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Collective synchronization of classical and quantum oscillators

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Abstract. Synchronization of weakly coupled oscillators is ubiquitous in biological and chemical complex systems. Recently, research on collective dynamics of many-body systems has been received much attention due to their possible applications in engineering. In this survey paper, we mainly focus on the large-time dynamics of several synchronization models and review state-of-art results on the collective behaviors for synchronization models. Following a chronological order, we begin our discussion with two classical phase models (Winfree and Kuramoto models), and two quantum synchronization models (Lohe and Schrödinger–Lohe models). For these models, we present several sufficient conditions for the emergence of synchronization using mathematical tools from dynamical systems theory, kinetic theory and partial differential equations in a unified framework.

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Contents

1. Introduction

The jargon “synchronization” is a compound word made of syn (the same) and chronous (time) in Greek, and it represents a phenomenon in which rhythms of weakly coupled oscillators are adjusted to the common frequency due to their weak interactions between oscillators. The phase-locking and frequency synchrony in an ensemble of oscillators are ubiquitous in coupled oscillator models in various scientific areas such as biology, engineering, physics, and social sciences, etc. Despite of its natural appearance [15] in our nature, its scientific report was a rather recent event compared to long history of human beings. In the middle of seventeenth century, Christian Huygens observed an anti-phase synchronization of two pendulum clocks hanging over the common bar on the wall, and reported his observation to Royal Academy of Sciences with the title “Of an odd kind of sympathy”. This was the first event that synchronization became a science topic, and Huygens’ naked eye observation was finally experimentally confirmed by a group of physicists in Georgia Tech [14] in the early 21st century. After Huygens’ observation, synchronization has been reported on and off in early 20th century by several scientists such as Lord Rayleigh, Edward Appleton, and Balthasar van der Pol, to name a few (see a book [76] for a brief history of synchronization). In particular, Nobert Wiener was interested in the synchronous behaviors of α-rhythms arising from the coupled neurons in human brain. In [86], he had discussed a synchronization phenomenon of α-rhythms, but it was pretty immature to verify and study such a collective phenomenon using an advanced technology such as MRI (see [80]). However, Wiener's enthusiasm on synchronization of α-rhythm was transferred to Arthur Winfree. As his bachelor thesis, he tried to explain Wiener's curiosity using a mathematical model. He proposed the first mathematical model for synchronization and succeed to explain synchronization phenomenon using this model, which was published later as a paper in [88]. Soon after Winfree's work, a physicist Yoshiki Kuramoto introduced another simple ODE model in [58]. In a thermodynamic limit in [57], he showed that the kinetic Kuramoto model exhibits a phase-transition like phenomenon in a bifurcation diagram in asymptotic order parameter vs. coupling strength. Furthermore, he derived an explicit representation for the critical coupling strength in his self-consistent analysis. However, some of tacit assumptions in self-consistent analysis are still unclear to verify [1, 79]. The Kuramoto model contains two competing mechanisms: intrinsic randomness and nonlinear coupling. Collective dynamics such as the emergence of phase-locked states arises from these competing. Recently, the synchronization of nonlinear oscillators became an active research area in different disciplines such as biology, nonlinear dynamics, statistical physics, and sociology, etc, due to their possible engineering applications, for example, in power grid system and UAV (unmanned aerial vehicles).

The purpose of this paper is to review state-of-art results on the collective dynamics of synchronization models for classical and quantum oscillator systems and discuss the relations between models. In particular, we address the question on the validity of models (or justification of the model). Since synchronization models in literature are phenomenological, and they are not derived from the first principle of physics, the validity of the model for a given situation has been based on the numerical simulations and heuristic arguments. Thus, the following question

> "Under what conditions on initial data and parameters in the model, can we expect some desired collective behaviors of the model?"

is quite natural from the modeling viewpoint. We do not try to present all existing results on the synchronization. Instead, our choices of topics are very subjective depending on authors' experience and taste. Even for one synchronization, namely the Kuramoto model, there is a rather extensive review papers [1] on the mathematical and physical results of the Kuramoto and its variant models up to 2005. However in last ten years, the number of literature on synchronization was increased to more than a thousand. Hence, it will be an almost impossible job to present a summary of all known results. Recently, Döfler and Bullo wrote an interesting survey article [30] focused on the synchronization in complex networks of Kuramoto oscillators. What we present on the Kuramoto model in Section 3 can be regarded as orthogonal results after Döfler and Bullo. Compared with previous survey articles, our main focus is to present synchronization estimates for several synchronization models in a unified framework, which complements existing review papers [1, 12, 30, 79]. There are many synchronization models in literature and most of them need mathematical treatment, and at present there are few analytically tracktable models. In this review paper, we focus on mainly four models, namely the

Winfree model and the Kuramoto model for classical synchronization, and the Lohe model and the Schrödinger–Lohe model for quantum synchronization. For these models, we review sufficient frameworks leading to the collective behaviors. In this paper, we have not discussed any possible applications of the presented theoretical works to other research fields such as biology, control engineering, economics, physics, politics, and sociology, etc., but we expect that mathematical results on the synchronization models can provide a rigorous foundation for more refined modelings, e.g., consensus algorithms in engineering and social sciences such as the design of distributed power grids and networks [31,33,62], distributed controls of robot networks [16], herding phenomenon of volatilities in financial markets [3,8,10,17,73] and clustering and swarming phenomena in biological and social complex systems [11,12,38,54], to name a few.

