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The adjoint Reidemeister torsion for the connected sum of knots

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Abstract. Let K be the connected sum of knots $K_1, \dots, K_n$. It is known that the $\mathrm{SL}_2(\mathbb{C})$-character variety of the knot exterior of K has a component of dimension ≥ 2 as the connected sum admits a so-called bending. We show that there is a natural way to define the adjoint Reidemeister torsion for such a high-dimensional component and prove that it is locally constant on a subset of the character variety where the trace of a meridian is constant. We also prove that the adjoint Reidemeister torsion of K satisfies the vanishing identity if each K_i does so.

1. Introduction

Let M be a compact oriented 3-manifold with tours boundary and $\mathcal{X}(M)$ be the character variety of irreducible representations $\pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$. It happens very often that $\mathcal{X}(M)$ has a component of dimension 1. For instance, if the interior of M admits a hyperbolic structure of finite volume, then the distinguished component is 1-dimensional [15] and if M contains no closed essential surface, then every component is 1-dimensional [3].

Once we fix a simple closed curve μ on the boundary torus ∂M, the *adjoint Reidemeister torsion* is defined as a meromorphic function on each 1-dimensional component of $\mathcal{X}(M)$ under some mild assumptions [6, 14]. It enjoys fruitful interaction with quantum field theory and carries several conjectures consequently. See, for instance, [5, 7, 12]. Recently, it has been conjectured in [8] that the adjoint Reidemeister torsion satisfies a certain vanishing identity with respect to the trace function as follows.

Conjecture 1.1. Suppose that the character variety $\mathcal{X}(M)$ consists of 1-dimensional components and that the interior of M admits a hyperbolic structure of finite volume. Then for generic $c \in \mathbb{C}$ we have

$$\sum_{[\rho] \in \mathcal{X}^c_\mu(M)} \frac{1}{\tau_\mu(M;\rho)} = 0 \tag{1}$$

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where $\mathcal{X}^c_\mu(M)$ is the pre-image of $c \in \mathbb{C}$ under the trace function $\mathcal{X}(M) \to \mathbb{C}$ of $\mu \subset \partial M$ and $\tau_\mu(M;\rho)$ is the adjoint Reidemeister torsion associated to μ and a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$.

As mentioned earlier, there are several 3-manifolds satisfying the conditions required in Conjecture 1.1. However, there are also several examples of 3-manifolds with torus boundary whose character varieties have high-dimensional components. The simplest one might be (the knot exterior of) the connected sum of knots. See [2, 4, 13] for other examples. Two immediate problems when we consider Conjecture 1.1 for such 3-manifolds are that

(P1) the adjoint Reidemeister torsion is not defined for a component of dimension ≥ 2;

(P2) the sum in equation (1) does not make sense as the level set $\mathcal{X}^c_\mu(M)$ is no longer finite.

Related to these problems, we address the following question.

Question 1.2. *Is the adjoint Reidemeister torsion defined and locally constant on $\mathcal{X}^c_\mu(M)$?*

If the answer of Question 1.2 is positive, then the sum in equation (1) makes sense for M in an obvious way: we understand the sum by taking one representative on each connected component of $\mathcal{X}^c_\mu(M)$.

The main purpose of the paper is to investigate Question 1.2 and Conjecture 1.1 for the connected sum of knots. To state our results, let K be the connected sum of knots $K_1, \ldots, K_n$ in S^3 with a meridian μ. We denote by M and M_j the knot exteriors of K and K_j, respectively. For technical reasons, we assume that for $1 \leq j \leq n$

(C) the level set $\mathcal{X}^c_\mu(M_j)$ consists of finitely many μ-regular characters with the canonical longitude having trace other than ± 2 for generic $c \in \mathbb{C}$.

For example, one may choose K_j as a two-bridge knot or a torus knot. It is known that the character variety $\mathcal{X}(M)$ has a component of dimension ≥ 2 as the connected sum admits a so-called *bending*. We refer to [10, 11, 13] for details on the bending construction. Our main theorems are as follows.

Theorem 1.3. *Let K be the connected sum of knots $K_1, \ldots, K_n$ in S^3 satisfying the above condition* (C) *and μ be a meridian. Then there is a natural way to define the adjoint Reidemeister torsion on $\mathcal{X}^c_\mu(M)$ for generic $c \in \mathbb{C}$ which is locally constant.*

Theorem 1.4. *Let K be the connected sum of knots $K_1, \ldots, K_n$ in S^3 satisfying the above condition* (C) *and μ be a meridian. Then the knot exterior M of K satisfies equation* (1) *if each M_j does so.*

It is proved in [18] that every hyperbolic two-bridge knot satisfies equation (1) for a meridian μ, hence we obtain the following corollary.

Corollary 1.5. *The knot exterior of the connected sum of hyperbolic two-bridge knots satisfies equation* (1) *for a meridian* μ.

Remark 1.6. Conjecture 1.1 was derived from the 3d–3d correspondence under the assumption that the interior of M admits a hyperbolic structure (see [8, Section 3] for details). It fails without the assumption, since torus knots are non-hyperbolic and do not satisfy equation (1). However, Theorem 1.4 and Corollary 1.5 suggest that one can relax the hyperbolicity condition, as the connected sum of knots is never hyperbolic.

The paper is organized as follows. In Section 2, we briefly recall basic definitions on the sign-refined Reidemeister torsion. We define the adjoint Reidemeister torsion for the connected sum of knots in Sections 3.1 and 3.2, and prove Theorems 1.3 and 1.4 in Section 3.3.

2. Review on the sign-refined Reidemeister torsion

2.1. The Reidemeister torsion of a chain complex

Let C_* be a chain complex of vector spaces over a field $\mathbb{F}$

$$C_* = (0 \to C_n \xrightarrow{\partial_n} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \to 0)$$

and $H_*(C_*)$ be the homology of C_*. For a basis c_* of C_* and a basis h_* of $H_*(C_*)$ the *Reidemeister torsion* is defined as follows. Here and throughout the paper, every basis and tuple is assumed to be ordered. For $0 \le i \le n$, we choose a lift $\tilde{h}_i$ of h_i to C_i and a tuple b_i of vectors in C_i such that $\partial_i b_i$ is a basis of $\partial_i C_i$. Then the tuple $c'_i = (\partial_{i+1} b_{i+1}, \widetilde{h_i}, b_i)$ is another basis of C_i. Letting A_i be the basis transition matrix taking c_i to c'_i, we have

$$\operatorname{tor}(C_*, c_*, h_*) = \prod_{i=0}^{n} \det A_i^{(-1)^{i+1}} \in \mathbb{F}^*.$$

Also, the *sign-refined Reidemeister torsion* is defined as

$$\operatorname{Tor}(C_*, c_*, h_*) = (-1)^{|C_*|} \operatorname{tor}(C_*, c_*, h_*) \in \mathbb{F}^*, \quad |C_*| = \sum_{i=0}^{n} \alpha_i(C_*)\beta_i(C_*)$$

where $\alpha_i(C_*) = \sum_{j=0}^{i} \dim C_j$ and $\beta_i(C_*) = \sum_{j=0}^{i} \dim H_j(C_*)$.

Suppose that we have a short exact sequence of chain complexes

$$0 \to C'_* \to C_* \to C''_* \to 0 \tag{2}$$

with bases c_*, c'_*, and c''_* of C_*, C'_*, and C''_*, respectively. It is proved in [17, Lemma 3.4.2] that if c_*, c'_*, and c''_* are *compatible* with respect to sequence (2), i.e., $c_* = (c'_*, c''_*)$, then

$$\operatorname{Tor}(C_*, c_*, h_*) = (-1)^{v+u} \operatorname{Tor}(C'_*, c'_*, h'_*) \operatorname{Tor}(C''_*, c''_*, h''_*) \operatorname{tor}(\mathcal{H}) \tag{3}$$

where h_*, h'_*, and h''_* are bases of $H_*(C_*)$, $H_*(C'_*)$, and $H_*(C''_*)$, respectively. Here

$$v = \sum_i \alpha_{i-1}(C'_*)\alpha_i(C''_*), \tag{4}$$

$$u = \sum_i ((\beta_i(C_*) + 1)(\beta_i(C'_*) + \beta_i(C''_*)) + \beta_{i-1}(C'_*)\beta_i(C''_*)), \tag{5}$$

and $\operatorname{tor}(\mathcal{H})$ is the Reidemeister torsion of the long exact sequence induced from (2) with respect to h_*, h'_*, and h''_*. We refer to [16, 17] for details.

