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Witten–Reshetikhin–Turaev invariants for 3-manifolds from Lagrangian intersections in configuration spaces

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Abstract. In this paper, we construct a topological model for the Witten–Reshetikhin–Turaev invariants for 3-manifolds coming from the quantum group $U_q(\mathrm{sl}(2))$, as graded intersection pairings of homology classes in configuration spaces. More precisely, for a fixed level $\mathcal{N} \in \mathbb{N}$, we show that the level $\mathcal{N}$ WRT invariant for a 3-manifold is a state sum of Lagrangian intersections in a covering of a *fixed* configuration space in the punctured disc. This model brings a new perspective on the structure of the level $\mathcal{N}$ Witten–Reshetikhin–Turaev invariant, showing that it is completely encoded by the intersection points between certain Lagrangian submanifolds in a fixed configuration space, with additional gradings which come from a particular choice of a local system. This formula provides a new framework for investigating the open question about categorifications of the WRT invariants.

1. Introduction

After the discovery of the Jones polynomial for [5] knots, the world of quantum invariants encountered a powerful development, provided by constructions due to Witten, Reshetikhin, and Turaev. More precisely, Witten [17] predicted the existence of an extension of the Jones polynomial to 3-manifolds, and Reshetikhin–Turaev [14] provided an algebraic construction of such invariants. They showed that the representation theory of the quantum group $U_q(\mathrm{sl}(2))$ leads to invariants for links coloured with finite-dimensional representations of this quantum group, called coloured Jones polynomials. Further on, for any level $\mathcal{N} \in \mathbb{N}$, one can use linear combinations of coloured Jones polynomials with colours less than $\mathcal{N}$ in order to get a 3-manifold invariant $\tau_{\mathcal{N}}$. However, there are open questions about the geometry and topology which are contained in the Witten–Reshetikhin–Turaev invariants. An active research area concerns categorifications for invariants of links and 3-manifolds. For instance, Khovanov homology, which is a categorification for the Jones polynomial for knots, was proved to be a powerful tool which contains much information [6, 7, 9, 12, 13, 15]. The story is different for the analogous invariants for 3-manifolds. There

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is an important open question about the existence of categorifications for Witten–Reshetikhin–Turaev invariants.

Our aim is to describe these invariants as intersection pairings between homology classes in coverings of configuration spaces. We refer to such a description as "topological model". The first topological model was constructed for the Jones polynomial by Bigelow [4], based on the work of Lawrence [8]. The main result of the paper shows that the level $\mathcal{N}$ WRT invariant is a state sum of graded intersections between Lagrangian submanifolds in a fixed configuration space. This provides a new framework for the study of these invariants and a starting point in investigating categorification questions.

In the first part of this article, Theorem 1.3, we generalise the author's previous work [1,2] constructing a topological model for coloured Jones polynomials coloured with different colours. Then, the translation of the algebraic definition of the WRT invariant using Theorem 1.3 would show that the WRT invariant $\tau_{\mathcal{N}}$ is a linear combination of Lagrangian intersections in various configuration spaces. Further on, the main part of the paper is geometric. We encode the coefficients of the coloured Jones polynomials coming from the Kirby colour by adding certain circles to the supports of the Lagrangian submanifolds as well as adding extra punctures to the punctured disc. Then, we show that we can move the whole intersection formula—which a priori would be in different configuration spaces—in a fixed configuration space, as presented in Theorem 1.6.

1.1. Homological tools

For $n, m \in \mathbb{N}$, we define $C_{n,m} = \mathrm{Conf}_m(\mathcal{D}_n)$ to be the unordered configuration space of m points in the n-punctured disc $\mathcal{D}_n$. We use two extra parameters $k, \bar{l} \in \mathbb{N}$ and define a local system:

$$\Phi : \pi_1(C_{n+1+3\bar{l},m}) \to \mathbb{Z}^n \oplus \mathbb{Z}^{\bar{l}} \oplus \mathbb{Z}.$$

The definition of this local system depends on the parameter k. Roughly speaking, the monodromy around each puncture gives us one variable, and the last $\mathbb{Z}$-component counts the winding of particles in the configuration space. The parameter k is used for orientation purposes: the monodromies of Φ around the first $n-k$ punctures and the last k punctures are counted with opposite orientations. In our model, l will be the number of link components. The extra $3l$ punctures will play an important role in the model for the WRT-invariants. We define $\tilde{C}^{-k}_{n+1+3\bar{l},m}$ to be the covering of $C_{n,m}$ corresponding to Φ. We use the homology of a quotient of this covering space (quotienting the first n components of the local system towards l variables, for $l \leqslant n$), as follows:

- Lawrence representations $H^{-k}_{n+1,m,\bar{l}}$ which are

$$\mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]\text{-modules.}$$

They come from the Borel–Moore homology of $\widetilde{C}^{-k}_{n+1+3\bar{l},m}$ and have an action of coloured braids on $n+1+3\bar{l}$ strands (Definition 3.15, Proposition 3.16).

- Dual Lawrence representations $H^{-k,\partial}_{n+1,m,\bar{l}}$ (Definition 3.15) (using the homology relative to the boundary of the same covering space)
- Graded intersection pairing (Proposition 3.17):

$$\langle\, , \rangle : H^{-k}_{n+1,m,\bar{l}} \otimes H^{-k,\partial}_{n+1,m,\bar{l}} \to \mathbb{Z}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}, y_1^{\pm 1}, \ldots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}].$$

Homology classes. We will construct certain classes in these homology groups, given by lifts of Lagrangian submanifolds in the base configuration space. These submanifolds are encoded by "geometric supports" which are sets of arcs in the punctured disc. The product of these arcs quotiented to the unordered configuration space gives the Lagrangian submanifolds. Then, the lifts in the covering will be encoded by sets of "paths to the base points" which are collections of arcs in the punctured disc, from the base point towards the geometric support.

Remark 1.1. The pairing is encoded in the base configuration space, and it is parametrised by the intersection points between the geometric supports of the homology classes, graded by monomials which are prescribed by the local system Φ.

1.2. Topological model coloured Jones polynomials

In the author's earlier work [1,2], a topological model for the coloured Jones polynomial for links coloured with the same colour was constructed (i.e., each component is coloured with the same colour). In the first part of this paper, we generalise this result and construct a topological model for coloured Jones polynomials for links coloured with different colours. Let L be an oriented framed link with framings $f_1, \ldots, f_l \in \mathbb{Z}$. We consider $\beta_n \in B_n$ a braid such that $L = \widehat{\beta_n}$ by braid closure. Now, let us fix a set of colours $N_1, \ldots, N_l \in \mathbb{N} \setminus \{0\}$ for the strands of the link. This colouring induces a colouring of the strands of the braid: $(C_1, \ldots, C_n)$.

We use the configuration space of $1 + \sum_{i=1}^{n}(C_i - 1)$ particles in the $(2n+1)$-punctured disc and a $\mathbb{Z}^{2n+1} \oplus \mathbb{Z}$ local system constructed as above, with $k = n$ and $\bar{l} = 0$. Then, we have the homologies

$$H^{-n}_{2n+1,1+\sum_{i=1}^{n}(C_i-1),0} \quad \text{and} \quad H^{-n,\partial}_{2n+1,1+\sum_{i=1}^{n}(C_i-1),0},$$

which are

$$\mathbb{Z}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}, d^{\pm 1}]\text{-modules.}$$

Definition 1.2 (Coloured homology classes). With the procedure described above, for any indices $i_1, \ldots, i_n \in \mathbb{N}$ such that $0 \leqslant i_k \leqslant C_k - 1$ for all $k \in \{1, \ldots, n\}$, we define two Lagrangian submanifolds and consider the classes given by their lifts in the covering, as presented in Figure 4.2:

$$\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n}_{2n+1,1+\sum_{i=1}^{n}(C_i-1),0} \quad \text{and} \quad \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n,\partial}_{2n+1,1+\sum_{i=1}^{n}(C_i-1),0}.$$

The set of such sequences of indices is denoted by $C(\bar{N})$.

Theorem 1.3 (Topological state sum model for coloured Jones polynomials for coloured links). *Let us fix a set of colours $N_1,\dots,N_l \in \mathbb{N}$. Then, the coloured Jones polynomial of L coloured with colours $N_1,\dots,N_l$ has the following model:*

$$J_{N_1,\dots,N_l}(L,q) = q^{\sum_{i=1}^{l}(f_i - \sum_{j\neq i} lk_{i,j})(N_i-1)}$$
$$\cdot \left(\sum_{\bar{i}\in C(\bar{N})} \left(\prod_{i=1}^{n} x_{C(i)}^{-1} \right) \cdot \left\langle (\beta_n \cup \mathbb{I}_{n+1})\, \mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \right\rangle \right)\Bigg|_{\psi^C_{q,N_1,\dots,N_l}}.$$

In this expression, $\psi^C_{q,N_1,\dots,N_l}$ is the specialisation of variables to one variable from formula (2.2), and

$$C : \{1,\dots,2n\} \to \{1,\dots,l\}$$

is the colouring presented in Definition 2.4.

Note that this formula is a state sum of intersections in a configuration space where the number of particles depends on the choice of individual colours $N_1,\dots,N_l$ for colouring the link.

1.3. Topological model for WRT invariants

The second part of the paper is devoted to the construction of a topological model for the Witten–Reshetikhin–Turaev 3-manifold invariants. Let us fix a level $\mathcal{N} \in \mathbb{N}$, and let us consider the $2\mathcal{N}$th root of unity $\xi = e^{\frac{2\pi i}{2\mathcal{N}}}$. We will use the description of closed oriented 3-manifolds as surgeries along framed oriented links. In turn, we will look at links as closures of braids. Suppose that the corresponding link has l components and the braid has n strands.

We start with the construction of the homology classes in this context. This time we use a covering of the configuration space of $n(\mathcal{N}-2)+l+1$ particles in the $(2n+3l+1)$-punctured disc and a $\mathbb{Z}^{2n+1} \oplus \mathbb{Z}^{3l} \oplus \mathbb{Z}$ local system constructed as above, with $k=n$ and $\bar{l}=l$. We consider the homology groups

$$H^{-n}_{2n+1,n(\mathcal{N}-2)+l+1,l} \quad \text{and} \quad H^{-n,\partial}_{2n+1,n(\mathcal{N}-2)+l+1,l},$$

which are

$$\mathbb{Z}[x_1^{\pm 1},\dots,x_l^{\pm 1}, y_1^{\pm 1},\dots,y_l^{\pm 1}, d^{\pm 1}]\text{-modules.}$$

$\mathcal{F}_{\bar{i}}^{\mathcal{N}} \in H^{-n}_{2n+1,n(\mathcal{N}-2)+l+1,l}$ (Definition 5.2)

$\mathcal{L}_{\bar{i}}^{\mathcal{N}} \in H^{-n,\partial}_{2n+1,n(\mathcal{N}-2)+l+1,l}$ (Definition 5.4)

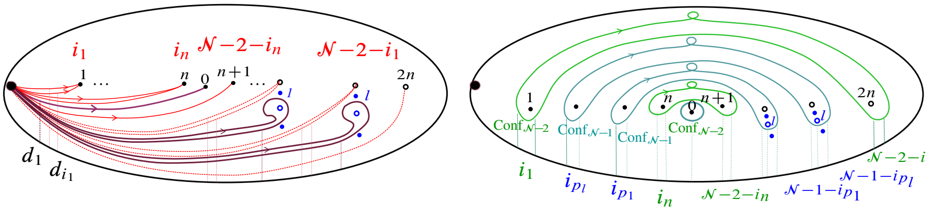

Figure 1.1. WRT homology classes.

Definition 1.4 (Homology classes). Let us fix a set of indices $i_1, \ldots, i_n \in \{0, \ldots, \mathcal{N} - 2\}$ and denote by

$$\bar{i} := (i_1, \ldots, i_n).$$

We consider the classes given by the geometric supports from Figure 1.1.

Denote by $p_1, \ldots, p_l$ a sequence of strands of the braid that correspond to different components of the link, and denote by f_{p_i} the framing of the component associated to p_i.