The rest of the paper is organized as follows. In Section 2, we present five synchronization models and discuss relationships between them. In Section 3, as the first mathematical model for synchronization, we discuss collective behaviors of the Winfree model such as partial and complete oscillator deaths, and partial and complete phase-locking. In Section 4. we present the Kuramoto model which is known to be the most well-known prototype phase model in the synchronization community and present a sufficient framework leading to the complete synchronization. We also discuss its kinetic counterpart and present well-posedness issues in measure and BV settings. In Section 5, we present the Lohe matrix-valued ODE model for a quantum synchronization. In two complex dimensions, 2×2 unitary matrix can be expanded as a linear combination of Pauli's matrices. Then, we can reduce the Lohe model into five equations, one for phase and four for coefficients in Pauli's expansion, and we study dynamic features of the special model. After this special case, we present a complete synchronization estimate in any dimension. In Section 6, we present an infinite-dimensional Lohe model, namely, the Schrödinger–Lohe model which can be used in the synchronization of the Schrödinger equation. Finally, Section 7 is devoted to the summary of this paper and discuss possible future directions.

Before we move on to the next sections, we should tell you the style of presentation in this survey paper. Due to the limitation of pages and vast literature in the synchronization models, we have to restrict our focus on issues to be discussed to some topics. Because of this, presentation might be rather dry and formal, so some parts can be boring to read. In any case, we try to give an overall view on the current status of the state-of-art results, and some feeling and intuitions why some results can be true and some brief outline of proofs for some major theorems. If you find some topics are interesting enough, we recommend to look at the references therein.

2. Preliminaries

In this section, we discuss three types of agent (oscillator)-based synchronization models which are designed for each oscillators, namely *pulse-coupled one*, *phase-coupled one* and *state-coupled one*, and then introduce five specific models in a chronological order. The phase-coupled models and state-coupled models will be discussed in detail in the following four sections. We will also discuss relationships between these models. In literature [1, 76, 88], several oscillator-based phenomenological models were proposed to investigate the collective behaviors of an ensemble of coupled oscillators, e.g., flashing of fireflies [15] and synchronous firing of cardiac pacemaker [75], etc. Phenomenological synchronization models can be classified into three categories, pulse-coupled, phase-coupled and state-coupled ones. In the following three subsections, we discuss these three types of models.

2.1. A pulse-coupled model. For the pulse-coupled models [69, 75], a basic dynamic entity to be monitored is an action potential of each oscillator, think of coupled pacemaker cells in your heart. These pacemaker cells themselves are beating rhythmically, thus they can be understood as oscillators. In fact, our regular heart beating is due to their synchronous collective behavior. One of such pulse-coupled model is an integrate-and-fire model. This model consists of two steps (dynamic and firing steps). In the first dynamic step, each action potential increases in time, until it reaches some threshold value. In the second firing steps, once it reaches that value, then it suddenly drops to some lower value and at the same time, it makes other neighboring cell's action potentials boost to increase. By repeating this process over time, action potentials of pacemaker cells become synchronized in finite time.

2.1.1. The Peskin model. As an concrete example of a pulse-coupled model, we present the Peskin model for coupled pacemaker cells that is similar to integrate-and-fire model in neuroscience [69, 75]. Let $x_j = x_j(t) \in [0, 1]$ be a normalized voltage-like action potential of the j-th cell and here the threshold value is assumed to be unity. Then, the Peskin model reads as follows.

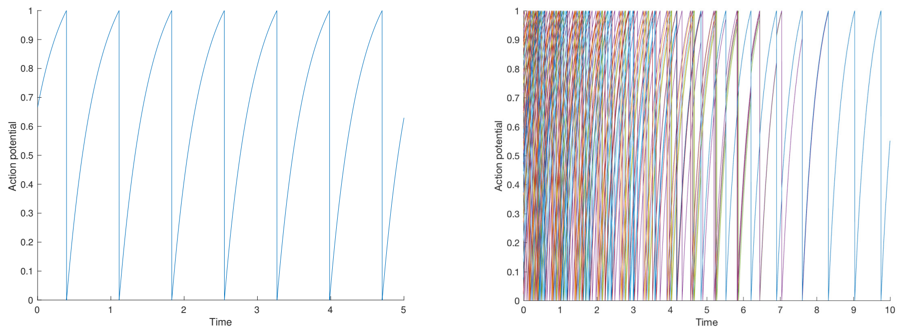
$$\begin{cases} \dot{x_j} = -\delta_j x_j + S_j: & \text{integrating step,} \\ \text{If } \ x_j(t_*) = 1, \ \text{then } x_j(t_*+) = 0 \text{ and} & \\ x_k(t_*+) = \min\{1, x_k(t_*) + \varepsilon_{jk}\} \text{ for } k \neq j: & \text{firing step,} \end{cases} \tag{2.1}$$