2.2. The adjoint Reidemeister torsion of a CW-complex

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$ and fix a basis of $\mathfrak{g}$ as

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that the Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is given by

$$\left\langle \begin{pmatrix} b & a \\ c & -b \end{pmatrix}, \begin{pmatrix} b' & a' \\ c' & -b' \end{pmatrix} \right\rangle = 8bb' + 4ac' + 4ca'.$$

Let X be a finite CW-complex and $\rho\colon \pi_1(X) \to \mathrm{SL}_2(\mathbb{C})$ be a representation. We consider a cochain complex

$$C^*(X; \mathfrak{g}_\rho) = \operatorname{Hom}_{\mathbb{Z}[\pi_1 X]}(C_*(\widetilde{X}; \mathbb{Z}), \mathfrak{g})$$

where $\widetilde{X}$ is the universal cover of X. Here $\mathfrak{g}$ is viewed as a $\mathbb{Z}[\pi_1(X)]$-module through the adjoint action $\operatorname{Ad}\rho\colon \pi_1(X) \to \operatorname{Aut}(\mathfrak{g})$ associated to ρ. We denote the cohomology of $C^*(X; \mathfrak{g}_\rho)$ by $H^*(X; \mathfrak{g}_\rho)$ and call it the *twisted cohomology*. Note that $H^0(X; \mathfrak{g}_\rho)$ coincides with the set of invariant vectors in $\mathfrak{g}$ under the $\pi_1(X)$-action.

Let $c_1, \dots, c_n$ be all the cells of X and fix their order by $c_X = (c_1, \dots, c_n)$. We assume that each cell c_i has a preferred orientation and a lift $\tilde{c}_i$ to $\widetilde{X}$. We define an element $c_i^{(k)} \in C^*(X; \mathfrak{g}_\rho)$ for $1 \leq i \leq n$ and $1 \leq k \leq 3$ by assigning $\tilde{c}_i$ to e_k and every cell of $\widetilde{X}$ that is not a lift of c_i to 0. Then the tuple

$$\mathbf{c}_X = (c_1^{(1)}, c_1^{(2)}, c_1^{(3)}, \ldots, c_n^{(1)}, c_n^{(2)}, c_n^{(3)})$$

is a basis of $C^*(X; \mathfrak{g}_\rho)$. We refer to it as the *geometric basis*.

Let $C_*(X; \mathbb{R})$ be the ordinary chain complex of X with the real coefficient. Note that the tuple c_X is a basis of $C_*(X; \mathbb{R})$. For an orientation o_X of the $\mathbb{R}$-vector space $H_*(X; \mathbb{R})$ we define

$$\varepsilon(o_X) = \operatorname{sgn}(\operatorname{Tor}(C_*(X; \mathbb{R}), c_X, h_X)) \in \{\pm 1\}$$

where h_X is any basis of $H_*(X; \mathbb{R})$ positively oriented with respect to o_X and $\operatorname{sgn}(x)$ is the sign of $x \in \mathbb{R}^*$.

Definition 2.1. For a basis $\mathbf{h}_X$ of $H^*(X; \mathfrak{g}_\rho)$ and an orientation o_X of $H_*(X; \mathbb{R})$ the *adjoint Reidemeister torsion* is defined as

$$\tau(X; \rho, \mathbf{h}_X, o_X) = \varepsilon(o_X) \cdot \operatorname{Tor}(C^*(X; \mathfrak{g}_\rho), \mathbf{c}_X, \mathbf{h}_X) \in \mathbb{C}^*.$$

The above definition does not depend on the order, orientations, and lifts of c_i's. Moreover, it does not depend on the choice of a basis of $\mathfrak{g}$ if the Euler characteristic of X is zero.

Note that every notion in this section associated to ρ is invariant under conjugating ρ up to an appropriate isomorphism. In particular, the adjoint Reidemeister torsion is invariant under the conjugation.

Example 2.2. Let Σ be a 2-torus with a usual CW-structure: one 0-cell p, two 1-cells μ and λ, and one 2-cell Σ as in Figure 1 (left). We choose their lifts (to the universal cover of Σ) as in Figure 1 (right) and fix an order of the cells by $c_\Sigma = (p, \mu, \lambda, \Sigma)$. Let o_Σ be the orientation of $H_*(\Sigma; \mathbb{R})$ induced from c_Σ so that $\varepsilon(o_\Sigma) = 1$.

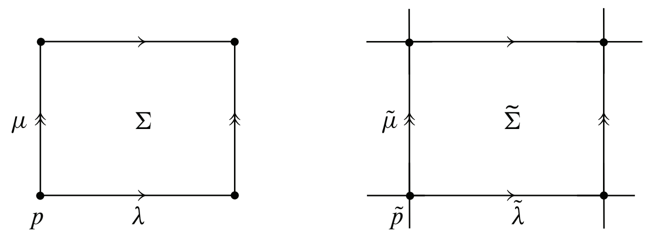

Figure 1. The cells of a 2-torus and their lifts.

Let $\rho\colon \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ be a representation with $\operatorname{tr}\rho(\mu) \neq \pm 2$. Up to conjugation we have

$$\rho(\mu) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(\lambda) = \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}$$

for some $m \neq \pm 1$ and $l \in \mathbb{C}^*$. With respect to the geometric basis, the boundary maps $\delta^0\colon C^0(\Sigma; \mathfrak{g}_\rho) \to C^1(\Sigma; \mathfrak{g}_\rho)$ and $\delta^1\colon C^1(\Sigma; \mathfrak{g}_\rho) \to C^2(\Sigma; \mathfrak{g}_\rho)$ are given by

$$\begin{aligned} \delta^0 &= \begin{pmatrix} \operatorname{Ad}\rho(\mu) - I_3 \\ \operatorname{Ad}\rho(\lambda) - I_3 \end{pmatrix}, \\ \delta^1 &= (\operatorname{Ad}\rho(\lambda) - I_3 \quad I_3 - \operatorname{Ad}\rho(\mu)). \end{aligned}$$

Here I_k is the identity matrix of size k. It follows that $\dim H^i(\Sigma; \mathfrak{g}_\rho) = 1$ for $i = 0, 2$, $\dim H^i(\Sigma; \mathfrak{g}_\rho) = 2$ for $i = 1$, and $\dim H^i(\Sigma; \mathfrak{g}_\rho) = 0$ otherwise. Note that $H^0(\Sigma; \mathfrak{g}_\rho)$ is spanned by $e_2 \in \mathfrak{g}$, as it is invariant under the actions of μ and λ. Let $P = \frac{1}{8} e_2 \in H^0(\Sigma; \mathfrak{g}_\rho)$ and define maps

$$\begin{aligned} \psi^0 &\colon C^0(\Sigma; \mathfrak{g}_\rho) \to \mathbb{C}, && \alpha \mapsto \langle \alpha(\tilde{p}), P \rangle, \\ \psi^1 &\colon C^1(\Sigma; \mathfrak{g}_\rho) \to \mathbb{C}^2, && \alpha \mapsto (\langle \alpha(\tilde{\mu}), P \rangle,\ \langle \alpha(\tilde{\lambda}), P \rangle), \\ \psi^2 &\colon C^2(\Sigma; \mathfrak{g}_\rho) \to \mathbb{C}, && \alpha \mapsto \langle \alpha(\tilde{\Sigma}), P \rangle. \end{aligned}$$

One easily checks that ψ^i induces an isomorphism $H^i(\Sigma; \mathfrak{g}_\rho) \to \mathbb{C}$ for $i = 1, 3$ and φ^2 induces an isomorphism $H^2(\Sigma; \mathfrak{g}_\rho) \to \mathbb{C}^2$. For simplicity, we use the same notation ψ^i for these isomorphisms. We choose a basis $\mathbf{h}^i_\Sigma$ of $H^i(\Sigma; \mathfrak{g}_\rho)$ by the pre-image of the standard basis of $\mathbb{C}$ ($\mathbb{C}^2$ if $i = 2$) under ψ^i. Explicitly, we have $\mathbf{h}^0_\Sigma = p^{(2)}$, $\mathbf{h}^1_\Sigma = (\mu^{(2)}, \lambda^{(2)})$, and $\mathbf{h}^2_\Sigma = \Sigma^{(2)}$. Choosing a tuple b^i of vectors in $C^i(\Sigma; \mathfrak{g}_\rho)$ as $b^0 = (p^{(1)}, p^{(3)})$, $b^1 = (\lambda^{(1)}, \lambda^{(3)})$, and $b^2 = \emptyset$, we obtain

$$\tau(\Sigma; \rho, \mathbf{h}_\Sigma, o_\Sigma) = -1 \cdot (m^2 - 1)(m^{-2} - 1) \cdot (-(m^2 - 1)(m^{-2} - 1))^{-1} = 1.$$

Note that a different choice of $P \in H^0(\Sigma; \mathfrak{g}_\rho)$ changes the basis $\mathbf{h}_\Sigma$ but still we have $\tau(\Sigma; \rho, \mathbf{h}_\Sigma, o_\Sigma) = 1$.

