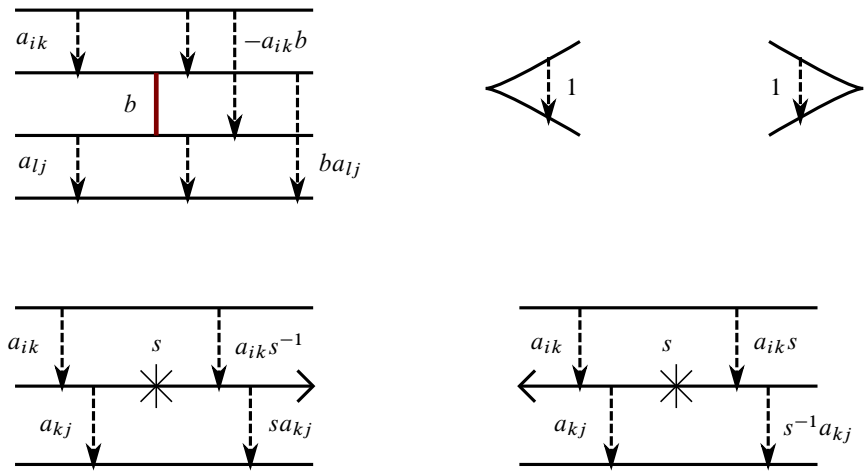


Figure 1. The coefficient rules for 1-dimensional MCFs.

(A4) When x_0 has a *basepoint* on sheet S_k with coefficient s (for spin basepoints $s = -1$), let Δ be the diagonal matrix with s (resp., s^{-1}) at position k on the diagonal if π_x is orientation preserving (resp., reversing) at S_k; the remaining diagonal entries of Δ are 1's. Then,

$$A_+ = \Delta A_- \Delta^{-1}.$$

Remark 2.3. (1) Note that since the $A_\pm$ are upper triangular, (A1) implies that

$$a^{\pm}_{k,k+1} = 0$$

when x_- and x_+ border a crossing between sheets S_k and S_{k+1}.

(2) As the differentials, d_x, are locally constant, one can equivalently (as done in [42]) assign complexes (C_ν, d_ν) to the *connected components*, R_ν, of $M^{\mathcal{C}}_{\text{reg}}$.

See Figure 1 for an illustration of the coefficients that appear in the axioms (A2)–(A4). In our figures, we indicate a coefficient in a differential, $\langle d_x S_j, S_i \rangle = a_{i,j}$, as a dotted arrow at x from S_i to S_j labeled with $a_{i,j}$; handleslides are indicated by solid dark red line segments labeled with coefficients.

An example of an MCF for a 0-graded Legendrian knot in J^1S^1 appears in Figure 2.

Remark 2.4. The above definition of MCF over a ring R is similar to the definition from [26] with added flexibility allowed for basepoints and spin points and a simplified axiom at cusps. For Legendrians in $J^1\mathbb{R}$, the MCFs considered in [26] become a special case of Definition 2.2 by choosing $*$ and ξ so that on each component of Λ

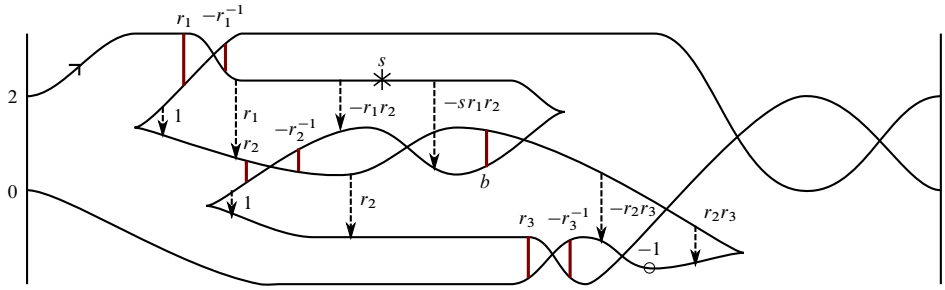


Figure 2. An MCF for a 0-graded Legendrian knot $\Lambda \subset J^1S^1$. The Maslov potential μ is indicated at the left. The coefficients $r_1, r_2, r_3, s \in \mathbb{F}^*$ and $b \in \mathbb{F}$ must satisfy $-sr_1r_2 = 1$ and $r_2r_3 = 1$. The differentials d_x have $d_x = 0$ at $x = 0$ and are then uniquely determined by (A1)–(A4). Some selected coefficients $a_{ij} = \langle d_x S_j, S_i \rangle$ are indicated by the dotted arrows.

a single $*_i$ point is placed next to some chosen right cusp, and spin basepoints are placed next to all remaining right cusps.

2.3. 2-dimensional Morse complex families

Consider now a ρ-graded Legendrian surface

$$(\Lambda, \mu) \subset J^1M$$

and a commutative, unital ring R. We allow for the case where $\partial M \neq \emptyset$ with Λ properly embedded as in Section 2.1.

Definition 2.5. • A *homology structure*, $* = \{*_1, \dots, *_r\}$ is a collection of co-orientable curves in Λ satisfying the following:

(1) The $*_i$ are disjoint from one another except at endpoints and all $*_i$ are transverse to $\partial\Lambda$ as well as to the inverse image of the singular set of $\pi_{xz}(\Lambda)$.

(2) The arc components of $*$ have their endpoints either at boundary points of Λ, or at trivalent vertices where three arc endpoints are simultaneously located. The intersections of arcs at trivalent vertices are pairwise transverse.

- A *combinatorial spin structure*, $\xi = \{\xi_1, \dots, \xi_s\}$, is a collection of disjoint embedded curves in Λ whose boundary consists of the set of all swallowtail points of Λ and some number of points on $\partial\Lambda$.
- A multiplicative (resp., additive) *co-oriented coefficient* for a co-orientable curve $C \subset \Lambda$ is an assignment of coefficients α and β in R^* (resp., in R) to the two distinct co-orientations of C such that $\beta = \alpha^{-1}$ (resp., $\beta = -\alpha$.)

Remark 2.6. Note that the difference of two combinatorial spin structures is a relative cycle in $C_1(\Lambda, \partial\Lambda; \mathbb{Z}/2)$ so that combinatorial spin structures up to equivalence form an affine space over $H_1(\Lambda, \partial\Lambda; \mathbb{Z}/2) = H^1(\Lambda; \mathbb{Z}/2)$, just like geometric spin structures. The role of ξ in the definition of MCFs for Legendrian surfaces is analogous to the choice of geometric spin structure required in lifting the coefficients of the Legendrian contact homology DGA to $\mathbb{Z}$; cf. [11].

Definition 2.7. An *MCF* for (Λ, μ) over R is a tuple $\mathcal{C} = (\{d_x\}, H, *, \xi)$, when char $R \neq 2$, or $\mathcal{C} = (\{d_x\}, H, *)$, when char $R = 2$, consisting of the following:

(1) A homology structure, $*$, with *multiplicative* co-oriented coefficients in R^* assigned to the curves $*_i$ such that at each trivalent vertex of $*$ the coefficients s_1, s_2, s_3 of the three arcs with respect to a cyclically consistent choice of co-orientations (i.e., either all counter-clockwise or all clockwise in coordinates around the vertex) satisfy $s_1 s_2 s_3 = 1$.

(2) A combinatorial spin structure, ξ, when char $R \neq 2$. All curves in ξ are assigned the (multiplicative) coefficient -1.

(3) A stratified handleslide set,

$$H = H_0 \sqcup H_{-1}.$$

- The H_0 is a collection of formal *handleslides curves* that are immersed curves, $h : X \to M$, with $X = S^1$ or $[0, 1]$ equipped with *additive* co-oriented coefficients $b \in R$ and upper and lower endpoint lifts,

$$u, l : X \to \Lambda, \quad \pi_x \circ u = \pi_x \circ l = h,$$

  satisfying

$$z(u(t)) > z(l(t)) \quad \text{and} \quad \mu(u(t)) = \mu(l(t)) \in \mathbb{Z}/\rho$$

  for all $t \in \mathrm{Int}(X)$. Moreover, on $\mathrm{Int}(M)$, the image of u and l is disjoint from cusp edge and swallowtail points. The endpoints of handleslide curves are subject to the requirements specified in (B2) below.

- The H_{-1} is a set of *super-handleslide points* in M equipped with (i) co-orientations (in M, i.e., a choice of orientation of M near the point), (ii) coefficients $b \in R$, and (iii) upper and lower endpoint lifts to points $u, l \in \Lambda \setminus \Lambda_{\text{cusp}}$ with $z(u) > z(l)$ and $\mu(u) = \mu(l) - 1$.

Except at endpoints as allowed by the Axiom (B2) below, the curves and points in H_0 and H_{-1} are assumed to be self-transverse and transverse to one another and to the base projections of $*$, ξ, and singularities of $\pi_{xz}(\Lambda)$. We will refer to a curve in H_0 or a point in H_{-1} having upper and lower endpoint lifts on sheets S_i and S_j, $i < j$, as an (i, j)-*(super)-handleslide*. Note that for curves in H_0 this terminology

only makes sense locally since the numbering of the sheets of Λ can change when passing crossings or cusps of Λ.

Let $M^{\mathcal{C}}_{\text{reg}} \subset M$ consist of those points not coinciding with the base projection of any singularity of $\pi_{xz}(\Lambda)$ (crossing, cusp, or swallowtail point), homology or spin structure curve, or with any handleslide arc or super-handleslide point. We write

$$M^{\mathcal{C}}_{\text{sing}} := M \setminus M^{\mathcal{C}}_{\text{reg}}$$

and refer to this subset as the set of *base singularities* of $\mathcal{C}$.

(4) A locally constant collection, $\{d_x\}$, of upper triangular differentials $d_x : C_x \to C_x$ of degree $+1$ mod ρ defined for each $x \in M_{\text{reg}}$.

The data is subject to the following axioms.

(B1) The axioms (A1)–(A4) are required to hold along transverse slices to base projections of crossings arcs, cusp arcs, handleslide curves in H_0, and curves in $*$ or ξ. Here, the role of the matrices A_- and A_+, as well as the coefficients of handleslides, is specified by the orientation of the transverse slice (which specifies a co-orientation for handleslide arcs). For homology curves, $*_i$, a choice of co-orientation of $*_i$ is used to specify the coefficient s and the (local) orientation of the one-dimensional slice of Λ that appears in (A4).

(B2) The only possible endpoints for handleslide arcs are as follows:

- (Trivalent vertices) Three handleslide arcs with the same upper and lower endpoints can all share a common endpoint at a trivalent vertex provided that the sum of the coefficients is equal to 0 with respect to cyclically consistent co-orientations at the vertex. See Figure 3.
- (Commutation relation) Whenever an (i, j)-handleslide intersects a (j, k)-handleslide arc at a point $x \in M$, a third (i, k)-handleslide arc has an endpoint at x. The coefficients of the three handleslide arcs are related as in Figure 4.
- (Super handleslides) Suppose a (k, l)-super-handleslide point at $p \in M$ has coefficient c, and let $A = (a_{ij})$ be the matrix of d_x in a region of M_{reg} adjacent to c. Then, for each $i < k$, a single (i, l)-handleslide arc with coefficient $a_{ik}c$ has an endpoint at p, and for each $l < j$, a single (k, j)-handleslide arc with coefficient ca_{lj} has an endpoint at p. The coefficients of the handleslide arcs are with respect to their counter-clockwise co-orientations around p as specified by the co-orientation of p itself. See Figure 5.
- (Swallowtail points) Handleslide arcs and spin structures in a neighborhood of swallowtail points are required to appear as in Figure 6.
- (Boundary points) Handleslide arcs (resp., homology and spin structure arcs) may have endpoints at transverse intersections with ∂M (resp., $\partial\Lambda$). Moreover, in a collar neighborhood of M where $\pi_{xz}(\Lambda)$ appears as a product, these arcs appear as products (of points on M or Λ with the collar coordinate) as well.

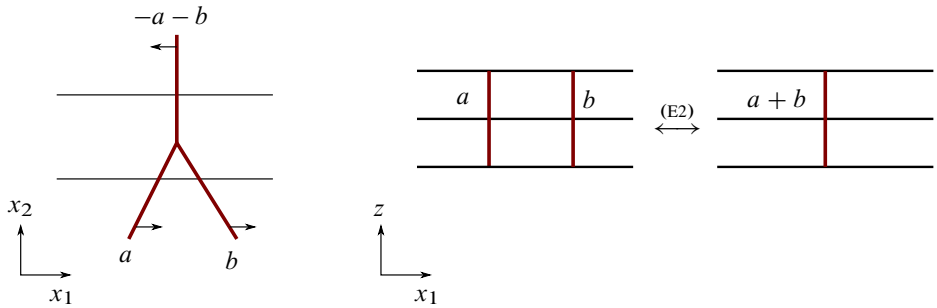


Figure 3. Handleslide coefficients in the trivalent vertex axiom. In Figures 3–6, the left column depicts the base projection together with two horizontal lines that indicate slices of the front projection appearing in the right two columns. The handleslide coefficients in the front projection slices are always written with respect to the left-to-right co-orientation. The label (E2) reflects the enumeration from Section 4.1.

(B3) Handleslide coefficients change when a handleslide endpoint passes a homology or spin structure curve as indicated in Figure 7.

Example 2.8. An example of a 0-graded MCF for a Legendrian pair of pants $\Lambda \subset J^1(S^1 \times [0,1])$ is pictured in Figure 8. The differentials d_x vanish for $x = (x_1, x_2)$ where Λ is 2-sheeted and are uniquely determined by the axioms for other values of x. Results in Section 4 will ease considerations involving the differentials d_x when constructing MCFs.

Remark 2.9. (1) It may be worthwhile to emphasize that for handleslide arcs the (additive) co-oriented coefficients are with respect to co-orientations in M, while for homology structure curves $*_i$, the co-orientations are in Λ. When char $R = 2$, the co-orientation can be ignored when considering coefficients of handleslides, but this is not so for the $*_i$. If M (resp., Λ) is oriented, one could equivalently use orientations rather than co-orientations for specifying coefficients of handleslide (resp., $*_i$) arcs. However, we will make use of non-orientable Legendrians; e.g., when ρ is odd, the cobordisms constructed in the proofs of Theorems 1.2 and 1.3 may be non-orientable.

(2) Note that the coefficients a_{ik} and a_{lj} of d_x that appear in (B2) (Super handleslides) are not affected by the handleslide arcs with endpoints at p and hence are independent of the choice of the point x near p. If one associates the matrices $C = cE_{k,l}$ to the (k,l)-super-handleslide at p with coefficient c and $B_m = b_m E_{i_m,j_m}$ for $1 \le m \le M$ to the (i_m, j_m)-handleslides with endpoints at p where the b_m are their (counterclockwise) coefficients, then the axiom at handleslide points is equivalent to the identity

$$A_x C + C A_x = \sum_m B_m = (I + B_m) \cdots (I + B_2)(I + B_1) - I. \tag{2.1}$$

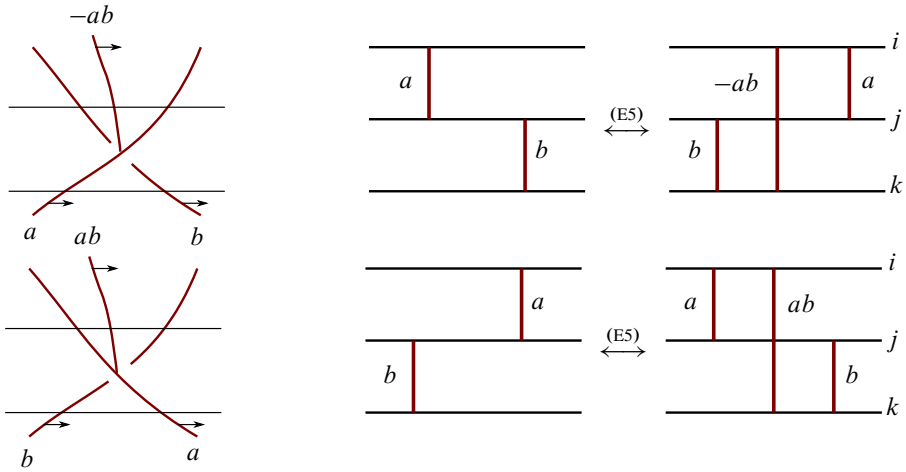


Figure 4. Coefficients in the commutation axiom.

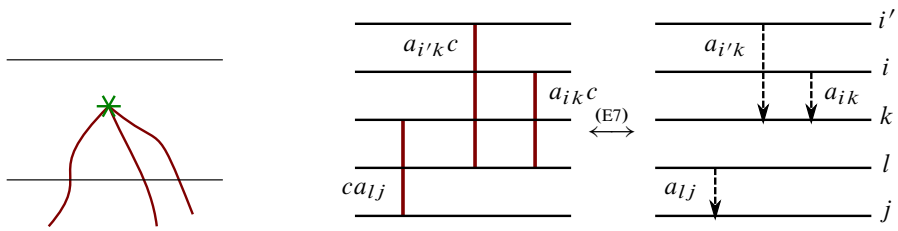


Figure 5. Coefficients in the super-handleslide axiom.

(3) The configuration of handleslide arcs with endpoints at a swallowtail arc specified in (B2) (swallowtail points) is essentially the only one allowed in order for the 1-dimensional axioms (A1)–(A3) to hold when passing through a transverse slice containing the swallowtail sheets in the right column of Figure 5. Moreover, the effect on coefficients of d_x when passing through this slice from left to right is to negate all coefficients of arrows beginning or ending on the swallowtail point, and this explains the necessity of spin arc endpoints at swallowtail points. See Section 4.1.1.

Note that for any 1-dimensional Legendrian

$$\Lambda \subset J^1M$$

with MCF, $\mathcal{C}$, there is a *product MCF* on $\Lambda \times [0,1]$ that we will denote by $\mathcal{C} \times [0,1]$ where all of the data $(\{d_x\}, H, *, \xi)$ are constant in the $[0,1]$ direction. Here, the co-oriented coefficients on H and $*$ are assigned from those of $\mathcal{C}$ using the co-orientations arising from the orientations of M and Λ.

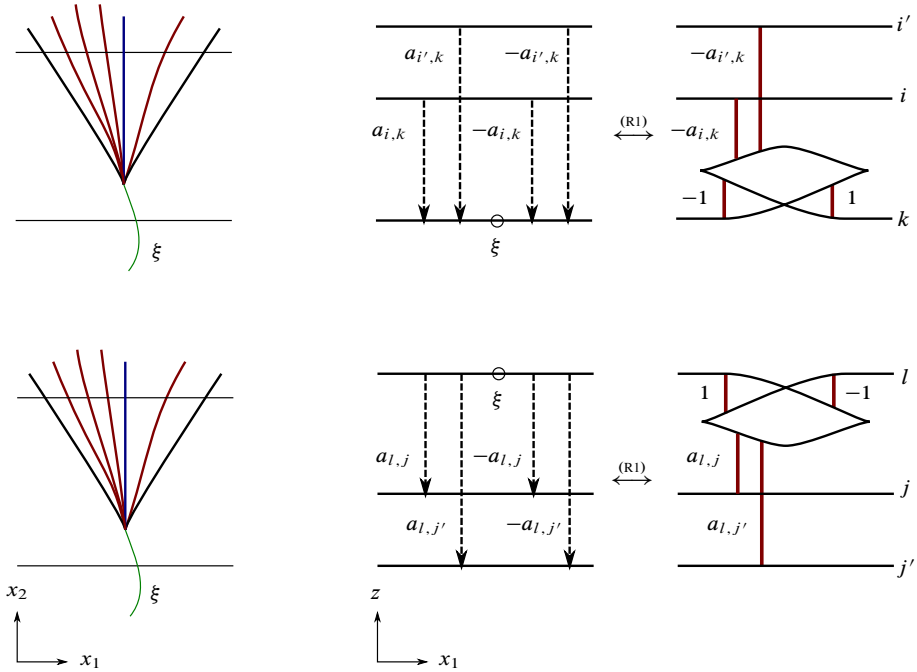


Figure 6. For an upward (resp., downward) swallowtail point, s_0, a spin arc ξ on sheet S_k (resp., S_l) outside of the swallowtail region, i.e., where the additional swallowtail cusp sheets do not exist, has an endpoint at s_0 (when char $\mathbb{F} \neq 2$). Within the swallowtail region, two $(k+1, k+2)$-handleslide arcs (resp., $(l, l+1)$-handleslide arcs), one on each side of the crossing arc, have their endpoint at s_0. In addition, for each $i < k$ (resp., $l < j$), an (i, k)- (resp., $(l+2, j+2)$-) handleslide arc with coefficient $-a_{i,k}$ (resp., $a_{l,j}$) inside the swallowtail region has endpoint at s_0 where the matrix coefficients $a_{i,k} = \langle d_x S_k, S_i \rangle$ (resp., $a_{l,j} = \langle d_x S_j, S_l \rangle$) are computed with x to the left of ξ.

2.4. Continuation maps and monodromy

Let $\mathcal{C}$ be an MCF for a Legendrian $\Lambda \subset J^1 M$ of dimension 1 or 2, and let $\sigma : [0,1] \to M$ be a path that is transverse to the base singularities of $\mathcal{C}$. That is, σ passes through base projections of cusps, crossings, handleslide arcs, and basepoint arcs transversally at a finite collection of times $0 < s_1 < s_2 < \cdots < s_N < 1$, and σ does not intersect codimension 2 base singularities of $\mathcal{C}$. On each of the intervals in $[0,1] \setminus \{s_1, s_2, \ldots, s_N\}$, the complexes $(C_{\sigma(s)}, d_{\sigma(s)})$ do not change. Choose $0 = s'_0 < s'_1 < \cdots < s'_N = 1$ with $s'_{i-1} < s_i < s'_i$ for $1 \leq i \leq N$, and write $x_i = \sigma(s'_i)$. For each i, the axioms (A1)–(A4) give rise to a quasi-isomorphism:

$$f_i : (C_{x_{i-1}}, d_{x_{i-1}}) \to (C_{x_i}, d_{x_i})$$

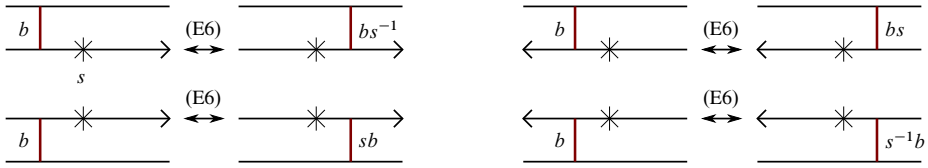


Figure 7. The arrows should be interpreted as the co-orientation of the homology curve $*_i$.