where S_j and δ_j are positive intrinsic parameters of pacemaker cells satisfying the relation $S_j > \delta_j$ so that the action potentials reach to unity in finite time. In the original Peskin's model [75], he considered only identical cells, whose intrinsic properties are the same. In a cardiac pacemaker, the SA node is beating with its intrinsic rate 60–100 bpm, so the pacemakers are not identical. Moreover, the parameter ε_{ij} in the firing step may depend on the connected pair of cells. If we consider some ε_{ij} to be 0, then the pulse-coupling is not all-to-all any more. In the case of fireflies, each one can affect only nearest ones, so the assumption of all-to-all coupling is also not natural. However, for the mathematical treatment, we are forced to consider an ensemble of identical pacemaker cells with all-to-all couplings:

$$\delta_j = \delta, \quad S_j = S, \quad \varepsilon_{jk} = \varepsilon_j > 0, \quad 1 \le j, k \le N. \tag{2.2}$$

Peskin [75] showed that system (2.1)–(2.2) with $N = 2$ exhibits finite-time synchronization for any initial configurations and conjectured that this will be true for any number of cells and generic initial configurations (see Figure 1 for $N = 50$). Finally, Mirollo and

Strogatz showed that Peskin's conjecture is in fact true for any finite N and any initial configuration up to Lebesgue measure zero. Their result can be summarized as follows.



(a) Time-evolution of 1 neuron (b) Time-evolution of 50 neurons

Figure 1. $N = 50,\ \delta = 2.5,\ S = 3,\ \varepsilon_j = 0.002,\ \ \Delta t = 0.001$, 4th order Runge–Kutta method

Theorem 2.1 ([69]). *Let x_j be a solution to (2.1)–(2.2). Then except a measure zero set of initial configurations, a.e. initial configurations lead to the complete synchronization in finite time, i.e., there exists a positive constant $T \in (0, \infty)$ such that*

$$|x_i(t) - x_j(t)| = 0, \quad t \geq T,\ 1 \leq i, j \leq N.$$

Remark 2.2. (i) Mirollo and Strogatz's arguments in the proof do not characterize the measure zero set explicitly. Thus, for a given initial configuration $\mathcal{X}_0 = (x_{10}, \ldots, x_{N0})$, it is not clear at all whether it belongs to this measure zero set or not. Hence, for a practical purpose, it will be very interesting to provide this measure zero set equipped with explicit conditions (e.g., see Theorem 4.7 for the Kuramoto model in Section 4).

(ii) As far as authors know, rigorous justification of the synchronization for the non-identical oscillators is still open.

2.2. Phase-coupled models. For a phase-coupled model, the basic dynamic entity to be monitored is phase for the state of an oscillator. One way to visualize an ensemble of oscillators is to regard them as point rotors moving on the unit circle $\mathbb{S}^1 \in \mathbb{C}$. More precisely, let $x_j = e^{\mathrm{i}\theta_j}$ be the position of the j-th rotor with an natural frequency Ω_j. When there are no mutual interactions between rotors, each rotor will move along the circle with their given natural frequency Ω_j which is assumed to be time-independent random variable extracted from some given distribution function $g = g(\Omega)$:

$$\dot{\theta}_j = \Omega_j, \quad t > 0, \quad \text{i.e.,} \ \ \theta_j(t) = \theta_j(0) + t\Omega_j, \quad t \geq 0,\ 1 \leq j \leq N. \tag{2.3}$$

Hence, the motion of ensemble is completely integrable. However, when oscillators begin to interact, interaction effects need to be incorporated to the motion of the free flow (2.3).

Then, how to incorporate such interaction effects will be the topics to be discussed in the following subsections. To put our discussion on a more general setting, we consider a time-independent network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \mathcal{C})$, where $\mathcal{N}, \mathcal{E}$, and $\mathcal{C} = (c_{ij})$ denote the set of nodes, edges, and capacity matrix representing the degree of communication capacity between nodes, respectively. Assume that oscillators are located on the nodes of the network, and their mutual interactions are made through the connecting edges with a capacity.

Suppose that the state of the j-th oscillator is represented by a complex number $z_j(t) = r_j(t)e^{i\theta_j(t)}$. Then, the state of oscillator will be determined by two real valued quantities, namely amplitude r_j and phase θ_j. Throughout the paper, we will consider weakly coupled oscillators, i.e., amplitudes keep to be constant, say $r_j = 1$. Thus, for weakly coupled oscillators, we only need to consider the variation of phase θ_j. As long as there is no confusion, we will use the terminology phase model instead of phase-coupled model for simplicity of presentation. Before we move on to the description of phase models, we first recall the notions of phase-locked and phase-locked state in the following definitions.