2.3. The adjoint Reidemeister torsion of a knot exterior

Let M be the knot exterior of a knot $K \subset S^3$ with any given triangulation. It is well known that $\dim H_i(M; \mathbb{R}) = 1$ for $i = 0, 1$ and $\dim H_i(M; \mathbb{R}) = 0$ otherwise. We choose the orientation o_M of $H_*(M; \mathbb{R})$ induced from a basis $h_M = (\mathrm{pt}, \mu)$ of $H_*(M; \mathbb{R})$ where pt is a point in M and μ is a meridian of K oriented arbitrarily.

Let $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a representation of the knot group. For the sake of simplicity, we assume that

$$m \neq \pm 1 \quad \text{and} \quad \Delta_K(m^2) \neq 0,$$

where m is an eigenvalue of $\rho(\mu)$ and Δ_K is the Alexander polynomial of K. It follows that if ρ is reducible, then it should be abelian (see e.g. [1]). Therefore, ρ is either irreducible (Section 2.3.1) or abelian (Section 2.3.2).

2.3.1. Irreducible representations. Suppose that ρ is irreducible. In this case we further assume that ρ is μ-regular [14, Definition 3.21], i.e., $\dim H^1(M;\mathfrak{g}_\rho) = 1$ and the inclusion $\mu \hookrightarrow M$ induces an injective map $H^1(M;\mathfrak{g}_\rho) \to H^1(\mu;\mathfrak{g}_\rho)$. We choose an element $P \in H^0(\Sigma;\mathfrak{g}_\rho)$, where $\Sigma = \partial M$, and define maps

$$\psi^1\colon C^1(M;\mathfrak{g}_\rho) \to \mathbb{C}, \quad \alpha \mapsto \langle \alpha(\tilde{\mu}), P\rangle,$$
$$\psi^2\colon C^2(M;\mathfrak{g}_\rho) \to \mathbb{C}, \quad \alpha \mapsto \langle \alpha(\widetilde{\Sigma}), P\rangle,$$

where $\tilde{\mu}$ and $\widetilde{\Sigma}$ are lifts of μ and Σ (to the universal cover of M) respectively satisfying $\tilde{\mu} \subset \widetilde{\Sigma}$. Here the boundary torus Σ is oriented as in Stokes' theorem. It is proved in [14] that the μ-regularity implies that ψ^i induces an isomorphism $H^i(M;\mathfrak{g}_\rho) \to \mathbb{C}$ for $i = 1, 2$. We define

$$\tau_\mu(M;\rho) = \tau(M;\rho,\mathbf{h}_M, o_M) \tag{6}$$

where $\mathbf{h}^i_M$ is a basis of $H^i(M;\mathfrak{g}_\rho)$ given by the pre-image of the standard basis of $\mathbb{C}$ under ψ^i. Note that a different choice of $P \in H^0(\Sigma;\mathfrak{g}_\rho)$ changes the basis $\mathbf{h}_M$ but not the value of $\tau_\mu(M;\rho)$.

2.3.2. Abelian representations. Suppose that ρ is abelian. This case might not be that interesting, as it essentially reduces to the Alexander polynomial. However, we present it explicitly as a setup for Section 3.

Lemma 2.3. *We have* $\dim H^i(M;\mathfrak{g}_\rho) = 1$ *for* $i = 0, 1$ *and* $\dim H^i(M;\mathfrak{g}_\rho) = 0$ *otherwise.*

Proof. We choose any Wirtinger presentation of the knot group

$$\pi_1(M) = \langle g_1, \dots, g_n \,|\, r_1, \dots, r_{n-1}\rangle.$$

Recall that the corresponding 2-dimensional cell complex X consists of one 0-cell p, n 1-cells $g_1, \dots, g_n$, and $n-1$ 2-cells $r_1, \dots, r_{n-1}$. It is known that X is simple homotopy equivalent to M and thus we may use X instead of M. We choose a lift of the base point p arbitrarily and the lifts of other cells accordingly. Then with respect to the geometric basis, the boundary maps

$$\delta^0\colon C^0(X;\mathfrak{g}_\rho) \to C^1(X;\mathfrak{g}_\rho)$$

and

$$\delta^1\colon C^1(X;\mathfrak{g}_\rho) \to C^2(X;\mathfrak{g}_\rho)$$

are given as

$$\delta^0 = \begin{pmatrix} \Phi(g_1 - 1) \\ \vdots \\ \Phi(g_n - 1) \end{pmatrix}$$

and

$$\delta^1 = \begin{pmatrix} \Phi(\partial r_1/\partial g_1) & \cdots & \Phi(\partial r_1/\partial g_n) \\ \vdots & \ddots & \vdots \\ \Phi(\partial r_{n-1}/\partial g_1) & \cdots & \Phi(\partial r_{n-1}/\partial g_n), \end{pmatrix}$$

where Φ is the $\mathbb{Z}$-linear extension of $\operatorname{Ad}\rho$ and $\partial r_j/\partial g_i$ denotes the Fox free differential. Similarly, as in Example 2.2, we have

$$\rho(g_1) = \cdots = \rho(g_n) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \quad m \neq \pm 1$$

up to conjugation. It is clear that $\operatorname{Im}\delta^0 \simeq \mathbb{C}^2$, $\operatorname{Ker}\delta^0 \simeq \mathbb{C}$ and $\dim H^0(X;\mathfrak{g}_\rho) = 1$. On the other hand, δ^1 is surjective since $\Delta_K(1) \neq 0$ and $\Delta_K(m^{\pm 2}) \neq 0$. It follows that $\dim H^2(X;\mathfrak{g}_\rho) = 0$ and $\dim H^1(X;\mathfrak{g}_\rho) = 1$, since the Euler characteristic of X is zero. Explicitly, the twisted cohomology of X is generated by

$$\alpha \in C^0(X;\mathfrak{g}_\rho) \text{ such that } \alpha(\tilde{p}) = e_2, \tag{7a}$$

$$\alpha \in C^1(X;\mathfrak{g}_\rho) \text{ such that } \alpha(\widetilde{g_i}) = e_2 \quad \text{for all } 1 \leq i \leq n. \tag{7b}$$

■

Once again, we choose an element $P \in H^0(\Sigma;\mathfrak{g}_\rho) = H^0(M;\mathfrak{g}_\rho)$ and define

$$\psi^0 \colon C^0(M;\mathfrak{g}_\rho) \to \mathbb{C}, \quad \alpha \mapsto \langle \alpha(\tilde{p}), P\rangle,$$
$$\psi^1 \colon C^1(M;\mathfrak{g}_\rho) \to \mathbb{C}, \quad \alpha \mapsto \langle \alpha(\tilde{\mu}), P\rangle,$$

where $\tilde{p}$ and $\tilde{\mu}$ are lifts of p and μ (to the universal cover of M) respectively satisfying $\tilde{p} \subset \tilde{\mu}$. It is clear from equation (7) that ψ^i induces an isomorphism $H^i(M;\mathfrak{g}_\rho) \to \mathbb{C}$ for $i = 0, 1$. We define

$$\tau_\mu(M;\rho) = \tau(M;\rho,\mathbf{h}_M, o_M)$$

where $\mathbf{h}^i_M$ is a basis of $H^i(M;\mathfrak{g}_\rho)$ given by the pre-image of the standard basis of $\mathbb{C}$ under ψ^i. In fact, one can compute that

$$\tau_\mu(M;\rho) = \frac{\Delta_K(m^2)\Delta_K(m^{-2})}{(m - m^{-1})^2}$$

up to sign, but we would not use this fact in this paper.

3. The connected sum of knots

Let K be the connected sum of knots $K_1, \dots, K_n$ in S^3. We denote by M and M_j the knot exteriors of K and K_j, respectively. It is known that the JSJ decomposition of M consists of a composing space and $M_1, \dots, M_n$.