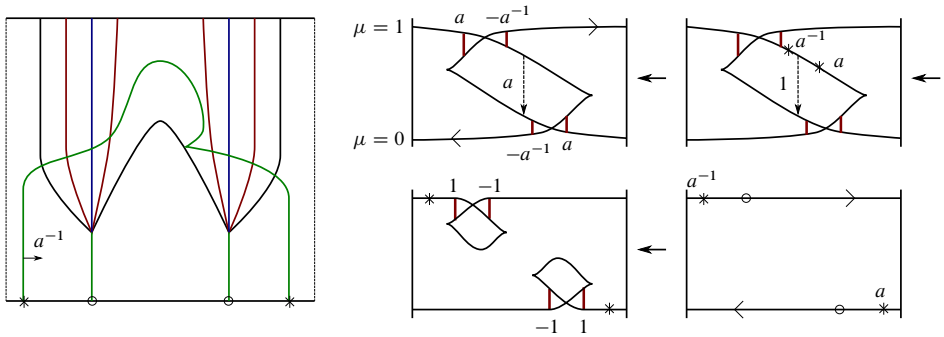


Figure 8. An example of a 2-dimensional MCF $\mathcal{C}$ for $\Lambda \subset J^1(S^1 \times [0,1])$ defined for any choice of $a \in \mathbb{F}^*$. On the left, the base projection of cusp (black), crossing (blue), handleslide (red), and homology and spin structure (green) curves are pictured. Slices of the front projection at different values of $x_2 \in [0,1]$ appear on the right, with the arrows indicating the positive x_2 direction. Except for the homology structure, all coefficients are with respect to left-to-right co-orientations. The coefficient of the homology curve with respect to the global co-orientation indicated in the base projection is a^{-1}. However, the coefficients of homology basepoints in the front slices are written with respect to the co-orientation specified by the orientation of the slice of Λ.

defined as follows depending on the type of singularity of $\mathcal{C}$ that occurs at s_i:

(1) When s_i is a *left cusp*, i.e., when two new sheets in positions k and $k+1$ appear as s increases past the cusp point at s_i, we have

$$f_i(S_j) = \begin{cases} S_j & \text{if } j < k, \\ S_{j+2} & \text{if } j \geq k. \end{cases} \tag{2.2}$$

(2) When s_i is a *right cusp*, i.e., sheets S_k and S_{k+1} that exist at x_{i-1} meet at the cusp point as s increases, we have

$$f_i(S_j) = \begin{cases} S_j & \text{if } j < k, \\ S_{j-2} & \text{if } j > k+1, \\ 0 & \text{if } j = k, k+1. \end{cases} \tag{2.3}$$

(3) When s_i is a *crossing* between sheets S_k and S_{k+1}, we have

$$f_i(S_j) = \begin{cases} S_{k+1} & \text{if } j = k, \\ S_k & \text{if } j = k+1, \\ S_j & \text{if } j \neq k, k+1. \end{cases} \tag{2.4}$$

(4) When s_i is a *handleslide* with upper and lower endpoints sheets k and l (with $k < l$) and coefficient $a \in R$ with respect to the co-orientation specified by σ, we have

$$f_i(S_j) = \begin{cases} aS_k + S_l & \text{if } j = l, \\ S_j & \text{if } j \neq l. \end{cases} \tag{2.5}$$

(5) When s_i is a *basepoint* on sheet k with coefficient $t \in R^*$ with respect to the co-orientation specified by locally lifting σ to Λ, we have

$$f_i(S_j) = \begin{cases} tS_k & \text{if } j = k, \\ S_j & \text{if } j \neq k. \end{cases} \tag{2.6}$$

Composing all of the chain maps as they occur along σ, we get a *continuation map*:

$$f_\sigma : (C_{x_0}, d_{x_0}) \to (C_{x_1}, d_{x_1}), \quad f_\sigma = f_N \circ \cdots \circ f_1.$$

We refer to the homology $H^*(C_x, d_x)$ at $x \in M^{\mathcal{C}}_{\text{reg}}$ as the *fiber cohomology* of $\mathcal{C}$ at x. The continuation maps induce *continuation isomorphisms* on the fiber cohomologies:

$$\phi_\sigma : H^*(C_{x_0}, d_{x_0}) \xrightarrow{\cong} H^*(C_{x_1}, d_{x_1}).$$

Example 2.10. The MCF from Example 2.8 has $H^0(C_x, d_x) \cong H^1(C_x, d_x) \cong \mathbb{F}$. Viewing $S^1 = [0,1]/\{0,1\}$, for any $x_2 \in [0,1]$, the horizontal loop $\sigma : [0,1] \to S^1 \times [0,1]$, $\sigma(t) = ([t], x_2)$ has continuation isomorphism $\phi_\sigma = -a^{-1}\text{id}$.

The fundamental properties of this construction are summarized as follows.

Proposition 2.11. *Let $\sigma, \tau : [0,1] \to M$ be paths transverse to $M^{\mathcal{C}}_{\text{sing}}$.*

(1) *If σ and τ are path homotopic (i.e., homotopic with endpoints fixed), then f_σ and f_τ are chain homotopic.*

(2) *If $\sigma(1) = \tau(0)$, then $f_{\sigma * \tau} = f_\tau \circ f_\sigma$.*

In particular, the continuation isomorphisms provide a well-defined anti-homomorphism:

$$\Phi_{\Lambda,\mathcal{C}} : \pi_1(M, x_0) \to \text{GL}(H^*(C_{x_0}, d_{x_0})), \quad [\sigma] \mapsto \phi_\sigma,$$

i.e., $\Phi_{\Lambda,\mathcal{C}}([\sigma_1] \cdot [\sigma_2]) = \Phi_{\Lambda,\mathcal{C}}([\sigma_2]) \cdot \Phi_{\Lambda,\mathcal{C}}([\sigma_1])$.

The anti-homomorphism $\Phi_{\Lambda,\mathcal{C}}$ is called the *monodromy representation* of $(\Lambda, \mathcal{C})$.

Remark 2.12. The proposition is established with $R = \mathbb{F}_2$ coefficients in [42, Section 4.2], and a proof using the same method carries through in the present setting. For completeness, and because of the more general coefficients, we have included a proof in Section 4.1 below.

3. MCFs and augmented Legendrian cobordism

This section begins with notions of equivalence and cobordism for MCFs. As augmented Legendrian cobordism is most naturally defined through augmentations of the Legendrian contact homology DGA, we observe a bijection between MCFs and augmentations up to cobordism using the results from [35]. While the bijection is only proven with $\mathbb{F}_2$ coefficients, we include a discussion about extending the bijection to general coefficient fields in Section 3.3.

3.1. Equivalence of MCFs and augmented Legendrian cobordism

Note that when M has boundary, any MCF, $\mathcal{C}$, for a Legendrian surface $\Lambda \subset J^1M$ can be restricted to a boundary component $B \subset \partial M$ to produce a 1-dimensional MCF, denoted by $\mathcal{C}|_B$, for $\Lambda|_B$. As a slight extension of this construction, when

$$\Lambda \subset J^1([0,1] \times [0,1]),$$

we can also restrict MCFs from Λ to the Legendrian tangles sitting above any of the boundary edges $[0,1] \times \{i\}$ or $\{i\} \times [0,1]$ for $i = 0, 1$.

Definition 3.1. Let $(\Lambda, *, \xi)$ be a 1-dimensional Legendrian link or tangle in J^1M equipped with fixed collections of homology and spin basepoints, $*$ and ξ. We say that two ρ-graded MCFs, $\mathcal{C}_0$ and $\mathcal{C}_1$, for $(\Lambda, *, \xi)$ are *equivalent* if there exists a ρ-graded 2-dimensional MCF $\mathcal{C}$ on the trivial cobordism $\Lambda \times [0,1]$ having

- homology and spin structure $* \times [0,1]$ and $\xi \times [0,1]$,
- $\mathcal{C}|_{M \times \{i\}} = \mathcal{C}_i$ for $i = 0, 1$.
- In the case where $M = [0,1]$, we require that the handleslide set of $\mathcal{C}$ is disjoint from a neighborhood of $\partial M \times [0,1]$.

The terminology used in the following definitions is justified by the connection between MCFs and augmentations of the Legendrian contact homology dg-algebra; see Section 3.2 below.

Definition 3.2. A *(ρ-graded) augmented Legendrian* in J^1M is a pair $(\Lambda, \mathcal{C})$ consisting of a (ρ-graded) Legendrian (Λ, μ) in J^1M with a Morse complex family $\mathcal{C}$.

Often we suppress the explicit reference to ρ from terminology with the understanding that all augmented Legendrians under consideration are ρ-graded with respect to some fixed $\rho \geq 0$.

Definition 3.3. Two augmented Legendrians $(\Lambda_0, \mathcal{C}_0)$ and $(\Lambda_1, \mathcal{C}_1)$ in J^1M, where M is S^1 or $[0,1]$, are *cobordant* if there is an augmented Legendrian cobordism $(\Lambda, \mathcal{C})$ in $J^1(M \times [0,1])$ such that in neighborhoods of $M \times \{i\}$ for $i = 0, 1$, $(\Lambda, \mathcal{C})$ agrees with the product MCF, $(\Lambda_i \times [0,1], \mathcal{C}_i \times [0,1])$. When

$$M = [0,1],$$

we require in addition that there is a neighborhood N of $\partial M \times I$ such that

- in $N \times \mathbb{R}_z$ the front projection of Λ consists of n non-singular, non-intersecting sheets;
- all curves and points in H, $*$, and ξ have their base projections disjoint from N. In particular, the differentials $\{d_x\}$ are constant in x on both components of $\partial M \times I$.

Note that cobordism of augmented Legendrians defines an equivalence relation.

3.2. MCFs and augmentations of the Legendrian contact homology DGA

Recall that for compact Legendrian submanifolds in 1-jet spaces the *Legendrian contact homology dg-algebra* (abbrv. *LCH DGA*) was constructed by Ekholm, Etnyre, and Sullivan in [10, 12]. In its most basic form, the LCH algebra of $\Lambda \subset J^1M$, denoted by $(\mathcal{A}(\Lambda), \partial)$, is a free associative $\mathbb{F}_2$-algebra with identity generated by the Reeb chords of Λ and graded by $\mathbb{Z}/\rho\mathbb{Z}$ when Λ is ρ-graded. The degree -1 differential, ∂, is defined by a count of rigid holomorphic disks. When Λ is equipped with a spin structure, the coefficients can be upgraded from $\mathbb{F}_2$ to $\mathbb{Z}$; see [11], and there are various ways to incorporate the group ring $\mathbb{F}_2[H_1(\Lambda)]$ or $\mathbb{Z}[H_1(\Lambda)]$ into $(\mathcal{A}(\Lambda), \partial)$ by paying attention to the boundary values of holomorphic disks; see, e.g., [10, 16, 17] or [13, Section 2] for the "fully non-commutative" DGA.

A *ρ-graded augmentation* of Λ to $\mathbb{F}$ is a unital homomorphism of differential algebras

$$\varepsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0), \quad \varepsilon(1) = 1, \quad \varepsilon \circ \partial = 0$$

that preserves the $\mathbb{Z}/\rho$-grading, i.e.,

$$\varepsilon(x) \neq 0 \Rightarrow |x| = 0 \pmod{\rho}.$$

Two ρ-graded augmentations, ε_1 and ε_2, are *DGA homotopic* if there exists an $(\varepsilon_1, \varepsilon_2)$-derivation

$$H : \mathcal{A}(\Lambda) \to \mathbb{F}, \quad H(xy) = H(x)\varepsilon_2(y) + (-1)^{|x|}\varepsilon_1(x)H(y)$$

of degree 1 mod ρ such that $\varepsilon_1 - \varepsilon_2 = H \circ \partial$. Here, when ρ is odd, it is required that $\operatorname{char} \mathbb{F} = 2$. Denote the set of all ρ-graded augmentations of Λ to $\mathbb{F}$ by $\mathrm{Aug}_\rho(\Lambda; \mathbb{F})$, and denote the set of DGA homotopy classes by $\mathrm{Aug}_\rho(\Lambda; \mathbb{F})/\sim$.

Remark 3.4. When considering augmentations to $\mathbb{F}_2$, for computing $\mathrm{Aug}_\rho(\Lambda; \mathbb{F}_2)$ or $\mathrm{Aug}_\rho(\Lambda; \mathbb{F}_2)/\sim$, we can work with the version of $\mathcal{A}$ generated by Reeb chords only, without needing to make use of homology classes in $H_1(\Lambda)$. This is because any $\mathbb{F}_2$-valued augmentation must map all elements of $H_1(\Lambda) \subset \mathcal{A}^\times$ to $\mathbb{F}_2^\times = \{1\}$.

Consider now a ρ-graded Legendrian cobordism $\Sigma \subset J^1(M \times I)$ with $M = \mathbb{R}$ or S^1 between Λ_0 and Λ_1. As in [14], after modifying Σ near $M \times \partial I$ to have Morse minimum ends and working with an appropriate class of almost complex structures, the LCH DGA of Σ is well defined and contains the DGAs of Λ_0 and Λ_1 as sub-DGAs. See also [34, Section 5.3] and [35, Section 2.5]. Thus, we have the diagram

$$\mathcal{A}(\Lambda_1) \hookrightarrow \mathcal{A}(\Sigma) \hookleftarrow \mathcal{A}(\Lambda_0).$$

For augmentations $\varepsilon_0 \in \mathrm{Aug}(\Lambda_0; \mathbb{F})$ and $\varepsilon_1 \in \mathrm{Aug}(\Lambda_1; \mathbb{F})$, we say that $(\Lambda_0, \varepsilon_0)$ and $(\Lambda_1, \varepsilon_1)$ are *cobordant* if there exists a Legendrian cobordism Σ from Λ_0 and Λ_1 with an augmentation ε such that $\varepsilon|_{\mathcal{A}(\Lambda_i)} \simeq \varepsilon_i$ (DGA homotopy).

The works [23, 25] established over $\mathbb{F}_2$ a bijection between MCFs and DGA homotopy classes of augmentations for Legendrians in $J^1\mathbb{R}$; see [35] for the case of Legendrians in J^1S^1. A correspondence between MCFs and augmentations for closed Legendrian surfaces is established in [38]. In [34], it is shown that a bijection arises as well at the level of cobordism classes. More precisely, for $M = \mathbb{R}$ or S^1, let $\mathrm{Cob}_\rho^{\mathrm{MCF}}(J^1M; \mathbb{F}_2)$ (resp., $\mathrm{Cob}_\rho^{\mathrm{Aug}}(J^1M; \mathbb{F}_2)$) denote the set of cobordism classes of pairs $(\Lambda, \mathcal{C})$ (resp., (Λ, ε)) consisting of ρ-graded Legendrians in J^1M equipped with MCFs (resp., augmentations) defined over $\mathbb{F}_2$. Let $\mathrm{MCF}_\rho(\Lambda; \mathbb{F}_2)/\sim$ denote the set of equivalence classes of ρ-graded MCFs for a Legendrian $\Lambda \subset J^1M$.

Theorem 3.5. *Let $M = \mathbb{R}$ or S^1. For any $\Lambda \subset J^1M$ after modifying Λ by a Legendrian isotopy induced by a planar isotopy of $\pi_{xz}(\Lambda)$, there is a bijection:*

$$\Phi : \mathrm{MCF}_\rho(\Lambda; \mathbb{F}_2)/\sim \xrightarrow{\cong} \mathrm{Aug}_\rho(\Lambda; \mathbb{F}_2)/\sim.$$

Moreover, the maps Φ induce a bijection:

$$\mathrm{Cob}_\rho^{\mathrm{MCF}}(J^1M; \mathbb{F}_2) \cong \mathrm{Cob}_\rho^{\mathrm{Aug}}(J^1M; \mathbb{F}_2).$$

Proof. The bijection Φ is from [35, Proposition 5.17]. The fact that the Φ collectively induce a bijection on cobordism classes follows from [35, Corollary 4.12 and Proposition 5.19] and the characterization of induced augmentation sets in terms of Morse minimum cobordisms obtained in [35, Proposition 2.15]. The reader is referred to the ArXiv version of this paper [33] for more details. ∎

3.3. Remarks about general $\mathbb{F}$

We expect that the bijections from Theorem 3.5 should remain valid with an arbitrary field $\mathbb{F}$ provided that one works with an appropriate version of the LCH DGA. The approach taken in [35] is extendable to the case when $\operatorname{char}\mathbb{F} = 2$ as follows. With the formulation of MCFs over $\mathbb{F}$ used in this article, the bijections

$$\Phi : \mathrm{MCF}_\rho(\Lambda;\mathbb{F})/\sim \xrightarrow{\cong} \mathrm{Aug}_\rho(\Lambda;\mathbb{F})/\sim$$

will take equivalence classes of MCFs of Λ with a fixed collection of homology basepoints, $* = \{*_1, \dots, *_r\}$, to the set of DGA homotopy classes of augmentations of the multi-basepointed DGA, $\mathcal{A}(\Lambda, *)$, as defined in [31]. Recalling that $\mathcal{A}(\Lambda, *)$ has one invertible generator $t_i^{\pm 1}$ for each $*_i$, we remark that we require DGA homotopy operators to satisfy $H(t_i) = 0$, and under the bijection, the coefficient $s_i \in \mathbb{F}^\times$ associated to the basepoint $*_i$ as part of an MCF $\mathcal{C}$ satisfies $\varepsilon_i(t_i) = s_i^{-1}$. We also clarify the meaning of cobordism for augmentations of the multi-basepointed DGAs. When considering augmentations $\varepsilon_i : \mathcal{A}(\Lambda_i, *^i) \to \mathbb{F}$ for $i = 0, 1$, one should allow Morse minimum cobordisms $\Sigma \subset J^1(M \times [0, 1])$ equipped with homology structures $*$ that agree with $*^i \times [0, 1]$ near the boundary components $\Lambda_i \times \{i\}$ of Σ for $i = 0, 1$. A 2-dimensional analog of the multi-basepointed DGA, $\mathcal{A}(\Sigma, *)$, then results by associating invertible generators $u_i^{\pm 1}$ to the (co-orientations of the) edges of $*$, subject to the relations $u_1 u_2 u_3 = 1$ (assuming cyclically consistent co-orientation) at trivalent vertices. The differential of Reeb chords is modified to record intersections of the boundary of holomorphic disks with $*$ using appropriate factors of $u_i^{\pm 1}$. The pairs $(\Lambda_i, \varepsilon_i)$, $i = 0, 1$, should then be declared cobordant if there exists some $(\Sigma, *)$ with an augmentation

$$\alpha : \mathcal{A}(\Sigma, *) \to \mathbb{F}$$

satisfying $h_i^* \alpha \simeq \varepsilon_i$, where

$$h_i : \mathcal{A}(\Lambda_i, *^i) \to \mathcal{A}(\Sigma, *)$$

is the DGA map that is the inclusion on Reeb chords and maps $t_i \mapsto u_j^{\pm 1}$ appropriately. In fact, a closely related construction is used in the context of embedded Lagrangian cobordisms in the work of Gao–Shen–Weng [20, Section 2.4].

The proof of Theorem 3.5 above, essentially from [35], factors through the cellular DGA from [38–40] (defined over $\mathbb{F}_2$) and the 2-dimensional correspondence between MCFs and augmentations of the cellular DGA (over $\mathbb{F}_2$) from [42]. In establishing the isomorphism between the cellular DGA and the LCH DGA, the relevant rigid holomorphic disks, or rather their degenerations to gradient flow trees, are all enumerated in [38,39]. Thus, it is not difficult to record intersections between their boundaries and the curves that make up a homology structure. When this is done, the isomorphism

between the cellular DGA and the LCH DGA may be upgraded to incorporate homology structures, and the arguments of [35, 42] may be carried forward. As this would take us somewhat far afield, we do not consider these details in the present article.

We expect that the bijection continues to hold as well when $\operatorname{char}\mathbb{F} \neq 2$, although it is not clear that the method of proof from [35] can be pushed through with the readily available techniques. The cellular DGA can be formulated with signs by making use of a combinatorial spin structure in the presence of swallowtail points; see [41] for a formulation in all dimensions for Legendrians with at worst cusp singularities. However, it is more difficult to extend the isomorphism with LCH to include signs. The key issue is to be able to explicitly compute the orientation sign of a holomorphic disk from the combinatorics of the corresponding GFT. See [13, 28] for progress on this difficult technical problem.

4. Constructions of MCFs

In this section, we establish some tools for efficiently constructing MCFs and augmented Legendrian cobordisms that we will apply in later sections. In Section 4.1, we present a complete list of local moves on 1-dimensional slices that can be used to build up an arbitrary augmented Legendrian cobordism out of elementary ones. For ease of use, it is helpful that one can typically perform these moves without having to pay undue attention to the differentials $\{d_x\}$; see Proposition 4.1. In Section 4.2, we illustrate the moves by building up the (useful but non-elementary) D_4^- cobordism. Sections 4.3 and 4.4 establish standard forms for handleslide sets and MCFs up to equivalence that simplify later arguments. Finally, Section 4.5 records that MCFs can always be extended over cobordisms arising from Legendrian isotopies.

4.1. 2-dimensional MCFs via movies of slices

A 2-dimensional augmented Legendrian, $(\Sigma, \mathcal{C})$, in $J^1(M \times [0,1])$ can be built up out of elementary cobordisms that modify the 1-dimensional slices in a local way as described in the following moves including equivalence and basepoint moves as shown in Figure 9, Legendrian isotopy moves as shown in Figure 10, and Legendrian cobordism moves as shown in Figure 11. As above, coefficients of handleslides that intersect a slice $M \times \{t\}$ are always specified with respect to the co-orientation arising from the standard orientation of M (left to right in figures); co-orientations for basepoints are indicated by arrows in the figures.

Equivalence moves: (E1) Two adjacent (i, j)-handleslides with coefficients a and $-a$ can be cancelled or created.

(E2) Two adjacent (i, j)-handleslides with coefficients a and b can be interchanged with a single (i, j)-handleslide with coefficient $a + b$.

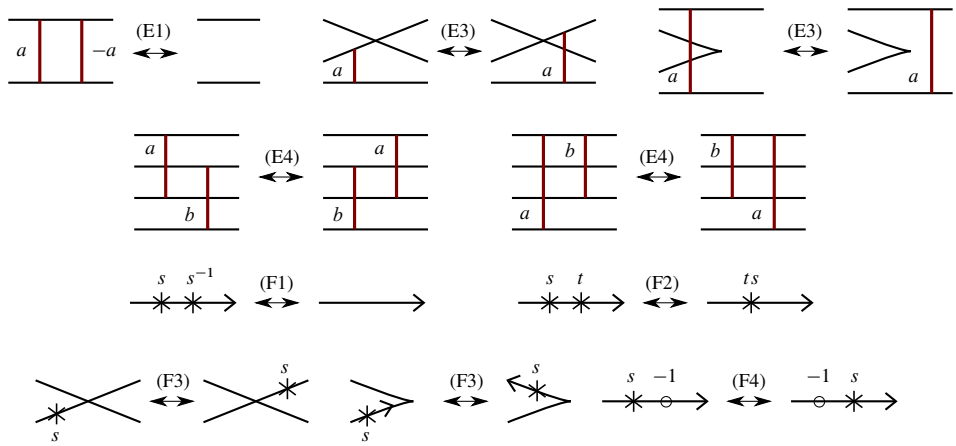


Figure 9. Examples of equivalence and basepoint moves. Note that moves (E2), (E5), (E6), and (E7) are pictured in the figures in Section 2.3.

(E3) An (i, j)-handleslide can be moved past the x-value of a crossing or a cusp as long as the endpoints trace out a continuous path on $\Lambda \setminus \Lambda_{\text{cusp}}$.

(E4) An (i, j)-handleslide can be commuted with a (k, l)-handleslide as long as $j \neq k$ and $i \neq l$.

(E5) Commuting an (i, j)-handleslide and a (j, k)-handleslide results in the appearance of a new (i, k)-handleslide with coefficient as in Figure 4.