Definition 2.3. Let $\Theta(t) = (\theta_1(t), \dots, \theta_N(t))$ be a state of a phase model.

(1) The state Θ is phase-locked if the phase differences are constant in time:

$$\theta_i(t) - \theta_j(t) = \theta_{ij}^{\infty} : \text{constant}, \quad 1 \le i, j \le N.$$

(2) The state Θ approaches a phase-locked state asymptotically if the phase differences approach to some fixed value as $t \to \infty$:

$$\exists \lim_{t\to\infty} |\theta_i(t) - \theta_j(t)|, \quad 1 \le i, j \le N.$$

In the sequel, we will consider two phase models, the Winfree model and the Kuramoto model.

2.2.1. The Winfree model. As aforementioned in the introduction, Arthur Winfree [88] introduced the first mathematical model for synchronization as his bachelor thesis in the middle of 1960's. In his thesis, he proposed a phase model for weakly coupled oscillators.

Let $\theta_j = \theta_j(t)$ and Ω_j be the phase and natural frequency of the j-th Winfree oscillator, respectively. In the presence of mutual interactions, the free dynamics (2.3) should be supplemented by adding suitable frequency perturbations in the R.H.S. of (2.3):

$$\dot{\theta}_j = \Omega_j + \omega_j^{\text{per}}, \quad i = 1, \dots, N. \tag{2.4}$$

Thus an interesting question is how to model such frequency perturbations ω_j^{per} using some kind of reasonable rules (axioms). For this, Winfree introduced two real-valued functions employed in the modeling of interactions:

$$I = I(\theta) : \text{an influence function and}$$
$$S = S(\theta) : \text{an sensitivity (or response) function.}$$

and two rules for phase interactions:

($\mathcal{A}1$) The stimulus I_c impinging on the j-th oscillator is given as a weighted sum of the influences contributed by phases of all other oscillators in the ensemble:

$$I_c(\Theta) := \sum_{k=1}^{N} c_{kj} I(\theta_k).$$

In this case, two assumptions are implicitly made. First, the influences of an individual are assumed to be propagated without attenuation, in a time much shorter than the average period of the oscillators. Second, they are additive in their effect.

($\mathcal{A}2$) The frequency perturbation ω_j^{per} of the j-th oscillator is proportional to the product of the sensitivity $S(\theta_j)$ and the average stimulus $I_c(\Theta)$:

$$\omega_j^{\text{per}} = KS(\theta_j)I_c(\Theta) = K\sum_{k=1}^{N} c_{kj} S(\theta_j) I(\theta_k). \tag{2.5}$$

Combining (2.4) and (2.5), Winfree proposed the following coupled ODE system:

$$\dot{\theta}_j = \Omega_j + K\sum_{k=1}^{N} c_{kj} S(\theta_j) I(\theta_k), \quad j = 1, \dots, N. \tag{2.6}$$

As an explicit example, if we set

$$c_{kj} = \frac{1}{N}, \quad S(\theta) = -\sin\theta, \quad I(\theta) = 1 + \cos\theta,$$

then, we have a special case of the Winfree model:

$$\dot{\theta}_j = \Omega_j - \frac{K}{N}\sum_{k=1}^{N} \sin\theta_j (1 + \cos\theta_k), \quad j = 1, \dots, N. \tag{2.7}$$

As an obvious generalization of (2.6), we can also think of a generalized Winfree model [58]: for $j = 1, \dots, N$, $\ell \geq 1$,

$$\dot{\theta}_j = \Omega_j + K_1\sum_{k=1}^{N} c_{kj}^1 S_1(\theta_j) I_1(\theta_k) + \cdots + K_\ell \sum_{k=1}^{N} c_{kj}^\ell S_\ell(\theta_j) I_\ell(\theta_k). \tag{2.8}$$

In general, the Winfree model (2.6) does not have any conservation laws (constraints) except the number of oscillators. Furthermore, it is not a gradient flow except the special choice of influence and sensitivity functions as in (2.7). Due to the lack of conservation laws, there are not many tools to investigate the dynamics of the Winfree model. This is why there are few literature [36, 44, 45, 49, 65, 70, 72, 77, 78, 87, 88] dealing with the Winfree model, compared to the enormous amount of literature on the Kuramoto model. However, the Winfree model exhibits many interesting asymptotic dynamics compared to the Kuramoto model. We next study the gradient flow structure of the Winfree model with special choices of S and I.

