

A letter from Kharkov to Moscow

V. Drinfeld* to V. Schechtman

Abstract. This naive, non-technical, and enthusiastic old letter could help people starting to learn deformation theory or derived algebraic geometry.

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Dear Vadik,

Here is the promised letter.

Let there be given a DG Lie algebra \mathfrak{g} . Shift the grading on \mathfrak{g} down by 1 (that is, what had degree i now has degree $i - 1$), dualize the resulting complex and use that to generate a free supercommutative algebra (without unit). We get $\mathfrak{g}^* \oplus \Lambda^2 \mathfrak{g}^* \oplus \Lambda^3 \mathfrak{g}^* \oplus \dots$. On this structure, besides the differential arising from the differential in \mathfrak{g} there is also the Chevalley differential, arising from the commutator in \mathfrak{g} . Using their sum as the total differential, we get the commutative DG algebra $C^*(\mathfrak{g})$. If you don't pay attention to the fact that, generally speaking, $V^{**} \neq V$, then it represents the functor $\text{Hom}(C^*(\mathfrak{g}), B) = \text{MC}(\mathfrak{g} \otimes B)$, where B is any commutative DG algebra and $\text{MC}(\mathfrak{g} \otimes B)$ is the set of elements of $\mathfrak{g} \otimes B$ of degree 1 satisfying the Maurer–Cartan equations.

*Translation from the Russian by Keith Conrad, who thanks Maria Gordina, Irina Nickolaeva and Dmitri Orlov for their help.

Comment added by the author, 16 June 2014: At the time of my correspondence with Schechtman I felt that I was trying to understand something known rather than inventing new things. Maybe my feeling was correct. I wouldn't be surprised if everything from my correspondence with Schechtman is already in Quillen's article "Rational Homotopy Theory" (maybe except for the word "operad"). See especially Section 7 of his article. Anyway, the role of Quillen's article was crucial.

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Rigorously, it's better to define the covariant functor $F: \text{DGLie} \rightarrow \text{DGCocom}$, where $F(\mathfrak{g})$ is the cofree commutative coalgebra generated by the shifted \mathfrak{g} , with the understood differential. For any cocommutative DG algebra A we have

$$\text{Hom}(A, F(\mathfrak{g})) = \text{MC}(\text{Hom}_k(A, \mathfrak{g})).$$

Moreover we have the functor $G: \text{DGCocom} \rightarrow \text{DGLie}$ such that

$$\text{Hom}(G(A), \mathfrak{g}) = \text{MC}(\text{Hom}_k(A, \mathfrak{g})).$$

Here $G(A)$ is the free Lie (super)algebra generated by the complex A with grading shifted up, and it's sufficient to define the differential on the generators, that is, on A , and it is equal to the sum of the differential $A \rightarrow A$ and the comultiplication $A \rightarrow \text{Sym}^2(A) \subset G(A)$ (in case of characteristic 2, you have to write Γ^2 instead of Sym^2).

Since F and G are adjoints, we have morphisms $A \rightarrow FG(A)$ and $GF(\mathfrak{g}) \rightarrow \mathfrak{g}$.

Theorem 1. *These are quasi-isomorphisms.*

This can all be ‘‘Sugawarized’’. A DG Sugawara–Lie algebra is, by definition, a \mathbb{Z} -graded space \mathfrak{g} plus a differential of degree 1 on the cofree cocommutative coalgebra generated by \mathfrak{g} with grading shifted down by 1 (the square of the differential is 0). Essentially this differential is defined by the assignment of mappings $\mathfrak{g} \rightarrow \mathfrak{g}$, $\Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$, $\Lambda^3 \mathfrak{g} \rightarrow \mathfrak{g}, \dots$ (the differential, commutator, and ‘‘higher’’ operations), and setting the square of a differential to 0 determines an identity that all these operations must satisfy. A Sugawara morphism is defined as a morphism of the corresponding commutative DG coalgebras (a morphism of these DG coalgebras automatically takes primitive elements, i.e., \mathfrak{g} , to primitive elements). Thus, the functor $F_{\text{Sug}}: \text{DGLie–Sug} \rightarrow \text{DGCocom}$ is tautological. In the same way we can also introduce DGCocom and $G_{\text{Sug}}: \text{DGCocom} \rightarrow \text{DGLie}$. If you consider F_{Sug} as a functor $\text{DGLie–Sug} \rightarrow \text{DGCocom–Sug}$ and G_{Sug} as a functor $\text{DGCocom–Sug} \rightarrow \text{DGLie–Sug}$, then they are adjoints: if $\mathfrak{g} \in \text{DGLie–Sug}$ and $A \in \text{DGCocom–Sug}$, then

