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Quantized representations of knot groups

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Abstract. We propose a new non-commutative generalization of the representation variety and the character variety of a knot group. Our strategy is to reformulate the construction of the algebra of functions on the space of representations in terms of Hopf algebra objects in a braided category (braided Hopf algebra). The construction works under the assumption that the algebra is braided commutative. The resulting knot invariant is a module with a coadjoint action. Taking the coinvariants yields a new quantum character variety that may be thought of as an alternative to the skein module. We give concrete examples for a few of the simplest knots and links.

1. Introduction

The discovery of the Jones polynomial brought us a new method to study knots and links, but its relation to the geometric properties of the knot complement was unclear at that moment. After Witten's interpretation in terms of $\mathrm{SU}(2)$ Chern–Simons's theory, R. Kashaev [10] observed a precise relation between quantum invariants and the hyperbolic volume of the knot complement. This was reinterpreted as a relation between the colored Jones invariant and the hyperbolic volume by H. Murakami and the first author in [16]. Moreover, it was observed by Q. Chen and T. Yang in [4] that such relation also holds for the Witten–Reshetikhin–Turaev invariant of closed 3-manifolds. These relations between quantum invariants and hyperbolic volumes are not rigorously proved yet in general and are known as the volume conjecture. In some sense, the volume conjecture means that the colored Jones invariants represent a quantization of the hyperbolic volume. Viewing the hyperbolic structure as a particular flat $\mathrm{SL}(2,\mathbb{C})$ connection, the above was given an interpretation in terms of topological quantum field theory with gauge group $\mathrm{SL}(2,\mathbb{C})$; see [6].

Once we got a relation like the volume conjecture, it is natural to think about quantization of other geometric properties. For example, if the knot is hyperbolic, its complement will be isometric to a quotient of hyperbolic space $\mathbb{H}^3/\Gamma$. The discrete subgroup Γ is isomorphic to the fundamental group of the knot complement (the

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knot group). Using a suitable quantization of the Lie group $\mathrm{SL}(2,\mathbb{C})$ and its discrete subgroup Γ, then the geometric structure of the complement of K may be quantized. More generally, we construct a quantum deformation of the space of representations of the knot group into some linear algebraic group.

Our construction is not 3-dimensional but $(2+1)$-dimensional, as the construction of quantum invariants including the Jones polynomial and the Kontsevich invariant [17]. For a knot K, these invariants are obtained from a braid b whose closure is isotopic to K. The braid b is interpreted as an isotopy of a punctured disk, where the punctured disk is 2-dimensional and the deformation parameter is 1-dimensional. For the Jones polynomial, the braid group action is given by the quantum R-matrix, which comes from the monodromy matrix of conformal field theory. For the Kontsevich invariant, the braid group action is given by the Kontsevich's iterated integral. In both cases, we first consider such action of b, and then take the "quantum trace" of these actions to get an invariant of K.

The starting point for this paper is the space of G representations of the knot group, where G is a linear algebraic group whose coordinate ring has a natural cocommutative Hopf algebra structure. Presenting the knot as a closed braid and interpreting the braid group action in terms of Hopf algebra, we get a description of the representation space that is suitable for generalization. Replacing the coordinate ring of G by a braided Hopf algebra and redoing the exact same construction while taking care of the braiding allows us to quantize the space of representations. To construct a certain "trace", we need evaluation and coevaluation maps, which we do not know how to construct for our case with Hopf algebras and braided Hopf algebras because they might be infinite dimensional. Instead of taking a trace, we just take the b invariant part of the algebra corresponding to the thickened punctured disk and then show that it is independent of the choice of b.

We start by briefly recalling the construction of the space of representations of the knot group Γ_K into a group G that we aim to generalize/quantize in this work. The space of representations is described by an ideal in a tensor power of the coordinate algebra $\mathbb{C}[G]$. The coordinate algebra is generated by the matrix entries, and any presentation of Γ_K allows us to express the relations as polynomial equations in these matrix entries.

This construction works for any finitely presented group and any affine algebraic group and is independent of the chosen presentation; see [2, Proposition 8.2]. However, it is not clear how to generalize this ideal in a non-commutative deformation (i.e., quantizing) because one would need some way to order the variables that no longer commute.

For a knot K, presented as the closure of a braid b, the Wirtinger presentation tells us all relations are given by conjugation. Viewing the relations as equations on the matrix elements of our representation defines an ideal I_b as follows. To prepare our

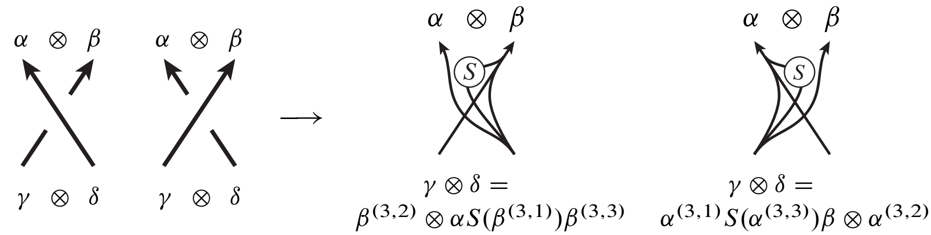


Figure 1. Dual of the relations of the Wirtinger presentation at crossings and their diagrammatic presentations, read top to bottom.

generalization to the non-commutative world, we construct the submodule I_b using the commutative Hopf algebra structure of the coordinate ring $\mathbb{C}[G]$:

$$\begin{aligned}
&\Delta : \mathbb{C}[G] \to \mathbb{C}[G]^{\otimes 2} \text{ with } \Delta(f)(a_1 \otimes a_2) = f(a_1 a_2) && \text{(comultiplication)},\\
&S : \mathbb{C}[G] \to \mathbb{C}[G] \text{ with } S(f)(a) = f(a^{-1}) && \text{(antipode)},\\
&\varepsilon : \mathbb{C}[G] \to \mathbb{C} \text{ with } \varepsilon(f) = f(e) \ (e : \text{the identity of } G) && \text{(counit)}.
\end{aligned}$$

If the braid b is a product of the standard generators $\sigma_1, \dots, \sigma_n$, say

$$b = \sigma_{i_k}^{\varepsilon_k} \sigma_{i_{k-1}}^{\varepsilon_{k-1}} \cdots \sigma_{i_2}^{\varepsilon_2} \sigma_{i_1}^{\varepsilon_1},$$

the ideal I_b is generated by

$$\psi_b - \mathrm{id} : \mathbb{C}[G]^{\otimes n} \to \mathbb{C}[G]^{\otimes n} \quad (i = 1, 2, \dots, n),$$

where ψ_b is given by

$$\psi_b(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) = \big(\psi_{\sigma_{i_k}^{\varepsilon_k}} \circ \psi_{\sigma_{i_{k-1}}^{\varepsilon_{k-1}}} \circ \cdots \circ \psi_{\sigma_{i_2}^{\varepsilon_2}} \circ \psi_{\sigma_{i_1}^{\varepsilon_1}}\big)(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n),$$

and $\psi_{\sigma_i^{\pm 1}}$ is given by

$$\begin{aligned}
&\psi_{\sigma_i}(\alpha_1 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n)\\
&\qquad = \big(\alpha_1 \otimes \cdots \otimes \alpha_{i+1}^{(3,2)} \otimes \alpha_i S(\alpha_{i+1}^{(3,1)})\alpha_{i+1}^{(3,3)} \otimes \cdots \otimes \alpha_n\big),\\
&\psi_{\sigma_i^{-1}}(\alpha_1 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n)\\
&\qquad = \big(\alpha_1 \otimes \cdots \otimes \alpha_i^{(3,1)} S(\alpha_i^{(3,3)})\alpha_{i+1} \otimes \alpha_i^{(3,2)} \otimes \cdots \otimes \alpha_n\big).
\end{aligned} \tag{1.1}$$

Here, we use Sweedler's notation, i.e., the tensor $\alpha^{(3,1)} \otimes \alpha^{(3,2)} \otimes \alpha^{(3,3)}$ means $(\Delta \otimes \mathrm{id})(\Delta(\alpha)) = \sum \alpha^{(3,1)} \otimes \alpha^{(3,2)} \otimes \alpha^{(3,3)}$.

As already mentioned, each generator just acts by conjugation as in the Wirtinger presentation. A diagrammatic interpretation of (1.1) is given in Figure 1. The diagrams

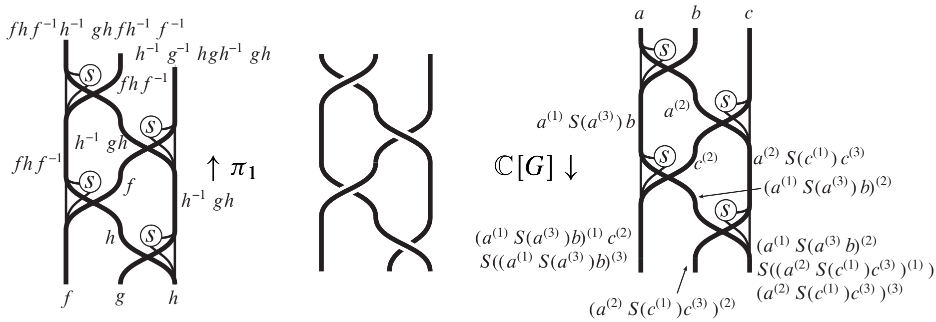


Figure 2. The braid $(\sigma_2\sigma_1^{-1})^2$ interpreted as a Hopf diagram in the group algebra of π_1 (left, read bottom to top) or interpreted in $\mathbb{C}[G]$ (right, read top to bottom).

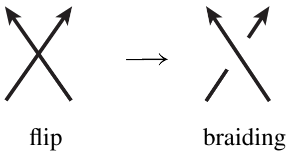


Figure 3. Generalize the flip to the braiding.

should be read top to bottom where each strand represents a copy of the algebra, the Y-shape represents the multiplication, the upside-down Y represents the coproduct, and the S represents the antipode; see also Figure 6. In Figure 2, we showed what happens in the case of the braid $(\sigma_2\sigma_1^{-1})^2$ whose closure is the figure eight knot. Notice that reading the diagrams bottom to top and interpreting the Y-shape as the coproduct in the group algebra of the knot group recovers the corresponding Wirtinger presentation.

The construction of I_b which we sketched above works not only for $\mathbb{C}[\mathrm{SL}(2,\mathbb{C})]$ but also for any commutative Hopf algebra. Our main result is that it also works for braided commutative (braided) Hopf algebras.

A braided Hopf algebra A is a generalization of a Hopf algebra where the braiding is used instead of the usual flip sending $x \otimes y$ to $y \otimes x$ as in Figure 3. Braided commutativity is a generalization of the commutativity property of usual Hopf algebras, which is given in Definition 2.4.

To generalize the above construction of the ideal I_b to get a space of A representations, we modify the relation at the crossing as in Figure 4. Our main result is to define a module I_b and show that the quotient of $A^{\otimes n}$ divided by I_b only depends on the knot K; see Theorem 4.3. In the final example at the end of the paper, we will return to the figure eight knot and show what our construction amounts to in this case.

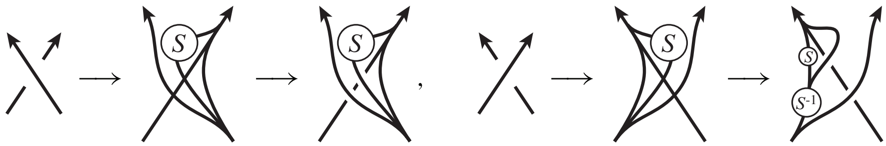


Figure 4. Relations of the braided Wirtinger presentation, read top to bottom.

An important example of a braided Hopf algebra is BSL(2); it is the braided one-parameter deformation of the coordinate ring of $\mathrm{SL}(2, \mathbb{C})$; see [12]. By applying the above construction, we get the space of BSL(2) representations which is a quantization of the $\mathrm{SL}(2, \mathbb{C})$ representation space of K.

