

Actions of \mathfrak{sl}_2 on algebras appearing in categorification

Ben Elias and You Qi

Abstract. We prove that many of the recently-constructed algebras and categories which appear in categorification can be equipped with an action of \mathfrak{sl}_2 by derivations. The \mathfrak{sl}_2 representations which appear are filtered by tensor products of coVerma modules. In a future paper, we will address the implications of the \mathfrak{sl}_2 structure for categorification.

Contents

1. Introduction	733
2. Representations of polynomial \mathfrak{sl}_2 -algebras	748
3. On decompositions in the presence of \mathfrak{sl}_2	760
4. \mathfrak{sl}_2 action on the KLR algebra	771
5. \mathfrak{sl}_2 action on the Hecke category	781
6. \mathfrak{sl}_2 action on Lauda’s categorification of $U_q(\mathfrak{sl}_2)$	795
7. \mathfrak{sl}_2 action on the thick calculus $\check{\mathcal{U}}$	803
References	804

1. Introduction

1.1. \mathfrak{sl}_2 structures

Here are several 2-categories with graded 2-morphism spaces which play fundamental roles in categorical representation theory in type A :

- The Khovanov–Lauda–Rouquier category $\mathcal{U}^+(\mathfrak{gl}_n)$, which categorifies the positive half of the quantum group of \mathfrak{gl}_n [15, 31].
- the Lauda’s category $\mathcal{U}(\mathfrak{sl}_2)$, which categorifies the entire quantum group of \mathfrak{sl}_2 [22];
- the thickened category $\check{\mathcal{U}}(\mathfrak{sl}_2)$ of Khovanov–Lauda–Mackaay–Stosic, which also categorifies the entire quantum group of \mathfrak{sl}_2 [17];

2020 *Mathematics Subject Classification.* Primary 16W25; Secondary 17B60.

Keywords. Categorification, $\mathfrak{sl}(2)$ action, KLR algebra, Hecke category.

- the diagrammatic Hecke category $\mathcal{H}(S_n)$, which categorifies the Iwahori–Hecke algebra of S_n [6].

They are each defined by generators and relations over some base ring, using the technology of planar diagrammatics. Since the relations only have integral coefficients, we can and will assume the base ring is the integers.

The first main theorem of this paper says that these categories have a surprising new structure.

Theorem 1.1. *Each of the categories listed above admits an action of \mathfrak{sl}_2 by derivations, compatible with the graded 2-category structure. Moreover, divided powers of the raising and lowering operators act on the integral form.*

More precisely, we will define three operators $\{\mathbf{d}, \mathbf{h}, \mathbf{z}\}$ on the morphism spaces in these categories, and each operator $\mathbf{x} \in \{\mathbf{d}, \mathbf{h}, \mathbf{z}\}$ will satisfy the Leibniz rule

$$\mathbf{x}(f \circ g) = \mathbf{x}(f) \circ g + f \circ \mathbf{x}(g), \quad \mathbf{x}(f \otimes g) = \mathbf{x}(f) \otimes g + f \otimes \mathbf{x}(g)$$

with respect to both vertical (\circ) and horizontal (\otimes) composition. The triple $(\mathbf{d}, \mathbf{h}, -\mathbf{z})$ will act as an \mathfrak{sl}_2 triple on each morphism space. A graded preadditive category equipped with an action of \mathfrak{sl}_2 by derivations will be called an \mathfrak{sl}_2 -category. An \mathfrak{sl}_2 -category is analogous to a dg-category, but for an unusual kind of homological algebra; morphism spaces in both categories are modules over a Hopf algebra. The concept of an \mathfrak{sl}_2 -category does not fit precisely into the framework of Hopfological algebra [14, 29], but we hope to address this in a follow-up paper.

Let us discuss these three operators in turn.

The degree +2 operator \mathbf{d} has been the central topic of study in the recent programme which attempts to categorify key objects in representation theory (e.g. quantum groups and Hecke algebras) at a root of unity. For each of the categories above, the operator \mathbf{d} has been defined in previous works [8–10, 18]. There is a large family of degree +2 derivations one could place on each of these categories, but \mathbf{d} is unique (up to duality, see Remark 1.2) in satisfying certain key properties important for categorification. There is still no geometric understanding for the existence and importance of \mathbf{d} , though see [1] for more on the connection to Steenrod operations. We will not discuss categorification at a root of unity any further in this paper, though we will have much to say in the next paper.

The degree 0 operator \mathbf{h} is the degree operator. It multiplies any homogeneous morphism by a scalar, equal to its degree. Thus, the weight grading for \mathfrak{sl}_2 matches the ordinary grading in these categories.

The degree -2 operator \mathbf{z} is new in this paper. It is also remarkably easy to define, once you know that it exists. Most of the generating morphisms of these categories live in the minimal degree in their respective morphism spaces, and \mathbf{z} must send them

to zero for degree reasons. From these considerations, it is rather simple to verify Theorem 1.1, and even to prove that \mathbf{z} is the unique derivation of degree -2 (up to scalar). Note that $-\mathbf{z}$ is the lowering operator in the \mathfrak{sl}_2 triple, while we prefer to discuss \mathbf{z} because it eliminates many signs from the formulas.

To reiterate, proving Theorem 1.1 is quite easy. However, as far as we are aware there was no expectation at all that these categories should admit \mathfrak{sl}_2 actions, and no one had bothered to look for a degree -2 derivation. We discovered it by accident, as part of an effort to explain certain “hard Lefschetz style” phenomena which appeared in the study of \mathbf{d} .

Remark 1.2. Each of these categories admits a duality functor, a contravariant automorphism $f \mapsto \bar{f}$ which flips diagrams upside-down. The operators \mathbf{h} and \mathbf{z} both intertwine with duality, but \mathbf{d} does not. Instead, duality intertwines \mathbf{d} with another derivation $\bar{\mathbf{d}}$, where

$$\bar{\mathbf{d}}(f) := \overline{(\mathbf{d}(\bar{f}))}.$$

Then $(\bar{\mathbf{d}}, \mathbf{h}, -\mathbf{z})$ is another \mathfrak{sl}_2 triple acting on the category. Readers familiar with the Jacobson–Morozov theorem might be surprised by the existence of two different \mathfrak{sl}_2 triples which share the same lowering and degree operators, but we should reiterate that these representations of \mathfrak{sl}_2 are all infinite dimensional.

Remark 1.3. In Section 4, we actually equip the Khovanov–Lauda–Rouquier category $\mathcal{U}^+(\mathfrak{g})$ with an \mathfrak{sl}_2 action for any (oriented) simply laced root datum. Meanwhile, it was proven in [10, Proposition 6.8] that there is no “good” degree $+2$ derivation on the Hecke category in simply laced types outside of finite and affine type A , so we do not expect an \mathfrak{sl}_2 action either. We have not yet tried to place an \mathfrak{sl}_2 action on other categorifications of interest though we expect the phenomenon to be common in certain categories related to equivariant constructible sheaves, see Remark 1.6. We certainly expect there to be an \mathfrak{sl}_2 action on $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{U}(\mathfrak{sl}_n)$ in general, and it is reasonable to expect that one exists on quantum Heisenberg 2-categories (see, e.g., [25, Section 4]).

1.2. The polynomial ring

If A and B are algebras equipped with an action of \mathfrak{sl}_2 by derivations, and M and N are bimodules equipped with a compatible action of \mathfrak{sl}_2 , then the space of bimodule morphisms $\text{Hom}(M, N)$ is naturally equipped with an \mathfrak{sl}_2 action as well. This is analogous to the internal Hom between two chain complexes, which is itself another chain complex.

There is a well-known action of \mathfrak{sl}_2 on the ring $R_n = \mathbb{Z}[x_1, \dots, x_n]$, where $\deg x_i = 2$ for all i (which determines the \mathbf{h} action), and where

$$\mathbf{d} = \sum_i x_i^2 \frac{\partial}{\partial x_i}, \quad \mathbf{z} = \sum_i \frac{\partial}{\partial x_i}. \tag{1}$$

Here, \mathfrak{sl}_2 appears as a subalgebra of the Witt Lie algebra acting on polynomials.¹ Note that \mathbf{z} acts trivially on the subring generated by the roots $(x_i - x_j)$, and \mathbf{d} does not preserve this subring. For all the categories above, morphism spaces have polynomial subalgebras which play an important role. In all cases, these polynomial subalgebras are preserved by the \mathfrak{sl}_2 action, and the two different \mathfrak{sl}_2 triples $(\mathbf{d}, \mathbf{h}, -\mathbf{z})$ and $(\bar{\mathbf{d}}, \mathbf{h}, -\mathbf{z})$ restrict to the same standard \mathfrak{sl}_2 triple on the polynomial ring.

Most of the categories we are discussing have full faithful embeddings into the category of bimodules over polynomial rings. For example, the Hecke category \mathcal{H} is equivalent (after base change) to the category of Soergel bimodules, certain (R_n, R_n) bimodules. To give another example, the nilHecke algebra NH_n is isomorphic to $\text{End}_{R_n^{\mathfrak{S}_n}}(R_n)$. However, we wish to emphasize that this realization in terms of bimodules does not equip NH_n or \mathcal{H} with an \mathfrak{sl}_2 action! One must still choose an action of \mathfrak{sl}_2 on the bimodules in question; this choice is not unique and is rather subtle in practice. Even a free module of rank 1 over R_n admits many compatible \mathfrak{sl}_2 actions.

Example 1.4. In [18, equations (65) and (66), p. 44], a one-parameter family of degree +2 derivations \mathbf{d}_a is defined on NH_n . We define the lowering operator \mathbf{z} on NH_n below in (4). For any scalar a , $(\mathbf{d}_a, \mathbf{h}, -\mathbf{z})$ is an \mathfrak{sl}_2 triple. A major point in [18] is that only two of these raising operators, $\mathbf{d}_{\pm 1}$, have the desired properties for categorification. The subtle properties of \mathbf{d}_a are invisible from the perspective of the polynomial ring R_n , since $(\mathbf{d}_a, \mathbf{h}, -\mathbf{z})$ restricts to the standard \mathfrak{sl}_2 triple on R_n for all a .

Remark 1.5. In [19], Khovanov and Rozansky use the action of the positive half² of the Witt Lie algebra on R_n to place an action of this same algebra on triply graded knot homology, which is built using Hochschild homology of Soergel bimodules. This (positive half of the) Witt action is an important precursor to our \mathfrak{sl}_2 action, though

¹The Witt Lie algebra is generated by differential operators $L_k = x^{k+1} \frac{\partial}{\partial x}$ for all $k \in \mathbb{Z}$, and it acts on the space of Laurent polynomials $\mathbb{Z}[x, x^{-1}]$. The subalgebra generated by L_k for $k \geq -1$ preserves the subring of ordinary polynomials $\mathbb{Z}[x]$. The action on a polynomial ring in n variables is just the n -fold tensor product of the action on the polynomial ring in one variable. Note that the operators L_k are quite different from the divided powers $\mathbf{d}^{(k)}$ or $\mathbf{z}^{(-k)}$.

²This positive half includes all operators L_k for $k \geq 0$. Their action extends to include L_{-1} as well, though they did not note this.

just as in the previous remark, an action on the polynomial ring does not determine a unique action on the Hecke category itself.

Remark 1.6. The \mathfrak{sl}_2 action (even on the polynomial ring) currently lacks a geometric motivation. The raising operator is related to general homological operations (e.g. Steenrod squares), though the connection is subtle, see [1] (and also see [21] for more on Steenrod operations and Soergel bimodules). However, the lowering operator does not seem to arise from a general construction. Though a good explanation is missing, one thing is clear: \mathfrak{sl}_2 acts on the \mathbb{C}^* -equivariant cohomology of a point. Geometric constructions of these categories involve perverse sheaves which are equivariant over an algebraic group. We suspect that there is a relationship between the \mathfrak{sl}_2 action and the existence of a copy of \mathbb{C}^* (i.e. \mathbb{G}_m) inside the algebraic group which is acting trivially. Insisting upon equivariance for a trivial action often leads to extra “homological” operations of higher degree, such as the “log of monodromy” maps from [2].

For example, the Hecke category studies $B \times B$ -equivariant sheaves on G , and any element of $Z(G) \cap B$ will act the same way on both sides, so its antidiagonal copy in $B \times B$ will act trivially. When $G = \mathrm{GL}_n(\mathbb{C})$, there is a central copy of \mathbb{C}^* in the torus, and there is also an \mathfrak{sl}_2 action on the Hecke category. When $G = \mathrm{SL}_n(\mathbb{C})$, there is no center, and \mathfrak{sl}_2 does not act (cf. [10, Proposition 6.9])! As a shadow of this fact, the reader can already verify that \mathfrak{sl}_2 acts on $\mathbb{Z}[x_1, \dots, x_n]$, but it does not have a (non-trivial) S_n -invariant action on the subring generated by $(x_i - x_j)$ for $i < j$. This contrasts the B -equivariant cohomology of a point for the Borel subgroup in $\mathrm{GL}_n(\mathbb{C})$ versus $\mathrm{SL}_n(\mathbb{C})$.

Remark 1.7. There are a number of (typically non-monoidal) categories which also play major roles in categorical representation theory in type A , and for which the operator \mathbf{d} has already been studied [20, 30].

- The cyclotomic quotients of Khovanov–Lauda–Rouquier categories, which categorify irreducible representations of the quantum group.
- Webster’s categories, which categorify tensor products of irreducible representations.

However, neither the cyclotomic quotients nor Webster’s categories admit actions of \mathfrak{sl}_2 . For example, cyclotomic quotients are quotients by an ideal inside $\mathcal{U}^+(\mathfrak{gl}_n)$ which is preserved by \mathbf{d} but not by \mathbf{z} . This suggests some nuance in how one should interpret modules over \mathfrak{sl}_2 -categories.

Remark 1.8. Other important categories are the cell subquotients of $\mathcal{H}(S_n)$, which categorify the irreducible modules over the Hecke algebra, and the Schur quotients of quantum group categorifications [26]. They arise from monoidal ideals generated by identity maps of various objects, and identity maps are killed by the \mathfrak{sl}_2 action.

Thus, the \mathfrak{sl}_2 action on $\mathcal{H}(S_n)$ descends to its cell subquotients, and the \mathfrak{sl}_2 action on $\mathcal{U}(\mathfrak{gl}_2)$ descends to the Schur quotient.

1.3. Contrasting algebra and representation theory

Given an \mathfrak{sl}_2 -category, we can forget some structure and study it in two ways.

- Forgetting the \mathfrak{sl}_2 action, we can study the category algebraically. We can study the splitting of objects into direct summands, the Jacobson radical, and so forth.
- Forgetting the algebra structure, we can study Hom spaces as representations of \mathfrak{sl}_2 . We can ask about their characters, their finite-dimensional subrepresentations, and so forth.

There seems to be an incredible connection between the structure of these categories as algebras and as \mathfrak{sl}_2 representations. It is so astounding to the authors that it needs to be showcased immediately. We hope this example will whet the appetite, and drum up excitement for our \mathfrak{sl}_2 action.

First, we present a toy example. For ease of discussion,³ let us work over a field \mathbb{k} of characteristic zero. The ring $\mathbb{k}[x]$ has graded Jacobson radical (x) , and the quotient by this ideal is \mathbb{k} . So, we have a short exact sequence

$$0 \rightarrow (x) \rightarrow \mathbb{k}[x] \rightarrow \mathbb{k} \rightarrow 0 \tag{2}$$

of $\mathbb{k}[x]$ -modules, and \mathbb{k} is the graded semisimplification of $\mathbb{k}[x]$. The submodule (x) is also preserved by the raising operator \mathbf{d} , so this is a short exact sequence of $U(\mathfrak{b}^+)$ -modules, where \mathfrak{b}^+ is the Lie algebra inside \mathfrak{sl}_2 generated by \mathbf{d} and \mathbf{h} . However, (x) is not preserved by \mathbf{z} . On the other hand, $\mathbb{k} \subset \mathbb{k}[x]$ is a subalgebra, and is also preserved by the \mathfrak{sl}_2 action. So, we have a short exact sequence

$$0 \rightarrow \mathbb{k} \rightarrow \mathbb{k}[x] \rightarrow Q \rightarrow 0 \tag{3}$$

of \mathfrak{sl}_2 -modules, where Q is the quotient module. Note that Q is simple, so it has no finite-dimensional submodules.

The short exact sequences (2) and (3) live in different categories, but they are both sequences of vector spaces. They split each other, in that the first map of (3) will give a section for the quotient map of (2). Consequently, we can identify Q with (x) as a vector space and as a complementary direct summand to \mathbb{k} . In this fashion, the maximal finite-dimensional \mathfrak{sl}_2 -submodule \mathbb{k} forms a semisimple subalgebra which maps isomorphically to the semisimplification (i.e. the quotient by the Jacobson radical).

³We could make most of the same statements over \mathbb{Z} , but our use of terms like the Jacobson radical would be inappropriate.

Let us reproduce the same behavior in a more interesting example. The nilHecke algebra NH_n is the endomorphism algebra of the object E^n inside $\mathcal{U}^+(\mathfrak{sl}_2)$. Put together, the nilHecke algebras form a monoidal category with morphism algebra $\text{NH} = \bigoplus_{n \geq 0} \text{NH}_n$, which is monoidally generated by morphisms depicted as a dot and a crossing. The \mathfrak{sl}_2 -module structure is defined on the generators below, and is extended to the whole category using the Leibniz rule:

$$\begin{aligned} \mathbf{d} \left(\begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \right) &= 2 \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array}, & \mathbf{d} \left(\begin{array}{c} \text{---} \\ \diagup \ \diagdown \\ \text{---} \end{array} \right) &= - \begin{array}{c} \text{---} \\ \bullet \ \diagdown \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \ \bullet \\ \text{---} \end{array}, \\ \mathbf{z} \left(\begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \right) &= \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}, & \mathbf{z} \left(\begin{array}{c} \text{---} \\ \diagup \ \diagdown \\ \text{---} \end{array} \right) &= 0. \end{aligned} \tag{4}$$

It is well known that NH_n is the endomorphism algebra of the polynomial ring

$$R = R_n = \mathbb{k}[x_1, x_2, \dots, x_n]$$

over its subalgebra R^{S_n} of invariant polynomials. The \mathfrak{sl}_2 -action on R_n is S_n -equivariant and thus descends to an \mathfrak{sl}_2 action on R^{S_n} . By the Chevalley theorem, R is free over R^{S_n} of rank $n!$. Choosing a basis, one obtains an isomorphism between NH_n and a matrix algebra

$$\text{NH}_n \cong \text{Mat}_{n!}(R^{S_n}).$$

On [18, p. 56] one can find a basis of NH_n which corresponds to the basis of matrix entries, and results about the action of \mathbf{d} on this basis.

The (graded) Jacobson radical of R^{S_n} is the ideal $R_+^{S_n}$ spanned by positive degree elements. The Jacobson radical of NH_n is therefore

$$\text{Jac}(\text{NH}_n) \cong \text{Mat}_{n!}(R_+^{S_n}).$$

Consequently, there is a short exact sequence

$$0 \rightarrow \text{Jac}(\text{NH}_n) \rightarrow \text{NH}_n \rightarrow \text{ss}(\text{NH}_n) \rightarrow 0, \tag{5}$$

where $\text{ss}(\text{NH}_n) \cong \text{Mat}_{n!}(\mathbb{k})$ is the semisimplification of NH_n , viewed as a quotient. This is a short exact sequence of NH_n -modules.

Now, let us examine the \mathfrak{sl}_2 structure on NH_n . The ideal $R_+^{S_n} \subset R^{S_n}$ is preserved by \mathbf{d} and \mathbf{h} but not by \mathbf{z} , since $\mathbf{z}(e_1) = n \cdot \text{id}$. Consequently, $\text{Jac}(\text{NH}_n)$ is preserved by \mathbf{d} and \mathbf{h} but not by \mathbf{z} , so there is no induced \mathfrak{sl}_2 structure on $\text{ss}(\text{NH}_n)$. The short exact sequence (5) is not a short exact sequence of \mathfrak{sl}_2 representations. However, $\text{Mat}_{n!}(\mathbb{k})$ is not just a quotient of $\text{NH}_n \cong \text{Mat}_{n!}(R^{S_n})$, it is also a subring. See Section 4.3 for the proof of the following theorem.

Theorem 1.9. *The maximal finite-dimensional \mathfrak{sl}_2 -subrepresentation of NH_n is a subalgebra, which maps isomorphically onto the semisimplification of NH_n . Moreover, under the identification of NH_n with $\text{Mat}_{n!}(R^{S_n})$ used in [18, Proposition 3.24], this subalgebra is precisely $\text{Mat}_{n!}(\mathbb{k})$.*

Example 1.10. When $n = 2$, the following morphisms correspond to the matrix entries in $\text{Mat}_2(\mathbb{k})$.

$$\left(\begin{array}{c|c} \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \\ \hline \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \end{array} \right) \quad (6)$$

The reader should confirm that this four-dimensional subspace of NH_2 is preserved by \mathbf{z} and \mathbf{d} , and is isomorphic as an \mathfrak{sl}_2 -representation to $V \otimes V^*$, where V is the standard representation of \mathfrak{sl}_2 .

Thus, one has a short exact sequence of \mathfrak{sl}_2 -representations

$$0 \rightarrow \text{Mat}_{n!}(\mathbb{k}) \rightarrow \text{NH}_n \rightarrow Q \rightarrow 0 \quad (7)$$

where Q is defined as this quotient. This is not a short exact sequence of NH_n -modules, since $\text{Mat}_{n!}(\mathbb{k})$ is a subring but not an ideal. However, both sequences (5) and (7) are sequences of \mathbb{k} -modules, and split each other. Thus, we can identify Q with the Jacobson radical as a vector space complementary to $\text{Mat}_{n!}(\mathbb{k})$. Said another way, the finite-dimensional part of the \mathfrak{sl}_2 -representation NH_n is precisely a complement to the Jacobson radical!

To summarize, there is a splitting of $U(\mathfrak{b}^+)$ -modules

$$\text{NH}_n = \text{Mat}_{n!}(R^{S_n}) \cong \text{Mat}_{n!}(R_+^{S_n}) \oplus \text{Mat}_{n!}(\mathbb{k}).$$

When viewed as modules over NH_n or its subring R , this is not a splitting but a filtration, with $\text{Mat}_{n!}(R_+^{S_n})$ being the submodule. When viewed as modules over \mathfrak{sl}_2 , this is not a splitting but a filtration, with $\text{Mat}_{n!}(\mathbb{k})$ being the submodule.

Remark 1.11. We come across this situation often in this paper. A vector space (resp. \mathbb{Z} -module) X is isomorphic to a direct sum $A \oplus B \oplus C$ as vector spaces. However, when X is equipped with additional structure (e.g. an action of \mathfrak{sl}_2 , or of R_n), there is no longer a direct sum decomposition. Instead, the subspaces A and $A \oplus B$ are preserved by the action. In this context we use the slightly imprecise phrasing that the *splitting becomes/is a filtration* in the presence of the extra structure. We call A the *subobject* and C the *quotient object*. The objects A , B , and C were not equipped a priori with any additional structure, and they inherit this structure only by the canonical identification of vector spaces with subquotients in the filtration $0 \subset A \subset A \oplus B \subset X$.

Note that NH_n is infinite dimensional and even infinitely generated as an \mathfrak{sl}_2 module (when $n > 1$), so this kind of representation does not conform to most familiar regimes (though they are direct limits of representations in category \mathcal{O}). Thankfully, NH_n has finite-dimensional weight spaces, with weights bounded below. It is easy to prove that such an \mathfrak{sl}_2 -representation contains a unique maximal finite-dimensional

subrepresentation, which we call its *core*. Equivalently, the core consists of all vectors on which \mathbf{d} acts nilpotently. From the Leibniz rule, one can see that the core must be closed under multiplication. Basic facts about the core are proven in Section 2. The example of NH_n is supposed to demonstrate that the core of an \mathfrak{sl}_2 -category has dramatic significance to the algebraic structure of the category.

One nice feature of the core is that it can be found using basic linear algebra. One need only compute the kernel of \mathbf{d} , and then use the lowering operator \mathbf{z} to produce the rest. This can be done in each Hom space independently, and does not involve the composition of morphisms. This simplicity is in contrast to other attempts to find complements to the Jacobson radical, by computing inclusion and projection maps to indecomposable summands. This involves much more complicated linear algebra and in-depth knowledge of the category.

We do not wish the reader to expect that the core of an \mathfrak{sl}_2 -algebra always projects isomorphically to the semisimplification, as this is false in more complicated examples, see Section 3.4. However, we conjecture that the map from the core to the semisimplification is injective, at least in geometrically-motivated 2-categories.

Conjecture 1.12. *In the 2-categories discussed in this paper, the core of each morphism space intersects the Jacobson radical trivially (every morphism in the core is split), so the map from the core to the semisimplification is injective.*

For example, one implication of this conjecture is that any degree zero morphism in the kernel of \mathbf{d} must split. If p is the projection map to an indecomposable direct summand and \mathbf{d} acts nilpotently on p , then every non-zero $\mathbf{d}^k(p)$ is also a projection map.

Remark 1.13. This conjecture is already quite deep, indicating the non-degeneracy of certain “local intersection forms” upon restriction to finite-dimensional \mathfrak{sl}_2 representations. One expects the eventual proof of this conjecture to come from a new kind of Hodge-theoretic argument. In all the examples computed to date, local intersection forms have satisfied (an analogue of) the Hodge–Riemann bilinear relations upon restriction to the core. Note that the raising operator \mathbf{d} differs from the Lefschetz operator of left multiplication studied in [11], and \mathbf{d} is not even a Lefschetz operator in the traditional sense.

1.4. Filtrations on morphism spaces

Hopefully, we have convinced the reader that the study of these categories as \mathfrak{sl}_2 -modules, and in particular the study of their finite-dimensional submodules, is of great interest. Morphism spaces in these categories are free modules over a polynomial

ring, as well as being modules over \mathfrak{sl}_2 . This is a useful tool in our effort to understand the \mathfrak{sl}_2 -module structure on these morphism spaces.

Definition 1.14. The *polynomial \mathfrak{sl}_2 -algebra* (R_n, \mathfrak{sl}_2) is the polynomial ring $R_n = \mathbb{Z}[x_1, \dots, x_n]$ equipped with the \mathfrak{sl}_2 action given in (1). We also let R_n denote the base change of R_n to any commutative base ring \mathbb{k} .

An (R_n, \mathfrak{sl}_2) -module is an R_n -module M which is also an \mathfrak{sl}_2 -module, satisfying a Leibniz rule. For $\mathbf{x} \in \{\mathbf{d}, \mathbf{h}, \mathbf{z}\}$, if we write \mathbf{x}_M for the action on M and \mathbf{x}_R for the action on R_n , the Leibniz rule states that

$$\mathbf{x}_M(r \cdot m) = \mathbf{x}_R(r) \cdot m + r \cdot \mathbf{x}_M(m).$$

Now, we ask: what kinds of (R_n, \mathfrak{sl}_2) -modules appear as morphism spaces in categories of interest? We know that morphism spaces will be free over R_n , meaning that they are a direct sum of rank 1 free modules, but one should not expect such a splitting to be respected by the \mathfrak{sl}_2 structure. The punchline will be: morphism spaces have (R_n, \mathfrak{sl}_2) -filtrations which are split over $U(\mathfrak{b}^-)$, whose subquotients are rank 1 free modules over R_n . Before stating the result, let us investigate rank 1 free modules, which are easy to classify.