Definition 1.5 (Lagrangian intersection in the configuration space). For a multi-index $i_1, \ldots, i_n \in \{0, \ldots, \mathcal{N} - 2\}$, we consider the following Lagrangian intersection:

$$\begin{cases} \Lambda_{\bar{i}}(\beta_n) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}, y_1^{\pm 1}, \ldots, y_l^{\pm 1}, d^{\pm 1}], \\ \Lambda_{\bar{i}}(\beta_n) := \prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i} - \sum_{j \neq p_i} lk_{p_i,j})} \cdot \prod_{i=1}^{n} x_{C(i)}^{-1} \langle (\beta_n \cup \mathbb{I}_{n+3l+1}) \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \rangle. \end{cases}$$

The main result shows that the $\mathcal{N}$th WRT invariant $\tau_{\mathcal{N}}(M)$ comes from a state sum of specialisations of these intersections, which take place in the configuration space

$$\mathrm{Conf}_{n(\mathcal{N}-2)+l+1}(\mathcal{D}_{2n+3l+1}).$$

Theorem 1.6 (Topological state sum model for the Witten–Reshetikhin–Turaev quantum invariants). *Let $\mathcal{N} \in \mathbb{N}$ be a fixed level and M a closed oriented 3-manifold. We consider L a framed oriented link with l components such that M is obtained by surgery along L. Also, let $\beta_n \in B_n$ such that $L = \widehat{\beta_n}$ as above. Then, the $\mathcal{N}$th Witten–Reshetikhin–Turaev invariant has the following model:*

$$\tau_{\mathcal{N}}(M) = \frac{\{1\}_{\xi}^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{i_1,\ldots,i_n=0}^{\mathcal{N}-2} \left(\sum_{\substack{1 \leqslant N_1,\ldots,N_l \leqslant \mathcal{N}-1 \\ i_k \leqslant C_k - 1}} \Lambda_{\bar{i}}(\beta_n) \Big|_{\psi^C_{\xi,N_1,\ldots,N_l}} \right).$$

In this expression, $\psi^{C}_{\xi,N_1,\dots,N_l}$ is a specialisation of variables to complex numbers (see relation (2.2)). The coefficients that appear in the above formula are presented in Notation 5.7.

Remark 1.7 (Intersections in various configuration spaces). For a fixed colour $\mathcal{N}$, the algebraic definition of the WRT invariant $\tau_{\mathcal{N}}(M)$ is given by a certain linear combination of $J_{N_1,\dots,N_l}(L,\xi)$ for all $N_1,\dots,N_l \in \{1,\dots,\mathcal{N}-1\}$. Then, Theorem 1.3 would interpret this invariant as follows:

$$\begin{aligned}&\tau_{\mathcal{N}}(M) \text{ is a linear combination over all } N_1,\dots,N_l \in \{1,\dots,\mathcal{N}-1\}\\ &\qquad\text{and all } \bar{i}=(i_1,\dots,i_n) \text{ with } 0 \leqslant i_k \leqslant C_k-1, k \in \{1,\dots,n\} \text{ of}\\ &\qquad\left\langle(\beta_n \cup \mathbb{I}_{n+1})\, \mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)}\right\rangle\Big|_{\psi^{C}_{\xi,N_1,\dots,N_l}}.\end{aligned}$$

Each term above is an intersection in the configuration space of $1+\sum_{i=1}^{n}(C_i-1)$ particles in the $(2n+1)$-punctured disc, which depends on the choice of colours $N_1,\dots,N_l$.

This means that the translation of the algebraic definition of the WRT invariant following Theorem 1.3 shows that $\tau_{\mathcal{N}}(M)$ is a linear combination of Lagrangian intersections in different configuration spaces $C_{2n+1,k}$, where the number of particles k varies between 1 and $(n-1)(\mathcal{N}-1)+1$.

Remark 1.8 (Intersection in a fixed configuration space). A feature of the model presented in Theorem 1.6 is that it globalises all these intersections from above, showing that the $\mathcal{N}$th WRT invariant is given by states of certain Lagrangian intersections in a fixed ambient space, as follows:

$$\begin{aligned}&\tau_{\mathcal{N}}(M) \text{ is a scalar times the state sum over all multi-indices}\\ &\qquad i_1,\dots,i_n \in \{0,\dots,\mathcal{N}-2\} \text{ of specialisations of the intersection}\\ &\qquad\Lambda_{\bar{i}}(\beta_n) \text{ associated to } N_1,\dots,N_l \in \{1,\dots,\mathcal{N}-1\} \text{ such that}\\ &\qquad i_k \leqslant C_k-1, k \in \{1,\dots,n\};\ \text{namely, } \Lambda_{\bar{i}}(\beta_n)\big|_{\psi^{C}_{\xi,N_1,\dots,N_l}}.\end{aligned}$$

All the intersections $\Lambda_{\bar{i}}(\beta_n)$ above are constructed from the classes

$$(\beta_n \cup \mathbb{I}_{n+3l+1})\mathcal{F}_{\bar{i}}^{\mathcal{N}} \quad \text{and} \quad \mathcal{L}_{\bar{i}}^{\mathcal{N}}$$

and take place in the fixed configuration space of $n(\mathcal{N}-2)+l+1$ points in the $(2n+3l+1)$-punctured disc.

Remark 1.9 (Encoding the Kirby colour). Now, we discuss the coefficients which appear in the algebraic definition of the Witten–Reshetikhin–Turaev invariant for a 3-manifold. This formula is given as linear combinations of coloured Jones polynomials

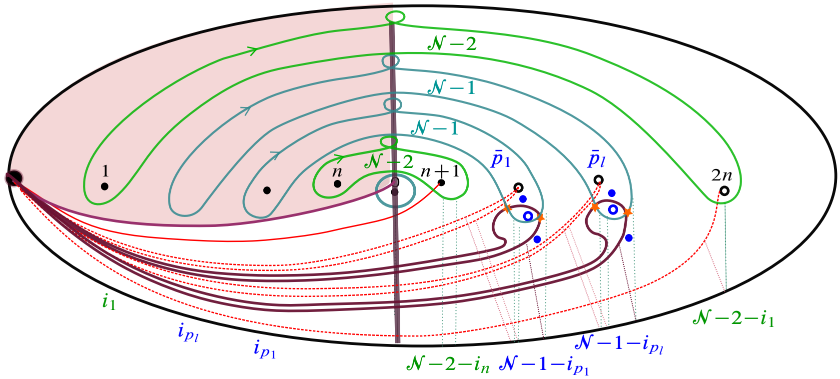

Figure 1.2. Lagrangian intersection encoding the Kirby colour.

of the underlying link L, and the coefficients come from the so-called Kirby colour and they are quantum integers.

Theorem 1.6 provides a globalised formula for $\tau_{\mathcal{N}}(M)$ and does not require individual coloured Jones polynomials. For each multi-index $\bar{i}$ bounded by the level, we consider the intersection form $\Lambda_{\bar{i}}(\beta_n)$ between globalised classes in a covering of the configuration space, which does not depend on any colouring. Then, we have to add up its specialisations, corresponding to colours which are "bigger" than the index $\bar{i}$.

The coefficients coming from the Kirby colour are encoded in the homology classes. Geometrically, they are given precisely by the special l purple circles and l blue figure eights from the supports of the homology classes, and they correspond to the orange intersection points in Figure 1.2.

1.4. Structure of the WRT invariants

Following this remark together with the fact that the intersection pairing is encoded by graded geometric intersections in the base configuration space (Remark 1.1), we conclude that we have a topological formula for the WRT invariant which is obtained from the intersection points between the following geometric supports:

$$(\beta_n \cup \mathbb{I}_{n+3l+1})\, \mathcal{F}_{\bar{i}}^{\mathcal{N}} \cap \mathcal{L}_{\bar{i}}^{\mathcal{N}}$$

for all choices of indices $i_1, \dots, i_n \in \{1, \dots, \mathcal{N}-2\}$, graded using the local system Φ.

Remark 1.10. In this way, we see that the WRT invariant at level $\mathcal{N}$ is completely encoded by the set of intersection points between certain Lagrangian submanifolds in the configuration space of $n(\mathcal{N}-2)+l+1$ points in the $(2n+3l+1)$-punctured disc. The number of particles is fixed, and it is determined by the level of the invariant $\mathcal{N}$, the number of components of the link l, and the number of strands of the braid n.

1.5. Questions underlying topological information

Our main motivation for this work is the understanding of the underlying topology which is carried by the Witten–Reshetikhin–Turaev invariants. The structural description presented above, provided by intersections between Lagrangians in a fixed configuration space, brings a new approach to investigating further questions concerning categorifications for these quantum invariants. Two questions that need to be solved for such categorifications concern the symplectic setup that one decides to choose and also how to take care of the denominator that appears in the intersection formula for the WRT-invariant $\tau_{\mathcal{N}}$ from Theorem 1.6.

Structure of the paper. This article has four main parts. In Section 3, we introduce the homological setting that we use as well as the particular choice of a local system and the corresponding covering space and homology groups. Then, in Section 4, we construct certain homology classes, and using those, we prove the topological intersection formula for the coloured Jones polynomials for links. After that, Section 5 has two parts. First, in Section 5.1, we construct a sequence of homology classes in a fixed covering space and use them to define a state sum formula. Then, we prove in Section 5.2 that it leads to a topological model for the $\mathcal{N}$th Witten–Reshetikhin–Turaev invariant. In the last part, Section 6, we present the formula for these invariants in the particular case where we have 3-manifolds which are given by surgeries along knots.

2. Notations

In the next sections, we will change the variables from the ring of Laurent polynomials using certain specialisations of coefficients. For this, we use the following definition.

Notation 2.1 (Specialisation). Let N be a module over a ring R. Let R' be another ring, and suppose that we have a specialisation of the coefficients, meaning a morphism:

$$\psi : R \to R'.$$

We denote by

$$N|_{\psi} := N \otimes_R R'$$

the specialisation of the module N by the function ψ.

Definition 2.2 (Quantum numbers). Let us define the following quantum numbers:

$$\{x\}_q := q^x - q^{-x}, \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}.$$

Definition 2.3 (Specialisations of coefficients). For a set of l colours $N_1, \dots, N_l \in \mathbb{N}$ and a colouring $C : \{1, \dots, n\} \to \{1, \dots, l\}$, we consider the specialisation of coefficients as follows:

$$\begin{array}{ccc} \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}] & \xrightarrow{f_C\ (3.3)} & \mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}] \\ & \searrow \quad \swarrow{\scriptstyle \psi^C_{q,N_1,\dots,N_l}\ (3.4)} & \\ & \mathbb{Z}[q^{\pm 1}] & \end{array}$$

Definition 2.4 (Our setting: specialisation corresponding to a braid closure). We will use this change of coefficients in the situation where n is replaced by $2n$, and these $2n$ points inherit a colouring with l colours coming from a braid closure of a braid with n strands:

$$C : \{1, \dots, 2n\} \to \{1, \dots, l\}. \tag{2.1}$$

For our model, we will use the function f_C associated to the colouring from (2.1). Further on, we define the specialisation of coefficients:

$$\psi^C_{q,N_1,\dots,N_l} : \mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_l^{\pm 1}, d^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}]$$
$$\begin{cases} \psi^C_{q,N_1,\dots,N_l}(x_i) = q^{N_i - 1}, \quad i \in \{1, \dots, l\} \\ \psi^C_{q,N_1,\dots,N_l}(y_i) = q^{N_i}, \\ \psi^C_{q,N_1,\dots,N_l}(d) = q^{-2}. \end{cases} \tag{2.2}$$

Notation 2.5. In the formulas from the paper, we denote by $C_i := N_{C(i)}$.

3. Definition of the local system and homology groups

In order to construct the classes that will lead to the 3-manifold invariants, we will use the homology of certain coverings of the configuration space in the punctured disc. The construction of the covering space will be more involved than the one used in [2]. More specifically, we will consider two types of punctures and use a subtle local system which counts the monodromies around these punctures in different manners.

For the following part, let us fix $\bar{l}, k \in \mathbb{N}$. Also, we consider a "weight" $m \in \mathbb{N}$. We start with the unordered configuration space of m points in the punctured disc with $n + 1 + 3\bar{l}$ punctures $\mathcal{D}_{n+1+3\bar{l}}$, denoted by

$$C_{n+1+3\bar{l},m}.$$

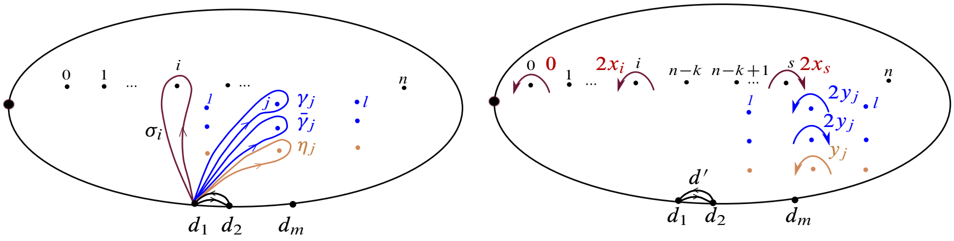

Figure 3.1. Local system.

Also, we fix

$$d_1, \ldots, d_m \in \partial \mathcal{D}_{n+1+3\bar{l}},$$

and let $\mathbf{d} = (d_1, \ldots, d_m)$ to be our base point in the configuration space. Now, we define a certain local system on this configuration space. For this, we use the homology of this configuration space, which has the following description.

Proposition 3.1. *Let us suppose that $m \geqslant 2$. Let us consider $[\,] : \pi_1(C_{n+1+3\bar{l},m}) \to H_1(C_{n+1+3\bar{l},m})$ be the abelianisation map. Then, the homology has the following form:*

$$H_1(C_{n+1+3\bar{l},m}) \simeq \underset{\langle[\sigma_i]\rangle}{\mathbb{Z}^{n+1}} \oplus \underset{\langle[\gamma_j],[\bar{\gamma}_j]\rangle}{\mathbb{Z}^{2\bar{l}}} \oplus \underset{\langle[\eta_j]\rangle}{\mathbb{Z}^{\bar{l}}} \oplus \underset{\langle[\delta]\rangle}{\mathbb{Z}}, \quad i \in \{0, \ldots, n+1\},\ j \in \{1, \ldots, \bar{l}\}.$$

The five types of generators are presented in Figure 3.1.