Let ${A_b}^A$ be the Ad invariant subspace, i.e.,

$${A_b}^A = \{x \in A^{\otimes n}/I_{d(b)} \mid \mathrm{Ad}(x) = x \otimes 1\}.$$

We call ${A_b}^A$ the *quantum A character variety* of K. If $A = \mathrm{BSL}(2)$, we also call it the *quantum* SL(2) *character variety*. Note that ${A_b}^A$ is not an algebra but an Ad-comodule. So, the quantum character variety is not a variety in the usual sense.

In the special case of SL(2), the quantum character variety which we just defined seems to be equal to the skein module of the knot complement, which is often viewed as a quantization of the SL(2) character variety [11].

Our construction of quantum $\mathrm{SL}(2, \mathbb{C})$ character variety seems to be generated by quantum traces as in [7]. A more detailed discussion of our quantization of the quantum $\mathrm{SL}(2, \mathbb{C})$ character variety will appear in a forthcoming paper. More generally, it seems plausible that our construction is related to the skein module defined for any ribbon category and any 3-manifold in [8, Definition 2.2].

A similar definition of a quantum analogue of the character variety is given by Habiro [9]. It would also be interesting to compare our quantization with the quantization based on ideal triangulations given in [5] and also with the quantization procedure of [1].

This paper is organized as follows. In Section 2, we introduce braided Hopf algebras with a focus on the braided commutative case. We also introduce braided Hopf diagrams to visualize morphisms between tensor powers of a braided Hopf algebra. In Section 3, we construct a representation of the braid group B_n in $\mathrm{End}(A^{\otimes n})$ for any braided Hopf algebra A. Here we use the braided version of the Wirtinger presentation given in Figure 4.

In Section 4, we define the space of A representations of (the group of) a knot K for any braided Hopf algebra A satisfying braided commutativity. Let b be a braid in B_n whose closure $\hat{b}$ is isotopic to K, and add n strands to represent elements of the fundamental group twined to b as in the Hopf diagram d_1 shown in Figure 5. Let I_b

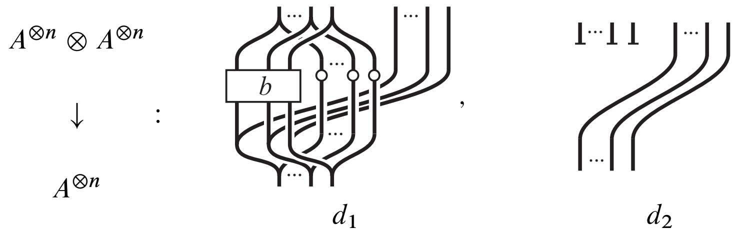


Figure 5. Braided Hopf diagrams of d_1 and d_2, where the space of A representations $A_b = A^{\otimes n}/\mathrm{Im}(d_1 - d_2)$.

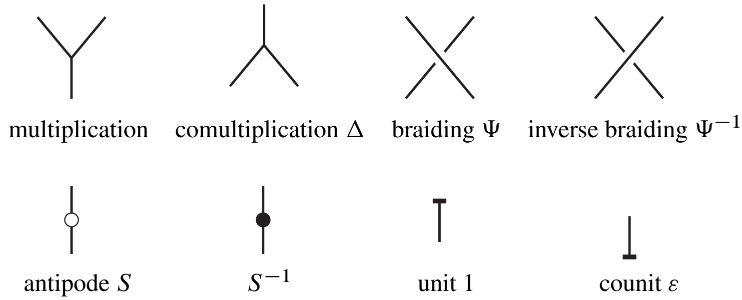


Figure 6. The operations of the braided Hopf algebra A.

be the image of the map corresponding to $d_1 - d_2$; the space of A representations is defined as $A^{\otimes n}/I_b$. We show that this space only depends on the isotopy type of $\hat{b}$.

In Section 5, we illustrate our constructions by applying them to the trefoil knot, the Hopf link, and the figure eight knot.

2. Braided Hopf algebra and braided commutativity

2.1. Braided Hopf algebra

A braided Hopf algebra is a version of a Hopf algebra having an extra operation called braiding. It may also be viewed as a Hopf object in a braided monoidal category. Such algebras are quite common in that they can be produced from any quasi-triangular Hopf algebra by transmutation [15]. These structures also go by the name braided group.

Definition 2.1. An algebra A over a field k is called a *braided Hopf algebra* if it is equipped with following linear maps described by the diagrams in Figure 6, satisfying

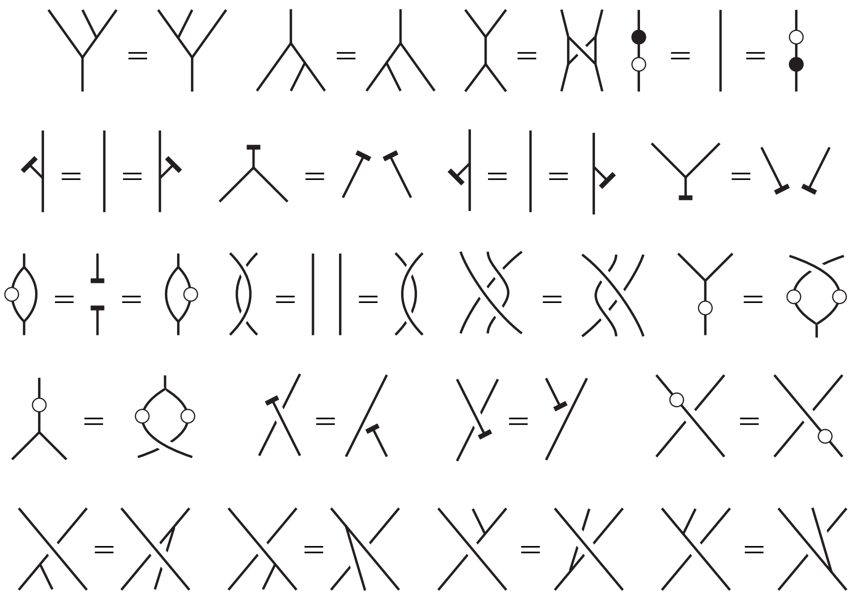


Figure 7. The relations of a braided Hopf algebra, read from top to bottom.

the relations given in Figure 7:

multiplication	$\mu : A \otimes A \to A$,	comultiplication	$\Delta : A \to A \otimes A$,
unit	$1 : k \to A$,	counit	$\varepsilon : A \to k$,
antipode	$S : A \to A$,	braiding	$\Psi : A \otimes A \to A \otimes A$.

Definition 2.2. A diagram expressing a linear mapping from $A^{\otimes m}$ to $A^{\otimes n}$ built from a combination of the Hopf algebra operations given in Figure 6 is called a *braided Hopf diagram*. Let $\mathrm{BHD}(m, n)$ denote the set of braided Hopf diagrams expressing linear homomorphisms from $A^{\otimes m}$ to $A^{\otimes n}$.

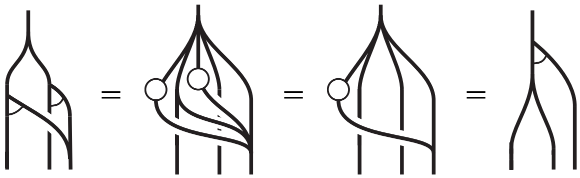
2.2. Adjoint coaction

A k-vector space M is called a right A-comodule if there is a linear map

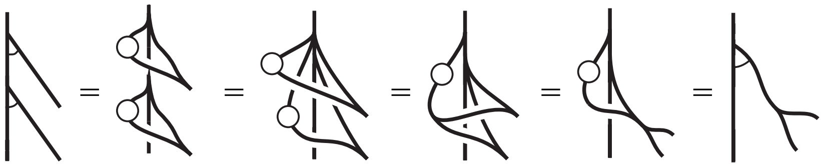
$$\Delta : M \to M \otimes A,$$

satisfying the coassociativity

$$(\Delta \otimes \mathrm{id})(\Delta) = (\mathrm{id} \otimes \Delta)(\Delta).$$



(2.1): Commutativity of adjoint and comultiplication.

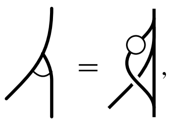


(2.2): Adjoint and multiplication.

Figure 8. Graphical proof of Proposition 2.3.

Then, A itself is a right A-comodule with the following adjoint coaction $\mathrm{ad} : A \to A \otimes A$:

$$\mathrm{ad}(x) = (\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})(S \otimes \Delta)\Delta(x),$$



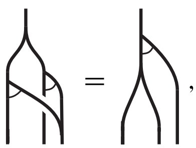
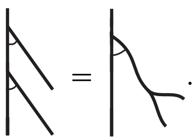
where $\mu : A \otimes A \to A$ is the multiplication of A, i.e., $\mu(x \otimes y) = xy$.

Proposition 2.3 (Cf. [14, Proposition A.1]). *Adjoint coaction satisfies the following relations:*

$$(\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\mathrm{ad} \otimes \mathrm{ad})\Delta(x) = (\Delta \otimes \mathrm{id})\mathrm{ad}(x), \tag{2.1}$$

$$(\mathrm{ad} \otimes \mathrm{id})\mathrm{ad} = (\mathrm{id} \otimes \Delta)\mathrm{ad}, \tag{2.2}$$

$$(\varepsilon \otimes \mathrm{id})\mathrm{ad} = 1 \circ \varepsilon \qquad (\mathrm{id} \otimes \varepsilon)\mathrm{ad} = \mathrm{id}, \tag{2.3}$$

(2.1):  , (2.2):  .

Proof. Relations (2.1) and (2.2) are proved by the graphical computation in Figure 8. Relations (2.3) come from the properties of the unit 1 and the counit ε. ∎

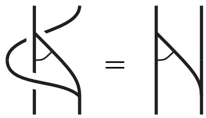


Figure 9. Braided commutativity in terms of braided Hopf diagrams.

2.3. Braided commutativity

We introduce the notion of braided commutativity, which implies the compatibility of the adjoint coaction with respect to the multiplication μ, the braiding Ψ, and the antipode S.

Definition 2.4. The braided Hopf algebra A is *braided commutative* if it satisfies

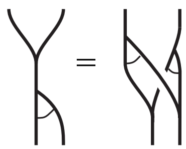
$$(\mathrm{id} \otimes \mu)(\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{ad})\Psi = (\mathrm{id} \otimes \mu)(\mathrm{ad} \otimes \mathrm{id}).$$

This relation is explained graphically in Figure 9.

Braided commutativity was introduced in [13], and it is shown there that many interesting braided Hopf algebras have this property. For example, transmutation procedure always produces braided commutative braided Hopf algebras. In the remainder of this section, we assume that A is braided commutative.

Proposition 2.5. *The adjoint coaction commutes with the multiplication, i.e.,*

$$(\mathrm{ad} \circ \mu) = (\mu \otimes \mu)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\mathrm{ad} \otimes \mathrm{ad}). \tag{2.4}$$



Proof. Relation (2.4) is proved by the graphical computation in Figure 10. At the second to last equality, we use braided commutativity. In the rest of this paper, an equality using braided commutativity is denoted by $\underset{\text{(bc)}}{=}$. ■

Proposition 2.6. *The adjoint coaction commutes with the braiding Ψ as follows:*

$$\begin{aligned}&(\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\mathrm{ad} \otimes \mathrm{ad})\Psi \\ &\quad = (\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\mathrm{ad} \otimes \mathrm{ad}).\end{aligned}$$

Proof. This relation comes from braided commutativity as explained in Figure 11. ■

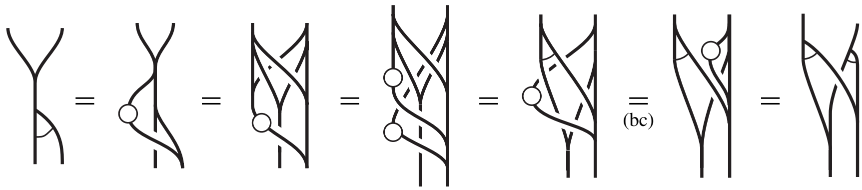


Figure 10. Adjoint is an algebra homomorphism.