(E6) An (i, j)-handleslide may be commuted past a homology or spin basepoint provided its coefficient changes as in Figure 7.

(E7) A group of handleslides is created or removed by passing a super-handleslide point as in Figure 5.

Basepoint moves: (F1) Two adjacent homology or spin basepoints with inverse coefficients, s and s^{-1}, with respect to a consistent co-orientation can be cancelled or created.

(F2) Two adjacent homology basepoints with coefficients, s then t, with respect to a consistent co-orientation can be replaced with a single basepoint with coefficient ts, and vice versa.

(F3) Basepoints can be moved continuously (in Λ) through cusps or crossings without change to coefficients.

(F4) Homology basepoints can be commuted with spin basepoints.

Legendrian isotopy moves: (R0) A pair of front singularities (two crossings, two cusps, or a crossing and a cusp) appearing in separate z-intervals in the front projection can have their x-coordinates pass one another.

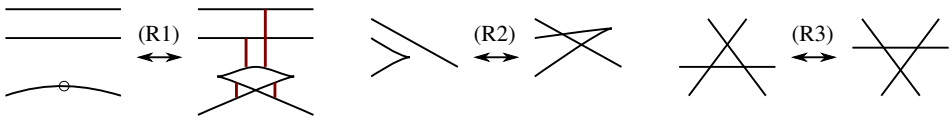


Figure 10. Legendrian Reidemeister moves. Reflections of (R1) (vertical) and (R2) (horizontal and vertical) are allowed. The coefficients of handleslides in the (R1) move are as in Figure 6.

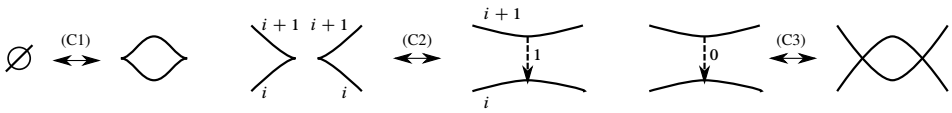


Figure 11. The cobordism moves for MCFs. To perform $\xleftarrow{(C2)}$, it is required that $d_x S_{k+1} = S_k$, and S_k and S_{k+1} must not appear in the differentials of any other sheets. For $\xrightarrow{(C2)}$, the Maslov potentials must take the same values, $i, i+1 \in \mathbb{Z}/\rho$, near the left and right cusps. To perform $\xrightarrow{(C3)}$, S_k must not appear in $d_x S_{k+1}$, although S_k and S_{k+1} may appear in the differentials of other sheets. No handleslides may appear in the x-intervals where these local moves are performed. In particular, there cannot be any handleslides inside the unknot or clasp in moves (C1) and (C3).

(R1) The standard Legendrian Reidemeister Move 1 can be performed provided the handleslides and spin basepoints are positioned on the two sides of the move as in Figure 6.

(R2) and (R3) Legendrian Reidemeister Moves 2 and 3 can be performed in x-intervals that do not contain handleslides.

Legendrian cobordism moves: (C1) (Unknot move) A standard Legendrian unknot component, U, disjoint from other components in the front projection can be added or removed provided that there are no handleslide endpoints or basepoints on U.

(C2) (Pinch move) Two sheets, S_k and S_{k+1}, defined near $x = x_0$ with no other sheets appearing between them (in the z-direction) such that

$$d_x S_{k+1} = S_k$$

and $\langle d_x S_i, S_k \rangle = \langle d_x S_i, S_{k+1} \rangle = 0$ for $i \neq k, k+1$ can be interchanged with a left cusp followed by a right cusp having equal Maslov potential values (in $\mathbb{Z}/\rho$) on their upper sheets as in Figure 11.

(C3) (Clasp move) Two consecutive crossings involving the same sheets, and without any handleslides located between them, can be removed. Conversely, for two adjacent sheets with

$$\langle d_x S_{k+1}, S_k \rangle = 0,$$

two consecutive crossings can be added near x.

In all of the moves, the augmented Legendrians $(\Lambda_0, \mathcal{C}_0)$ and $(\Lambda_1, \mathcal{C}_1)$ appearing on the two sides of the move are identical outside an interval $(x_a, x_b) \subset M$ where the move takes place, and no singularities of the MCFs $(\Lambda_i, \mathcal{C}_i)$ (crossings, cusps, handleslides, or basepoints) besides those referred to in the statement of the move appear in the interval (x_a, x_b). However, any number of other sheets of Λ, not pictured in the figures, without singularities in the front projection may appear as long as their positions are allowed by the statement of the move.

For each move, there is a corresponding *elementary augmented Legendrian cobordism* $(\Sigma, \mathcal{C}) \subset J^1(M \times [0,1])$ from $(\Lambda_0, \mathcal{C}_0)$ to $(\Lambda_1, \mathcal{C}_1)$ such that the MCF $(\Sigma, \mathcal{C})$ either (i) contains a unique codimension 2 base singularity of $(\Sigma, \mathcal{C})$ or (ii) contains a single critical point of the t-coordinate function, $M \times [0,1] \to [0,1]$, restricted to some codimension 1 base singularity of $(\Sigma, \mathcal{C})$.

Singularity of $(\Sigma, \mathcal{C})$	Move
Two cusps, crossings, handleslides, or basepoints with the same x-coordinate	(E3), (E4), (E5), (E6), (F3), (F4), (R0)
Local max or min of t restricted to a handleslide, basepoint, cusp, or crossing arc	(E1), (F1), (C1)–(C3)
Codimension 2 front singularity of Σ	(R1)–(R3)
Trivalent vertex of handleslide or basepoint arcs	(E2), (F2)
Super-handleslide point	(E7)

We have not specified explicitly how the differentials $\{d_x\}$ for $x \in (x_a, x_b)$ are changed when a move occurs. The following proposition guarantees their existence.

Proposition 4.1. *Let $(\Lambda_0, \mathcal{C}_0) \subset J^1 M$ be an augmented Legendrian corresponding to one side (either one) of one of the above moves.*

(1) *There exists an augmented Legendrian $(\Lambda_1, \mathcal{C}_1)$ together with a cobordism $(\Sigma, \mathcal{C}) \subset J^1(M \times [0,1])$ between $(\Lambda_0, \mathcal{C}_0)$ and $(\Lambda_1, \mathcal{C}_1)$ such that the singular sets of $(\Lambda_1, \mathcal{C}_1)$ and $(\Sigma, \mathcal{C})$ are as specified in the description of the move and the corresponding elementary cobordism.*

(2) *The family of differentials $\{d_x\}$ for $(\Lambda_1, \mathcal{C}_1)$ and $(\Sigma, \mathcal{C})$ are uniquely determined by $(\Lambda_0, \mathcal{C}_0)$.*

(3) *The continuation maps f_{σ_0} and f_{σ_1} associated to the paths*

$$\sigma_0 = [x_a, x_b] \times \{0\}$$

and

$$\sigma_1 = (\{x_a\} \times [0,1]) * ([x_a, x_b] \times \{1\}) * (\{x_a\} \times [0,1])^{-1}$$

are chain homotopic.

Note that for the equivalence moves (E1)–(E7) over $\mathbb{F}_2$ a detailed proof of (1) appears in [23, Proposition 3.8].

Proof. Within the rectangle $[x_a, x_b] \times [0, 1]$, the preceding table indicates how the elementary cobordism Σ, as well as its the handleslide set $H = H_0 \sqcup H_{-1}$, and its homology and spin structures, $*$ and ξ, are all specified by the different moves. Outside of this rectangle, $(\Sigma, \mathcal{C})$ agrees with the product cobordism. Thus, it just needs to be verified that differentials $\{d_{(x,t)}\}$ can be defined in $[x_a, x_b] \times [0, 1]$ to agree with those from $\mathcal{C}_0$ when $t = 0$ and satisfy the axioms (A1)–(A4). Observe the following in all cases:

(i) Each component of $(M \times [0, 1])^{\mathcal{C}}_{\text{reg}}$ in $[x_a, x_b] \times [0, 1]$ borders at least one of $[x_a, x_b] \times \{0\}$ or $[x_a, x_b] \times \{1\}$.

(ii) Out of these components, only the regions that border $\{x_a\} \times [0, 1]$ or $\{x_b\} \times [0, 1]$ intersect both $[x_a, x_b] \times \{0\}$ and $[x_a, x_b] \times \{1\}$.

(iii) Each codimension 1 base singularity of $(\Sigma, \mathcal{C})$ (crossing, cusp, handleslide, homology arc, or spin arc) intersects at least one of $M \times \{0\}$ and $M \times \{1\}$.

Thus, if one can define the differentials for $\mathcal{C}_1$ so that

(a) axioms (A1)–(A4) hold along $[x_a, x_b] \times \{1\}$,

(b) $\mathcal{C}_1$ agrees with $\mathcal{C}_0$ at $x = x_a$ and $x = x_b$,

then, after extending the differentials of $\mathcal{C}_0$ and $\mathcal{C}_1$ to all of $[x_a, x_b] \times [0, 1]$ to be constant in the components of $(M \times [0, 1])^{\mathcal{C}}_{\text{reg}}$, it follows that the axioms (A1)–(A4) hold everywhere for the 2-dimensional MCF $(\Sigma, \mathcal{C})$. This would complete the construction. Moreover, since (a) and (b) must hold, d_{x_a} and (A1)–(A4) uniquely specify the remaining differentials along $[x_a, x_b] \times \{1\}$ so that the uniqueness statement (2) of the proposition follows.

The existence of the differentials $\{d_x\}$ for $\mathcal{C}_1$ satisfying (a) and (b), as well as the existence of the chain homotopy from (3) of the proposition, is verified by a case-by-case check. We explain the procedure that should be followed in general and then examine the more involved cases in detail. Let $d^0_{x_a}$ and $d^0_{x_b}$ denote the differentials from $\mathcal{C}_0$ at x_a and x_b. Choose a sequence of x-values

$$x_a = x_0 < x_1 < \cdots < x_N = x_b$$

such that a single singularity, s, of $(\Lambda_1, \mathcal{C}_1)$ (crossing, cusp, handleslide, or basepoint) occurs in $[x_{i-1}, x_i]$ for $1 \leq i \leq N$. Inductively construct the differentials $d_{x_0}, \ldots, d_{x_N}$ for $\mathcal{C}_1$, starting with $d_{x_0} = d^0_{x_a}$. Next, d_{x_i} is determined by $d_{x_{i-1}}$ by the singularity in $[x_{i-1}, x_i]$ as uniquely specified by the axioms (A1)–(A4). This goes as follows:

- In the case of a *crossing*, *handleslide*, or *basepoint*, define

$$d_{x_i} = f_i d_{x_{i-1}} f_i^{-1}, \tag{4.1}$$

where f_i is the continuation map from (2.4), (2.5), or (2.6). In the case of a crossing, it is important to check that

$$\langle d_{x_{i-1}} S_{k+1}, S_k \rangle = 0$$

so that d_{x_i} will still be upper triangular.

- In the case of a *left cusp*, use the continuation map (2.2) to identify

$$C_{x_i} \cong C_{x_{i-1}} \oplus C_{\text{cusp}}, \quad \text{where } C_{\text{cusp}} = \text{Span}_R\{S_k, S_{k+1}\},$$

and define d_{x_i} to be the direct sum

$$d_{x_i} = d_{x_{i-1}} \oplus d_0, \quad \text{where } d_0 S_{k+1} = S_k. \tag{4.2}$$

- In the case of a *right cusp*, it is important to verify that with respect to the direct sum decomposition

$$C_{x_{i-1}} = C_{x_i} \oplus C_{\text{cusp}}$$

the differential $d_{x_{i-1}}$ has the form $d_{x_{i-1}} = d' \oplus d_0$. Then, define $d_{x_i} = d'$.

During this constructive procedure, the required condition at crossings and right cusps is seen to hold by either the hypothesis on the move in (C1)–(C3) or by the fact that the axioms are known to be valid for $\mathcal{C}_0$. Finally, one needs to verify that the resulting differential d_{x_N} agrees with $d^0_{x_b}$ from $\mathcal{C}_0$ and check that the two continuation maps f_{σ_0} and f_{σ_1} are chain homotopic. (These two checks are usually somewhat interrelated.) Note that since there are no singularities along the segments $\{x_a\} \times [0, 1]$ and $\{x_b\} \times [0, 1]$, f_{σ_0} and f_{σ_1} are the composition

$$f_{\sigma_0} = f^0_{N_0} \circ \cdots \circ f^0_1, \quad f_{\sigma_1} = f^1_{N_1} \circ \cdots \circ f^1_1$$

of the maps (2.2)–(2.6) from Section 2.4 associated to the sequence of singularities appearing along $[x_a, x_b] \times \{0\}$ and $[x_a, x_b] \times \{1\}$, respectively.

We comment now on the specifics of the above procedure for individual moves.

When there are no cusps in $[x_a, x_b] \times [0, 1]$. From (4.1), we have

$$d_{x_N} = f_{\sigma_1} d^0_{x_a} f^{-1}_{\sigma_1},$$

and we know that

$$d^0_{x_b} = f_{\sigma_0} d^0_{x_a} f^{-1}_{\sigma_0}.$$

In all cases except for the super-handleslide move (E7), that is for the moves (E1), (E2), (E4)–(E6), (F1), (F2), (F4), (R3), (C3) and the non-cusp cases of (E3), (F3), (R0), quick matrix computations show that $f_{\sigma_0} = f_{\sigma_1}$; e.g., (E5) follows from the

"Steinberg relation", $[aE_{ij}, bE_{jk}] = abE_{ik}$, and (R3) from the celebrated braid relation for the permutation matrices,

$$Q_{(k\,k+1)}Q_{(k+1\,k+2)}Q_{(k\,k+1)} = Q_{(k+1\,k+2)}Q_{(k\,k+1)}Q_{(k+1\,k+2)}.$$

This simultaneously verifies that $d_{x_N} = d^0_{x_b}$ and f_{σ_0} and f_{σ_1} are chain homotopic.

In the case of the super-handleslide move (E7), using the matrix reformulation of the axiom (B2) discussed in Remark 2.9, the product $(I + B_m)\cdots(I + B_1)$ that appears in (2.1) is the matrix of the continuation map, f, on the side of the (E7) move where the handleslides appear, while I is the matrix for the other continuation map. Thus, equation (2.1) translates to the identity

$$d_{x_a}K + Kd_{x_a} = f - \mathrm{id}, \tag{4.3}$$

where $K(S_l) = cS_k$ and $K(S_i) = 0$ for $i \neq l$, and we simplified notation to $d_{x_a} = d^0_{x_a}$. This and the fact that $d^2_{x_a} = 0$ imply that

$$\begin{aligned} fd_{x_a} &= d_{x_a}Kd_{x_a} + Kd^2_{x_a} + d_{x_a} \\ &= d^2_{x_a}K + d_{x_a}Kd_{x_a} + d_{x_a} = d_{x_a}f. \end{aligned}$$

In the case that $f_{\sigma_0} = \mathrm{id}$, we have

$$d_{x_N} = f_{\sigma_1}d_{x_a}f^{-1}_{\sigma_1} = fd_{x_a}f^{-1} = d_{x_a} = d_{x_b},$$

and in the case that $f_{\sigma_0} = f$, we have

$$d_{x_N} = f_{\sigma_1}d_{x_a}f^{-1}_{\sigma_1} = \mathrm{id}\,d_{x_a}\,\mathrm{id}^{-1} = d_{x_a} = fd_{x_a}f^{-1} = f_{\sigma_0}d_{x_a}f^{-1}_{\sigma_0} = d_{x_b}$$

as required. Moreover, (4.3) shows that f_{σ_1} and f_{σ_0} are chain homotopic.

When there are cusps in $[x_a, x_b] \times [0, 1]$. In all cases except for (C2), the continuation maps are equal, $f_{\sigma_0} = f_{\sigma_1}$. The verification for moves (E3), (R0), (R2), and (C1) is routine. Consulting the bottom row of Figure 1 is helpful when verifying the cusp case of move (F3).

For the pinch move (C2), write

$$C_{x_a} = C_{x_b} = C' \oplus C_{\mathrm{cusp}},$$

where C_{cusp} is spanned by S_{k+1} and S_k and C' by the remaining strands. Using the hypothesis that when the move is performed in the $\leftarrow$ direction the differential $d_{x_a} = d_{x_b}$ from $\mathcal{C}_0$ splits as in (4.2), the construction of the differentials is clear. The continuation maps, f_0 and f_1, associated to the sides of the move where the cusps do and do not exist, respectively, have the block matrix form

$$f_0 = \begin{bmatrix} \mathrm{id}_{C'} & 0 \\ 0 & 0 \end{bmatrix}, \quad f_1 = \begin{bmatrix} \mathrm{id}_{C'} & 0 \\ 0 & \mathrm{id}_{C_{\mathrm{cusp}}} \end{bmatrix}$$

so that setting $K(S_k) = S_{k+1}$ and $K(S_i) = 0, i \neq k$, we have the required homotopy:

$$f_1 - f_0 = d_{x_b} K + K d_{x_a}.$$

Finally, we turn to the (R1) move. See [42, Proposition 6.2] for the case of $\mathbb{F}_2$ coefficients. We consider the upward swallowtail; the downward case is similar. First, suppose that the direction of the (R1) move is such that the cusps exist in $(\Lambda_1, \mathcal{C}_1)$ but not $(\Lambda_0, \mathcal{C}_0)$. Let S_k be the sheet of $(\Lambda_0, \mathcal{C}_0)$ that contains the spin basepoint ξ that terminates at the swallowtail point. In between the cusps of Λ_1, the sheets numbered S_k, S_{k+1}, S_{k+2} form the three sheets involved with the swallowtail point. As usual, the sheets are numbered from top to bottom at each value of x so that the way the bottom two sheets are numbered changes at the crossing. Let $A_- = (a^-_{i,j})$ and $A_+ = (a^+_{i,j})$ denote the matrices of d_{x_a} and d_{x_b}, respectively; because of the spin basepoint, these matrices are related by

$$A_+ \Delta_k = \Delta_k A_-,$$

where Δ_i denotes the diagonal matrix with -1 in position i and 1's at all other diagonal entries. Let

$$\begin{aligned} H_S &= (I - a^-_{1,k} E_{1,k}) \cdots (I - a^-_{k-1,k} E_{k-1,k})(I - E_{k+1,k+2}) \\ &= I - \sum_{i<k} a^-_{i,k} E_{i,k} - E_{k+1,k+2}, \\ H_T &= I + E_{k+1,k+2} \end{aligned}$$

denote the products of matrices associated to the handleslides of $(\Lambda_1, \mathcal{C}_1)$ appearing to the left and right sides of the crossing, respectively. To establish that the differentials can be extended along $[x_a, x_b] \times \{1\}$ to agree with d_{x_a} and d_{x_b} from $\mathcal{C}_0$, it needs to be shown that

$$\widehat{A}_+ (H_T Q H_S) = (H_T Q H_S) \widehat{A}_-, \tag{4.4}$$

where $Q = Q_{(k+1\, k+2)}$ is the permutation matrix and $\widehat{A}_\pm$ is the block matrix obtained from inserting two new rows and columns at positions k and $k+1$ with the 2×2 block $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ placed on the diagonal. Note that $\widehat{A}_+ = \Delta_{k+2} \widehat{A}_- \Delta_{k+2}$. Computing

$$(H_T Q H_S) S_j = \begin{cases} S_j, & j \notin \{k, k+1, k+2\}, \\ S_k - \sum_{i<k} a^-_{i,k} S_i, & j = k, \\ S_{k+1} + S_{k+2}, & j = k+1, \\ -S_{k+2}, & j = k+2, \end{cases} \tag{4.5}$$

one checks (4.4) by verifying the equality when both sides are applied to S_j. In detail, when $j \notin \{k, k+1, k+2\}$, since $\widehat{A}_- S_j \in \mathrm{Span}(\{S_1, \ldots, S_n\} \setminus \{S_k, S_{k+1}\})$ and

$(H_T Q H_S)$ agrees with Δ_{k+2} when restricted to this subspace, we compute

$$\begin{aligned}(H_T Q H_S)\widehat{A}_- S_j &= \Delta_{k+2}\widehat{A}_- S_j \\ &= \Delta_{k+2}\widehat{A}_-(\Delta_{k+2}S_j) \\ &= \widehat{A}_+ S_j = \widehat{A}_+(H_T Q H_S)S_j.\end{aligned}$$

When $j = k$,

$$(H_T Q H_S)\widehat{A}_- S_k = 0,$$

and

$$\widehat{A}_+(H_T Q H_S)S_k = \widehat{A}_+\Big(S_k - \sum_{i<k} a^-_{ik} S_i\Big) = 0 - \sum_{l<i<k} a^-_{li} a^-_{ik} S_l = 0$$

since $A_-^2 = 0$. When $j = k+1$,

$$(H_T Q H_S)\widehat{A}_- S_{k+1} = (H_T Q H_S)S_k = S_k - \sum_{i<k} a^-_{i,k} S_i,$$

and

$$\begin{aligned}\widehat{A}_+(H_T Q H_S)S_{k+1} &= \widehat{A}_+(S_{k+1} + S_{k+2}) \\ &= S_k + \Delta_{k+2}\widehat{A}_-\Delta_{k+2}S_{k+2} \\ &= S_k - \sum_{i<k} a^-_{i,k} S_i.\end{aligned}$$

Finally, when $j = k+2$,

$$(H_T Q H_S)\widehat{A}_- S_{k+2} = \widehat{A}_- S_{k+2},$$

and

$$\widehat{A}_+(H_T Q H_S)S_{k+2} = \Delta_{k+2}\widehat{A}_-\Delta_{k+2}(-S_{k+2}) = \widehat{A}_- S_{k+2}.$$

Moreover, (continuing to blur the distinction between a linear map and its matrix), the continuation maps are

$$f_{\sigma_1} = p \circ (H_T Q H_S) \circ \iota \qquad \text{and} \qquad f_{\sigma_0} = \Delta_k$$

where ι and p are, respectively, the inclusion and projection maps from (2.2) and (2.3). Using (4.5), it is then easy to check that $f_{\sigma_1} = f_{\sigma_0}$.

If the (R1) move occurs in the opposite direction so that the cusps exist above Λ_0 and not above Λ_1, then the same calculation as above computes A_+ in terms of A_- as $A_+ = \Delta_k A_- \Delta_k$. Thus, the differentials d_{x_a} and d_{x_b} satisfy (A4) at the spin basepoint ξ of $\mathcal{C}_1$. ∎

A pleasant consequence of Proposition 4.1 is that one can construct augmented Legendrian cobordisms by presenting a sequence of 1-dimensional slices with handleslides and basepoints indicated and related as in the above moves, but *without needing to explicitly specify the families of differentials* $\{d_x\}$ at every step. One just needs to know that the initial slice $(\Lambda_0, \mathcal{C}_0)$ is equipped with differentials making $\mathcal{C}_0$ into a valid MCF. Note that occasionally, while constructing such a movie, partial information is needed about some of the differentials $\{d_x\}$ in order to perform one of the moves, e.g., (E7), (R1), (C2), and (C3). Typically, the relevant data can be reconstructed directly from the handleslides and basepoints at the current slice; provided that d_x is known for at least one value of x in the slice, the axioms (A1)–(A4) uniquely specify the rest of the differentials.