Proposition 2.4. *Suppose that the network structure $\mathcal{N}$ is symmetric:*

$$c_{kj} = c_{jk} \quad \text{for all } 1 \le k, j \le N,$$

and the influence and sensitivity functions are analytic, and let $\Theta = (\theta_1, \dots, \theta_N)$ be a solution to (2.6). Then, the Winfree model (2.6) is a gradient flow if and only if the influence and sensitivity functions satisfy

$$S(\theta) = I'(\theta),$$

i.e., the Winfree model takes the form of a gradient flow:

$$\dot{\Theta} = -\nabla_\Theta V(\Theta), \quad \text{where } V(\Theta) := -\sum_{k=1}^{N} \Omega_k \theta_k - \frac{K}{2} \sum_{k,l=1}^{N} c_{kl} I(\theta_k) I(\theta_l).$$

2.2.2. The Kuramoto model. Consider one Landau–Stuart (L–S) oscillator whose state $z \in \mathbb{C}$ is governed by the following complex-valued ODE:

$$\dot{z} = (1 - |z|^2 + \mathrm{i}\Omega) z. \tag{2.9}$$

Then, the equation (2.9) can be rewritten in terms of r and θ:

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = \Omega. \tag{2.10}$$

Note that it is easy to see that $r = 0$ and $r = 1$ are unstable and stable equilibrium point for the first equation in (2.10), respectively, i.e., the unit circle $r = 1$ is a stable limit-cycle for (2.9), whereas the phase dynamics is a linear motion in t, i.e., $\theta(t) = \theta_0 + \Omega t$. Let $z_j \in \mathbb{C}$ be the position of the j-th L–S oscillator, whose dynamics is governed by (2.9). We now consider a diffusive coupling of N L–S oscillators:

$$\dot{z}_j = (1 - |z_j|^2 + \mathrm{i}\Omega_j) z_j + K \sum_{k=1}^{N} c_{kj} (z_k - z_j), \quad j = 1, \dots, N, \tag{2.11}$$

where K is the positive coupling strength and Ω_j is the natural frequency of the j-th L–S oscillator. Then, it follows from numerical simulations that z_j approaches to a limit cycle $|z_j| = r_\infty$ for some range of K. To fix the idea, we assume that

$$r_\infty = 1 \quad \text{and} \quad z_j = e^{\mathrm{i}\theta_j}.$$

We now substitute the ansatz $z_j = e^{\mathrm{i}\theta_j}$ into (2.11) to obtain the Kuramoto model:

$$\dot{\theta}_j = \Omega_j + K \sum_{k=1}^{N} c_{kj} \sin(\theta_k - \theta_j). \tag{2.12}$$

Note that the R.H.S. of (2.12) is 2π-periodic, hence system (2.12) can be understood as a dynamical system on tori, but it can also be viewed as a dynamical system on $\mathbb{R}^N$ by lifting from tori to $\mathbb{R}^N$. Throughout the paper, we treat the Kuramoto model as a dynamical system in $\mathbb{R}^N$.

Proposition 2.5 ([47,81]). *Suppose that the network structure $\mathcal{N}$ is symmetric:*

$$c_{kj} = c_{jk} \quad \textit{for all } 1 \le k, j \le N,$$

and let $\Theta = (\theta_1, \dots, \theta_N)$ be a solution to (2.12). Then, the following assertions hold:

(1) *The total phase satisfies a balanced law:*

$$\frac{d}{dt} \sum_{k=1}^{N} \theta_k = \sum_{k=1}^{N} \Omega_k. \tag{2.13}$$

(2) *The system (2.12) is a gradient flow:*

$$\dot{\Theta} = -\nabla_\Theta V(\Theta),$$

where the analytic potential $V(\Theta)$ is given by

$$V(\Theta) := -\sum_{k=1}^{N} \Omega_k \theta_k + \frac{K}{2N} \sum_{k,l=1}^{N} c_{kl} \big(1 - \cos(\theta_k - \theta_l)\big).$$

Remark 2.6. The assertions of Proposition 2.5 have several implications.

(i) If the sum of natural frequencies is not zero, then it follows from (2.13) that $\sum_{j=1}^N \dot{\theta}_j \neq 0$. Thus, there will be no equilibrium for the Kuramoto flow. Hence we need to consider a relaxed equilibrium, namely relative equilibrium which means that the relative phases are time-invariant.

(ii) Since the Kuramoto system is a gradient flow system with analytic potential, the uniform boundedness of state is equivalent to the existence of limit $\lim_{t\to\infty} \Theta(t)$ (Theorem 2.7), which excludes the possibility of chaotic dynamics. Thus, in order to prove the existence of the phase-locked state (see Definition 2.3), we only need to show the uniform boundedness of fluctuations with respect to averaged motion. This is one of the key idea to the resolution of the complete synchronization problem for the Kuramoto model in [29,40]

Theorem 2.7 ([47]). *Let $\Theta = \Theta(t)$ be a uniformly bounded global solution to (2.12) with zero sum of natural frequencies in $\mathbb{R}^N$:*

$$\sup_{0 \le t < \infty} \|\Theta(t)\|_\infty < \infty, \quad \textit{where } \|\cdot\|_\infty := \|\cdot\|_{\ell^\infty}.$$

Then the phase configuration $\Theta(t)$ and the frequency vector $\dot{\Theta}(t)$ converge to a phase locked state and the zero vector, respectively as $t \to \infty$, i.e., there exists a phase locked state Θ^∞ such that

$$\lim_{t\to\infty} \|\Theta(t) - \Theta^\infty\|_\infty = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\dot{\Theta}(t)\|_\infty = 0.$$

Remark 2.8. In dynamical systems theory, uniform boundedness does not generally imply convergence. This is essentially due to the gradient flow structure of the Kuramoto flow with an analytical potential.