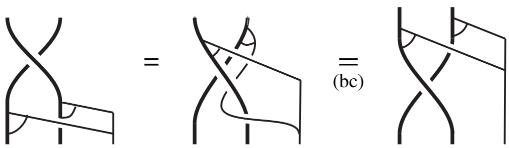


Figure 11. The adjoint coaction is commutative with the braiding.

Proposition 2.7. *The adjoint coaction commutes with the antipode S, i.e.,*

$$\mathrm{ad} \circ S = (S \otimes \mathrm{id}) \circ \mathrm{ad}.$$

Proof. This relation comes from braided commutativity as explained in Figure 12. Braided commutativity is used in the second equality. In the fourth equality, we used the antipode axiom, and in the final equation, the axiom relating S and multiplication are used. ∎

Proposition 2.8. *The adjoint* ad *and the antipode S satisfy* $\mu \circ (\mathrm{id} \otimes S) \circ \mathrm{ad} = S^2$.

Proof. This comes from the equalities of diagrams in Figure 13. ∎

3. Representation of braid groups

In this section, we recall the representation of the braid group B_n to $\mathrm{End}(A^{\otimes n})$ constructed by using the adjoint action of A. To construct representations of braid groups, A is not required to be braided commutative. However, for the distributivity of the representation given by Proposition 3.3, A has to be braided commutative.

3.1. Representation of generators

The braid group B_n is defined by the following generators and relations:

$$B_n = \big\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i (|i-j| \geq 2) \big\rangle.$$

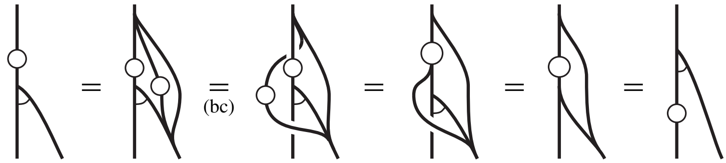


Figure 12. The adjoint ad commutes with S.

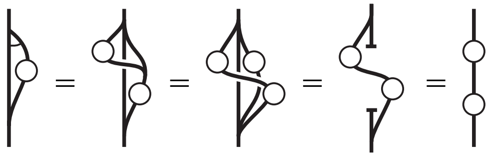


Figure 13. Proof for $\mu \circ (\mathrm{id} \otimes S) \circ \mathrm{ad} = S^2$.

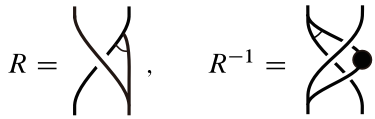


Figure 14. R, R^{-1} for crossings.

We define a braided Hopf diagram corresponding to the braid generators by generalizing the definition of R_{ad} in [3], which is based on [18]. These are braided versions of the Wirtinger presentation for the fundamental group of a knot complement. For $\sigma_i^{\pm 1} \in B_n$, let

$$\rho(\sigma_i^{\pm 1}) = \mathrm{id}^{\otimes(i-1)} \otimes R^{\pm 1} \otimes \mathrm{id}^{\otimes(n-i-1)} \in \mathrm{End}(A^{\otimes n}), \tag{3.1}$$

where $R^{\pm 1} : A^{\otimes 2} \to A^{\otimes 2}$ is given in Figure 14.

In the rest, we use the maps Δ_i, μ_i, S_i, Ψ_i, and ε_i acting on $A^{\otimes n}$. They are given by the following:

$$\begin{aligned}
\Delta_i &= \mathrm{id}^{i-1} \otimes \Delta \otimes \mathrm{id}^{n-i}, & \mu_i &= \mathrm{id}^{i-1} \otimes \mu \otimes \mathrm{id}^{n-i-1},\\
S_i &= \mathrm{id}^{i-1} \otimes S \otimes \mathrm{id}^{n-i}, & \Psi_i &= \mathrm{id}^{i-1} \otimes \Psi \otimes \mathrm{id}^{n-i-1},\\
\varepsilon_i &= \mathrm{id}^{i-1} \otimes \varepsilon \otimes \mathrm{id}^{n-i}.
\end{aligned}$$

We also use the generalized multiplication $\boldsymbol{\mu}^{(m)} : A^{\otimes m} \otimes A^{\otimes m} \to A^{\otimes m}$ and the generalized coproduct $\boldsymbol{\Delta}^{(m)} : A^{\otimes m} \to A^{\otimes m} \otimes A^{\otimes m}$ given by the diagrams in Figure 15.

3.2. Adjoint coaction

We define an adjoint coaction $\mathrm{Ad} : A^{\otimes n} \to A^{\otimes n} \otimes A$ as in Figure 16.

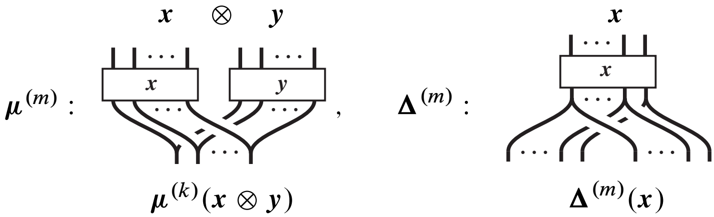


Figure 15. The generalized multiplication $\mu^{(m)} : A^{\otimes m} \otimes A^{\otimes m} \to A^{\otimes m}$ and the generalized coproduct $\Delta^{(m)} : A^{\otimes m} \to A^{\otimes m} \otimes A^{\otimes m}$.

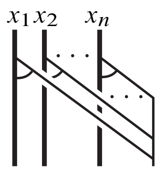


Figure 16. Adjoint coaction $\mathrm{Ad} : A^{\otimes n} \to A^{\otimes n} \otimes A$.

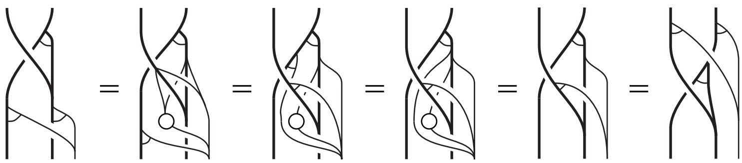


Figure 17. Commutativity of Ad and R.

Proposition 3.1. *The adjoint coaction* Ad *commutes with* $\rho(b)$ *for* $b \in B_n$, *i.e.,*

$$\mathrm{Ad} \circ \rho(b) = (\rho(b) \otimes \mathrm{id}) \circ \mathrm{Ad}. \tag{3.2}$$

Proof. This comes from the following commutativity of R and Ad:

$$\mathrm{Ad} \circ R = (R \otimes \mathrm{id}) \circ \mathrm{Ad}. \tag{3.3}$$

This is proved by the graphical computation in Figure 17. ∎

3.3. Representation of braid groups

Now, we construct a representation of braid groups in $\mathrm{End}(A^{\otimes n})$.

Theorem 3.2 ([18, Proposition 1]). *The map* ρ *defined for generators of* B_n *in* (3.1) *extends to an algebra homomorphism from the group algebra* $\mathbb{C} B_n$ *to* $\mathrm{End}(A^{\otimes n})$.

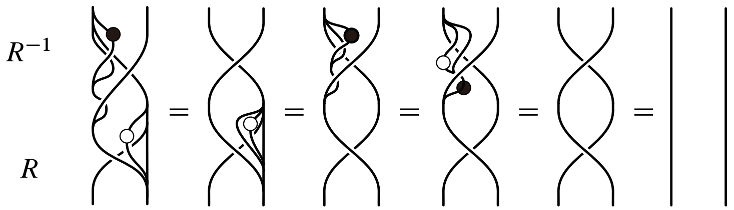


Figure 18. $RR^{-1} = \mathrm{id} \otimes \mathrm{id}$.

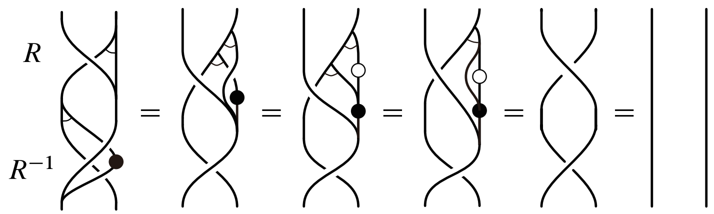


Figure 19. $R^{-1}R = \mathrm{id} \otimes \mathrm{id}$.

Proof. We first show that $\rho(\sigma_i)\rho(\sigma_i^{-1}) = \rho(\sigma_i^{-1})\rho(\sigma_i) = 1$ by proving

$$RR^{-1} = R^{-1}R = \mathrm{id} \otimes \mathrm{id}$$

using the graphical computation in Figures 18 and 19. The braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ comes from

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R),$$

which is shown by the graphical computation in Figure 20. We also have $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $j - i \geq 2$ since

$$R_i R_j = \mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-j-2)} \otimes R \otimes \mathrm{id}^{\otimes(n-j-1)} = R_j R_i,$$

where $R_i = \mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)}$. Hence, the relations of B_n are all satisfied. ∎

3.4. Distributivity of $\rho(b)$

The representation $\rho(b)$ is distributive over the multiplication as follows.

Proposition 3.3. *Assume that A is braided commutative. For $\boldsymbol{x}, \boldsymbol{y} \in A^{\otimes n}$, we have*

$$\rho(b)\boldsymbol{\mu}^{(n)}(\boldsymbol{x} \otimes \boldsymbol{y}) = \boldsymbol{\mu}^{(n)}(\rho(b)\boldsymbol{x} \otimes \rho(b)\boldsymbol{y}).$$

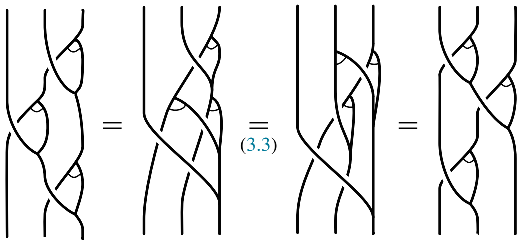


Figure 20. Braid relation $(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R)$.

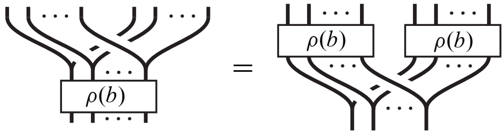


Figure 21. The representation ρ is distributive.

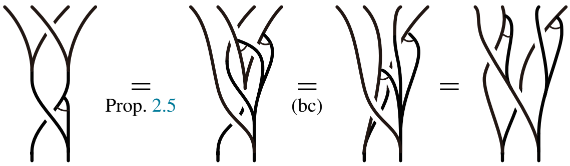


Figure 22. R is distributive over the multiplication.

This relation is visualized in Figure 21.

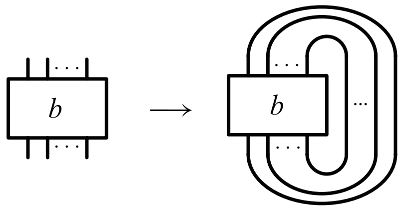
Proof. It is enough to show that

$$R\boldsymbol{\mu}^{(2)}(\boldsymbol{x} \otimes \boldsymbol{y}) = \boldsymbol{\mu}^{(2)}(R \otimes R)(\boldsymbol{x} \otimes \boldsymbol{y})$$

for the multiplication $\boldsymbol{\mu}^{(2)} : A^{\otimes 2} \otimes A^{\otimes 2} \to A^{\otimes 2}$ and $\boldsymbol{x} = x_1 \otimes x_2,\ \boldsymbol{y} = y_1 \otimes y_2 \in A^{\otimes 2}$, which is proved graphically in Figure 22. ■

4. Space of braided Hopf algebra representations of a knot

Throughout this section, A is a braided Hopf algebra that is braided commutative. For any knot K, we construct the space of A representations of K as a quotient of $A^{\otimes n}$ by a module I_b determined by a braid $b \in B_n$ whose closure is K. The number n and the



b $\qquad\qquad$ $\hat{b}$: the closure of b

Figure 23. Braid closure.