In fact, we can use this method to contruct arbitrary augmented Legendrian cobordisms.

Proposition 4.2. *Let* $M = \mathbb{R}, [0, 1]$, *or* S^1.

(i) *Two MCFs for a Legendrian* $\Lambda \subset J^1M$ *are equivalent if and only if they can be related by the equivalence moves (E1)–(E7).*

(ii) *Two augmented Legendrians* $(\Lambda_0, \mathcal{C}_0)$ *and* $(\Lambda_1, \mathcal{C}_1)$ *are cobordant if and only if they can be related by a sequence of elementary moves (E1)–(E7), (F1)–(F4), (R0)–(R3), and (C1)–(C3).*

Proof. After a mild wiggling, any augmented Legendrian cobordism can be cut up into the elementary cobordisms described in the above table. [Applying a C^∞-small isotopy of Σ induced by an isotopy of the base space $M \times [0, 1]$ rel. boundary arranges that base singularities of $(\Sigma, \mathcal{C})$ are positioned generically with respect to the t coordinate. In addition a C^0-small modification of the handleslide arcs near endpoints at super-handleslides arranges that the arcs approach their endpoints from the negative t-direction as in (E7).] For an equivalence, the underlying Legendrian cobordism is the product cobordism with basepoints remaining fixed, and so, only the elementary cobordisms corresponding to moves (E1)–(E7) appear. ∎

Remark 4.3. Note that using a combination of moves (F1) and (F2) (resp., (E1) and (E2)), a homology basepoint with coefficient 1 (resp., a handleslide with coefficient 0) can be added or removed by a cobordism (resp., equivalence).

In addition, Proposition 4.1 allows us to prove the statements about the continuation maps from Proposition 2.11.

Proof of Proposition 2.11. Note that (2) is clear from the definition. With Propositions 4.1 and 4.2 in hand, (1) follows as in the proof for the $\mathbb{F}_2$-coefficient case found in [42, Proposition 4.7]. Let $(\Lambda, \mathcal{C}) \subset J^1M$ be an augmented Legendrian, and let

$$\sigma, \tau : [0, 1] \to M$$

be path homotopic and transverse to the singular set of $(\Lambda, \mathcal{C})$, $M^{\mathcal{C}}_{\text{sing}}$. Let

$$I : [0,1] \times [0,1] \to M, \quad I(s,t) = \sigma_t(s)$$

with $\sigma_0 = \sigma$, $\sigma_1 = \tau$, $\sigma_t(0) = \sigma(0)$, and $\sigma_t(1) = \sigma(1)$, be a path homotopy between σ and τ that is also transverse to $M^{\mathcal{C}}_{\text{sing}}$; i.e., I is transverse to the base singularities of all codimensions. We can then pull back $(\Lambda, \mathcal{C})$ along I (in the expected manner) to get an augmented Legendrian cobordism

$$I^*(\Lambda, \mathcal{C}) \subset J^1([0,1] \times [0,1])$$

such that f_σ and f_τ are the continuation maps associated to the $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ slices of the cobordism. From Proposition 4.2 and Proposition 4.1 (3), we see that f_σ and f_τ are chain homotopic. ∎

4.1.1. Uniqueness of the (R1) handleslides. We make an observation that simplifies considerations involving the (R1) move. Clearly, the (R1) move can always be performed in the $\to$ direction: if the spin basepoint is not already present at the desired location of the (R1) move, then a pair of spin basepoints can be created using (F1) so that the (R1) move can be performed (leaving one additional spin point not involved in the move). At first glance, it may appear that there is a restriction on whether the $\leftarrow$ (R1) move can be performed, since the handleslides that appear in the interval $[x_a, x_b]$ where the move occurs must match the right column of Figure 6 with coefficients determined by d_{x_a}. In fact, the correctness of the coefficients is automatic once all other handleslides and basepoints have been removed from $[x_a, x_b]$.

Lemma 4.4. *Suppose that we have a 1-dimensional MCF $(\Lambda_0, \mathcal{C}_0)$ where in*

$$[x_a, x_b] \subset M$$

the Legendrian Λ_0 appears as in the top row and right side of (R1) in Figure 6, and let $x = x_l$ and $x = x_r$ be the locations of the left and right cusps. Suppose further that

- *there are no basepoints in $[x_a, x_b]$;*
- *all handleslides in $[x_a, x_b]$ are located in (x_l, x_r) and are either $(k+1, k+2)$-handleslides (these can appear on both sides of the crossing) or (i, k) handleslides for $i < k$, where the sheets that meet the cusp points are labeled $k, k+1$ in (x_l, x_r).*

Then, after commutation (E4) moves and grouping handleslides with common endpoints together using (E2) moves, the handleslides can be made to appear with coefficients as in Figure 6.

Proof. Clearly, the handleslides can be rearranged with (E4) and (E2) moves to have their locations as in Figure 6. For $i < k$, let $a_{i,k} = \langle d_{x_a} S_k, S_i \rangle$ and let $b_{i,k}$ be the coefficient of the (i,k)-handleslide in (x_r, x_l), where $b_{i,k} = 0$ means there is no handleslide. Let b_- and b_+ denote the coefficients of the $(k+1, k+2)$-handleslides that appear to the left and right of the crossings. Using (A1)–(A3) to determine from d_{x_a} the differential $d_{x_r^-}$ with $x_r^- < x_r$ just preceding the right cusp, we compute (consulting Figure 1 is helpful here)

$$\langle d_{x_r^-} S_{k+1}, S_k \rangle = -b_-, \quad \langle d_{x_r^-} S_{k+2}, S_k \rangle = 1 + b_- b_+,$$
$$\langle d_{x_r^-} S_{k+1}, S_i \rangle = a_{i,k} - b_{i,k} b_-.$$

Thus, (A4) implies that $-b_- = 1$, $1 + b_- b_+ = 0$, and $a_{i,k} - b_{i,k} b_- = 0$ so that the coefficients indeed must have the form $b_- = -1$, $b_+ = 1$, and $b_{i,k} = -a_{i,k}$. ∎

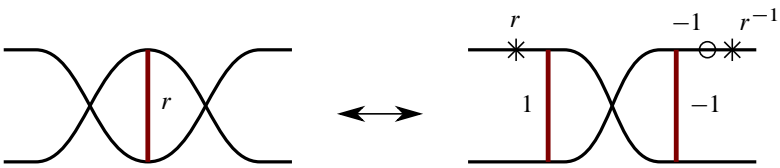
In particular, when constructing an augmented Legendrian cobordism via a movie of slices, if one wants to perform the ← (R1) move, it is enough to first arrange that the handleslides near the move are as in Lemma 4.4. This can always be done; see Proposition 4.15 below.

Remark 4.5. A similar statement holds for a downward swallowtail point as in the bottom row of Figure 6. We leave the precise formulation to the reader.

4.2. The D_4^- cobordism

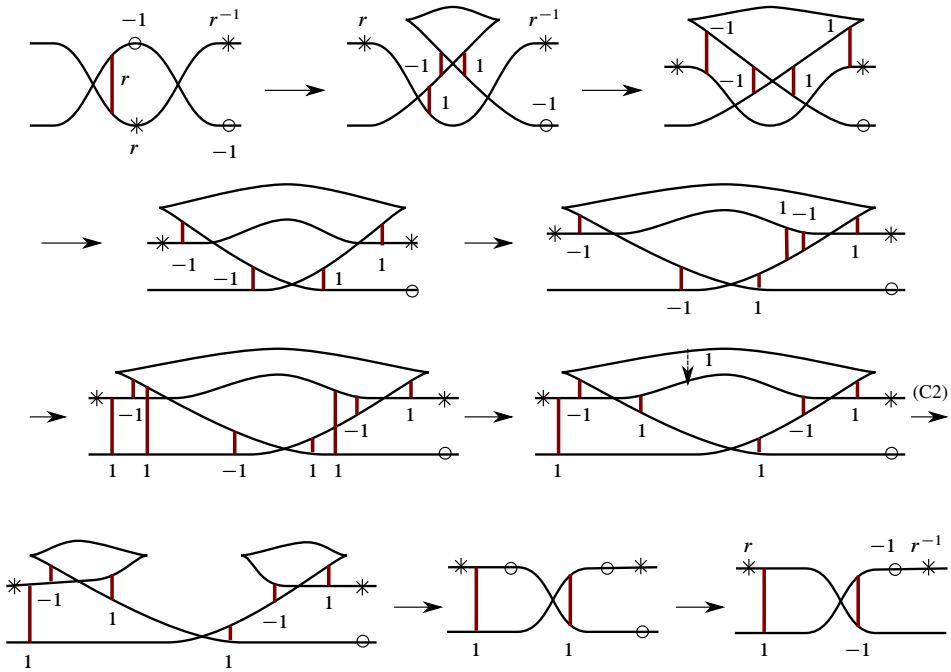
As an example of the above method, we combine several elementary moves to establish a (non-elementary) augmented cobordism move, useful later in Section 6, that corresponds to passing a desingularized D_4^- point on an augmented Legendrian surface. The (non-generic) D_4^- singularity has proven to be useful for the efficient construction of Lagrangian fillings and cobordisms; cf. [7, 45]. See, e.g., loc. cit. or [5] for a description of the D_4^- singularity and its desingularization.

Proposition 4.6. *For any $r \in R^*$, there is an augmented Legendrian cobordism:*



In the pictured region, the cobordism surface Σ has a standard desingularization of the D_4^- singularity and a single homology arc with endpoints as pictured; outside of the pictured region, the cobordism is a product.

Proof. Construct the cobordism via a sequence of elementary moves as indicated by the following:



Let $A_- = (a^-_{i,j})$ denote the matrix of d_x at the far left side of the pictured interval, and let S_k and S_{k+1} be the numbering of the two pictured sheets at this location. The construction starts by using (F1) to create a new pair of homology basepoints with coefficients r and r^{-1} and a new pair of spin basepoints (with coefficients -1 and denoted by $\circ$). After the first (R1) move for each $i < k$, there is an additional (i,k)-handleslide (not pictured) connecting S_i to the upper sheet of the cusps with coefficients $-a^-_{i,k+1}$. Immediately after their appearance, we convert each of these via a trivalent vertex into two (i,k)-handleslides with coefficients $-a^-_{i,k}r^{-1}$ and $-a^-_{i,k+1} + a^-_{i,k}r^{-1}$ positioned near the left and right cusps, respectively. (These handleslides with different values of i all commute with one another, so there is no issue in the repositioning.) At the location, $x = x_c$, where sheets S_k and S_{k+1} are connected by the dotted arrow just before the $\leftarrow$ (C2) move occurs, the sheets S_k and S_{k+1} form a subcomplex with $d_{x_c}(S_{k+1}) = S_k$ that splits off from the rest of (C_{x_c}, d_{x_c}) as a direct sum. [This follows from the identity (4.4) established in the proof of Proposition 4.1; in notation of (4.4) $\widehat{A}_-$ is now the matrix of d_x immediately to the right of the left cusp, and $\widehat{A}_+$ is the matrix of d_{x_c} at the dotted arrow. Note also that the handleslides and crossing that appear between x_l and x_c are consistent with the product $H_S Q H_T$ appearing in (4.4) with the coefficients $-a^-_{i,k}r^{-1}$ of the

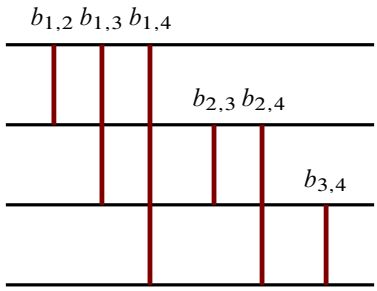


Figure 12. A properly ordered handleslide collection. The continuation map has matrix $I + \sum_{i<j} b_{i,j} E_{i,j}$.

(i,k)-handleslides equal to the negative of the coefficient $\langle d_{x'} S_k, S_i \rangle$ with x' just before the left cusp.] Thus, the required hypothesis is satisfied so that the $\leftarrow$ direction of the pinch move (C2) can be performed as pictured. Moreover, the subsequent (R1) moves can then be carried out using Lemma 4.4. ∎

4.3. Properly ordered handleslide collections

It will be convenient to have a standard positioning (up to equivalence) for a collection of handleslides in a region where a 1-dimensional Legendrian has no front singularities.

Definition 4.7. Let $(\Lambda, \mathcal{C}) \subset J^1 M$ be a 1-dimensional augmented Legendrian, and let $[x_a, x_b] \subset M$ be an interval where the front projection of Λ consists of n parallel strands without crossings or cusps. We say the handleslides of $\mathcal{C}$ appearing in the interval $[x_a, x_b]$ are *properly ordered* if

(i) for each $1 \leq i < j \leq n$ there is at most one (i, j)-handleslide in $[x_a, x_b]$;

(ii) the handleslides appear in lexicographical order with respect to their indices; i.e., whenever an (i_1, j_1)-handleslide appears before (to the left of) an (i_2, j_2)-handleslide, we have either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$.

See Figure 12.

If they are not properly ordered, then we can always arrange the handleslides in $[x_a, x_b]$ to become properly ordered by an equivalence that leaves $\mathcal{C}$ unchanged outside of $[x_a, x_b]$.

Proposition 4.8. *Let $(\Lambda, \mathcal{C}) \subset J^1[x_a, x_b]$ be an augmented Legendrian without crossings or cusps in the interval $[x_a, x_b] \subset M$.*

(1) *There is an equivalence of MCFs, $\mathcal{C} \sim \mathcal{C}'$, such that the handleslides of $\mathcal{C}'$ are properly ordered.*

(2) *If $\mathcal{C}$ has no basepoints and the handleslides are properly ordered, then the continuation map $f_{[x_a,x_b]}$ has matrix $I + \sum_{i<j} b_{i,j} E_{i,j}$, where $b_{i,j}$ is the coefficient of the (i,j)-handleslide of $\mathcal{C}$, and is 0 if no such handleslide exists.*

Proof. The proof of (1) is a straightforward induction on the number of handleslides of $\mathcal{C}$; cf. [23, Section 6e1]. The proof of (2) is a calculation: the matrix of $f_{[x_a,x_b]}$ is the product of handleslide matrices $I + b_{i,j} E_{i,j}$ in the order that the handleslides appear from *right to left*, and all of the $(b_{i_2,j_2} E_{i_2,j_2})(b_{i_1,j_1} E_{i_1,j_1})$ terms vanish since $i_1 < j_2$. ∎

4.4. *SR*-form MCFs

Brad Henry introduced in [23] a standard form for MCFs of Legendrian knots in $\Lambda \subset J^1\mathbb{R}$ called the SR-form. MCFs in SR-form have their handleslides in standard positions near the switches and returns of a normal ruling of Λ, and every MCF is equivalent to one in SR-form. Here, we extend this definition and result to MCFs in $J^1[0,1]$ or J^1S^1.

4.4.1. Generalized normal rulings. Let $M = [0,1]$ or S^1. A *ρ-graded generalized normal ruling*, σ, of a ρ-graded Legendrian link Λ is a family of involutions,

$$\sigma_{x_0} : \Lambda \cap \pi_x^{-1}(\{x_0\}) \to \Lambda \cap \pi_x^{-1}(\{x_0\}), \quad \sigma_{x_0}^2 = \mathrm{id},$$

defined for all $x_0 \in M$ not equal to the base projection of any crossing or cusp of $\pi_{xz}(\Lambda)$ and subject to several requirements. We refer the reader to [29] for the thorough definition and give a briefer summary here. For each x_0, the involution σ_{x_0} partitions the sheets of Λ at x_0, $S_1, \ldots, S_n \in C_{x_0}$, into some number of *fixed-point sheets* and some sets of *paired sheets*. The fixed-point sheets satisfy $\sigma_{x_0}(S_i) = S_i$, and the paired sheets form two element subsets $\{S_i, S_j\}$ with $\sigma_{x_0}(S_i) = S_j$. The ρ-graded condition is as follows: if S_i and S_j are paired and $z(S_i) > z(S_j)$, then the Maslov potential satisfies

$$\mu(S_i) = \mu(S_j) + 1.$$

At cusp points, the two sheets that meet at the cusp must be paired by ρ for x_0 near the cusp point. At crossings of $\pi_{xz}(\Lambda)$, the two sheets that meet at the crossing cannot be paired for x_0 near the crossing. Crossings where the pairing of sheets changes are called *switches*. At switches, the vertical ordering (z-coordinate) of the two crossing strands and their pairs (if they exist) must satisfy the normality condition pictured in Figure 13. The other crossings of $\pi_{xz}(\Lambda)$ are called *departures* and *returns*, depending on whether the normality condition is satisfied to the left or to the right of the crossing. By convention, crossings between fixed-point sheets are considered to be returns, but not departures or switches. See Figure 14 for an example.

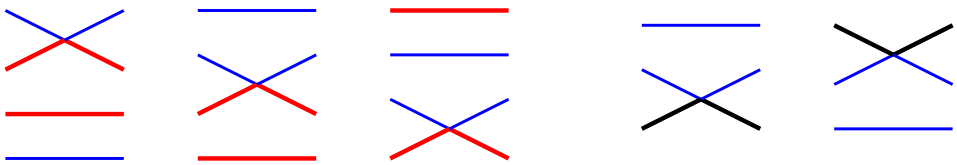


Figure 13. The normality condition at switches. Sheets of the same color are paired. The strands pictured in black are fixed-point sheets. Crossings between fixed-point sheets are not considered to be switches. (There may be any number of other strands above, below and in between the pictured ones.)

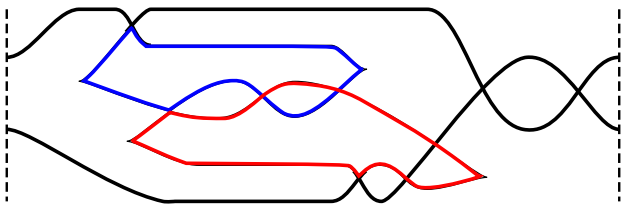


Figure 14. A generalized normal ruling of a Legendrian link in J^1S^1. The black strands are fixed-point sheets. From left to right, the crossings are two switches, a departure, a return, a switch, a departure, and three returns.

Remark 4.9. The terminology *generalized normal ruling* is from [29] where its purpose is to highlight the difference from ordinary normal rulings where the involutions σ_{x_0} are required to be fixed point free, e.g., in [18,37]. In this article, we will not have much need for the fixed point free requirement and will sometimes use the shortened terminology *normal ruling* to refer to any generalized normal ruling whether or not the σ_{x_0} have fixed points.

4.4.2. Barannikov normal form. Let C be a $\mathbb{Z}/\rho$-graded $\mathbb{F}$-vector space with basis $\{e_1, \dots, e_n\}$, and let

$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$$

be an involution satisfying $|e_i| = |e_j| + 1$ if $\sigma(j) = i$ and $i < j$. A differential $d_\sigma : C^* \to C^{*+1}$ is said to be in *Barannikov normal form with respect to* σ if

$$d(e_j) = \begin{cases} \alpha_{i,j} e_i & \text{if } i = \sigma(j) < j, \\ 0 & \text{else,} \end{cases}$$

with $\alpha_{i,j} \neq 0$.

Proposition 4.10. *For any upper triangular differential*

$$d : C^* \to C^{*+1}, \quad d^2 = 0, \quad d(e_j) \in \mathrm{Span}_{\mathbb{F}}\{e_1, \dots, e_{j-1}\} \text{ for all } j,$$

there exists a unique involution $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ *and a (non-unique) grading preserving upper triangular change of basis of the form*

$$\Phi(e_j) = e_j + \sum_{i<j} \beta_{i,j} e_j$$

such that $\Phi \circ d \circ \Phi^{-1}$ *is in Barannikov normal form with respect to* σ.

Proof. The existence of Φ and σ can be proven using induction on n. The involution σ is uniquely determined by the homology of the subquotients $H^*(C_{\leq j}/C_{\leq i})$, where $C_{\leq j}$ is the sub-complex spanned by $e_1, \dots, e_j$. See [6, Lemma 2] for a detailed proof. ∎

Remark 4.11. In [6], Barannikov also analyzed the way that the involution σ can change when the order of two generators is interchanged. This is the key ingredient in a construction of normal rulings from generating families due to Chekanov and Pushkar; see, e.g., [19, 37] for more discussion. Henry generalized this in [23, 37] to the following map from equivalence classes of MCFs to normal rulings.

Given an MCF $\mathcal{C}$ for a 1-dimensional Legendrian $\Lambda \subset J^1M$, define a family of involutions $\sigma(\mathcal{C})_{x_0} : \Lambda \cap \pi_x^{-1}(\{x_0\}) \to \Lambda \cap \pi_x^{-1}(\{x_0\})$ for $x_0 \in M_{\text{reg}}^{\mathcal{C}}$ by taking $\sigma(\mathcal{C})_{x_0}$ to be the involution of the Barannikov normal form of $d_{x_0} : C_{x_0} \to C_{x_0}$.

Proposition 4.12. *The construction*

$$\mathcal{C} \mapsto \sigma(\mathcal{C})$$

defines a mapping from the set of ρ*-graded MCFs to the set of* ρ*-graded generalized normal rulings of* Λ*. Moreover, the mapping is well defined on equivalence classes of MCFs.*

Proof. See [23, Lemma 3.14 and Proposition 3.15] for Legendrians in $J^1\mathbb{R}$ with $\mathbb{F}_2$ coefficients. See also [29, Section 5.B] for a related statement for Legendrians in J^1S^1 again over $\mathbb{F}_2$. Barannikov's result [6, Lemma 4] establishes the normality condition at switches. Since this statement is valid over any *field*, the proof goes through over $\mathbb{F}$. ∎

4.4.3. *SR*-form MCFs with respect to a generalized normal ruling. Let $\Lambda \subset J^1M$ with $M = [0, 1]$ or $S^1 = [0, 1]/\{0, 1\}$. For each crossing, c, choose a small open interval $I_c \subset M$ containing the crossing and no other singularities of $\pi_{xz}(\Lambda)$. In the case $M = [0, 1]$, choose small half-open intervals I_0 and I_1 containing $x = 0$ and 1 and no crossings or cusps of Λ; when $M = S^1$, choose such an open interval I_0 containing $x = 0$. The intervals should be chosen small enough as to all be disjoint from one another.