Definition 1.15. Let $p = \sum a_i x_i$ be a linear polynomial in R_n , and let $\Sigma(p) \in \mathbb{Z}$ be an integer whose image in \mathbb{k} agrees with $\sum a_i$. There is a free rank one graded R_n -module $R_n\langle p \rangle$ with generator 1_p living in degree $\Sigma(p)$. We define an (R_n, \mathfrak{sl}_2) -module structure on $R_n\langle p \rangle$ by setting

$$\mathbf{d}(g \cdot 1_p) = \mathbf{d}(g) \cdot 1_p + gp \cdot 1_p, \quad \mathbf{z}(g \cdot 1_p) = \mathbf{z}(g) \cdot 1_p \tag{8}$$

for any (homogeneous) $g \in R$.

The formulas (8) are determined by the Leibniz rule from the action on the generator 1_p :

$$\mathbf{d}(1_p) = p \cdot 1_p, \quad \mathbf{z}(1_p) = 0.$$

Note that $\Sigma(p) = \mathbf{z}_R(p)$, though Σ is more descriptive notation. In Proposition 2.26 we prove that every (R_n, \mathfrak{sl}_2) -module which is free of rank one as a graded R_n -module is isomorphic to $R_n\langle p \rangle$ for some p . If $p \neq p'$, then $R_n\langle p \rangle$ and $R_n\langle p' \rangle$ are non-isomorphic.

Definition 1.16. Let M be an (R_n, \mathfrak{sl}_2) -module which is free and finitely generated as a graded R_n -module. Thus, there is a finite set I such that $M = \bigoplus_{i \in I} M_i$ as graded R_n -modules, and each M_i is free of rank 1 over R_n . A *downfree filtration* on M is a splitting into free rank one R_n -modules as above, where

- (1) each M_i is preserved by \mathbf{z} ;

(2) the indexing set I can be equipped with a partial order, so that $\mathbf{d}(M_i) \subset \bigoplus_{j \leq i} M_j$ for all $i \in I$.

A homogeneous basis of M as an R_n -module is called *downfree* if it induces a downfree filtration.

Definition 1.17. Let M be an (R_n, \mathfrak{sl}_2) -module equipped with a downfree filtration. In particular, $\bigoplus_{j \leq i} M_j$ is an (R_n, \mathfrak{sl}_2) -submodule, and M is equipped with an I -indexed filtration by (R_n, \mathfrak{sl}_2) -submodules, where the subquotients are free of rank 1 over R_n . Each subquotient must be isomorphic as an (R_n, \mathfrak{sl}_2) -module to $R_n\langle p_i \rangle$ for some unique $p_i \in R_n$. The multiset of linear polynomials $\{p_i\}_{i=1}^n$ will be called the *downfree character* of M , with respect to this filtration.

The second main theorem of this paper says that well-known bases of morphism spaces in the categories of interest are actually downfree, and computes their downfree characters.

Example 1.18. The nilHecke algebra NH_2 on two strands is free of rank $(1 + q^{-2})$ as a left R_2 -module, spanned by the identity and the crossing. Since id is killed by \mathbf{d} and \mathbf{z} , it generates an \mathfrak{sl}_2 -submodule $R \cdot \text{id} \cong R\langle 0 \rangle$. Now

$$\mathbf{d}\left(\text{crossing}\right) = \text{parallel strands} - 2 \text{crossing}.$$

In the quotient by $R \cdot \text{id}$, \mathbf{d} will send the crossing X to $-2x_1 X$. So, the basis $\{\text{id}, X\}$ is downfree, and the downfree character of NH_2 is $\{0, -2x_1\}$. Note that

$$\{\Sigma(0), \Sigma(-2x_1)\} = \{0, -2\}$$

which matches the degrees of this graded basis. If instead we had chosen the raising operator $\bar{\mathbf{d}}$, we would have gotten downfree character $\{0, -2x_2\}$.

Note that NH_2 is also free as a right R_2 module, with the same basis. Because

$$\mathbf{d}\left(\text{crossing}\right) = -\text{parallel strands} - 2 \text{crossing},$$

the basis is downfree with character $\{0, -2x_2\}$. The right module character for \mathbf{d} matches the left module character for $\bar{\mathbf{d}}$ because they are related by duality, which also swaps the left and right action of R_n .

Remark 1.19. For the Hecke category, morphism spaces will also be R_n -bimodules, but the duality functor will not interchange these actions. A given basis may induce four different characters, based on whether one selects the left or right action of R_n , and whether one chooses \mathbf{d} or $\bar{\mathbf{d}}$.

More generally, associated to any element $w \in S_n$ and any reduced expression of w , one can construct the corresponding diagram in NH_n built from crossings, and

this element in NH_n is independent of the choice of reduced expression. We denote it $\psi_w \in NH_n$. Then the elements $\{\psi_w\}_{w \in S_n}$ form a basis of NH_n as a left or right R_n -module, which we call the *nilCoxeter basis*. It is not hard to compute that

$$\mathbf{d}(R_n \cdot \psi_w) \subset \bigoplus_{y \leq w} R_n \cdot \psi_y, \quad \mathbf{z}(R_n \cdot \psi_w) \subset R_n \cdot \psi_w.$$

Theorem 1.20. *The nilCoxeter basis of NH_n over R_n is a downfree basis, with partial order given by the Bruhat order on S_n .*

This is proven in Theorem 4.5, which also contains an explicit formula for the downfree character, and the generalization to all simply-laced KLR algebras.

Similarly, we can study the Hecke category $\mathcal{H}(S_n)$. Now, R_n is the endomorphism ring of the monoidal identity, so all morphism spaces are naturally R_n -bimodules. Following ideas of Libedinsky [24], Elias and Williamson in [12] define the *double leaves basis*, a basis of morphism spaces as left R_n -modules, which is indexed by coterminal Bruhat strolls. Let us summarize Theorem 5.25, which contains an explicit formula for the downfree character.

Theorem 1.21. *The double leaves basis of morphism spaces in $\mathcal{H}(S_n)$ over R_n is downfree, with partial order given by the lexicoBruhat order on coterminal Bruhat strolls.*

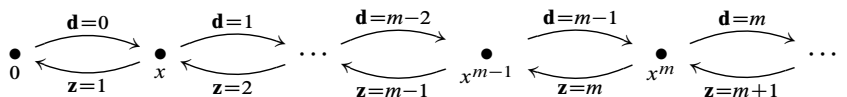
The natural bases of morphism spaces of $\mathcal{U}(\mathfrak{sl}_2)$ and $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ are also downfree (conjecturally, since we do not prove it here), though over a different base ring. Any given (non-zero) morphism space in $\mathcal{U}(\mathfrak{sl}_2)$ has $2n$ points on the boundary (n oriented in and n oriented out). This morphism space is free of rank $n!$ over $R_n \otimes \Lambda$, where Λ is the ring of symmetric functions acting by bubbles, and R_n acts by dots on the inward-oriented strands. We place an \mathfrak{sl}_2 structure on Λ in Section 6.2; the lowering operator \mathbf{z} depends on the choice of ambient weight. The combinatorics involved in describing the downfree character have not been developed.

1.5. What kinds of \mathfrak{sl}_2 -modules appear?

Having just described the kinds of (R_n, \mathfrak{sl}_2) -modules which appear in practice, we can ask about what these modules look like as \mathfrak{sl}_2 -representations, with an eye towards understanding their cores.

In this paper, all \mathfrak{sl}_2 representations have weights which are bounded below rather than above, so Verma modules $\Delta(k)$ are defined by inducing from $U(\mathfrak{b}^-)$ rather than $U(\mathfrak{b}^+)$. Let L_k denote the irreducible weight representation of *lowest weight* k for all k , a quotient of $\Delta(k)$ and submodule of $\nabla(k)$. Note that L_k is finite dimensional if and only if $k \leq 0$.

Example 1.22. Consider $\mathbb{Z}[x]$ as an \mathfrak{sl}_2 -module. With its usual basis of monomials, the module looks like this:



Hence, $\mathbb{Z}[x] \cong \nabla(0)$. This is a coVerma module, with the trivial module (spanned by the identity element) as a submodule. The quotient by the trivial submodule is isomorphic to $\Delta(2)$. Note that $\Delta(2) \not\cong \nabla(2)$ when we work over \mathbb{Z} or in finite characteristic.

As both a ring and as an \mathfrak{sl}_2 -module, we have $\mathbb{Z}[x_1, \dots, x_n] \cong \mathbb{Z}[x]^{\otimes n}$. The following proposition is very easy to prove.

Proposition 1.23 (See Proposition 2.26). *For $p = \sum a_i x_i \in R_n$, there is an isomorphism of \mathfrak{sl}_2 modules*

$$R_n \langle p \rangle \cong \nabla(a_1) \otimes \cdots \otimes \nabla(a_n).$$

In particular, $\text{Core}(R_n \langle p \rangle)$ is non-zero if and only if $a_i \in \mathbb{Z}_{\leq 0}$ for all i , in which case

$$\text{Core}(R_n \langle p \rangle) \cong L_{a_1} \otimes \cdots \otimes L_{a_n}.$$

Example 1.24. Suppose $M = R_n \langle p \rangle$ is generated in degree -2 . If $p = -2x_1$, then $\text{Core}(M)$ is three dimensional; if $p = -x_1 - x_2$, then $\text{Core}(M)$ is four dimensional; and if $p = -3x_1 + x_2$, then $\text{Core}(M) = 0$. This illustrates why the character of an (R_n, \mathfrak{sl}_2) -module is more useful than the graded degree.

Suppose one has an (R_n, \mathfrak{sl}_2) -module with a downfree filtration, and one knows the downfree character. By the proposition above, one knows the core of the associated graded module. A priori, this does not make it any easier to determine the core of M , because a finite-dimensional submodule of a subquotient of M need not lead to a finite-dimensional submodule of M itself. However, extensions between (R_n, \mathfrak{sl}_2) -modules are even more limited than extensions between their underlying \mathfrak{sl}_2 -modules, and sometimes the downfree character of M will determine the core of M ! Let us illustrate this with the following result.

Theorem 1.25. *Let $n = 1$, so that $R_n = \mathbb{Z}[x]$. Let $M = \mathbb{Z}[x] \langle ax \rangle \oplus \mathbb{Z}[x] \langle bx \rangle$ be a $(\mathbb{Z}[x], \mathfrak{sl}_2)$ -module with a downfree filtration, where $\mathbb{Z}[x] \langle bx \rangle$ is the submodule. If the downfree filtration does not split, then $b = a + 2$. Unless $a = 0$ and $b = 2$, we have $\text{Core}(M) \cong \text{Core}(\mathbb{Z}[x] \langle ax \rangle) \oplus \text{Core}(\mathbb{Z}[x] \langle bx \rangle)$.*

Proof. (Sketch) Let 1_a denote the generator of $\mathbb{Z}[x] \langle ax \rangle$, living in degree a . Then

$$\mathbf{z}(1_a) = 0, \quad \mathbf{d}(1_a) = ax1_a + m \tag{9}$$

for some $m \in \mathbb{Z}[x]\langle bx \rangle$. For the \mathfrak{sl}_2 relations to hold, we need $\mathbf{z}(m) = 0$. But the kernel of \mathbf{z} inside $\mathbb{Z}[x]\langle b \rangle$ is just the span of the generator 1_b . Thus, m is a scalar multiple of 1_b , and for degree reasons $b = a + 2$.

Even the associated graded has no core unless $a \leq 0$, so assume $a \leq 0$. There are no extensions between $\nabla(a)$ and $\nabla(a + 2)$ unless $a = 0$, by the usual theory of central characters. ■

Remark 1.26. The lack of extensions between $\nabla(a)$ and $\nabla(a + 2)$ does not mean that the downfree filtration splits. The splitting as an \mathfrak{sl}_2 -module and the splitting as an R_1 -module are not compatible.

Remark 1.27. When $a = 0$ and $b = 2$, one can find an extension of $\mathbb{Z}[x]\langle 0 \rangle$ by $\mathbb{Z}[x]\langle 2x \rangle$ with zero core, whereas the core of the associated graded would be one-dimensional.

This theorem implies that, for a two-step downfree filtration in one variable, the difference between the core of the original module and its associated graded is at most one copy of the trivial module. Similarly, one can prove that a three-step filtration can remove a copy of L_0 or L_{-1} , but not L_{-k} for $k \geq 2$.

Remark 1.28. The situation is more complex in more than one variable, because polynomials in the roots $(x_i - x_j)$ are killed by \mathbf{z} , and this allows for more extensions (such polynomials times 1_b are valid choices for m in (9)). See Section 5.2 for an example. Interestingly, many of these extensions do not admit integrally-defined divided powers! Keeping track of divided powers and integrality properties does seem to rigidify the possible extensions.

The (R_n, \mathfrak{sl}_2) -modules admitting a downfree filtration form a reasonably nice category which we feel is important to study. We hope to provide a methodical study of (R_n, \mathfrak{sl}_2) -modules in future work, and provide only the basics in this paper.

1.6. Conclusion

In Section 2 we provide some basic results and definitions related to \mathfrak{sl}_2 -categories and their modules, and the special case of polynomial rings. In the subsequent sections we examine the categories $\mathcal{U}^+(\mathfrak{g})$ (in simply laced type), \mathcal{H} , $\mathcal{U}(\mathfrak{sl}_2)$, and $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ in turn, constructing the derivation \mathbf{z} , establishing the \mathfrak{sl}_2 action, and verifying the claims made in this introduction about downfree filtrations. In Section 6.1 and Section 6.2 we discuss the \mathfrak{sl}_2 action on symmetric polynomials and symmetric functions, which may be of independent interest.

In Section 5, in order to prove results about the downfree filtration on the Hecke category, we need to establish some basic properties of light leaves and double leaves.

Aside from this, the proofs in this paper are all relatively straightforward computations.

We find this new \mathfrak{sl}_2 structure to be extremely tantalizing. In a follow-up paper we will introduce an \mathfrak{sl}_2 -enrichment of the categories of interest, where multiplicity spaces are naturally finite-dimensional \mathfrak{sl}_2 representations. We also introduce a conjectural approach to defining a triangulated category from an \mathfrak{sl}_2 -category, analogous to the derived categories associated to p -dg algebras in [14, 29], whose Grothendieck group is naturally a module over $\mathbb{Z}[q + q^{-1}]$, the Grothendieck ring of finite-dimensional \mathfrak{sl}_2 representations. We will state a number of conjectures about cores and Jacobson radicals, which translate into the existence of nice bases for the triangulated Grothendieck group. If true, these conjectures would produce a new kind of categorification and a new kind of canonical basis, where structure coefficients are naturally unimodal, being the graded dimensions of \mathfrak{sl}_2 representations. For example, the Grothendieck group of the diagrammatic Hecke category would be a variant of the Hecke algebra with base ring $\mathbb{Z}[q + q^{-1}]$. Our conjectures would also imply several open conjectures about categorification at a root of unity, e.g., that the p -dg Grothendieck group of the diagrammatic Hecke category is the Hecke algebra at a root of unity.

While some aspects of the material developed in the follow-up paper are quite technical and lengthy to explain, other aspects are accessible and directly computational. We continue our introduction in Section 3 by explaining the basics of the theory of \mathfrak{sl}_2 decompositions. In particular, this discussion further motivates the study of the core, and of *iterated cores* that one obtains by combining the algebra structure and the \mathfrak{sl}_2 structure on an \mathfrak{sl}_2 -category. We discuss examples in the diagrammatic Hecke category in Section 3.4, and state our new \mathfrak{sl}_2 -adapted Soergel categorification conjecture in Section 3.5.

Remark 1.29. This final remark is for those readers familiar with p -dg algebras and categorification at a root of unity. By forgetting the lowering operator and restricting from $U(\mathfrak{sl}_2)$ to $U(\mathfrak{b}^+)$, one obtains a p -dg structure on these categories. The biggest problem in computing the p -dg Grothendieck group is to prove that any object has a fantastic filtration whose subquotients are certain p -dg-indecomposable objects. Historically this has been done by computing idempotent decompositions explicitly, but this method becomes intractable quickly.

Our \mathfrak{sl}_2 -adapted Soergel categorification conjecture essentially states that Bott–Samelson objects have \mathfrak{sl}_2 -fantastic filtrations, whose subquotients are “indecomposable objects” tensored with a multiplicity space which is a finite-dimensional \mathfrak{sl}_2 representation. Because a finite-dimensional \mathfrak{sl}_2 representation has a filtration by one-dimensional $U(\mathfrak{b}^+)$ -modules, this will yield a filtration by p -dg indecomposable objects (with multiplicity one); an \mathfrak{sl}_2 -fantastic filtration will be a p -dg fantastic filtra-

tion. However, \mathfrak{sl}_2 -fantastic filtrations are more restrictive and include more structure, which ironically makes them easier to find. Once you compute the highest weight vector, you can apply \mathbf{z} to find a basis for the rest of the representation; this was a tool which was not previously available.

2. Representations of polynomial \mathfrak{sl}_2 -algebras

Most of the results in this section are relatively straightforward, but because we are not aware of any literature on the topic, we provide some details.

2.1. \mathfrak{g} -algebras and Leibniz exercises

Definition 2.1. Let \mathbb{k} be a commutative domain, and let \mathfrak{g} be a Lie algebra over \mathbb{k} . A \mathfrak{g} -algebra is a \mathbb{k} -algebra A equipped with an action of \mathfrak{g} by derivations. We sometimes write (A, \mathfrak{g}) for this structure. One can define a \mathfrak{g} -category similarly, and \mathfrak{g} will act on each Hom space. A *monoidal \mathfrak{g} -category* is a \mathfrak{g} -category with the additional requirement for each $\mathbf{x} \in \mathfrak{g}$ that

$$\mathbf{x}(f \otimes g) = \mathbf{x}(f) \otimes g + f \otimes \mathbf{x}(g).$$

By the interchange law, it is equivalent to require that

$$\mathbf{x}(f \otimes 1) = \mathbf{x}(f) \otimes 1, \quad \mathbf{x}(1 \otimes f) = 1 \otimes \mathbf{x}(f).$$

By default, in this paper $\mathbb{k} = \mathbb{Z}$. Before continuing, let us address the practical question of what it takes to place a \mathfrak{g} -algebra structure on A . Here are two simplifying lemmas. The first reduces the data required to define a single derivation.

Lemma 2.2. *Let A be a \mathbb{k} -algebra given by generators and relations. To define a derivation \mathbf{x} on A , it suffices to specify $\mathbf{x}(a)$ for each generator of A , and to check that \mathbf{x} preserves all the relations of A (what this means precisely is stated in the proof).*

Proof. Let S be the generating set of A , and $F\langle S \rangle$ the free algebra on these generators. Any assignment $\mathbf{x}: S \rightarrow F\langle S \rangle$ will extend to a unique derivation on $F\langle S \rangle$, using the Leibniz rule to define the action of \mathbf{x} on a word in S . If I is the ideal in $F\langle S \rangle$ generated by the relations, and \mathbf{x} sends the generating relations to elements of I , then \mathbf{x} preserves I by the Leibniz rule. Thus, \mathbf{x} descends to a derivation on A . ■

The second lemma reduces the work required to check that a collection of derivations gives an action of a particular Lie algebra \mathfrak{g} .

Lemma 2.3. *Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be three derivations on an algebra A , and suppose that*

$$[\mathbf{x}, \mathbf{y}](a) = \mathbf{z}(a) \tag{10}$$

holds for a generating set of elements $a \in A$. Then it holds for all $a \in A$.

Proof. The equation (10) is clearly linear in a , so it remains to check that if (10) holds for a and b , then it holds for the product ab . We compute

$$\begin{aligned} [\mathbf{x}, \mathbf{y}](ab) &= \mathbf{x}(\mathbf{y}(a)b + a\mathbf{y}(b)) - \mathbf{y}(\mathbf{x}(a)b + a\mathbf{x}(b)) \\ &= \mathbf{x}\mathbf{y}(a)b + \mathbf{y}(a)\mathbf{x}(b) + \mathbf{x}(a)\mathbf{y}(b) + a\mathbf{x}\mathbf{y}(b) \\ &\quad - \mathbf{y}\mathbf{x}(a)b - \mathbf{x}(a)\mathbf{y}(b) - \mathbf{y}(a)\mathbf{x}(b) - a\mathbf{y}\mathbf{x}(b) \\ &= [\mathbf{x}, \mathbf{y}](a)b + a[\mathbf{x}, \mathbf{y}](b) = \mathbf{z}(a)b + a\mathbf{z}(b) \\ &= \mathbf{z}(ab). \end{aligned} \quad \blacksquare$$

Here is one more practical consideration.

Definition 2.4. Let \mathbf{x} be a derivation on a \mathbb{Z} -algebra A . The *divided powers* of \mathbf{x} are the operators

$$\mathbf{x}^{(k)} := \frac{\mathbf{x}^k}{k!},$$

which a priori send A to $A \otimes_{\mathbb{Z}} \mathbb{Q}$. We say that the *divided powers* $\mathbf{x}^{(k)}$ are *defined over \mathbb{k}* or just *defined integrally*⁴ if there are operators $\mathbf{x}^{(k)}$ defined in A (without any base change) such that

$$k! \cdot \mathbf{x}^{(k)} = \mathbf{x}^k, \quad \mathbf{x}^{(k)} \mathbf{x}^{(\ell)} = \binom{k+\ell}{k} \mathbf{x}^{(k+\ell)}.$$

Lemma 2.5. *Let \mathbf{x} be a derivation on a \mathbb{Z} -algebra A , and suppose that $\mathbf{x}^{(k)}(a)$ is defined integrally for a generating set of elements in A . Then $\mathbf{x}^{(k)}$ is defined integrally (on all of A).*

Proof. Again, $\mathbf{x}^{(k)}$ is a linear operator. It is an easy exercise in the Leibniz rule (and the binomial theorem) that

$$\mathbf{x}^{(k)}(ab) = \sum_{i+j=k} \mathbf{x}^{(i)}(a)\mathbf{x}^{(j)}(b).$$

In particular, if $\mathbf{x}^{(k)}$ is defined on a and b for all k , then it is defined on ab . \blacksquare

We write \otimes for $\otimes_{\mathbb{k}}$.

⁴We use the words “defined integrally” in general, but for most of our applications in this paper $\mathbb{k} = \mathbb{Z}$.

Corollary 2.6. *Let A and B be \mathfrak{g} -algebras. Then the tensor product algebra $A \otimes B$, with its tensor product \mathfrak{g} -action, is a \mathfrak{g} -algebra. If the divided powers of $\mathbf{x} \in \mathfrak{g}$ are defined integrally on A and on B , then they are defined integrally on $A \otimes B$.*

Proof. Since $A \otimes B$ is generated by elements of A and elements of B , Lemmas 2.3 and 2.5 imply the result. ■

Remark 2.7. The \mathfrak{g} -action on an algebra A can be more conceptually understood in the language of a module-algebra over a Hopf algebra H (see, e.g., [28] for more details). Let H be a Hopf algebra over \mathbb{k} . We will always assume that H is a projective \mathbb{k} -module. The comultiplication on H is denoted Δ in Sweedler’s notation:

$$\Delta(h) = \sum h_1 \otimes h_2,$$

for any $h \in H$, and the counit map is denoted $\varepsilon: H \rightarrow \mathbb{k}$. A \mathbb{k} -algebra A is called an H -module algebra if A is an H -module, and the multiplication map of A is compatible with the H -action: for any $a, b \in A$ and $h \in H$

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_A = \varepsilon(h)1_A,$$

where 1_A is the identity element of A . When H and A are graded (super)algebras, the notions should be adapted so that the H action on A respects the graded (super)algebra structures.

As a particular case, for a \mathfrak{g} -algebra A , one may take H to be the universal enveloping algebra of a Lie algebra \mathfrak{g} over \mathbb{k} , with the H -action on A induced by derivations of \mathfrak{g} on A . Similarly, taking $H = \mathbb{k}[d]/(d^2)$ to be the graded superalgebra of dual numbers, where the degree of d is set to be 1 and

$$\Delta(d) = d \otimes 1 + 1 \otimes d, \quad \varepsilon(d) = 0,$$

one recovers the usual notion of a differential graded algebra as an H -module algebra.

2.2. Modules over \mathfrak{g} -algebras

Definition 2.8. Let (A, \mathfrak{g}) be a \mathfrak{g} -algebra. An (A, \mathfrak{g}) -module is an A -module M which is equipped with a \mathfrak{g} action. For $\mathbf{x} \in \mathfrak{g}$ write \mathbf{x}_M for the action of \mathbf{x} on M . We require the compatibility

$$\mathbf{x}_M(a \cdot m) = \mathbf{x}_A(a) \cdot m + a \cdot \mathbf{x}_M(m)$$

for any $a \in A$ and $m \in M$. A morphism of (A, \mathfrak{g}) -modules is an A -module morphism $M \rightarrow N$ which intertwines the action of \mathfrak{g} . These morphisms form a space denoted $\text{Hom}_{(A, \mathfrak{g})}(M, N)$.

This definition is completely analogous to the definition of a dg-module over a dg-algebra. Instead of just one differential, we keep track of a \mathfrak{g} action. Just as for dg-modules, we can consider the internal Hom space.

Definition 2.9. Let M and N be two (A, \mathfrak{g}) -modules. Then the space of A -module maps $\text{Hom}_A(M, N)$ can be equipped with the structure of a \mathfrak{g} -module, where

$$\mathbf{x}_{\text{Hom}}(\phi)(m) := \mathbf{x}_N(\phi(m)) - \phi(\mathbf{x}_M(m)).$$

Proposition 2.10. If A is commutative, then $\text{Hom}_A(M, N)$ is also an A -module. The actions of A and \mathfrak{g} are compatible, making $\text{Hom}_A(M, N)$ into an (A, \mathfrak{g}) -module.

Proof. We only need to check the Leibniz rule. We compute

$$\begin{aligned} \mathbf{x}_{\text{Hom}}(a \cdot \phi)(m) &= \mathbf{x}_N(a\phi(m)) - a\phi(\mathbf{x}_M(m)) \\ &= \mathbf{x}_A(a)\phi(m) + a\mathbf{x}_N(\phi(m)) - a\phi(\mathbf{x}_M(m)) \\ &= (\mathbf{x}_A(a) \cdot \phi)(m) + (a \cdot \mathbf{x}_{\text{Hom}}(\phi))(m). \end{aligned}$$

The result follows. ■

Lemma 2.11. Let M be an A -module with a presentation. To give M the structure of an (A, \mathfrak{g}) -module, it suffices to define $\mathbf{x}_M(m)$ for all generators $m \in M$ and all $\mathbf{x} \in \mathfrak{g}$, and to check the relations. For a given $\mathbf{x} \in \mathfrak{g}$, the divided powers of \mathbf{x}_M are defined integrally if and only if they are defined integrally on the generators of M .

Proof. This is entirely analogous to Lemmas 2.2, 2.3, and 2.5. We leave the proof to the reader. ■

Remark 2.12. Let A be an H -module algebra as in Remark 2.7, one may form the smash product ring $A\#H$, which, as an algebra, is isomorphic to the tensor product algebra $A \otimes_{\mathbb{k}} H$, whose multiplication is given by

$$(a \otimes h) \cdot (b \otimes k) = \sum a(h_1 \cdot b) \otimes h_2k,$$

for any $a, b \in A$ and $h, k \in H$. The abelian category of $A\#H$ -modules is acted upon by the abelian monoidal category of \mathbb{k} -projective H -representations. Let us denote this action by \boxtimes . Given an $A\#H$ -module M and a \mathbb{k} -projective H -representation V , $M \boxtimes V$ is isomorphic to $M \otimes_{\mathbb{k}} V$, with any element $a \otimes h \in A\#H$ acting on $x \otimes v \in M \otimes_{\mathbb{k}} V$ by

$$(a \otimes h) \cdot (x \otimes v) := \sum ((a \otimes h_1) \cdot x) \otimes h_2v.$$

By construction, $(-) \boxtimes V$ is an exact functor on $A\#H$ -modules.