$$\begin{aligned} \text{Hom}_{\text{Sug}}(F_{\text{Sug}}(\mathfrak{g}), A) &= \text{Hom}(GF_{\text{Sug}}(\mathfrak{g}), G_{\text{Sug}}(A)) \\ &= \text{Hom}(F_{\text{Sug}}(\mathfrak{g}), FG_{\text{Sug}}(A)) \\ &= \text{Hom}_{\text{Sug}}(\mathfrak{g}, G_{\text{Sug}}(A)). \end{aligned}$$

Therefore we have Sugawara morphisms $\mathfrak{g} \rightarrow GF_{\text{Sug}}(\mathfrak{g})$ and $FG_{\text{Sug}}(A) \rightarrow A$.

Theorem 1'. *These are quasi-isomorphisms.*

Proof of theorems. First we show Theorem 1' implies Theorem 1. For example, if \mathfrak{g} is a genuine DG Lie algebra, then we have a morphism $GF(\mathfrak{g}) \rightarrow \mathfrak{g}$ and a Sugawara morphism $\mathfrak{g} \rightarrow GF(\mathfrak{g})$, for which the composite $\mathfrak{g} \rightarrow GF(\mathfrak{g}) \rightarrow \mathfrak{g}$ is the identity (since the Sugawara morphism $\mathfrak{g} \rightarrow GF(\mathfrak{g})$ comes from the actual morphism $F(\mathfrak{g}) \rightarrow FGF(\mathfrak{g})$ and the composite $F \rightarrow FGF \rightarrow F$ is the identity; this is a general property of adjoint functors). Therefore if $\mathfrak{g} \rightarrow GF(\mathfrak{g})$ is a quasi-isomorphism then $GF(\mathfrak{g}) \rightarrow \mathfrak{g}$ is also a quasi-isomorphism. We prove now Theorem 1', for example, that $\mathfrak{g} \rightarrow GF_{\text{Sug}}(\mathfrak{g})$ is a quasi-isomorphism. By definition $GF_{\text{Sug}}(\mathfrak{g})$ is the free Lie algebra generated by $\mathfrak{g} \oplus \Lambda^2\mathfrak{g} \oplus \Lambda^3\mathfrak{g} \cdots$. The differential on $\mathfrak{g} \oplus \Lambda^2\mathfrak{g} \oplus \cdots$ maps $\Lambda^k\mathfrak{g} \rightarrow \bigoplus_{i=1}^k \Lambda^i\mathfrak{g}$ and the differential on $GF_{\text{Sug}}(\mathfrak{g})$ is the sum of two terms, one of which comes from the differential on $\mathfrak{g} \oplus \Lambda^2\mathfrak{g} \oplus \Lambda^3\mathfrak{g} \oplus \cdots$ and the second comes from the standard comultiplication on $\mathfrak{g} \oplus \Lambda^2\mathfrak{g} \oplus \Lambda^3\mathfrak{g} \oplus \cdots$. We introduce on $\mathfrak{g} \oplus \Lambda^2\mathfrak{g} \oplus \Lambda^3\mathfrak{g} \oplus \cdots$ the filtration

$$\mathfrak{g} \subset \mathfrak{g} \oplus \Lambda^2\mathfrak{g} \subset \mathfrak{g} \oplus \Lambda^2\mathfrak{g} \oplus \Lambda^3\mathfrak{g} \subset \cdots,$$

where elements of $\Lambda^k\mathfrak{g}$ are given degree k . Extend this filtration to the free Lie algebra generated by $\mathfrak{g} \oplus \Lambda^2\mathfrak{g} \oplus \cdots$. We get a filtration of the complex $GF_{\text{Sug}}(\mathfrak{g})$ by subcomplexes and the first term in the filtration is \mathfrak{g} .