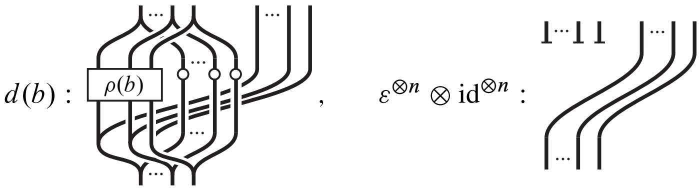


Figure 24. The braided Hopf diagrams for $d(b)$ and $\boldsymbol{\varepsilon}_n \otimes \mathrm{id}^{\otimes n}$.

module I_b depend on the choice of the braid b, but it is shown that if b and b' have isotopic closures, then $A^{\otimes n}/I_b$ and $A^{\otimes n'}/I_{b'}$ are isomorphic as Ad comodules.

4.1. Knots as braid closures

Let K be a knot in S^3; then it is known that there is a braid $b \in B_n$ for certain $n \in \mathbb{N}$ such that K is isotopic to the closure of b. The closure of b is obtained by connecting the top points and bottom points of b as in Figure 23 and is denoted by $\hat{b}$.

4.2. Space of A representations

For $b \in B_n$, let $d(b)$ be the braided Hopf diagram given in Figure 24 (left). Then, $d(b)$ is an element in $\mathrm{Hom}(A^{\otimes 2n}, A^{\otimes n})$. Let us assign $x_1, x_2, \ldots, x_n, y_1, \ldots, y_n \in A$ to the top points of $d(b)$; let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_n$, $\boldsymbol{y} = y_1 \otimes \cdots \otimes y_n$, and $d(b)(\boldsymbol{x} \otimes \boldsymbol{y})$ be the image of $\boldsymbol{x} \otimes \boldsymbol{y}$ by $d(b)$, which is an element in $A^{\otimes n}$. Let

$$I_{d(b)} = \mathrm{Im}\big(d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}\big).$$

Definition 4.1. A submodule I of $A^{\otimes n}$ is called an Ad-*comodule* if $\mathrm{Ad}(I) \subset I \otimes A$. A morphism between two Ad-comodules I and J is a module map $f : I \to J$ that commutes with Ad in the sense that $\mathrm{Ad} \circ f = (f \otimes \mathrm{id}_A) \circ \mathrm{Ad}$.

Proposition 4.2. *$I_{d(b)}$ is an* Ad*-comodule of $A^{\otimes n}$.*

Proof. From (2.1), (2.4) and (3.2), we have

$$\mathrm{Ad} \circ d(b) = (d(b) \otimes \mathrm{id}) \circ \mathrm{Ad} \quad \text{and} \quad \mathrm{Ad} \circ (\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}) = (\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes (n+1)}) \circ \mathrm{Ad}.$$

Therefore,

$$\mathrm{Ad} \circ (d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}) = (d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes (n+1)}) \circ \mathrm{Ad}$$

and the image of $\mathrm{Ad} \circ (d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})$ is contained in $I_{d(b)} \otimes A$. ■

Let $A_b = A^{\otimes n}/I_{d(b)}$; then A_b is an Ad-comodule of $A^{\otimes n}$ and it satisfies the following.

Theorem 4.3. *If the closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic, then A_{b_1} and A_{b_2} are isomorphic* Ad*-comodules. In other words, A_b is an invariant of the knot (or link) $\hat{b}$.*

Definition 4.4. The Ad-comodule A_b is called *the space of A representations* of the closure $\hat{b}$.

The Ad-comodule structure on A_b allows us to pass to the coinvariants. This should generalize the conjugation invariant functions on the representation variety, and hence, we introduce the following definition.

Definition 4.5. We say the *quantum A character variety* of K is the module of coinvariants ${A_b}^A$ under the coaction of Ad on A_b.

It should be noted that our quantum A character variety is not an algebra but only a module.

4.3. Equivalent pairs

To prepare our proof of the main theorem, Theorem 4.3, we introduce the notion of equivalent pairs of Hopf diagrams.

Definition 4.6. Suppose that $b \in B_n$ is a braid. The braided Hopf diagrams $d_1, d_2 \in \mathrm{BHD}(m, n)$ are called *b-equivalent* if $A^{\otimes n}/(d_1 - d_2)(A^{\otimes m})$ is isomorphic to A_b as Ad-comodules. When d_1 and d_2 are b-equivalent, we will denote this by $d_1 \sim_b d_2$.

For example, we always have $d(b) \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. Let M_n be the kernel of $\varepsilon^{\otimes n}$ in $A^{\otimes n}$. For $d \in \mathrm{BHD}(2n, n)$, we denote the induced map from $(A^{\otimes n}/M_n) \otimes A^{\otimes n}$ to $A^{\otimes n}/d(M_n \otimes A^{\otimes n})$ by $\bar{d}$.

Lemma 4.7. *If $d \in \mathrm{BHD}(2n, n)$ satisfies $d(1^{\otimes n} \otimes \boldsymbol{y}) = \boldsymbol{y}$, then the image of $d - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$ is equal to $d(M_n \otimes A^{\otimes n})$. Especially, $I_{d(b)} = d(b)(M_n \otimes A^{\otimes n})$.*

Proof. The assumption $d(1^{\otimes n} \otimes \boldsymbol{y}) = \boldsymbol{y}$ implies that

$$(d - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})(1^{\otimes n} \otimes \boldsymbol{y}) = 0.$$

Since $A^{\otimes n} = \mathbb{C}(1^{\otimes n}) \oplus M_n$ and $\varepsilon^{\otimes n}(M_n) = 0$, we get

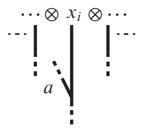
$$\begin{aligned}(d - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})(A^{\otimes n} \otimes A^{\otimes n}) &= (d - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})(M_n \otimes A^{\otimes n}) \\ &= d(M_n \otimes A^{\otimes n}).\end{aligned}$$

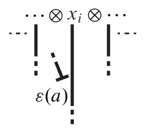
The last statement comes from the fact that $d(b)(1^{\otimes n} \otimes \boldsymbol{y}) = \boldsymbol{y}$. ∎

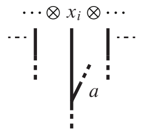
4.4. Equivalences at the top of braided Hopf diagrams (BHD)

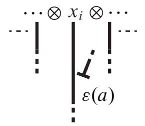
Let d be an element of $\mathrm{BHD}(2n, n)$ which is b equivalent to $\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. The following proposition shows that we can modify d at certain kinds of multiplications, adjoints, and braidings near the top of the diagram d so that the corresponding $\bar{d}$ does not change.

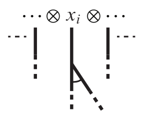
Proposition 4.8. *Let $d_1, d_2 \in \mathrm{BHD}(2n, n)$ be a pair of braided Hopf diagrams shown below. Assuming that $d_j(1^{\otimes n} \otimes \boldsymbol{y}) = \boldsymbol{y}$ for $j = 1, 2$, we have $d_1 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$ if and only if $d_2 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. In the pictures, the index i is in $\{1, 2, \ldots, n\}$.*

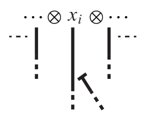
HmL d_1 

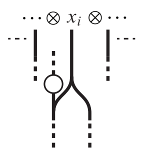
$\longleftrightarrow$ d_2 

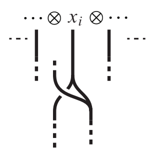
HmR d_1 

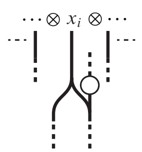
$\longleftrightarrow$ d_2 

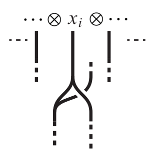
Ha d_1 

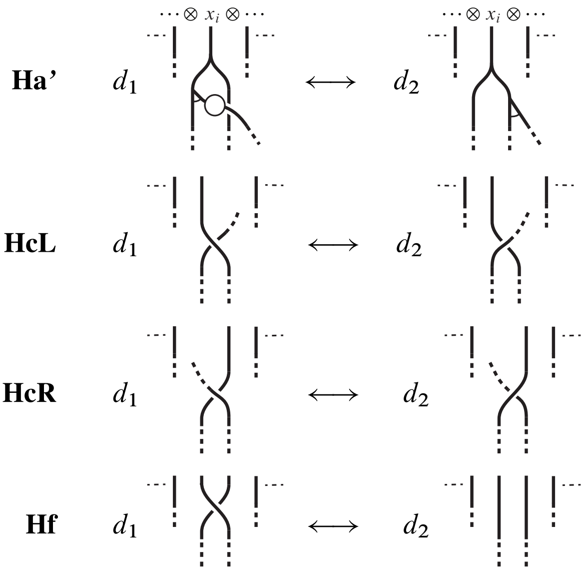
$\longleftrightarrow$ d_2 

HmL’ d_1 

$\longleftrightarrow$ d_2 

HmR’ d_1 

$\longleftrightarrow$ d_2 



Proof. **HmL, HmR:** Let d_1, d_2 be the braided Hopf diagrams of **HmL**, $I_1 = d_1(M_n \otimes A^{\otimes n})$ and $I_2 = d_2(M_n \otimes A^{\otimes n})$. Let $\overline{d_1}$ be the map from $A^{\otimes 2n}/\ker \varepsilon_i$ to $A^{\otimes n}/I_1$ induced by d_1. The multiplication of x_i by a in the diagram d_1 is the multiplication of $\bar{x}_i \in A/M_1$ by a, which is equal to $\varepsilon(a)\bar{x}_i$. Therefore, $\bar{d}_1 = \bar{d}_2$, where $\bar{d}_2$ is the map from $A^{\otimes 2n}/\ker \varepsilon_i$ to $A^{\otimes n}/I_1$ given by d_2. This implies that $I_2 \subset I_1$.

On the other hand, let $\widetilde{d_2}$ be the map from $A^{\otimes 2n}/\ker \varepsilon_i$ to $A^{\otimes n}/I_2$ induced by d_2. The multiplication of a and x_i in the diagram d_1 is the multiplication of $\bar{x}_i \in A/M_1$ by a, which is equal to $\varepsilon(a)\bar{x}_i$. Therefore, $\tilde{d}_2 = \tilde{d}_1$, where $\tilde{d}_1$ is the map from $A^{\otimes 2n}/\ker \varepsilon_i$ to $A^{\otimes n}/I_2$ given by d_1. This implies $I_1 \subset I_2$. Hence, we get $I_1 = I_2$. This means that $I_1 = I_{d(b)}$ if and only if $I_2 = I_{d(b)}$, which implies that $d_1 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$ if and only if $d_2 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$ by Lemma 4.7. The proof for d_1, d_2 in **HmR** is similar.

Ha: Let d_1, d_2 be the braided Hopf diagrams of **Ha**, $I_1 = d_1(M_n \otimes A^{\otimes n})$ and $I_2 = d_2(M_n \otimes A^{\otimes n})$.

Since $(\varepsilon \otimes \mathrm{id})\mathrm{ad}(M_1) = (1 \otimes 1)\varepsilon(M_1) = 0$, $\mathrm{ad}(M_1)$ is contained in $M_1 \otimes A$. Hence, ad induces the map $\overline{\mathrm{ad}}$ from A/M_1 to $(A/M_1) \otimes A$. On the other hand, A/M_1 is spanned by 1 and $\mathrm{ad}(1) = 1 \otimes 1$, $\overline{\mathrm{ad}}(\bar{x}) = \bar{x} \otimes 1$ for $\bar{x} \in A/M_1$. This relation means that $\bar{d}_1 = \overline{d_2}$ as a map from $A^{\otimes 2n}/\ker \varepsilon_i$ to $A^{\otimes n}/I_1$. Therefore, $I_2 \subset I_1$. By exchanging the role of d_1 and d_2 as in the case of **HmL**, we get $I_1 \subset I_2$. Hence, $I_1 = I_2$. Therefore, $d_1 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$ if and only if

$$d_2 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}.$$

HmL', HmR': Let d_1, d_2 be the braided Hopf diagrams of **HmL'**. Assume that $d_1 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. In the pictures, the symbol $a \leftrightarrow b$ means that a and b are both b

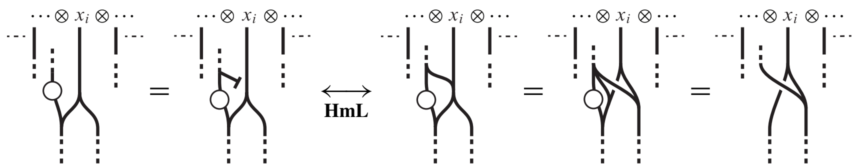


Figure 25. Proof for **HmL'**.