We continue with the augmentation map

$$\varepsilon : H_1(C_{n+1+3l,m}) \to \underset{\langle x_i\rangle}{\mathbb{Z}^n} \oplus \underset{\langle y_j\rangle}{\mathbb{Z}^l} \oplus \underset{\langle d'\rangle}{\mathbb{Z}}$$

given by

$$\begin{cases} \varepsilon(\sigma_0) = 0, \\ \varepsilon(\sigma_i) = 2x_i, & i \in \{1, \ldots, n-k\}, \\ \varepsilon(\sigma_i) = -2x_i, & i \in \{n-k+1, \ldots, n\}, \\ \varepsilon(\gamma_j) = 2y_j, & j \in \{1, \ldots, \bar{l}\}, \\ \varepsilon(\bar{\gamma}_j) = -2y_j, & j \in \{1, \ldots, \bar{l}\}, \\ \varepsilon(\eta_j) = -y_j, & j \in \{1, \ldots, \bar{l}\}, \\ \varepsilon(\delta) = d'. \end{cases}$$

Definition 3.2 (Local system). We use the local system given by the composition of the above morphisms:

$$\Phi : \pi_1(C_{n+1+3\bar{l},m}) \to \mathbb{Z}^n \oplus \mathbb{Z}^{\bar{l}} \oplus \mathbb{Z}$$
$$\langle x_j \rangle \ \langle y_j \rangle \ \langle d' \rangle, \ i \in \{1, \ldots, n\}, \ j \in \{1, \ldots, \bar{l}\}$$
$$\Phi = \varepsilon \circ [\,].$$

Definition 3.3 (Covering of the configuration space). Let $\widetilde{C}^{-k}_{n+1+3\bar{l},m}$ be the covering of $C_{n+1+3\bar{l},m}$ corresponding to the local system Φ.

Also, let us fix a base point $\tilde{\mathbf{d}} \in \widetilde{C}^{-k}_{n+1+3\bar{l},m}$ in the fibre over the base point $\mathbf{d}$.

3.1. Input of the construction

We will use the homologies of this covering space.

Remark 3.4. The morphism Φ is not surjective, and its image is the following:

$$\mathrm{Im}(\Phi) = (2\mathbb{Z})^n \oplus \mathbb{Z}^{2\bar{l}} \oplus \mathbb{Z}^{\bar{l}} \oplus \mathbb{Z} \subseteq \mathbb{Z}^n \oplus \mathbb{Z}^{2\bar{l}} \oplus \mathbb{Z}^{\bar{l}} \oplus \mathbb{Z}.$$

So, the group of deck transformations of the covering $\widetilde{C}^{-k}_{n+1+3\bar{l},m}$ is

$$(2\mathbb{Z})^n \oplus \mathbb{Z}^{2\bar{l}} \oplus \mathbb{Z}^{\bar{l}} \oplus \mathbb{Z},$$

and the homology of the covering is given by modules over its group ring:

$$\mathbb{Z}[x_1^{\pm 2}, \ldots, x_n^{\pm 2}, y_1^{\pm 1}, \ldots, y_{\bar{l}}^{\pm 1}, d'^{\pm 1}].$$

We consider the inclusion:

$$\iota : \mathbb{Z}[x_1^{\pm 2}, \ldots, x_n^{\pm 2}, y_1^{\pm 1}, \ldots, y_{\bar{l}}^{\pm 1}, d'^{\pm 1}] \subseteq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_{\bar{l}}^{\pm 1}, d'^{\pm 1}],$$

and from now on, we work with the homology of the covering tensored over ι with the group ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_{\bar{l}}^{\pm 1}, d'^{\pm 1}]$. So, these homology groups become modules over

$$\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_{\bar{l}}^{\pm 1}, d'^{\pm 1}].$$

Also, for computational purposes, we will use the variable $d := -d'$. We recall that our intersection pairing involves homology classes that are given by submanifolds in configuration spaces. In the next part, we will use a property that allows one to encode the sign of the geometric intersections in the configuration space by signs of the local intersections in the punctured disc, if we replace the variable d' by $-d$ in our formulas (see [2, Remark 3.4.3] and [4, Section 3] for a detailed explanation of this sign computation).

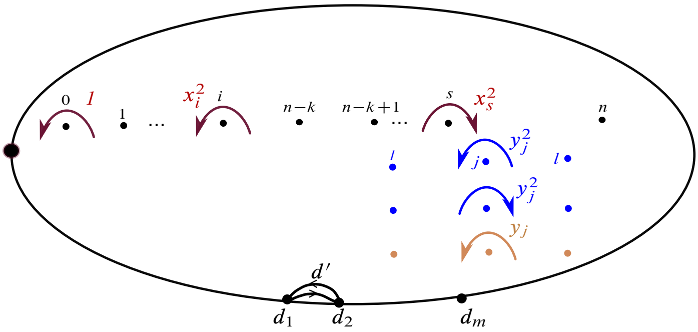

Figure 3.2. Local system via the group ring.

We define the following:

$$\gamma : \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d'^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]$$

$$\begin{cases} \gamma(x_i) = x_i \\ \gamma(y_j) = y_j \\ \gamma(d') = -d. \end{cases}$$

Using this notation, the homology groups of the covering become modules over

$$\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]. \tag{3.1}$$

Remark 3.5. Since we passed from groups to group rings, in the multiplicative notation, the local system evaluates the loops using the monodromies from Figure 3.2. The five types of generators are presented in Figure 3.2.

3.2. Homology groups

We will use a version of the relative homology of this covering space. First, let us fix $S^- \subseteq \partial \mathcal{D}_{n+1+3\bar{l}}$ to be the semicircle on the boundary of the punctured disc given by points with negative x-coordinate. Also, let $w \in S^- \subseteq \partial \mathcal{D}_{n+1+3\bar{l}}$ be a point on the boundary of the disc. We will use part of the Borel–Moore homology of the covering space $\tilde{C}^{-k}_{n+1+3\bar{l},m}$ which comes from the Borel–Moore homology of the base space $C_{n+1+3\bar{l},m}$ which is twisted by the local system.

Let us define C^- to be the part of the boundary of the configuration space $C_{n+1+3\bar{l},m}$ which is given by configurations of points where at least one of them belongs to the set S^-. Also, let P^- be the part of the boundary of $\tilde{C}^{-k}_{n+1+3\bar{l},m}$ which is given by the fibre over the subset C^-.

Notation 3.6. We will split the infinity part of the configuration space into two subsets. The complete definition and details of this construction are presented in [3, Remark 7.5], for the case where the surface that we use is the closed 2-disc minus half of its boundary and minus $n+1+3\bar{l}$ open discs (with pairwise disjoint closures) from its interior. Then, we consider two homology groups, as follows:

- Let $H_m^{lf,\infty,-}(\widetilde{C}^{-k}_{n+1+3\bar{l},m}, P^-;\mathbb{Z})$ be the homology relative to part of the infinity which is given by the open boundary of $\widetilde{C}^{-k}_{n+1+3\bar{l},m}$ containing the configurations that project in the base space to a multipoint which touches a puncture from the punctured disc and also relative to the boundary part P^-.
- Let $H_m^{lf,\Delta}(\widetilde{C}^{-k}_{n+1+3\bar{l},m}, \partial;\mathbb{Z})$ be the homology which is relative to the boundary of the covering $\widetilde{C}^{-k}_{n+1+3\bar{l},m}$ which is not in P^- and Borel–Moore with respect to collisions of points in the configuration space.

Remark 3.7. In general, the Borel–Moore homology of a covering space is different from the twisted Borel–Moore homology of the base space. In our situation, we will work with the homology of the covering space rather than the twisted homology of the base space, and for this, we will use the following properties.

Proposition 3.8 ([3, Theorem E]). *There are natural injective maps:*

$$\iota : H_m^{lf,\infty,-}(C_{n+1+3\bar{l},m}, C^-;\mathcal{L}_\Phi) \to H_m^{lf,\infty,-}(\widetilde{C}^{-k}_{n+1+3\bar{l},m}, P^-;\mathbb{Z}),$$
$$\iota^\partial : H_m^{lf,\Delta}(C_{n+1+3\bar{l},m}, \partial;\mathcal{L}_\Phi) \to H_m^{lf,\Delta}(\widetilde{C}^{-k}_{n+1+3\bar{l},m}, \partial;\mathbb{Z}),$$

where $\mathcal{L}_\Phi$ is the rank 1 *local system associated to* Φ [3, Definition 2.7].

In the next part, we use the subspaces of these homologies of the covering which come from the twisted homology of the base configuration space, using Proposition 3.8.

Definition 3.9 (Homology groups). We consider the submodules in the homologies of the covering space (which, as above, are modules over $\mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1},y_1^{\pm 1},\dots,y_{\bar{l}}^{\pm 1}, d^{\pm 1}]$):

- $\mathcal{H}^{-k}_{n+1,m,\bar{l}} \subseteq H_m^{lf,\infty,-}(\widetilde{C}^{-k}_{n+1+3\bar{l},m}, P^{-1};\mathbb{Z})$ and
- $\mathcal{H}^{-k,\partial}_{n+1,m,\bar{l}} \subseteq H_m^{lf,\Delta}(\widetilde{C}^{-k}_{n+1+3\bar{l},m}, \partial;\mathbb{Z})$

given by the images of the homology with twisted coefficients in the homology of the covering space via the inclusions ι and ι^∂ respectively.

The definition of these homology groups is rather subtle, but for our situation we will work with very precise classes given by submanifolds in the configuration space.

Moreover, in the next part we will see that there is a geometric intersection pairing between these homologies of the covering space with respect to different parts of its boundary. This geometric pairing comes from a Poincaré–Lefschetz-type duality for twisted homology [3, Proposition 3.2] combined with a relative cap product for twisted homology [3, Lemma 3.3]. They induce an intersection pairing at the level of these homology groups, following [3].

Proposition 3.10 ([3, Proposition 7.6]). *There exists a topological intersection pairing:*

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathcal{H}^{-k}_{n+1,m,\bar{l}} \otimes \mathcal{H}^{-k,\partial}_{n+1,m,\bar{l}} \to \mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}, y_1^{\pm 1},\dots,y_{\bar{l}}^{\pm 1}, d^{\pm 1}].$$

3.3. Computing the intersection pairing

In the next section, we will use the precise form of this intersection pairing, so we will briefly explain its formula, which is presented in [3, Section 7]. We will work with the coefficients that belong to the group ring from (3.1). For this, we introduce the following notation.

Notation 3.11. Let $\tilde{\Phi}$ be the morphism induced by the local system Φ, which takes values in the group ring of $\mathbb{Z}^n \oplus \mathbb{Z}^{\bar{l}} \oplus \mathbb{Z}$:

$$\tilde{\Phi} : \pi_1(C_{n+1+3\bar{l},m}) \to \mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}, y_1^{\pm 1},\dots,y_{\bar{l}}^{\pm 1}, d'^{\pm 1}].$$

Then, taking into account the change of variables γ, we define

$$\bar{\Phi} : \pi_1(C_{n+1+3\bar{l},m}) \to \mathbb{Z}[x_1^{\pm 1},\dots,x_n^{\pm 1}, y_1^{\pm 1},\dots,y_{\bar{l}}^{\pm 1}, d^{\pm 1}],$$
$$\bar{\Phi} = \gamma \circ \tilde{\Phi}.$$

This morphism will be used for the computation of the intersection pairing, as follows. Let us consider two classes $H_1 \in \mathcal{H}^{-k}_{n+1,m,\bar{l}}$ and $H_2 \in \mathcal{H}^{-k,\partial}_{n+1,m,\bar{l}}$. We suppose that these classes are given by the lifts $\tilde{X}_1, \tilde{X}_2$ of two immersed submanifolds $X_1, X_2 \subseteq C_{n+1+3\bar{l},m}$. Also, we assume that X_1 and X_2 have a transverse intersection, in a finite number of points.

Proposition 3.12 (Intersection pairing from intersections in the base space and the local system). *For each intersection point $x \in X_1 \cap X_2$, we define a certain loop and denote it by $l_x \subseteq C_{n+1+3\bar{l},m}$.*

(a) *Construction of l_x*

We suppose that we have the paths $\gamma_{X_1}, \gamma_{X_2}$ which start in **d**, *they end on X_1, X_2 respectively and that $\tilde{\gamma}_{X_1}(1) \in \tilde{X}_1$ and $\tilde{\gamma}_{X_2}(1) \in \tilde{X}_2$. Further on, we choose two paths $\delta_{X_1}, \delta_{X_2} : [0,1] \to C_{n+3\bar{l},m}$ with the property:*

$$\begin{cases} \mathrm{Im}(\delta_{X_1}) \subseteq X_1; \delta_{X_1}(0) = \gamma_{X_1}(1); \quad \delta_{X_1}(1) = x, \\ \mathrm{Im}(\delta_{X_2}) \subseteq X_2; \delta_{X_2}(0) = \gamma_{X_2}(1); \quad \delta_{x_2}(1) = x. \end{cases}$$

The composition of these paths gives us the loop:

$$l_x = \gamma_{X_1} \circ \delta_{X_1} \circ \delta_{X_2}^{-1} \circ \gamma_{X_2}^{-1}.$$

Also, let α_x be the sign of the geometric intersection between M_1 and M_2 in the base configuration space at the point x.