3.1. A composing space

Let $D_1, \dots, D_n$ be mutually disjoint discs in the interior of a disc D^2 and let

$$W = D^2 \setminus \operatorname{int}(D_1 \sqcup \cdots \sqcup D_n)$$

be a planar surface. Here $\operatorname{int}(X)$ denotes the interior of X. Let $Y = W \times S^1$, called a *composing space*, having $n+1$ boundary tori $\Sigma_j = \partial D_j \times S^1$ $(1 \le j \le n)$ and $\Sigma = \partial D^2 \times S^1$. Letting $\mu = \{\mathrm{pt}\} \times S^1$ and $\lambda_j = \partial D_j \times \{\mathrm{pt}\}$, we have

$$\pi_1(Y) = \langle \mu, \lambda_1, \dots, \lambda_n \mid [\mu, \lambda_1] = \cdots = [\mu, \lambda_n] = 1 \rangle.$$

One can check that $H_0(Y;\mathbb{R}) \simeq \mathbb{R}$ is generated by a point $p \in Y$, $H_1(Y;\mathbb{R}) \simeq \mathbb{R}^{n+1}$ is generated by $\mu, \lambda_1, \dots, \lambda_n$, and $H_2(Y;\mathbb{R}) \simeq \mathbb{R}^n$ is generated by $\Sigma_1, \dots, \Sigma_n$. We choose the orientation o_Y of $H_*(Y;\mathbb{R})$ induced from a basis $h_Y = (p, \mu, \lambda_1, \dots, \lambda_n, \Sigma_1, \dots, \Sigma_n)$ of $H_*(Y;\mathbb{R})$. Here we orient μ, λ_j, and Σ_j as in Example 2.2 and Stokes' theorem.

Let $\rho: \pi_1(Y) \to \mathrm{SL}_2(\mathbb{C})$ be a representation with $\operatorname{tr} \rho(\mu) \neq \pm 2$ and $\operatorname{tr} \rho(\lambda_j) \neq \pm 2$ for some $1 \le j \le n$. Since μ commutes with all λ_j's, we have up to conjugation

$$\rho(\mu) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(\lambda_j) = \begin{pmatrix} l_j & 0 \\ 0 & l_j^{-1} \end{pmatrix} \tag{8}$$

for some $m \neq \pm 1$ and $l_j \in \mathbb{C}^*$. Note that there is no relation among $m, l_1, \dots, l_n$.

Proposition 3.1. *We have*

$$\dim H^i(Y; \mathfrak{g}_\rho) = \begin{cases} 1 & \text{if } i = 0, \\ n+1 & \text{if } i = 1, \\ n & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first compute the twisted cohomology of W. Since W retracts to the wedge sum V of n circles $\lambda_1, \dots, \lambda_n$ (with the basepoint p), we may consider V instead of W:

$$\mathfrak{g} \simeq C^0(V; \mathfrak{g}_\rho) \xrightarrow{\delta^0} C^1(V; \mathfrak{g}_\rho) \simeq \mathfrak{g}^n, \quad \delta^0 = \begin{pmatrix} \operatorname{Ad} \rho(\lambda_1) - I_3 \\ \vdots \\ \operatorname{Ad} \rho(\lambda_n) - I_3 \end{pmatrix}.$$

From equation (8) with the assumption that $\operatorname{tr}\rho(\lambda_j) \neq \pm 2$ for some $1 \leq j \leq n$, we see

$$\dim H^i(W;\mathfrak{g}_\rho) = \dim H^i(V;\mathfrak{g}_\rho) = \begin{cases} 1 & \text{if } i = 0, \\ 3n-2 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, we assume that $l_1 \neq \pm 1$ and choose a basis $\mathbf{h}^i_W$ of $H^i(W;\mathfrak{g}_\rho)$ as

$$\mathbf{h}^0_W = p^{(2)}, \quad \mathbf{h}^1_W = (\underbrace{\lambda_1^{(2)}, \dots, \lambda_n^{(2)}}_{n}, \underbrace{\lambda_2^{(1)}, \dots, \lambda_n^{(1)}}_{n-1}, \underbrace{\lambda_2^{(3)}, \dots, \lambda_n^{(3)}}_{n-1}).$$

Here we choose a lift of p arbitrarily and determine the lifts of other cells accordingly. Recall (see Section 2.2) that the notations $p^{(k)}$ and $\lambda_j^{(k)}$ make sense after we fix lifts of p and λ_j.

We decompose Y into two copies Y_1 and Y_2 of $W \times I$ where I is an interval. It is clear that both Y_1 and Y_2 retract to W and $Y_1 \cap Y_2 = W \sqcup W$. From the short exact sequence

$$\begin{aligned} 0 &\to C^*(Y;\mathfrak{g}_\rho) \to C^*(Y_1;\mathfrak{g}_\rho) \oplus C^*(Y_2;\mathfrak{g}_\rho) \\ &\to C^*(W;\mathfrak{g}_\rho) \oplus C^*(W;\mathfrak{g}_\rho) \to 0, \end{aligned} \tag{9}$$

we obtain

$$\begin{aligned} \mathcal{H}\colon 0 &\to H^0(Y;\mathfrak{g}_\rho) \xrightarrow{f_0} H^0(W;\mathfrak{g}_\rho) \oplus H^0(W;\mathfrak{g}_\rho) \\ &\xrightarrow{g_0} H^0(W;\mathfrak{g}_\rho) \oplus H^0(W;\mathfrak{g}_\rho) \xrightarrow{d_0} H^1(Y;\mathfrak{g}_\rho) \\ &\xrightarrow{f_1} H^1(W;\mathfrak{g}_\rho) \oplus H^1(W;\mathfrak{g}_\rho) \xrightarrow{g_1} H^1(W;\mathfrak{g}_\rho) \oplus H^1(W;\mathfrak{g}_\rho) \\ &\xrightarrow{d_1} H^2(Y;\mathfrak{g}_\rho) \to 0. \end{aligned} \tag{10}$$

The map g_i in the above sequence sends $(x, y) \in H^i(W;\mathfrak{g}_\rho) \oplus H^i(W;\mathfrak{g}_\rho)$ to $(x - y,\ y - \mu_*(x))$ where μ_* denotes the action on $H^i(W;\mathfrak{g}_\rho)$ induced from $\mu = \{\mathrm{pt}\} \times S^1$. More precisely, fixing identifications $H^0(W;\mathfrak{g}_\rho) \simeq \mathbb{C}$ and $H^1(W;\mathfrak{g}_\rho) \simeq \mathbb{C}^{3n-2}$ with respect to $\mathbf{h}_W$, the matrix expressions of g_0 and g_1 are given by

$$g_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad g_1 = \left(\begin{array}{c|ccc} & -I_n & 0 & 0 \\ I_{3n-2} & 0 & -m^2 I_{n-1} & 0 \\ & 0 & 0 & -m^{-2} I_{n-1} \\ \hline -I_{3n-2} & & I_{3n-2} & \end{array}\right)$$

where I_k is the identity matrix of size k. In particular, $\operatorname{Ker} g_0$ is generated by $\mathrm{e}_1 + \mathrm{e}_2$ and $\operatorname{Ker} g_1$ is generated by $\mathrm{e}_1 + \mathrm{e}_{3n-1}, \dots, \mathrm{e}_n + \mathrm{e}_{4n-1}$. Here e_k is a unit vector whose coordinates are all zero, except one at the k-th coordinate. It follows that

$$\dim \operatorname{Im} g_0 = 1, \quad \dim \operatorname{Im} g_1 = 5n - 4,$$

and

$$\begin{aligned} \dim H^0(Y; \mathfrak{g}_\rho) &= \dim \operatorname{Im} f_0 = \dim \operatorname{Ker} g_0 = 1, \\ \dim H^2(Y; \mathfrak{g}_\rho) &= 2 \dim H^1(W; \mathfrak{g}_\rho) - \dim \operatorname{Im} g_1 = n. \end{aligned}$$

Furthermore, we have $\dim H^1(Y; \mathfrak{g}_\rho) = n + 1$, since the Euler characteristic of Y is zero. ∎

It is geometrically natural to choose a basis $\mathbf{h}_Y^i$ of $H^i(Y; \mathfrak{g}_\rho)$ as

$$\begin{aligned} \mathbf{h}_Y^0 &= p^{(2)}, \\ \mathbf{h}_Y^1 &= (\mu^{(2)}, \lambda_1^{(2)}, \dots, \lambda_n^{(2)}), \\ \mathbf{h}_Y^2 &= (\Sigma_1^{(2)}, \dots, \Sigma_n^{(2)}). \end{aligned}$$

Alternatively, we may describe the basis $\mathbf{h}_Y$ as follows (as in Example 2.2). Let $P = \frac{1}{8} e_2 \in H^0(Y; \mathfrak{g}_\rho)$ and consider the isomorphisms

$$\begin{aligned} &\psi^0 \colon H^0(Y; \mathfrak{g}_\rho) \to \mathbb{C}, && \alpha \mapsto \langle \alpha(\tilde{p}), P \rangle, \\ &\psi^1 \colon H^1(Y; \mathfrak{g}_\rho) \to \mathbb{C}^{n+1}, && \alpha \mapsto (\langle \alpha(\tilde{\mu}), P \rangle, \langle \alpha(\tilde{\lambda}_1), P \rangle, \dots, \langle \alpha(\tilde{\lambda}_n), P \rangle), \\ &\psi^2 \colon H^2(Y; \mathfrak{g}_\rho) \to \mathbb{C}^n, && \alpha \mapsto (\langle \alpha(\widetilde{\Sigma}_1), P \rangle, \dots, \langle \alpha(\widetilde{\Sigma}_n), P \rangle). \end{aligned}$$

Then the basis $\mathbf{h}_Y^i$ maps to the standard basis of $\mathbb{C}$, $\mathbb{C}^{n+1}$, or $\mathbb{C}^n$ under ψ^i accordingly.