In this language, an (A, \mathfrak{g}) -module is no other than a module over $A\#H$, where H is the universal enveloping algebra of \mathfrak{g} over \mathbb{k} . It follows readily that the category of (A, \mathfrak{g}) -modules constitute an abelian category, with a categorical module structure over the monoidal category of projective H -representations.

2.3. Weights

When $\mathfrak{g} = \mathfrak{sl}_2$ we change notation to impose one additional condition: that \mathbf{h} acts semisimply.

Definition 2.13. A (weight) \mathfrak{sl}_2 -algebra (over \mathbb{k}) is a \mathbb{k} -algebra A equipped with an action of \mathfrak{sl}_2 by derivations, on which the Cartan element $\mathbf{h} \in \mathfrak{sl}_2$ acts diagonalizably (with eigenvalues in \mathbb{k}). A (weight) (A, \mathfrak{sl}_2) -module is an A -module M with a compatible \mathfrak{sl}_2 action, on which the Cartan element $\mathbf{h} \in \mathfrak{sl}_2$ acts diagonalizably. A divided powers \mathfrak{sl}_2 -algebra is an \mathfrak{sl}_2 -algebra where the divided powers of \mathbf{d} and \mathbf{z} are integrally defined, and similarly for a divided powers (A, \mathfrak{sl}_2) -module.

We let $(\mathbf{d}, \mathbf{h}, -\mathbf{z})$ be the standard \mathfrak{sl}_2 triple, and we write \mathbf{x}_A or \mathbf{x}_M for the action of \mathbf{x} on A or M , where $\mathbf{x} \in \{\mathbf{d}, \mathbf{h}, \mathbf{z}\}$.

Lemma 2.14. Equip A (resp. M) with a \mathbb{k} -grading by the eigenvalues of \mathbf{h} , the usual weight grading. Then A is a graded algebra (resp. and M is a graded module).

Proof. For homogeneous elements a_1 and m_2 of weights d_1 and d_2 respectively, the Leibniz rule for \mathbf{h} implies that a_1m_2 has weight $d_1 + d_2$. ■

Most often in this paper the eigenvalues of \mathbf{h} will be integers, and our rings and modules will be \mathbb{Z} -graded. Only in this section on generalities, and to some extent in Section 6.2, will we care about more general weights.

Remark 2.15. For all the examples we study in this paper, the eigenvalues live in the image of the map $\mathbb{Z} \rightarrow \mathbb{k}$. Even when this map is not injective, all our examples can be compatibly \mathbb{Z} -graded. In this context, a few statements need to be modified in the obvious way. For example, for an element $a \in \mathbb{k}$ we will often write “ $a \in \mathbb{Z}$ ” to indicate that a is in the image of $\mathbb{Z} \rightarrow \mathbb{k}$, but in the \mathbb{Z} -graded context one should instead choose a preimage $a \in \mathbb{Z}$. Without further ado, we assume that our examples are \mathbb{Z} -graded even in finite characteristic.

Let us make one remark about divided powers.

Definition 2.16. Let $U = U_{\mathbb{Z}}(\mathfrak{sl}_2)$ denote the idempotent divided powers form of the enveloping algebra of \mathfrak{sl}_2 . A definition can be found, for instance, in [17, Section 3.1] (by specializing the q to be 1 in the quantum setting). All modules over U

will be assumed to be weight modules, so that the idempotent 1_n acts by multiplication by n for $n \in \mathbb{Z}$.

Note that U acts on any \mathfrak{sl}_2 -module where divided powers exist and the weights are integers. An action of U on A and B extends to an action on $A \otimes B$ (Corollary 2.6) since U is a Hopf algebra. The proof that U is a Hopf algebra is essentially the same as the proof of Lemma 2.5. A divided powers \mathfrak{sl}_2 -algebra over \mathbb{Z} is the same thing as an algebra in the category of (weight) U -modules.

2.4. On representations of \mathfrak{sl}_2 over the integers

The reader may be familiar with the properties of category \mathcal{O} , but many things are different and slightly unfamiliar when working over the integers. The goal of this section is to make precise what we mean by Verma and coVerma modules, and to warn the reader of some pitfalls. Because we study \mathbf{z} instead of $-\mathbf{z}$, and bounded-below modules rather than bounded-above modules, some signs may differ from expectations.

Definition 2.17. For $k \in \mathbb{k}$ we will define modules $\Delta(k)$ and $\nabla(k)$ over \mathfrak{sl}_2 as follows:

$$\Delta(k): \begin{array}{ccccccc} & \xrightarrow{\mathbf{d}=1} & & \xrightarrow{\mathbf{d}=2} & \cdots & \xrightarrow{\mathbf{d}=m-1} & \xrightarrow{\mathbf{d}=m} & \xrightarrow{\mathbf{d}=m+1} & \cdots \\ \bullet_0 & \xleftarrow{\mathbf{z}=k} & \bullet_1 & \xleftarrow{\mathbf{z}=k+1} & \cdots & \xleftarrow{\mathbf{z}=k+m-2} & \bullet_{m-1} & \xleftarrow{\mathbf{z}=k+m-1} & \bullet_m & \xleftarrow{\mathbf{z}=k+m} & \cdots \end{array} \tag{11.1}$$

$$\nabla(k): \begin{array}{ccccccc} & \xrightarrow{\mathbf{d}=k} & & \xrightarrow{\mathbf{d}=k+1} & \cdots & \xrightarrow{\mathbf{d}=k+m-2} & \xrightarrow{\mathbf{d}=k+m-1} & \xrightarrow{\mathbf{d}=k+m} & \cdots \\ \bullet_0 & \xleftarrow{\mathbf{z}=1} & \bullet_1 & \xleftarrow{\mathbf{z}=2} & \cdots & \xleftarrow{\mathbf{z}=m-1} & \bullet_{m-1} & \xleftarrow{\mathbf{z}=m} & \bullet_m & \xleftarrow{\mathbf{z}=m+1} & \cdots \end{array} \tag{11.2}$$

Let us rephrase these pictures in formulas. The module $\Delta(k)$ has a free \mathbb{Z} -basis $\{v_{k,m}\}_{m \geq 0}$, corresponding to the dots labeled by m in the picture above, where $v_{k,m}$ has weight $k + 2m$. Set $v_{k,m} := 0$ for $m < 0$. We have

$$\mathbf{d}(v_{k,m}) = (m + 1)v_{k,m+1}, \quad \mathbf{z}(v_{k,m+1}) = (m + k)v_{k,m}. \tag{12}$$

Similarly, $\nabla(k)$ has a free \mathbb{Z} -basis $\{w_{k,m}\}$, where $w_{k,m}$ has weight $k + 2m$. Set $w_{k,m} := 0$ for $m < 0$. We have

$$\mathbf{d}(w_{k,m}) = (m + k)w_{k,m+1}, \quad \mathbf{z}(w_{k,m+1}) = (m + 1)w_{k,m}. \tag{13}$$

Lemma 2.18. *When $a \in \mathbb{Z}$, divided powers are defined integrally on $\Delta(a)$ and $\nabla(a)$. More generally, divided powers are defined in \mathbb{k} whenever the a -binomial coefficients are defined in \mathbb{k} for all $l \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$: these are the elements*

$$\binom{a+l}{m} := \frac{(a+l)(a+l-1)\cdots(a+l+1-m)}{m!}.$$

Proof. It is a simple computation that

$$\mathbf{d}^{(l)} v_{k,m} = \binom{m+l}{l} v_{k,m+l}, \quad \mathbf{z}^{(l)} w_{k,m} = \binom{m}{l} w_{k,m-l}, \tag{14.1}$$

$$\mathbf{z}^{(l)} v_{k,m} = \binom{m+k-1}{l} v_{k,m-l}, \quad \mathbf{d}^{(l)} w_{k,m} = \binom{m+k+l-1}{l} w_{k,m+l}. \tag{14.2}$$

■

For example, suppose $\mathbb{k} = \mathbb{Z}[y]$ and $a = y$. Then

$$\binom{a+7}{3} = \frac{(y+7)(y+6)(y+5)}{3!},$$

is not an element of \mathbb{k} , so divided powers are not defined integrally on $\Delta(a)$ or $\nabla(a)$.

For the rest of this section, $\mathbb{k} = \mathbb{Z}$, so we can think of Verma and coVerma modules as U -modules.

Let $v_k := v_{k,0}$ and $w_k := w_{k,0}$. Note that

$$\mathbf{d}^{(m)}(v_k) = v_{k,m}, \quad \mathbf{z}^{(m)}(w_{k,m}) = w_k.$$

The module $\Delta(k)$ is generated by v_k over U , and the divided powers $\mathbf{d}^{(m)}$ applied to this generator give the basis. Meanwhile, $\nabla(k)$ is infinitely-generated over U , but it is “co-generated” by w_k , as the divided powers $\mathbf{z}^{(m)}$ bring every basis element to w_k .

Given any U -module M which is free as a \mathbb{Z} -module, $2M \subset M$ will be a proper submodule. Simplicity, in its naive sense as when working over a field, is not as useful a concept.

Definition 2.19. A morphism of U -modules is called **h-split** if it is a split morphism of \mathbb{Z} -modules for each weight space. Let $U\text{-spmod}$ denote the category whose objects are U -modules where weight spaces are free over \mathbb{Z} , and whose morphisms are **h-split** maps. If M is a module in $U\text{-spmod}$, a U -submodule is called **h-split** if the inclusion map is **h-split**, and similarly for quotient modules.

Note that any isomorphism, or more generally any genuinely split map (over U), is automatically **h-split**. However, $U\text{-spmod}$ is not an additive category, as the sum of **h-split** morphisms need not be **h-split**. After all, $\text{id} + \text{id} = 2 \text{id}$. Nonetheless, if f is any **h-split** morphism, then its kernel and cokernel naturally live in $U\text{-spmod}$, so $U\text{-spmod}$ shares some features with an abelian category.

When restricting to **h-split** morphisms, Verma and coVerma modules have features which resemble those in the familiar category \mathcal{O} .

Proposition 2.20. *The modules $\Delta(k)$ and $\nabla(k)$ are indecomposable for all $k \in \mathbb{Z}$. When $k > 0$, the modules $\Delta(k)$ and $\nabla(k)$ have no \mathfrak{h} -split submodules or quotients. There are \mathfrak{h} -split short exact sequences for all $k \leq 0$ given by*

$$0 \rightarrow \nabla(-k + 2) \rightarrow \Delta(k) \rightarrow W(k) \rightarrow 0, \tag{15.1}$$

$$0 \rightarrow W^\vee(k) \rightarrow \nabla(k) \rightarrow \Delta(-k + 2) \rightarrow 0, \tag{15.2}$$

where $W(k)$ and $W^\vee(k)$, the so-called Weyl and dual Weyl modules, are defined by these short exact sequences. Aside from those given in (15), there are no other \mathfrak{h} -split submodules or quotients of $\Delta(k)$ and $\nabla(k)$ for $k \leq 0$.

Proof. Since all weight spaces are free of rank 1 over \mathbb{Z} , an \mathfrak{h} -split submodule is determined by which weight spaces it includes. Now, the usual arguments (analyzing which arrows have zero coefficient) imply that only certain collections of weight spaces can give a submodule. One need only verify that when $k \leq 0$ the submodule of $\Delta(k)$ generated in degree $-k + 2$ is $\nabla(-k + 2)$, which one can do directly from the pictures (11) or the corresponding formulas. ■

What is more interesting is the following observation.

Proposition 2.21. *If \mathbb{k} contains \mathbb{Q} as a subring, then $\nabla(k) \cong \Delta(k)$ if and only if $k > 0$. Otherwise, $\nabla(k) \cong \Delta(k)$ if and only if $k = 1$.*

Note a major difference between this setting and the usual category \mathcal{O} in characteristic zero. Normally, $\Delta(k) \cong \nabla(k)$ for all $k > 0$, and one reverses the placement of $\Delta(-k + 2)$ and $\nabla(-k + 2)$ in (15); they are isomorphic, so it does not matter, but it is often done to help illustrate the Verma and coVerma resolutions of simple modules. Over \mathbb{Z} , one is not permitted to swap $\Delta(-k + 2)$ and $\nabla(-k + 2)$, and the Weyl and dual Weyl modules do not have Verma or coVerma resolutions. This fact will be of great importance in the sequel.

Proof. Clearly, $\nabla(1) \cong \Delta(1)$ by sending $w_{1,m} \mapsto v_{1,m}$, since the formulas (12) and (13) agree. When $k \leq 0$, clearly $\nabla(k) \not\cong \Delta(k)$ since one has a finite rank submodule and the other does not. So, suppose $k > 0$. What happens over \mathbb{Q} is well known, so assume that there is some $a \geq 1$ such that $\{1, \dots, a\}$ is invertible in \mathbb{k} and $a + 1$ is not invertible. We claim that the image of $\mathbf{z}^{(a)}$ in degree k is different when comparing $\nabla(k)$ to $\Delta(k)$, which implies they are non-isomorphic. Clearly, w_k is in the image of $\mathbf{z}^{(a)}$ inside $\nabla(k)$, whereas only the span of $(a + 1)v_k$ is in the image of $\mathbf{z}^{(a)}$ inside $\Delta(k)$. ■

Remark 2.22. If M is any module with a lowest weight vector v in weight k , then there is a morphism $\Delta(k) \rightarrow M$ sending $v_k \mapsto v$. This morphism will not necessarily

be **h**-split. For example, the natural morphism $\Delta(k) \rightarrow \nabla(k)$ is not **h**-split except when $k = 1$.

2.5. Rank-one modules over polynomial rings

Let $R_1 = \mathbb{k}[x]$, equipped with its standard \mathfrak{sl}_2 structure from (1), where $\mathbf{d} = x^2 \frac{\partial}{\partial x}$ and $\mathbf{z} = \frac{\partial}{\partial x}$. By our choice of convention, $\mathbf{h} = 2x \frac{\partial}{\partial x}$.

Proposition 2.23. *As an \mathfrak{sl}_2 -module, $R_1 \cong \nabla(0)$. In particular, divided powers are well defined, so R_1 is a U -module.*

Proof. If we send $w_{0,m} \mapsto x^m$, we can confirm (13) easily. ■

Remark 2.24. One is tempted to say that the ideal (x) is isomorphic to $\Delta(2)$, but one must be careful with this statement. Ideals are usually thought of as submodules, so that (x) is identified with a subset of R_1 . This subset is an R_1 -submodule but not a U -submodule, because it is not preserved by \mathbf{z} . This was discussed at more length in the introduction, see the toy example from Section 1.3.

Definition 2.25. Let $a \in \mathbb{k}$ and let $R_1\langle a \rangle$ denote the free rank one graded R_1 module with generator 1_a . We give it an (R_1, \mathfrak{sl}_2) -module structure by setting

$$\mathbf{d}(1_a) = ax \cdot 1_a, \quad \mathbf{z}(1_a) = 0, \quad \mathbf{h}(1_a) = a1_a,$$

and extending these operators to all of $R_1\langle a \rangle$ by the Leibniz rule.

That $R_1\langle a \rangle$ is well defined is a consequence of the following proposition.

Proposition 2.26. *The following statements hold.*

- (i) *Any (R_1, \mathfrak{sl}_2) -module structure on the rank-one free module R_1 is isomorphic to $R_1\langle a \rangle$ for a unique $a \in \mathbb{k}$.*
- (ii) *As an \mathfrak{sl}_2 -module, $R_1\langle a \rangle \cong \nabla(a)$. In particular, divided powers are well defined when $a \in \mathbb{Z}$.*

Proof. For a rank-one module, denote by v a generator as an R_1 -module. Then, the module is also generated by v as an (R_1, \mathfrak{sl}_2) -module. For degree reasons, there must be $a, a' \in \mathbb{k}$ such that

$$\mathbf{d}(v) = axv, \quad \mathbf{h}(v) = a'v, \quad \mathbf{z}(v) = 0.$$

The commutation relation $[\mathbf{d}, -\mathbf{z}] = \mathbf{h}$ applied to the generator v shows that

$$a'v = \mathbf{h}v = \mathbf{z}\mathbf{d}v - \mathbf{d}\mathbf{z}v = av,$$

implying that $a' = a$.

For the second statement, if we send $w_{a,m} \mapsto x^m \cdot 1_a$, we can confirm (13) easily. ■

Now, we let $R_n = \mathbb{k}[x_1, \dots, x_n]$ be the polynomial \mathfrak{sl}_2 -algebra, as in Definition 1.14.

Lemma 2.27. *As an algebra and an \mathfrak{sl}_2 -module we have $R_n \cong R_1 \otimes \dots \otimes R_1$. In particular, divided powers are well defined.*

Proof. The isomorphism with the tensor product is obvious. See Corollary 2.6 for the rest. ■

We recall a definition from the introduction.

Definition 2.28. Let $p = \sum a_i x_i$ be a linear polynomial in R_n , and let $\Sigma(p) = \sum a_i \in \mathbb{k}$. Note that $\Sigma(p) = \mathbf{z}_R(p)$, though Σ is more descriptive notation. There is a free rank one graded R_n -module $R_n\langle p \rangle$ with generator 1_p . We define an (R_n, \mathfrak{sl}_2) -module structure on $R_n\langle p \rangle$ by setting

$$\mathbf{d}(1_p) = p \cdot 1_p, \quad \mathbf{z}(1_p) = 0, \quad \mathbf{h}(1_p) = \Sigma(p)1_p,$$

and extending by the Leibniz rule.

Proposition 2.29. *As an \mathfrak{sl}_2 -module,*

$$R_n\langle p \rangle \cong R_1\langle a_1 \rangle \otimes \dots \otimes R_1\langle a_n \rangle \cong \nabla(a_1) \otimes \dots \otimes \nabla(a_n). \tag{16}$$

In particular, divided powers are well defined when $a_i \in \mathbb{Z}$ for all i .

Proof. It is easily verified that the map $1_p \mapsto 1_{a_1} \otimes \dots \otimes 1_{a_n}$ induces an isomorphism. ■

Proposition 2.30. *Every (R_n, \mathfrak{sl}_2) -module which is free of rank one as a graded R_n -module is isomorphic to $R_n\langle p \rangle$ for some p .*

Proof. Let M be such a module, and name the generator 1_M . Then $\mathbf{z}(1_M) = 0$, and $\mathbf{d}(1_M) = p \cdot 1_M$ for some linear polynomial p , both for degree reasons. The remaining structure follows from the Leibniz rule. The fact that the degree of 1_M must be $\Sigma(p)$ follows from the fact that

$$\mathbf{z}\mathbf{d}(1_M) = \mathbf{z}(p) \cdot 1_M = \Sigma(p)1_M. \quad \blacksquare$$

Remark 2.31. For more examples of \mathfrak{sl}_2 -algebras, see Section 6.1 and Section 6.2.

2.6. The core

We will eventually be interested in settings where the base ring \mathbb{k} is \mathbb{Z} or \mathbb{Q} or $\mathbb{Z}[y]$ or of finite characteristic. Thus, we are careful in this section to make general statements.

Definition 2.32. An $\mathfrak{sl}_2(\mathbb{k})$ module is called *bounded* if

- it is a weight module, with weights in \mathbb{Z} ,
- the set of weights with non-zero weight spaces is bounded below,
- and each non-zero weight space is free of finite rank over \mathbb{k} .

Proposition 2.33. *Suppose that $\mathbb{Z} \rightarrow \mathbb{k}$ is injective and \mathbb{k} is Noetherian. Any bounded \mathfrak{sl}_2 representation M will have a maximal submodule which is finitely generated over \mathbb{k} , which we call the core of M and denote $\text{Core}(M)$. It satisfies*

$$\text{Core}(M) = \{m \in M \mid \mathbf{d}^N(m) = 0 \text{ for some } N \in \mathbb{N}\}. \tag{17}$$

Note that the core may be zero.

Proof. For the sake of this proof, define $\text{Core}(M)$ using (17). Note that $\text{Core}(M)$ is a weight module. Clearly, \mathbf{d} acts nilpotently on any submodule which is finitely generated as a \mathbb{k} -module, so $\text{Core}(M)$ contains all such submodules. We need only prove that $\text{Core}(M)$ is finitely generated over \mathbb{k} .

We now argue that $\text{Core}(M)$ is locally finitely generated, i.e., any element is contained in an \mathfrak{sl}_2 submodule which is finitely generated over \mathbb{k} . Suppose that $m \in M$ homogeneous is acted upon nilpotently by \mathbf{d} . By the PBW theorem, the span of $\{\mathbf{z}^a \mathbf{h}^b \mathbf{d}^c \cdot m\}_{a,b,c \geq 0}$ is an \mathfrak{sl}_2 subrepresentation containing m . Only finitely many pairs (a, c) will give a non-zero result, and the span of $\{\mathbf{z}^a \mathbf{h}^b \mathbf{d}^c \cdot m\}_{b \geq 0}$ agrees with the span of the single vector $\mathbf{z}^a \mathbf{d}^c \cdot m$. Thus, m is contained in a subrepresentation which is finite rank over \mathbb{k} .

If $\text{Core}(M)$ is not finitely generated over \mathbb{k} , then by the Noetherian hypothesis it must have non-zero elements in infinitely many weight spaces (lest it be contained in a finite rank \mathbb{k} -module). Because $\text{Core}(M)$ is locally finitely generated, it must have highest weight vectors in arbitrarily high weights. We now argue that $\text{Core}(M)$ also has (lowest weight) vectors in arbitrary low weights, a contradiction because M is bounded. This last argument is standard, we are just being careful because we have placed very few assumptions on \mathbb{k} .

Suppose that $\text{Core}(M)$ has a highest weight vector v in weight $k > 0$. Let $\Delta'(k)$ be the bounded-above Verma module with highest weight k . Then v induces a non-zero map of $\mathfrak{sl}_2(\mathbb{k})$ -modules $\phi: \Delta'(k) \rightarrow \text{Core}(M)$. Since the target is a submodule of a free \mathbb{k} -module, it is torsion free over \mathbb{k} . Since each weight space in $\Delta'(k)$ is free of rank 1 over \mathbb{k} , the kernel of ϕ is divisible, so it is spanned by a subset of the weight

spaces in $\Delta'(k)$. Now, standard arguments imply that the kernel of ϕ is zero in weight $-k$, so $\text{Core}(M)$ is non-zero in weight $-k$. ■

Remark 2.34. When \mathbb{k} is a PID, one can use Smith normal form to deduce that the kernel of a map between free (finite rank) modules is a split summand of the source. One can use this to prove that $\text{Core}(M)$ is a **h**-split submodule of M . We imagine this is always true (even when \mathbb{k} is a Dedekind domain or something unusual) but we do not have the knowledge or examples to say one way or another.

Lemma 2.35. *There is a left-exact functor Core from bounded \mathfrak{sl}_2 -modules to \mathfrak{sl}_2 -modules which are finitely generated over \mathbb{k} , sending M to $\text{Core}(M)$, and restricting any morphism to the core.*

Proof. The image under an \mathfrak{sl}_2 -intertwiner of a finitely generated module is a finitely generated submodule. Thus, the core is sent to the core under any \mathfrak{sl}_2 -intertwiner. Injective maps restrict to injective maps, so Core is left-exact. ■

Remark 2.36. Note that Core is not right exact. For example, for $a \leq 0$ the canonical map $\Delta(a) \rightarrow W(a)$ is surjective, but $\text{Core}(\Delta(a)) = 0$ while $\text{Core}(W(a)) = W(a)$. In this sense, the core of a quotient can be larger than the core of the original module. Of course, the core of a quotient can be smaller too, as the quotient map could kill part of the core.

Proposition 2.37. *Let $p = \sum a_i x_i \in R_n$ with $a_i \in \mathbb{Z}$. If $a_i > 0$ for some i , then $\text{Core}(R_n \langle p \rangle) = 0$. If $a_i \leq 0$ for all i , then*

$$\text{Core}(R_n \langle p \rangle) \cong W_{a_1}^\vee \otimes \cdots \otimes W_{a_n}^\vee. \tag{18}$$

Proof. We use the identification (16) of $R_n \langle p \rangle$ with a tensor product of coVermas.

Let M be any \mathfrak{sl}_2 module and consider $\nabla(a) \otimes M$. Any vector v in this tensor product can be written uniquely as

$$\sum_{j \geq 0} w_{a,j} \otimes m_j$$

for some $m_j \in M$, with finitely many m_j being non-zero. Let J be the maximal value of j for which $m_j \neq 0$. Then

$$\mathbf{d}^N \left(\sum_{j \geq 0} w_{a,j} \otimes m_j \right) = \mathbf{d}^N(w_{a,J}) \otimes m_J + \sum_{j' < J+N} w_{a,j'} \otimes n_{j'}$$

for some $n_{j'} \in M$. So, long as $\mathbf{d}^N(w_{a,J}) \neq 0$, $\mathbf{d}^N(v) \neq 0$. But by the formula from (14), $\mathbf{d}^N(w_{a,J})$ is never zero when $a > 0$. In fact, it is never zero when $J > -a$.

As a consequence, we deduce that $\nabla(a) \otimes M$ has zero core when $a > 0$. Similar arguments prove that $\Delta(a) \otimes M$ has no core for any $a \in \mathbb{Z}$. This also follows from the familiar idea that tensoring a Verma module with anything yields a module with a Verma filtration, and modules with a Verma filtration have no finite-dimensional submodules.

This proves that $\text{Core}(R_n\langle p \rangle) = 0$ if any $a_i > 0$. If $a_i \leq 0$ for all i , then the tensor product of the submodules $W_{a_i}^\vee$ clearly lives inside $\text{Core}(R_n\langle p \rangle)$. Since $\nabla(a_i)/W_{a_i}^\vee \cong \Delta(-a_i + 2)$, the quotient of $R_n\langle p \rangle$ by the tensor product of coweyl modules has a filtration whose subquotients are isomorphic to $M \otimes \Delta(a)$ for some a . If \mathbf{d} has no nilpotents on the associated graded of a filtered module, it has no nilpotents on the whole module. This proves that the core of $R_n\langle p \rangle$ is not bigger than expected. ■

Corollary 2.38. *Let $p = \sum a_i x_i$. Then the elements*

$$\mathbb{B} := \{x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \mid 0 \leq b_i \leq -a_i\}$$

form a basis for the core of $R_n\langle p \rangle$.

Proof. This follows from the case $n = 1$, where it is straightforward. ■

Finally, we mention one more useful result. The same argument will show that the core of a tensor product contains the tensor product of the cores.

Proposition 2.39. *In an \mathfrak{sl}_2 -category where all morphism spaces are bounded \mathfrak{sl}_2 representations, the core is a subcategory. In a monoidal \mathfrak{sl}_2 -category with this property, the core is a monoidal subcategory.*

Proof. We need to prove that the core contains all identity maps (which is obvious) and is closed under composition (both horizontal and vertical). We use the description of the core as those elements on which \mathbf{d} acts nilpotently. If $\mathbf{d}^{N_1}(f) = 0$ and $\mathbf{d}^{N_2}(g) = 0$, then $\mathbf{d}^{N_1+N_2}(f \circ g) = 0$ and $\mathbf{d}^{N_1+N_2}(f \otimes g) = 0$, by the Leibniz rule. ■

3. On decompositions in the presence of \mathfrak{sl}_2

In this quasi-introductory section we discuss the nature of direct sum decompositions in the presence of an \mathfrak{sl}_2 action, and the possible implications for the Grothendieck group. Afterwards, we will reconnect these ideas to the concept of the core. For simplicity, we work with categories over a field \mathbb{k} rather than the integers.

The idea has already arisen in categorification at a root of unity (which studies categories with a derivation \mathbf{d}) to study a direct sum decomposition not with the traditional use of idempotents, but by studying filtrations on representable functors instead. This is a guiding principle below.