So far, we have discussed two classical oscillator models where the phase of an oscillator at each node is uniquely determined. However, if we imagine that a quantum oscillator is located on the node of a network, then the phase of quantum oscillator will not be determined by a single valued function. We next introduce such a matrix-valued ODE model and a PDE model which describes the evolution of unitary transformation.

2.3. A state coupled model. In this subsection, we discuss two state coupled models for quantum synchronization. One is finite-dimensional and the other is infinite-dimensional. These models were originated from two papers [63,64] by a physicist, Max Lohe. As you can see them in the sequel, the motivation for these models are not clear at present, but in some special case, these models can be reduced to the Kuramoto model.

2.3.1. The Lohe model. Suppose that Lohe oscillators with finite-dimensional state space is distributed over the network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \mathcal{C})$, and let U_j and U_j^* be a $d \times d$ unitary matrix representing the state of the Lohe oscillator on the j-th node and its Hermitian conjugate, respectively, and let H_j be a $d \times d$ Hermitian matrix whose eigenvalues correspond to the natural frequencies of the Lohe oscillator at node j. Then, the Lohe model reads as follows.

$$\mathrm{i}\dot{U}_j U_j^* = H_j - \frac{\mathrm{i}K}{2} \sum_{k=1}^{N} c_{kj} \left(U_j U_k^* - U_k U_j^* \right), \quad j = 1, \ldots, N, \tag{2.14}$$

where K is the uniform, nonnegative coupling strength and c_{kl} is real.

Proposition 2.9 ([63,64]). *Then, the following assertions hold:*

(1) *Let $\{U_j\}$ be a solution to (2.14) with initial data $\{U_j^0\}$. Then, $U_j U_j^*$ is conserved along the Lohe flow:*

$$U_j(t) U_j^*(t) = U_j^0 U_j^{*0}, \quad t \geq 0, \ 1 \leq i \leq N.$$

(2) *The Lohe system (2.14) is invariant under right-translation by a unitary matrix in the sense that if $L \in U(d)$ and $V_j = U_j L$, then V_j satisfies*

$$\begin{aligned} &\mathrm{i}\dot{V}_j V_j^* = H_j - \frac{\mathrm{i}K}{2} \sum_{k=1}^{N} c_{kj} \left(V_j V_k^* - V_k V_j^* \right), \quad j = 1, \ldots, N, \ t > 0, \\ &V_j(0) = U_j^0 L. \end{aligned}$$

Remark 2.10. The first assertion shows that the unitarity of U_j is preserved along the Lohe flow; hence, all components of U_j are bounded a priori by the unity, and the matrix valued ODE in (2.14) can be regarded as a d^2-coupled system of first-order ODEs. Thus, the standard Cauchy–Lipschitz theory for local solutions and the a priori uniform bound of the components of U yield a unique global solution to (2.14).

2.3.2. The Schrödinger–Lohe model. In this part, we discuss a quantum synchronization for an infinite state space $d = \infty$. We first note that the Lohe model can be rewritten as another equivalent form. By multiplying U_j and using $U_j^* U_j = \mathrm{Id}$, we have

$$\mathrm{i}\dot{U}_j - H_j U_j = \frac{\mathrm{i}K}{2}\sum_{k=1}^{N} c_{kj}\left(U_k - U_j U_k^* U_j\right). \tag{2.15}$$

Note that L.H.S. is the free Schrödinger equation in a finite-dimensional form and R.H.S. is a nonlinear Lohe coupling. Based on this simple observation, the straightforward candidate for the Lohe model in an infinite-dimensional state space will be

$$\mathrm{i}\partial_t \psi_j + \frac{1}{2}\Delta\psi_j - V_j\psi_j = \frac{\mathrm{i}K}{2}\sum_{k=1}^{N} c_{kj}\left(\psi_k - \frac{\langle\psi_j,\psi_k\rangle}{\langle\psi_j,\psi_j\rangle}\psi_j\right), \quad (x,t)\in\mathbb{R}^d\times\mathbb{R}, \tag{2.16}$$

where K and $V_j = V_j(x,t)$ are nonnegative coupling strength and a real-valued potential function acting on the j-th oscillator, respectively, and $\langle\cdot,\cdot\rangle$ is the standard inner product in $L^2(\mathbb{R}^d_x)$. The Planck constant $\hbar$ and mass m are assumed to be unity so that they do not appear in system (2.16).