(b) *Intersection form*

Then, the intersection pairing can be computed from the set of loops l_x and the local system:

$$\langle\!\langle H_1, H_2 \rangle\!\rangle = \gamma\Big(\sum_{x \in X_1 \cap X_2} \alpha_x \cdot \Phi(l_x)\Big) \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]. \quad (3.2)$$

Remark 3.13 (Our situation). For actual computations, in the case where the homology classes come from products of one-dimensional submanifolds quotiented in the configuration space, one can directly compute the previous intersection from a sum without the change of coefficients given by γ. More precisely, in this case, we will replace the local system Φ by $\bar{\Phi}$ in the previous formula, and for the sign contribution, count just the product of local orientations in the disc around each component of the intersection point x (instead of keeping track of the sign of orientations in the configuration space α_x).

We will see such an example of such computation in Section 5.2.

3.4. Specialisations given by colourings

Definition 3.14 (Change of coefficients). For the next part, we suppose that we have a colouring C of the n punctures of the disc $\{1, \dots, n\}$ into l colours:

$$C : \{1, \dots, n\} \to \{1, \dots, l\}.$$

We will work in the situation where

$$\bar{l} = 0 \quad \text{or} \quad \bar{l} = l.$$

Let us fix $\bar{l}$ components $\bar{p}_1, \dots, \bar{p}_{\bar{l}} \in \{1, \dots, n\}$.

In our situation, the $3\bar{l}$ blue points will be split into groups of 3 points, each group being positioned underneath a puncture labelled by $\bar{p}_i$, for $i \in \{1, \dots, \bar{l}\}$. Then, the colouring C induces a colouring of the $3\bar{l}$ blue points.

We define the corresponding change of variables, where we change the first $n + \bar{l}$ components

$$x_1, \dots, x_n, y_1, \dots, y_{\bar{l}}$$

from the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]$ to $l + \bar{l}$ variables, denoted by $x_1, \dots, x_l, y_1, \dots, y_{\bar{l}}$, as follows:

$$f_C : \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]$$
$$\begin{cases} f_C(x_i) = x_{C(i)}, & i \in \{1, \dots, n\}, \\ f_C(y_j) = y_{C(\bar{p}_j)}, & j \in \{1, \dots, \bar{l}\}. \end{cases} \tag{3.3}$$

Now, we will change the coefficients of the homology groups using the function f_C.

Definition 3.15 (Homology groups). Let us define the homologies which correspond to these coefficients, given by

- $H^{-k}_{n+1,m,\bar{l}} := \mathcal{H}^{-k}_{n+1,m,\bar{l}}|_{f_C}$,
- $H^{-k,\partial}_{n+1,m,\bar{l}} := \mathcal{H}^{-k,\partial}_{n+1,m,\bar{l}}|_{f_C}$.

They are modules over $\mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]$.

Now, we look at braids with $n + 1 + 3\bar{l}$ strands whose action on the punctured disc preserves the colouring of the punctures (given by C). We denote the set of such braids by $B^C_{n+1+3\bar{l}}$.

Proposition 3.16 ([3]). *There is a braid group action (which comes from the mapping class group action) which is compatible with the action of deck transformations at the homological level:*

$$B^C_{n+1+3\bar{l}} \curvearrowright H^{-k}_{n+1,m,\bar{l}} \quad (\textit{as a module over } \mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}]).$$

Proposition 3.17 ([3]). *There is also a topological intersection pairing:*

$$\langle\, , \rangle : H^{-k}_{n+1,m,\bar{l}} \otimes H^{-k,\partial}_{n+1,m,\bar{l}} \to \mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}],$$

whose method of computation is the same as the one presented in Proposition 3.12, specialised using the change of coefficients f_C:

$$\langle\, , \rangle = \langle\!\langle\, , \rangle\!\rangle|_{f_C}.$$

Definition 3.18 (Specialisation of coefficients). Let $N_1, \dots, N_l \in \mathbb{N}$ a sequence of natural numbers. We define the specialisation of coefficients given by

$$\psi^C_{q,N_1,\dots,N_l} : \mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_{\bar{l}}^{\pm 1}, d^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}]$$
$$\begin{cases} \psi^C_{q,N_1,\dots,N_l}(x_i) = q^{N_i - 1}, & i \in \{1, \dots, l\}, \\ \psi^C_{q,N_1,\dots,N_l}(y_i) = q^{N_i}, & i \in \{1, \dots, \bar{l}\}, \\ \psi^C_{q,N_1,\dots,N_l}(d) = q^{-2}. \end{cases} \tag{3.4}$$

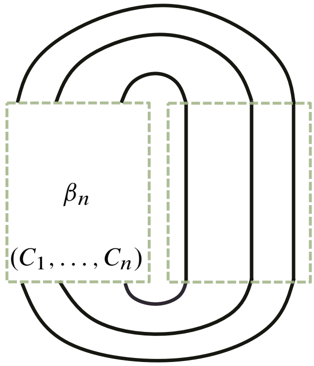

Figure 4.1. Colouring of the braid.

4. Coloured Jones polynomials for framed links

In this section, we show a topological intersection formula for coloured Jones polynomials for links whose components are coloured with different colours.

Let us start with $L = K_1 \cup \cdots \cup K_l$, a framed oriented link with framings $f_1, \ldots, f_l \in \mathbb{Z}$. Let us choose $\beta_n \in B_n$ a braid such that $L = \widehat{\beta_n}$. We also fix a set of colours $N_1, \ldots, N_l \in \mathbb{N}$.

Notation 4.1. For a natural number $M \in \mathbb{N}$, we denote by V_M the M-dimensional representation of the quantum group $U_q(\mathrm{sl}(2))$.

We colour the components of the link L with the representations $V_{N_1}, \ldots, V_{N_l}$ and denote the coloured Jones polynomial of this framed link by $J_{N_1,\ldots,N_l}(L, q)$ (as in [16]). Also, for the further notations, we consider

$$\bar{N} := (N_1, \ldots, N_l).$$

Definition 4.2 (Induced colourings). (a) Colourings of the braid. The colouring of the link given by $\bar{N}$ induces a colouring of the strands of the braid, and we denote the corresponding colours by $(C_1, \ldots, C_n)$, as presented in Figure 4.1.

Now, we look at the link as the closure of the braid β_n together with n straight strands, and so, we have an associated colouring of $2n$ points $C : \{1, \ldots, 2n\} \to \{1, \ldots, l\}$. This means that we have the following colours on the $2n$ points:

$$\bar{C}^{\bar{N}} := (C_1, \ldots, C_n, C_n, \ldots, C_1).$$

(b) Set of states. We consider the following indexing set:

$$C(\bar{N}) := \left\{\bar{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n \mid 0 \leqslant i_k \leqslant C_k - 1, \forall k \in \{1, \ldots, n\}\right\}.$$

4.1. Homology classes

Now that we have the induced colouring of the braid and the corresponding indexing set $C(\bar{N})$, we can present the homology groups that we will use. More specifically, we will use the configuration space of $1+\sum_{i=1}^{n}(C_i-1)$ points on the $(2n+1)$-punctured disc. Then, we consider the covering coming from the local system Φ associated to the parameters:

$$n \to 2n; \quad m \to 1+\sum_{i=1}^{n}(C_i-1); \quad \bar{l} \to 0; \quad k \to -n.$$

We use the corresponding homology groups

$$H^{-n}_{2n+1,1+\sum_{i=1}^{n}(C_i-1),0} \quad \text{and} \quad H^{-n,\partial}_{2n+1,1+\sum_{i=1}^{n}(C_i-1),0}.$$

For the following part, since the third component is zero, we will just erase it from the indices of the homology groups. Now, we are ready to define the homology classes that will be used in the intersection model. The classes will be prescribed by a couple given by the following:

- A *geometric support*, meaning a *set of arcs in the punctured disc*. The image of the product of these arcs in the configuration space gives us a submanifold which has half of the dimension of the configuration space.
- A set of *paths to the base point*, which start in the base points from the punctured disc and end on these curves. The set of these paths gives a path in the configuration space, from $\mathbf{d}$ to the submanifold mentioned above.

Then, we lift the path to a path in the covering space, starting from $\tilde{\mathbf{d}}$, and then we lift the submanifold through the end point of this path. The detailed construction of such homology classes is presented in [2, Section 5].

Definition 4.3 (Homology classes). For any set of indices $\bar{i}=(i_1,\ldots,i_n)\in C(\bar{N})$, we define two homology classes, given by the geometric supports from Figure 4.2.

Proposition 4.4. *The geometric support for the dual class, shown in the right-hand side of Figure 4.2, leads to a well-defined homology class in the covering.*

Proof. We notice that the local system $\bar{\Phi}$ counts the monodromy around the symmetric points $(i, 2n+1-i)$ with opposite signs. Also, the relative winding of configurations of C_i-1 particles on a loop around a figure eight vanishes, as shown in Figure 4.3. This means that the configuration space on a figure eight around these punctures lifts to a submanifold in the covering. On the other hand, the local system Φ is chosen to have trivial monodromy around the puncture labelled by 0, so the circle which goes around this has a well-defined lift. Using this argument for all the

$$\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n}_{2n+1,1+\sum_{i=1}^{n}}(C_i-1) \quad \text{and} \quad \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n,\partial}_{2n+1,1+\sum_{i=1}^{n}(C_i-1)}$$

↓ lifts

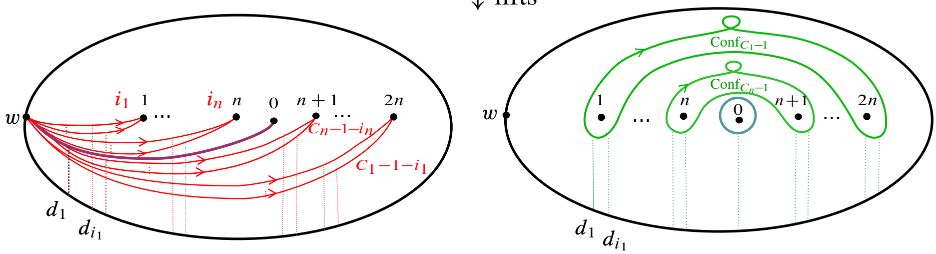

Figure 4.2. Lagrangians for link invariants coloured with different colours.

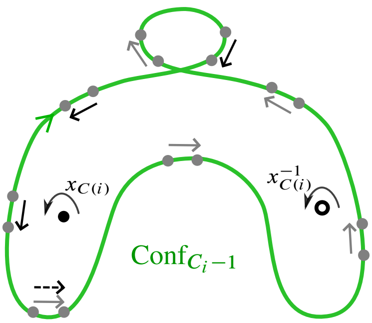

Figure 4.3. Monodromy of configurations on a figure eight.

figure eights and the little circle around the puncture 0, we conclude that we have a well-defined lift which gives a homology class in the covering

$$\mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n,\partial}_{2n+1,1+\sum_{i=1}^{n}(C_i-1)}. \qquad \blacksquare$$

In the next part, we use the specialisation of coefficients:

$$\psi^{C}_{q,N_1,\dots,N_l} : \mathbb{Z}[x_1^{\pm 1},\dots,x_l^{\pm 1},d^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}]$$
$$\begin{cases} \psi^{C}_{q,N_1,\dots,N_l}(x_i) = q^{N_i-1}, & i \in \{1,\dots,l\}, \\ \psi^{C}_{q,N_1,\dots,N_l}(d) = q^{-2}. \end{cases}$$

4.2. Intersection model

Now, we show that the coloured Jones polynomial of a link coloured with the colours $N_1,\dots,N_l$ can be obtained from an intersection pairing which uses the classes

$$\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)} \quad \text{and} \quad \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \quad \text{for all } \bar{i} \in C(\bar{N}),$$

as stated in Theorem 1.3. We recall the formula below.

Theorem 4.5 (Topological state sum model for coloured Jones polynomials for coloured links). *We have the following intersection model for the coloured Jones polynomials of a link L:*

$$J_{N_1,\dots,N_l}(L,q) = q^{\sum_{i=1}^{l}(f_i-\sum_{j\neq i} lk_{i,j})(N_i-1)}$$
$$\cdot \Bigg(\sum_{\bar{i}\in C(\bar{N})} \Big(\prod_{i=1}^{n} x_{C(i)}^{-1}\Big) \cdot \big\langle (\beta_n \cup \mathbb{I}_{n+1})\, \mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \big\rangle \Bigg)\Bigg|_{\psi^C_{q,N_1,\dots,N_l}}.$$

In this formula, we denote by $(lk_{i,j})_{i,j\in\{1,\dots,l\}}$ *the linking matrix of the link L.*

Proof. The proof of this intersection formula is a generalisation of the strategy used in the model for coloured Jones polynomials coloured with the same colour, presented in [2], based on arguments from [1]. We outline the main steps as follows.