It remains to prove that all factors of this filtration, except the first, are quasi-isomorphic to 0. For example, the third factor is $\Lambda^3\mathfrak{g} \oplus (\mathfrak{g} \otimes \Lambda^2\mathfrak{g}) \oplus \text{Lie}_3(\mathfrak{g})$, where $\text{Lie}_3(\mathfrak{g})$ is the component of degree 3 of the free Lie algebra generated by \mathfrak{g} . The differential here is the sum of two terms, of which the first preserves the summands $\Lambda^3\mathfrak{g}$, $\mathfrak{g} \otimes \Lambda^2\mathfrak{g}$, and $\text{Lie}_3(\mathfrak{g})$, and the second acts thus: $0 \rightarrow \Lambda^3\mathfrak{g} \rightarrow \mathfrak{g} \otimes \Lambda^2\mathfrak{g} \rightarrow \text{Lie}_3(\mathfrak{g}) \rightarrow 0$. It's sufficient to prove the acyclicity of the second term in the differential. But here the initial Sugawara–Lie algebra structure doesn't matter (you can just consider it to be 0): we simply need to show that if A is a cofree supercommutative coalgebra, then, introducing the usual differential on the free Lie algebra generated by $A[-1]$ (where $[-1]$ is the shift of the grading) we get a complex whose cohomology sits in the lowest dimension. This fact is apparently standard (our complex reduces to the cotangent complex of the algebra A^* or, perhaps, $k \oplus A^*$). The fact that $FG_{\text{Sug}}(A) \rightarrow A$ is a quasi-isomorphism is proved even more simply (the fact needed for this from the cohomology of free Lie algebras is quite standard). \square

Remarks. (1) If \mathfrak{g} is a DG Sugawara–Lie algebra and A is a cocommutative DG coalgebra then

$$\text{Hom}(A, F_{\text{Sug}}(\mathfrak{g})) = \text{MC}(\text{Hom}_k(A, \mathfrak{g})).$$

Here $\text{Hom}_k(A, \mathfrak{g})$ is a DG Sugawara–Lie algebra and the Maurer–Cartan equation has to be written in terms of higher operations: $d\omega + \frac{1}{2!}[\omega, \omega] + \frac{1}{3!}[\omega, \omega, \omega] + \cdots = 0$.

- (2) If as A for use the chain complex of a simply connected space, tensored with \mathbb{Q} , then $G(A)$ is a DG Lie algebra whose cohomology is (homotopy groups) $\otimes \mathbb{Q}$. Of course the grading has to be shifted so that k -dimensional chains in A get degree $-k$ and the k -dimensional homotopy groups get degree $1 - k$. See R.M. Hain “Iterated integrals and homotopy periods” (Nauka 1988).¹

¹The original book, in English, is Mem. Amer. Math. Soc. **47** no. 291 (1984).

Questions. (1) Where are these facts written down (I did not find them in the book of Hain after a cursory glance)?

(2) Why are Sugawara operads acyclic?

(3) What is the nature of the duality between commutative DG algebras and DG Lie algebras? There is an analogous duality between associative DG algebras and, again, associative DG algebras. Is there a general concept of duality of operads?²

Let's pass to the deformation theory. Since there is no general theory (to devise one is one of the problems), let's consider two typical examples. In each of them DG Lie algebras arise.

Example 1. Deformations of associative algebras.