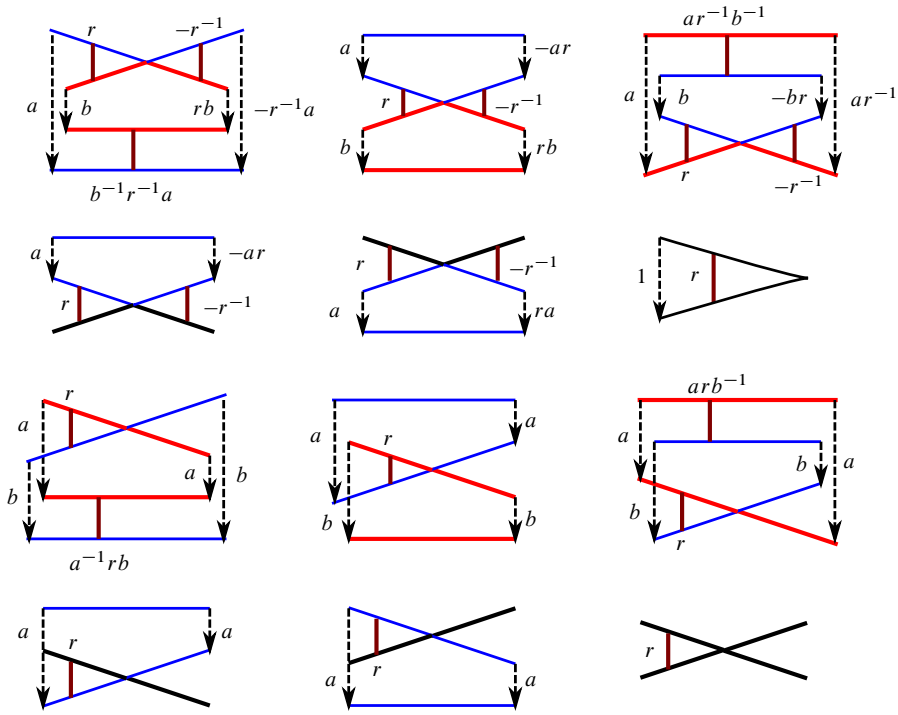


Figure 15. Handleslides near switches, right cusps, and returns for an SR-form MCF.

Definition 4.13. Let σ be a (ρ-graded) generalized normal ruling for Λ. An MCF $\mathcal{C}$ for Λ is in *SR-form with respect to* σ, if for some choice of intervals as above, the following conditions are satisfied:

(1) For all x not belonging to any of the intervals I_0, I_1, or I_c, the differential $d_x : C_x \to C_x$ is in Barannikov normal form with respect to σ_x.

(2) The only handleslides of $\mathcal{C}$ are arranged as follows.

  (i) In the intervals I_c when c is a switch or return collections of handleslides appear as in Figure 15.

  (ii) If $M = [0, 1]$, properly ordered handleslide collections appear in I_0 and I_1.

  (iii) If $M = S^1$, a properly ordered handleslide collection appears in I_0.

  (iv) If $\rho = 1$, i.e., if $(\Lambda, \mathcal{C})$ is 1-graded, then handleslides may appear at right cusps as in Figure 15.

Handleslide coefficients are required to be as indicated in Figure 15, and we emphasize that in the case of returns or right cusps $r = 0$ is allowed; cf. Remark 4.3.

As an example, the MCF from Figure 2 is in SR-form with respect to the normal ruling from Figure 14.

Proposition 4.14. *Any ρ-graded MCF, $\mathcal{C}$, for a 1-dimensional Legendrian $\Lambda \subset J^1M$ is equivalent to an SR-form MCF, $\mathcal{C}'$, for a ρ-graded generalized normal ruling, σ, of Λ. Moreover, if $M = [0,1]$ and $d_0 : C_0 \to C_0$ is already in Barranikov normal form, then $\mathcal{C}'$ can be constructed without any handleslide collection in the interval I_0 containing $x = 0$.*

Proof. For Legendrians in $J^1\mathbb{R}$ with $\mathbb{F}_2$ coefficients, this is proven in [23, Section 6]. In our proof, we pay particular attention to the changes required for Legendrians in $J^1[0,1]$ or J^1S^1 with $\mathbb{F}$ coefficients.

Let $(\Lambda, \mathcal{C}) \subset J^1M$ with $M = [0,1]$ or S^1. Beginning at $x = 0$, we apply the Proposition 4.10 to find an upper-triangular isomorphism $\Phi : C_0 \to C_0$ and a Barannikov normal form differential d_{σ_0} with $\Phi d_0 = d_\sigma \Phi$, where

$$\sigma_0 : \{S_1, \dots, S_{n_0}\} \to \{S_1, \dots, S_{n_0}\}$$

is an involution of the sheets of Λ at $x = 0$. Using Proposition 4.8, there exists a properly ordered handleslide collection $\mathcal{H}_0$ whose continuation map is Φ. Now, applying a sequence of $\leftarrow$ (E1) moves, we can modify $\mathcal{C}$ in the interval I_0 to begin with two handleslide collections $\mathcal{H}_0$ followed by $\mathcal{H}_1$ where $\mathcal{H}_1$ is the reflection of $\mathcal{H}_0$ across a vertical line with coefficients negated. We leave $\mathcal{H}_0$ in the interval I_0 and then follow Henry's procedure of "sweeping" $\mathcal{H}_1$ past the singularities of $(\Lambda, \mathcal{C})$ from left to right, identifying the generalized normal ruling σ as we go. Observe, if d_0 was already in Barranikov normal form, then we can take $\Phi = \mathrm{id}$ so that $\mathcal{H}_0 = \emptyset$ as claimed.

Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be so that x_i with $1 \le i \le N-1$ are the x-coordinates of the singularities of $(\Lambda, \mathcal{C})$. Suppose inductively that we have constructed a normal ruling σ of Λ on $[x_0, x_i)$ and an equivalence from $\mathcal{C}$ to $\mathcal{C}_i$ such that $\mathcal{C}_i$ satisfies the conditions of Definition 4.13 from x_0 up to the location of a properly ordered collection of handleslides $\mathcal{H}_i$ that is located in a small interval of the form $(x_i - \varepsilon, x_i)$. To extend this situation up to x_{i+1}, we consider some cases.

Case 1. x_i is a *handleslide*. Then, we simply group the new handleslide into $\mathcal{H}_i$, apply Proposition 4.8 (1) to produce a new properly ordered collection $\mathcal{H}_{i+1}$, and then slide $\mathcal{H}_{i+1}$ over to x_{i+1}.

Case 2. x_i is a *crossing* c between sheets S_k and S_{k+1}. We sweep the handleslides in $\mathcal{H}_i$ past x_i using (E3)–(E5) to become $\mathcal{H}_{i+1}$. If $\mathcal{H}_i$ has no $(k, k+1)$-handleslide, this can be done without issue, and σ is extended so that c is not a switch. If $\mathcal{H}_i$ does have a $(k, k+1)$-handleslide with coefficient $r \in \mathbb{F}^*$, it cannot be slid past the crossing; instead, it is left behind adjacently to the left of the crossing. All other

handleslides slide past the crossing to form part of $\mathcal{H}_{i+1}$. In doing so, additional handleslides may be produced from (E5) when the $(k, k+1)$-handleslide is passed, but these additional handleslides can then themselves be moved into $\mathcal{H}_{i+1}$. Next, if the normality condition for σ holds (resp., does not hold) at the left side of c, then σ extends so that c is a switch (resp., return). Move (E1) is then applied to produce the remaining required handleslides (with coefficients determined by r and $d_{x_i-\varepsilon}$) as in Figure 15. The use of the (E1) move produces handleslides in pairs, and the extra handleslides that are not needed for Figure 15 are then moved into $\mathcal{H}_{i+1}$, leaving the handleslides in I_c in the required form.

An important point is that with the handleslides and coefficients as in Figure 15, and with $d_{x_i-\varepsilon}$ known to be Barannikov form, a computation shows that d_x is also in Barannikov form for x to the right of I_c up to $\mathcal{H}_{i+1}$. The way the Barannikov form differential changes is indicated in Figure 15.

Case 3. x_i is a *basepoint*. We use (E6) to sweep all of the handleslides past the basepoint, changing their coefficients appropriately.

Case 4. x_i is a *left cusp*. We use (E3) to continuously sweep $\mathcal{H}_i$ past the cusp. The new cusp sheets are paired by σ.

Case 5. x_i is a *right cusp* connecting sheets S_k and S_{k+1}. If there is a $(k, k+1)$-handleslide, $h_{k,k+1}$, (this is only possible if $\rho = 1$), then pull it out of the left side of $\mathcal{H}_i$. Next, note that for any x-value within $\mathcal{H}_i$ we have $\langle d_x S_{k+1}, S_k\rangle = 1$. [This is true just to the left of the cusp by (A2), and passing handleslides cannot change the matrix coefficients of d_x corresponding to vertically adjacent sheets.] For each $i < k$, if there is an $(i, k+1)$-handleslide, $h_{i,k+1}$, in $\mathcal{H}_i$ with coefficient $b_{i,k+1} \neq 0$, then we apply a (i, k)-super-handleslide point, aka move $\leftarrow$(E7), at its location with coefficient $-b_{i,k+1}$. Because $\langle d_x S_{k+1}, S_k\rangle = 1$, this produces a new $(i, k+1)$-handleslide with coefficient $-b_{i,k+1}$ that we use to cancel $h_{i,k+1}$ and possibly some additional (i', k)- and (i, j)-handleslides with $i' < i$ and $k+1 < j$. Similarly, for each $k+1 < j$, if there is a $h_{k,j}$ handleslide in $\mathcal{H}_i$, then we use a $(k+1, j)$-super-handleslide to cancel it, possibly introducing some additional (i, j)- and $(k+1, j')$-handleslides with $i < k$ and $j < j'$. At this point, there are no $(i, k+1)$- or (k, j)-handleslides left in $\mathcal{H}_i$, so the only handleslides with endpoints on the cusp sheets are $h_{k,k+1}$ (off to the left, should it exist), and some number of (i, k)- and $(k+1, j)$-handleslides. We now sweep all other handleslides past the right cusp using (E3). Possibly, some additional (i, k)- and $(k+1, j)$-handleslides are created during this process, (but it is important to note that no new $(i, k+1)$- or (k, j)-handleslides appear). Now, we use (E2) to group the remaining (i, k)- and $(k+1, j)$-handleslides so that for each $i < k$ and $k+1 < j$ there is at most one such handleslide. We argue that the coefficients, $b_{i,k}$ and $b_{k+1,j}$, of all such handleslides actually have to be 0.

The reason is that, since d_x is in Barannikov form for x to the left of $h_{k,k+1}$ and $\mathcal{H}_i$, $\langle d_x S_{k+1}, S_k\rangle$ is the only non-zero matrix coefficient involving S_k and S_{k+1}. Thus, for such d_x, we have $\langle d_x S_{k+1}, S_i\rangle = 0$ for all $i < k$. Therefore, one sees that after passing through the remaining handleslides of $\mathcal{H}_i$, we get $\langle d_{x'} S_{k+1}, S_i\rangle = b_{i,k}$ for x' immediately to the left of the cusp. Similarly, for $k+1 < j$ we get

$$\langle d_{x'} S_k, S_j\rangle = -b_{k+1,j}.$$

Now, (A3) tells us that these matrix coefficients of $d_{x'}$ have to be zero. At this point, the only handleslide remaining to the left of the right cusp is $h_{k,k+1}$ (should it exist), and its positioning is as allowed in Figure 15.

Once we get to $x_i = x_N$, the construction is completed as follows: in the $J^1[0,1]$ case, $\mathcal{H}_N$ just becomes the properly ordered handleslide collection at $x = 1$. In the J^1S^1 case, we combine $\mathcal{H}_N$ and $\mathcal{H}_0$ into a single group of handleslides and properly order them with Proposition 4.8. ∎

4.5. Augmented cobordism and Legendrian isotopy

Recall that when $\Lambda_t, 0 \le t \le 1$ is a Legendrian isotopy in J^1M there is a Legendrian cobordism $\Sigma : \Lambda_0 \to \Lambda_1$ in $J^1(M \times [0,1])$ such that the t slices of the front projection of Σ coincide with the front projections of the Λ_t. See, e.g., [4, Section 3.3].

Proposition 4.15. *Let $(\Lambda_0, \mathcal{C}_0) \subset J^1M$ be an augmented Legendrian, and let Λ_t, $0 \le t \le 1$ be a generic Legendrian isotopy with associated Legendrian cobordism $\Sigma \subset J^1([0,1] \times M)$. Then, there exist MCFs $\mathcal{C}$ and $\mathcal{C}_1$ for Σ and Λ_1, respectively, such that $(\Sigma, \mathcal{C})$ is an augmented Legendrian cobordism from $(\Lambda_0, \mathcal{C}_0)$ to $(\Lambda_1, \mathcal{C}_1)$.*

Proof. See item (B2) in [35, Section 7.2] for a proof over $\mathbb{F}_2$. From Proposition 4.1, we know that the MCF can be extended during a Reidemeister move of type (R0), (R2), or (R3) as long as there are no handleslides or basepoints in the interval where the move occurs. Using the equivalence and basepoint moves (E1)–(E7) and (F1)–(F4), it is straightforward to arrange this. [Considerations may be simplified by first applying Proposition 4.14 to arrange that $\mathcal{C}_0$ is in SR-form. For the (R2) $\leftarrow$ move reflected in the horizontal direction, in the case that there is a handleslide at the second crossing (necessarily a return of any normal ruling), a super-handleslide move (E7) can be applied to remove the handleslide as in [35].] To perform (R1) $\rightarrow$, we can first apply (F1) to create the necessary spin basepoint. To perform (R1) $\leftarrow$, note that it is enough to arrange for handleslides to be positioned as in the statement of Lemma 4.4, as then the coefficients are correct in order for the move to be performed. With $\mathcal{C}_0$ in SR-form, the handleslides are in the correct positions, except in the case where $\rho = 1$ when there may be an additional handleslide adjacent to the right cusp and connecting

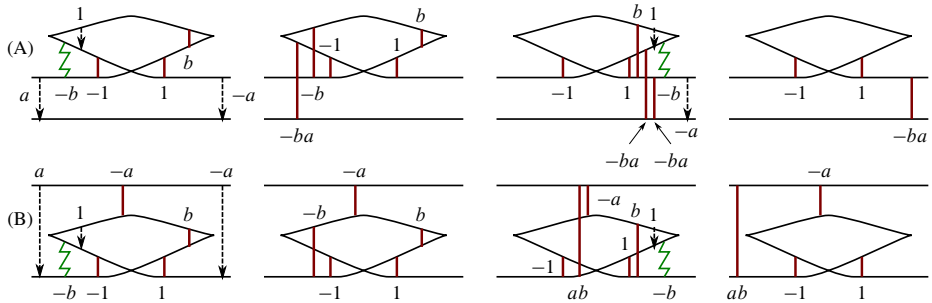


Figure 16. Moving a handleslide at a right cusp in an SR-form MCF away from an (R1) move. Two cases (A) and (B) are pictured depending on the configuration of the normal ruling at the (R1) move. The green lightning bolts are super-handleslides. A similar cobordism (not pictured) applies in the case where the bottom strand of the (R1) move is a fixed-point strand. The vertical reflection is similar.

the two cusp sheets. In this case, Figure 16 indicates how to move this handleslide out of the way, should it exist, arranging the positioning of Lemma 4.4. ∎

5. Cobordism invariants of augmented Legendrians

In this section, we record a complete set of cobordism invariants for ρ-graded augmented Legendrians in J^1S^1 and (full) n-tangles in $J^1[0,1]$. The spin invariant, defined in the case when $\operatorname{Char}\mathbb{F} \neq 2$ and ρ is even, is an element of $\mathbb{Z}/2$ determined by the combinatorial spin structure of $(\Lambda, \mathcal{C})$. A richer invariant, the graded monodromy matrix, arises from the continuation maps of Section 2.4. In the J^1S^1 case, only the conjugacy class of the monodromy matrix is a well-defined invariant.

5.1. Full augmented Legendrian n-tangles

Let $n \geq 0$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n) \in (\mathbb{Z}/\rho)^n$. We say that a ρ-graded Legendrian n-tangle, $\Lambda \subset J^1[0,1]$, has *boundary Maslov potential* $\vec{\mu}$ if on the n boundary points of Λ at $x = 0$ and $x = 1$ the Maslov potential satisfies

$$\mu(S_i^0) = \mu(S_i^1) = \mu_i \quad \text{for } 1 \leq i \leq n.$$

Definition 5.1. An augmented Legendrian n-tangle $(\Lambda, \mathcal{C})$ is *full* if the differentials of $\mathcal{C}$ at $x = 0$ and $x = 1$ are both 0; i.e., $d_0 : C_0 \to C_0$ and $d_1 : C_1 \to C_1$ have $d_0 = d_1 = 0$. For fixed n and $\vec{\mu}$, we write $\operatorname{Leg}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$ for the set of ρ-graded full augmented Legendrian n-tangles with boundary Maslov potential $\vec{\mu}$ and $\operatorname{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$ for the corresponding set of cobordism classes.

Observe that concatenation induces a well-defined operation on $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$:

$$[(\Lambda_1,\mathcal{C}_1)]\cdot[(\Lambda_2,\mathcal{C}_2)] = [(\Lambda_1 * \Lambda_2, \mathcal{C}_1 * \mathcal{C}_2)],$$

with Λ_1 placed to the left of Λ_2 when forming $\Lambda_1 * \Lambda_2$; since the augmented tangles are full the differentials from $\mathcal{C}_1$ and $\mathcal{C}_2$ agree where Λ_1 and Λ_2 are joined. Moreover, for $(\Lambda,\mathcal{C}) \in \mathrm{Leg}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$, we can form its *closure*, $(\widehat{\Lambda},\widehat{\mathcal{C}}) \subset J^1S^1$, by identifying $x=0$ with $x=1$ and gluing corresponding sheets of Λ. This provides a well-defined closure map on cobordism classes, $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F}) \to \mathrm{Cob}^{\rho}(J^1S^1;\mathbb{F})$.

Proposition 5.2. *For any ρ and $\vec{\mu}$, $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$ is a group with respect to concatenation.*

Proof. Clearly, the augmented tangle consisting of n parallel strands is an identity element $\mathbf{1}_n$. The inverse for an augmented n-tangle $(\Lambda,\mathcal{C})$ in $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$ is provided by the *reverse*, $(\overline{\Lambda},\overline{\mathcal{C}})$, obtained from reflecting the front projection of $(\Lambda,\mathcal{C})$ across the vertical line $x=1/2$ and negating (resp., inverting) all coefficients of handleslides (resp., basepoints). [A cobordism from $(\Lambda,\mathcal{C}) * (\overline{\Lambda},\overline{\mathcal{C}})$ to $\mathbf{1}_n$ arises from applying a combination of the moves (E1) $\rightarrow$, (F1) $\rightarrow$, (C3) $\leftarrow$, (C2) $\rightarrow$, and (C1) $\leftarrow$ at the location where the concatenation occurs to successively cancel each singularity of $(\Lambda,\mathcal{C})$ with its reflection in $(\overline{\Lambda},\overline{\mathcal{C}})$. Moreover, $(\overline{\Lambda},\overline{\mathcal{C}}) * (\Lambda,\mathcal{C}) = (\overline{\Lambda},\overline{\mathcal{C}}) * (\overline{\overline{\Lambda}},\overline{\overline{\mathcal{C}}}) \sim \mathbf{1}_n$ follows also.] ∎

5.2. Spin invariants

When $\mathrm{Char}\,\mathbb{F} \neq 2$ and the grading $\rho \in \mathbb{Z}_{\geq 0}$ is even (in particular, this implies that ρ-graded cobordisms are orientable), we can associate a *spin invariant*, $\xi(\Lambda,\mathcal{C}) \in \mathbb{Z}/2$, to an augmented Legendrian with coefficients in $\mathbb{F}$ as follows:

- For $(\Lambda,\mathcal{C}) \subset J^1S^1$, define $\xi(\Lambda,\mathcal{C})$ as the mod 2 sum

$$\xi(\Lambda,\mathcal{C}) = \#(\text{spin basepoints}) + \#(\text{right cusps}) + \#(\text{components of } \Lambda).$$

- For $(\Lambda,\mathcal{C}) \in \mathrm{Leg}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$, define

$$\xi(\Lambda,\mathcal{C}) = \xi(\widehat{\Lambda},\widehat{\mathcal{C}}) + n.$$

Proposition 5.3. *The spin invariant $\xi(\Lambda,\mathcal{C}) \in \mathbb{Z}/2$ depends only on the cobordism class of $(\Lambda,\mathcal{C})$.*

Proof. From Proposition 4.2, it suffices to establish that $\xi(\Lambda,\mathcal{C})$ is unchanged when one of the moves (E1)–(E7), (F1)–(F4), (R0)–(R3), and (C1)–(C3) is performed. All of the quantities involved in the definition of ξ are unchanged by any of the

moves (E1)–(E7), (F2)–(F4), (R0), (R2)–(R3), and (C3). In move (F1), the number of spin basepoints increases or decreases by 2. In (R1), a spin basepoint disappears but is replaced with a new right cusp. In (C1), the number of right cusps and components both decrease by 1 when the unknotted component is removed. Finally, when (C2) $\to$ is performed, the number of right cusps decreases by 1, and the number of components must either decrease or increase by 1 depending on whether the two cusps belonged to different components or the same component before the move. [In the case that they belong to the same component, it is crucial that the grading ρ is even so that the two parallel strands on the right side of (C2) have opposite orientations. If the orientations were the same, then the number of components would not be changed.] ■

Remark 5.4. The reason for the addition of n in defining ξ in the $J^1[0,1]$ case is so that the spin invariant will provide a group homomorphism $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F}) \to \mathbb{Z}/2$.

5.3. Monodromy matrices

Given $\vec{\mu} = (\mu_1, \ldots, \mu_n)$, we write

$$\mathbf{n} : \mathbb{Z}/\rho \to \mathbb{Z}_{\geq 0}, \quad \mathbf{n}(l) = \#\{i \mid \mu_i = l\}.$$

For $(\Lambda, \mathcal{C}) \in \mathrm{Leg}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$, $\mathbf{n}$ is the *graded dimension* of the fiber cohomology

$$\mathbf{n}(l) = \dim H^l(C_x, d_x), \quad l \in \mathbb{Z}/\rho,$$

for $x \in [0,1]^{\mathcal{C}}_{\mathrm{reg}}$. [This is clear at $x = 0$ or 1 since $d_0 = d_1 = 0$ as $(\Lambda, \mathcal{C})$ is full, and the fiber homology is independent of x with isomorphisms provided by the continuation maps.] For $m = 0, 1$, we have an ordered basis $[S_1^m], \ldots, [S_n^m] \in H^*(C_m, d_m)$ giving rise to ordered bases in each graded component $H^l(C_m, d_m)$ for $l \in \mathbb{Z}/\rho$. To $(\Lambda, \mathcal{C})$ we associate the *graded monodromy matrix*

$$\mathbf{M}_{\Lambda,\mathcal{C}} \in \mathrm{GL}(\mathbf{n}, \mathbb{F}) := \prod_{l \in \mathbb{Z}/\rho} \mathrm{GL}(\mathbf{n}(l), \mathbb{F}),$$

that is, the matrix with respect to the above bases of the continuation isomorphism (as defined in Section 2.4),

$$\phi_{\Lambda,\mathcal{C}} : H^*(C_0, d_0) \to H^*(C_1, d_1),$$

associated to the path $\sigma(t) = t$ from 0 to 1 in $[0,1]$. That is, $\mathbf{M}_{\Lambda,\mathcal{C}}(l) \in \mathrm{GL}(\mathbf{n}(l), \mathbb{F})$ is the matrix of

$$\phi_{\Lambda,\mathcal{C}} : H^l(C_0, d_0) \to H^l(C_1, d_1)$$

in cohomological grading l.