Proposition 2.11. *Let $\Psi = (\psi_1,\dots,\psi_N)$ be a solution to (2.16) with $\|\psi_j(0)\|_2 < \infty$. Then, we have*

$$\|\psi_j(t)\|_2 = \|\psi_j(0)\|_2, \quad t \geq 0.$$

Note that, on the manifold $\|\psi_j\|_2 = 1$, system (2.16) can be rewritten as

$$\mathrm{i}\partial_t\psi_j = -\frac{1}{2}\Delta\psi_j + V_j\psi_j + \frac{\mathrm{i}K}{2}\sum_{k=1}^{N} c_{kj}\Big(\psi_k - \langle\psi_j,\psi_k\rangle\psi_j\Big),$$

which has structural similarity to (2.15).

2.4. Relations between aforementioned models. In this subsection, we discuss the relations between aforementioned four synchronization models. In all relations, we will see that the Kuramoto model plays a role of a backbone.

2.4.1. From Winfree to Kuramoto. We first recall a trigonometric identity:

$$\sin(\theta_k - \theta_j) = \sin\theta_k\cos\theta_j - \cos\theta_k\sin\theta_j \quad \text{and} \quad c_{kj} = \frac{1}{N}$$

to see that

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\cos\theta_j\sin\theta_k + \frac{K}{N}\sum_{k=1}^{N}(-\sin\theta_j)\cos\theta_k.$$

If we now set

$$\ell = 2, \quad K_1 = K_2 = K, \quad (S_1(\theta), I_1(\theta)) = (\cos\theta, \sin\theta),$$

and

$$(S_2(\theta), I_2(\theta)) = (-\sin\theta, \cos\theta),$$

then, the Kuramoto model (2.12) becomes a special case of a generalized Winfree model (2.8). Note that the pair of sensitivity and influence functions are orthogonal in a standard inner product in $\mathbb{R}^2$.

$$\langle (S_1(\theta), I_1(\theta)) \cdot (S_2(\theta), I_2(\theta)\rangle = 0.$$

On the other hand, the Kuramoto model can also be derived from the Winfree in a low-coupling regime. For this, we follow heuristic arguments in [57, 74].

Consider the Winfree model with all-to-all interaction:

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^{N} I(\theta_k)S(\theta_j), \tag{2.17}$$

where I and S are smooth periodic functions with the common period T. We now consider the situation where the natural frequencies Ω_j of each oscillator are close to some fixed frequencies, say unity and the coupling strengths are sufficiently weak:

$$\Omega_j = 1 + \varepsilon\omega_j, \quad K = \varepsilon\kappa, \quad 0 < \varepsilon \ll 1. \tag{2.18}$$

Then, at the zeroth order in ε, we obtain

$$\dot{\theta}_j^0(t) = 1.$$

Thus, we can expect the solution of the equation (2.17) to be of the perturbative form:

$$\theta_j(t) = t + \theta_j^1(t).$$

We substitute this ansatz and (2.18) into the equation (2.17) to obtain

$$1 + \dot{\theta}_j^1(t) = 1 + \varepsilon\omega_j + \frac{\varepsilon\kappa}{N}\sum_{k=1}^{N} I(t + \theta_k^1(t))S(t + \theta_j^1(t)),$$

or equivalently,

$$\dot{\theta}_j^1(t) = \varepsilon\omega_j + \frac{\varepsilon\kappa}{N}\sum_{k=1}^{N} I(t + \theta_k^1(t))S(t + \theta_j^1(t)). \tag{2.19}$$

Because $S(\theta)$ and $I(\theta)$ are bounded and ω_j are constants, for sufficiently small ε, the change of phase for each oscillator would be very small in one period. In other words, the variation of θ_j^1 is much slower than that of θ_j. Then, we have

$$\theta_j^1(t),\ \dot{\theta}_j^1(t) \sim \mathcal{O}(\varepsilon), \quad t \in [t_0, t_0 + T].$$

Due to the small change of θ_j, the terms $\dot{\theta}_j^1(t)$ and $\theta_j^1(t)$ can be regarded as constants in the time interval $[t_0, t_0 + T]$. More precisely we have

$$1 + \dot{\theta}_j^1(t) \simeq 1 + \dot{\theta}_j^1(t_0), \quad \theta_j^1(t) \simeq \theta_j^1(t_0), \quad t \in [t_0, t_0 + T].$$

Then, we take integral of the equation (2.19) to obtain:

$$\begin{aligned}
1+\dot\theta_j^1(t_0) &= \frac{1}{T}\int_{t_0}^{t_0+T}\big(1+\dot\theta_j^1(t_0)\big)\,dt\\
&\simeq \frac{1}{T}\int_{t_0}^{t_0+T}\big(1+\dot\theta_j^1(t)\big)\,dt\\
&= \frac{1}{T}\int_{t_0}^{t_0+T}\Big(1+\varepsilon\omega_j+\frac{\varepsilon\kappa}{N}\sum_{k=1}^N I\big(t+\theta_k^1(t)\big)S\big(t+\theta_j^1(t)\big)\Big)\,dt\\
&\simeq 1+\varepsilon\omega_j+\frac{\varepsilon\kappa}{NT}\sum_{k=1}^N\int_{t_0}^{t_0+T} I\big(t+\theta_k^1(t_0)\big)S\big(t+\theta_j^1(t_0)\big)\,dt\\
&= 1+\varepsilon\omega_j+\frac{\varepsilon\kappa}{NT}\sum_{k=1}^N\int_{t_0+\theta_j^1(t_0)}^{t_0+\theta_j^1(t_0)+T} I\big(t+\theta_k^1(t_0)-\theta_j^1(t_0)\big)S(t)\,dt.
\end{aligned}$$