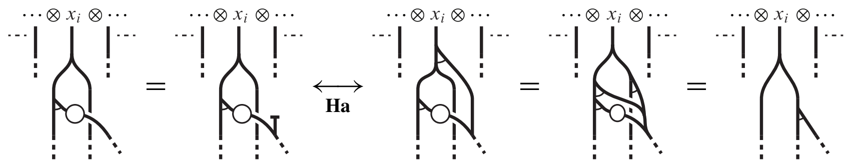


Figure 26. Proof for **Ha'**.

equivalent to $\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. By **HmL**, we have a sequence of equalities as in Figure 25 as a map from $(A^{\otimes n}/M_n) \otimes A^{\otimes n}$ to A_b. Hence,

$$d_1(M_m \otimes A^{\otimes n}) = d_2(M_n \otimes A^{\otimes n}) \quad \text{and} \quad d_2 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}.$$

The proof for **HmR'** is similar.

Ha': Let d_1, d_2 be the braided Hopf diagrams of **Ha'**. Assume that

$$d_1 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}.$$

By **Ha**, we have a sequence of equalities in Figure 26 as a map from $(A^{\otimes n}/M_n) \otimes A^{\otimes n}$ to A_b. In the third and fourth equalities, we used Proposition 2.3. Hence,

$$d_1(M_m \otimes A^{\otimes n}) = d_2(M_n \otimes A^{\otimes n}) \quad \text{and} \quad d_2 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}.$$

The opposite direction is proved similarly.

HcL, HcR: Let d_1, d_2 be the braided Hopf diagrams of **HcL**. Assume that

$$d_1 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}.$$

We have a sequence of equalities as in Figure 27 as a map from $A^{\otimes n}/M_n \otimes A^{\otimes n}$ to A_b. Hence, $d_1(M_n \otimes A^{\otimes n}) = d_2(M_n \otimes A^{\otimes n})$ and $d_2 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. The proof for **HcR** is given in Figure 28.

Hf: The relation **Hf** comes from the facts that Ψ is an automorphism of $A^{\otimes 2}$ and $\varepsilon^{\otimes 2} \circ \Psi = \varepsilon^{\otimes 2}$. ∎

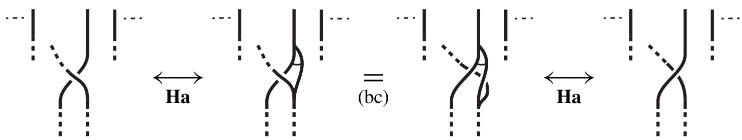


Figure 27. Proof for **HcL**. Recall that bc stands for braided commutativity.

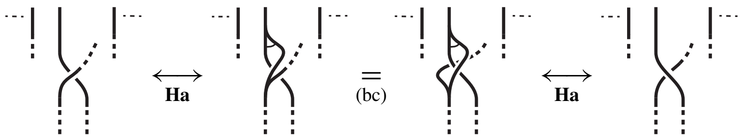


Figure 28. Proof for **HcR**.

Proposition 4.9. *Let d be an element of* $\mathrm{BHD}(2n, n)$ *which is b-equivalent to* $\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. *Let d' be an element of* $\mathrm{BHD}(2n, 2n)$ *which gives an isomorphism from $A^{\otimes 2n}$ to $A^{\otimes 2n}$ such that*

$$(\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}) \circ d' = \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}.$$

Then, $d \circ d' \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. Especially, for the diagram d_{ij} having an arc connecting the $(n+j)$-th strand to the i-th strand as in Figure 29, then

$$d \circ d_{ij} \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}.$$

Moreover, for the diagrams d_1 and d_2 in Figure 29, $d_1 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$ if and only if $d_2 \sim_b \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$.

Proof. Since d' is an isomorphism, the image of $(d - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}) \circ d' = d \circ d' - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$ is equal to $I_{d(b)}$. Therefore, $d \circ d'$ is b equivalent to $\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. The diagram d_{ij} is an isomorphism since adding the antipode S to the arc connecting the $(n+j)$-th strand to the i-th strand of d_{ij}, we get the inverse of d_{ij}.

By adding d_{ij} to the top of d_2, we get d_1. This implies the last statement of the proposition. ■

4.5. Another expression of $d(b)$

Below we will show an alternative way of expressing the braided Hopf diagram $d(b)$ viewed as a map $\overline{d(b)} : A^{\otimes n}/M_n \otimes A^{\otimes n} \to A_b$. Although this expression for $\overline{d(b)}$ will not be used in this text, we include it because it corresponds more naturally to the closed braid of b. It also suggests that the construction given here for a closed braid may extend to a plat presentation of a knot. We will elaborate on this point in a future publication.

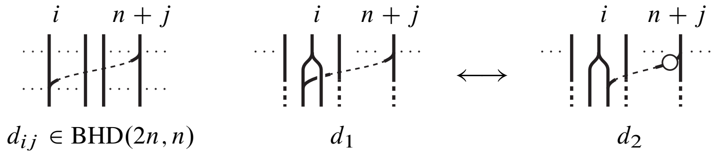


Figure 29. The diagram d_{ij} having an arc connecting the $(n+j)$-th strand to the i-th strand and the move comes from this diagram.

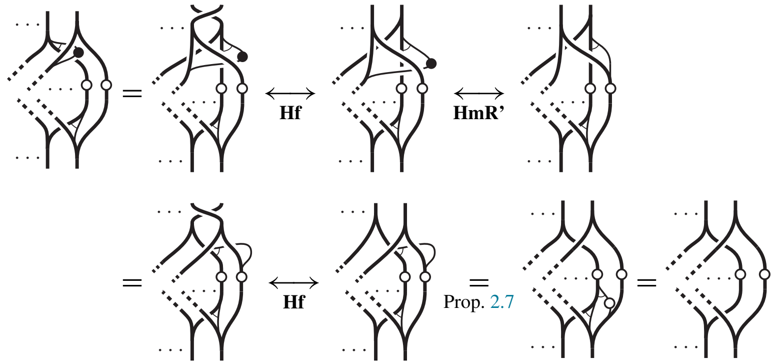


Figure 30. Cancellation of the topmost σ_{2n-2}^{-1} and the bottom-most σ_{2n-2}.

Proposition 4.10. *For $b \in B_n$, the map $\overline{d(b)}$ induced by $d(b)$ satisfies*

$$\begin{aligned}\overline{d(b)} = {} & (\mu_1\mu_3\cdots\mu_{2n-1}) \\ & \circ \big((\sigma_{2n-2}\sigma_{2n-4}\cdots\sigma_2)(\sigma_{2n-3}\sigma_{2n-5}\cdots\sigma_3)\cdots(\sigma_{n+1}\sigma_{n-1})\sigma_n\big) \\ & \circ (\boldsymbol{\mu}^{(n)} \otimes \mathrm{id}^{\otimes n}) \circ (\mathrm{id}^{\otimes n} \otimes \Psi^{(n)}) \circ (b \otimes S^{\otimes n} \otimes \mathrm{id}^{\otimes n}) \\ & \circ \big(\sigma_n(\sigma_{n-1}^{-1}\sigma_{n+1}^{-1})\cdots(\sigma_3^{-1}\sigma_5^{-1}\cdots\sigma_{2n-3}^{-1})(\sigma_2^{-1}\sigma_4^{-1}\cdots\sigma_{2n-2}^{-1})\big) \\ & \circ (\Delta_{2n-1}\Delta_{2n-3}\cdots\Delta_1)\end{aligned}$$

as a map from $A^{\otimes n}/M_n \otimes A^{\otimes n}$ to A_b, where $\Psi^{(n)}$ is defined as

$$\Psi^{(n)} = (\Psi_n\Psi_{n-1}\cdots\Psi_1)(\Psi_{n+1}\Psi_n\cdots\Psi_2)\cdots(\Psi_{2n-1}\Psi_{2n-2}\cdots\Psi_n).$$

$\Psi^{(n)}$ is a composition of n^2 maps Ψ_j; see also Figure 37.

Proof. We first remove the first and last σ_{2n-2} as in Figure 30. Then, remove σ_{2n-3}, $\sigma_{2n-4}, \ldots, \sigma_{2n}$ similarly along the rightmost string. After doing these operations, do similar operations along the string next to the rightmost string. Repeat these operations for all σ's along the strings with the antipode S. ∎

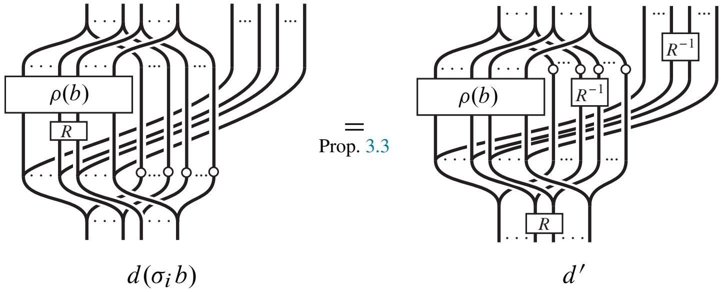


Figure 31. Deform $d(b\sigma_i)$ to d' as a map from $A^{\otimes n}/M_n$ to $A_{b\sigma_i}$.

4.6. Markov moves

It is known that the closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic in S^3 if and only if there is a sequence of the following two types of moves connecting b_1 to b_2. These moves are called the Markov moves, and such b_1 and b_2 are called *Markov equivalent*.

First Markov move (MI). $bb' \leftrightarrow b'b$ for $b, b' \in B_n$.

Second Markov move (MII). $b \in B_n \leftrightarrow \sigma_n^{\pm 1} b \in B_{n+1}$.

Theorem 4.11. *The quotient algebras A_{b_1} and A_{b_2} are isomorphic if b_1 and b_2 are Markov equivalent.*

The proof of this theorem consists of Propositions 4.12, 4.14, and 4.15 in the following two subsections.

4.7. Invariance under the MI move

First, we show that the quotient algebra keeps its structure when we apply an MI move.

Proposition 4.12. *For $b_1, b_2 \in B_n$, $A_{b_2 b_1}$ is isomorphic to $A_{b_1 b_2}$.*

This comes from the following lemma.

Lemma 4.13. *For $b \in B_n$, $A_{\sigma_i b}$ is isomorphic to $A_{b\sigma_i}$. Also, $A_{\sigma_i^{-1} b}$ is isomorphic to $A_{b\sigma_i^{-1}}$.*

Proof. As a map from $A^{\otimes n}/M_n \otimes A^{\otimes n}$ to $A_{b\sigma_i}$, $d(\sigma_i b)$ is deformed as in Figures 31 and 32, whereas R and R^{-1} are given in Figure 14. In Figure 32, R^{-1} in Figure 31 is moved upward and switched to the left strands by using the moves in Proposition 4.8. After we apply the transformation suggested in Figure 32 to Figure 31, we

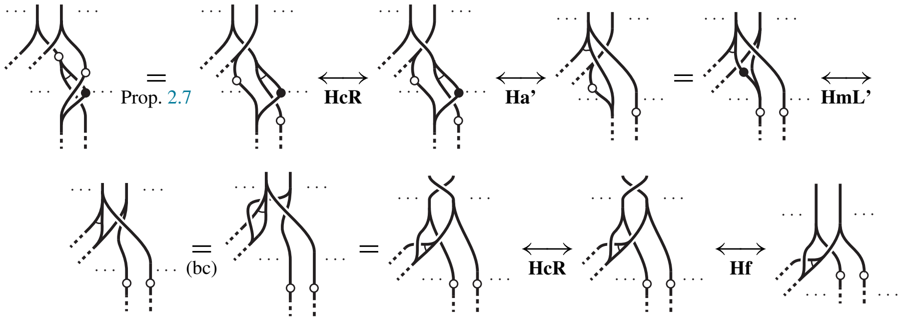


Figure 32. Deform d' as a map from $(A^{\otimes n}/M_n) \otimes A^{\otimes n}$ to $A_{b\sigma_i}$.