Proposition 3.2. $\tau(Y; \rho, \mathbf{h}_Y, o_Y) = (-1)^{n-1}(m - m^{-1})^{2n-2}$.

Proof. Recall that Y decomposes into two copies Y_1 and Y_2 of $W \times I$ with $Y_1 \cap Y_2 = W \sqcup W$ and that W retracts to V, the wedge sum of n circles $\lambda_1, \dots, \lambda_n$ with the base point p.

We construct $V \times I$ from two copies of V (regarding them as $V \times \partial I$) by adding cells $p \times I, \lambda_1 \times I, \dots, \lambda_n \times I$. Choose cell orders of V, $V \times I$, and $V \times S^1$ as

- $c_V = (p, \lambda_1, \dots, \lambda_n)$,
- $c_{V \times I} = (c_V, c_V, c_{\widetilde{V}})$ where $c_{\widetilde{V}} = (p \times I, \lambda_1 \times I, \dots, \lambda_n \times I)$,
- $c_{V \times S^1} = (c_V, c_V, c_{\widetilde{V}}, c_{\widetilde{V}})$.

Then the basis transition between $(c_{V \times I}, c_{V \times I})$ and $(c_V, c_V, c_{V \times S^1})$ is an even permutation. On the other hand, for

$$h_{V \times S^1} = (p, \mu, \lambda_1, \dots, \lambda_n, \Sigma_1, \dots, \Sigma_n) \, (= h_Y)$$

a straightforward computation shows that

$$
\begin{aligned}
&\mathrm{Tor}(C_*(V\times S^1;\mathbb{R}), c_{V\times S^1}, h_{V\times S^1})\\
&= (-1)^{|C_*(V\times S^1;\mathbb{R})|}\det\begin{pmatrix} I_n & 0\\ I_n & I_n\end{pmatrix}^{-1}\det\left(\begin{array}{c|c|c|c} -I_n & 0 & I_n & 0\\ I_n & 0 & 0 & 0\\ \hline \multirow{2}{*}{0} & 1 & \multirow{2}{*}{0} & 0\\ & 1 & & 1\end{array}\right)\det\begin{pmatrix} 1 & 1\\ -1 & 0\end{pmatrix}^{-1}\\
&= 1.
\end{aligned}
\tag{11}
$$

Note that $|C_*(V\times S^1;\mathbb{R})|$ is obviously even.

We choose any triangulation of Y and cell orders c_Y, c_{Y_i}, and c_W according to $c_{V\times S^1}$, $c_{V\times I}$, and c_V, respectively. Applying formula (3) to the short exact sequence (9), we obtain

$$1 = (-1)^{v+u}\,\mathrm{Tor}(C^*(Y;\mathfrak{g}_\rho), \mathbf{c}_Y, \mathbf{h}_Y)\ \mathrm{tor}(\mathcal{H})$$

after canceling out the torsion terms for $W \simeq Y_i$. Here $\mathrm{tor}(\mathcal{H})$ is the Reidemeister torsion of the long exact sequence (10) with respect to $\mathbf{h}_Y$ and $\mathbf{h}_W$. Note that the basis transition between $(\mathbf{c}_{Y_1}, \mathbf{c}_{Y_2})$ and $(\mathbf{c}_Y, \mathbf{c}_W, \mathbf{c}_W)$ is an even permutation. One easily checks from definitions (4) and (5) that $v \equiv 0$ and $u \equiv \sum_i \beta_i(C^*(Y;\mathfrak{g}_\rho)) \equiv n-1$ in modulo 2. To simplify notations, we rewrite sequence (10) as

$$
\begin{array}{ccccccccccccccccc}
0 & \longrightarrow & \mathcal{H}^0 & \xrightarrow{f_0} & \mathcal{H}^1 & \xrightarrow{g_0} & \mathcal{H}^2 & \xrightarrow{d_0} & \mathcal{H}^3 & \xrightarrow{f_1} & \mathcal{H}^4 & \xrightarrow{g_1} & \mathcal{H}^5 & \xrightarrow{d_1} & \mathcal{H}^6 & \longrightarrow & 0\\
 & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
 & & \mathbb{C} & \xrightarrow{f_0} & \mathbb{C}^2 & \xrightarrow{g_0} & \mathbb{C}^2 & \xrightarrow{d_0} & \mathbb{C}^{n+1} & \xrightarrow{f_1} & \mathbb{C}^{6n-4} & \xrightarrow{g_1} & \mathbb{C}^{6n-4} & \xrightarrow{d_1} & \mathbb{C}^n & &
\end{array}
$$

where the two rows are identified with respect to $\mathbf{h}_Y$ and $\mathbf{h}_W$. We choose a tuple b^i of vectors in $\mathcal{H}^i$ as

$$
\begin{aligned}
b^0 &= e_1,\\
b^1 &= e_1,\\
b^2 &= e_1,\\
b^3 &= (e_2, e_3, \ldots, e_{n+1}),\\
b^4 &= (e_{n+1}, e_{n+2}, \ldots, e_{6n-4}),\\
b^5 &= (e_1, e_2, \ldots, e_n),\\
b^6 &= \emptyset
\end{aligned}
$$

where e_k is a unit vector whose coordinates are all zero, except one at the k-th coordinate. Then the basis transition matrix A_i at $\mathcal{H}^i$ (see Section 2.2) is given by

$$
\begin{aligned}
A_0 &= I_1,\\
A_1 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\\
A_2 &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},\\
A_3 &= I_{n+1}\\
A_4 &= \left(\begin{array}{cc|c} I_n & 0 & \\ 0 & I_{2n-2} & 0 \\ \hline I_n & 0 & \\ 0 & 0 & I_{3n-2} \end{array}\right),\\
A_5 &= \left(\begin{array}{c|ccc|c} 0 & -I_n & 0 & 0 & I_n \\ & 0 & -m^2 I_{n-1} & 0 & 0 \\ I_{2n-2} & 0 & 0 & -m^{-2} I_{n-1} & \\ \hline 0 & & & & \\ -I_{2n-2} & & I_{3n-2} & & 0 \end{array}\right),\\
A_6 &= I_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\operatorname{tor}(\mathcal{H}) &= -\det A_5^{-1}\\
&= (-1)^{n-1} \det \left(\begin{array}{c|ccc} I_{2n-2} & 0 & -m^2 I_{n-1} & 0 \\ & 0 & 0 & -m^{-2} I_{n-1} \\ \hline 0 & & & \\ -I_{2n-2} & & I_{3n-2} & \end{array}\right)^{-1}\\
&= (-1)^{n-1} \det \begin{pmatrix} (1-m^2) I_{n-1} & 0 \\ 0 & (1-m^{-2}) I_{n-1} \end{pmatrix}^{-1}\\
&= (m-m^{-1})^{2-2n}.
\end{aligned}
$$

Note that the third equation follows from the determinant formula for a block matrix. We conclude that

$$
\operatorname{Tor}(C^*(Y;\mathfrak{g}_\rho), \mathbf{c}_Y, \mathbf{h}_Y) = (-1)^{n+1}(m-m^{-1})^{2n-2}.
$$

This completes the proof, since we have $\varepsilon(o_Y) = 1$ from equation (11). ∎

Remark 3.3. We have $\tau(Y;\rho,\mathbf{h}_Y,o_Y) = 1$ for $n = 1$. This agrees with the computation given in Example 2.2, as Y retracts to a 2-torus when $n = 1$.

3.2. The knot exterior of the connected sum

The composing space Y has $n+1$ boundary tori $\Sigma_1, \dots, \Sigma_n$ and Σ. For $1 \leq j \leq n$ we glue the knot exterior M_j of $K_j \subset S^3$ to Y by using a homeomorphism $\partial M_j \to \Sigma_j$ that maps the meridian and canonical longitude of K_j to μ and λ_j, respectively. The resulting manifold M is a compact 3-manifold with $\partial M = \Sigma$ and is the knot exterior of the connected sum of $K_1, \dots, K_n$. We refer to [9, Example IX.21] for details. We choose the orientation o_M of $H_*(M;\mathbb{R})$ as in Section 2.3, i.e., the one induced from the basis $h_M = (\mathrm{pt}, \mu)$ of $H_*(M;\mathbb{R})$.