3.1. Actions on partial idempotent completions

It is traditional when studying (pre)additive categories over a field to take (the additive closure and then) the Karoubi envelope. One obtains a Krull–Schmidt category, whose split Grothendieck group is spanned by isomorphism classes of indecomposable objects. Then one uses direct sum decompositions to find relations in the Grothendieck group. One way to motivate the Karoubi envelope is that it naturally appears when studying modules over the category: representable modules $\text{Hom}(X, -)$ are projective, and thus so are their direct summands $\text{Hom}(X, -) \circ e$ for any idempotent $e \in \text{End}(X)$.

However, for a category with an \mathfrak{sl}_2 action, there is no induced action of \mathfrak{sl}_2 on the Karoubi envelope!

Let X be an object and $e \in \text{End}(X)$ an idempotent. The subspace

$$\text{Hom}(X, -) \circ e \subset \text{Hom}(X, -)$$

is not closed under \mathfrak{sl}_2 in general. For each $\mathbf{x} \in \{\mathbf{d}, \mathbf{z}, \mathbf{h}\}$ one can try to define an operator $\underline{\mathbf{x}}$ on $\text{Hom}(X, -) \circ e$ by truncating the usual action, to wit

$$\underline{\mathbf{x}}(f) := \mathbf{x}(f)e \quad \text{for } f \in \text{Hom}(X, -) \circ e. \tag{19}$$

Each operator $\underline{\mathbf{x}}$ satisfies the Leibniz rule. However, these operators will not necessarily satisfy the \mathfrak{sl}_2 relations.

Definition 3.1. An \mathfrak{sl}_2 *submodule idempotent* is an idempotent satisfying $e\mathbf{x}(e) = 0$ for all $\mathbf{x} \in \{\mathbf{d}, \mathbf{h}, \mathbf{z}\}$. An \mathfrak{sl}_2 *quotient idempotent* instead satisfies $\mathbf{x}(e)e = 0$. An object which has no non-trivial \mathfrak{sl}_2 submodule idempotents is called \mathfrak{sl}_2 -*indecomposable*.

By applying the Leibniz rule to $\mathbf{x}(e^2) = \mathbf{x}(e)$, one deduces that e is an \mathfrak{sl}_2 submodule idempotent if and only if $\mathbf{x}(e) = \mathbf{x}(e)e$. Hence, $\text{Hom}(X, -) \circ e$ is closed under \mathbf{x} , and the formula (19) is another way to write the restriction of the \mathfrak{sl}_2 action to this submodule.

It is also easy to deduce that e is an \mathfrak{sl}_2 submodule idempotent if and only if $(1 - e)$ is an \mathfrak{sl}_2 quotient idempotent. In this case, $\text{Hom}(X, -) \circ (1 - e)$ is not closed under the \mathfrak{sl}_2 action. However, using the identification

$$\text{Hom}(X, -) \circ (1 - e) \cong \text{Hom}(X, -) / \text{Hom}(X, -) \circ e,$$

the \mathfrak{sl}_2 action induced on this quotient agrees with the formulas from (19).

Let e be an \mathfrak{sl}_2 submodule idempotent. Looking at representable functors out of the partial idempotent completion which adjoins the images of e and $1 - e$, we have a short exact sequence of \mathfrak{sl}_2 representations

$$0 \rightarrow \text{Hom}(X, -) \circ e \rightarrow \text{Hom}(X, -) \rightarrow \text{Hom}(X, -) \circ (1 - e) \rightarrow 0, \tag{20}$$

which splits as modules for the underlying additive category. In this way, the direct sum decomposition $X \cong \text{Im}(e) \oplus \text{Im}(1 - e)$ in the additive category becomes a filtration for \mathfrak{sl}_2 . We still refer to the idempotents e and $(1 - e)$ as complements.

Definition 3.2. An \mathfrak{sl}_2 decomposition of an object X is an ordered sequence of idempotents e_1, e_2, \dots, e_n such that

- (1) the set $\{e_1, \dots, e_n\}$ is a complete collection of orthogonal idempotents: $e_i e_j = e_j e_i = 0$ for $i \neq j$, and $\sum_{i=1}^n e_i = \text{id}_X$;
- (2) for any $k \leq n$, the idempotent $e_1 + e_2 + \dots + e_k$ is an \mathfrak{sl}_2 submodule idempotent.

In this case, the idempotents e_i are called \mathfrak{sl}_2 subquotient idempotents.

Similarly, the formula (19) defines an \mathfrak{sl}_2 action on $\text{Hom}(X, -) \circ e$ for any \mathfrak{sl}_2 subquotient idempotent. For the analogous definition in the context of p -dg algebras, see [9, Section 4].

3.2. Future goals, and cofibrance

The goal is to find a setting where certain idempotent decompositions give rise to relations on the Grothendieck group, much as they did in the split Grothendieck group of the underlying additive category. Another goal is for this Grothendieck group to be naturally a module over the Grothendieck ring of finite-dimensional \mathfrak{sl}_2 representations. Not every idempotent decomposition should give rise to a relation. For a general idempotent, the subspaces $\text{Hom}(X, -) \circ e$ and $\text{Hom}(X, -) \circ (1 - e)$ are entangled by the \mathfrak{sl}_2 action, and no filtration (20) is present; one does not expect a relation in the Grothendieck group. However, even for submodule idempotents, one does not always expect a relation, because of a technical point we now address.

The program of categorification at a root of unity, which accounts for the action of \mathfrak{d} but not \mathfrak{z} , has an established framework for Grothendieck groups like the one we seek. There is a triangulated category associated to a category with an action of \mathfrak{d} , constructed using Hopfological algebra [14,29]. In this triangulated category, the short exact sequence (20) does not always give rise to a distinguished triangle, but it does when the module $\text{Hom}(X, -) \circ e$ is cofibrant. In particular, the appropriate category to study is not the partial idempotent completion which adjoins all \mathfrak{sl}_2 subquotient idempotents, but only those which are cofibrant.

There are typically two ways to prove that the image of an idempotent is cofibrant. The first is to show that it is isomorphic to a representable module, which happens when the idempotent factors compatibly with \mathbf{d} :

$$e = i \circ p, \quad p \circ i = \text{id}_Y \text{ for some object } Y, \quad \mathbf{d}(p) \circ i = 0. \tag{21}$$

The second method, the *two-out-of-three principle*, uses the fact that if two out of three terms in (20) are cofibrant, then so is the third. Thus, the complement of a representable module in a representable module is cofibrant. Adjoining its image, one can assume it is representable for future decompositions.

So, to summarize, one expects relations on the Grothendieck group coming from cofibrant short exact sequences like (20), and cofibrance can be confirmed by factoring the idempotent as a projection and inclusion, and checking $\mathbf{d}(p) \circ i = 0$. For a longer discussion of these thorny issues, see [9, Section 4].

Everything below in this section should be taken with a grain of salt. The authors’ expectation is that there is a triangulated category one can build from any category with \mathfrak{sl}_2 action, where short exact sequences coming from cofibrant \mathfrak{sl}_2 submodule idempotents give rise to distinguished triangles. If one can factor an idempotent as $e = i \circ p$, where i and p have degree zero, then it will be cofibrant if $\mathbf{x}(p) \circ i = 0$ for all $\mathbf{x} \in \mathfrak{sl}_2$. However, there are additional cofibrant objects coming from representable modules in the enriched category.

Definition 3.3. Let \mathcal{C} be an \mathfrak{sl}_2 -category. The *\mathfrak{sl}_2 -enriched category* is the \mathfrak{sl}_2 -category $\mathcal{C}_{\text{enrich}}$ defined as follows. Its objects are formal tensor products $B \boxtimes V$ (see Remark 2.12 for an explanation of the notation), for an object B of \mathcal{C} and a finite-dimensional representation V of \mathfrak{sl}_2 . We set

$$\text{Hom}_{\mathcal{C}_{\text{enrich}}}(B \boxtimes V, B' \boxtimes V') := \text{Hom}_{\mathcal{C}}(B, B') \otimes \text{Hom}_{\mathfrak{sl}_2}(V, V') \tag{22}$$

as an \mathfrak{sl}_2 -module. Composition is given by $(f \otimes \phi) \circ (g \otimes \psi) = (f \circ g) \otimes (\phi \circ \psi)$. There is a monoidal action of the category of finite-dimensional \mathfrak{sl}_2 representations on $\mathcal{C}_{\text{enrich}}$, where on objects one has $(B \boxtimes V) \otimes V' := B \boxtimes (V \otimes V')$.

One should think about $B \boxtimes V$ as though it were the direct sum of $\dim(V)$ copies of B in \mathcal{C} (with the appropriate grading shifts). Then $\text{Hom}_{\mathcal{C}_{\text{enrich}}}(B \boxtimes V, B' \boxtimes V')$ can be viewed appropriately as matrices of morphisms between B and B' . However, the action of \mathfrak{sl}_2 on V and V' is used to twist the action on these matrices of morphisms. If $\dim(V) > 1$, then $B \boxtimes V$ is decomposable as an object in the underlying category \mathcal{C} , but can be \mathfrak{sl}_2 -indecomposable when V is indecomposable.

We develop the enriched category more formally in the sequel. This method of enrichment is similar to techniques one encounters when studying the equivariantization–deequivariantization principle, see, e.g., [3, Section 4]. In the next section

we discuss a practical method to find an \mathfrak{sl}_2 submodule idempotent whose image is $B \boxtimes V$, generalizing the equations in (21).

Representable modules from the enriched category should also be considered as cofibrant. Taking the \mathfrak{sl}_2 -enriched category is analogous to taking the additive closure of a preadditive category, and adjoining the images of cofibrant \mathfrak{sl}_2 subquotient idempotents is analogous to taking the Karoubi envelope. Both are operations one typically does before attempting to take a Grothendieck group. Below, we refer to the partial idempotent completion which adds cofibrant \mathfrak{sl}_2 subquotient idempotents as the *cofibrant Karoubi envelope*.

At the moment, there are only conjectural ideas for how to build a triangulated category from an \mathfrak{sl}_2 -enriched category with an appropriate Grothendieck group. These ideas (\mathfrak{sl}_2 -omological algebra) will be explored in the sequel. However, one can explicitly state conjectures for the expected behavior of the 2-categories studied in this paper, which would be required for this triangulated category to be well behaved, such as our \mathfrak{sl}_2 -adapted Soergel categorification conjecture in Section 3.5 below.

3.3. Decompositions and the core

For sanity, throughout this section we work over a field of characteristic zero. Recall the divided powers operators $\mathbf{x}^{(k)} := \frac{\mathbf{x}^k}{k!}$ for $k \geq 1$. It may help the reader to read this section concurrently with the following one, which gives examples of the general phenomena discussed here.

Suppose we have objects X and B in an \mathfrak{sl}_2 category, and we wish to construct an \mathfrak{sl}_2 submodule idempotent whose image is $B \boxtimes V$ where, for example, V is the three-dimensional irreducible module for \mathfrak{sl}_2 . In the underlying additive category, we would need to prove that

$$B(-2) \oplus B(0) \oplus B(+2) \overset{\oplus}{\subset} X,$$

which we would do by constructing inclusion and projection maps. Specifically, we would find maps $p_j: X \rightarrow B$ of degree j for $j \in \{-2, 0, 2\}$, and inclusion maps i_j of degree $-j$, satisfying

$$p_j \circ i_k = \delta_{jk} \text{id}_B. \tag{23}$$

We set $e_j = i_j \circ p_j$ and $e = \sum e_j$.

Suppose can find maps p_{-2} and i_2 of degree -2 , satisfying the equations

$$\mathbf{z}(p_{-2}) = 0 = \mathbf{z}(i_{-2}), \quad \mathbf{d}^3(p_{-2}) = 0, \quad \mathbf{d}^{(2)}(p_{-2}) \circ i_{-2} = \text{id}_B. \tag{24}$$

Let us set

$$p_0 = \mathbf{d}(p_{-2}), \quad p_2 = \mathbf{d}^{(2)}(p_{-2}).$$

Then $\mathbf{d}(p_2) = 0$, and $\{p_{-2}, p_0, p_2\}$ is the standard basis for V (styled as a Weyl module) sitting isomorphically inside the core of $\text{Hom}(X, B)$. Similarly, one can define

$$i_0 = -\frac{1}{2}\mathbf{d}(i_2), \quad i_{-2} = \mathbf{d}^{(2)}(i_2).$$

However, we make no assumption that $\mathbf{d}^3(i_2) = 0$! Were this true, then $\{i_2, -i_0, i_{-2}\}$ would be the dual standard basis for V (styled as a dual Weyl module) sitting inside $\text{Hom}(B, X)$. In practice, $\text{Hom}(B, X)$ may have no finite-dimensional \mathfrak{sl}_2 subrepresentations.

Applying \mathbf{d} to the equality $p_2i_2 = \text{id}_B$ we obtain

$$-2p_2i_0 = \mathbf{d}(p_2)i_2 + p_2\mathbf{d}(i_2) = 0,$$

so $p_2i_0 = 0$. Similarly, applying \mathbf{d} and \mathbf{z} repeatedly we can deduce that (23) holds.

Moreover, let $\phi = \mathbf{d}(i_{-2})$, which measures the failure of the span of $\{i_2, i_0, i_{-2}\}$ to be a finite-dimensional subrepresentation. Again, using the Leibniz rule, we deduce that

$$p_j \circ \phi = 0 \quad \text{for all } j \in \{-2, 0, 2\}.$$

We leave the reader to verify that

$$\begin{aligned} e_2\mathbf{d}(e_2) &= i_2p_2\mathbf{d}(i_2)p_2 = -2i_2p_2i_0p_2 = 0, \\ e_2\mathbf{z}(e_2) &= i_2p_2i_2\mathbf{z}(p_2) = i_2p_0 \neq 0. \end{aligned}$$

In particular, e_2 is a \mathbf{d} -submodule idempotent and a \mathbf{z} -quotient idempotent, but not an \mathfrak{sl}_2 subquotient idempotent. Similarly, e_{-2} is a \mathbf{d} -quotient idempotent and a \mathbf{z} -submodule idempotent, and e_0 is a subquotient for both, but neither of these are \mathfrak{sl}_2 subquotient idempotents. However, recalling that $e = \sum e_j$, we have

$$e\mathbf{d}(e) = 0, \quad e\mathbf{z}(e) = 0.$$

Thus, e is an \mathfrak{sl}_2 submodule idempotent. We claim that $\text{Hom}(X, -) \circ e$ is a submodule of $\text{Hom}(X, -)$ which is isomorphic as an \mathfrak{sl}_2 -module to $\text{Hom}(B, -) \boxtimes V$.

Remark 3.4. The inclusion and projection maps i_j and p_j form an Fc-filtration on $B \boxtimes V$ for \mathbf{d} in the sense of [9, Section 4.7]. They also form an Fc-filtration for \mathbf{z} , with the opposite order.

The reader may imagine how this generalizes to other irreducible representations of \mathfrak{sl}_2 . Finding projection and inclusion maps of negative-most degree which satisfy the analogue of (24) is our primary tool for constructing cofibrant \mathfrak{sl}_2 -submodule idempotents. As one can see, this is intimately related with finding the core of $\text{Hom}(X, B)$, and proving that the maps within are split.

Remark 3.5. It is not obvious that every \mathfrak{sl}_2 -submodule idempotent of X whose image is isomorphic to $B \boxtimes V$ should be constructible in this fashion, and this is a slightly thorny and technical issue, not suitable for this brief overview. We explore this question in the sequel.

Remark 3.6. That projection and inclusion maps of negative-most degree are in the kernel of \mathbf{z} is a relatively ubiquitous phenomenon. Maps of negative-most degree are constructed from diagrams without polynomials, which by the formulas in this paper are in the kernel of \mathbf{z} . Also, maps of negative-most degree often split when this degree is non-positive (for the diagrammatic Hecke category in characteristic zero, this is related to the Soergel conjecture, and further to hard Lefschetz on multiplicity spaces, see [13]).

Remark 3.7. One can define a *Lefschetz form* on degree $-k$ elements of $\text{Hom}(X, B)$, which sends $(f, g) \mapsto f \circ \mathbf{d}^k(\bar{g})$, where \bar{g} is the vertical flip of g . This form is valued in $\text{End}(B)$. If B is absolutely indecomposable, then $\text{End}(B)$ is local with maximal ideal \mathfrak{m} and quotient field $\text{End}(B)/\mathfrak{m} \cong \mathbb{k}$. In this case one can define a Lefschetz form valued in \mathbb{k} .

In the prototypical example above, when $i_2 = \overline{p_{-2}}$, then the assumption that i_2 pairs non-trivially against $\mathbf{d}^2(p_{-2})$ is the non-degeneracy of a Lefschetz form. In the examples we have computed, this Lefschetz form satisfies the Hodge–Riemann bilinear relations. Thus, the existence of cofibrant \mathfrak{sl}_2 subquotient idempotents is related to the Hodge theory of the operator \mathbf{d} , when restricted to the core of $\text{Hom}(X, B)$.

The entire discussion above relates to proving that an \mathfrak{sl}_2 subquotient idempotent has the appropriate factorization behavior, which implies it is cofibrant. What about the other \mathfrak{sl}_2 subquotient idempotents in an \mathfrak{sl}_2 decomposition?

If we wish to continue decomposing X , we need to examine the complement of $B \boxtimes V$. If it is not \mathfrak{sl}_2 -indecomposable, we seek a cofibrant \mathfrak{sl}_2 submodule summand isomorphic to $B' \boxtimes V'$ in this complement, which would be an \mathfrak{sl}_2 subquotient summand of X . To apply our techniques above one does not necessarily need the core of $\text{Hom}(X, B')$ to be non-trivial; one needs the core of $\text{Hom}(\text{Im}(1 - e), B')$ to be non-trivial instead. In other words, we want a projection map $p'_j : X \rightarrow B'$ of non-positive degree for which $\mathbf{z}(p'_j) = 0$, and while $\mathbf{d}^{-j+1}(p'_j)$ might not be zero, it must be zero modulo $\text{Hom}(X, -) \circ e$. These secondary projection maps p'_j live in the core relative to e , while e was generated by projection maps in a genuine core. Thus, these projection maps live in an *iterated core*: the core of some morphism space, considered modulo an \mathfrak{sl}_2 -invariant ideal generated by elements in the core of some other morphism space (and so forth).

In the sequel, we will carefully define the notion of an \mathfrak{sl}_2 -*antastic filtration*, which is effectively an \mathfrak{sl}_2 decomposition where each subquotient idempotent e_i is

constructed using projection maps which are in the core modulo the ideal generated by $e_1 + \dots + e_{i-1}$. For an \mathfrak{sl}_2 -antastic filtration, the subquotient idempotents e_i have cofibrant images. We also permit \mathfrak{sl}_2 -antastic filtrations where all but one of the idempotents factors as above. The remaining idempotent will be cofibrant by the two-out-of-three principle.

3.4. Behavior in examples

To elaborate on the ideas of the previous section, and on the relationship between the core and the semisimplification, we discuss several miracles which occur in the diagrammatic Hecke category $\mathcal{H}(S_n)$, and try to avoid distracting the reader with the details. All these examples were computed explicitly in [10], and we give more precise references for each example. Only the raising operator \mathbf{d} was studied in [10], but adding \mathbf{z} into the mix is relatively easy (often it is zero for degree reasons). We work over a field \mathbb{k} .

We begin with an example where everything is as easy as possible, studied in [10, Sections 4.3 and 4.4].

Example 3.8. There is an object B_s (for any simple reflection s) satisfying

$$B_s B_s \cong B_s(-1) \oplus B_s(+1) \tag{25}$$

in $\mathcal{H}(S_n)$. Since the endomorphism ring of B_s is a local ring with quotient field \mathbb{k} , the semisimplification of $\text{End}(B_s B_s)$ is isomorphic to $\text{Mat}_2(\mathbb{k})$. The degree -1 projection map $p_1: B_s B_s \rightarrow B_s(-1)$ is uniquely determined up to scalar, and the miracle is that $\mathbf{d}^2(p_1) = 0$.

In close analogy to Example 1.10, the core of the \mathfrak{sl}_2 action on $\text{End}(B_s B_s)$ is a subalgebra isomorphic to $\text{Mat}_2(\mathbb{k})$, which is isomorphic to $V \otimes V^*$ as an \mathfrak{sl}_2 representation. The matrix entries have the form $i_j p_k$ for $j, k \in \{1, 2\}$, where i_j and p_k are particular inclusion and projection maps for the decomposition (25). Moreover, the core of $\text{Hom}(B_s B_s, B_s)$ is precisely the span of p_1 and p_2 , with $\mathbf{d}(p_1) = p_2$ and $\mathbf{d}(p_2) = 0$. Similarly, the core of $\text{Hom}(B_s, B_s B_s)$ is the span of i_1 and i_2 with $\mathbf{d}(i_2) = i_1$ and $\mathbf{d}(i_1) = 0$.

Note however that neither primitive idempotent $e_1 = i_1 p_1$ nor $e_2 = i_2 p_2 = 1 - e_1$ is an \mathfrak{sl}_2 submodule idempotent. We have $e_1 \mathbf{z}(e_1) = 0$ but $e_1 \mathbf{d}(e_1) \neq 0$, and vice versa for e_2 . Thus, $B_s B_s$ is \mathfrak{sl}_2 -indecomposable.

In the enriched category, there is an isomorphism $B_s B_s \cong B_s \boxtimes \mathbb{k}^2$, where \mathbb{k}^2 is the standard representation of \mathfrak{sl}_2 .

The next example was done in great detail in [10, Sections 6.4 and 6.5, clarified further in Section 6.8].

Example 3.9. In the Hecke category of S_3 , there is a specific object⁵ X which splits as a direct sum of two non-isomorphic indecomposable objects Y and Z . Let J denote the Jacobson radical of the category. It is the case that $\text{End}(Y)/J \cong \mathbb{k}$ and $\text{End}(Z)/J \cong \mathbb{k}$, so that $\text{End}(X)/J$ is two dimensional, spanned by the two idempotents which project to these summands. However, the core of $\text{End}(X)$ is one-dimensional, spanned by the identity map, and neither primitive idempotent is killed by \mathbf{d} .

The decomposition $X \cong Y \oplus Z$ implies that

$$\text{Hom}(X, -) \cong \text{Hom}(Y, -) \oplus \text{Hom}(Z, -),$$

or more precisely

$$\text{Hom}(X, -) \cong \text{Hom}(Y, -) \circ p_Y \oplus \text{Hom}(Z, -) \circ p_Z, \tag{26}$$

where p_Y and p_Z are the projection maps.

Consider $\text{Hom}(X, Y)$ and $\text{Hom}(X, Z)$. Both are supported in non-negative degrees, and are spanned in degree zero by their respective projection maps p_Y and p_Z . Thus, for degree reasons, $\mathbf{z}(p_Y) = 0$ and $\mathbf{z}(p_Z) = 0$, and if either Hom space has a non-zero core, it must be a trivial module spanned by the projection map.

The first miracle is that

$$\mathbf{d}(p_Y) = 0.$$

As a consequence, $\text{Hom}(Y, -) \circ p_Y \subset \text{Hom}(X, -)$ is preserved by the \mathfrak{sl}_2 action, making the decomposition (26) a filtration with respect to \mathfrak{sl}_2 . In particular, this submodule $\text{Hom}(Y, -) \circ p_Y$ is generated by the core of $\text{Hom}(X, Y)$.

Meanwhile, $\mathbf{d}(p_Z) \neq 0$, and $\text{Core}(\text{Hom}(X, Z)) = 0$. The second miracle is that

$$\mathbf{d}(p_Z) \in \text{Hom}(Y, Z) \cdot p_Y. \tag{27}$$

We have already observed by $\text{Hom}(Y, Z) \cdot p_Y$ is preserved by \mathfrak{sl}_2 inside $\text{Hom}(X, Z)$, and (27) implies that the image of p_Z spans the core of the quotient module $\text{Hom}(X, Z)/(\text{Hom}(Y, Z) \cdot p_Y)$.

Hence, the splitting of (26) as modules over \mathcal{H} becomes a filtration with respect to the \mathfrak{sl}_2 action, with $\text{Hom}(Y, -) \circ p_Y$ being the submodule. Moreover, each layer of the filtration is generated by its core modulo the ideal generated by the previous part of the filtration.

In contrast, if i_Y is the corresponding inclusion map in $\text{Hom}(Y, X)$, we have $\mathbf{d}(i_Y) \neq 0$. This explains why the corresponding idempotent $e_Y = i_Y p_Y$ is not killed by \mathbf{d} . However, we have the miracles

$$\mathbf{d}(i_Z) = 0, \quad \mathbf{d}(i_Y) \in i_Z \circ \text{Hom}(Y, Z).$$

⁵Here $X = B_s B_t B_s$, $Y = B_s$, and $Z = B_s t s$.

Working with contravariant representable functors instead, we see that the decomposition

$$\text{Hom}(-, X) = i_Y \circ \text{Hom}(-, Y) \oplus i_Z \circ \text{Hom}(Z, -)$$

of functors is a filtration for \mathfrak{sl}_2 , where this time $i_Z \circ \text{Hom}(Z, -)$ is the submodule.

Another consequence of these miracles is that $e_Y \mathbf{d}(e_Y) = 0$, so e_Y is an \mathfrak{sl}_2 subquotient idempotent with complement e_Z . This equips X with an \mathfrak{sl}_2 -antastic filtration.

In the above example, one could not find sufficiently many projection maps in the core to split X into indecomposable direct summands. However, one could construct *iterated cores* by taking the core, quotienting by the ideal it generates, taking the core of what remains, quotienting by the ideal it generates, and so forth. One could find sufficiently many projection maps for X within the iterated cores. Similar to Conjecture 1.12, we conjecture that iterated cores do not intersect the Jacobson radical (in the 2-categories we consider in this paper). A thorough development of iterated cores and a formal statement of this conjecture will be found in the sequel to this paper.

Iterated cores are still insufficient to split any object into indecomposable direct summands, as seen in the next example, which is studied in [10, Section 9].

Example 3.10. In the Hecke category of S_8 , there is an object⁶ X which splits as a direct sum of two non-isomorphic indecomposable objects Y and Z . The object Z is technically not in the category but in the Karoubi envelope. However, there is no splitting of $\text{Hom}(X, -)$ or $\text{Hom}(-, X)$ into direct summands as a left module over \mathcal{H} which is filtered with respect to \mathfrak{sl}_2 ; neither e_Y nor e_Z is an \mathfrak{sl}_2 subquotient idempotent. These two additive summands are entangled by the \mathfrak{sl}_2 action. At the same time, the cores of both $\text{Hom}(X, Y)$ and $\text{Hom}(Y, X)$ are zero. We cannot even define an \mathfrak{sl}_2 structure on $\text{Hom}(Z, -)$ or $\text{Hom}(-, Z)$.

In the example above, the \mathfrak{sl}_2 structure cannot be used to find a complement for the Jacobson radical, and X is \mathfrak{sl}_2 -indecomposable. This may be a feature rather than a bug! In characteristic 2 the splitting $X \cong Y \oplus Z$ does not hold, and X is indecomposable; perhaps this is detected by the \mathfrak{sl}_2 action.

3.5. The \mathfrak{sl}_2 -adapted Soergel categorification conjecture

We assume the reader is familiar with the ordinary Soergel categorification theorem, see [7, Sections 5.5 and 11] or [12, Theorems 3.15 and 6.25].

⁶Here X is the Bott–Samelson bimodule associated to the sequence 35246135724635, with top summand Z , and Y is the indecomposable Soergel bimodule associated to the element 232565.