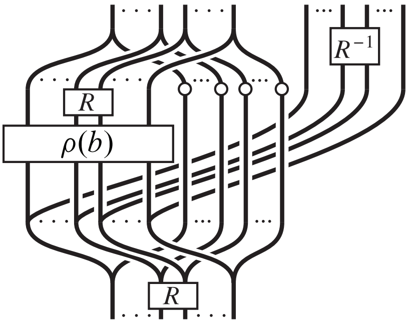


Figure 33. Result of the deformation of $d(\sigma_n b)$ in Figure 32.

get Figure 33. It follows that $\sigma_i d(b\sigma_i)(\mathrm{id}^{\otimes n} \otimes \sigma_i^{-1}) \sim_{\sigma_i b} \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. Since σ_i^{-1} is an automorphism of $A^{\otimes n}$, we have $\sigma_i d(b\sigma_i) \sim_{b\sigma_i} \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}$. This implies that

$$
\begin{aligned}
I_{d(\sigma_i b)} &= \big(\sigma_i d(b\sigma_i) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}\big)(A^{\otimes n} \otimes A^{\otimes n}) \\
&= \big(\sigma_i d(b\sigma_i) - \sigma_i \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}\big)(A^{\otimes n} \otimes A^{\otimes n}) \\
&= \sigma_i \big(d(b\sigma_i) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}\big)(A^{\otimes n} \otimes A^{\otimes n}) = \sigma_i I_{d(b\sigma_i)}.
\end{aligned}
$$

Hence, the left multiplication of σ_i^{-1} induces an isomorphism from $A^{\otimes n}/I_{d(\sigma_i b)}$ to $A^{\otimes n}/I_{d(b\sigma_i)}$.

For $\sigma_i^{-1} b$, we have

$$
A_{b\sigma_i^{-1}} = A_{\sigma_i \sigma_i^{-1} b \sigma_i^{-1}} \cong A_{\sigma_i^{-1} b \sigma_i^{-1} \sigma_i} = A_{\sigma_i^{-1} b}
$$

since $A_{b\sigma_i} \cong A_{\sigma_i b}$. ∎

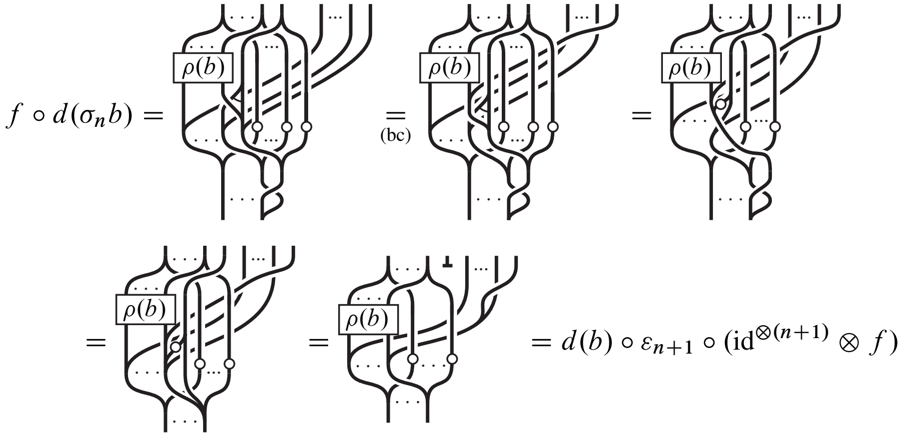


Figure 34. The map $f \circ d(\sigma_n b)$ is equal to $d(b) \circ \varepsilon_{n+1} \circ (\mathrm{id}^{\otimes(n+1)} \otimes f)$.

4.8. Invariance under MII move

We show that the quotient algebra keeps its structure by MII move. We first compare A_b and $A_{\sigma_n b}$.

Proposition 4.14. *For $b \in B_n$, the* Ad*-comodules A_b and $A_{\sigma_n b}$ are isomorphic.*

Proof. Let f be the linear surjection from $A^{\otimes(n+1)}$ to $A^{\otimes n}$ defined by

$$f = \mu_n \circ \Psi_n^{-1}.$$

We first show that $f(I_{d(\sigma_n b)}) \subset I_{d(b)}$. From Figure 34, we know that

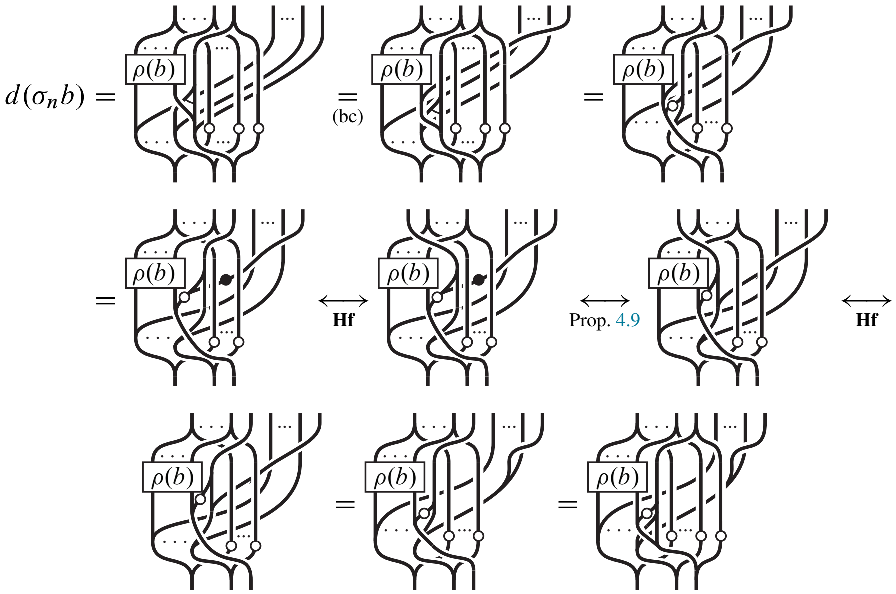
$$f(d(\sigma_n b)(\boldsymbol{x} \otimes \boldsymbol{y})) = d(b)(\varepsilon_{n+1}(\boldsymbol{x}) \otimes f(\boldsymbol{y})),$$

and this means that $f(I_{d(\sigma_n b)}) \subset I_{d(b)}$. So, f induces a map $\bar{f}$ from $A_{d(\sigma_n b)}$ to $A_{d(b)}$.

Next, we show that $\bar{f}$ is an isomorphism. Since f is surjective, it is enough to show that $\bar{f}$ is injective. For this, we check that $\ker f$ is contained in $I_{d(\sigma_n b)}$. For $\boldsymbol{x} \in A^{\otimes(n+1)}$ and $\boldsymbol{y} \in \ker f$, Figure 35 shows that

$$d(\sigma_n b)(\boldsymbol{x} \otimes \boldsymbol{y}) - d(\sigma_n b)(\boldsymbol{x} \otimes f(\boldsymbol{y}) \otimes 1) \in I_{d(\sigma_n b)},$$

and hence, $d(\sigma_n b)(\boldsymbol{x} \otimes \boldsymbol{y}) \in I_{d(\sigma_n b)}$ since $\boldsymbol{y} \in \ker f$. Moreover, $d(\sigma_n b)(\boldsymbol{x} \otimes \boldsymbol{y}) - (\varepsilon^{\otimes(n+1)})(\boldsymbol{x})\boldsymbol{y} \in I_{d(\sigma_n b)}$ by the definition of $I_{d(\sigma_n b)}$, hence we get $(\varepsilon^{\otimes(n+1)})(\boldsymbol{x})\boldsymbol{y} \in I_{d(\sigma_n b)}$. Here $\boldsymbol{x}$ is an arbitrary element of $A^{\otimes(n+1)}$, so $\boldsymbol{y}$ must be an element of $I_{d(\sigma_n b)}$. ∎



$$= d(\sigma_n b) \circ (\mathrm{id}^{\otimes(n+1)} \otimes f \otimes 1)$$

Figure 35. The two maps $d(\sigma_n b)$ and $d(\sigma_n b) \circ (\mathrm{id}^{\otimes(n+1)} \otimes f \otimes 1)$ are equal as maps from $A^{\otimes 2(n+1)}$ to $A_{\sigma_n b}$.

Next, we compare A_b and $A_{\sigma_n^{-1} b}$.

Proposition 4.15. *For $b \in B_n$, the* Ad*-comodules A_b and $A_{\sigma_n^{-1} b}$ are isomorphic.*

Proof. Let g be the linear surjection from $A^{\otimes(n+1)}$ to $A^{\otimes n}$ defined by

$$g = \mu_n \circ S_n^{-2} \circ \Psi_n^{-1}.$$

Then, Figure 36 shows that $g(I_{d(\sigma_n^{-1} b)}) \subset I_{d(b)}$. To obtain the second equality in that figure, we slide down through the g part at the bottom to cancel the strand containing the right-most antipode, using the antipode axiom and anti-multiplicativity of S^{-1}. The third equality similarly slides the top-rightmost strand through g.

Next, Figure 38 shows that $\ker g \subset I_{d(\sigma_n^{-1} b)}$, using an argument similar to that of the previous proposition. It follows that g induces an isomorphism from $A_{\sigma_n^{-1} b}$ to A_b. ∎

Instead of $I_{d(b)}$, we could also consider the Ad-comodule $J_{d(b)}$ given by the image of

$$\boldsymbol{\mu}^{(n)} \circ \left(\boldsymbol{\Psi}^{(n)}\right)^{-1} \circ \left(\mathrm{id}^{\otimes n} \otimes \left(\rho(b) \circ \left(\boldsymbol{\theta}^{(n)}\right)^{-1} \circ \left(S^{-2}\right)^{\otimes n}\right)\right) - \boldsymbol{\mu}^{(n)}, \tag{4.1}$$

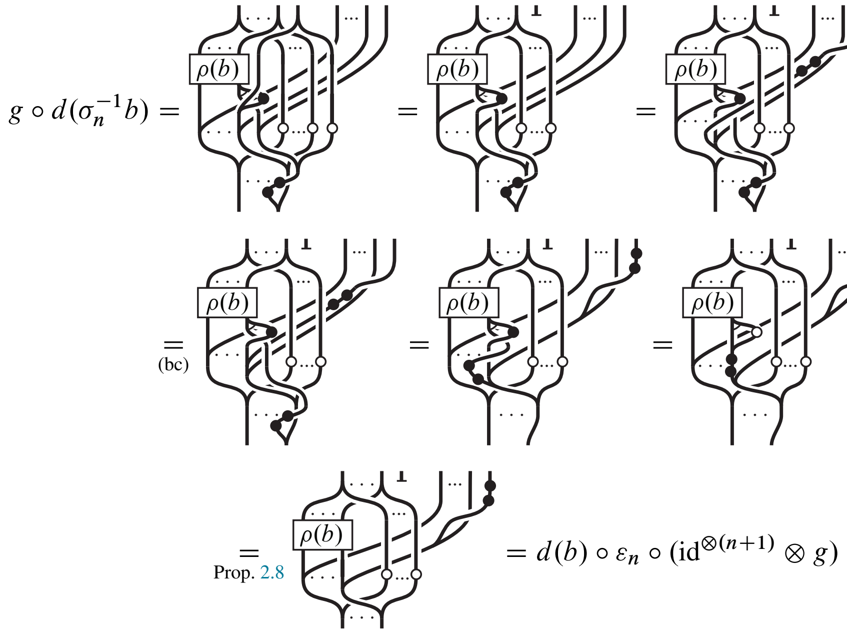


Figure 36. The map $g \circ d(\sigma_n^{-1} b)$ is equal to $d(b) \circ \varepsilon_n \circ (\mathrm{id}^{\otimes(n+1)} \otimes g)$.