Let $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. We denote by m and l_j eigenvalues of $\rho(\mu)$ and $\rho(\lambda_j)$ respectively as in equation (8). For simplicity we assume that

$$m \neq \pm 1 \quad \text{and} \quad \Delta_{K_j}(m^2) \neq 0 \quad \text{for all } 1 \leq j \leq n, \tag{12}$$

where Δ_{K_j} is the Alexander polynomial of K_j. It follows that each restriction $\rho_j\colon \pi_1(M_j) \to \mathrm{SL}_2(\mathbb{C})$ of ρ is either irreducible or abelian. We further assume that if ρ_j is irreducible, then

$$l_j \neq \pm 1 \quad \text{and} \quad \rho_j \text{ is } \mu\text{-regular.} \tag{13}$$

Without loss of generality, we assume that $\rho_1, \dots, \rho_k$ are abelian and $\rho_{k+1}, \dots, \rho_n$ are irreducible where k should be less than n, otherwise ρ becomes abelian. In particular, $l_j \neq \pm 1$ for some $1 \leq j \leq n$.

Proposition 3.4. *We have*

$$\dim H^i(M;\mathfrak{g}_\rho) = \begin{cases} n-k & \text{if } i = 1,2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From the short exact sequence

$$0 \to C^*(M;\mathfrak{g}_\rho) \to \bigoplus_{j=1}^{n} C^*(M_j;\mathfrak{g}_\rho) \oplus C^*(Y;\mathfrak{g}_\rho) \to \bigoplus_{j=1}^{n} C^*(\Sigma_j;\mathfrak{g}_\rho) \to 0, \tag{14}$$

we have

$$\begin{aligned} \mathscr{G}\colon 0 \to H^0(M;\mathfrak{g}_\rho) &\xrightarrow{F_0} \bigoplus_{j=1}^{n} H^0(M_j;\mathfrak{g}_\rho) \oplus H^0(Y;\mathfrak{g}_\rho) \xrightarrow{G_0} \bigoplus_{j=1}^{n} H^0(\Sigma_j;\mathfrak{g}_\rho) \\ \xrightarrow{D_0} H^1(M;\mathfrak{g}_\rho) &\xrightarrow{F_1} \bigoplus_{j=1}^{n} H^1(M_j;\mathfrak{g}_\rho) \oplus H^1(Y;\mathfrak{g}_\rho) \xrightarrow{G_1} \bigoplus_{j=1}^{n} H^1(\Sigma_j;\mathfrak{g}_\rho) \\ \xrightarrow{D_1} H^2(M;\mathfrak{g}_\rho) &\xrightarrow{F_2} \bigoplus_{j=1}^{n} H^2(M_j;\mathfrak{g}_\rho) \oplus H^2(Y;\mathfrak{g}_\rho) \xrightarrow{G_2} \bigoplus_{j=1}^{n} H^2(\Sigma_j;\mathfrak{g}_\rho) \\ \xrightarrow{D_2} H^3(M;\mathfrak{g}_\rho) &\to 0. \end{aligned} \tag{15}$$

With respect to the bases $\mathbf{h}_{\Sigma_j}$, $\mathbf{h}_{M_j}$, and $\mathbf{h}_Y$ given in Example 2.2 and Sections 2.3 and 3.1, the map G_0 in sequence (15) can be identified with

$$G_0 \colon \mathbb{C}^{k+1} \to \mathbb{C}^n, \quad (x_1, \dots, x_k, y) \mapsto (x_1 - y, \dots, x_k - y, \underbrace{-y, \dots, -y}_{n-k}). \tag{16}$$

It follows that

$$\dim \operatorname{Im} G_0 = \dim \operatorname{Ker} D_0 = k + 1$$

and

$$\dim \operatorname{Im} D_0 = n - k - 1.$$

Also, the matrix expression of

$$G_1 \colon \mathbb{C}^{2n+1} \to \mathbb{C}^{2n}$$

is given by

$$G_1 = \left(\begin{array}{ccc|c|ccc} 1 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \kappa_1 & & 0 & 0 & -1 & & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & & 1 & -1 & 0 & & 0 \\ 0 & \cdots & \kappa_n & 0 & 0 & \cdots & -1 \end{array}\right)$$

where $\kappa_j = \langle \mathbf{h}^1_{M_j}(\widetilde{\lambda_j}), P \rangle$. Since we obtain an invertible matrix (of size $2n$) from G_1 by deleting the $(n+1)$-st column, we have $\dim \operatorname{Im} G_1 = 2n$ and $\dim \operatorname{Ker} G_1 = \dim \operatorname{Im} F_1 = 1$. ∎

We let $P = \frac{1}{8} e_2 \in H^0(Y; \mathfrak{g}_\rho)$ and define maps

$$\begin{aligned} \psi^1 \colon H^1(M; \mathfrak{g}_\rho) &\to \mathbb{C}, & \alpha &\mapsto \langle \alpha(\tilde{\mu}), P \rangle, \\ \psi^2 \colon H^2(M; \mathfrak{g}_\rho) &\to \mathbb{C}^{n-k}, & \alpha &\mapsto (\langle \alpha(\widetilde{\Sigma}_{k+1}), P \rangle, \dots, \langle \alpha(\widetilde{\Sigma}_n), P \rangle). \end{aligned}$$

Lemma 3.5. *ψ^1 induces an isomorphism $H^1(M; \mathfrak{g}_\rho)/\operatorname{Im} D_0 \to \mathbb{C}$ and ψ^2 is an isomorphism.*

Proof. It is clear that ψ^1 is compatible with the isomorphism $H^1(M_j; \mathfrak{g}_\rho) \to \mathbb{C}$, $\alpha \mapsto \langle \alpha(\tilde{\mu}), P \rangle$ for $1 \le j \le n$. In particular, ψ^1 is surjective. On the other hand, it follows from sequence (15) that an element of $\operatorname{Im} D_0$ maps to the trivial element in $H^1(M_j; \mathfrak{g}_\rho)$ under the restriction map $H^1(M; \mathfrak{g}_\rho) \to H^1(M_j; \mathfrak{g}_\rho)$. Therefore, ψ^1 induces a map $H^1(M; \mathfrak{g}_\rho)/\operatorname{Im} D_0 \to \mathbb{C}$ which is an isomorphism since $\dim H^1(M; \mathfrak{g}_\rho) = n - k$ and $\dim \operatorname{Im} D_0 = n - k - 1$. The second claim that ψ^2 is an isomorphism is obvious from sequence (15). ∎

Recall that the basis $\mathbf{h}^0_{\Sigma_j}$ of $H^0(\Sigma_j;\mathfrak{g}_\rho)$ gives us an isomorphism

$$\bigoplus_{j=1}^{n} H^0(\Sigma_j;\mathfrak{g}_\rho) \simeq \mathbb{C}^n.$$

Denoting by $(\mathrm{e}_1,\dots,\mathrm{e}_n)$ the standard basis of $\mathbb{C}^n$, we choose a basis of $\operatorname{Im} D_0$ as

$$(D_0(\mathrm{e}_{k+1}),\dots,D_0(\mathrm{e}_{n-1})).$$

Note that equation (16) shows that the above tuple is indeed a basis of $\operatorname{Im} D_0$. We then extend it to a basis $\mathbf{h}^1_M$ of $H^1(M;\mathfrak{g}_\rho)$ by adding an element ξ at the first position which maps to the standard basis of $\mathbb{C}$ under ψ^1:

$$\mathbf{h}^1_M = (\xi, D_0(\mathrm{e}_{k+1}),\dots,D_0(\mathrm{e}_{n-1})).$$

We also choose a basis $\mathbf{h}^2_M$ of $H^2(M;\mathfrak{g}_\rho)$ by the pre-image of the standard basis of $\mathbb{C}^{n-k}$ under ψ^2. With the above choice of $\mathbf{h}_M$, we define the adjoint Reidemeister torsion for the connected sum as follows.

Definition 3.6. Let M be the knot exterior of the connected sum of knots $K_1,\dots,K_n$ in S^3 and $\rho\colon \pi_1(M)\to \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation satisfying conditions (12) and (13). We define the *adjoint Reidemeister torsion* (associated to ρ and the meridian μ) as

$$\tau_\mu(M;\rho) = \tau(M;\rho,\mathbf{h}_M,o_M). \tag{17}$$

Note that it reduces to the standard definition (6) of a knot exterior when $n=1$.