Step 1. We consider the homology classes

$$\bar{\mathcal{F}}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n}_{2n,\sum_{i=1}^{n}(C_i-1)} \quad \text{and} \quad \bar{\mathcal{L}}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n,\partial}_{2n,\sum_{i=1}^{n}(C_i-1)}$$

which have the same geometric support as the classes $\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}$ and $\mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)}$ except that we remove the 1-dimensional part which is supported around the puncture labelled by 0, namely, the purple segment and the blue circle (see a similar argument in [2, Step 2, Section 6]). Then, we have that

$$\big\langle (\beta_n \cup \mathbb{I}_{n+1})\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \big\rangle = d^{-\sum_{k=1}^{n} i_k} \big\langle (\beta_n \cup \mathbb{I}_n)\bar{\mathcal{F}}_{\bar{i}}^{(C_1,\dots,C_n)}, \bar{\mathcal{L}}_{\bar{i}}^{(C_1,\dots,C_n)} \big\rangle.$$

This means that we want to prove the following:

$$J_{N_1,\dots,N_l}(L,q) = q^{\sum_{i=1}^{l}(f_i-\sum_{j\neq i} lk_{i,j})(N_i-1)}$$
$$\cdot \Bigg(\sum_{\bar{i}\in C(\bar{N})} \prod_{i=1}^{n} x_{C(i)}^{-1} \cdot d^{-\sum_{k=1}^{n} i_k} \big\langle (\beta_n \cup \mathbb{I}_n)\, \bar{\mathcal{F}}_{\bar{i}}^{(C_1,\dots,C_n)}, \bar{\mathcal{L}}_{\bar{i}}^{(C_1,\dots,C_n)} \big\rangle \Bigg)\Bigg|_{\psi^C_{q,N_1,\dots,N_l}}.$$

Step 2. For the next part of the proof, we follow step by step the correspondence to the Reshetikhin–Turaev definition of the coloured Jones polynomials. More specifically, the caps of the diagram correspond to the sum of the classes $\bar{\mathcal{F}}_{\bar{i}}^{(C_1,\dots,C_n)}$ over all $\bar{i} \in C(\bar{N})$. Further on, the braid action on the quantum side and on the homological side correspond, using the identification due to Martel [10].

Step 3. In the end, the caps of the diagram require that, after the braid group action, we evaluate just the components which are symmetric with respect to the middle of the disc. More precisely, this means that the indices corresponding to the points k and $2n+1-k$ should sum up to the colour $C_k - 1$. This is encoded geometrically by the intersection with the dual class $\bar{\mathcal{L}}_{\bar{i}}^{(C_1,\dots,C_n)}$.

On the algebraic side, one should also encode an extra coefficient which corresponds to the caps of the diagram. We refer to the details of the argument for a single colour as they are presented in [1, Section 5 and Section 7], except that here we have a different local system. The fact that the change of the local system does not affect the flow of the proof follows by a similar computation as the one from [2, Step 3, Section 6]. The main points are as follows.

This coefficient is given by the pivotal structure, more specifically by the action of the element K^{-1} from the quantum group. Now, for a set of indices $i_1, \ldots, i_n$, the K^{-1} action on the corresponding tensor monomial is given by

$$q^{-\sum_{k=1}^{n}((C_k-1)-2i_k)} = \left(\prod_{k=1}^{n} q^{-(C_k-1)}\right) \cdot q^{\sum_{k=1}^{n} 2i_k}.$$

This coefficient is precisely the specialisation:

$$\psi^{C}_{N_1,\ldots,N_l}\left(\left(\prod_{i=1}^{n} x^{-1}_{C(i)}\right) \cdot d^{-\sum_{k=1}^{n} i_k}\right).$$

The remaining coefficient which appears in the formula comes from the framing contribution of the components of the link L. ■

5. WRT from intersections in configuration spaces

In this part, we pass towards invariants for 3-manifolds and aim to construct the intersection model for the Witten–Reshetikhin–Turaev invariants, as presented in Theorem 1.6. Let us fix a level $\mathcal{N} \in \mathbb{N}$. As in the previous section, we start with a framed oriented link with l components, which is the closure of a braid with n strands.

Definition 5.1 (Choice of l points). Let us choose l strands of the braid β_n, which all belong to different components of the link, and denote their indices by $p_1, \ldots, p_l$. Also, we look at the $2n+1$ punctured disc and denote the symmetric images of these points with respect to the middle axis by $\bar{p}_1, \ldots, \bar{p}_l$.

This time we will use the homology of the covering of the configuration space of $n(\mathcal{N}-2)+l+1$ particles in the punctured disc with $2n+3l+1$ punctures, associated to the parameters

$$n \to 2n+1; \quad m \to n(\mathcal{N}-2)+l+1; \quad l = \bar{l}; \quad k \to -n.$$

More precisely, we will work with the homology groups

$$H^{-n}_{2n+1,n(\mathcal{N}-2)+l+1,l} \quad \text{and} \quad H^{-n,\partial}_{2n+1,n(\mathcal{N}-2)+l+1,l}.$$

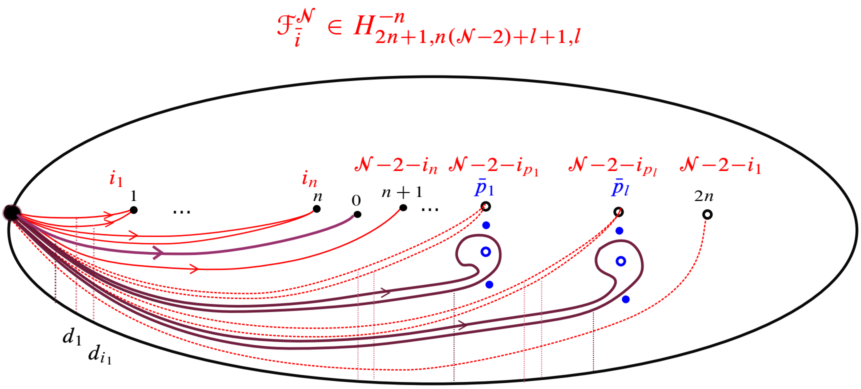

Figure 5.1. Homology classes for level $\mathcal{N}$ WRT invariants.

On the picture, we consider $3l$ blue punctures in the punctured disc such that they are split into triples which lie below the privileged punctures $\bar{p}_1, \ldots, \bar{p}_l$, as in Figure 5.1. Now, we are ready to define the main tools in our construction, which are the homology classes in the above homologies.

5.1. Homology classes

Definition 5.2 (First homology class). For a set of indices $i_1, \ldots, i_n \in \{0, \ldots, \mathcal{N}-2\}$, we denote $\bar{i} := (i_1, \ldots, i_n)$, and we consider the class given by the geometric support from Figure 5.1.

Remark 5.3. When we take one of the circles from the above picture, its lift has a non-trivial monodromy, so this corresponds to an arc which starts and ends in the fibre over w. This shows that the lift of the geometric support from Figure 5.1 will lead to a well-defined homology class in the homology relative to the fibre P^{-1}.

Definition 5.4 (Second homology class). Also for each choice of indices $i_1, \ldots, i_n \in \{0, \ldots, \mathcal{N}-2\}$, we consider the geometric support given by the product of configuration spaces on the figure eights (and of the circle from the middle of the disc) from Figure 5.2. We define the associated homology class as shown in Figure 5.2.

Proposition 5.5. *The homology class $\mathcal{L}_{\bar{i}}^{\mathcal{N}}$ is well defined.*

Proof. First of all, the figure eights from the above picture have trivial monodromy around the punctures. This comes from the fact that the local system has opposite monodromies around the symmetric points of the punctured disc. Also, it has opposite monodromies around two blue punctures which are displayed on the vertical direction (and lie in one of the discs bounded by a figure eight). Secondly, since we are working

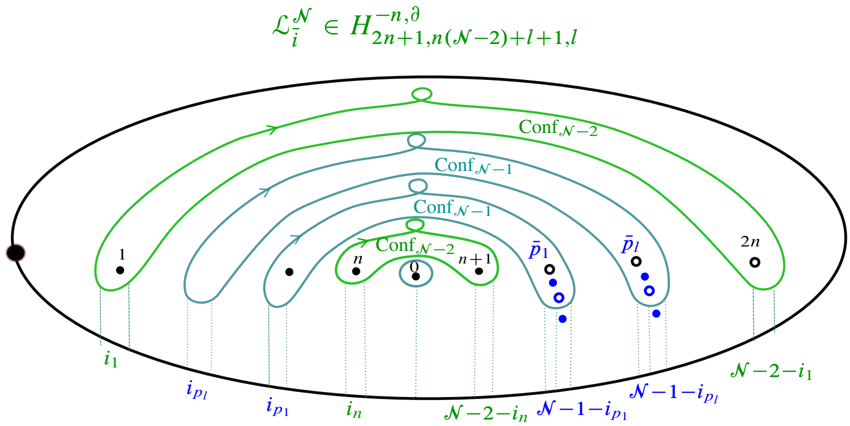

Figure 5.2. Dual homology classes for level $\mathcal{N}$ WRT invariants.

on a figure eight, the relative winding of configurations of $\mathcal{N}-1$ (or $\mathcal{N}-2$) particles vanishes when we do a loop around this figure eight, as shown in Figure 5.3.

On the other hand, we recall that Φ is chosen to have trivial monodromy around the puncture labelled by 0, so the circle which goes around this has a well-defined lift. Using this argument for all the configurations on figure eights and the little circle around the puncture 0, we conclude that we have a well-defined lift which gives the homology class $\mathscr{L}_{\bar{i}}^{\mathcal{N}}$. ■

We recall the definition of the specialisation of coefficients which is associated to this context:

$$\psi^{C}_{q,N_1,\dots,N_l} : \mathbb{Z}[x_1^{\pm 1},\dots,x_l^{\pm 1},y_1^{\pm 1},\dots,y_l^{\pm 1},d^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}]$$

$$\begin{cases} \psi^{C}_{q,N_1,\dots,N_l}(x_i) = q^{N_i-1}, & i \in \{1,\dots,l\}, \\ \psi^{C}_{q,N_1,\dots,N_l}(y_i) = q^{N_i}, \\ \psi^{C}_{q,N_1,\dots,N_l}(d) = q^{-2}. \end{cases}$$

5.2. WRT from intersections in configuration spaces

Definition 5.6 (Kirby colour). For $\mathcal{N} \in \mathbb{N}$, the Kirby colour corresponding to the quantum group $U_\xi(\mathrm{sl}(2))$ (see [16]) is given by

$$\Omega := \sum_{N=1}^{\mathcal{N}-1} q \dim(V_N) \cdot V_N = \sum_{N=1}^{\mathcal{N}-1} [N]_\xi \cdot V_N.$$

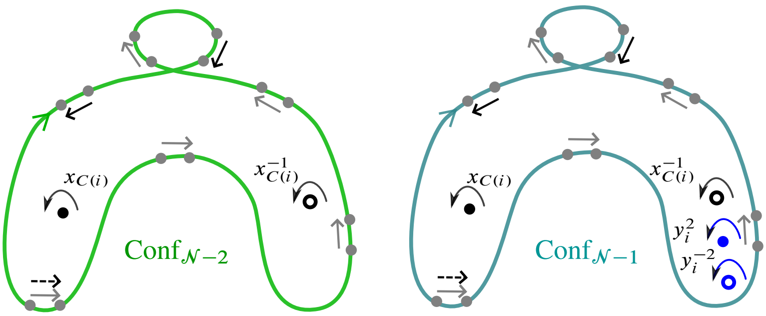

Figure 5.3. Monodromy of configurations on a figure eight.

Notation 5.7. We denote by b_+, b_-, and b the number of positive, negative, and zero eigenvalues of the linking matrix of L. Also, we consider

$$\begin{aligned}\Delta_+ &= J_\Omega(\mathcal{U}_+, \xi),\\ \Delta_- &= J_\Omega(\mathcal{U}_-, \xi),\\ \mathcal{D} &= |\Delta_+|,\end{aligned}$$

where $\mathcal{U}_+$ and $\mathcal{U}_-$ are the unknot with framing $+1$ and -1, respectively [16].

We will use the homological classes constructed above together with the specialisation of coefficients in order to prove the main result, which we recall below.