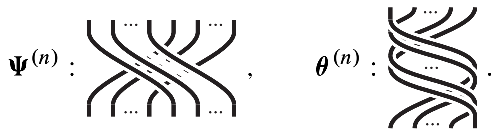


Figure 37. $\boldsymbol{\Psi}^{(n)}$ and $\boldsymbol{\theta}^{(n)}$.

where $\boldsymbol{\Psi}^{(n)}$ is the braiding of two bunches of n strands and $\boldsymbol{\theta}^{(n)}$ is the full twist given in Figure 37. The following proposition shows these are in fact isomorphic.

Proposition 4.16. *$I_{d(b)} \cong J_{d(b)}$ as* Ad*-comodules of $A^{\otimes n}$.*

Proof. The deformation of Figure 39 shows that $J_{d(b)} \subset I_{d(b)}$. On the other hand, the deformation of Figure 40 shows that the image of

$$\begin{aligned}\boldsymbol{\mu}^{(n)} &\circ (\boldsymbol{\mu}^{(n)} \otimes \mathrm{id}^{\otimes n}) \circ (\mathrm{id}^{\otimes n} \otimes \boldsymbol{\Psi}^{(n)}) \circ (b \otimes S^{\otimes n} \otimes \mathrm{id}^{\otimes n}) \\ &\circ (\boldsymbol{\Delta}^{(n)} \otimes \mathrm{id}^{\otimes n}) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n}\end{aligned}$$

is contained in $J_{d(b)}$. This means that $I_{d(b)} \subset J_{d(b)}$. ∎

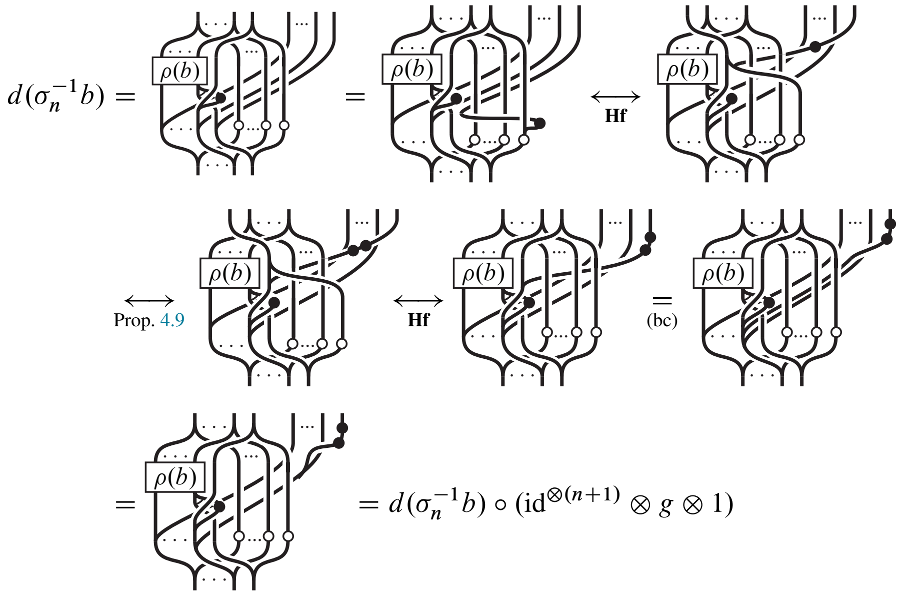


Figure 38. The map $d(\sigma_n^{-1}b)$ is equal to $d(\sigma_n^{-1}b) \circ (\mathrm{id}^{\otimes(n+1)} \otimes g \otimes 1)$.

4.9. Spanning set of I_b

Theorem 4.17. *Let X be a set of generators of A, and $x_i = 1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(n-i)}$ for $x \in X$. Then, the* Ad*-comodule $I_{d(b)}$ in $A^{\otimes n}$ is spanned by*

$$\{d(b)(x_i \otimes \boldsymbol{y}) - \varepsilon(x)\boldsymbol{y} \mid x \in X,\ i = 1, \dots, n,\ \boldsymbol{y} \in A^{\otimes n}\}.$$

Proof. Let I' be the Ad-comodule spanned by $\{d(b)(x_i \otimes \boldsymbol{y}) - \varepsilon(x)\boldsymbol{y} \mid x \in X,\ i = 1, \dots, n,\ \boldsymbol{y} \in A^{\otimes n}\}$. Since $I' \subset I_{d(b)}$ is obvious, we show that $I_{d(b)} \subset I'$. If $(d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})(\boldsymbol{x} \otimes \boldsymbol{y})$ and $(d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})(\boldsymbol{x}' \otimes \boldsymbol{y})$ are contained in I' for any $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{y}$ in $A^{\otimes n}$; then Figure 42 shows that $d(b)(\boldsymbol{\mu}^{(n)}(\boldsymbol{x} \otimes \boldsymbol{x}') \otimes \boldsymbol{y})$ is equal to $\varepsilon^{\otimes n}(\boldsymbol{\mu}^{(n)}(\boldsymbol{x} \otimes \boldsymbol{x}'))\boldsymbol{y}$ modulo I'. Hence, $(d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})(\boldsymbol{\mu}^{(n)}(\boldsymbol{x} \otimes \boldsymbol{x}') \otimes \boldsymbol{y})$ is contained in I', and this implies that $I_{d(b)} \subset I'$. In the figure, $d(b)'$ means the part of $d(b)$ given by Figure 41. ∎

5. Examples

Let A be a finitely generated braided commutative braided Hopf algebra, and let X be a set of generators of A. We construct explicit equations describing the space of A representations for the trivial knot, the Hopf link, the trefoil knot, and the figure eight knot.

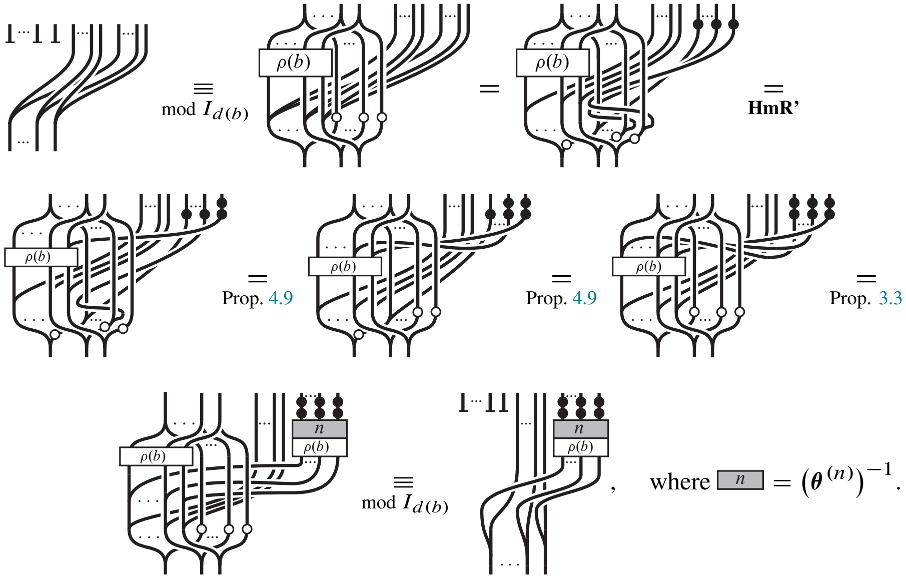


Figure 39. Deformations to show that $\mu^{(n)}(x \otimes y)$ is equal to $\mu^{(n)} \circ (\Psi^{(n)})^{-1} \circ (\mathrm{id}^{\otimes n} \otimes (\rho(b) \circ (\theta^{(n)})^{-1} \circ S^{-2}))(x \otimes y)$ modulo $I_{d(b)}$.

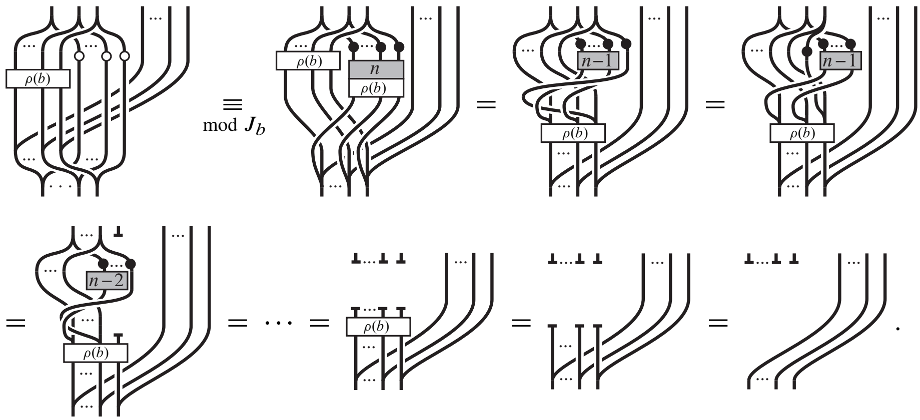


Figure 40. Deformations to show that $d(b)(x \otimes y)$ is equal to $(\varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes n})(x \otimes y)$ modulo J_b. In the third equality, we applied the antipode axiom to cancel the right-most antipode after shifting the box with the $n-1$ twist of the way to the far right.

5.1. Trivial knot

Let $I = I_{d(1)}$ be the image of $d(1) - \varepsilon \otimes \mathrm{id}$. Then, the space of A representations for the trivial knot A_1 is given by $A_1 = A/I$. Since $d(1)$ can be deformed as in

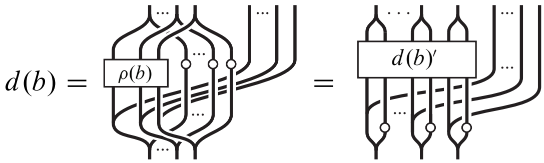


Figure 41. The part $d(b)'$ of the braided Hopf diagram $d(b)$.

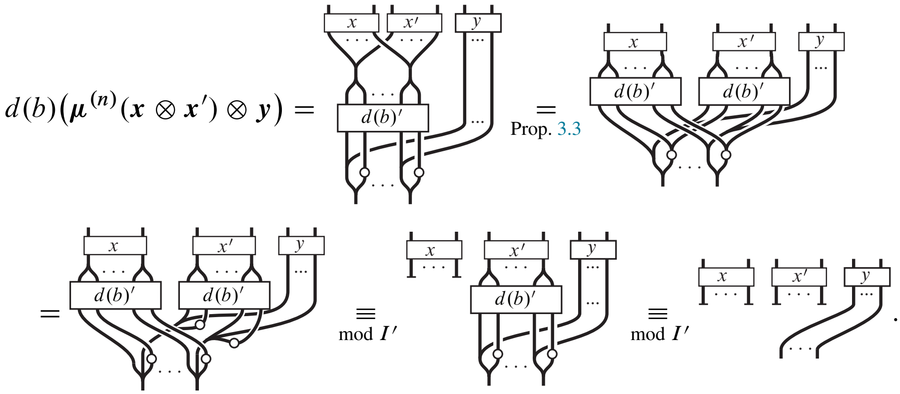


Figure 42. Computation of $d(b)(\boldsymbol{\mu}^{(n)}(\boldsymbol{x} \otimes \boldsymbol{x}') \otimes \boldsymbol{y})$ modulo I'.

Figure 43, we have $d(1)(x \otimes y) - d(1)(x \otimes S^{-2}(y)) \in I$. We also have $d(1)(x \otimes y) - \varepsilon(x)y \in I$ and $d(1)(x \otimes S^{-2}(y)) - \varepsilon(x)S^{-2}(y) \in I$; hence, $y - S^{-2}(y) \in I$. So, in the quotient space A_1, S^2 acts trivially. Moreover, from relation (4.1), we have

$$\mu(x \otimes y) = \mu \circ \Psi(x \otimes y).$$

This implies that A_1 is a commutative algebra.