Remark 5.5. When it is convenient, we can also view $\mathbf{M}_{\Lambda,\mathcal{C}}$ as the single $n \times n$ matrix $M_{\Lambda,\mathcal{C}} \in \mathrm{GL}(n, \mathbb{F})$ of $\phi_{\Lambda,\mathcal{C}}$ considered as a linear map

$$\bigoplus_l H^l(C_0, d_0) \to \bigoplus_l H^l(C_1, d_1).$$

The individual $\mathbf{M}_{\Lambda,\mathcal{C}}(l)$ are the sub-matrices of $M_{\Lambda,\mathcal{C}}$ obtained from the rows and columns corresponding to sheets with Maslov potential $\mu(S_m^k) = l$. This amounts to embedding $\mathrm{GL}(\mathbf{n}, \mathbb{F}) = \prod_{l \in \mathbb{Z}/\rho} \mathrm{GL}(\mathbf{n}(l), \mathbb{F}) \subset \mathrm{GL}(n, \mathbb{F})$ as the subgroup of degree 0 linear automorphisms.

Proposition 5.6. *If $(\Lambda_i, \mathcal{C}_i)$ for $i = 0, 1$ are cobordant full augmented Legendrian n-tangles in $J^1[0, 1]$, then their monodromy matrices are equal, $\mathbf{M}_{\Lambda_0,\mathcal{C}_0} = \mathbf{M}_{\Lambda_1,\mathcal{C}_1}$.*

Proof. Given a cobordism $(\Sigma, \mathcal{C}) \subset J^1([0, 1] \times [0, 1])$ from $(\Lambda_0, \mathcal{C}_0) \subset J^1([0, 1] \times \{0\})$ to $(\Lambda_1, \mathcal{C}_1) \subset J^1([0, 1] \times \{1\})$, the continuation maps for $(\Sigma, \mathcal{C})$ associated to the two paths around the boundary of $[0, 1] \times [0, 1]$ from $(0, 0)$ to $(1, 1)$ agree with $\phi_{\Lambda_0,\mathcal{C}_0}$ and $\phi_{\Lambda_1,\mathcal{C}_1}$, respectively. [The continuation maps for the vertical edges $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$ are the identity map, because these edges do not intersect any singularities of $(\Sigma, \mathcal{C})$.] Thus, Proposition 2.11 implies that $\phi_{\Lambda_0,\mathcal{C}_0} = \phi_{\Lambda_1,\mathcal{C}_1}$. ∎

For an augmented Legendrian $(\Lambda, \mathcal{C})$ in $J^1(S^1)$, we have a monodromy map

$$\phi_{\Lambda,\mathcal{C}} : H^*(C_0, d_0) \to H^*(C_0, d_0),$$

that is, the continuation map on homology for the loop

$$\sigma : [0, 1] \to S^1 = [0, 1]/\{0, 1\}, \quad \sigma(t) = t.$$

In this case, we form the *graded monodromy matrix* $\mathbf{M}_{\Lambda,\mathcal{C}} \in \mathrm{GL}(\mathbf{n}, \mathbb{F})$, where $\mathbf{n}$ is the graded dimension of the fiber cohomology of $\mathcal{C}$, by making a choice of basis of $H^l(C_0, d_0)$ for each $l \in \mathbb{Z}/\rho$. Note that $\mathbf{M}_{\Lambda,\mathcal{C}}$ is well defined up to conjugacy in $\mathrm{GL}(\mathbf{n}, \mathbb{F})$.

Proposition 5.7. *If two augmented Legendrians $(\Lambda_0, \mathcal{C}_0)$ and $(\Lambda_1, \mathcal{C}_1)$ in J^1S^1 are cobordant, then their fiber cohomologies have the same graded dimension $\mathbf{n}$ and their monodromy matrices $\mathbf{M}_{\Lambda_0,\mathcal{C}_0}$ and $\mathbf{M}_{\Lambda_1,\mathcal{C}_1}$ are conjugate in $\mathrm{GL}(\mathbf{n}, \mathbb{F})$.*

Proof. If $(\Sigma, \mathcal{C}) \subset J^1(S^1 \times [0, 1])$ is a cobordism between $(\Lambda_0, \mathcal{C}_0)$ and $(\Lambda_1, \mathcal{C}_1)$, then the continuation map, ϕ_{σ_0}, associated to the path $\sigma_0(t) = (0, t)$ provides an isomorphism between the fiber cohomologies of $\mathcal{C}_0$ and $\mathcal{C}_1$. Moreover, using Proposition 2.11, we have

$$\phi_{\Lambda_0,\mathcal{C}_0} = \phi_{\sigma_0}^{-1} \circ \phi_{\Lambda_1,\mathcal{C}_1} \circ \phi_{\sigma_0}.$$

Choosing bases, we see that the monodromy matrices are conjugate. ∎

6. Computation of cobordism classes in $J^1[0,1]$

As in previous sections, the augmented Legendrians under consideration, $(\Lambda, \mathcal{C})$, are ρ-graded Legendrians equipped with MCFs. In this section, we establish the following classification of full augmented Legendrian n-tangles in $J^1[0,1]$ up to cobordism. Recall the notations $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F})$ for the set of cobordism classes of full augmented n-tangles with boundary Maslov potential $\vec{\mu}$ with concatenation; $\xi(\Lambda, \mathcal{C})$ and $\mathbf{M}_{\Lambda,\mathcal{C}}$ are the spin and monodromy matrix invariants. We use $\mathrm{GL}(\mathbf{n}, \mathbb{F})^{\mathrm{op}}$ to denote $\mathrm{GL}(\mathbf{n}, \mathbb{F})$ with the opposite operation, $\mathbf{A} \cdot \mathbf{B} = \mathbf{BA}$.

Theorem 6.1. *Let $n \geq 0$ and $\vec{\mu} \in (\mathbb{Z}/\rho)^n$.*

(1) *When* $\mathrm{Char}\,\mathbb{F} \neq 2$ *and ρ is even, the map*

$$\begin{aligned} \Phi : \mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F}) &\xrightarrow{\cong} \mathrm{GL}(\mathbf{n}, \mathbb{F})^{\mathrm{op}} \times \mathbb{Z}/2 \\ [(\Lambda, \mathcal{C})] &\mapsto (\mathbf{M}_{\Lambda,\mathcal{C}}, \xi(\Lambda, \mathcal{C})) \end{aligned}$$

is a group isomorphism.

(2) *When* $\mathrm{Char}\,\mathbb{F} = 2$*, except in the case when $n = 0$ and $\rho = 1$, the map*

$$\begin{aligned} \Phi : \mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F}) &\xrightarrow{\cong} \mathrm{GL}(\mathbf{n}, \mathbb{F})^{\mathrm{op}} \\ [(\Lambda, \mathcal{C})] &\mapsto \mathbf{M}_{\Lambda,\mathcal{C}} \end{aligned}$$

is a group isomorphism.

(3) *When $n = 0$ and $\rho = 1$,* $\mathrm{Cob}^{\rho}_{\vec{\mu}}(J^1[0,1];\mathbb{F}) = \{[(U, \mathcal{C}_b)] \mid b \in \mathbb{F}\}$*, where $(U, \mathcal{C}_b)$ is the standard Legendrian unknot having a single handleslide with coefficient b as in Figure 19.*

Remark 6.2. In Theorem 6.1 (3), it should be emphasized that it is not known whether the $[(U, \mathcal{C}_b)]$ are all distinct or not; for any $b \in \mathbb{F}$, the monodromy matrix is 0. We conjecture that they are distinct simply because we have not been able to construct a cobordism between $(U, \mathcal{C}_{b_1})$ and $(U, \mathcal{C}_{b_2})$ when $b_1 \neq b_2$.

The proof of Theorem 6.1 (appearing at the end of the section) is based on several ingredients. From Proposition 5.6, we have that Φ is a well-defined map. Next, Proposition 6.3 will establish the surjectivity of Φ. Injectivity requires more work and is established according to the following outline. After a preparatory discussion about positive braids, we introduce a class of standard-form augmented Legendrian n-tangles that for $n \geq 1$ are certain *simple positive permutation braids* with special MCFs. To show that Φ is injective, we show that (i) any two standard-form augmented Legendrians with the same monodromy matrix and spin invariant (when defined) are cobordant (in fact, isotopic), and (ii) an arbitrary full augmented Legendrian

n-tangle is cobordant to one in standard form. See Corollary 6.14 and Proposition 6.15, respectively.

6.1. Surjectivity of Φ

We call a (full augmented) Legendrian n-tangle in $J^1[0,1]$ without cusps a (*full augmented*) *Legendrian n-braid*.

Proposition 6.3. *For any boundary Maslov potential $\vec{\mu}$, and any graded matrix $\mathbf{P} \in \mathrm{GL}(\mathbf{n}, \mathbb{F})$, there exists a full augmented Legendrian n-braid $(\Lambda, \mathcal{C}) \in \mathrm{Leg}^{\rho}_{\vec{\mu}}(J^1[0,1]; \mathbb{F})$ with monodromy matrix $\mathbf{M}_{\Lambda,\mathcal{C}} = \mathbf{P}$. Moreover, when* $\mathrm{Char}\,\mathbb{F} \neq 2$ *and ρ is even, the spin invariant $\xi(\Lambda, \mathcal{C}) \in \mathbb{Z}/2$ may be arbitrary.*

Proof. First, considering the case where the grading is concentrated in a single degree, we prove that any matrix $P \in \mathrm{GL}(n, \mathbb{F})$ can be realized as the monodromy matrix of a full augmented Legendrian $(\Lambda, \mathcal{C}) \subset J^1[0,1]$ whose boundary strands all have the same grading, $\mu_i = l$. Given $P \in \mathrm{GL}(n, \mathbb{F})$, the usual Gauss–Jordan elimination algorithm decomposes P into a product of the following types of elementary matrices used in the algorithm for row interchanges, row additions, and row scaling:

(i) $Q_{(i\,j)}$ for $i < j$: obtained from interchanging the i-th and j-th rows of the identity matrix;

(ii) $I + bE_{i,j}$ for $i < j$, $b \in \mathbb{F}$;

(iii) $\Delta_i(s)$ for $1 \leq i \leq n$, $s \in \mathbb{F}$: diagonal matrices with s at the (i,i)-position and 1's at the other diagonal entries.

[Note that $I + bE_{j,i}$ with $i < j$ can be obtained as $Q_{(i\,j)}(I + bE_{i,j})Q_{(i\,j)}$.] Any such elementary matrix is the monodromy matrix of a full augmented Legendrian n-tangle: (i) A single crossing between the i-th and $(i+1)$-th strands with no handleslides has monodromy matrix $Q_{(i\,i+1)}$, and so, arbitrary permutation matrices, e.g., the $Q_{(i\,j)}$, can be realized by concatenating crossings. (ii) The matrix $I + bE_{i,j}$ is the monodromy matrix of a trivial n-braid having a single handleslide with coefficient b connecting the i-th and j-th strands. (iii) A basepoint on the i-th strand with coefficient s (with the left-to-right co-orientation) has monodromy matrix $\Delta_i(s)$. Concatenating appropriately then produces $(\Lambda, \mathcal{C})$ with

$$M_{\Lambda,\mathcal{C}} = P.$$

Now, for general $\vec{\mu}$ and $\mathbf{P} \in \mathrm{GL}(\mathbf{n}, \mathbb{F})$, we proceed as follows. For each graded part $\mathbf{P}(l)$ of $\mathbf{P}$, from Step 1, we have an augmented braid $(\Lambda(l), \mathcal{C}(l))$ with all the braid strands having grading $n(l)$ and with monodromy matrix $\mathbf{P}(l)$. Stack the $(\Lambda(l), \mathcal{C}(l))$ on top of one another so that the boundary strand gradings are non-decreasing from

top to bottom, and call this augmented braid $(\widetilde{\Lambda}, \widetilde{\mathcal{C}})$; its graded monodromy matrix is $\mathbf{P}$. The boundary Maslov potential $\vec{\mu}'$ of $(\widetilde{\Lambda}, \widetilde{\mathcal{C}})$ is a reordering of $\vec{\mu}$. We then form a concatenation:

$$(\Lambda, \mathcal{C}) := (B, \mathcal{C}_B) * (\widetilde{\Lambda}, \widetilde{\mathcal{C}}) * (\overline{B}, \overline{\mathcal{C}_B}).$$

Here, B is a positive braid with left (resp., right) boundary Maslov potential $\vec{\mu}$ (resp., $\vec{\mu}'$) such that all the crossings of the B are between strands with different potentials; the MCF $\mathcal{C}_B$ on B has no handleslides and all $d_x = 0$; and $(\overline{B}, \overline{\mathcal{C}_B})$ denotes the reverse of $(B, \mathcal{C}_B)$. The concatenation $(\Lambda, \mathcal{C})$ now has the correct boundary Maslov potential $\vec{\mu}$ and has graded monodromy matrix $\mathbf{P}$ since the crossings of B and $\overline{B}$ between strands with different potentials do not affect the graded components of the monodromy matrix.

For the remaining claim regarding $\xi(\Lambda, \mathcal{C})$, note that adding a pair of consecutive basepoints consisting of a spin point and a homology basepoint with coefficient -1 will change the value of the spin invariant without affecting the monodromy matrix. ■

6.2. Aside: Simple positive permutation braid words

A Legendrian n-braid may be written as a product (left-to-right concatenation) of positive braid generators, $\sigma_1, \ldots, \sigma_{n-1}$, of the braid group $\mathcal{B}_n$ where we view σ_i as the Legendrian n-tangle consisting of a single crossing between strands i and $i+1$. Moreover, it is a result of Garside [21, Theorem 4] that two positive braid words w_1 and w_2 are equal in $\mathcal{B}_n$ if and only if they can be related using the braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \ge 2$ in the monoid generated by the σ_i *without ever needing to introduce inverses of the* σ_i. As the relations arise from Legendrian isotopies, we see that two positive braid words that are equal in $\mathcal{B}_n$ represent Legendrian isotopic Legendrian braids.

A *positive permutation braid* is a positive braid such that each pair of strands intersects at most once. There is a one-to-one correspondence between positive permutation braids (as elements of $\mathcal{B}_n$) and elements of the symmetric group S_n; see [15]. In particular, for each $\pi \in S_n$, there is a *Legendrian permutation braid* B_π well defined up to Legendrian isotopy. Here, we use the convention[1] that the permutation $\pi \in S_n$ associated to a braid in B_n is the permutation of strands read from *left to right*; i.e., the strand that starts at position i at $x = 0$ ends in position $\pi(i)$ at $x = 1$.

One way to characterize Legendrian permutation braids is that if a braid is not a permutation braid, then after a Legendrian isotopy it contains a product σ_i^2, i.e., an eye-shaped region as in the clasp move (C3). This follows from [15, Proposition 2.4],

[1] This convention results in the map from B_n to S_n being an anti-homomorphism rather than a homomorphism.

by making use of the above correspondence from positive braids in B_n to Legendrian isotopy classes.

Lemma 6.4 ([15]). *Let B_π be a Legendrian permutation braid for the permutation $\pi \in S_n$. If the i-th and $(i+1)$-th strands at $x = 1$ intersect in the braid, then there is a Legendrian isotopy from B_π to $B'\sigma_i$, where B' is another Legendrian permutation braid.*

Corollary 6.5. *If a Legendrian braid B is not a permutation braid, then it is Legendrian isotopic to $B'\sigma_i^2 B''$ for some braid generator σ_i and Legendrian braids B' and B''.*

Proof. Write B as a braid word $B = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}$. If B is not a positive permutation braid, there is a k such that $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{k-1}}$ is a positive permutation braid but $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ is not. Thus, the i_k and i_k+1 strands (as labelled at the right side) cross in $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{k-1}}$. By Lemma 6.4, $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{k-1}}$ is Legendrian isotopic to $B'\sigma_{i_k}$, and thus, B is Legendrian isotopic to the required form. ∎

Definition 6.6. We call a Legendrian permutation braid *simple* if it is *not* possible to arrange for the product $\sigma_{i+1}\sigma_i\sigma_{i+1}$ to appear in its braid word by applying a sequence of commutation relations, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \neq 1$.

A similar notion called *reduced* positive permutation braid (which instead forbids $\sigma_i\sigma_{i+1}\sigma_i$) is used by Kálmán in [27].

Proposition 6.7. *Any Legendrian permutation braid is Legendrian isotopic to one that is simple.*

Proof. Start with any positive permutation braid word w. If w is not simple, then after a sequence of commutation relations the product $\sigma_{i+1}\sigma_i\sigma_{i+1}$ shows up. Change it to $\sigma_i\sigma_{i+1}\sigma_i$ via the braid relation and then repeat the process. Note that the braid relation decreases the sum of all the subscripts j of braid generators σ_j appearing in the word, while the commutation relation preserves this sum. Thus, the procedure must stop at some point, resulting in a simple braid word. ∎

Intuitively, a Legendrian permutation braid is simple if it only admits type (b) but not type (a) triangles in Figure 17. The following lemma makes this statement precise.

Lemma 6.8. *Let B_π be a simple Legendrian permutation braid for a permutation $\pi \in S_n$. Denote by $c_{i,j}$ a crossing between the strands labeled i and j at the left boundary of B_π. For any $i < j < k$ such that $\pi(k) < \pi(j) < \pi(i)$, the crossing $c_{i,j}$ appears to the left of the crossing $c_{j,k}$ (as in the braid shown in Figure 17 (b)).*

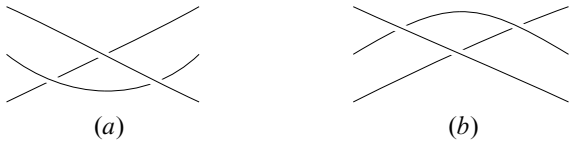


Figure 17. Simple braids only admit the type (b) triangles but not type (a) triangles.

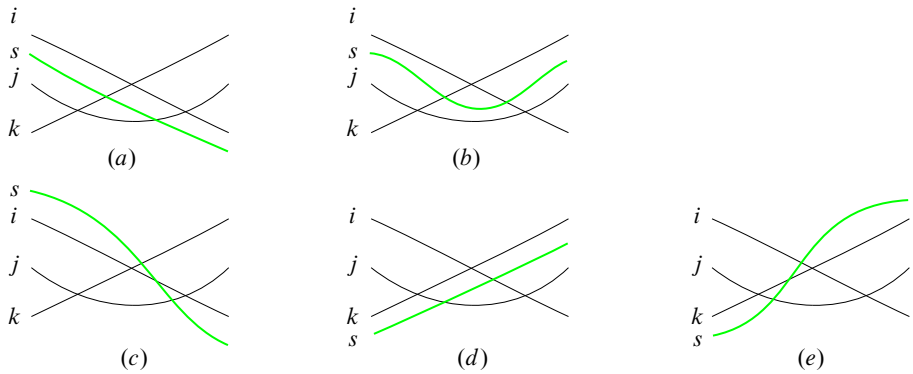


Figure 18. Possible cases for a strand s to pass the triangular region formed by strands i, j, and k.

Proof. Suppose that $B_\pi = \sigma_{i_1} \cdots \sigma_{i_m}$, and assume that $\sigma_{i_1} \cdots \sigma_{i_{\ell-1}}$ satisfies the above condition but $\sigma_{i_1} \cdots \sigma_{i_\ell}$ does not. Say the crossing σ_{i_ℓ} is $c_{i,j}$ with $i < j$. Among all the strands k' such that the triple (i, j, k') violates the condition in Lemma 6.8 (i.e., $i < j < k'$ and $\pi(k') < \pi(j) < \pi(i)$, but the crossing $c_{i,j}$ happens on the right of the crossing $c_{j,k'}$); take the strand such that the crossing $c_{j,k'}$ is the closest one to the crossing $c_{i,j}$ in the braid word B_π; call this strand k. Thus, the strands i, j, and k form a triangular region as shown in Figure 17 (a).

We claim that no other strand intersects the triangular region. The claim can be proved by contradiction. If there was a strand s that intersects the triangular region, there are three cases as follows.

- The strand s enters the triangular region from the upper left edge. If the strand goes out of the triangular region by crossing the bottom edge as shown in Figure 18 (a), which implies that $s < j$, then the triple (s, j, k) violates the condition and thus contradicts the choice of $c_{i,j}$. Otherwise, the strand s leaves the triangular region along the upper right edge as shown in Figure 18 (b). Then, (i, s, k) forms another triple that violates the condition and again contradicts the choice of $c_{i,j}$.
- The strand s enters the triangular region from the upper right edge, which implies that $s < i$ and thus $s < j < k$. The strand s can only leave the triangular region along the bottom edge as shown in Figure 18 (c). Therefore, we have $\pi(s) > \pi(j)$

and thus $\pi(s) > \pi(k)$, which implies that strand s and strand k intersect. Thus, the triple (s, j, k) violates the condition and contradicts the choice of $c_{i,j}$.

- The strand s enters the triangular region from the bottom edge as shown in Figure 18 (d) and (e), which implies that $s > j$ and $\pi(s) < \pi(j)$. Thus, $s > i$ and $\pi(s) < \pi(i)$, so the triple (i, j, s) violates the condition. This contradicts the choice of k.

With the claim proven, we know that both $c_{j,k}$ and $c_{i,j}$ correspond to the braid generator σ_{i_ℓ} and $c_{i,k}$ corresponds to the braid generator $\sigma_{i_\ell - 1}$. Moreover, $B_\pi = B_1\sigma_{i_\ell}B_2\sigma_{i_\ell-1}B_3\sigma_{i_\ell}B_4$, where B_2 and B_3 are products of σ_t with $t \neq i_\ell - 1, i_\ell, i_\ell + 1$. Thus, applying commutations, we arrive at $B_1B_2\sigma_{i_\ell}\sigma_{i_\ell-1}\sigma_{i_\ell}B_3B_4$, showing that B_π is not simple. ∎

Using the lemma, we can prove a uniqueness statement for simple braid words.

Proposition 6.9. *Any two simple Legendrian permutation braids with the same permutation $\pi \in S_n$ are related by commutation relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geq 2$.*

Proof. Suppose that we have two simple Legendrian permutation braids B_1, B_2 that give the same permutation π. We can assume that they are different on the first braid letter, say, $B_1 = \sigma_i B_1'$, $B_2 = \sigma_k B_2'$, and $i < k$. By considering the location of the crossing $c_{k,k+1}$ in B_1, we will show that one can use the commutation relation to move $c_{k,k+1}$ to be the first crossing of B_1. One can then repeat the argument on the new B_1' and B_2' to inductively complete the proof.

First, note that $k \neq i + 1$. [If $k = i + 1$, then we would have $\pi(i + 2) < \pi(i + 1) < \pi(i)$. Then, Lemma 6.8 would fix an order for the crossings $c_{i,i+1}$ and $c_{k,k+1}$ in any simple permutation braid for π so that they cannot both be the first crossing of a simple permutation braid.] In order to show that $c_{k,k+1}$ can be moved to the leftmost of B_1, we only need to prove that no strand intersects the strands k or $k + 1$ on the left of $c_{k,k+1}$ in B_1. On that one hand, since $c_{k,k+1}$ is the left most crossing of B_2, Lemma 6.8 prevents any strand s for $s < k$ from intersecting strand k in any simple positive permutation braid for π. On the other hand, as $c_{k,k+1}$ is a crossing in B_1, Lemma 6.8 prevents any strand s for $s > k + 1$ to intersect strand $k + 1$ on the left of $c_{k,k+1}$ in B_1. These two facts permit us to move $c_{k,k+1}$ to the leftmost in B_1 using the commutation relations. ∎

6.3. Standard-form augmented braids

Definition 6.10. For $n \geq 1$, a full augmented Legendrian n-braid denoted by $(B_\pi, \mathcal{C}) \in \mathrm{Leg}^\rho_{\vec{\mu}}(J^1[0, 1]; \mathbb{F})$ is called a *standard form* in case of the following:

- B_π is a simple Legendrian permutation braid for a permutation $\pi \in S_n$.