Conjecture 3.11. *Let us work in the \mathfrak{sl}_2 -enriched category of $\mathcal{H}(S_n)$. There are objects D_y in the cofibrant Karoubi envelope, which satisfy and are inductively defined by the following properties.*

- (1) *Let \underline{w} be a reduced expression for an element $w \in S_n$. The Bott–Samelson object $\text{BS}(\underline{w})$ has an \mathfrak{sl}_2 -antastic filtration, and all but one subquotient in the filtration is isomorphic to $D_y \boxtimes V$ for some finite-dimensional \mathfrak{sl}_2 representation V and some $y < w$ in S_n . The remaining subquotient is therefore cofibrant. This last subquotient is \mathfrak{sl}_2 -indecomposable and not isomorphic to $D_y \boxtimes V$ for any $y < w$ and any V , and we denote it by $D_{\underline{w}}$.*
- (2) *The morphisms known as rex moves produce isomorphisms between $D_{\underline{w}}$ and $D_{\underline{w}'}$ for any two reduced expressions for the same element w . We let D_w denote an object in this isomorphism class.*
- (3) *Any Bott–Samelson object $\text{BS}(\underline{x})$, whether the expression \underline{x} is reduced or not, has an \mathfrak{sl}_2 -antastic filtration whose subquotients are isomorphic to $D_y \boxtimes V$ for various finite-dimensional V and various $y \in S_n$.*

The ordinary Soergel categorification theorem can be stated in almost identical fashion, except using ordinary direct sum decompositions (with grading shifts) rather than \mathfrak{sl}_2 -antastic filtrations. This is used to define the indecomposable Soergel bimodule B_w for $w \in S_n$. Note that D_w need not be isomorphic to B_w in the underlying additive category. In example 3.10, letting $w = 35246135724635$ which is short for $s_3s_5s_2 \dots$, the Bott–Samelson bimodule $X = \text{BS}(\underline{w})$ is \mathfrak{sl}_2 -indecomposable, so $D_w = X$. However, D_w is not indecomposable, and its top indecomposable summand B_w was called Z above.

If this conjecture holds, then, in some conjectural triangulated category, we expect that the objects D_w will descend to a basis in the triangulated Grothendieck group, as a free module over the Grothendieck ring of finite-dimensional \mathfrak{sl}_2 representations. Because an \mathfrak{sl}_2 -antastic filtration descends to an Fc-filtration for \mathbf{d} , we will obtain an analogous result for the p -dg Hecke category, as a free module over the cyclotomic ring $\mathbb{Z}[\zeta]$ where ζ^2 is a primitive p -th root of unity.

The Grothendieck ring of this triangulated category should provide a $\mathbb{Z}[q + q^{-1}]$ -form of the Hecke algebra. The structure coefficients for the basis $[D_w]$ are unimodal polynomials, because they are the graded dimensions of finite-dimensional \mathfrak{sl}_2 representations. This does not imply the unimodality of Kazhdan–Lusztig structure coefficients, since $D_w \neq B_w$.

We think of Conjecture 3.11 as stating that the Hecke category “has enough” \mathfrak{sl}_2 -antastic filtrations, or that it “has enough” split maps in the iterated cores. When trying to generalize this conjecture to the other 2-categories studied in this paper, it is not easy to rigorously state what it means to “have enough,” without pinning

down implicitly some suspected collection of objects, like $\{D_w\}$, which generate the category. We have failed to find a concise way to discuss the generalizations of Conjecture 3.11, so we leave further discussion to the next paper where these ideas are developed more formally.

4. \mathfrak{sl}_2 action on the KLR algebra

4.1. Definition of the action

Definition 4.1. Let $\mathcal{U}^+(\mathfrak{g})$ denote the Khovanov–Lauda–Rouquier category associated to an oriented simply laced root datum. A presentation by generators and relations can be found in [18, Section 4.1].

For the reader familiar with KLR algebras: a KLR algebra is determined by the polynomials Q_{ij} in the double crossing relation, for each pair of vertices i and j in the (oriented) Dynkin diagram. If i and j are not connected by an edge, $Q_{ij} = 0$ as usual. If an edge is oriented from i to j , then $Q_{ij} = x_i - x_j$. In our action of \mathfrak{sl}_2 below, we will have

$$\mathbf{d}(x_i) = x_i^2, \quad \mathbf{z}(x_i) = 1, \tag{28}$$

for all⁷ i . It is crucial here that $\mathbf{z}(Q_{ij}) = 0$, so one cannot use the alternative choice $Q_{ij} = x_i + x_j$.

Remark 4.2. The original definition of Khovanov and Lauda [15] used $Q_{ij} = x_i + x_j$, while the signed variant was introduced in [16, p. 15]. It was correctly predicted on [16, p. 17] that this signed variant matches Lusztig’s geometric categorification of the quantum group. As shown explicitly by Lauda in [23, Proposition 3.4], different choices of Q_{ij} yield isomorphic categories when the Dynkin diagram is simply connected. This isomorphism will rescale the polynomials x_i by various signs. Evidently one could transfer our \mathfrak{sl}_2 action along Lauda’s isomorphisms, but they would no longer satisfy (28).

Definition 4.3. Equip $\mathcal{U}^+(\mathfrak{g})$ with an \mathfrak{sl}_2 action as follows. The derivation \mathbf{d} is the derivation ∂_1 defined in [18, Definition 4.13]. The derivation \mathbf{z} sends a dot to 1, and kills all crossings. In formulas, where red and blue are adjacent colors and red and

⁷Do not confuse the index i , which represents a node in the Dynkin diagram, with the indices in $R_n = \mathbb{k}[x_1, \dots, x_n]$. For each node i one has an i -colored nilHecke algebra, containing a polynomial ring R_n which is the n -fold tensor product of $\mathbb{k}[x_i]$.

green are distant, we have

$$\begin{aligned}
 \mathbf{d}(\text{dot on top}) &= 2 \text{dot on bottom}, & \mathbf{d}(\text{cross}) &= -\text{dot on top-left} - \text{dot on bottom-right}, \\
 \mathbf{d}(\text{cross with blue/red}) &= \text{dot on top-left}, & \mathbf{d}(\text{cross with green/red}) &= 0, \\
 \mathbf{z}(\text{dot on top}) &= \text{dot on bottom}, & \mathbf{z}(\text{cross}) &= 0, \\
 \mathbf{z}(\text{cross with blue/red}) &= 0, & \mathbf{z}(\text{cross with green/red}) &= 0.
 \end{aligned}$$

One extends the derivation to all diagrams using the monoidal Leibniz rule.

Theorem 4.4. *The action of \mathfrak{sl}_2 on $\mathcal{U}^+(\mathfrak{g})$ given in Definition 4.3 is well defined. The divided power operators $\mathbf{d}^{(k)}$ and $\mathbf{z}^{(k)}$ are well defined in the integral form for all $k \geq 0$, making $\mathcal{U}^+(\mathfrak{g})$ into a divided powers \mathfrak{sl}_2 -algebra.*

Proof. First we need to check that the derivations \mathbf{d} and \mathbf{z} are well defined. By Lemma 2.2, we need only check that they satisfy the relations of $\mathcal{U}^+(\mathfrak{g})$. That \mathbf{d} preserves the relations was checked⁸ in [18, Section 4.1]. Let us check that \mathbf{z} preserves the relations.

Any diagram without dots is sent to zero by \mathbf{z} . Any relation which is a linear combination of diagrams without dots is therefore preserved by \mathbf{z} . The remaining relations are these:

$$\text{identity} = \text{dot on top-left} - \text{dot on bottom-right}. \tag{29}$$

Applying \mathbf{z} will kill the identity map, and send the other diagrams to the crossing,

$$\mathbf{z}(\text{dot on top-left}) = \text{cross} = \mathbf{z}(\text{dot on bottom-right}),$$

so this relation is preserved. The vertical flip is also a relation, and preserved for the same reason.

Similarly, \mathbf{z} will preserve the relation

$$\text{cross with blue/red} = \text{cross with blue/red and dot on top-left}, \tag{30}$$

because

$$\mathbf{z}(\text{cross with blue/red and dot on top-left}) = \text{cross with blue/red} = \mathbf{z}(\text{cross with blue/red}).$$

⁸In [18, end of Section 4.1], a multiparameter family of degree +2 derivations is defined, and it is checked throughout the entirety of that section that the relations are preserved by each derivation in this family. The differentials in [18, Definition 4.13] are special members of this family, cf. [18, Proposition 4.11].

Variants of this relation (e.g., when colors are distant, or the vertical flip) are preserved by \mathbf{z} for the same reason.

The final relation with dots involves two adjacent colors:

$$\begin{array}{c} \text{crossing} \\ \text{with dots} \end{array} = \pm \left(\begin{array}{c} \text{parallel lines} \\ \text{with dots} \end{array} - \begin{array}{c} \text{parallel lines} \\ \text{with dots} \end{array} \right). \tag{31}$$

The sign is determined by the orientation of the edge connecting these two colors, and will not affect the rest of the computation. Applying \mathbf{z} to the left-hand side yields zero. The right-hand side is $Q_{ij} = \pm(x_i - x_j)$, and $\mathbf{z}(x_i - x_j) = 0$.

Hence, \mathbf{z} is a well-defined derivation. Now, we wish to check that $(\mathbf{d}, \mathbf{h}, -\mathbf{z})$ is an \mathfrak{sl}_2 triple, and also that divided powers are well defined integrally. By Lemmas 2.3 and 2.5, we need only check this on the generators.

Certain Hom spaces are clearly identified as (R, \mathfrak{sl}_2) -modules for various polynomial rings R . For example, when i and j are adjacent colors, $\text{Hom}(\mathcal{E}_i \mathcal{E}_j, \mathcal{E}_j \mathcal{E}_i) \cong \mathbb{Z}[x_i, x_j]$ as left $\mathbb{Z}[x_i, x_j]$ -modules. By construction, \mathbf{z} kills the generator of the free rank 1 module, and \mathbf{d} multiplies it by x_j , so the result is precisely the module $\mathbb{Z}[x_i, x_j]\langle x_j \rangle$. (We should also confirm that the generator lives in degree $+1 = \Sigma(x_j)$.) We have already proven in Proposition 2.29 that this is a well-defined \mathfrak{sl}_2 -representation with integrally-defined divided powers. Here are several such isomorphisms (here i and j are adjacent colors, and i and k are distant):

$$\begin{aligned}
 \text{End}(\mathcal{E}_i) &\cong \mathbb{Z}[x_i]\langle 0 \rangle, \\
 \text{Hom}(\mathcal{E}_i \mathcal{E}_k, \mathcal{E}_k \mathcal{E}_i) &\cong \mathbb{Z}[x_i, x_k]\langle 0 \rangle, \\
 \text{Hom}(\mathcal{E}_i \mathcal{E}_j, \mathcal{E}_j \mathcal{E}_i) &\cong \mathbb{Z}[x_i, x_j]\langle x_j \rangle.
 \end{aligned}$$

This takes care of all generators but the crossing in $\text{End}(\mathcal{E}_i \mathcal{E}_i)$.

It is easy to verify that

$$\mathbf{z}\mathbf{d} \left(\text{crossing} \right) = -2 \text{crossing}$$

as desired. It is easy to verify that

$$\mathbf{d}^2 \left(\text{crossing} \right) = 2 \text{crossing} \quad \text{and} \quad \mathbf{d}^3 \left(\text{crossing} \right) = 0.$$

One can see this also from the matrix of (6). In particular, when acting on the (i, i) crossing, $\mathbf{d}^{(2)}$ is well defined integrally and $\mathbf{d}^{(k)} = 0$ for all $k \geq 3$. ■

4.2. Action on the basis of crossings

It was proven by in [15, Theorem 2.5] that each morphism space in $\mathcal{U}^+(\mathfrak{g})$ is a free left (or right) module over the polynomial subring generated by dots, and that a basis can be constructed using diagrams built entirely out of crossings.

More precisely, let $\underline{i} = (i_1, \dots, i_n)$ and $\underline{j} = (j_1, \dots, j_n)$ be sequences of colors, both having length n . Let $(S_n)_{\underline{i}}^{\underline{j}}$ denote the subset of S_n consisting of those permutations w for which $i_k = j_{w(k)}$ for all $1 \leq k \leq n$. This subset is empty unless the number of strands of each color agrees in \underline{i} and \underline{j} . If this happens, and there are a_m strands with the color m , then restricting $w \in (S_n)_{\underline{i}}^{\underline{j}}$ to the strands of color m we get a permutation in S_{a_m} . This induces a bijection between $(S_n)_{\underline{i}}^{\underline{j}}$ and $S_{a_1} \times \dots \times S_{a_d}$, and this bijection preserves the Bruhat order. In each case, the Bruhat order is generated by the operation which removes a single crossing between two same-colored strands, when the result is still a reduced expression.

Pick a reduced expression of $w \in (S_n)_{\underline{i}}^{\underline{j}}$, and let ψ_w denote the crossing diagram for that reduced expression, with the strands colored to represent a morphism in $\text{Hom}(\mathcal{E}_{\underline{i}}, \mathcal{E}_{\underline{j}})$. Then, as a left module over $R_{\underline{j}}$ (dots acting on the target sequence \underline{j}), or as a right module over $R_{\underline{i}}$ (dots acting on the source sequence \underline{i}), we have

$$\{\psi_w\}_{w \in (S_n)_{\underline{i}}^{\underline{j}}} \text{ is a basis for } \text{Hom}(\mathcal{E}_{\underline{i}}, \mathcal{E}_{\underline{j}}).$$

Theorem 4.5. *For any $w \in (S_n)_{\underline{i}}^{\underline{j}}$ we have*

$$\mathbf{z}(\psi_w) = 0, \quad \mathbf{d}(\psi_w) \in \text{Span}\{\psi_v\}_{v \in (S_n)_{\underline{i}}^{\underline{j}}, v \leq w}.$$

Thus, $\{\psi_w\}$ induces a downfree filtration of $\text{Hom}(\mathcal{E}_{\underline{i}}, \mathcal{E}_{\underline{j}})$, parametrized by $(S_n)_{\underline{i}}^{\underline{j}}$ with its Bruhat order. In the associated graded, the span of ψ_w is a copy of $R_n\langle p(w) \rangle$, where a formula for $p(w)$ is given in (33) or (34), depending on whether we study the left $R_{\underline{j}}$ -action or the right $R_{\underline{i}}$ -action.

Proof. Let us first prove the statement in the context of left modules over $R_{\underline{j}}$. Clearly, \mathbf{z} kills ψ_w as desired. The result is easiest to see using a different formula for the action of \mathbf{d} on same-colored crossings, namely

$$\mathbf{d}\left(\begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array}\right) = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} - 2 \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array}. \tag{32}$$

We also recall

$$\mathbf{d}\left(\begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array}\right) = \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array}, \quad \mathbf{d}\left(\begin{array}{c} \text{---} \diagdown \text{---} \\ \text{---} \diagup \text{---} \end{array}\right) = 0.$$

When we apply \mathbf{d} to ψ_w , we take the sum of \mathbf{d} applied to each crossing: up to linear combinations, this will either add a dot to the northwest of a crossing, or remove the crossing.

The first term on the right and side of (32) says to remove a same-colored crossing. This produces an expression for an element $v < w$ which is smaller in the Bruhat order. Note that this may not be a reduced expression for v , and even if it is, it may not be the chosen reduced expression ψ_v , so relations (such as (31)) must be applied

to rewrite this as a linear combination of basis diagrams. This rewriting process may produce diagrams which are even lower in the Bruhat order (cf. the adjacent-colored Reidemeister III relation [15, (2.8)]), but it is well known that the relations of the KLR algebra can be used to simplify an arbitrary diagram to a basis diagram without ever going upwards in the Bruhat order. We state this as Lemma 4.7 below. So, whenever we remove a crossing, we get something in the R_j -span of $\psi_{v'}$ for $v' < w$ in the Bruhat order.

Now, suppose a dot is placed to the northwest of a crossing. This dot should be forced to the top of the diagram (using (29) or (30)). On its way up, (29) can produce an error term where a same-colored crossing is removed; once again, this will be in the span of diagrams which are smaller in the Bruhat order. To see what polynomial remains on the top, we need only count the number of times a dot appeared on each strand.

For the k -th strand on top (colored j_k), we count the ways a dot can be added to that strand. We get coefficient -2 each time that strand is the northwest strand of a same-colored crossing, and $+1$ each time that strand is the northwest strand of an adjacent-colored crossing. Note that the k -th strand on top will cross the ℓ -th strand on top, and be the northwest strand, if and only if $k < \ell$ and $w^{-1}(\ell) < w^{-1}(k)$. Consequently, let

$$p(w) = \sum_{k=1}^n x_k \cdot \left(-2 \cdot \#\{\ell > k \mid w^{-1}(\ell) < w^{-1}(k), j_\ell = j_k\} + \#\{\ell > k \mid w^{-1}(\ell) < w^{-1}(k), j_\ell \text{ adjacent to } j_k\} \right). \tag{33.1}$$

Hence, $p(w)$ represents the action of \mathbf{d} on the associated graded in the downfree filtration. We may write $p(w)$ more succinctly using the dot product on the set of colors (see [15, top of p3]) by the formula

$$p(w) = \sum_{k=1}^n \left(\sum_{\ell > k, w^{-1}(\ell) < w^{-1}(k)} -j_k \cdot j_\ell \right) x_k. \tag{33.2}$$

Suppose we instead study this morphism space as a right R_i -module. Now, it helps to use the following formulas instead:

$$\begin{aligned} \mathbf{d} \left(\begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \right) &= - \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} - 2 \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array}, \\ \mathbf{d} \left(\begin{array}{c} \text{---} \diagdown \text{---} \\ \text{---} \diagup \text{---} \end{array} \right) &= \begin{array}{c} \text{---} \diagdown \text{---} \\ \text{---} \diagup \text{---} \end{array}. \end{aligned}$$

The k -th strand on bottom receives a dot whenever it is the southeast strand in a crossing, and if it crosses the ℓ -th strand on bottom, then $\ell < k$ and $w(\ell) > w(k)$. So,

let

$$p(w) = \sum_{k=1}^n x_k \cdot \left(-2 \cdot \#\{\ell < k \mid w(\ell) > w(k), i_\ell = i_k\} + \#\{\ell < k \mid w(\ell) > w(k), i_\ell \text{ adjacent to } i_k\} \right), \tag{34.1}$$

$$p(w) = \sum_{k=1}^n \left(\sum_{\ell < k, w(\ell) > w(k)} -i_\ell \cdot i_k \right) x_k. \tag{34.2}$$

Then this $p(w)$ describes the associated graded as a right R_i -module. ■

Remark 4.6. Applying the anti-involution which flips each diagram upside-down, we get the analogous result for the \mathfrak{sl}_2 -triple $(\bar{\mathbf{d}}, \mathbf{h}, -\mathbf{z})$. This will swap the left and right action, so it will also swap (34) and (33).

In the proof above, we used the following statement.

Lemma 4.7. *Let D be a crossing diagram corresponding to a subexpression of a reduced expression for $w \in S_n$, where we remove one crossing. Then D is in the left R_j -span (or the right R_i -span) of $\{\psi_v\}_{v \leq w}$.*

This lemma is considered obvious by most people in the field (including us), for which reason it is difficult to cite. The result is stated without proof in [15, (2.33)], for example. It would not be hard to prove this lemma directly, but it would be several annoying pages without much payoff. We will continue the tradition and not prove the lemma. One could also deduce the result from the main theorem in [4].

While Theorem 4.5 gives control over the associated graded, it does not describe the upper-triangular terms in \mathbf{d} explicitly. This was because rewriting non-reduced expressions or the wrong reduced expression in terms of the basis can be complicated. However, for the nilHecke algebra it is easy: non-reduced expressions are zero, and all reduced expressions for the same element are equal. For the nilHecke algebra it is not hard to give a precise formula for $\mathbf{d}(\psi_w)$.

Theorem 4.8. *Inside NH_n we have*

$$\mathbf{d}(\psi_w) = p(w)\psi_w + \sum_{v <_1 w} (1 + 2m_{v,w})\psi_v. \tag{35}$$

Here, $v <_1 w$ means that $v < w$ in the Bruhat order and $\ell(v) = \ell(w) - 1$. The integers $m_{v,w}$ will be described in the proof.

Note that we study the left action of R_n in this proof. We leave the adaptations for the right action to the reader.

Proof. To any element $w \in S_n$, one can associate its inversion set, the set of pairs (a, b) with $1 \leq a < b \leq n$ such that $w(a) > w(b)$. Each crossing in a reduced expression for w produces a single inversion (a, b) , and we refer to it as the (a, b) -crossing. In other words, if (a, b) is in the inversion set of w then the a th strand and the b th strand (on bottom) will eventually cross exactly once, and this crossing can be identified with the pair (a, b) . Note that b will be the strand on the northwest of the (a, b) crossing; we always write our inversions (a, b) in order so that $a < b$.

Pick a triple $a < b < c$ and consider the ordered set $\{(a, b) < (a, c) < (b, c)\}$, called the *packet* of the triple (a, b, c) . A quick examination (try to draw it) should convince the reader that the inversion set of w intersected with this packet is either a prefix or a suffix. Moreover, for any reduced expression for w , the crossings in a packet either appear in lexicographic order (if a prefix) or antilexicographic order (if a suffix). Only when the entire packet is contained in the inversion set of w can they appear in either lexicographic or antilexicographic order (because the full set is both a prefix and a suffix), and the braid relation $sts = tst$ swaps lexicographic for antilexicographic. For example, if (b, c) is not an inversion, but (a, c) is, then (a, b) must also be an inversion, and (a, b) must occur below (a, c) in a reduced expression for w . These ideas are the start of Manin–Schechtmann’s theory of higher Bruhat orders, see [27].

If $v <_1 w$ then the inversion set of v is equal to that of w with exactly one inversion removed. Let (a, b) be this inversion. Then (with regard to the left action of R_n) we set

$$m_{v,w} := \#\{c \mid a < c < b \text{ and } w(b) < w(a) < w(c)\}.$$

In this formula, the condition that $w(c) > w(a)$ is equivalent to the condition that $w(c) > w(b)$. After all, if $w(b) < w(c) < w(a)$ then the packet of (a, c, b) will be entirely contained in the inversion set of w , and removing (a, b) will not yield either a prefix or a suffix, so it will not yield a reduced expression.

We now argue that (35) holds.

Applying \mathbf{d} to ψ_w as in the proof of Theorem 4.5, we get a sum of diagrams where either a dot is added or a crossing is removed. We have already computed that when all dots reach the top, the overall polynomial is $p(w)$. We need to compute the coefficient with which the (a, b) crossing is removed, for some (a, b) in the inversion set of w . Any crossing which occurs lower down in ψ_w , and for which the a or b strand is the northwest-southeast strand, will produce a dot which will eventually be forced through the (a, b) crossing. Thus, the coefficient involved in the removal of the (a, b) crossing is

- +1 from (32),
- +2 if some (c, b) crossing occurs below the (a, b) crossing,
- −2 if some (c, a) crossing occurs below the (a, b) crossing.

So, let c be any other strand. If some (c, a) crossing occurs below the (a, b) crossing, then clearly $c < a < b$. Since both (c, a) and (a, b) crossings appear, so must (c, b) , and since (c, a) happens before (a, b) , they come in lexicographic order. So, (c, b) also occurs below (a, b) . Then the overall contribution to the coefficient is $-2 + 2 = 0$. Thus, no contribution of -2 can occur without being canceled by a contribution of $+2$. Conversely, if $c < a < b$ and a (c, b) crossing occurs before an (a, b) crossing, then the crossings appear in lexicographic order, meaning that a (c, a) crossing must have come first.

If some (c, b) crossing occurs below the (a, b) crossing, but a (c, a) crossing does not also occur below the (a, b) crossing, then we must have $a < c < b$. This will contribute $+2$ to the coefficient, and $m_{v,w}$ is exactly counting such contributions. ■

4.3. The nilHecke algebra and the matrix algebra

We wish to justify some of the claims made in the introduction. For sake of sanity we work in characteristic zero. Let $n \geq 1$.

Theorem 4.9. *Let $\mathbb{k} = \mathbb{Q}$. As an \mathfrak{sl}_2 representation, $\text{Mat}_{n!}(\mathbb{k})$ is a finite-dimensional subrepresentation of NH_n , and is isomorphic to $\text{End}_{\mathbb{Q}}(L_0 \otimes L_1 \otimes \cdots \otimes L_{n-1})$, where L_k is the irreducible representation of \mathfrak{sl}_2 of dimension $k + 1$. It is the core of NH_n .*

Most aspects of this theorem were proven in [18, Proposition 3.24 and preceding]. We give three proofs, mostly for pedagogical reasons. The example of $n = 3$ is done explicitly after the proofs, and it may help the reader to look at the proofs and the example simultaneously.

Proof. Let V denote the cohomology ring of the flag variety, thought of as the quotient of R_n by the ideal generated by positive degree elements of $R_n^{S_n}$. Then V is an $n!$ -dimensional graded vector space, equipped with a natural action of \mathfrak{d} . By [15, Example 2.2.3] there is a ring isomorphism $\text{NH}_n \cong \text{Mat}_{n!}(R_n^{S_n})$, and hence a vector space isomorphism $\text{NH}_n \cong \text{Mat}_{n!}(\mathbb{k}) \otimes_{\mathbb{k}} R_n^{S_n}$, where we identify $\text{Mat}_{n!}(\mathbb{k})$ with $\text{End}_{\mathbb{k}}(V)$. It was proven in [18, Proposition 3.24] that the action of \mathfrak{d} respects this tensor product decomposition. From this we can immediately deduce that \mathfrak{d} acts nilpotently on $\text{Mat}_{n!}(\mathbb{k}) \subset \text{NH}_n$, so this subalgebra is contained in the core of NH_n .

There are a number of ways to confirm that the core of NH_n is not bigger than $\text{Mat}_{n!}(\mathbb{k})$. Using the tensor decomposition $\text{NH}_n \cong \text{End}(V) \otimes R_n^{S_n}$, one can verify that nothing else is acted on nilpotently by \mathfrak{d} . Alternatively, the core of a (bounded below) \mathfrak{sl}_2 -algebra is a subalgebra by Proposition 2.39, and any subalgebra of $\text{Mat}_{n!}(R_n^{S_n})$ which properly contains $\text{Mat}_{n!}(\mathbb{k})$ is infinite dimensional.

Note that V is not naturally an \mathfrak{sl}_2 -representation: \mathfrak{sl}_2 acts on R_n but does not preserve the ideal generated by positive degree elements of $R_n^{S_n}$, so it does not act on

the quotient. However, we know that the core is some finite-dimensional \mathfrak{sl}_2 -representation, so its isomorphism class is determined by its graded dimension. ■

Here is a second, more computational proof.