5.2. Hopf link

The Hopf link is the closure of $b = \sigma_1^2$ in B_2. Let

$$I_1 = (d(\sigma_1^2) - \varepsilon^{\otimes 2} \otimes \mathrm{id}^{\otimes 2})((A \otimes 1) \otimes A^{\otimes 2})$$

and

$$I_2 = (d(\sigma_1^2) - \varepsilon^{\otimes 2} \otimes \mathrm{id}^{\otimes 2})((1 \otimes A) \otimes A^{\otimes 2}).$$

Then, $I_{\sigma_1^2} = I_1 + I_2$ by Theorem 4.17. Figure 44 shows that $I_2 = \sigma_1 \cdot I_1$, and so, we get $A_{\sigma_1^2} = A \otimes A/(I_1 + \sigma_1 \cdot I_1)$.

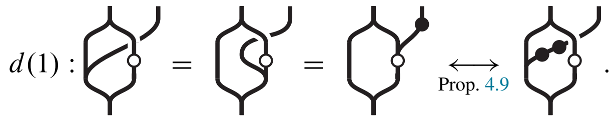


Figure 43. Deformation of $d(1)$ to show that $d(1)(x \otimes y) - d(1)(x \otimes S^{-2}(y)) \in I$.

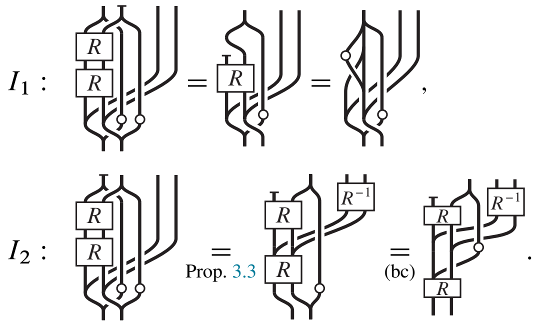


Figure 44. The modules I_1 and I_2 for the Hopf link.

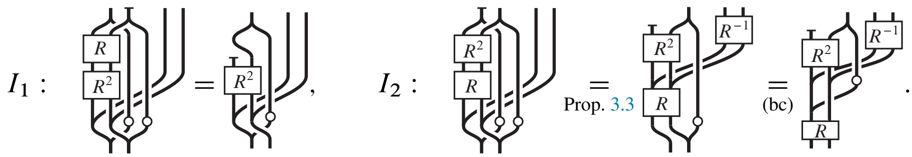


Figure 45. The modules I_1 and I_2 for the trefoil knot.

5.3. Trefoil knot

The trefoil knot is the closure of $b = \sigma_1^3$ in B_2, so it can be treated like the Hopf link that we considered above. In fact, a similar computation is valid for all closures of two strand braids. Let

$$I_1 = (d(\sigma_1^3) - \varepsilon^{\otimes 2} \otimes \mathrm{id}^{\otimes 2})((A \otimes 1) \otimes A^{\otimes 2})$$

and

$$I_2 = (d(\sigma_1^3) - \varepsilon^{\otimes 2} \otimes \mathrm{id}^{\otimes 2})((1 \otimes A) \otimes A^{\otimes 2}).$$

Then, $I_{\sigma_1^2} = I_1 + I_2$ by Theorem 4.17. Figure 45 shows that $I_2 = \sigma_1 \cdot I_1$, and so, we get

$$A_{\sigma_1^3} = A \otimes A / (I_1 + \sigma_1 \cdot I_1).$$

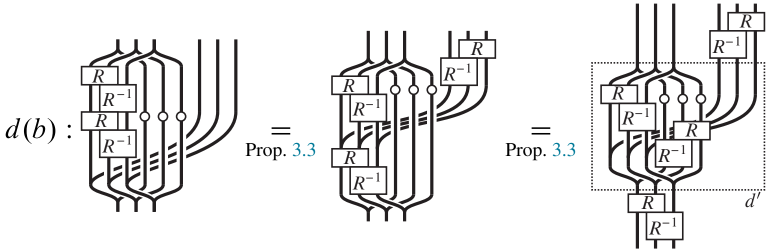


Figure 46. Deformation of $d(b)$ and the braided Hopf diagram d'.

5.4. Figure eight knot

The figure eight knot 4_1 is isotopic to the closure of the braid $b = \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$.

Let d' the braided Hopf diagram assigned in Figure 46, and let I' be the image of $d' - \varepsilon^{\otimes 3} \otimes \mathrm{id}^{\otimes 3}$. Then,

$$\begin{aligned}\sigma_2^{-1}\sigma_1 I' &= \sigma_2^{-1}\sigma_1 \mathrm{Im}(d' - \varepsilon^{\otimes 3} \otimes \mathrm{id}^{\otimes 3}) \\ &= \mathrm{Im}((d(b) - \varepsilon^{\otimes n} \otimes \mathrm{id}^{\otimes 3}) \circ (\mathrm{id}^{\otimes 3} \otimes \sigma_2^{-1}\sigma_1)) = I\end{aligned}$$

since $\sigma_2^{-1}\sigma_1$ is an automorphism of $A^{\otimes 3}$. Hence, A_b is isomorphic to $A^{\otimes 3}/I'$. Let

$$\begin{aligned}&I_1 = d'(A \otimes 1^{\otimes 2} \otimes A^{\otimes 3}), \quad I_2 = d'(1 \otimes A \otimes 1 \otimes A^{\otimes 3}), \\ &I_3 = d'(1^{\otimes 2} \otimes A \otimes A^{\otimes 3});\end{aligned}$$

then $I' = I_1 + I_2 + I_3$.

We first look at I_3. Let φ_3 be the map from $A^{\otimes 3}$ to $A^{\otimes 2}$ defined by

$$\varphi_3(x_1 \otimes x_2 \otimes x_3) = \mu_1 \circ \mu_3 \circ \Psi_2 \circ (\mathrm{id}^{\otimes 2} \otimes \mathrm{ad}) \circ \Psi_2 \circ S_2^2(x_1 \otimes x_2 \otimes x_3).$$

See Figure 47. The same figure shows that

$$x_1 \otimes x_2 \otimes x_3 - \varphi_3(x_1 \otimes x_2 \otimes x_3) \in I'. \tag{5.1}$$

Therefore, $A_b \cong A^{\otimes 2}/(\varphi_3(I_1) + \varphi_3(I_2) + \varphi_3(I_3))$, where $A^{\otimes 2} = A \otimes \mathbb{C} \otimes A$.

Next, we look at I_2. Let φ_2 be a map from $A^{\otimes 3}$ to $A^{\otimes 2}$ defined by

$$\begin{aligned}\varphi_2(x_1 \otimes x_2 \otimes x_3) = \mu_2 \circ \Psi_2 \circ \Psi_1 \circ \mu_3 \circ \mu_2 \circ \Psi_3 \circ \Psi_2^{-1} \circ \mu_3 \circ S_3^{-1} \\ \circ (\mathrm{id} \otimes \mathrm{ad} \otimes \mathrm{id}^{\otimes 2}) \circ S_2 \circ S_1^2 \circ (\mathrm{ad} \otimes \mathrm{id}^{\otimes 2})(x_1 \otimes x_2 \otimes x_3)\end{aligned}$$

as in Figure 48. The same figure shows that $x_1 \otimes x_2 \otimes x_3 - 1 \otimes \varphi_2(x_1 \otimes x_2 \otimes x_3) \in I'$. Combining (5.1), we get

$$x_1 \otimes x_2 \otimes x_3 - \varphi_3(1 \otimes \varphi_2(x_1 \otimes x_2 \otimes x_3)) \in I'.$$

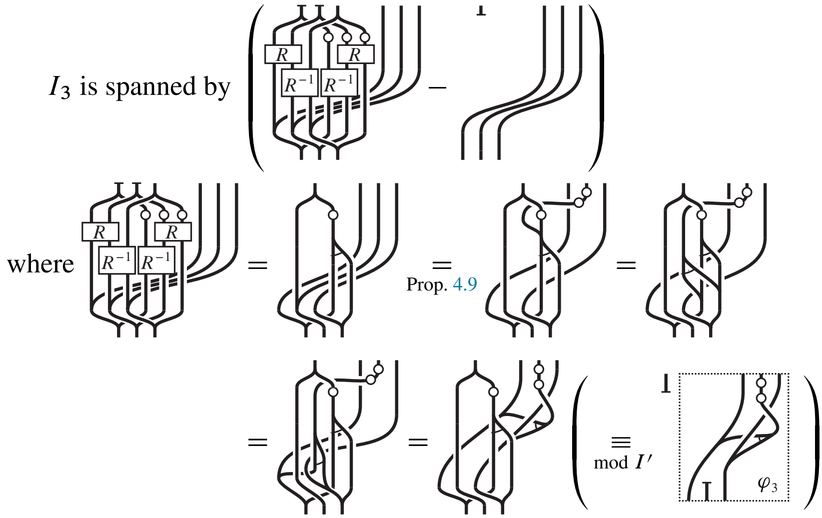


Figure 47. The subspace I_3.

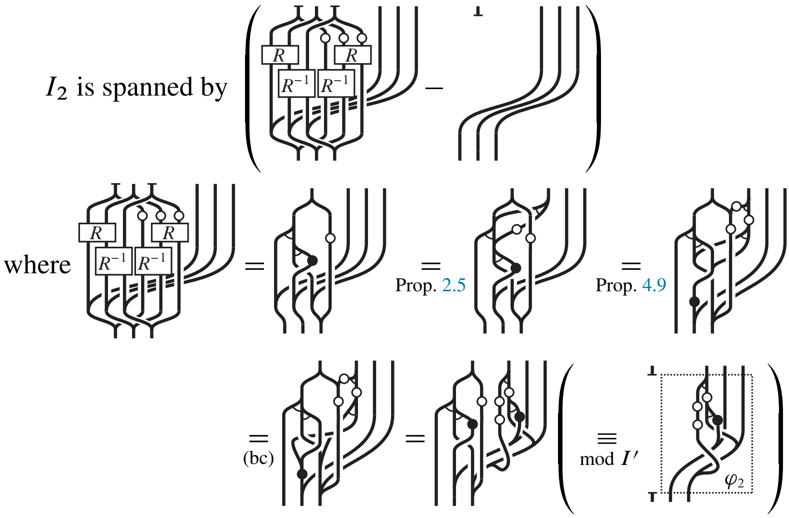
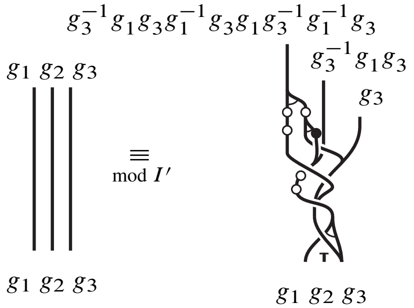


Figure 48. The subspace I_2.

This relation is presented graphically in Figure 49. Reading the diagram bottom to top and interpreting it in the group algebra, we find the following presentation of $\pi_1(S^3 \setminus 4_1)$:

$$\pi(S^3 \setminus 4_1) = \langle g_1, g_3 \mid g_3^{-1} g_1 g_3 g_1^{-1} g_3 g_1 g_3^{-1} g_1^{-1} g_3 g_1^{-1} \rangle.$$



Relations of $\pi_1(S^3 \setminus 4_1)$ obtained from the picture:

$$g_1 = g_3^{-1}g_1g_3g_1^{-1}g_3g_1g_3^{-1}g_1^{-1}g_3$$
$$g_2 = g_3^{-1}g_1g_3$$
$$g_3 = g_3$$

Figure 49. $x_1 \otimes x_2 \otimes x_3 \equiv \varphi_3(1 \otimes \varphi_2(x_1 \otimes x_2 \otimes x_3)) \bmod I'$.

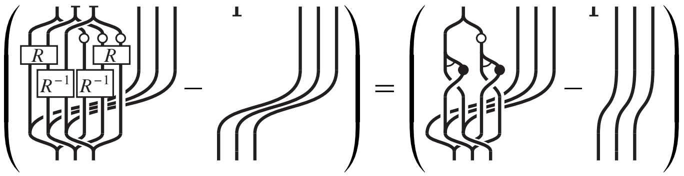
I_1 is spanned by 

Figure 50. The subspace I_1.

Finally, the subspace I_1 is spanned by the image of the map given in Figure 50.

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