Figure 19. (Left) The 1-graded augmented Legendrian unknots $(U, \mathcal{C}_b)$, $b \in \mathbb{F}$. (Right) A Legendrian unknot with spin invariant 1.

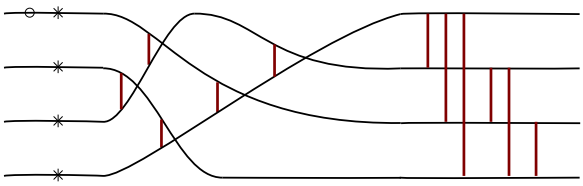


Figure 20. An example of a standard-form full augmented braid with permutation $\pi = (1\,3\,2\,4)$. Coefficients of handleslides (resp., basepoints) are not pictured, but they may be any elements of $\mathbb{F}$ (resp., $\mathbb{F}^*$).

- Handleslides of $\mathcal{C}$ are located as follows: (i) a handleslide (possibly with coefficient 0) appears directly to the left of each crossing of B_π, connecting the crossing strands, and (ii) a *properly ordered handleslide collection* (see Definition 4.7), $\mathcal{H}_1$, appears to the right of all crossings of B_π.
- Near $x = 0$, each strand contains one homology basepoint (possibly with coefficient 1), and the top strand contains either one or zero spin basepoints located to the left of all homology basepoints.

For $n = 0$, we say an augmented Legendrian 0-tangle, $(\Lambda, \mathcal{C})$, is a *standard form* if either

(i) $\operatorname{Char}\mathbb{F} \neq 2$, and either $\Lambda = \emptyset$ or $(\Lambda, \mathcal{C})$ is the augmented Legendrian unknot with spin invariant 1 as pictured in Figure 19,

(ii) $\operatorname{Char}\mathbb{F} = 2$, $\rho \neq 1$, and $\Lambda = \emptyset$, or

(iii) $\operatorname{Char}\mathbb{F} = 2$, $\rho = 1$, and $(\Lambda, \mathcal{C}) = (U, \mathcal{C}_b)$ is as in Figure 19 for some $b \in \mathbb{F}$.

See Figure 20 for an example of a standard-form augmented 4-braid. Note that for $n \geq 1$, the MCFs of standard forms are in SR-form with respect to the normal ruling σ_0 of B_π for which every strand is a fixed-point strand; all crossings of B_π are returns of σ_0.

Next, we compute the monodromy matrices of standard-form augmented Legendrians. In the statement, we use the perspective of Remark 5.5 and view the graded monodromy matrix $\mathbf{M}_{B_\pi,\mathcal{C}} \in \mathrm{GL}(\mathbf{n}, \mathbb{F})$ as a single matrix $M_{B_\pi,\mathcal{C}} \in \mathrm{GL}(n, \mathbb{F})$. We reprise the notation $c_{i,j}$ for a crossing between strands i and j (as labeled at $x = 0$) of a permutation braid.

Proposition 6.11. *For $n \geq 1$, the monodromy matrix of a standard-form full augmented n-braid $(B_\pi, \mathcal{C})$ satisfies*

$$M_{B_\pi,\mathcal{C}} = RT_\pi \Delta \Xi,$$

where

- *$R = I + \sum_{i<j} b_{i,j} E_{i,j}$ is upper-triangular with ones on the diagonal and $b_{i,j} \in \mathbb{F}$ is the coefficient of the (i, j)-handleslide in $\mathcal{H}_1$;*
- *T_π is obtained from the permutation matrix $Q_\pi = \sum_j E_{\pi(j),j}$ as*

$$T_\pi = Q_\pi + \sum_{(i,j)\in X_\pi} h_{i,j} E_{\pi(i),j}, \tag{6.1}$$

  where $X_\pi = \{(i, j) \mid i < j \text{ and } \pi(i) > \pi(j)\}$; i.e., the sum is over crossings of B_π, and $h_{i,j} \in \mathbb{F}$ is the coefficient of the handleslide to the left of $c_{i,j}$;
- *$\Delta = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, where $s_i \in \mathbb{F}^*$ is the coefficient of the homology basepoint on strand i with respect to the left-to-right co-orientation;*
- *$\Xi = \mathrm{diag}(\pm 1, 1, \ldots, 1)$ where the $(1, 1)$-entry is -1 (resp., 1) if the top sheet at $x = 1$ has (resp., does not have) a spin basepoint.*

Proof. Since the differentials of $\mathcal{C}$ are all zero, the continuation map on homology is the same as the chain level map which can be computed as

$$f_{\tau_4} \circ f_{\tau_3} \circ f_{\tau_2} \circ f_{\tau_1},$$

where from left to right τ_1 is the interval containing the spin basepoint (if it exists), τ_2 contains the homology basepoints, τ_3 contains the crossings of B_π and their associated handleslides, and τ_4 contains the collection of handleslides $\mathcal{H}_1$. It is clear from the definition (see (2.6)) that the matrices of f_{τ_1} and f_{τ_2} are Ξ and Δ, respectively, and Proposition 4.8 (2) shows that the matrix of f_{τ_4} is R.

It remains to show that the matrix of f_{τ_3} is

$$T_\pi = Q_\pi + \sum_{(i,j)\in X_\pi} h_{i,j} E_{\pi(i),j}.$$

We use induction on the word length of the braid B_π. When the word length is 0, the braid is a trivial braid and the monodromy matrix is the identity matrix, which satisfies the formula.

For the inductive step, suppose that $B_\pi = B_{\pi'}\sigma_k$, and let $\mathcal{C}'$ be the restriction of $\mathcal{C}$ to the part of the interval τ_3 containing $B_{\pi'}$. The monodromy matrix of $(B_{\pi'}, \mathcal{C}')$ has the stated form, $T_{\pi'}$, and the monodromy matrix for $(B_\pi, \mathcal{C})$ is

$$M = Q_{(k\,k+1)}(I + hE_{k,k+1})T_{\pi'},$$

(a) (b)

Figure 21. Two full augmented braids with the same monodromy matrix. In both cases, all handleslide coefficients are 1.

where $h := h_{\pi^{-1}(k+1),\pi^{-1}(k)}$ is the coefficient of the handleslide before σ_k, or zero if it does not exist,

$$\begin{aligned} M &= Q_{(k\,k+1)}(I + hE_{k,k+1})\Big(Q_{\pi'} + \sum_{(i,j)\in X_{\pi'}} h_{i,j}E_{\pi'(i),j}\Big) \\ &= (Q_{(k\,k+1)} + hE_{k+1,k+1})\Big(Q_{\pi'} + \sum_{(i,j)\in X_{\pi'}} h_{i,j}E_{\pi'(i),j}\Big). \end{aligned}$$

Note from Lemma 6.8 that there is no crossing between the strand $(\pi')^{-1}(k+1)$ and any strand s for $s > (\pi')^{-1}(k+1)$. Thus, for $(i,j) \in X_{\pi'}$, the term $\pi'(i)$ cannot be $k+1$, so we have

$$\begin{aligned} M &= Q_\pi + hE_{k+1,(\pi')^{-1}(k+1)} + \sum_{(i,j)\in X_{\pi'}} h_{i,j}E_{\pi(i),j} \\ &= Q_\pi + hE_{\pi(\pi^{-1}(k+1)),\pi^{-1}(k)} + \sum_{(i,j)\in X_{\pi'}} h_{i,j}E_{\pi(i),j} \\ &= Q_\pi + \sum_{(i,j)\in X_\pi} h_{i,j}E_{\pi(i),j}. \end{aligned}$$

This completes the proof. ∎

Remark 6.12. (i) The matrices T_π have the following property: besides the 1's from the permutation matrix Q_π, all additional non-zero entries appear in positions (i,j) that are both to the right of the 1 from Q_π in row i and below the 1 from Q_π in column j.

(ii) If B_π is allowed to be a Legendrian permutation braid that is *not simple*, then the monodromy matrix is not in the form shown in Proposition 6.11. For example, Figure 21 shows two full augmented braids with the same monodromy matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The braid in part (b) is simple and the monodromy matrix satisfies the formula in Proposition 6.11, while the braid in part (a) is not simple and the monodromy matrix does not have the form of T_π.

Next, we prove a uniqueness statement for matrix decompositions as in Proposition 6.11 without the spin term, Ξ. This is related to the Bruhat decomposition.

Lemma 6.13. *Let $R_i, T_i, \Delta_i \in \mathrm{GL}(n, \mathbb{F})$ for $i = 1, 2$ be such that*

- *R_i is upper triangular with 1's on the diagonal;*
- *T_i has the form (6.1) for some permutation $\pi_i \in S_n$;*
- *Δ_i is diagonal.*

If $R_1T_1\Delta_1 = R_2T_2\Delta_2$, then $R_1 = R_2$, $T_1 = T_2$ (in particular, $\pi_1 = \pi_2$), and $\Delta_1 = \Delta_2$.

Proof. Assume that $R_1T_1\Delta_1 = R_2T_2\Delta_2$, and write

$$T_1 = RT_2\Delta \quad \text{where } R = R_1^{-1}R_2 \text{ and } \Delta = \Delta_2\Delta_1^{-1}.$$

Case 1. $\pi_1 = \pi_2$. Write $\pi = \pi_1 = \pi_2$, and assume for contradiction that $R \neq I$. Choose $i < j$ such that $r_{i,j} \neq 0$ and $r_{i,j'} = 0$ for all $j < j' \leq n$ where $R = (r_{i,j})$. Then, the $(i, \pi^{-1}(j))$-entry of $RT_2\Delta$ is non-zero. [This is because the $(j, \pi^{-1}(j))$-entry of $T_2\Delta$ is the top most non-zero entry in $\mathrm{col}_{\pi^{-1}(j)}(T_2\Delta)$; see Remark 6.12.] This contradicts the fact that the $(i, \pi^{-1}(j))$-entry of T_1 is zero. Thus, $R = I$, and considering the $(\pi(j), j)$-entries in the identity $T_1 = T_2\Delta$ shows that $\Delta = I$ also.

Case 2. $\pi_1 \neq \pi_2$. This case leads to a contradiction. Let k be chosen such that $\pi_1^{-1}(k) \neq \pi_2^{-1}(k)$. Without loss of generality, we may assume that $\pi_1^{-1}(k) > \pi_2^{-1}(k)$; i.e., the 1 in $\mathrm{row}_k Q_{\pi_1}$ is to the right of the 1 in $\mathrm{row}_k Q_{\pi_2}$. Now, we have

$$\mathrm{row}_k T_1 = \mathrm{row}_k(T_2\Delta) + \sum_{k<i} r_{k,i}\mathrm{row}_i(T_2\Delta). \tag{6.2}$$

Note that the set of i such that $k < i$ and $r_{k,i} \neq 0$ is non-empty since $\mathrm{row}_k T_1 \neq \mathrm{row}_k(T_2\Delta)$ as the first non-zero entries in these rows are located in the same position as in Q_{π_1} and Q_{π_2}, respectively. Among these values of i, let i' be the one having $\pi_2^{-1}(i)$ smallest, i.e., such that the first non-zero entry of $\mathrm{row}_i(T_2\Delta)$ is farthest to the left. Note that we must have $\pi_2^{-1}(i') < \pi_2^{-1}(k)$ as well. [Otherwise, all entries below the (non-zero) $(k, \pi_2^{-1}(k))$-entry of $T_2\Delta$ and having non-zero $r_{k,i}$ coefficient in (6.2) would be 0. This would show that the $(k, \pi_2^{-1}(k))$-entry of T_1 is non-zero, contradicting the fact that $\pi_2^{-1}(k) < \pi_1^{-1}(k)$.] Thus, the $\pi_2^{-1}(i')$ entry of the row vector on the right side of (6.2) is non-zero, contradicting the fact that the corresponding entry on the left vanishes since $\pi_2^{-1}(i') < \pi_1^{-1}(k)$. ∎

Corollary 6.14. *Suppose that either $n \geq 1$ or $n = 0$ and $\rho \neq 1$. Two standard-form ρ-graded augmented Legendrian n-tangles with the same spin invariant (when defined) and monodromy matrix are cobordant.*

Proof. In the special case of 0-tangles, this is obvious from the definition. Given two such standard-form augmented n-braids $(B_{\pi_1}, \mathcal{C}_1)$ and $(B_{\pi_2}, \mathcal{C}_2)$ with $n \geq 1$, computing their monodromy matrices as in Proposition 6.11, we have

$$R_1 T_1(\Delta_1 \Xi_1) = R_2 T_2(\Delta_2 \Xi_2).$$

Applying Lemma 6.13, we see that $R_1 = R_2$, $T_1 = T_2$ (in particular, $\pi_1 = \pi_2$), and $\Delta_1 \Xi_1 = \Delta_2 \Xi_2$. Moreover, since $\pi_1 = \pi_2$, the closures of B_{π_1} and B_{π_2} have the same number of components. Therefore, the equality of spin invariants, $\xi(B_{\pi_1}, \mathcal{C}_1) = \xi(B_{\pi_2}, \mathcal{C}_2)$, implies that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same number of spin basepoints (this number being 1 or 0). Thus, $\Xi_1 = \Xi_2$, so $\Delta_1 = \Delta_2$ as well. As the entries of the individual matrices R, T_π, Δ, and Ξ in Proposition 6.11 together uniquely determine a standard form $(B_\pi, \mathcal{C})$ (up to applying commutation moves to braid crossings, in particular, up to cobordism), this completes the proof. ∎

6.4. Constructing cobordisms to standard form

The final ingredient before proving Theorem 6.1 is the following.

Proposition 6.15. *Any full augmented Legendrian n-tangle is cobordant to one in standard form.*

We establish Proposition 6.15 in two steps. First, Proposition 6.16 handles the case $n = 0$ and shows that when $n \geq 1$ we can remove cusps by an augmented cobordism; this transforms a given tangle with $n \geq 1$ into an augmented Legendrian braid. Then, in Proposition 6.17, we show that any augmented Legendrian braid is cobordant to one in standard form. Clearly, the combination of these two statements proves Proposition 6.15.

Proposition 6.16. *Assume that either ρ is even or* $\operatorname{char} \mathbb{F} = 2$.

(1) *When $n \geq 1$, any* full *augmented Legendrian n-tangle in $J^1[0,1]$ is cobordant to one without cusps, i.e., to an augmented Legendrian braid.*

(2) *Any augmented* 0*-tangle is cobordant to a standard-form* 0*-tangle.*

Proof. We apply the inductive procedure from [35, Section 7.3], used there for constructing cobordisms from augmented Legendrians in $\mathbb{R}^3$ over $\mathbb{F}_2$ to the empty set, and address the important new issues arising from (i) the setting of n-tangles instead of links in $\mathbb{R}^3$ and (ii) the coefficients in a general field $\mathbb{F}$ together with the presence of homology and spin basepoints.

The statement is established by induction on the number of cusps, c, with the $c = 0$ case being tautological. Let $(\Lambda, \mathcal{C}) \subset J^1[0,1]$ be an n-tangle with $n \geq 0$ such that Λ has at least one cusp, and suppose that (1) and (2) are established for Legendrians with fewer cusps. Let l be the left cusp of Λ with largest x-coordinate. We can write Λ as a product of tangles in the form

$$\Lambda = Xl\sigma_{k-1}\sigma_{k-2}\cdots\sigma_{k-s}\sigma_{k+1}\sigma_{k+2}\cdots\sigma_{k+t}Y, \tag{6.3}$$

where the sheets of l are numbered k and $k+1$ and $s, t \geq 0$. (Initially, we can take $s = t = 0$, i.e., $\Lambda = XlY$.) Then, we apply induction on the word length $|Y|$, i.e., the number of crossings and right cusps in Y, to establish (1) and (2) for Legendrians of the form (6.3) with the current value of c. In our $J^1[0,1]$ setting, it is crucial to note that the base case $|Y| = 0$ is vacuous since Y contains at least one right cusp. [This is because the two sheets of l cancel in the homology of (C_x, d_x), so immediately to the right of l the number of sheets of Λ is larger than the total dimension of $H^*(C_x, d_x) \cong H^*(C_1, d_1)$ which is n since $\mathcal{C}$ is *full*. Thus, there must be a right cusp between l and $x = 1$.]

The inductive (on $|Y|$) step is carried out by writing

$$Y = zY'$$

and considering cases (numbered 1–10 in [35]) based on the type (right cusp or crossing) and location of the singularity of z. In all cases, it is shown how to construct an augmented cobordism from $(\Lambda, \mathcal{C})$ to an augmented Legendrian $(\Lambda', \mathcal{C}')$ such that Λ' either can be written in the form (6.3) with Y replaced by Y' or has fewer left cusps than Λ. For many of the cases (specifically for Cases 1, 4, 5b, 6, 7, 9, and 10b) the underlying Legendrian cobordism is a Legendrian isotopy, and in our setting of $\mathbb{F}$ coefficients, we apply Proposition 4.15 to extend $\mathcal{C}$ over the cobordism. Cases 3 and 10a are possibilities for z that cannot actually occur since Λ would have a zig-zag stabilization and hence could not have any MCF. This point remains valid for MCFs over $\mathbb{F}$.

In Cases 2, 5c, 8, and 10c, the first step in [35] is to apply an equivalence to put $\mathcal{C}$ into SR-form with respect to a normal ruling for Λ. In our setting, we first move all homology and spin basepoints away from the $l\sigma_{k-1}\sigma_{k-2}\cdots\sigma_{k-s}\sigma_{k+1}\sigma_{k+2}\cdots\sigma_{k+t}z$ part of Λ, and then we apply Proposition 4.14 to arrange the SR-form. We should keep in mind that the normal ruling may now be *generalized*. The construction of the cobordism then depends on whether the crossing σ_{k-1} that appears next to l is a departure or a switch. [Note: Even with a *generalized* normal ruling, it is still impossible for σ_{k-1} to be a return since the cusp strands are not fixed-point strands.] If the crossing is a departure, the cusp tangency move pictured in Figure 22 is applied, noting that SR-form MCFs have no handleslides at departures. If the crossing is a switch,

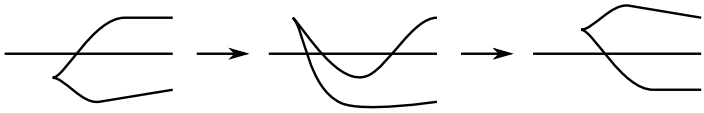


Figure 22. The cusp tangency move is a combination of (R2) and the clasp move (C3). (No handleslides are present.)

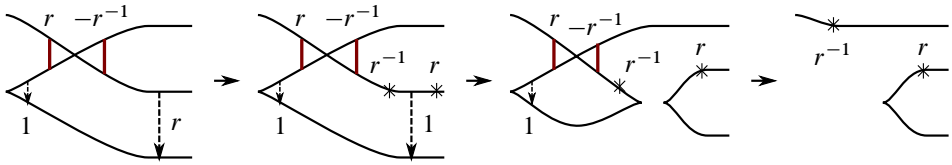


Figure 23. Cobordism to remove a switch next to a left cusp. The coefficient r is non-zero. There may be one additional unpictured handleslide at the switch with upper endpoint on the lower sheet of the cusp, if the switch is in the first configuration pictured in Figure 15.

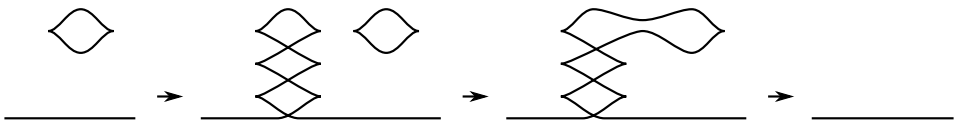


Figure 24. Cobordism to remove an unknot, U, in the presence of at least one additional component, Λ'. Note that U may contain basepoints or (if $\rho = 1$) handleslides. The 1st and 3rd arrows are augmented cobordisms constructed from Legendrian isotopies via Proposition 4.15. The (R1) moves are performed so that the Maslov potentials agree when applying the (C2) $\rightarrow$ move.

then a cobordism involving the pinch move (C2) is applied to remove the switch. The modification of this cobordism to $\mathbb{F}$ coefficients, allowing for the first handleslide coefficient at the switch to be arbitrary, makes use of an added homology curve and is pictured in Figure 23. Note that because the differentials of an SR-form MCF are in Barannikov normal form, the pictured arrow at the location of the pinch move, $x = x_0$, is the only matrix coefficient of d_{x_0} involving the sheets of the pinch move. Thus, the pinch move (C2) $\leftarrow$ can be applied once the basepoints are introduced to make the coefficient in the differential become 1.

In the last remaining Case 5A, the left cusp l is part of a Legendrian unknot, U, disjoint from the rest of Λ. The new issue here is that U may contain basepoints and, if $\rho = 1$, a handleslide in its interior. If Λ has another component, Λ', then we use the cobordism pictured in Figure 24 to absorb U into Λ'. This decreases the number of cusps of Λ by 1 so that the outer induction on c applies. In the remaining case where $\Lambda = U$, we must have $n = 0$. Using (F1)–(F4), we move all of the homology

and spin basepoints together so that we have at most 1 basepoint of each type. The coefficient of the homology basepoint is necessarily $(-1)^{s_0}$, where s_0 is the number of spin basepoints. [This is necessary in order for (A2) to be satisfied at both the left and right cusps of U.] If $s_0 = 1$ and $\Lambda = U$, we are done, as $(\Lambda, \mathcal{C})$ is now the standard form pictured on the right side of Figure 19. [There is no handleslide since the presence of spin basepoints implies $\operatorname{char} \mathbb{F} \neq 2$ so that we have required that ρ is even.] If $s_0 = 0$, then the homology basepoint has coefficient 1 and can be erased; see Remark 4.3. Then, if U has no handleslide, we apply (C1) $\leftarrow$ to obtain a cobordism to $\emptyset$. If there is a handleslide, then $\rho = 1$ and $(\Lambda, \mathcal{C})$ has the standard form pictured on the left side of Figure 19. ∎

Proposition 6.17. *Any full augmented Legendrian n-braid $(B_0, \mathcal{C}_0)$ in $J^1[0,1]$ with $n \geq 1$ is cobordant to a standard form $(B_\pi, \mathcal{C})$.*

Proof. *Step* 1. Construct a cobordism from $(B_0, \mathcal{C}_0)$ to $(B_\pi, \mathcal{C}_1)$, where B_π is a simple permutation braid.

To ease considerations, we begin by using (F1)–(F4) to move all basepoints (homology and spin) to the far left side of B_0. According to Corollary 6.5, if B_0 is not a positive permutation braid, we can apply a Legendrian isotopy to B_0 to arrange for σ_i^2 to appear in its braid word. As usual, Proposition 4.15 makes this Legendrian isotopy into an augmented cobordism. One can then remove either one or two crossings from the σ_i^2 part of B via an augmented Legendrian cobordism: if there is a handleslide in the middle of the two crossings and connecting the crossing strands with coefficient $r \in \mathbb{F}^*$, then we apply the D_4^- cobordism from Proposition 4.6, reducing the number of crossings by 1. If not, a clasp move (C3) $\leftarrow$ applies to remove both crossings. Repeating this procedure inductively, we arrive at $(B_1, \mathcal{C}_1)$ with B_1 a permutation braid, and then applying Proposition 6.7, we may arrange that $B_1 = B_\pi$ is a simple permutation braid.