Proof. Let

$$\delta = (n - 1)x_1 + (n - 2)x_2 + \cdots + 2x_{n-2} + 1x_{n-1} \in R_n.$$

Consider the (R_n, \mathfrak{sl}_2) -module $R_n\langle -\delta \rangle$. As an \mathfrak{sl}_2 representation, Proposition 2.37 states that

$$\text{Core}(R_n\langle -\delta \rangle) \cong L_{n-1} \otimes L_{n-2} \otimes \cdots \otimes L_0,$$

which is a representation of dimension $n!$. The same is true for $R_n\langle -\delta' \rangle$ where

$$\delta' = 1x_2 + 2x_3 + \cdots + (n - 1)x_n,$$

though it may be nicer to order the tensor products in a fashion respecting the indices on the polynomials:

$$\text{Core}(R_n\langle -\delta' \rangle) \cong L_0 \otimes \cdots \otimes L_{n-2} \otimes L_{n-1}.$$

In fact, a basis for $\text{Core}(R_n\langle -\delta \rangle)$ is also a basis for R_n as a free module over $R_n^{S_n}$. From Schubert theory, it is well known that the polynomials

$$\mathbb{B} = \{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq n - i \text{ for all } i\}$$

form a basis for R_n over $R_n^{S_n}$. Meanwhile, by Proposition 2.37 and Corollary 2.38, they also form a basis for the core of $R_n\langle -\delta \rangle$. In similar fashion, for $\text{Core}(R_n\langle -\delta' \rangle)$ we can use the basis

$$\mathbb{B}' = \{x_1^{b_1} \cdots x_n^{b_n} \mid 0 \leq b_i \leq i - 1 \text{ for all } i\}.$$

There is a map of R_n -bimodules

$$R_n \otimes_{\mathbb{k}} R_n \rightarrow \text{NH}_n, \quad f \otimes g \mapsto f\psi_{w_0}g.$$

To make this a map of graded R_n -bimodules, we need to shift the source so that $1 \otimes 1$ lives in degree $-n(n - 1)$. Keeping track of the action of \mathbf{d} , we get an \mathfrak{sl}_2 -intertwiner

$$\phi: R_n \otimes_{\mathbb{k}} R_n\langle -\delta_l - \delta'_r \rangle \rightarrow \text{NH}_n. \tag{36}$$

Here we think of $R_n \otimes_{\mathbb{k}} R_n$ as a polynomial ring in $2n$ variables, the left variables and the right variables, where δ_l uses the left variables and δ'_r uses the right variables. This map ϕ is known to be surjective, see e.g. the matrix basis of NH_n described in [17, Proposition 2.16] or [18, Proposition 3.3]. It is not injective, because it factors

through the quotient $R_n \otimes_{R_n^{S_n}} R_n$, though $\text{Ker } \phi$ must be an \mathfrak{sl}_2 -submodule. In fact, by counting graded dimensions, one can verify that

$$R_n \otimes_{R_n^{S_n}} R_n \langle -\delta_l - \delta'_r \rangle \rightarrow \text{NH}_n$$

is an isomorphism.

Again, by Proposition 2.37 and Corollary 2.38, we know that

$$\text{Core}(R_n \otimes_{\mathbb{K}} R_n \langle -\delta_l - \delta'_r \rangle) \cong \text{End}(L_{n-1} \otimes L_{n-2} \otimes \cdots \otimes L_0)$$

as \mathfrak{sl}_2 -representations, being spanned by $\mathbb{B} \otimes \mathbb{B}'$. However, we are interested in the core of the quotient $R_n \otimes_{R_n^{S_n}} R_n$, and the core of a quotient module can be both bigger and smaller than the original, see Remark 2.36. Thankfully, we also know that $\mathbb{B} \otimes \mathbb{B}'$ is a basis of $R_n \otimes_{R_n^{S_n}} R_n$ as an $R_n^{S_n}$ module, and goes to a basis of NH_n over $R_n^{S_n}$, see again [17, Proposition 2.16]. So, the map ϕ from (36) is injective on the core.

As in the other proof, once one knows that the core of NH_n contains $\mathbb{B} \otimes \mathbb{B}'$ or $\text{Mat}_{n!}(\mathbb{K})$, there are a number of ways to confirm that it is not bigger. ■

Remark 4.10. Note that $\mathbb{B} \otimes \mathbb{B}'$ is not quite the matrix basis of

$$\text{Mat}_{n!}(\mathbb{K}) \subset \text{Mat}_{n!}(R_n^{S_n}),$$

though it has the same span. To get the matrix basis one must use dual bases for R_n over $R_n^{S_n}$, such as the Schubert and dual Schubert bases.

Here is a sketch of a third proof that the core contains

$$\text{End}_{\mathbb{Q}}(L_0 \otimes L_1 \otimes \cdots \otimes L_{n-1}).$$

We ignore the remaining parts of the theorem.

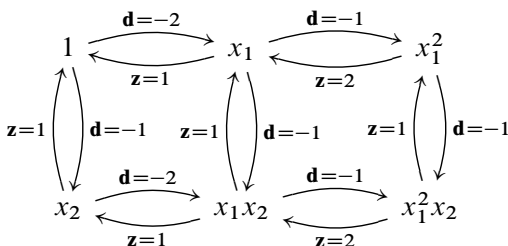
Proof. (Sketch) By restricting from R_n to $R_n^{S_n}$, $R_n \langle -\delta \rangle$ is an $(R_n^{S_n}, \mathfrak{sl}_2)$ -module of rank $n!$ with the \mathfrak{sl}_2 -stable basis \mathbb{B} . As was shown in [18, Section 3.1] (see the discussion there around equations (63) and (64)), the isomorphism

$$\text{NH}_n \cong \text{End}_{R_n^{S_n}}(R_n \langle -\delta \rangle),$$

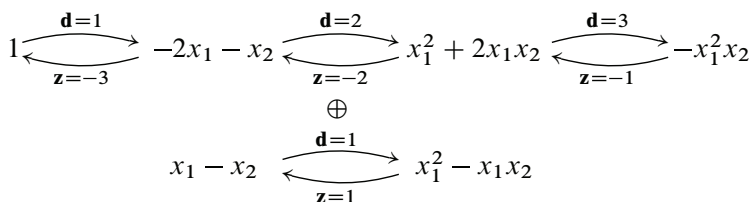
equips the nilHecke algebra with an \mathfrak{sl}_2 -action, which agrees with Definition 4.3.

We have already argued in Section 2.6 that the core of a tensor product contains the tensor product of the cores. Since endomorphism rings are particular kinds of tensor products (though one must be careful when taking tensor products of infinite-dimensional representations in this way), one deduces that the core of an endomorphism ring contains the endomorphism ring of the cores. Thus, $\text{Core}(\text{NH}_n)$ contains $\text{End}_{\mathbb{Q}}(\text{Core}(R_n \langle -\delta \rangle))$, which is $\text{End}_{\mathbb{Q}}(L_0 \otimes L_1 \otimes \cdots \otimes L_{n-1})$. ■

Example 4.11. Let $n = 3$. Here is a basis of $\text{Core}(R_3\langle -2x_1 - x_2 \rangle)$, making it clear the isomorphism with $L_2 \otimes L_1$:



Here is another basis making clear the isomorphism with $L_3 \oplus L_1$:



For those interested in integral structure, this second basis is not a \mathbb{Z} -basis, as the determinant of the change of basis matrix is 3. To obtain analogous bases of $R_3\langle -x_2 - 2x_3 \rangle$ just swap x_1 and x_3 .

Now, NH_3 is isomorphic to a 6×6 matrix algebra over $R_3^{S_3}$, and the matrix entries correspond to ψ_{w_0} with certain polynomials on top and on bottom, see e.g. [18, Proposition 3.3]. The polynomials on top and those on bottom must be dual bases for R_3 over $R_3^{S_3}$ with respect to the Demazure operator ∂_{w_0} . However, one can get a basis over $R_3^{S_3}$ (not necessarily a matrix basis) but choosing any two bases for R_3 over $R_3^{S_3}$, and placing them on top and bottom of ψ_{w_0} . Choosing your favorite bases for $\text{Core}(R_3\langle -2x_1 - x_2 \rangle)$ and $\text{Core}(R_3\langle -x_2 - 2x_3 \rangle)$ respectively for the top and bottom, the \mathfrak{sl}_2 structure is transparent.

5. \mathfrak{sl}_2 action on the Hecke category

5.1. Definition of the action

Definition 5.1. Let $\mathcal{H} = \mathcal{H}(S_n)$ denote the diagrammatic Hecke category associated to the action of S_n on $R_n = \mathbb{Z}[x_1, \dots, x_n]$. A presentation by generators and relations can be found in [6].

Definition 5.2. Equip \mathcal{H} with an \mathfrak{sl}_2 action as follows. The derivation \mathbf{d} is the derivation defined in [10, Theorem 2.5], where $g_i = x_i$. The derivation \mathbf{z} kills all diagrams

without polynomials, and sends x_i to 1. In the formulas below, blue represents the simple reflection $s_i = (i, i + 1)$. Red represents s_{i+1} , and green represents some color distant from blue:

$$d \left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \right) = x_i \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}, \quad d \left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \right) = x_{i+1} \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}, \quad (37.1)$$

$$d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}, \quad d \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}, \quad (37.2)$$

$$d \left(\begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \bullet \quad \bullet \\ \text{---} \end{array}, \quad d \left(\begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} \right) = - \begin{array}{c} \text{---} \\ \bullet \quad \bullet \\ \text{---} \end{array}, \quad (37.3)$$

$$d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 0, \quad d \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = 0, \quad (37.4)$$

$$d \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \bullet \\ \diagdown \quad \diagup \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \bullet \\ \text{---} \end{array}, \quad (37.5)$$

$$d \left(\begin{array}{c} \text{---} \\ x_i \\ \text{---} \end{array} \right) = x_i^2, \quad z \left(\begin{array}{c} \text{---} \\ x_i \\ \text{---} \end{array} \right) = \text{---}, \quad (37.6)$$

$$z \left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \right) = 0, \quad z \left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \right) = 0, \quad z \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 0, \quad z \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = 0, \quad (37.7)$$

$$z \left(\begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} \right) = 0, \quad z \left(\begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} \right) = 0, \quad (37.8)$$

$$z \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 0, \quad z \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = 0, \quad z \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = 0. \quad (37.9)$$

One extends the derivation to all diagrams using the monoidal Leibniz rule.

Theorem 5.3. *The action of \mathfrak{sl}_2 on \mathcal{H} given in Definition 5.2 is well defined. The divided power operators $\mathbf{d}^{(k)}$ and $\mathbf{z}^{(k)}$ are defined integrally for all $k \geq 0$, making \mathcal{H} into a divided powers \mathfrak{sl}_2 -algebra.*

Proof. As in the proof of Theorem 4.4, we will use the lemmas of Section 2.1 to reduce the amount of work we need to do. This time we do it tacitly.

First we need to check that the derivations \mathbf{d} and \mathbf{z} preserve the relations of \mathcal{H} . That \mathbf{d} preserves the relations was checked in [10, Theorems 2.1, 2.3]. Let us check that \mathbf{z} preserves the relations.

Any diagram without polynomials is sent to zero by \mathbf{z} . Any relation which is a linear combination of diagrams without polynomials is therefore preserved by \mathbf{z} . There is only one relation that remains, the polynomial forcing relation. Below, blue represents s_i , and j is arbitrary:

$$\left[\begin{array}{c} \text{---} \\ | \\ x_j \text{ --- } s_i(x_j) \\ | \\ \text{---} \end{array} \right] = \partial_i(x_j) \begin{array}{c} \text{---} \\ \bullet \\ \bullet \\ \text{---} \end{array}. \tag{38}$$

Note that $\partial_i(x_j)$ is a scalar. This relation is preserved by \mathbf{z} since $\mathbf{z}(x_i) = 1$ for all i , so \mathbf{z} kills the left-hand side, and the right-hand side is a diagram without polynomials so is also killed by \mathbf{z} . Hence, \mathbf{z} is a well-defined derivation.

Now, we wish to check that $(\mathbf{d}, \mathbf{h}, -\mathbf{z})$ is an \mathfrak{sl}_2 triple, and also that divided powers are defined integrally. We can check these properties on the generators. Since \mathbf{d} and \mathbf{z} raise or lower the degree appropriately, we need only check that $[\mathbf{z}, \mathbf{d}] = \mathbf{h}$. For each of the non-polynomial generators ϕ , $\mathbf{z}(\phi) = 0$, and it is very easy to confirm that

$$\mathbf{z}(\mathbf{d}(\phi)) = (\deg \phi) \cdot \phi = \mathbf{h}(\phi).$$

Meanwhile, for $\text{End}(\mathbb{1}) = R_n$, the action of \mathfrak{sl}_2 is the standard one. This confirms that the \mathfrak{sl}_2 action is well defined. That $\mathbf{d}^{(k)}$ is defined integrally was checked in [10, Section 8]. That $\mathbf{z}^{(k)}$ is defined integrally on the polynomial ring $\text{End}(\mathbb{1})$ was checked in Lemma 2.27. That $\mathbf{z}^{(k)}$ is defined integrally on the other generators is easy, since it is zero for $k \geq 1$. ■

5.2. Reminders: Rex moves and lower terms

Definition 5.4. For $w \in S_n$, let $I_{<w}$ denote the two-sided ideal spanned by all morphisms which factor through reduced expressions for elements $v \in S_n$ with $v < w$.

Given two reduced expressions $\underline{w}, \underline{w}'$ for the same element $w \in S_n$, Matsumoto’s theorem states that they can be connected by a sequence of braid relations. To such a sequence of braid relations there is a corresponding diagram built from 4-valent and 6-valent vertices, having source \underline{w} and target \underline{w}' , which we call a *rex move*. There are many potential sequences of braid relations which go from \underline{w} to \underline{w}' , and the corresponding rex moves in \mathcal{H} are not equal.

Lemma 5.5. Let $\underline{w}, \underline{w}'$ be reduced expressions for $w \in S_n$. For any two rex moves $\underline{w} \rightarrow \underline{w}'$, their difference lies in $I_{<w}$.

Proof. This is proven in [12, Lemma 7.4]. ■

Rex moves are always in the kernel of \mathbf{z} , as any diagram is, but are not always in the kernel of \mathbf{d} , see (37.5). Two different rex moves with the same source and target can have different values of \mathbf{d} . For example, let $s = s_i$ and $t = s_{i+1}$, and consider the reduced expression (s, t, s) . The identity map of this reduced expression is a rex move which is killed by \mathbf{d} . Meanwhile, the path $(s, t, s) \rightarrow (t, s, t) \rightarrow (s, t, s)$ gives the doubled 6-valent vertex, which is not in the kernel of \mathbf{d} . The reader versed with this diagrammatic calculus should have no trouble verifying that

$$\mathbf{d} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) = - \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = - \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right). \tag{39}$$

Note at least that \mathbf{d} sends the doubled 6-valent vertex to the ideal $I_{<sts}$ of lower terms.

Remark 5.6. The right-hand side of (39) is a left R_n -linear combination of double leaves, and double leaves form a basis for morphisms as a left (or right) R_n -module. Note the non-trivial root α_t which appears. This gives an example of the kind of behavior discussed in Remark 1.28.

5.3. Reminders: Subexpressions and light leaves

Let $\underline{x} = (s_{x_1}, \dots, s_{x_d})$ be an expression of length d , and $\mathbf{e} \subset \underline{x}$ be a subexpression. To \mathbf{e} we associate a Bruhat stroll $(1 = w_0, w_1, \dots, w_d)$ as in [12, Section 2.4], where to get from w_{i-1} to w_i we multiply by either s_{x_i} or by 1 depending on \mathbf{e} . Note that the Bruhat stroll determines the subexpression, and vice versa. We refer to w_d as the *terminus* of \mathbf{e} . We let $E(\underline{x}, w)$ denote the set of subexpressions $\mathbf{e} \subset \underline{x}$ with terminus w . Following ideas of Libedinsky [24], to each $\mathbf{e} \in E(\underline{x}, w)$ we associate in [12, Construction 6.1] a morphism $LL_{\mathbf{e}}: \underline{x} \rightarrow \underline{w}$ called a *light leaf*, whose target is some reduced expression for w (depending on \mathbf{e}). Flipping this light leaf upside-down, we get a morphism $\Gamma\Gamma_{\mathbf{e}}: \underline{w} \rightarrow \underline{x}$.

Let us remark on some important features of the light leaf construction. If \underline{v} is a reduced expression for some $v \in S_n$, then it has a unique subexpression with terminus v , the *top subexpression*. Any rex move starting at v , including the identity map, is a valid light leaf for the top subexpression, and all light leaves for the top subexpression are rex moves. Let us note that any other subexpression of \underline{v} has terminus $v' < v$, so its light leaf lies in the ideal $I_{<v}$.

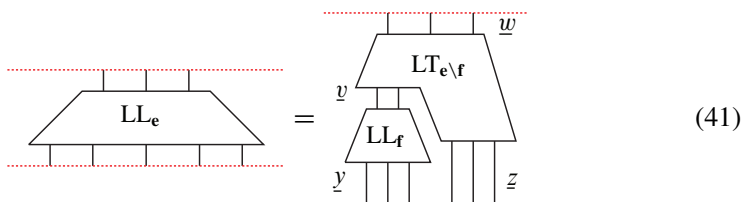
If $\underline{x} = \underline{v}\underline{z}$ is a concatenation of two smaller sequences, we can restrict a subsequence $\mathbf{e} \subset \underline{x}$ to a subsequence $\mathbf{f} \subset \underline{v}$. Suppose that \underline{v} is a reduced expression for some $v \in S_n$, and \mathbf{e} restricts to the top subsequence $\mathbf{f} \subset \underline{v}$. Then we refer to $LL_{\mathbf{e}}$ as a *light*

tail, and since it is determined by the restriction of \mathbf{e} to \underline{z} , we may use the notation $LT_{\mathbf{e} \setminus \mathbf{f}}$.

Finally, suppose that $\underline{x} = \underline{y}\underline{z}$ is a concatenation of two smaller sequences, and $\mathbf{e} \in E(\underline{x}, w)$ restricts to $\mathbf{f} \in E(\underline{y}, v)$. Then $LL_{\mathbf{f}}$ is a map from \underline{y} to \underline{v} for some reduced expression for v . Meanwhile, there is a light tail $\underline{v}\underline{z} \rightarrow \underline{w}$ determined by the subexpression $\mathbf{e} \setminus \mathbf{f}$ of \underline{z} . The inductive construction of light leaves states that

$$LL_{\mathbf{e}} = LT_{\mathbf{e} \setminus \mathbf{f}} \circ (LL_{\mathbf{f}} \otimes \text{id}_{\underline{z}}). \tag{40}$$

Schematically, we draw



When \underline{z} has length 1, we think of $LT_{\mathbf{e} \setminus \mathbf{f}}$ as being a single *tier* of the light leaf algorithm. When \underline{z} has length d , the light tail is built inductively from the last d tiers. This is discussed in [12, Remark 6.4].

Because light tails will be important in some proofs below, let us introduce some terminology.

Definition 5.7. Let $w \in S_n$. Relative to w , we call a sequence of simple reflections $\underline{z} = (s_{z_1}, \dots, s_{z_m})$ a *tail expression*, and a sequence $\mathbf{e} \subset \underline{z}$ of 0s and 1s a *tail subexpression*. A tail (sub)expression is the same thing as a (sub)expression, but we interpret its Bruhat stroll differently, and we decorate the subsequence with U s and D s accordingly. There is a unique $v \in S_n$ such that $vs_{z_1}^{\mathbf{e}_1} s_{z_2}^{\mathbf{e}_2} \dots s_{z_m}^{\mathbf{e}_m} = w$. The *tail Bruhat stroll* associated to w and $\mathbf{e} \subset \underline{z}$ is the Bruhat stroll which starts at v rather than 1, and wends its way to w . We call v the *start* of the tail Bruhat stroll.

Fix $v, w \in S_n$ and a reduced expression \underline{v} for v . For a tail expression \underline{z} , there is a bijection between ordinary subexpressions $\mathbf{e} \subset \underline{v}\underline{z}$ whose restriction to \underline{v} is the top subsequence $\mathbf{f} \subset \underline{v}$, and tail subexpressions $(\mathbf{e} \setminus \mathbf{f}) \subset \underline{z}$ with start v and terminus w . This bijection is natural over the choice of reduced expression for \underline{v} , in the obvious sense. Henceforth (and unlike the previous paragraph), we always use $(\mathbf{e} \setminus \mathbf{f}) \subset \underline{z}$ as notation for a tail subexpression, even though \mathbf{e} and \mathbf{f} themselves need not have been chosen. Whenever we choose another sequence \underline{y} and consider the concatenation $\underline{y}\underline{z}$, and whenever $\mathbf{f} \in E(\underline{y}, v)$, we will then set \mathbf{e} to be the subexpression of $\underline{y}\underline{z}$ whose restriction to \underline{y} is \mathbf{f} and whose restriction to \underline{z} is $\mathbf{e} \setminus \mathbf{f}$.

5.4. Reminders: Double leaves

For two sequences \underline{x} and \underline{x}' , subexpressions $\mathbf{e} \subset \underline{x}$ and $\mathbf{f} \subset \underline{x}'$ are called *coterminal* if they have the same terminus w , and we refer to $(\mathbf{e}, \mathbf{f}, w)$ as a *coterminal triple* subordinate to $(\underline{x}, \underline{x}')$. Sometimes we omit w from the triple, writing only (\mathbf{e}, \mathbf{f}) . To each coterminal triple we associate a *double leaf* morphism in $\text{Hom}(\underline{x}, \underline{x}')$,

$$DL_{\mathbf{e}, \mathbf{f}} := \Gamma \Gamma_{\mathbf{f}} \circ N \circ LL_{\mathbf{e}}, \tag{42}$$

where N is some rex move from the target of $LL_{\mathbf{e}}$ to the source of $\Gamma \Gamma_{\mathbf{f}}$, both being reduced expressions for w . By taking one double leaf for each coterminal triple, one obtains a basis for $\text{Hom}(\underline{x}, \underline{x}')$ as a left (or right) R_n -module. Double leaves are always diagrams without polynomials, so they are killed by \mathbf{z} .

Light leaves and double leaves are not determined only by the subexpressions \mathbf{e} and \mathbf{f} . There are many choices of rex moves in the construction of each light leaf, as well as the choice of rex move N in (42). In particular, the composition $N \circ LL_{\mathbf{e}}$ of a rex move with a light leaf is, itself, another valid choice of a light leaf associated to \mathbf{e} . In this way, we could remove the rex move N from (42), absorbing it into the light leaf $LL_{\mathbf{e}}$ (or into the upside-down light leaf $\Gamma \Gamma_{\mathbf{f}}$).

When one speaks about the double leaves *basis*, one must choose one amongst the many possible double leaves for each coterminal triple to be a basis element. There are many different double leaves bases. When we speak of light leaves or double leaves, we typically refer to the set of all possible maps produced by the algorithm, with the flexibility of using arbitrary rex moves. When we speak of a *distinguished* light leaf or double leaf, we must have fixed one for each subexpression, and we refer to that one. In this way we can separate in our language between the rigid choices one must make to get a basis, and the flexible choices which are sufficient for a spanning set.

There is a filtration by the spans of certain double leaves with regard to a certain partial order, for which the image of a double leaf in the associated graded does not depend on the choice of rex moves! This result will be proven in the next section. Thankfully, \mathbf{d} will preserve this (downfree) filtration, and one can easily compute its action on the associated graded. One cannot expect much more from the combinatorics of subexpressions: since \mathbf{d} acts non-trivially on rex moves, one should not expect a formula for the \mathbf{d} action on a double leaf which is independent of the choice of rex move.

5.5. The lexicoBruhat order

In this section we develop some technology for working with light leaves and double leaves. This technology is not original to this paper: it is part of work in progress [5]

by the first author, and much was known (but not written in the literature) previously to the experts.⁹

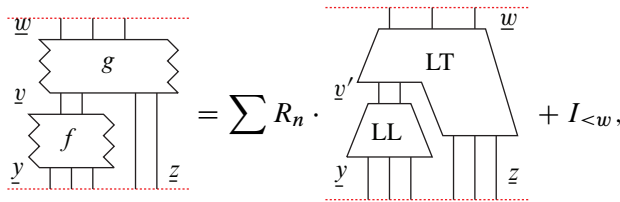
To study double leaves it helps to first study light leaves. From the fact that distinguished double leaves form a basis for $\text{Hom}(\underline{x}, \underline{w})$, we deduce¹⁰ that distinguished in $E(\underline{x}, w)$ form a basis for $\text{Hom}(\underline{x}, \underline{w})/I_{<w}$. Until further notice, we will be studying this Hom space, modulo lower terms. In the following proposition we discuss not the span of particular light leaves, but the span of all light leaves which are constructible by the non-deterministic algorithm of [12, Construction 6.1]. In other words, in this proposition we are agnostic to the choice of rex moves.

Proposition 5.8. *Let \underline{w} be a reduced expression for some $w \in S_n$. Let $\underline{x} = \underline{y}\underline{z}$ be the concatenation of a sequence \underline{y} of length k and a sequence \underline{z} of length $d - k$. Each subexpression of \underline{x} restricts to a subexpression of \underline{y} . For $v \in S_n$, let $X_{\leq v} \subset E(\underline{x}, w)$ denote the subset which restricts to $E(\underline{y}, v')$ for some $v' \leq v$. Let $H_{\leq v}$ denote the subspace of $\text{Hom}(\underline{x}, \underline{w})$ spanned (over the left action of R_n) by all possible light leaves corresponding to subexpressions in $X_{\leq v}$.*

Let $f: \underline{y} \rightarrow \underline{v}$ be any morphism whose target is a reduced expression \underline{v} for $v \in S_n$. Then

$$g \circ (f \otimes \text{id}_{\underline{z}}) \in H_{\leq v} \pmod{I_{<w}} \tag{43}$$

for all $g \in \text{Hom}(\underline{v}\underline{z}, \underline{w})$. Schematically, we have



where $v' \leq v$.

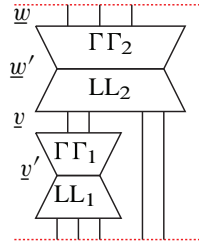
Proof. We suppose the result is true for all elements less than v in the Bruhat order, and deduce the result for v . Then, by induction, the result will be true for any $v \in S_n$

⁹The technology we present is closely related to ideas developed by Elias and Williamson in their early attempts to prove that double leaves span. The eventual proof in [12] used a different inductive proof of spanning. In the proof that double leaves are linearly independent, [12] used the path dominance order on triples, which is a stronger partial order than the lexicoBruhat order.

¹⁰This was actually proven first, and used to deduce that distinguished double leaves form a basis, see [12, Proposition 7.6].

(without the need to check any base case). Note that everything else (i.e. \underline{y} , \underline{z} , \underline{w} , k) is unchanging in this induction.

We know that $f \in \text{Hom}(\underline{y}, v)$ and $g \in \text{Hom}(\underline{vz}, \underline{w})$ are both in the left R_n -span of double leaves. So, up to taking left linear combinations over R_n , our morphism has the following form:



The source \underline{w}' of $\Gamma\Gamma_2$ is a reduced expression for some element $w' \leq w$. Thus, either $\Gamma\Gamma_2 \in I_{<w}$, or $\Gamma\Gamma_2$ is a rex move, and can be absorbed into LL_2 . We assume henceforth that $\Gamma\Gamma_2$ is the identity map, and $\underline{w}' = \underline{w}$.

The source \underline{v}' of $\Gamma\Gamma_1$ is a reduced expression for some element $v' \leq v$. If $v' < v$, then we can replace g with $LL_2 \circ (\Gamma\Gamma_1 \otimes \text{id}_{\underline{z}})$ and f with LL_1 , and we have a diagram of the form (43) but for v' instead of v . By induction, this lives in $H_{\leq v'}$ modulo $I_{<w}$. So, we reduce to the case when $v' = v$, in which case $\Gamma\Gamma_1$ is a rex move, and can be absorbed into LL_1 . Thus, we have reduced to the following case:

(44)

Now, LL_2 is associated to some subexpression $\mathbf{e} \subset \underline{vz}$, which restricts to some subexpression $\mathbf{f} \subset \underline{v}$. If \mathbf{f} is the top subexpression (the unique subexpression with terminus v) then (44) is itself a light leaf by (41). If \mathbf{f} is any other expression, then $LL_{\mathbf{f}}$ sends \underline{v} to a reduced expression for an element $v' < v$. Now, we again can refactor our diagram, letting $f = LL_{\mathbf{f}} \circ LL_1$, and letting g denote $LT_{\mathbf{e} \setminus \mathbf{f}}$, and can use the inductive hypothesis for v' . ■

Our main application of Proposition 5.8 will be to a *mistaken light leaf*, or a *light leaf with error*. Suppose that $x = \underline{y}\underline{z}$ is a concatenation, and $\mathbf{e} \subset x$. In order to construct $LL_{\mathbf{e}}$ as in (40), we have already constructed the light leaf $LL_{\mathbf{f}}$ associated to the restriction of \mathbf{e} to \underline{y} , and the light tail $LT_{\mathbf{e} \setminus \mathbf{f}}$ associated to the restriction of \mathbf{e} to \underline{z} . However, instead of gluing these together (along the reduced expression \underline{v}) as in (40),

we make an error and insert some morphism $E \in I_{<v}$ as in the following picture:



The result is a mistaken light leaf; if \underline{y} has length k , we say the mistake happened in the k -th tier of the light leaf, and we denote the mistaken light leaf by $LL_e^{oops,k}$.