Theorem 5.8 (Topological state sum model for the Witten–Reshetikhin–Turaev quantum invariants). *Let M be a closed oriented 3-manifold and L a framed oriented link with l components such that M is obtained by surgery along L. Let us choose a braid β_n such that $L = \widehat{\beta_n}$. Now, for $i_1, \ldots, i_n \in \{0, \ldots, \mathcal{N}-2\}$, we consider the following Lagrangian intersection:*

$$\begin{cases}\Lambda_{\bar{i}}(\beta_n) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}, y_1^{\pm 1}, \ldots, y_l^{\pm 1}, d^{\pm 1}],\\ \Lambda_{\bar{i}}(\beta_n) := \prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i} - \sum_{j \neq p_i} lk_{p_i,j})} \cdot \prod_{i=1}^{n} x_{C(i)}^{-1} \langle (\beta_n \cup \mathbb{I}_{n+3l+1}) \, \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \rangle.\end{cases}$$

Then, the $\mathcal{N}$th Witten–Reshetikhin–Turaev invariant has the following model:

$$\tau_{\mathcal{N}}(M) = \frac{\{1\}_\xi^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{i_1,\ldots,i_n=0}^{\mathcal{N}-2} \left(\sum_{\substack{1 \leqslant N_1,\ldots,N_l \leqslant \mathcal{N}-1\\ \bar{i} \in C(N_1,\ldots,N_l)}} \Lambda_{\bar{i}}(\beta_n) \Big|_{\psi^C_{\xi, N_1,\ldots,N_l}} \right).$$

Proof. The Witten–Reshetikhin–Turaev quantum invariant at level $\mathcal{N}$ is defined using the coloured Jones polynomials of the link L whose components are coloured with the Kirby colour Ω (see [11, 16]):

$$\tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot J_{\Omega,\ldots,\Omega}(L,\xi).$$

This means that the invariant is given by the following linear combination of coloured Jones polynomials, with colours less than $\mathcal{N}-1$:

$$\tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{1\leqslant N_1,\ldots,N_l\leqslant \mathcal{N}-1} [N_1]_\xi \cdot \ldots \cdot [N_l]_\xi \cdot J_{N_1,\ldots,N_l}(L,\xi). \quad \blacksquare$$

Step I (WRT invariant as a sum of intersections in various configuration spaces)

Now, we remind the topological formula for the coloured Jones polynomials, which is presented in Theorem 1.3:

$$J_{N_1,\ldots,N_l}(L,q) = q^{\sum_{i=1}^{l}(f_i-\sum_{j\neq i} lk_{i,j})(N_i-1)}$$
$$\cdot \left(\sum_{\bar{i}\in C(\bar{N})} \left(\prod_{i=1}^{n} x_{C(i)}^{-1}\right) \cdot \left\langle (\beta_n \cup \mathbb{I}_{n+1})\, \mathcal{F}_{\bar{i}}^{(C_1,\ldots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\ldots,C_n)} \right\rangle \right)\Bigg|_{\psi^C_{q,N_1,\ldots,N_l}}.$$

We notice that the variables $x_{C(p_1)}, \ldots, x_{C(p_l)}$ correspond to the special strands of the braid $p_1, \ldots, p_l$ which are all associated to different components of the link. More precisely, we have that

$$(C_{p_1}, \ldots, C_{p_l}) = (N_1, \ldots, N_l)$$

as unordered families. We recall the notation $C_{p_i} = N_{C(p_i)}$. Further on, the variables are specialised in the following manner:

$$\psi^C_{q,N_1,\ldots,N_l}(x_{C(p_i)}) = q^{N_{C(p_i)}-1} = q^{C_{p_i}-1}, \quad \forall\, i \in \{1,\ldots,l\}.$$

This remark allows us to encode the framing correction, and we obtain the following formula:

$$J_{N_1,\ldots,N_l}(L,q) = \left(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i}-\sum_{j\neq p_i} lk_{p_i,j})} \prod_{i=1}^{n} x_{C(i)}^{-1} \right.$$
$$\left. \cdot \sum_{\bar{i}\in C(\bar{N})} \left\langle (\beta_n \cup \mathbb{I}_{n+1})\, \mathcal{F}_{\bar{i}}^{(C_1,\ldots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\ldots,C_n)} \right\rangle \right)\Bigg|_{\psi^C_{q,N_1,\ldots,N_l}}$$

(here, we used the notations from the statement of Theorem 1.6 for the framings).

This means that the 3-manifold invariant is given by the expression presented below:

$$\tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{1 \leqslant N_1,\dots,N_l \leqslant \mathcal{N}-1} [N_1]_\xi \cdot \dots \cdot [N_l]_\xi \\ \cdot \left(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i} - \sum_{j \neq p_i} lk_{p_i,j})} \prod_{i=1}^{n} x_{C(i)}^{-1} \cdot \sum_{\bar{i} \in C(\bar{N})} \left\langle (\beta_n \cup \mathbb{I}_{n+1}) \right. \right. \\ \left. \left. \cdot \mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \right\rangle \right) \Bigg|_{\psi^C_{\xi,N_1,\dots,N_l}}. \tag{5.1}$$

Step II (Construction of homology classes in a fixed configuration space)

In this part, we focus on each intersection pairing which occurs in the above formula. Such intersection is given by the following expression:

$$\left\langle (\beta_n \cup \mathbb{I}_{n+1})\, \mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}, \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \right\rangle.$$

This pairing comes from an intersection in the configuration space of $1 + \sum_{i=1}^{n}(C_i - 1)$ points in the $(2n+1)$-punctured disc, and the homology classes belong to the homology groups:

$$\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n}_{2n+1,1+\sum_{i=1}^{n}(C_i-1)}; \quad \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)} \in H^{-n,\partial}_{2n+1,1+\sum_{i=1}^{n}(C_i-1)}.$$

We would like to arrive at an intersection in a configuration space where the number of particles does not depend on the individual components given by the set $(C_1,\dots,C_n)$.

In order to achieve this, we use the property that all components of this set are bounded by the level of the 3-manifold invariant; more precisely, we have that

$$0 \leqslant i_k \leqslant C_k - 1 \leqslant \mathcal{N} - 2, \quad \forall\, k \in \{1,\dots,n\}.$$

Now, let us investigate the geometric supports of the two classes. For any k, we remark that the geometric support of $\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)}$ has

- i_k curves ending in the kth puncture,
- $C_k - i_k - 1$ curves ending in the $(2n-k+1)$st puncture.

Based on these remarks, we will “complete” each index, which is associated to a colour $C_k - 1$, up to $\mathcal{N} - 2$. We will do this using the property that the action of the braid $(\beta_n \cup \mathbb{I}_{n+1})$ is trivial on the right-hand side of the $(2n+1)$-punctured disc.

More precisely, for each $k \in \{1,\dots,n\}$, let us add $\mathcal{N} - C_k - 1$ extra segments/configuration points on the part of the geometric supports of the classes

$$\mathcal{F}_{\bar{i}}^{(C_1,\dots,C_n)} \quad \text{and} \quad \mathcal{L}_{\bar{i}}^{(C_1,\dots,C_n)},$$

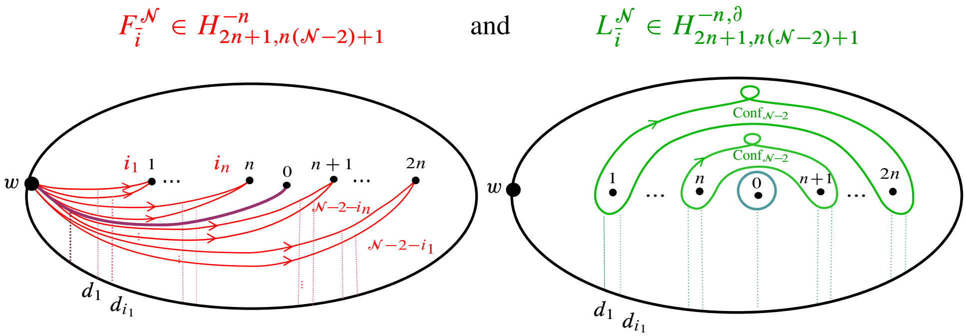

Figure 5.4. Classes corresponding to the level $\mathcal{N}$ and multi-index $\bar{i}$.

which end/go around the puncture $2n+1-k$. Thanks to this change, the new geometric supports have in total $\mathcal{N}-2$ curves/configuration points which end/go around symmetric punctures of the punctured disc.

Definition 5.9 (Level $\mathcal{N}$ homology classes). Following this procedure, we consider the homology classes given by the geometric supports which are presented in Figure 5.4 and denote them by

$$F_{\bar{i}}^{\mathcal{N}} \in H^{-n}_{2n+1,n(\mathcal{N}-2)+1} \quad \text{and} \quad L_{\bar{i}}^{\mathcal{N}} \in H^{-n,\partial}_{2n+1,n(\mathcal{N}-2)+1}.$$

Further on, we show that the change of the classes does not affect the outcome of the intersection pairing.

Proposition 5.10 (Equality of intersection pairings in different configuration spaces). *For any choice of indices*

$$0 \leqslant i_k \leqslant C_k - 1, \quad k \in \{1, \ldots, n\},$$

we have the following relation between intersection pairings:

$$\left\langle (\beta_n \cup \mathbb{I}_{n+1}) \mathscr{F}_{\bar{i}}^{(C_1,\ldots,C_n)}, \mathscr{L}_{\bar{i}}^{(C_1,\ldots,C_n)} \right\rangle = \left\langle (\beta_n \cup \mathbb{I}_{n+1}) F_{\bar{i}}^{\mathcal{N}}, L_{\bar{i}}^{\mathcal{N}} \right\rangle.$$

Proof. This relation can be seen from the formula of the graded intersection form. The pairing is encoded by the intersection points between the geometric supports of the classes in the base configuration space, which are graded by certain coefficients coming from the local system.

We denote the geometric support of a class $\mathscr{C}$ by $s\mathscr{C}$. Further on, we notice that the intersections

$$\left((\beta_n \cup \mathbb{I}_{n+1}) s\mathscr{F}_{\bar{i}}^{(C_1,\ldots,C_n)}\right) \cap s\mathscr{L}_{\bar{i}}^{(C_1,\ldots,C_n)}, \tag{5.2a}$$

$$\left((\beta_n \cup \mathbb{I}_{n+1}) sF_{\bar{i}}^{\mathcal{N}}\right) \cap sL_{\bar{i}}^{\mathcal{N}} \tag{5.2b}$$

have the same support in the left-hand side of the disc, and they differ by the fact that the second pair has more intersection points in the right-hand side of the disc. This establishes a bijection between the intersection points from (5.2a) and the intersection points from (5.2b).

For the next part, let us fix an intersection point P from (5.2a) and denote by $\tilde{P}$ its correspondent in (5.2b). Now, we look at the monomials which are associated to these points. The loop in the configuration space which is associated to $\tilde{P}$ is obtained from the loop associated to P union with other

$$\sum_{k=1}^{n}(\mathcal{N}-C_k-1)$$

loops which pass through the extra intersection points in the right-hand side of the disc. However, we see that the extra loops are evaluated trivially by the local system since they do not twist or go around any puncture, and so they contribute with coefficients which are all 1. This concludes that the two intersection pairings lead to the same result. ■

Proposition 5.10 together with formula (5.1) show that we can obtain WRT invariant from intersections between the new homology classes, as follows:

$$\begin{aligned}\tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{1 \leqslant N_1,\dots,N_l \leqslant \mathcal{N}-1} [N_1]_\xi \cdot \dots \cdot [N_l]_\xi \\ \cdot \Bigg(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i} - \sum_{j \neq p_i} lk_{p_i,j})} \prod_{i=1}^{n} x_{C(i)}^{-1} \\ \cdot \sum_{\bar{i} \in C(\bar{N})} \left\langle (\beta_n \cup \mathbb{I}_{n+1})\, F_{\bar{i}}^{\mathcal{N}}, L_{\bar{i}}^{\mathcal{N}} \right\rangle \Bigg) \Bigg|_{\psi^C_{\xi,N_1,\dots,N_l}}.\end{aligned} \tag{5.3}$$

We arrived at a state sum model for $\tau_{\mathcal{N}}$ as intersections between homology classes which are given by geometric supports in a fixed ambient manifold, namely, the configuration space of $n(\mathcal{N}-2)+1$ points on the $2n+1$ punctured disc. Then, the "individual colours" from the initial formula appear in the specialisations of coefficients and also in the quantum numbers coming from the Kirby colour.

Encoding the coefficients of the Kirby colour. Pursuing this line, in the following parts, we aim to understand geometrically the coefficients which come from the Kirby colour and encode them by intersections between the homology classes.