Step 2. Move the handleslides of $\mathcal{C}_1$ into standard form position.

To do this, we simply apply Proposition 4.14. Note that since $(B_\pi, \mathcal{C}_1)$ is full, the differential at $x = 0$ is $d_0 = 0$ which is in Barannikov normal form. Thus, $\mathcal{C}_1$ is equivalent to an SR-form, $\mathcal{C}_2$, without any properly ordered handleslide collection at the $x = 0$ side of the braid.

Step 3. Move the basepoints of $(B_\pi, \mathcal{C}_2)$ into the standard form position.

For each strand of B_π, we move the homological and spin basepoints to the far left and collect them together so that each strand has at most 1 basepoint of each type. To arrange the standard form, we need to move all of the spin basepoints to the top strand. This is done using a sequence of cobordisms as in Figure 25. ∎

We now complete the proof of Theorem 6.1.

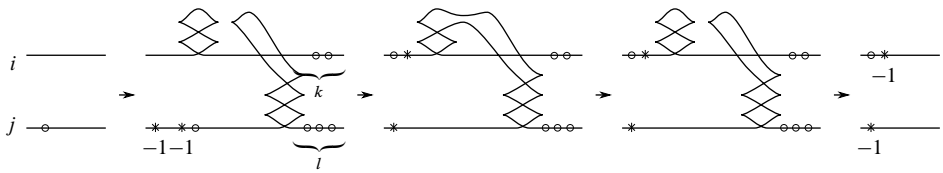


Figure 25. The figure shows how to move a spin basepoint near $x = 0$ up to the next adjacent sheet, introducing in the process an additional homology basepoint with coefficient -1 on both sheets. The Maslov potential on the two sheets is i and j, and $k, l > 0$ should be chosen so that $i + k = j + l \bmod \rho$. There are handleslides (unpictured) and new spin points arising from the $k + l$ (R1) moves that are performed. When the pair of basepoints, one spin and one homology, are moved together, the coefficients of differentials and handleslides remain unchanged. Thus, the (C2) and (R1) moves can all be reversed at the 4-th arrow removing the extra $k + l$ spin points that were created at the 1-st arrow.

Proof of Theorem 6.1. The statement (3) about the case when $n = 0$ and $\rho = 1$ is Proposition 6.16 (2). For (1) and (2), we need to show that Φ is an isomorphism.

Φ is bijective. The surjectivity of Φ was established in Proposition 6.3. To establish injectivity, assume that two full augmented n-tangles $(\Lambda_1, \mathcal{C}_1)$ and $(\Lambda_2, \mathcal{C}_2)$ have the same monodromy matrix $M_1 = M_2$ and (if $\operatorname{Char}\mathbb{F} \neq 2$) the same spin invariant. By Proposition 6.15, we may assume that $(\Lambda_i, \mathcal{C}_i)$ are in standard form, and then Corollary 6.14 is applied to show that $[(\Lambda_1, \mathcal{C}_1)] = [(\Lambda_2, \mathcal{C}_2)]$.

Φ is a homomorphism. Let $[(\Lambda_1, \mathcal{C}_1)], [(\Lambda_2, \mathcal{C}_2)] \in \operatorname{Cob}^{\rho}_{\mu}(J^1[0,1]; \mathbb{F})$. From Proposition 2.11, we get that $\mathbf{M}_{\Lambda_1 * \Lambda_2, \mathcal{C}_1 * \mathcal{C}_2} = \mathbf{M}_{\Lambda_2, \mathcal{C}_2} \mathbf{M}_{\Lambda_1, \mathcal{C}_1}$. In the case that $\operatorname{Char}\mathbb{F} \neq 2$, we also need to show that the spin invariant ξ defines a homomorphism to $\mathbb{Z}/2$. For the case $n = 0$, this is clear as the operation on the cobordism group is just disjoint union. When $n \geq 1$, using Proposition 6.16, we may assume that Λ_1 and Λ_2 are Legendrian braids. Clearly, the number of spin basepoints of $(\Lambda_1 * \Lambda_2, \mathcal{C}_1 * \mathcal{C}_2)$ is the sum of that of the factors. As there are no right cusps, we just need to show that

$$c + n = (c_1 + n) + (c_2 + n) \mod 2,$$

where c, c_1, and c_2 are the numbers of components of the closure of $\Lambda_1 * \Lambda_2$, Λ_1, and Λ_2. The number of components, $c(b)$, of the closure of a braid $b \in B_n$ is the number of cycles in the cycle decomposition of the image of b under the standard homomorphism $\theta : B_n \to S_n$. Moreover, one computes that $c(b) + n \bmod 2$ is just the parity of $\theta(b)$ and hence is additive under concatenation. [If $\theta(b)$ factors into disjoint cycles of length $\ell_1, \ldots, \ell_{c(b)}$, then it factors into $(\ell_1 - 1) + \cdots + (\ell_{c(b)} - 1)$ transpositions. Thus, parity of $\theta(b) = (\ell_1 - 1) + \cdots + (\ell_{c(b)} - 1) = (\ell_1 + \cdots + \ell_{c(b)}) - c(b) = n + c(b) \pmod 2$.] ∎

7. Computation of cobordism classes in J^1S^1

In this short section, we obtain the following cobordism classification for augmented Legendrians in J^1S^1.

Theorem 7.1. *Assume that either ρ is even or* $\operatorname{Char}\mathbb{F} = 2$ *and* $\rho \neq 1$. *Then, two ρ-graded augmented Legendrians in J^1S^1 over $\mathbb{F}$ are cobordant if and only if*

- *they have the same spin invariant (if* $\operatorname{Char}\mathbb{F} \neq 2$*);*
- *their fiber cohomologies have the same graded dimension,*

$$\mathbf{n} : \mathbb{Z}/\rho \to \mathbb{Z}_{\geq 0};$$

- *their monodromy matrices are conjugate in* $\mathrm{GL}(\mathbf{n}, \mathbb{F})$.

Moreover, with $\mathrm{Cl}(\mathrm{GL}(\mathbf{n}, \mathbb{F}))$ *denoting conjugacy classes in* $\mathrm{GL}(\mathbf{n}, \mathbb{F})$*, we have bijections:*

(1) *When* $\operatorname{Char}\mathbb{F} \neq 2$ *(ρ is even),*

$$\Psi : \mathrm{Cob}^{\rho}(J^1S^1; \mathbb{F}) \xrightarrow{\cong} \Big(\bigsqcup_{\mathbf{n}} \mathrm{Cl}(\mathrm{GL}(\mathbf{n}, \mathbb{F}))\Big) \times \mathbb{Z}/2$$
$$[(\Lambda, \mathcal{C})] \mapsto ([\mathbf{M}_{\Lambda,\mathcal{C}}], \xi(\Lambda, \mathcal{C})).$$

(2) *When* $\operatorname{Char}\mathbb{F} = 2$ *and* $\rho \neq 1$,

$$\Psi : \mathrm{Cob}^{\rho}(J^1S^1; \mathbb{F}) \xrightarrow{\cong} \bigsqcup_{\mathbf{n}} \mathrm{Cl}(\mathrm{GL}(\mathbf{n}, \mathbb{F}))$$
$$[(\Lambda, \mathcal{C})] \mapsto [\mathbf{M}_{\Lambda,\mathcal{C}}].$$

The proof will be given after the following lemma. Recall the closure operation from Section 5.1.

Lemma 7.2. *Any augmented Legendrian in* $(\Lambda, \mathcal{C}) \subset J^1S^1$ *is cobordant to the closure of a* full *augmented n-tangle in* $J^1[0, 1]$ *whose boundary Maslov potential,*

$$\vec{\mu} = (\mu_1, \ldots, \mu_n),$$

is non-decreasing from top to bottom.

Proof. Applying Proposition 4.14, we can arrange that $(\Lambda, \mathcal{C})$ is in SR-form. Then, we can choose some $x = x_0$ where d_{x_0} is in Barannikov normal form. That is, the sheets of Λ at $x = x_0$ are divided into some number of pairs, $\{S_i, S_j\}$ satisfying $\partial S_j = rS_i$ for $i < j$ and $r \in \mathbb{F}^*$, and some number of fixed-point sheets, S_k with $\partial S_k = 0$. Now, for a given pair $\{S_i, S_j\}$, we apply the cobordism in Figure 26. We repeat this procedure until only the fixed-point strands are left. At this point, the differential d_{x_0}

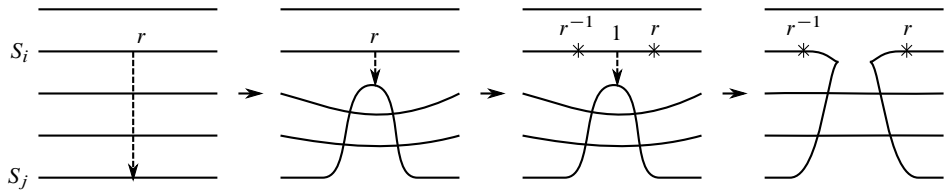


Figure 26. The differential satisfies $\langle d_{x_0} S_j, S_i \rangle = r \in \mathbb{F}^*$. Coefficients of basepoints are indicated with respect to the left-to-right co-orientation.

vanishes so that cutting along $x = x_0$ realizes $(\Lambda, \mathcal{C})$ as the closure of a full n-tangle. Moreover, since $d_{x_0} = 0$, we can apply clasp moves at $x = x_0$ to arrange that $\vec{\mu}$ is non-decreasing. ∎

Proof of Theorem 7.1. Surjectivity of Ψ follows from Theorem 6.1 since we can take the closure of an n-tangle with arbitrary monodromy matrix and spin invariant. To prove injectivity, suppose that $(\Lambda_1, \mathcal{C}_1)$ and $(\Lambda_2, \mathcal{C}_2)$ have equal spin invariants (if $\operatorname{Char} \mathbb{F} \neq 2$) and have monodromy matrices $\mathbf{M}_{\Lambda_i, \mathcal{C}_i} \in \mathrm{GL}(\mathbf{n}, \mathbb{F})$ that are conjugate:

$$\mathbf{M}_{\Lambda_1, \mathcal{C}_1} = P \mathbf{M}_{\Lambda_2, \mathcal{C}_2} P^{-1}.$$

Using Lemma 7.2, we can assume that each $(\Lambda_i, \mathcal{C}_i)$ is the closure of a full augmented n-tangle in $J^1[0, 1]$ that we will denote by the same notation and that the Λ_i have the non-decreasing boundary Maslov potential, $\vec{\mu}$, determined by $\mathbf{n}$. From Theorem 6.1, there exist two further full augmented n-tangles, $(\Lambda_P, \mathcal{C}_P)$ and $(\Lambda_{P^{-1}}, \mathcal{C}_{P^{-1}})$ in $\mathrm{Leg}^{\rho}_{\vec{\mu}}(J^1[0, 1]; \mathbb{F})$ with vanishing spin invariants (if $\operatorname{Char} \mathbb{F} \neq 2$) and monodromy matrices P and P^{-1}, respectively. Now, working with cobordism classes in $J^1 S^1$, we can compute

$$\begin{aligned}(\Lambda_2, \mathcal{C}_2) &= (\Lambda_P, \mathcal{C}_P) * (\Lambda_{P^{-1}}, \mathcal{C}_{P^{-1}}) * (\Lambda_2, \mathcal{C}_2) \\ &= (\Lambda_{P^{-1}}, \mathcal{C}_{P^{-1}}) * (\Lambda_2, \mathcal{C}_2) * (\Lambda_P, \mathcal{C}_P) \\ &= (\Lambda_1, \mathcal{C}_1),\end{aligned}$$

where the first and third equalities are from cobordisms in $J^1[0, 1]$ that arise from applying Theorem 6.1 and the second equality is an isotopy in $J^1 S^1$. ∎

Remark 7.3. When $\rho = 1$, the statement (and proof) that cobordism classes of 1-graded augmented Legendrians in $J^1 S^1$ are determined by the dimension of their fiber cohomology, $\mathbf{n}$, and the conjugacy class of the monodromy matrix remains true *provided that* $\mathbf{n} \neq 0$. As in the $J^1[0, 1]$ case, cf. Theorem 6.1, any 1-graded $(\Lambda, \mathcal{C}) \subset J^1 S^1$ with $\mathbf{n} = 0$ is cobordant to $(U, \mathcal{C}_b)$ for some $b \in \mathbb{F}$, but it is not known that the $(U, \mathcal{C}_b)$ are non-cobordant.

We close this section with a comparison between augmented Legendrian cobordism classes in J^1S^1 and ordinary Legendrian cobordism classes. The following is an easy consequence of the work of Arnold [2–4] and is likely well known, though we have not been able to locate an explicit discussion of the J^1S^1 case.[2]

Theorem 7.4. *Two oriented Legendrian links in J^1S^1 are oriented cobordant if and only if*

(i) *they have the same rotation number (taken to be the sum of the rotation numbers of the components);*

(ii) *they represent the same homology class in $H_1(J^1S^1)$.*

Proof. It is clear from various perspectives (e.g., moves on slices, or an argument with the Maslov class) that both the rotation number and the homology class are cobordism invariants. To prove the converse statement, we make use of Arnold's result that two Legendrians in $J^1\mathbb{R}$ are oriented cobordant if and only if they have the same rotation number. For Legendrians, $A, B \subset J^1S^1$, we write $A \sqcup B$ for A stacked above B in J^1S^1. Note that any $\Lambda \subset J^1S^1$ with $[\Lambda] = [\widehat{1_m}] \in H_1(J^1S^1)$ for $m \in \mathbb{Z}$ (positive m means oriented right; negative m means oriented left) is cobordant to $1_m \sqcup \Lambda'$ with $\Lambda' \subset J^1(0,1) \subset J^1S^1$ and $r(\Lambda) = r(\Lambda')$. To see this, use the cobordisms

$$\Lambda \sim 1_m \sqcup 1_{-m} \sqcup \Lambda \sim 1_m \sqcup \Lambda',$$

where both cobordisms are constructed as in Figure 26. [Without MCFs, the cobordism can be done between any two strands with opposite orientation. Moreover, since $1_{-m} \sqcup \Lambda$ is null homologous, the number of strands oriented left is equal to the number oriented right at any generic $x \in S_1$. Thus, we can remove all of the strands at some x resulting in Λ' as required.] Now, given any Λ_0, Λ_1 with the same homology class and rotation number, we have

$$\Lambda_0 \sim 1_m \sqcup \Lambda_0' \sim 1_m \sqcup \Lambda_1' \sim \Lambda_1,$$

where the second cobordism exists by the classification in $J^1\mathbb{R}$ since Λ_0' and Λ_1' have the same rotation number. ∎

As a consequence, there are many examples of augmented Legendrians $(\Lambda_i, \mathcal{C}_i)$, $i = 0, 1$ that are distinct in $\mathrm{Cob}^\rho(J^1S^1; \mathbb{F})$ with ρ even, but such that the underlying

[2] We remark that [4] does state the cobordism classification for Legendrians in $ST^*\mathbb{R}^2$, but despite the contactomorphism $ST^*\mathbb{R}^2 \cong J^1S^1$, the cobordism classification is *not* the same. The reason is that when working with Legendrians in $ST^*\mathbb{R}^2$ Arnold considers cobordisms that live in $ST^*(\mathbb{R}^2 \times [0,1])$, and this is not contactomorphic (or even homeomorphic) to $J^1(S^1 \times [0,1])$.

Legendrians Λ_0 and Λ_1 are oriented cobordant. In fact, in this case, Λ_0 and Λ_1 are oriented cobordant if and only if their fiber cohomologies have the same Euler characteristic. [The Euler characteristic determines the homology class of Λ in $H_1(J^1S^1) \cong \mathbb{Z}$; the existence of an even graded MCF implies the existence of an even graded generalized normal ruling, and as in [43, Section 4], it follows that the rotation number must be 0.] For example, the closure of the full 2-tangle, $(\mathbf{1}_2, \mathcal{C})$, with boundary Maslov potential $\mu = (0, 1)$ is nontrivial in $\mathrm{Cob}^\rho(J^1S^1; \mathbb{F})$, while $\widehat{\mathbf{1}_2}$ is oriented cobordant to $\emptyset$.

8. Augmented Legendrian surfaces with prescribed monodromy representation

As an application of the cobordism classification, we construct augmented Legendrian surfaces having arbitrary monodromy representations. Let M be a connected, compact surface, and fix a basepoint $x_0 \in \mathrm{Int}(M)$. We allow for cases where $\partial M \neq 0$ and/or M is non-orientable. Recall the definition of the monodromy representation from Section 2.4.

Theorem 8.1. *Assume that ρ is even or* $\mathrm{char}\,\mathbb{F} = 2$. *Let $\mathbf{n} : \mathbb{Z}/\rho \to \mathbb{Z}_{\geq 0}$ with $n = \sum_l \mathbf{n}(l)$ finite, and let $\Phi : \pi_1(M, x_0) \to \mathrm{GL}(\mathbf{n}, \mathbb{F})^{\mathrm{op}}$ be a group homomorphism. Then, there exists a ρ-graded augmented Legendrian surface $(\Sigma, \mathcal{C}) \subset J^1M$ whose monodromy representation satisfies $\Phi_{\Sigma,\mathcal{C}} = \Phi$.*

Proof. Fix a handle decomposition of M with a single 0-handle having x_0 at its center. Above the 0-handle, h_0, of M, define Σ so that the front projection consists of n non-singular, non-intersecting sheets, $S_1, \ldots, S_n$, with Maslov potential $\vec{\mu} = (\mu(S_1), \ldots, \mu(S_n))$ as specified by $\mathbf{n}$; i.e., there are n_l sheets with $\mu(S_i) = l$ and the Maslov potential is non-decreasing as i increases. Define $\mathcal{C}$ above h_0 to have $d_x \equiv 0$.

For each 1-handle h_1^i with core curve σ_i, we apply Theorem 6.1 to produce a full augmented n-tangle $(\Lambda_i, \mathcal{C}_i) \subset J^1\sigma_i$, with boundary Maslov potential $\vec{\mu}$ and vanishing spin invariant, having monodromy matrix $\mathbf{M}_{\Lambda_i,\mathcal{C}_i} = \Phi([\sigma_i])$. Define $(\Sigma, \mathcal{C})$ above h_1^i to be the product cobordism under the identification $J^1h_1^i \cong J^1(\sigma_i \times [0, 1])$.

For each 2-handle, $h_2 \cong D^2$, we already have $(\Sigma, \mathcal{C})$ defined above ∂D^2. Moreover, the monodromy matrix of $(\Lambda_\partial, \mathcal{C}_\partial) := (\Sigma, \mathcal{C})|_{\partial h_2}$ is the identity. [The attaching map $\gamma : S^1 \to M \setminus \mathrm{Int}(h_2)$ is homotopic in $M \setminus \mathrm{Int}(h_2)$ to a loop $\tilde{\gamma}$ that is product of the σ_i and their inverses. Since the continuation isomorphisms ϕ_{σ_i} along these loops have matrices $\Phi([\sigma_i])$, we get from Proposition 2.11 that the matrix of ϕ_γ is $\Phi([\tilde{\gamma}])$. The latter is the identity matrix since γ is null homotopic in M.] In addition, the spin invariant of $(\Lambda_\partial, \mathcal{C}_\partial)$ viewed as a full n-tangle is 0. Therefore, another application of

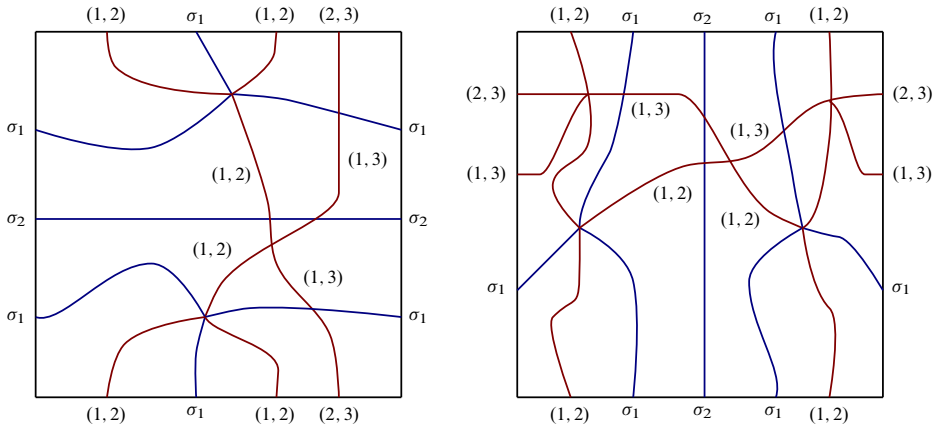


Figure 27. A blue curve labeled with σ_i denotes a crossing arc between sheets i and $i+1$. A red curve labeled with (i, j) denotes an (i, j)-handleslide with coefficient $1 \in \mathbb{F}_2$.

Theorem 6.1 produces a cobordism $(\Sigma_A, \mathcal{C}_A)$ from $(\Lambda_\partial, \mathcal{C}_\partial)$ to the identity augmented n-tangle. We then use $(\Sigma_A, \mathcal{C}_A)$ to fill $(\Sigma, \mathcal{C})$ into an annular neighborhood $A \subset h_2$ of ∂h_2, and complete the construction of $(\Sigma, \mathcal{C})$ by placing n non-intersecting sheets with vanishing d_x in the remaining disk $h_2 \setminus A$. By construction, on the generating set $[\sigma_i] \in \pi_1(M, x_0)$, we have $\Phi_{\Sigma,\mathcal{C}}([\sigma_i]) = \mathbf{M}_{\Lambda_i,\mathcal{C}_i} = \Phi([\sigma_i])$. ∎

Remark 8.2. In the case that $\operatorname{Char}\mathbb{F} = 2$, the augmented Legendrian $(\Sigma, \mathcal{C})$ can be constructed so that Σ is a Legendrian *n-weave* in the terminology of Casals and Zaslow [7]. To see this, note that above the 1-skeleton of M the $(\Lambda_i, \mathcal{C}_i)$ can be chosen with Λ_i a Legendrian braid (in fact, by Propositions 6.16 and 6.17, a standard form, permutation braid). Moreover, since we start with an augmented Legendrian braid above ∂h_2, the cobordism used to fill in $(\Sigma, \mathcal{C})$ over the 2-handle can be constructed using only Proposition 6.17. The underlying Legendrian cobordism involves only clasp moves and D_4^- singularities at Step 1 and Step 2, and since there are no spin basepoints the Legendrian remains unchanged during Step 3.

For example, Figure 27 presents a pair of three sheeted augmented Legendrian weaves over $\mathbb{F}_2$ in J^1T^2 with monodromy representations, Φ_1 and Φ_2, satisfying

$$\Phi_1(\mathbf{l}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Phi_1(\mathbf{m}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Phi_2(\mathbf{l}) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \Phi_2(\mathbf{m}) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\mathbf{l}$ and $\mathbf{m}$ are the generators of $\pi_1(T^2, (0,0))$ corresponding to the horizontal and vertical edges of the square, respectively.

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