Lemma 5.9. *With the same setup as for (45), the mistaken light leaf lies in the subspace $H_{<v}$ of $\text{Hom}(\underline{x}, \underline{w})/I_{<w}$.*

Proof. This is immediate from Proposition 5.8. ■

This lemma implies that $LL_e^{oops,k}$ is in the span of light leaves $LL_{e'}$ which, at the k -th step, factor through elements $< v$ rather than through v . However, we have not yet proven that the Bruhat stroll of e' does not go much higher than e ever went. Below we will get more control on mistaken light leaves, asserting that e' is less than e in some partial order on subexpressions. To get this additional control, we cannot merely apply Proposition 5.8, but must produce a more subtle version. The reader should think of Proposition 5.8 as a warm-up exercise; the same ideas factor into the proof of Theorem 5.13.

Definition 5.10. Let $\underline{x} = (s_{x_1}, \dots, s_{x_d})$ be an expression of length d , and let e and e' be two subexpressions of \underline{x} . Let $(1 = w_0, w_1, \dots, w_d)$ and $(1 = w'_0, \dots, w'_d)$ be their associated Bruhat strolls, see [12, Section 2.4]. If $e \neq e'$, then let $1 \leq k \leq d$ be the index such that $w_k \neq w'_k$ and $w_j = w'_j$ for all $j > k$; we call k the *index of last difference*. We say that $e < e'$ if $w_k < w'_k$ in the Bruhat order. We write $e \leq e'$ if either $e = e'$ or $e < e'$. We call this the *lexicoBruhat order on subexpressions*.

Lemma 5.11. *The lexicoBruhat order is a total order on the set of subexpressions of \underline{x} .*

Proof. Clearly, this relation is transitive, and $e \leq e' \leq e$ implies $e = e'$. Suppose that $e \neq e'$, and let k be the index of last difference. Then $w_k = w'_k s_{x_k}$, so w_k and w'_k are comparable in the Bruhat order. Hence, e and e' are comparable in the lexicoBruhat order. ■

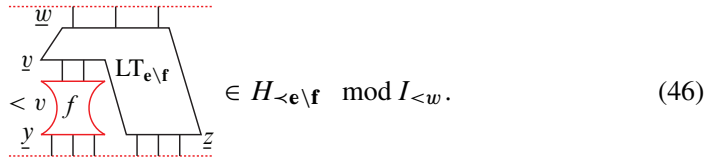
Remark 5.12. Fix w , and consider a tail expression \underline{z} . Clearly, the lexicoBruhat order on subexpressions can be extended to a total order on tail subexpressions of \underline{z} relative to w , in the obvious way.

Theorem 5.13. Fix $w \in S_n$ with a rex \underline{w} , and let \underline{z} be a tail expression of length m relative to w . Let $\mathbf{e} \setminus \mathbf{f} \subset \underline{z}$ be a tail subexpression with start v ; choose a rex \underline{v} for v , and a light tail $LT_{\mathbf{e} \setminus \mathbf{f}}: \underline{v}\underline{z} \rightarrow \underline{w}$. Let $f: \underline{y} \rightarrow \underline{v}$ be any morphism to \underline{v} which lives in $I_{<v}$. Then

$$LT_{\mathbf{e} \setminus \mathbf{f}} \circ (f \otimes id_{\underline{z}}) \in H_{<\mathbf{e} \setminus \mathbf{f}} \text{ mod } I_{<w},$$

where $H_{<\mathbf{e} \setminus \mathbf{f}}$ is the left R_n -span of light leaves $LL_{\mathbf{e}'}$: $\underline{y}\underline{z} \rightarrow \underline{w}$ whose tail subexpressions $\mathbf{e}' \setminus \mathbf{f}' \subset \underline{z}$ satisfy $(\mathbf{e}' \setminus \mathbf{f}') < (\mathbf{e} \setminus \mathbf{f})$ in the lexicoBruhat order.

Pictorially, we have



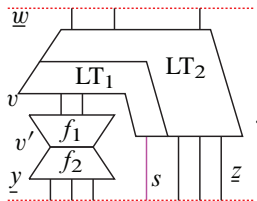
Before giving the proof, here is the immediate corollary.

Corollary 5.14. Fix $\underline{x} = \underline{y}\underline{z}$, where \underline{y} has length k . Let $\mathbf{e} \in E(\underline{x}, w)$, and let \mathbf{f} be the restriction of \mathbf{e} to \underline{y} . Let $LL_{\mathbf{e}}^{\text{oops},k}$ be any mistaken light leaf where the mistake happens in the k -th tier of the light leaf. Then

$$LL_{\mathbf{e}}^{\text{oops},k} \in H_{<\mathbf{e} \setminus \mathbf{f}} \text{ mod } I_{<w}.$$

Proof of Theorem 5.13. We prove this by induction on the length m of \underline{z} , and within each m , by induction on v . If $m = 0$ then $f \in I_{<w}$ and the result is trivial. If $m = 1$ then the result actually follows easily Proposition 5.8, but we will use essentially the same argument below for the inductive step.

Suppose that the result holds for \underline{z} , and let us extend \underline{z} by prepending a simple reflection s . Suppose we have $(\mathbf{e} \subset \mathbf{f}) \subset s\underline{z}$, with start v . Let $\mathbf{e} \setminus \mathbf{f}'$ be its restriction to \underline{z} , which has start x . The associated light tail has a first tier LT_1 associated to s , and the remaining light tail LT_2 associated to \underline{z} . Write $f = f_1 \circ f_2$ where the source of f_1 is a reduced expression for some element $v' < v$. We are analyzing the composition



Now, apply Proposition 5.8 to the subdiagram which is $LT_1 \circ (f_1 \otimes \text{id}_s)$. We get

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} \quad (47)$$

The right-hand side is supposed to schematically represent a linear combination (over the left action of R_n) of morphisms of two kinds. In the second diagram on the right-hand side of (47), the curvy red morphism is supposed to represent the “lower terms” of Proposition 5.8. Note that f_2 can be absorbed into the lower terms, and the result exactly has the form (46) but for \underline{z} rather than $s\underline{z}$. By induction, this morphism is in the span of $H_{<e \setminus f}$, which is a subspace of $H_{<e \setminus f}$, modulo $I_{<w}$.

In the first diagram in (47), v'' is some element $< v$, but since it is the target of a light leaf whose source is a rex for v' , we must also have $v'' \leq v' < v$. Meanwhile, the composition of the two light tails $LT_2 \circ (LT_1' \otimes \text{id}_z)$ is a light tail whose associated tail subexpression agrees with $e \setminus f$ on \underline{z} , but disagrees on s , going to a lower term. Let us resolve $LL \circ f_2$, noting that light leaves span the maps to v'' modulo lower terms.

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} \quad (48)$$

The first diagram on the right-hand side of (48) is a light leaf and lives in $H_{<e \setminus f}$. The second diagram is in $H_{<e \setminus f}$ by induction on v , since $v'' < v$. This proves (46) for $s\underline{z}$, completing the inductive step. ■

Corollary 5.15. Any two light leaves associated to the same subexpression $e \in E(\underline{x}, \underline{w})$ are equal in the space $\text{Hom}(\underline{x}, \underline{w})/I_{<w}$, modulo the left R_n -span of light leaves for smaller subexpressions in the lexicoBruhat order.

Proof. Let LL and LL' be the two light leaves associated to e . The only difference between LL and LL' is the choice of rex moves at various tiers in the light leaves algorithm. Let \underline{x} have length m . We can write $LL - LL'$ as a telescoping sum

$$LL - LL' = LL_0 - LL_1 + LL_1 - LL_2 + \cdots + LL_{m-1} - LL_m,$$

where $LL_0 = LL$, $LL' = LL_m$, each LL_k is a light leaf for \mathbf{e} , and LL_{k-1} differs from LL_k in the choice of rex move made at the k -th tier of the algorithm. For example, LL_{m-1} uses the first $m - 1$ tiers of LL' , but uses the last tier from LL .

The difference between two rex moves (between two reduced expressions for some $v \in S_n$) consists of lower terms (i.e. lives in the ideal $I_{<v}$). Thus, each difference $LL_k - LL_{k-1}$ is a mistaken light leaf, where the mistake happened in the k -th tier. By Corollary 5.13, $LL_k - LL_{k-1}$ is in the span of light leaves $LL_{\mathbf{e}'}$ with $\mathbf{e}' < \mathbf{e}$. ■

Remark 5.16. This corollary gives an alternate route to proving that distinguished light leaves span the space $\text{Hom}(\underline{x}, \underline{w})/I_{<w}$. Once one proves that light leaves span, this corollary proves that a single distinguished light leaf $LL_{\mathbf{e}}$ will have the same span as all light leaves associated to \mathbf{e} , modulo $I_{<w}$ and modulo light leaves for $\mathbf{e}' < \mathbf{e}$. By induction on the lexicoBruhat order, one deduces that distinguished light leaves have the same span as all light leaves modulo $I_{<w}$.

Let us quickly remark on the difference between the left R_n -span and the right R_n -span. Modulo lower terms, there is none! Thus, all the results above also apply to the right R_n -span.

Lemma 5.17. *The left R_n -span of any set of morphisms in $\text{Hom}(\underline{x}, \underline{w})$ agrees with the right R_n -span modulo $I_{<w}$.*

Proof. Any polynomial p can be forced through the reduced expression for w at the top of the diagram, using (38). The result is $w^{-1}(p)$ on the right-hand side, plus error terms where strands are broken. These error terms are all in $I_{<w}$. ■

Now, we bootstrap these results about light leaves to results about double leaves.

Definition 5.18. Let \underline{x} and \underline{y} be expressions. Suppose that $(\mathbf{e}, \mathbf{f}, w)$ and $(\mathbf{e}', \mathbf{f}', w')$ are two coterminal triples subordinate to $(\underline{x}, \underline{y})$. We say that $(\mathbf{e}, \mathbf{f}, w) \preceq (\mathbf{e}', \mathbf{f}', w')$ if $\mathbf{e} \preceq \mathbf{e}'$ and $\mathbf{f} \preceq \mathbf{f}'$. We call this the *lexicoBruhat order on triples*.

Suppose $w < w'$. If \underline{x} has length m then the index of last difference between \mathbf{e} and \mathbf{e}' is m , and $w < w'$, so $\mathbf{e} < \mathbf{e}'$. Similarly, $\mathbf{f} < \mathbf{f}'$. Thus $(\mathbf{e}, \mathbf{f}, w) < (\mathbf{e}', \mathbf{f}', w')$ whenever $w < w'$. In particular, the span of all double leaves associated to triples less than $(\mathbf{e}, \mathbf{f}, w)$ in the lexicoBruhat order will have $I_{<w}$ as a subspace.

The definition of a mistaken double leaf $DL_{\mathbf{e}, \mathbf{f}}^{\text{oops}}$ is similar to that of a mistaken light leaf. At one tier in either $LL_{\mathbf{e}}$ or $\Gamma\Gamma_{\mathbf{f}}$, one inserts an error term which goes lower in the Bruhat order than it should.

Theorem 5.19. *Any mistaken double leaf $DL_{\mathbf{e}, \mathbf{f}}^{\text{oops}}$ is in the left R_n span of double leaves associated to triples $(\mathbf{e}', \mathbf{f}', w')$ which are smaller than $(\mathbf{e}, \mathbf{f}, w)$ in the lexicoBruhat order.*

Proof. The error in $DL_{\mathbf{e},\mathbf{f}}^{\text{oops}}$ is made either in $LL_{\mathbf{e}}$ or in $\Gamma\Gamma_{\mathbf{f}}$, and the arguments are the same either way, so let us assume the error is made in $LL_{\mathbf{e}}$. Thus,

$$DL_{\mathbf{e},\mathbf{f}}^{\text{oops}} = \Gamma\Gamma_{\mathbf{f}} \circ LL_{\mathbf{e}}^{\text{oops}}.$$

By Corollary 5.13,

$$LL_{\mathbf{e}}^{\text{oops}} \in \left(\sum_{\mathbf{e}' < \mathbf{e}} R_n \cdot LL_{\mathbf{e}'} \right) + I_{<w}. \tag{49}$$

Since $I_{<w}$ is a two-sided ideal, $\Gamma\Gamma_{\mathbf{f}} \circ I_{<w} \subset I_{<w}$. Also, $I_{<w}$ lies in the span of smaller double leaves (as discussed a few paragraphs ago). Meanwhile,

$$\Gamma\Gamma_{\mathbf{f}} \circ LL_{\mathbf{e}'} = DL_{\mathbf{f},\mathbf{e}'},$$

and $(\mathbf{f}, \mathbf{e}', w) < (\mathbf{f}, \mathbf{e}, w)$ when $\mathbf{e}' < \mathbf{e}$. Thus, every term in the sum (49), when composed with $\Gamma\Gamma_{\mathbf{f}}$, is in the span of lower double leaves in the lexicoBruhat order on triples. ■

Corollary 5.20. *Any two double leaves associated to the same triple $(\mathbf{e}, \mathbf{f}, w)$ are equal in $\text{Hom}(x, y)$ modulo the span of double leaves for smaller triples in the lexicoBruhat order.*

Proof. The proof is the same as Corollary 5.15. ■

Remark 5.21. Once one proves that double leaves span, one deduces from Corollary 5.20 that distinguished double leaves have the same span. This gives an alternative proof of some of the results from [12, Chapter 7].

To obtain an analogue of Theorem 5.19 for the right R_n -action, one can either use Lemma 5.17 and modify the proof of the theorem accordingly, or one can use the following lemma.

Lemma 5.22. *For any coterminal triple $(\mathbf{e}, \mathbf{f}, w)$, the left and right R_n spans of $\{DL_{\mathbf{e}',\mathbf{f}'}\}_{(\mathbf{e}',\mathbf{f}') < (\mathbf{e},\mathbf{f})}$ agree.*

Proof. Consider any coterminal triple $(\mathbf{e}', \mathbf{f}', w') < (\mathbf{e}, \mathbf{f}, w)$. Clearly, $w' \leq w$. Using (38) to push a polynomial p across the reduced expression for w' in the middle of the double leaf, we get

$$DL_{\mathbf{e}',\mathbf{f}'} \cdot p - w'(p) \cdot DL_{\mathbf{e}',\mathbf{f}'} \in I_{<w'}. \tag{50}$$

However, $I_{<w'} \subset I_{<w}$ is already in the (right or left) span of lower double leaves. ■

5.6. Double leaves and the raising operator

The key result of this section will be that $\mathbf{d}(\text{DL}_{\mathbf{e},\mathbf{f}})$ is equal to $p_{\text{DL}}(\mathbf{e}, \mathbf{f}) \cdot \text{DL}_{\mathbf{e},\mathbf{f}}$ for some polynomial $p(\mathbf{e}, \mathbf{f})$, plus a linear combination of mistaken double leaves. By Theorem 5.19, this means that \mathbf{d} preserves the span of double leaves (or distinguished double leaves) associated to triples $\leq (\mathbf{e}, \mathbf{f}, w)$.

Definition 5.23. Let $\underline{y} = (s_{y_1}, \dots, s_{y_d})$ have length d , and let $\mathbf{e} \subset \underline{y}$ be a subexpression, with associated Bruhat stroll $(1 = w_0, w_1, \dots, w_d)$. Recall from [12, Section 2.4] that each index of \mathbf{e} can be decorated as either $U0, U1, D0$, or $D1$. Define linear polynomials

$$p_{\text{LL}}(\mathbf{e}) = \sum_{\mathbf{e}_k=U0} w_k(x_{y_i}) - \sum_{\mathbf{e}_k=D0} w_k(x_{y_k+1}),$$

$$p_{\Gamma\Gamma}(\mathbf{e}) = \sum_{\mathbf{e}_k=U0} w_k(x_{y_k+1}) - \sum_{\mathbf{e}_k=D0} w_k(x_{y_k}).$$

For a coterminal triple $(\mathbf{e}, \mathbf{f}, w)$, let

$$p_{\text{DL}}(\mathbf{e}, \mathbf{f}) = p_{\text{LL}}(\mathbf{e}) + p_{\Gamma\Gamma}(\mathbf{f}).$$

Proposition 5.24. Let $(\mathbf{e}, \mathbf{f}, w)$ be a coterminal triple subordinate to $(\underline{x}, \underline{y})$, and let $\text{DL}_{\mathbf{e},\mathbf{f}}$ be any double leaf for this triple. Then

$$\mathbf{z}(\text{DL}_{\mathbf{e},\mathbf{f}}) = 0, \quad \mathbf{d}(\text{DL}_{\mathbf{e},\mathbf{f}}) = p_{\text{DL}}(\mathbf{e}, \mathbf{f}) \cdot \text{DL}_{\mathbf{e},\mathbf{f}} + E,$$

where E is a linear combination of mistaken double leaves for $(\mathbf{e}, \mathbf{f}, w)$. Working instead with right R_n -modules we have

$$\mathbf{z}(\text{DL}_{\mathbf{e},\mathbf{f}}) = 0, \quad \mathbf{d}(\text{DL}_{\mathbf{e},\mathbf{f}}) = \text{DL}_{\mathbf{e},\mathbf{f}} \cdot w^{-1}(p_{\text{DL}}(\mathbf{e}, \mathbf{f})) + E.$$

Proof. Clearly, \mathbf{z} kills any double leaf, because it is a diagram without polynomials. Let us apply \mathbf{d} to a double leaf. This is a linear combination of terms where we have applied \mathbf{d} to each generator in the double leaf, and we analyze each term individually. We prove the result for the left R_n action, as the result for the right action follows by the proof of Lemma 5.22, see (50).

When we apply \mathbf{d} to a $2m$ -valent vertex in a rex move, it produces either zero (37.4), or a rex move with broken strands (37.5). A rex move with broken strands, in the context of the larger diagram, yields a mistaken double leaf.

Whenever $U0$ appears in \mathbf{e} , there is a dot colored s_i in $\text{LL}_{\mathbf{e}}$ for some $1 \leq i \leq n - 1$. When we apply \mathbf{d} to this dot, it gets multiplied by x_i , see (37.1). If $U0$ appears at the k -th step, then this polynomial x_i can then be forced to the left through the reduced expression for w_k , using (38). The result will be $w_k(x_i)$ on the left, plus terms with broken strands. These broken strands yield mistaken double leaves.

Similarly, whenever $U0$ appears in \mathbf{f} , there is a dot colored s_i in $\Gamma\Gamma_{\mathbf{f}}$. This time \mathbf{d} multiplies the dot by x_{i+1} , see (37.1). The rest of the argument is the same.

Whenever $D0$ appears in \mathbf{e} , there is a trivalent vertex colored s_i in $LL_{\mathbf{e}}$. The raising operator \mathbf{d} places $-x_{i+1}$ below this trivalent vertex, see (37.2). This can then be pulled left through a reduced expression for w_k , yielding $-w_k(x_{i+1})$ on the left, together with broken strands. Whenever $D0$ appears in \mathbf{f} , the argument is the same except that \mathbf{d} places $-x_i$ above the trivalent vertex in $\Gamma\Gamma_{\mathbf{f}}$ instead. Together, these contributions from $U0$ and $D0$ add up to $p_{DL}(\mathbf{e}, \mathbf{f})$.

Whenever $U1$ appears in \mathbf{e} or \mathbf{f} , the corresponding part of the light leaf is the identity map, which is killed by \mathbf{d} .

Whenever $D1$ appears in \mathbf{e} or \mathbf{f} (at the k -th step), the corresponding part of the light leaf is a cap or cup colored s_i . Caps and cups are sent to broken caps and cups by \mathbf{d} , see (37.3). Equivalently, one can break the s_i -colored line at the end of the reduced expression for w_{k-1} , and then apply a cap or cup. Thus, the result is a mistaken double leaf. ■

Theorem 5.25. *Pick a distinguished double leaf for each coterminal triple subordinate to $(\underline{x}, \underline{y})$. For any such triple $(\mathbf{e}, \mathbf{f}, w)$ we have*

$$\mathbf{z}(DL_{\mathbf{e},\mathbf{f}}) = 0, \quad \mathbf{d}(DL_{\mathbf{e},\mathbf{f}}) \in \text{Span}\{DL_{\mathbf{e}',\mathbf{f}'}\}_{(\mathbf{e}',\mathbf{f}') \leq (\mathbf{e},\mathbf{f})}.$$

Note that, by Lemma 5.22, this span does not depend on whether we work with left or right R_n -modules. Thus, distinguished double leaves induce a downfree filtration of $\text{Hom}(\underline{x}, \underline{y})$ as a left or right R_n -module, parametrized by coterminal triples with the lexicoBruhat order. In the associated graded, the left span of $DL_{\mathbf{e},\mathbf{f}}$ is a copy of $R_n\langle p_{DL}(\mathbf{e}, \mathbf{f}) \rangle$, see Definition 5.23. Working instead with right modules, the span of $DL_{\mathbf{e},\mathbf{f}}$ is a copy of $R_n\langle w^{-1}(p_{DL}(\mathbf{e}, \mathbf{f})) \rangle$.

Proof. This follows immediately from Proposition 5.24 and Theorem 5.19. ■

Remark 5.26. Replacing \mathbf{d} with $\bar{\mathbf{d}}$ will effectively swap the roles of \mathbf{e} and \mathbf{f} , obtaining the polynomial $p_{LL}(\mathbf{f}) + p_{\Gamma\Gamma}(\mathbf{e})$ for the left action of R_n .

6. \mathfrak{sl}_2 action on Lauda’s categorification of $U_q(\mathfrak{sl}_2)$

6.1. Actions on symmetric polynomials

The standard action of \mathfrak{sl}_2 on R_n (see (1)) is S_n equivariant, so it descends to an action of \mathfrak{sl}_2 on $R_n^{S_n}$. Recall that ∂_i is the Demazure operator,

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}},$$

which sends R_n to $R_n^{S_i}$.

Lemma 6.1. *For any $f \in R_n$ and any $1 \leq i \leq n - 1$ we have*

$$\partial_i(\mathbf{z}(f)) = \mathbf{z}(\partial_i(f)). \tag{51}$$

Remark 6.2. Note that the corresponding statement for \mathbf{d} is false! This lemma holds for \mathbf{z} effectively because \mathbf{z} kills the crossing in the nilHecke algebra, while \mathbf{d} does not kill the crossing.

Proof. Let us compute in NH_n , with its \mathfrak{sl}_2 structure from Definition 4.3. If X_i denotes the i -th crossing in NH_n , then we have

$$X_i f - s_i(f)X_i = \partial_i(f). \tag{52}$$

Applying \mathbf{z} to both sides, and using the fact that \mathbf{z} kills X_i , we get

$$X_i \mathbf{z}(f) - \mathbf{z}(s_i(f))X_i = \mathbf{z}(\partial_i(f)).$$

However, \mathbf{z} commutes with s_i , so we can apply (52) with f replaced by $\mathbf{z}(f)$ to get

$$X_i \mathbf{z}(f) - s_i(\mathbf{z}(f))X_i = \partial_i(\mathbf{z}(f)).$$

Combining these two equations, we deduce (51). ■

Corollary 6.3. *The operator \mathbf{z} commutes with the Demazure operator ∂_{w_0} associated to the longest element of S_n .*

Proof. The Demazure operator ∂_{w_0} can be defined as a composition of operators ∂_i along a reduced expression for w_0 . Now, apply (51). ■

Lemma 6.4. *Let e_k (resp. h_k, p_k) denote the elementary (resp. complete, power sum) symmetric polynomials in n variables, living in $R_n^{S^n}$. Set $e_{n+1} = 0$. Then*

$$\mathbf{d}(e_k) = e_k e_1 - (k + 1)e_{k+1}, \tag{53.1a}$$

$$\mathbf{d}(h_k) = (k + 1)h_{k+1} - h_k h_1, \tag{53.1b}$$

$$\mathbf{d}(p_k) = k p_{k+1}, \tag{53.1c}$$

and

$$\mathbf{z}(e_k) = (n + 1 - k)e_{k-1}, \tag{53.2a}$$

$$\mathbf{z}(h_k) = (n - 1 + k)h_{k-1}, \tag{53.2b}$$

$$\mathbf{z}(p_k) = \begin{cases} k p_{k-1} & \text{if } k \neq 1, \\ n p_0 & \text{if } k = 1. \end{cases} \tag{53.2c}$$

Moreover, let s_λ denote the Schur polynomial, where λ is a partition with at most n rows, and let $s_\lambda = 0$ when λ has more than n rows. Then

$$\mathbf{d}(s_\lambda) = \sum_{\mu=\lambda+\square} c(\square)s_\mu, \quad \mathbf{z}(s_\lambda) = \sum_{\lambda=\mu+\square} (c(\square) + n)s_\mu. \tag{53.3}$$

Here, $\lambda + \square$ is any partition obtained by adding a single box to λ , and $c(\square)$ is the content of that box.

Proof. The formulas for \mathbf{d} can be found in [9, (2.7)-(2.9)], except for $\mathbf{d}(p_k)$ which is easy to check. The formula for $\mathbf{z}(p_k)$ is also easy to check.

Let us compute $\mathbf{z}(h_k)$. Suppose that $\sum_{i=1}^n a_i = k - 1$. There are contributions to the coefficient of $x_1^{a_1} \cdots x_n^{a_n}$ inside $\mathbf{z}(h_k)$ coming from $x_i \cdot x_1^{a_1} \cdots x_n^{a_n}$ for each $1 \leq i \leq n$. We have

$$\mathbf{z}(x_i \cdot x_1^{a_1} \cdots x_n^{a_n}) = (a_i + 1)x_1^{a_1} \cdots x_n^{a_n} + \text{terms with different monomials.} \tag{54}$$

So, the overall contribution to the coefficient of $x_1^{a_1} \cdots x_n^{a_n}$ is

$$\sum_{i=1}^n (a_i + 1) = k - 1 + n.$$

We leave the computation of $\mathbf{z}(e_k)$ to the reader.

To compute $\mathbf{z}(s_\lambda)$, recall that one can define the Schur polynomial as

$$s_\lambda := \partial_{w_0}(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \cdot x_1^{n-1} x_2^{n-2} \cdots x_n^0).$$

By the above Corollary, \mathbf{z} commutes with ∂_{w_0} . Applying \mathbf{z} to the monomial inside the Demazure operator, we obtain

$$\sum_{i=1}^n (\lambda_i + n - i)x_1^{\lambda_1} \cdots x_i^{\lambda_i - 1} \cdots x_n^{\lambda_n} \cdot x_1^{n-1} x_2^{n-2} \cdots x_n^0. \tag{55}$$

If removing a box from the i -th row of λ yields a partition μ , then $\lambda_i - i$ is the content of that box. Applying ∂_{w_0} to the sum in (55), such terms yield

$$\sum_{\lambda=\mu+\square} (c(\square) + n)s_\mu$$

which is the desired answer. We need only prove that all the other terms in (55) are killed by ∂_{w_0} .