In order to give a vector space V the structure of an associative algebra, we have to pick an $f \in \text{Hom}(V \otimes V, V)$ that satisfies a certain quadratic relation. It's convenient to consider on $\bigoplus_{n=1}^{\infty} \text{Hom}(V^{\otimes n}, V)$ the structure of a Lie superalgebra, identifying $\bigoplus_{n=1}^{\infty} \text{Hom}(V^{\otimes n}, V)$ with the space of superderivations of the cofree coassociative supercoalgebra $\bigoplus_{n=1}^{\infty} V^{\otimes n}$. Then the condition on f can be written in the form $\frac{1}{2}[f, f] = 0$. Fix any f_0 such that $\frac{1}{2}[f_0, f_0] = 0$. Linearizing the equation to $\frac{1}{2}[f_0 + h, f_0 + h] = 0$ we get $[f_0, h] = 0$. The complex $\bigoplus_{n=1}^{\infty} \text{Hom}(V^{\otimes n}, V)$ arises with the differential $d = \text{ad } f_0$. This DG Lie algebra, as is known, is responsible for deformations. The precise assertion, subsuming all earlier assertions on this theme made in the literature, is the following. For simplicity we remove from our DG Lie algebra its zeroth component $\text{Hom}(V, V)$ (this corresponds to the fact that we are intending to get a variety of associative algebra structures on V not factored by the action of $\text{GL}(V)$), apply the functor F and dualize (in order to get an algebra, not a coalgebra). This algebra sits in degrees ≤ 0 . Take its 0-dimensional cohomology.

Claim: We get a maximal ideal of the ring of functions on the space of associative algebras completed at the point f_0 .

This is a tautology (especially if we note that $\frac{1}{2}[f_0 + h, f_0 + h] = dh + \frac{1}{2}[h, h]$).

It's natural to introduce the hypervariety of associative algebra structures on V such that

- (1) the algebra of functions on it is a DG algebra concentrated in degrees ≤ 0 , whose 0-dimensional cohomology is the algebra of functions on the usual variety,
- (2) the completions of this DG algebra are constructed as indicated above.

In this case the global object differs little from the local one: the corresponding algebra is simply a Chevalley complex, constructed from $\bigoplus_{n=1}^{\infty} \text{Hom}(V^{\otimes n}, V)$ (for which, since we are not completing, that Chevalley complex should be understood as a direct sum, not a direct product). This is seen more clearly in coordinates: if we simply write the

²See comment (1) at the end.

associativity condition $c_{ij}^r c_{rk}^\ell = c_{ir}^\ell c_{jk}^r$ then we now introduce new odd unknowns ξ_{ijk}^ℓ and write $c_{ij}^r c_{rk}^\ell - c_{ir}^\ell c_{jk}^r = d\xi_{ijk}^\ell$ etc.

This is all satisfied for the deformations of commutative algebras and Lie algebras (instead of cofree coassociative coalgebras it's necessary to use the corresponding Lie coalgebras and cocommutative coalgebras – again duality!)

In the previous example, for a complete picture it was necessary to factor out by the action of $GL(V)$. But BRST and Feigin teach that instead of a naive factorization (especially when the action is not free) it's better to adjoin odd variables. It's the same thing as applying our functor F on all DG Lie algebras (including the zeroth component)³. By the way, if k has nonzero characteristic, then there may be problems here since adjoining odd variables is coarser than accounting for the action of a group (since these “odd variables” depend only on the action of the Lie algebra).

Example 2. Let M be a smooth projective variety. How do we construct the DG Lie algebra responsible for the deformations of M ? Its cohomology is, of course, $H^i(M, \Theta)$, where Θ is the tangent bundle of M . The DG algebra itself is simpler to construct in the \mathbb{C} -analytic situation (Θ -valued forms of type $(0, q)$). In the case of an arbitrary field of characteristic 0 we have to use the Thom–Sullivan complex corresponding to some open covering, or construct a Sugawara structure on the usual Cech complex.

At first glance, in characteristic p there must arise the problem of commutative cochains, Steenrod operations, and all such things. The simplest effect of this type (in characteristic 2) would be the nonequality $[\alpha, \alpha] \neq 0, \alpha \in H^1(M, \Theta)$. It's possible, however, to show that if $\alpha \in H^1(M, \Theta)$ then $[\alpha, \alpha] = 0$. Moreover, it's possible to define an $\alpha^2 \in H^2(M, \Theta)$ for every $\alpha \in H^1(M, \Theta)$ so that $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + [\alpha, \beta]$. The proof uses the fact that in any characteristic, besides the Lie algebra of vector fields there is a *formal group of automorphisms*, and more precisely a Hopf algebra corresponding to this group. This is a Hopf algebra B such that its primitive elements are vector fields and, moreover, B is formally cosmooth in the sense that if from the coalgebra $\text{Ker}(B \xrightarrow{\epsilon} k)$ we construct a DG Lie algebra by applying the functor G , then this DG algebra has only one-dimensional cohomology (which are the primitive elements). Note that if the characteristic of k is not 0 then the universal enveloping algebra is not formally cosmooth.