For the moment, we have an intersection in the $(2n+1)$-punctured disc $\mathcal{D}_{2n+1}$ which takes values in the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, d^{\pm 1}]$ (Definition 3.15). Let us look at the terms which appear in formula (5.3). For a fixed set of colours $N_1, \dots, N_l$, we

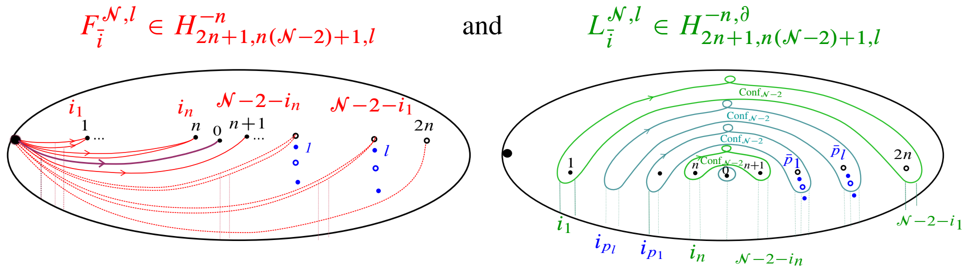

Figure 5.5. Homology classes from the disc with extra punctures.

have a state sum which is given by

$$[N_1]_\xi \cdot \ldots \cdot [N_l]_\xi \cdot \Bigg(\sum_{\bar{i} \in C(\bar{N})} \left\langle (\beta_n \cup \mathbb{I}_{n+1})\, F_{\bar{i}}^{\mathcal{N}}, L_{\bar{i}}^{\mathcal{N}} \right\rangle \Bigg) \Bigg|_{\psi^C_{\xi,N_1,\ldots,N_l}} =$$

$$\sum_{\bar{i} \in C(\bar{N})} [N_1]_\xi \cdot \ldots \cdot [N_l]_\xi \cdot \left(\langle (\beta_n \cup \mathbb{I}_{n+1})\, F_{\bar{i}}^{\mathcal{N}}, L_{\bar{i}}^{\mathcal{N}} \rangle \right) \Big|_{\psi^C_{\xi,N_1,\ldots,N_l}}.$$

Now, we want to understand topologically the term

$$[N_1]_\xi \cdot \ldots \cdot [N_l]_\xi \cdot \left(\langle (\beta_n \cup \mathbb{I}_{n+1})\, F_{\bar{i}}^{\mathcal{N}}, L_{\bar{i}}^{\mathcal{N}} \rangle \right) \Big|_{\psi^C_{\xi,N_1,\ldots,N_l}}$$

in a unified way which does not depend on the choice of individual colours $N_1, \ldots, N_l$. More precisely, we would like to see this term as a $\psi^C_{\xi,N_1,\ldots,N_l}$ specialisation of an intersection which does not depend on the individual colours.

Step III (Add extra punctures to the punctured disc)

We do this by adding $3l$ points to our punctured disc and work in $\mathscr{D}_{2n+3l+1}$. In this manner, we have a richer local system which carries monodromies around these additional punctures.

Definition 5.11 (Homology classes using the $(2n + 3l + 1)$-punctured disc). We consider the homology classes given by the geometric supports which are presented in Figure 5.5, in the configuration space in the $(2n + 3l + 1)$-punctured disc.

We recall that the homologies

$$H^{-n}_{2n+1,n(\mathcal{N}-2)+1,l} \quad \text{and} \quad H^{-n,\partial}_{2n+1,n(\mathcal{N}-2)+1,l}$$

are modules over $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}, y_1^{\pm 1}, \ldots, y_l^{\pm 1}, d^{\pm 1}]$, and in the next part, we will use the new variables $y_1, \ldots, y_l$. We remember that the monodromy of Φ around the extra blue punctures corresponding to $\bar{p}_1, \ldots, \bar{p}_l$ is evaluated through f_C with the variables (using Definition 3.2 and relation (3.3)):

$$\begin{pmatrix} y_1^2 \\ y_1^{-2} \\ y_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} y_l^2 \\ y_l^{-2} \\ y_l^{-1} \end{pmatrix}.$$

The order of these evaluations might vary, but the columns correspond exactly to the above triples. Then, these monodromies get evaluated through $\psi^C_{\xi, N_1, \ldots, N_l}$ to the following values:

$$\begin{pmatrix} \xi^{2N_1} \\ \xi^{-2N_1} \\ \xi^{-N_1} \end{pmatrix}, \ldots, \begin{pmatrix} \xi^{2N_l} \\ \xi^{-2N_l} \\ \xi^{-N_l} \end{pmatrix}.$$

Based on this remark, we notice that we can encode the quantum numbers using the variables $y_1, \ldots, y_l$, and we have the following relation:

$$[N_1]_\xi \cdot \ldots \cdot [N_l]_\xi = \{1\}_\xi^{-l} \left(\prod_{i=1}^{l} (y_i - y_i^{-1}) \right) \Bigg|_{\psi^C_{\xi, N_1, \ldots, N_l}}. \tag{5.4}$$

Lemma 5.12. *Using this property, we obtain a new intersection pairing, and the following relation holds:*

$$\begin{aligned} &[N_1]_\xi \cdot \ldots \cdot [N_l]_\xi \cdot (\langle (\beta_n \cup \mathbb{I}_{n+1}) \, F_{\bar{i}}^{\mathcal{N}}, L_{\bar{i}}^{\mathcal{N}} \rangle)\big|_{\psi^C_{q, N_1, \ldots, N_l}} \\ &\quad = \{1\}_\xi^{-l} \left(\prod_{i=1}^{l} (y_i - y_i^{-1}) \big\langle (\beta_n \cup \mathbb{I}_{n+3l+1}) \, F_{\bar{i}}^{\mathcal{N},l}, L_{\bar{i}}^{\mathcal{N},l} \big\rangle \right) \Bigg|_{\psi^C_{q, N_1, \ldots, N_l}}. \end{aligned} \tag{5.5}$$

Proof. Following equation (5.4), we have the equality of the coefficients which appear in both terms from above. Now, we notice that the addition of the extra punctures does not change the intersection forms, and so, we have

$$\big\langle (\beta_n \cup \mathbb{I}_{n+1}) F_{\bar{i}}^{\mathcal{N}}, L_{\bar{i}}^{\mathcal{N}} \big\rangle = \big\langle (\beta_n \cup \mathbb{I}_{n+3l+1}) \, F_{\bar{i}}^{\mathcal{N},l}, L_{\bar{i}}^{\mathcal{N},l} \big\rangle.$$

We can see this from the fact that the supports of the homology classes have the same intersection points, and so, the only question that we have concerns their gradings. The second intersection belongs to a covering where there could be potential monodromies around the $3l$-blue punctures. However, we remark that the loops which are associated to the intersection points in the right-hand side of the disc do not wind around the blue punctures, and so, they have trivial monodromies. ∎

Using this lemma together with the formula from equation (5.3), we conclude the following formula:

$$\tau_{\mathcal{N}}(M) = \frac{\{1\}_{\xi}^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{1 \leqslant N_1,\dots,N_l \leqslant \mathcal{N}-1} \left(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i} - \sum_{j \neq p_i} lk_{p_i,j})} \prod_{i=1}^{n} x_{C(i)}^{-1} \cdot \prod_{i=1}^{l} (y_i - y_i^{-1}) \sum_{\bar{i} \in C(\bar{N})} \left\langle (\beta_n \cup \mathbb{I}_{n+3l+1})\, F_{\bar{i}}^{\mathcal{N},l}, L_{\bar{i}}^{\mathcal{N},l} \right\rangle \right) \Bigg|_{\psi^C_{\xi,N_1,\dots,N_l}}. \tag{5.6}$$

We remark that we arrived at an expression given by graded intersections (the terms between the brackets) which do not depend anymore on the choice of colours $N_1, \dots, N_l$. After that, we have to specialise them using the change of coefficients $\psi^C_{\xi,N_1,\dots,N_l}$.

Step IV (Coefficients of the Kirby colour encoded by circles in the supports of the homology classes)

In the last part, we will show that we can encode the coefficients of the Kirby colour by adding l points to our configuration space and considering the classes which are obtained from the supports of the classes $F_{\bar{i}}^{\mathcal{N},l}$ and $L_{\bar{i}}^{\mathcal{N},l}$ by adding l extra circles.

More specifically, we prove that the pairing that arises from the homology classes $\mathcal{F}_{\bar{i}}^{\mathcal{N}}$ and $\mathcal{L}_{\bar{i}}^{\mathcal{N}}$ captures precisely the extra coefficients from equation (5.5). We recall that

$$\mathcal{F}_{\bar{i}}^{\mathcal{N}} \in H^{-n}_{2n+1,n(\mathcal{N}-2)+l+1,l} \quad \text{(Figure 5.1)},$$
$$\mathcal{L}_{\bar{i}}^{\mathcal{N}} \in H^{-n,\partial}_{2n+1,n(\mathcal{N}-2)+l+1,l} \quad \text{(Figure 5.2)}.$$

Proposition 5.13 (Encoding the Kirby colour). *For any index $\bar{i}$, we have*

$$\left(\prod_{k=1}^{l} (y_k - y_k^{-1}) \right) \left\langle (\beta_n \cup \mathbb{I}_{n+3l+1})\, F_{\bar{i}}^{\mathcal{N},l}, L_{\bar{i}}^{\mathcal{N},l} \right\rangle = \left\langle (\beta_n \cup \mathbb{I}_{n+3l+1})\, \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \right\rangle. \tag{5.7}$$

Proof. For the computation of these intersections, we will use the formulas for the intersection pairing presented in equation (3.2) and Remark 3.13, where for each intersection point we count the product of local orientations in the disc and multiply it with the evaluation of the local system $\bar{\Phi}$ on the associated loop in the configuration space.

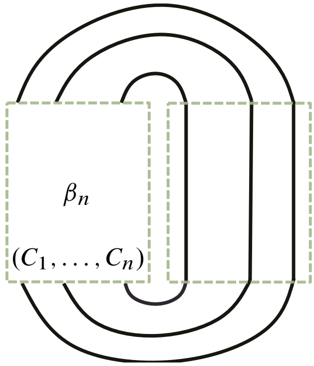

Figure 5.6. Colouring.

We notice that the two families of classes which lead to these intersection pairings have similar geometric supports except the fact that

- $s\mathcal{F}_{\bar{i}}^{\mathcal{N}}$ is constructed from $sF_{\bar{i}}^{\mathcal{N},l}$ by adding l extra circles;
- $s\mathcal{L}_{\bar{i}}^{\mathcal{N}}$ comes from $sL_{\bar{i}}^{\mathcal{N},l}$, but it has l extra points, one on each figure eight which goes around the punctures p_k for $k \in \{1, \ldots, l\}$.

Now, we look at the intersection points between the geometric supports which are obtained after we act with the braid:

$$\left((\beta_n \cup \mathbb{I}_{n+3l+1})sF_{\bar{i}}^{\mathcal{N},l}\right) \cap sL_{\bar{i}}^{\mathcal{N},l}, \tag{5.8a}$$

$$\left((\beta_n \cup \mathbb{I}_{n+3l+1})s\mathcal{F}_{\bar{i}}^{\mathcal{N}}\right) \cap s\mathcal{L}_{\bar{i}}^{\mathcal{N}}. \tag{5.8b}$$

These two intersections have the same components in the left-hand side of the punctured disc. The difference occurs in the right-hand side of it.

Let us denote by

$$(q_1, r_1), \ldots, (q_l, r_l)$$

the intersection points between the purple circles from $s\mathcal{F}_{\bar{i}}^{\mathcal{N}}$ and the blue figure eights from $s\mathcal{L}_{\bar{i}}^{\mathcal{N}}$ which intersect them (they are the orange points from Figure 5.6).

Also, let us look at the first pairing (5.8a); consider the set of intersection points between the supports of the homology classes and denote then as follows:

$$((\beta_n \cup \mathbb{I}_{n+1})sF_{\bar{i}}^{\mathcal{N},l}) \cap sL_{\bar{i}}^{\mathcal{N},l} = \{\bar{m}_1, \ldots, \bar{m}_s\}. \tag{5.9}$$

Here, each element is a multipoint in the configuration space, which has $n(\mathcal{N}-2)+1$ components.

Remark 5.14. The set of intersection points between the new classes (from (5.8b)) is obtained from the intersection points defined in (5.9), together with a choice of l

orange points which belong to different circles:

$$((\beta_n \cup \mathbb{I}_{n+3l+1})s\mathcal{F}_{\bar{i}}^{\mathcal{N}}) \cap s\mathcal{L}_{\bar{i}}^{\mathcal{N}} = \{\bar{m}_1, \ldots, \bar{m}_s\} \times \{q_1, r_1\} \times \cdots \times \{q_l, r_l\}. \quad (5.10)$$

Proof. Let P be a multipoint which belongs to the intersection

$$((\beta_n \cup \mathbb{I}_{n+3l+1})s\mathcal{F}_{\bar{i}}^{\mathcal{N}}) \cap s\mathcal{L}_{\bar{i}}^{\mathcal{N}}.$$

This means that it has exactly one component on each red curve and purple curve from $(\beta_n \cup \mathbb{I}_{n+3l+1})s\mathcal{F}_{\bar{i}}^{\mathcal{N}}$. In particular, it has exactly one point on each purple circle, which should be at the intersection with the dual support $s\mathcal{L}_{\bar{i}}^{\mathcal{N}}$.