If removing a box from the i -th row of λ is not a partition, it is because $\lambda_i = \lambda_{i+1}$, in which case $\lambda_i - 1 + n - i = \lambda_{i+1} + n - (i + 1)$. Thus, the coefficients of x_i and x_{i+1} in the i -th monomial from the sum in (55) will be equal. Thus, this monomial is invariant under s_i , and killed by ∂_i and hence by ∂_{w_0} . ■

Lemma 6.5. *Divided powers $\mathbf{d}^{(k)}$ and $\mathbf{z}^{(k)}$ are defined integrally on $R_n^{S_n}$ for all $k \geq 0$.*

Proof. These divided powers are defined on R_n , and $R_n^{S_n}$ is an \mathfrak{sl}_2 -invariant subring. ■

6.2. Actions on symmetric functions

It was observed in [9, end of Section 2.2] that the formulas for \mathbf{d} acting on symmetric polynomials were independent of the number of variables n , and could be extended to a derivation on the ring Λ of symmetric functions (in infinitely many variables). However, it is clear from (53.2) that the formulas for \mathbf{z} acting on symmetric polynomials are not independent of n . If we let the number of variables go to infinity, where should \mathbf{z} send $e_1 = x_1 + x_2 + \dots$, given that $\mathbf{z}(x_i) = 1$ for all i ? The appropriate answer is to treat n as a formal variable; this was suggested to the authors by M. Khovanov.

Definition 6.6. Let $\Lambda[y]$ denote the ring of symmetric functions, extended by a formal variable y of degree zero. That is, $\Lambda[y] = \mathbb{Z}[y, e_1, e_2, \dots]$ is a polynomial ring in infinitely many variables, where e_k has degree $2k$. We define an \mathfrak{sl}_2 action on the generators of $\Lambda[y]$ as follows:

$$\begin{aligned} \mathbf{d}(e_k) &= e_k e_1 - (k + 1)e_{k+1}, & \mathbf{d}(y) &= 0, \\ \mathbf{z}(e_k) &= (y + 1 - k)e_{k-1}, & \mathbf{z}(y) &= 0. \end{aligned} \tag{56}$$

We extend this action to all of $\Lambda[y]$ using the Leibniz rule.

Lemma 6.7. *Definition 6.6 gives a well-defined \mathfrak{sl}_2 action by derivations. The \mathfrak{sl}_2 action also satisfies the following formulas on familiar elements of $\Lambda[y]$:*

$$\mathbf{d}(e_k) = e_k e_1 - (k + 1)e_{k+1}, \tag{57.1a}$$

$$\mathbf{d}(h_k) = (k + 1)h_{k+1} - h_k h_1, \tag{57.1b}$$

$$\mathbf{d}(p_k) = k p_{k+1}, \tag{57.1c}$$

$$\mathbf{z}(e_k) = (y + 1 - k)e_{k-1}, \tag{57.2a}$$

$$\mathbf{z}(h_k) = (y - 1 + k)h_{k-1}, \tag{57.2b}$$

$$\mathbf{z}(p_k) = \begin{cases} k p_{k-1} & \text{if } k \neq 1, \\ y p_0 & \text{if } k = 1, \end{cases} \tag{57.2c}$$

and

$$\mathbf{d}(s_\lambda) = \sum_{\mu=\lambda+\square} c(\square)s_\mu, \quad \mathbf{z}(s_\lambda) = \sum_{\lambda=\mu+\square} (c(\square) + y)s_\mu. \tag{57.3}$$

For all $n \geq 1$ there is a surjective \mathfrak{sl}_2 -equivariant quotient map

$$\Psi_n: \Lambda[y] \rightarrow R_n^{S_n}, \quad e_k \mapsto \begin{cases} e_k & \text{if } k \leq n, \\ 0 & \text{if } k > n, \end{cases} \quad y \mapsto n.$$

Proof. The existence of Ψ_n for all n is immediate from the formulas for the actions of \mathbf{d} and \mathbf{z} . We can use this to efficiently check (57) using (53). For an example of this argument, consider $\mathbf{z}(h_k) \in \Lambda[y]$. This will be some $\mathbb{Z}[y]$ -linear combination of polynomials in the variables $\{h_i\}$. Because Ψ_n is equivariant, the coefficient of h_{k-1} is some polynomial which evaluates to $n - 1 + k$ after specializing $y \mapsto n$ for all $n \geq k$. Thus, this coefficient is precisely $y - 1 + k$. Similarly, the coefficient of $h_1 h_{k-2}$ is some polynomial which evaluates to zero after specializing $y \mapsto n$ for all $n \geq k$. Thus, this coefficient is zero. ■

Remark 6.8. For any $n \in \mathbb{Z}$, there is a quotient of $\Lambda[y]$ which sends $y \mapsto -n$. It is an \mathfrak{sl}_2 -algebra over \mathbb{Z} . When $n > 1$, the ideal in this quotient generated by e_k for $k > n$ is \mathfrak{sl}_2 -invariant, and the further quotient by this ideal gives the map Ψ_n . When $n \leq 0$, there are no non-trivial \mathfrak{sl}_2 -invariant ideals in the quotient.

Sadly, divided powers of \mathbf{z} are not defined integrally on $\Lambda[y]$. For example, we have $\mathbf{z}^{(2)}(e_5) = \frac{(y-3)(y-4)}{2} e_3$, and $\binom{y-3}{2} := \frac{(y-3)(y-4)}{2}$ is not in $\mathbb{Z}[y]$. The following lemma will imply that divided powers of \mathbf{z} are defined integrally whenever y is specialized to an integer, even a negative integer.

Lemma 6.9. *On $\Lambda[y]$, the divided powers $\mathbf{d}^{(m)}$ are defined integrally, but $\mathbf{z}^{(m)}$ are not defined integrally. However, if we base change over $\mathbb{Z}[y]$ to a larger base ring containing $\binom{y+k}{m}$ for any $k \in \mathbb{Z}$ and $m \geq 0$, then $\mathbf{z}^{(m)}$ is defined after base change.*

Proof. We need only check that divided powers are well defined on the generators. The formula for \mathbf{d} does not involve the variable y , so the fact that $\mathbf{d}^{(m)}(e_k)$ is defined integrally can be checked after applying Ψ_n for sufficiently large n . Using (56) it is easy to verify that

$$\mathbf{z}^{(m)}(e_k) = \binom{y + 1 - k}{m} e_{k-m}. \quad \blacksquare$$

Remark 6.10. Note that the divided powers $\mathbf{z}^{(m)}$ are always defined integrally on power sums p_k , without the need for binomial coefficients. However, power sums generate a different integral form for $\Lambda[y]$ than do the elementary or complete symmetric functions.

6.3. Rank-one modules and their cores

In Section 2.5 we examined (R_n, \mathfrak{sl}_2) -modules which were free of rank 1 over R_n . Now, we do the same for $(R_n^{S_n}, \mathfrak{sl}_2)$ -modules and $\Lambda[y]$ -modules. As before, the rank-one free modules are parametrized by degree two elements in the corresponding ring.

For each $a \in \mathbb{Z}$, define a rank-one module over $R_n^{S_n}$ called $R_n^{S_n}\langle ae_1 \rangle$, with generator 1_a living in degree an . Equip it with an \mathfrak{sl}_2 action by the formulas

$$\mathbf{d}(1_a) = ae_1 1_a, \quad \mathbf{z}(1_a) = 0. \tag{58}$$

For each $a \in \mathbb{Z}$, define a rank-one module over $\Lambda[y]$ called $\Lambda[y]\langle ae_1 \rangle$, with generator 1_a living in degree ay . Equip it with an \mathfrak{sl}_2 action by the same formulas (58). Note that this \mathfrak{sl}_2 -representation has weights not in \mathbb{Z} but in $\mathbb{Z}[y]$. After specializing y to an integer, this yields an \mathfrak{sl}_2 -representation with weights in \mathbb{Z} .

Proposition 6.11. *If $a > 0$, then $\text{Core}(R_n^{S_n}\langle ae_1 \rangle) = 0$. If $a \leq 0$, then*

$$\text{Core}(R_n^{S_n}\langle ae_1 \rangle) = \mathbb{Z}\langle e_1^{i_1} \dots e_n^{i_n} \mid i_1 + \dots + i_n \leq -a \rangle. \tag{59.1}$$

On the other hand, $\text{Core}(\Lambda[y]\langle ae_1 \rangle) = 0$ whenever $a \neq 0$, and

$$\text{Core}(\Lambda[y]) = \mathbb{Z}[y]. \tag{59.2}$$

Proof. That $\text{Core}(R_n^{S_n}\langle ae_1 \rangle) = 0$ when $a > 0$ follows from Proposition 2.37 and the left exactness of taking cores:

$$\text{Core}(R_n^{S_n}\langle ae_1 \rangle) \subset \text{Core}(R_n\langle e_1 \rangle) = 0.$$

The core computation (59.1) when $a < 0$ essentially follows from the proof of [9, Corollary 2.12]. There, we worked only with \mathbf{d} in the context of p -dg algebras, but exactly the same computation shows that the right-hand side of (59.1) is closed under \mathbf{d} . It is clearly closed under \mathbf{z} , making it a finite rank submodule. We also showed that the remainder of $R_n^{S_n}\langle ae_1 \rangle$ is acyclic as a p -complex, or in other words, after specialization to finite characteristic it splits into free modules over $\mathbb{F}_p[\mathbf{d}]/(\mathbf{d}^p)$. We get the result here by taking the limit $p \rightarrow \infty$. More concretely, let v be some element in $R_n^{S_n}\langle ae_1 \rangle$ not in the suspected core; it will be in the core if and only if there is some $N > 0$ such that $\mathbf{d}^N(v) = 0$. However, when p is sufficiently large (e.g. $p > \deg(v) + N$, p does not divide any coefficients of v), v descends to a non-zero element in a free module over $\mathbb{F}_p[\mathbf{d}]/(\mathbf{d}^p)$, and hence $\mathbf{d}^N(v) \neq 0$.

Similarly, (59.2) in the context of p -dg algebras was studied in [8, Proposition 3.8]. There, it is shown that $\Lambda\langle ae_1 \rangle$ is acyclic as a p -complex whenever $a \neq 0$. When $a = 0$, the augmentation ideal was proven to be acyclic. A similar argument to the previous paragraph, letting $p \rightarrow \infty$, will imply the desired results. ■

6.4. Action on \mathcal{U}

Definition 6.12. Let $\mathcal{U} = \mathcal{U}(\mathfrak{sl}_2)$ denote the categorification of quantum \mathfrak{sl}_2 , defined by Lauda in [22]. We will follow the review given in [8, Section 4.1].

Definition 6.13. Place an \mathfrak{sl}_2 action by derivations on \mathcal{U} as follows. The raising operator \mathbf{d} was defined¹¹ in [8, Definition 5.9]. The lowering operator \mathbf{z} kills all generators except for dots, and as usual, \mathbf{z} sends a dot to the identity map.

Theorem 6.14. *The action of \mathfrak{sl}_2 on \mathcal{U} given in Definition 6.13 is well defined. The divided power operators $\mathbf{d}^{(k)}$ and $\mathbf{z}^{(k)}$ are well defined in the integral form for all $k \geq 0$, making \mathcal{U} into a divided powers \mathfrak{sl}_2 -algebra. Moreover, inside a region labeled by $\lambda \in \mathbb{Z}$, the subalgebra of $\text{End}(\mathbb{1}_\lambda)$ generated by bubbles is preserved by the \mathfrak{sl}_2 action, and is isomorphic as an \mathfrak{sl}_2 -algebra to the specialization $\Lambda[y]/(y - \lambda)$. Under this isomorphism, the clockwise bubble of degree $2k$ is matched with the symmetric function h_k , and a counterclockwise bubble of degree $2k$ is matched with $(-1)^k e_k$.*

Proof. First we confirm the action of \mathfrak{sl}_2 on bubbles. That \mathbf{d} acts by the formulas in (57.1) was already proven in [8, Corollary 4.8]. Inside a region labeled λ , the clockwise bubble with no dots has degree $2(1 - \lambda)$. Thus, a degree $2k$ bubble has $\lambda + k - 1$ dots. Applying \mathbf{z} to a bubble with $\lambda + k - 1$ dots, we get $(\lambda + k - 1)$ times a bubble with $\lambda + k - 2$ dots. This matches the formula

$$\mathbf{z}(h_k) = (y + k - 1)h_{k-1}$$

after specializing $y = \lambda$. Note that a real bubble of negative degree is sent to another real bubble of negative degree, which is still zero.

The computation for counterclockwise bubbles is similar. Consequently, \mathbf{z} preserves the bubble relations: positivity of bubbles, normalization of bubbles, and the infinite Grassmannian relations. We already know that \mathbf{z} preserves the nilHecke relations, since the action on the nilHecke algebra is the same as that in Definition 4.3. We need to check the remaining relations.

The biadjointness relations [8, (4.1)] are easy to check.

Consider the “reduction to bubbles” or the “curl relation” [8, (4.6a)]. Up to a sign, the right-hand side is $\sum_{a+b=-\lambda} h_a x^b$ where the clockwise bubble h_a appears in region λ . We compute that

$$\begin{aligned} \mathbf{z}\left(\sum_{a+b=\lambda} h_a x^b\right) &= \sum_{a+b=-\lambda} b h_a x^{b-1} + (\lambda + a - 1)h_{a-1}b \\ &= \sum_{a+b=-\lambda-1} h_a x^b (b + 1 + \lambda + a) = 0. \end{aligned}$$

¹¹In [8, Definition 5.9] the raising operator was called ∂_1 .

Since \mathbf{z} kills the left-hand side (it has no dots), \mathbf{z} preserves the reduction to bubbles relation. The other curl relation [8, (4.6b)] is proved similar.

Consider the identity decomposition relation [8, (4.7b)]. Using very similar arguments, we compute that

$$\mathbf{z}\left(\sum_{a+b+c=-\lambda-1} x_1^a h_b x_2^c\right) = \sum_{a+b+c=-\lambda-2} x_1^a h_b x_2^c (a + 1 + \lambda + b + c + 1) = 0.$$

Hence, \mathbf{z} preserves the identity decomposition relation. That handles all the relations of \mathcal{U} .

We need to check that $[\mathbf{z}, \mathbf{d}] = \mathbf{h}$ on each of the generating morphisms of \mathcal{U} , which is immediate from the formulas of [8, Definition 5.9].

We need to check that divided powers are defined integrally on the generating morphisms of \mathcal{U} . For the generators inside a nilHecke algebra, this was already done in Theorem 4.4. The cups and caps are killed by \mathbf{z} . All that remains to check is that $\mathbf{d}^{(k)}$ is well defined on the cups and caps. A related question was pursued in [8, Lemma A.3 and preceding], which proves that $\mathbf{d}^p = 0$ in characteristic p . The proof was to argue that all the coefficients appearing in \mathbf{d}^k were multinomial coefficients times $k!$, and this is the same proof¹² needed to show that $\mathbf{d}^{(k)}$ is defined integrally. ■

6.5. Remarks on a downfree filtration

Let \mathcal{E} denote the upward strand, an object in \mathcal{U} , and \mathcal{F} the downward strand. Throughout this section we fix $\lambda \in \mathbb{Z}$ and let Λ denote the specialization of $\Lambda[y]$ at $y = \lambda$.

In [22, Section 8], Lauda proves that $\text{Hom}_{\mathcal{U}}(\mathbb{1}_\lambda \mathcal{E}^n, \mathbb{1}_\lambda \mathcal{E}^n)$ is isomorphic to $\Lambda \otimes \text{NH}_n$, where bubbles appear on the left of crossing diagrams. We can view crossing diagrams as a basis of this Hom space over $\Lambda \otimes R_n$, where R_n acts on the bottom. Then this is a downfree filtration over the base ring $\Lambda \otimes R_n$, with the same downfree character computed in (34). This is because applying \mathbf{d} to a crossing diagram will not create any bubbles.

Using adjunction, every morphism in the space $\text{Hom}_{\mathcal{U}}(\mathcal{E}^n \mathcal{F}^n \mathbb{1}_\lambda, \mathbb{1}_\lambda)$ is obtained by taking a morphism in $\text{Hom}_{\mathcal{U}}(\mathbb{1}_\lambda \mathcal{E}^n, \mathbb{1}_\lambda \mathcal{E}^n)$ and placing caps on top. Thus, crossing diagrams will form a basis for $\text{Hom}_{\mathcal{U}}(\mathcal{E}^n \mathcal{F}^n, \mathbb{1})$ under the action of Λ on the left and R_n acting on the inwardly-pointing boundary strands. This basis is again in bijection with S_n . However, the differential of a cap does introduce bubbles. We expect that this produces a downfree filtration over $\Lambda \otimes R_n$, but the formula for the downfree character is currently unknown, and will involve $e_1 \in \Lambda$.

¹²Admittedly, the proof in [8, Lemma A.3] is rather hand-wavy, but the result is still a relatively easy exercise.

For example, $\text{Hom}_{\mathcal{U}}(\mathcal{E}\mathcal{F}1_\lambda, 1_\lambda)$ has a basis with one diagram, given by the cap. By [8, Definition 5.9], \mathbf{d} sends this cup to itself with a dot minus itself with a degree two bubble. Thus, this Hom space, as a module over the \mathfrak{sl}_2 -algebra $R_1 \otimes \Lambda$, is isomorphic to $(R_1 \otimes \Lambda)\langle x_1 - e_1 \rangle$.

Similar arguments to those used in [22, Section 8] will produce a basis for any morphism space in \mathcal{U} . The basis will be a collection of reduced diagrams for oriented planar matchings with $2n$ boundary points (n oriented in, and n oriented out). It is a basis over the left action of Λ and the action of R_n by placing dots on the inwardly-pointing boundary strands. The basis is in bijection with S_n , though this bijection does not preserve the number of crossings in a diagram. We expect that this is a downfree basis, with a partial order coming from crossing removal; in this case, the partial order does not coincide with the usual Bruhat order on S_n . The downfree character is currently unknown.

7. \mathfrak{sl}_2 action on the thick calculus $\dot{\mathcal{U}}$

Definition 7.1. Let $\dot{\mathcal{U}}$ denote the 2-category defined in [17, Section 4].

It can be hard to determine from [17] what precisely the generators and relations of $\dot{\mathcal{U}}$ are, over and above the presentation of \mathcal{U} , as it is not stated explicitly there. We give the answer in [9, Proposition 5.2]. There are two new generators for each thickness a :

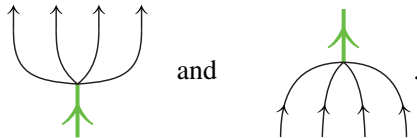
- the splitter

$$\mathcal{E}^{(a)} \rightarrow \mathcal{E}^{\otimes a}$$

- and the merger

$$\mathcal{E}^{\otimes a} \rightarrow \mathcal{E}^{(a)},$$

that is



They compose in one direction to be the endomorphism ψ_{w_0} inside NH_n , this is [9, (5.4a)]:

$$\text{[Diagram of splitter and merger composition]} = \text{[Diagram of crossing strands]} \tag{60}$$

In the other direction, they compose to the identity if sufficiently many dots are placed in between, which is [9, (5.4b)]. A more general version of [9, (5.4b)] is

$$\begin{array}{c}
 \text{Diagram of } f \text{ with 4 strands and arcs} \\
 = \\
 \text{Diagram of } \partial_{w_0} f \text{ with 2 strands}
 \end{array}
 \tag{61}$$

Definition 7.2. We place an \mathfrak{sl}_2 structure on $\dot{\mathcal{U}}$, extending the operator \mathbf{d} from [9, Definition 5.3], by asserting that \mathbf{z} kills the new generating splitter and merger.

Theorem 7.3. The \mathfrak{sl}_2 action on $\dot{\mathcal{U}}$ is well defined, and divided powers are defined integrally.

Proof. We need to check that \mathbf{z} preserves the relations. Clearly, it preserves (60) since both sides go to zero. Since \mathbf{z} commutes with ∂_{w_0} , it is easy to verify that it preserves (61) as well.

We need to check that $[\mathbf{z}, \mathbf{d}] = \mathbf{h}$ on the new generators. This is straightforward, since $\mathbf{z}(-\delta) = \mathbf{z}(-\delta')$ agrees with the degree of the splitter. In fact, $\text{Hom}(\mathcal{E}^{(a)}, \mathcal{E}^{\otimes a})$ is isomorphic to $R_a(-\delta)$ as a left (R_a, \mathfrak{sl}_2) -module, and $\text{Hom}(\mathcal{E}^{\otimes a}, \mathcal{E}^{(a)})$ is isomorphic to $R_a(-\delta')$ as a right (R_a, \mathfrak{sl}_2) -module. Thus, divided powers are well defined by Proposition 2.29. For more on δ and δ' see the second proof of Theorem 4.9. ■

Acknowledgments. The authors would like to thank Mikhail Khovanov for his interest, and his suggestion in Section 6.2.

Funding. Ben Elias was partially supported by NSF CAREER grant DMS-1553032 and NSF FRG grant DMS-1800498. This paper was completed while he was visiting the Institute of Advanced Study, where he was supported by NSF grant DMS-1926686. You Qi was partially supported by the NSF grant DMS-1947532.

References

[1] A. Beliakova and B. Cooper, [Steenrod structures on categorified quantum groups](#). *Fund. Math.* **241** (2018), no. 2, 179–207 Zbl 1432.17011 MR 3766567

[2] R. Bezrukavnikov and Z. Yun, [On Koszul duality for Kac-Moody groups](#). *Represent. Theory* **17** (2013), 1–98 Zbl 1326.20051 MR 3003920

- [3] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, [On braided fusion categories. I](#). *Selecta Math. (N.S.)* **16** (2010), no. 1, 1–119 Zbl [1201.18005](#) MR [2609644](#)
- [4] B. Elias, [A diamond lemma for Hecke-type algebras](#). *Trans. Amer. Math. Soc.* **375** (2022), no. 3, 1883–1915 Zbl [1497.20044](#) MR [4378083](#)
- [5] B. Elias, [On diagrammatics for semisimple monoidal categories and their integral forms](#). In preparation.
- [6] B. Elias and M. Khovanov, [Diagrammatics for Soergel categories](#). *Int. J. Math. Math. Sci.* (2010), article no. 978635 Zbl [1219.18003](#) MR [3095655](#)
- [7] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, [Introduction to Soergel bimodules](#). RSME Springer Ser. 5, Springer, Cham, 2020 Zbl [1507.20001](#) MR [4220642](#)
- [8] B. Elias and Y. Qi, [An approach to categorification of some small quantum groups II](#). *Adv. Math.* **288** (2016), 81–151 Zbl [1329.81238](#) MR [3436383](#)
- [9] B. Elias and Y. Qi, [A categorification of quantum \$\mathfrak{sl}\(2\)\$ at prime roots of unity](#). *Adv. Math.* **299** (2016), 863–930 Zbl [1355.81096](#) MR [3519482](#)
- [10] B. Elias and Y. Qi, [Categorifying Hecke algebras at prime roots of unity, part I](#). 2020, arXiv:[2005.03128](#), to appear in *Trans. Amer. Math. Soc.*
- [11] B. Elias and G. Williamson, [The Hodge theory of Soergel bimodules](#). *Ann. of Math. (2)* **180** (2014), no. 3, 1089–1136 Zbl [1326.20005](#) MR [3245013](#)
- [12] B. Elias and G. Williamson, [Soergel calculus](#). *Represent. Theory* **20** (2016), 295–374 Zbl [1427.20006](#) MR [3555156](#)
- [13] B. Elias and G. Williamson, [Relative hard Lefschetz for Soergel bimodules](#). *J. Eur. Math. Soc. (JEMS)* **23** (2021), no. 8, 2549–2581 Zbl [1475.20006](#) MR [4269421](#)
- [14] M. Khovanov, [Hopfological algebra and categorification at a root of unity: the first steps](#). *J. Knot Theory Ramifications* **25** (2016), no. 3, article no. 1640006 Zbl [1370.18017](#) MR [3475073](#)
- [15] M. Khovanov and A. D. Lauda, [A diagrammatic approach to categorification of quantum groups. I](#). *Represent. Theory* **13** (2009), 309–347 Zbl [1188.81117](#) MR [2525917](#)
- [16] M. Khovanov and A. D. Lauda, [A diagrammatic approach to categorification of quantum groups II](#). *Trans. Amer. Math. Soc.* **363** (2011), no. 5, 2685–2700 Zbl [1214.81113](#) MR [2763732](#)
- [17] M. Khovanov, A. D. Lauda, M. Mackaay, and M. Stošić, [Extended graphical calculus for categorified quantum \$\mathfrak{sl}\(2\)\$](#) . *Mem. Amer. Math. Soc.* **219** (2012), no. 1029 Zbl [1292.17013](#) MR [2963085](#)
- [18] M. Khovanov and Y. Qi, [An approach to categorification of some small quantum groups](#). *Quantum Topol.* **6** (2015), no. 2, 185–311 Zbl [1352.81038](#) MR [3354331](#).
- [19] M. Khovanov and L. Rozansky, [Positive half of the Witt algebra acts on triply graded link homology](#). *Quantum Topol.* **7** (2016), no. 4, 737–795 Zbl [1376.57015](#) MR [3593567](#)
- [20] M. Khovanov, Y. Qi, and J. Sussan, [p-DG cyclotomic nilHecke algebras](#). 2017, arXiv:[1711.07159](#)
- [21] N. Kitchloo, [Soergel bimodules, the Steenrod algebra, and triply graded link homology](#). 2018, arXiv:[1305.4725](#)
- [22] A. D. Lauda, [A categorification of quantum \$\mathfrak{sl}\(2\)\$](#) . *Adv. Math.* **225** (2010), no. 6, 3327–3424 Zbl [1219.17012](#) MR [2729010](#)

- [23] A. D. Lauda, [Parameters in categorified quantum groups](#). *Algebr. Represent. Theory* **23** (2020), no. 4, 1265–1284 Zbl [1495.17026](#) MR [4125578](#)
- [24] N. Libedinsky, [Sur la catégorie des bimodules de Soergel](#). *J. Algebra* **320** (2008), no. 7, 2675–2694 Zbl [1196.20005](#) MR [2441994](#)
- [25] A. Licata and A. Savage, [A survey of Heisenberg categorification via graphical calculus](#). *Bull. Inst. Math. Acad. Sin. (N.S.)* **7** (2012), no. 2, 291–321 Zbl [1281.20005](#) MR [3024895](#)
- [26] M. Mackaay, M. Stošić, and P. Vaz, [A diagrammatic categorification of the \$q\$ -Schur algebra](#). *Quantum Topol.* **4** (2013), no. 1, 1–75 Zbl [1272.81098](#) MR [2998837](#)
- [27] Y. I. Manin and V. V. Schechtman, [Arrangements of hyperplanes, higher braid groups and higher Bruhat orders](#). In *Algebraic number theory*, pp. 289–308, Adv. Stud. Pure Math. 17, Academic Press, Boston, MA, 1989 Zbl [0759.20002](#) MR [1097620](#)
- [28] S. Montgomery, [Hopf algebras and their actions on rings](#). CBMS Regional Conference Series in Mathematics 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993 Zbl [0793.16029](#) MR [1243637](#)
- [29] Y. Qi, [Hopfological algebra](#). *Compos. Math.* **150** (2014), no. 1, 1–45 Zbl [1343.16010](#) MR [3164358](#)
- [30] Y. Qi and J. Sussan, [\$p\$ -DG cyclotomic nilHecke algebras II](#). 2018, arXiv:[1811.04372](#)
- [31] R. Rouquier, [2-Kac-Moody algebras](#). 2008, arXiv:[0812.5023](#)

Received 30 March 2022.

Ben Elias

Department of Mathematics, University of Oregon, Fenton Hall, Room 210, Eugene, OR 97403, USA; belias@uoregon.edu

You Qi

Department of Mathematics, University of Virginia, 141 Cabell Drive, Kerchof Hall, P.O. Box 400137, Charlottesville, VA 22904, USA; yq2dw@virginia.edu