Conjecture. (from which we could get the DG Lie algebra responsible for deformations of M)

Let there be a cosimplicial Hopf algebra $A_0 \rightrightarrows A_1 \cdots$ for which all A_k are cocommutative and formally cosmooth. Let \mathfrak{g}_k be the primitive elements of A_k . Then on the complex $\mathfrak{g}_0 \rightarrow \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \rightarrow \cdots$ there is the structure of a DG Sugawara–Lie algebra (more precisely, a class of such structures, connected to each other by Sugawara isomorphisms).

Anyway, in characteristic 0 there is an honest DG Lie algebra, which means an honest commutative DG algebra (by the way, from the fact that $\dim H^*(M, \Theta) < \infty$ it follows

³See comment (2) at the end.

that this commutative DG algebra can be chosen pro-free with a finite number of generators). Let's suppose that $H^0(M, \Theta_M) = 0$ and, moreover, $\text{Aut}(M) = \{e\}$. Then we can speak about the moduli hyperspace completed at a given point. Question: how to define a moduli hyperspace globally? What is, generally, a hyper-ringed space, hyperscheme, etc.? How to glue them together? What does it mean to say "deformation of M with base $\text{Spec } B$," where B is a commutative DG algebra?⁴ If, let's say, B sits in degrees ≤ 0 , and B_0 is a complete local ring, then, probably, as in the usual case, a deformation of an open affine chart $U_i \subset M$ is trivial, but nontriviality occurs from the way they are glued. The difference from the usual theory of deformations must apparently arise from the following: let \tilde{U}_i be a trivially deformed U_i ; we must give for each pair (i, j) an isomorphism ϕ_{ij} between the open subset of \tilde{U}_i corresponding to $U_i \cap U_j$ and the open subset of \tilde{U}_j corresponding to $U_i \cap U_j$; in the usual theory it's required that $\phi_{ij}\phi_{jk} = \phi_{ik}$, but now we have to require that there is a homotopy between $\phi_{ij}\phi_{jk}$ and ϕ_{ik} and that there are higher homotopies.

I see two instructive examples. The first one is to compute the moduli hyperspace of abelian varieties (for rigidity, with a marked point). Here the DG Lie algebra is quasi-isomorphic to the DG algebra of Θ -valued constant $(0, q)$ -forms, in which the commutator is 0. Therefore the corresponding commutative DG algebra is isomorphic to its own cohomology, i.e., one simply needs to find some sheaf of supercommutative \mathbb{Z} -graded algebras $\mathcal{O} = \mathcal{O}_0 \oplus \mathcal{O}_1 \oplus \dots$ on the moduli space of abelian varieties, with \mathcal{O}_0 the usual sheaf of functions. From general considerations \mathcal{O} is locally freely generated over \mathcal{O}_0 by certain generators (finitely many in each dimension). From this \mathcal{O}_1 and $\mathcal{O}_2/\Lambda^2\mathcal{O}_1$ are easy to find. Question: can we describe the extension $0 \rightarrow \Lambda^2\mathcal{O}_1 \rightarrow \mathcal{O}_2 \rightarrow \mathcal{O}_2/\Lambda^2\mathcal{O}_1 \rightarrow 0$?

Second example: for a smooth variety M , find the part of the Hilbert hyperscheme parametrizing finite subschemes of degree 1 (!!!). Generally, if Y is a subscheme of M then the tangent space at Y in the Hilbert scheme is $\text{Hom}_{\mathcal{O}_M}(\mathcal{I}_Y, \mathcal{O}_Y)$, and its "higher analogues"⁵ are apparently $\text{Ext}_{\mathcal{O}_M}^i(\mathcal{I}_Y, \mathcal{O}_Y)$. The commutator in $\bigoplus_i \text{Ext}^{i-1}(\mathcal{I}_Y, \mathcal{O}_Y)$ is probably defined by the formula $[a, b] = axb - bxa$, where $x \in \text{Ext}^1(\mathcal{O}_Y, \mathcal{I}_Y)$ is the canonical element. In the case when Y is a point, a DG Lie algebra with commutator 0 is obtained, and the same problem arises as with the moduli hyperspace of abelian varieties.