Lemma 3.7. *Equation* (17) *does not depend on the choice of $P\in H^0(Y;\mathfrak{g}_\rho)$ and the order of indices of $\Sigma_{k+1},\dots,\Sigma_n$.*

Proof. If we replace P by cP for some $c\in\mathbb{C}^*$, then the basis transition matrices for $H^i(M;\mathfrak{g}_\rho)$ is $\frac{1}{c}I_{n-k}$ for both $i=1,2$ and thus $\tau_\mu(M;\rho)$ does not change. If we exchange two indices other than n, then the basis transition is clearly an odd permutation for both $H^1(M;\mathfrak{g}_\rho)$ and $H^2(M;\mathfrak{g}_\rho)$. Therefore, $\tau_\mu(M;\rho)$ does not change. If we exchange the index n with another one, then the basis transition for $H^2(M;\mathfrak{g}_\rho)$ is an odd permutation. On the other hand, since $\mathrm{e}_{k+1}+\cdots+\mathrm{e}_n\in \operatorname{Im} G_0 = \operatorname{Ker} D_0$ (see equation (16)), we have $D_0(\mathrm{e}_n) = -D_0(\mathrm{e}_{k+1})-\cdots-D_0(\mathrm{e}_{n-1})$. It follows that the basis transition matrix for $H^1(M;\mathfrak{g}_\rho)$ has determinant -1 and thus $\tau_\mu(M;\rho)$ does not change. ∎

Theorem 3.8. $\tau_\mu(M;\rho) = (m-m^{-1})^{2n-2}\tau_\mu(M_1;\rho_1)\dots\tau_\mu(M_n;\rho_n)$.

Proof. Choose any triangulation of M with any cell order c_M. We denote by c_Y (resp., c_{M_j} and c_{Σ_j}) the cell order restricted to Y (resp., M_j and Σ_j). Note that the Euler characteristics of M, M_j, Y, Σ_j are even. Consequently, we may assume that the number of i-dimensional cells in each of M, M_j, Y, and Σ_j is even by applying the barycentric subdivision once. Let $e = 1$ (resp., -1) if the basis transition between $(c_{\Sigma_1}, \ldots, c_{\Sigma_n}, c_M)$ and $(c_{M_1}, \ldots, c_{M_n}, c_Y)$ is an even (resp., odd) permutation.

Applying formula (3) to the short exact sequence (14), we obtain

$$\begin{aligned} e \cdot \prod_{j=1}^{n} &\operatorname{Tor}(C^*(M_j; \mathfrak{g}_\rho), \mathbf{c}_{M_j}, \mathbf{h}_{M_j}) \cdot \operatorname{Tor}(C^*(Y; \mathfrak{g}_\rho), \mathbf{c}_Y, \mathbf{h}_Y) \\ &= (-1)^{v+u} \operatorname{Tor}(C^*(M; \mathfrak{g}_\rho), \mathbf{c}_M, \mathbf{h}_M) \cdot \prod_{j=1}^{n} \operatorname{Tor}(C^*(\Sigma_j; \mathfrak{g}_\rho), \mathbf{c}_{\Sigma_j}, \mathbf{h}_{\Sigma_j}) \cdot \operatorname{tor}(\mathcal{G}). \end{aligned}$$

where $\operatorname{tor}(\mathcal{G})$ is the Reidemeister torsion of the long exact sequence (15) with respect to $\mathbf{h}_{M_j}$, $\mathbf{h}_Y$, $\mathbf{h}_{\Sigma_j}$, and $\mathbf{h}_M$. It is clear from definition (4) that v is even since the number of i-dimensional cells in each of M, M_j, Y and Σ_j is even for all i. Also, a direct computation from definition (5) gives that $u \equiv n$ in modulo 2. Recall Proposition 3.4 that there are two trivial terms $H^0(M; \mathfrak{g}_\rho)$ and $H^3(M; \mathfrak{g}_\rho)$ in $\mathcal{G}$. Ignoring these trivial terms, we rewrite $\mathcal{G}$ as

$$\begin{array}{ccccccccccccccccccc}
0 \longrightarrow & \mathcal{G}^0 & \xrightarrow{G_0} & \mathcal{G}^1 & \xrightarrow{D_0} & \mathcal{G}^2 & \xrightarrow{F_1} & \mathcal{G}^3 & \xrightarrow{G_1} & \mathcal{G}^4 & \xrightarrow{D_1} & \mathcal{G}^5 & \xrightarrow{F_2} & \mathcal{G}^6 & \xrightarrow{G_2} & \mathcal{G}^7 & \to 0 \\
& \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\
& \mathbb{C}^{k+1} & \overset{G_0}{\to} & \mathbb{C}^n & \overset{D_0}{\to} & \mathbb{C}^{n-k} & \overset{F_1}{\to} & \mathbb{C}^{2n+1} & \overset{G_1}{\to} & \mathbb{C}^{2n} & \overset{D_1}{\to} & \mathbb{C}^{n-k} & \overset{F_2}{\to} & \mathbb{C}^{2n-k} & \overset{G_2}{\to} & \mathbb{C}^n &
\end{array}$$

where the two rows are identified with respect to $\mathbf{h}_{M_j}$, $\mathbf{h}_Y$, $\mathbf{h}_{\Sigma_j}$, and $\mathbf{h}_M$. We choose a tuple b^i of vectors in $\mathcal{G}^i$ as

$$\begin{aligned} b^0 &= (e_1, e_2, \ldots, e_{k+1}), \\ b^1 &= (e_{k+1}, e_{k+2}, \ldots, e_{n-1}), \\ b^2 &= e_1 \\ b^3 &= (e_1, e_2, \ldots, e_n,\ e_{n+2}, e_{n+3}, \ldots, e_{2n+1}), \\ b^4 &= \emptyset, \\ b^5 &= (e_1, e_2, \ldots, e_{n-k}), \\ b^6 &= (e_{n-k+1}, e_{n-k+2}, \ldots, e_{2n-k}), \\ b^7 &= \emptyset \end{aligned}$$

where e_k is a unit vector whose coordinates are all zero, except one at the k-th coordinate. Then the basis transition matrix A_i at $\mathcal{G}^i$ (see Section 2.2) is given by

$$
\begin{aligned}
A_0 &= I_{k+1}, \\
A_1 &= \left(\begin{array}{c|c|c}
I_k & \begin{matrix}-1\\ \vdots \\ -1\end{matrix} & 0 \\ \hline
0 & \begin{matrix}-1\\ \vdots \\ -1\end{matrix} & I_{n-k-1} \\ \hline
0 & -1 & 0
\end{array}\right), \\
A_2 &= \begin{pmatrix} 0 & 1 \\ I_{n-k-1} & 0 \end{pmatrix}, \\
A_3 &= \left(\begin{array}{c|c|c}
\begin{matrix}1\\ \vdots \\ 1\end{matrix} & I_n & 0 \\ \hline
1 & 0 & 0 \\ \hline
\begin{matrix}\kappa_1\\ \vdots \\ \kappa_n\end{matrix} & 0 & I_n
\end{array}\right), \\
A_4 &= \left(\begin{array}{ccc|ccc}
1 & \cdots & 0 & 0 & \cdots & 0 \\
\kappa_1 & & 0 & -1 & & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & & 1 & 0 & & 0 \\
0 & \cdots & \kappa_n & 0 & \cdots & -1
\end{array}\right), \\
A_5 &= I_{n-k}, \\
A_6 &= \left(\begin{array}{c|c}
I_{n-k} & 0 \\ \hline
\begin{matrix}0 \\ I_{n-k}\end{matrix} & I_n
\end{array}\right), \\
A_7 &= -I_n.
\end{aligned}
$$

It follows that

$$\operatorname{tor}(\mathscr{G}) = (-1)^{n-k}(-1)^{n-k-1}(-1)^n(-1)^{\frac{n(n+1)}{2}}(-1)^n = (-1)^{\frac{n(n+1)}{2}+1}.$$

Therefore, we conclude that

$$
\begin{aligned}
&e \cdot \prod_{j=1}^{n} \operatorname{Tor}(C^*(M_j;\mathfrak{g}_\rho), \mathbf{c}_{M_j}, \mathbf{h}_{M_j}) \cdot \operatorname{Tor}(C^*(Y;\mathfrak{g}_\rho), \mathbf{c}_Y, \mathbf{h}_Y) \\
&\quad = (-1)^{\frac{n(n+1)}{2}+n+1} \operatorname{Tor}(C^*(M;\mathfrak{g}_\rho), \mathbf{c}_M, \mathbf{h}_M) \prod_{j=1}^{n} \operatorname{Tor}(C^*(\Sigma_j;\mathfrak{g}_\rho), \mathbf{c}_{\Sigma_j}, \mathbf{h}_{\Sigma_j}).
\end{aligned} \tag{18}
$$