We notice that for any fixed $k \in \{1, \ldots, l\}$ the kth purple circle intersects only one component from the dual support (given by the configuration space of $\mathcal{N} - 1$ points on the blue figure eight), and this is exactly in the two orange points which we denote by

$$\{q_k, r_k\}.$$

This means that P has exactly one component from each of these sets with two elements. Then, for the rest of the components of P, we should use the configuration space of $\mathcal{N} - 2$ points on the blue figure eights. So, the rest of the components of P belong to the red curves from the support

$$((\beta_n \cup \mathbb{I}_{n+3l+1})s\mathcal{F}_{\bar{i}}^{\mathcal{N}})$$

which are not circles intersected with the configuration spaces of $\mathcal{N} - 2$ particles on the figure eights. This intersection gives precisely a point belonging to

$$((\beta_n \cup \mathbb{I}_{n+1})sF_{\bar{i}}^{\mathcal{N},l}) \cap sL_{\bar{i}}^{\mathcal{N},l}.$$

This procedure establishes the desired bijection. ■

So far, we saw the correspondence at the level of sets. Now, we are interested in the coefficients coming from the local system. For this, we turn our attention towards the coefficients which are carried by the orange points.

In order to compute these coefficients, for each $k \in \{1, \ldots, l\}$, we have to look at the chosen point on the purple circle (q_k or r_k) and to evaluate the monodromy of the path associated to this point around the punctures of the disc. We have drawn in Figure 5.7 the paths associated to q_k and r_k (coloured in yellow). We recall that the counter-clockwise monodromies around the three blue punctures which are situated underneath $\bar{p}_k$ are evaluated by the variables

$$\begin{pmatrix} y_k^2 \\ y_k^{-2} \\ y_k^{-1} \end{pmatrix}.$$

Using this for the two paths from the picture, we see that the points $\{q_k, r_k\}$ carry the following coefficients:

$$\begin{aligned} &(q_k) \quad y_k^2 \cdot y_k^{-1} = y_k, \\ &(r_k) \quad -y_k^{-1}. \end{aligned}$$

The yellow loops associated to the orange intersection points will not add extra d components when we evaluate a loop corresponding to an intersection point in the configuration space. This is because these loops do not contribute to the relative winding in the configuration space. Also, the opposite sign comes from the local intersections in these two orange points.

We conclude that for any $k \in \{1, \ldots, l\}$ the two points $\{q_k, r_k\}$ contribute to the grading with the coefficients $\{y_k, -y_k^{-1}\}$. This property together with the correspondence presented in relation (5.10) shows that the extra coefficients that appear in the second intersection are precisely

$$\prod_{k=1}^{l} (y_k - y_k^{-1}),$$

which concludes the relation between the intersection pairings from (5.7). ■

Now, using this property together with the expression from equation (5.6), we obtain

$$\begin{aligned} \tau_{\mathcal{N}}(M) = &\frac{\{1\}_\xi^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \\ &\cdot \sum_{1 \leqslant N_1, \ldots, N_l \leqslant \mathcal{N}-1} \Bigg(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i} - \sum_{j \neq p_i} lk_{p_i,j})} \prod_{i=1}^{n} x_{C(i)}^{-1} \\ &\cdot \sum_{\bar{i} \in C(\bar{N})} \langle (\beta_n \cup \mathbb{I}_{n+3l+1}) \, \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \rangle \Bigg) \Bigg|_{\psi^C_{\xi, N_1, \ldots, N_l}} \\ = &\frac{\{1\}_\xi^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \\ &\cdot \sum_{1 \leqslant N_1, \ldots, N_l \leqslant \mathcal{N}-1} \sum_{\bar{i} \in C(\bar{N})} \Bigg(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i} - \sum_{j \neq p_i} lk_{p_i,j})} \\ &\cdot \prod_{i=1}^{n} x_{C(i)}^{-1} \cdot \langle (\beta_n \cup \mathbb{I}_{n+3l+1}) \, \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \rangle \Bigg) \Bigg|_{\psi^C_{\xi, N_1, \ldots, N_l}}. \end{aligned}$$

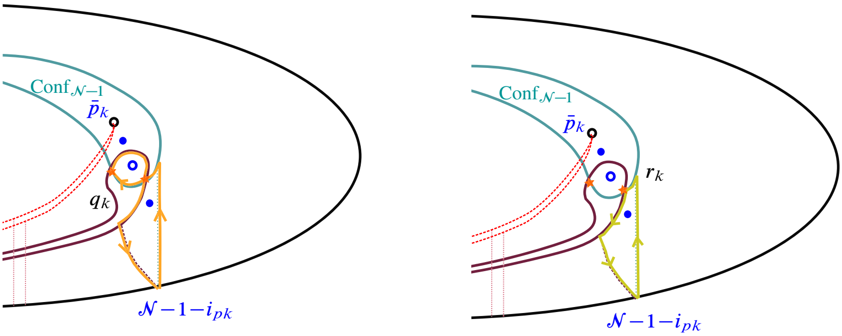

Figure 5.7. Paths associated to the intersection points.

Exchanging the two sums and taking care of the conditions which the colouring imposes on the multi-indices, we obtain the following formula:

$$\begin{aligned}
\tau_{\mathcal{N}}(M) &= \frac{\{1\}_{\xi}^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b+} \cdot \Delta_-^{b-}} \\
&\quad \cdot \sum_{i_1,\dots,i_n=0}^{\mathcal{N}-2} \sum_{\substack{\bar{N}=(N_1,\dots,N_l)\\ 1\leqslant N_1,\dots,N_l\leqslant \mathcal{N}-1\\ \bar{i}\in C(\bar{N})}} \Bigg(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i}-\sum_{j\neq p_i} lk_{p_i,j})} \\
&\quad \cdot \prod_{i=1}^{n} x_i^{-1} \cdot \langle (\beta_n \cup \mathbb{I}_{n+3l+1})\, \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \rangle \Bigg) \Bigg|_{\psi^C_{\xi,N_1,\dots,N_l}} \\
&= \frac{\{1\}_{\xi}^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b+} \cdot \Delta_-^{b-}} \cdot \sum_{i_1,\dots,i_n=0}^{\mathcal{N}-2} \Bigg(\sum_{\substack{1\leqslant N_1,\dots,N_l\leqslant \mathcal{N}-1\\ \bar{i}\in C(N_1,\dots,N_l)}} \Lambda_{\bar{i}}(\beta_n) \Big|_{\psi^C_{\xi,N_1,\dots,N_l}} \Bigg).
\end{aligned}$$

This relation concludes the proof of the main formula.

Corollary 5.15 (Detailed formula for the invariant). *We have the following topological model for the WRT invariant:*

$$\begin{aligned}
\tau_{\mathcal{N}}(M) &= \frac{\{1\}_{\xi}^{-l}}{\mathcal{D}^b \cdot \Delta_+^{b+} \cdot \Delta_-^{b-}} \cdot \sum_{i_1,\dots,i_n=0}^{\mathcal{N}-2} \sum_{\substack{\bar{N}=(N_1,\dots,N_l)\\ 1\leqslant N_1,\dots,N_l\leqslant \mathcal{N}-1\\ \bar{i}\in C(\bar{N})}} \Bigg(\prod_{i=1}^{l} x_{C(p_i)}^{(f_{p_i}-\sum_{j\neq p_i} lk_{p_i,j})} \\
&\quad \cdot \prod_{i=1}^{n} x_{C(i)}^{-1} \cdot \langle (\beta_n \cup \mathbb{I}_{n+3l+1})\, \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \rangle \Bigg) \Bigg|_{\psi^C_{\xi,N_1,\dots,N_l}}.
\end{aligned}$$

6. Topological model for the WRT invariants of 3-manifolds obtained as surgeries along knots

This section is devoted to the topological model constructed above, for the particular case where the link is actually a knot (this means that $l = 1$). Let us consider a knot K which is the closure of a braid with n strands $\beta_n \in B_n$.

In this case, we work in the covering of the configuration space of $n(\mathcal{N} - 2) + 2$ particles in the punctured disc with $2n + 4$ punctures, and we will use the homology groups

$$H^{-n}_{2n+1,n(\mathcal{N}-2)+2,1}$$

and

$$H^{-n,\partial}_{2n+1,n(\mathcal{N}-2)+2,1} \quad (\text{as } \mathbb{Z}[x^{\pm1}, y^{\pm1}, d^{\pm1}]\text{-modules}).$$

This means that we have 3 privileged blue points in the punctured disc.

Further on, we use the specialisation of coefficients associated to a colouring with one colour $N \in \mathbb{N}$, given by

$$\psi^C_{q,N} : \mathbb{Z}[x^{\pm1}, y^{\pm1}, d^{\pm1}] \to \mathbb{Z}[q^{\pm1}]$$
$$\begin{cases} \psi^C_{q,N}(x) = q^{N-1}, \\ \psi^C_{q,N}(y_i) = q^N, \\ \psi^C_{q,N}(d) = q^{-2}. \end{cases}$$

6.1. Homology classes

Definition 6.1. (a) First homology class. For any set $i_1, \ldots, i_n \in \{0, \ldots, \mathcal{N} - 2\}$, we consider the class given by the geometric support from Figure 6.1.

(b) Second homology class. The second homology class is given by the geometric support from Figure 6.2.

Corollary 6.2 (Topological model for the Witten–Reshetikhin–Turaev invariants of knot surgeries). *Let M be a closed oriented 3-manifold obtained by surgery along a knot K with framing $f \in \mathbb{Z}$. We choose a braid β_n such that*

$$K = \widehat{\beta_n}.$$

Further on, for $i_1, \ldots, i_n \in \{0, \ldots, \mathcal{N} - 2\}$, we consider the following Lagrangian intersection:

$$\begin{cases} \Lambda_{\bar{i}}(\beta_n) \in \mathbb{Z}[x^{\pm1}, y^{\pm1}, d^{\pm1}], \\ \Lambda_{\bar{i}}(\beta_n) := x^{f-w(\beta_n)} \cdot x^{-n} \langle (\beta_n \cup \mathbb{I}_{n+4})\, \mathcal{F}^{\mathcal{N}}_{\bar{i}}, \mathcal{L}^{\mathcal{N}}_{\bar{i}} \rangle. \end{cases}$$

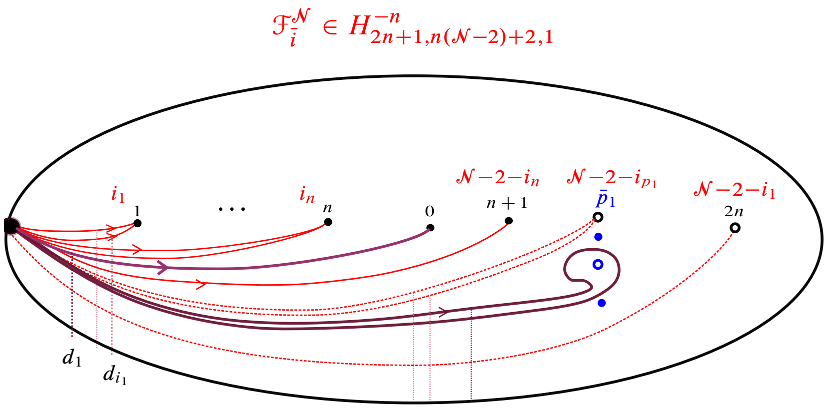

Figure 6.1. Homology classes for WRT invariants of surgeries along knots.

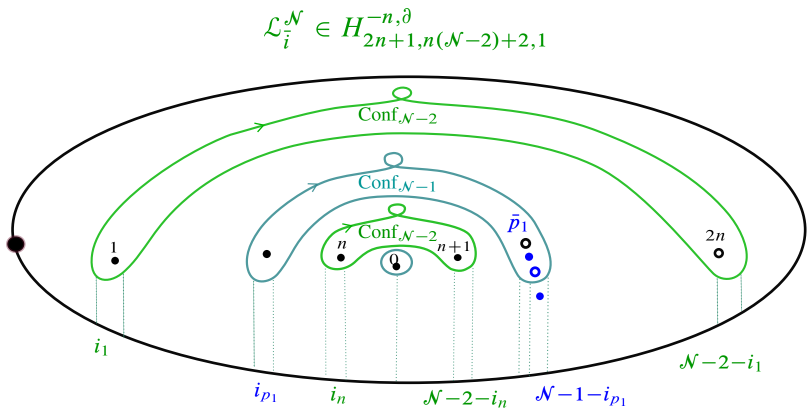

Figure 6.2. Dual homology classes for WRT invariants of surgeries along knots.

Here, $w(\beta_n)$ is the writhe of the braid. Then, the $\mathcal{N}$th Witten–Reshetikhin–Turaev invariant is obtained from these intersections as follows:

$$\tau_{\mathcal{N}}(M) = \frac{\{1\}_{\xi}^{-1}}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{i_1,\dots,i_n=0}^{\mathcal{N}-2} \left(\sum_{N=\max\{i_1+1,\dots,i_n+1\}}^{\mathcal{N}-1} \Lambda_{\bar{i}}(\beta_n) \Big|_{\psi^C_{\xi,N}} \right).$$

Remark 6.3. This tells as that the level $\mathcal{N}$ WRT invariant of a surgery along a knot which is the closure of a braid with n strands is obtained from states of graded intersections in the configuration space of $n(\mathcal{N}-2)+2$ points in the $(2n+4)$-punctured disc.

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