I'd like it if you and Sasha Beilinson and others thought about all these themes.

Sincerely,

Volodya

⁴See comment (3) on the next page.

⁵See comment (4) on the next page.

Extra comments and references added by B. Toën

- (1) Koszul duality for quadratic operads appears in [2, 3]. A general form of Koszul duality for E_n -algebras, $n \in [1, \infty]$ has been expressed as an equivalence of certain ∞ -categories in [4].
- (2) This is the infinitesimal analogue of taking the quotient stack by the natural action of $GL(V)$ (see [6] *Quotient stacks* in §3.3). See also [7] Prop. 2.2.6.8 and 2.2.6.9, for a construction of the global derived moduli stack of algebra structures whose tangent complex is the one described by Drinfeld.
- (3) It is possible to define a global derived stack (in the sense of [6] §3.3) of flat and proper schemes. By definition it consists of the ∞ -functor sending a commutative simplicial ring B to the ∞ -groupoid of derived schemes X , together with a flat and proper morphism $X \rightarrow \mathbf{Spec} B$. The resulting derived stack is not an Artin derived stack (in the sense of [6] Def. 3.1) because of the existence of formal deformations of smooth projective varieties which are not algebraizable (this can be fixed as usual by fixing polarizations). It is however a well defined derived stack with many nice infinitesimal properties, and in particular possesses a tangent complex and an obstruction theory. The cohomology groups of this tangent complex, at a point corresponding to a proper scheme M , are given by the higher cohomology spaces $H^i(M, \Theta_M)$. We refer to [5] for more results in this direction.
- (4) The higher structures described here correspond to the tangent complex of the derived quot scheme, as considered for instance in [1], rather than of the derived Hilbert scheme. For the structure sheaves of subschemes, the derived quot scheme and the derived Hilbert scheme share the same underlying scheme. However, the derived structures differ, as already noted in [1] (see also the footnote in [6] on page 12). This is a general phenomenon: a given classical moduli problem might have different natural derived extensions, depending on how one “thinks” of this moduli problem. Here the two different derived extensions correspond to the two natural ways of considering the structure sheaf of a sub-scheme, either as a quasi-coherent \mathcal{O} -module or as a quasi-coherent commutative \mathcal{O} -algebra. One could introduce a hierarchy of different derived extensions by also considering these structure sheaves as quasi-coherent E_n - \mathcal{O} -algebras for various n .

References

- [1] Ciocan-Fontanine, I; Kapranov, M. *Derived Quot schemes*. Ann. Sci. Ecole Norm. Sup. (4). **34** (3), (2001), 403–440.
- [2] Ginzburg, V.; Kapranov, M. *Koszul duality for operads*. Duke Math. J. **76** (1), (1994), 203–272.
- [3] Loday, J.L.; Vallette, B. *Algebraic operads*. Grundlehren der mathematischen Wissenschaften, Volume **346**, Springer-Verlag (2012).
- [4] Lurie, J. *Moduli problems for ring spectra*. Proceedings of the International Congress of Mathematicians. Volume II, 1099–1125, Hindustan Book Agency, New Delhi, 2010. (see also the expanded version *Formal moduli problems* available at: <http://www.math.harvard.edu/~lurie/>).

- [5] Pridham, J. *Derived moduli problems of schemes and sheaves*. J. of K-theory **10** (1), (2012), 41–85.
- [6] Toën, B. *Derived Algebraic Geometry*. EMS Surv. Math. Sci. **1** (2), (2014), 153–240
- [7] Toën, B.; Vezzosi, G. *Homotopical algebraic geometry. II. Geometric stacks and applications*. Mem. Amer. Math. Soc. **193** (902), (2008), x+224 pp.

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