On the other hand, applying formula (3) to the short exact sequence

$$0 \to \bigoplus_{j=1}^{n} C_*(\Sigma_j;\mathbb{R}) \to \bigoplus_{j=1}^{n} C_*(M_j;\mathbb{R}) \oplus C_*(Y;\mathbb{R}) \to C_*(M;\mathbb{R}) \to 0, \tag{19}$$

we have

$$\begin{aligned} &e \cdot \prod_{j=1}^{n} \mathrm{Tor}(C_*(M_j;\mathbb{R}), c_{M_j}, h_{M_j}) \cdot \mathrm{Tor}(C_*(Y;\mathbb{R}), c_Y, h_Y) \qquad (20) \\ &= (-1)^{u'+v'} \prod_{j=1}^{n} \mathrm{Tor}(C_*(\Sigma_j;\mathbb{R}), c_{\Sigma_j}, h_{\Sigma_j}) \cdot \mathrm{Tor}(C_*(M;\mathbb{R}), c_M, h_M) \cdot \mathrm{tor}(\mathscr{G}'), \end{aligned}$$

where $\mathrm{tor}(\mathscr{G}')$ is the Reidemeister torsion of the long exact sequence induced from (19) with respect to the bases h_{Σ_j}, h_{M_j}, h_Y, and h_M. Repeating similar computations, we have $u' \equiv v' \equiv 0 \mod 2$ and $\mathrm{tor}(\mathscr{G}') = (-1)^{\frac{n(n+1)}{2}}$. Then, from equation (20) we have

$$e \cdot \prod_{j=1}^{n} \varepsilon(o_{M_j}) \cdot \varepsilon(o_Y) = (-1)^{\frac{n(n+1)}{2}} \prod_{j=1}^{n} \varepsilon(o_{\Sigma_j}) \cdot \varepsilon(o_M). \tag{21}$$

Combining equations (18) and (21) with Example 2.2 and Proposition 3.2, we obtain the desired formula. ∎

3.3. Proofs of Theorems 1.3 and 1.4

Recall that $\mathcal{X}(M)$ is the character variety of irreducible representations $\pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ and $\mathcal{X}_\mu^c(M)$ is the pre-image of $c \in \mathbb{C}$ under the trace function $\mathcal{X}(M) \to \mathbb{C}$ of μ. We use the notations $\mathcal{X}(M_j)$ and $\mathcal{X}_\mu^c(M_j)$ similarly for $1 \le j \le n$. Since we assumed that

(C) the level set $\mathcal{X}_\mu^c(M_j)$ consists of finitely many μ-regular characters with the canonical longitude having trace other than ± 2 for generic $c \in \mathbb{C}$,

conditions (12) and (13) in Section 3.2 are satisfied for generic $c \in \mathbb{C}$. It follows that the adjoint Reidemeister torsion is well defined on the level set $\mathcal{X}_\mu^c(M)$ for generic $c \in \mathbb{C}$.

Lemma 3.9. *The connected components of $\mathcal{X}_\mu^c(M)$ are the pre-images of the restriction map (i.e., induced by the inclusions $M_j \to M$):*

$$\Phi\colon \mathcal{X}_\mu^c(M) \to \prod_{j=1}^{n} (\mathcal{X}_\mu^c(M_j) \sqcup \{[\alpha_j]\}) \setminus \{([\alpha_1], \ldots, [\alpha_n])\}$$

where $\alpha_j\colon \pi_1(M_j) \to \mathrm{SL}_2(\mathbb{C})$ is the abelian representation with $\mathrm{tr}(\alpha_j(\mu)) = c$, and $[\alpha_j]$ denotes its character.

Proof. We first prove that Φ is surjective. Let ρ_j be a representation $\pi_1(M_j) \to \mathrm{SL}_2(\mathbb{C})$ satisfying $\mathrm{tr}(\rho_j(\mu)) = c$ for $1 \le j \le n$. Since we assume that $c \neq \pm 2$, we can conjugate each ρ_j so that $\rho_1(\mu) = \rho_2(\mu) = \cdots = \rho_n(\mu)$. This is sufficient to extend these representations to $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ which is irreducible since at least one of ρ_j's is irreducible.

For a point p in the image of Φ, without loss of generality, we may assume $p = ([\alpha_1], \ldots, [\alpha_k], [\rho_{k+1}], \ldots, [\rho_n])$, where $\alpha_1 \ldots, \alpha_k$ are abelian and $\rho_{k+1}, \ldots, \rho_n$ are irreducible. To analyze the pre-image $\Phi^{-1}(p)$, consider two characters in $\Phi^{-1}(p)$, those are conjugacy classes of irreducible representations ρ and ρ' of $\pi_1(M)$. As $\mathrm{tr}(\rho(\mu)) = \mathrm{tr}(\rho'(\mu)) \neq \pm 2$, after conjugating we may assume that $\rho(\mu) = \rho'(\mu)$. Let $D \subset \mathrm{PSL}_2(\mathbb{C})$ denote the centralizer of $\rho(\mu) = \rho'(\mu)$. Since $\mathrm{tr}(\rho(\mu)) = \mathrm{tr}(\rho'(\mu)) \neq \pm 2$, the group D is conjugate to the group of diagonal matrices and thus $D \cong \mathbb{C}^*$. Let ρ_j and ρ'_j denote the respective restrictions of ρ and ρ' to $\pi_1(M_j)$. The assumption $\rho(\mu) = \rho'(\mu)$ implies that

- for $j = 1, \ldots, k$, $\rho_j = \rho'_j$. It is because of that the genericity assumption (12) implies that ρ_j and ρ'_j are abelian, and an abelian representation of a knot exterior is determined by the trace of μ;
- for $j = k+1, \ldots, n$, ρ'_j and ρ_j are conjugate by some matrix of D, because an irreducible representation is determined by its character.

Namely, ρ and ρ' differ by *bending* along some of the tori $\Sigma_{k+1}, \ldots, \Sigma_n$. Note that bending along all tori $\Sigma_{k+1}, \ldots, \Sigma_n$ simultaneously by the same matrix in D does not change the conjugacy class. It follows that the pre-image $\Phi^{-1}(p)$ is homeomorphic to

$$\underbrace{(D \times \cdots \times D)}_{n-k}/D \cong \underbrace{(\mathbb{C}^* \times \cdots \times \mathbb{C}^*)}_{n-k}/\mathbb{C}^* \cong (\mathbb{C}^*)^{n-k-1}.$$

As the pre-images of Φ are connected and the image is discrete, those pre-images are the connected components. ∎

From Theorem 3.8 and Lemma 3.9, we obtain Theorem 1.3: the adjoint Reidemeister torsion is locally constant on $\mathcal{X}^c_\mu(M)$. Note that the term $(m - m^{-1})^{2n-2}$ in Theorem 3.8 is the constant $(c^2-4)^{n-1}$ on $\mathcal{X}^c_\mu(M)$.

On the other hand, we have

$$\begin{aligned}
&(c^2-4)^{n-1} \sum_{[\rho] \in \mathcal{X}^c_\mu(M)} \frac{1}{\tau_\mu(M;\rho)} \\
&\quad = \prod_{j=1}^{n} \Bigg(\sum_{[\rho] \in \mathcal{X}^c_\mu(M_j)} \frac{1}{\tau_\mu(M_j;\rho)} + \frac{1}{\tau_\mu(M_j;\alpha_j)} \Bigg) - \prod_{j=1}^{n} \frac{1}{\tau_\mu(M_j;\alpha_j)} \\
&\quad = \sum_{J \subsetneq \{1,\ldots,n\}} \Bigg(\prod_{j \notin J} \sum_{[\rho] \in \mathcal{X}^c_\mu(M_j)} \frac{1}{\tau_\mu(M_j;\rho)} \Bigg) \cdot \prod_{j \in J} \frac{1}{\tau_\mu(M_j;\alpha_j)},
\end{aligned}$$

where J runs on all subsets of $\{1,\dots,n\}$ different from the whole set (J is the subset of indexes j such that the restriction to $\pi_1(M_j)$ is abelian, hence J may be empty but not the whole set). Note that each connected component of $\mathcal{X}^c_\mu(M)$ is homeomorphic to $(\mathbb{C}^*)^l$ for some $0 \le l \le n-1$ and that we understand the sum $\Sigma_{[\rho]\in\mathcal{X}^c_\mu(M)}$ by taking one representative on each connected component of $\mathcal{X}^c_\mu(M)$, which agrees with the ordinary sum for M_j. This completes the proof of Theorem 1.4, because we have assumed that for each $j = 1,\dots,n$:

$$\sum_{[\rho]\in\mathcal{X}^c_\mu(M_j)} \frac{1}{\tau_\mu(M_j;\rho)} = 0.$$

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