Thus we have

$$\dot\theta_j^1(t_0)\simeq\varepsilon\omega_j+\frac{\varepsilon\kappa}{NT}\sum_{k=1}^N\int_{t_0+\theta_j^1(t_0)}^{t_0+\theta_j^1(t_0)+T} I\big(t+\theta_k^1(t_0)-\theta_j^1(t_0)\big)S(t)\,dt. \tag{2.20}$$

For the special case for the pair (I,S):

$$I(x)=1+\cos x,\quad S(x)=-\sin x,$$

R.H.S. of (2.20) can be simplified as follows.

$$\begin{aligned}
-\frac{\varepsilon\kappa}{2N\pi}\sum_{k=1}^N\int_t^{t+2\pi}&\big(\cos(\tau+\theta_k^1(t)-\theta_j^1(t))+1\big)\sin\tau\,d\tau\\
&= -\frac{\varepsilon\kappa}{2N\pi}\sum_{k=1}^N\int_t^{t+2\pi}\cos\big(\tau+\theta_k^1(t)-\theta_j^1(t)\big)\sin\tau\,d\tau\\
&= \frac{\varepsilon\kappa}{2N\pi}\sum_{k=1}^N\int_t^{t+2\pi}\sin\big(\theta_k^1(t)-\theta_j^1(t)\big)\sin^2\tau\,d\tau\\
&= \frac{\varepsilon\kappa}{4N\pi}\sum_{k=1}^N\sin\big(\theta_k^1(t)-\theta_j^1(t)\big),
\end{aligned}\tag{2.21}$$

where we used the 2π-periodicity of S and I.

We now combine (2.20) and (2.21) to obtain

$$\dot\theta_j^1(t)\simeq\varepsilon\omega_j+\frac{\varepsilon\kappa}{4N\pi}\sum_{k=1}^N\sin(\theta_k^1(t)-\theta_j^1(t)),$$

which is the Kuramoto model. This means that when we consider short time period (e.g., one period), the Kuramoto model can be viewed as an approximation of Winfree model in small natural frequency and coupling strength regime. However, our main focus is the large-time dynamics of synchronization model in a large coupling strength regime. Thus, the large-time dynamics of the Winfree and Kuramoto are completely different, which are consistent with the above observations.

2.4.2. From Lohe to Kuramoto. For the one-dimensional case $d = 1$, the Lohe model (5.1) reduces to the Kuramoto model (2.12) as follows. Note that $d = 1$, 1×1 unitary matrix and 1×1 hermitian matrix correspond to a complex number with unit modulus and real number, respectively. Thus, we can set

$$U_j := e^{-\mathrm{i}\theta_j}, \quad H_j := \Omega_j. \tag{2.22}$$

Then, by direct calculation, we have

$$i\dot{U}_j U_j^* = \dot{\theta}_j, \quad U_j U_k^* - U_k U_j^* = e^{\mathrm{i}(\theta_k - \theta_j)} - e^{-\mathrm{i}(\theta_k - \theta_j)} = 2\mathrm{i}\sin(\theta_k - \theta_j).$$

Thus, system (5.1) becomes the Kuramoto model:

$$\dot{\theta}_j = \Omega_j + K\sum_{k=1}^{N} c_{kj}\sin(\theta_k - \theta_j), \quad t > 0.$$

2.4.3. From Schrödinger–Lohe to Kuramoto. In this part, we discuss the relationship between the Schrödinger–Lohe model and the Kuramoto model. Consider a spatially homogeneous S–L model (2.16) with constant potential $V_j = -\Omega_j$ and all-to-all couplings:

$$\mathrm{i}\partial_t \psi_j = -\Omega_j \psi_j + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(\psi_k - \frac{\langle \psi_j, \psi_k\rangle}{\langle \psi_j, \psi_j\rangle}\psi_j\right). \tag{2.23}$$

We set the following ansatz for $\psi_j(t) = e^{\mathrm{i}\theta_j(t)}$ and substitute this ansatz into (2.23) to obtain

$$-\dot{\theta}_j \psi_j = -\Omega_j \psi_j + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\Big(\psi_k - \langle \psi_j, \psi_k\rangle \psi_j\Big).$$

Then, we take an inner product of the above relation with ψ_i. We use the relation $\langle \psi_j, \psi_k\rangle = \cos(\theta_j - \theta_k) + \mathrm{i}\sin(\theta_j - \theta_k)$ and compare the resulting relation to obtain the Kuramoto model:

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j).$$

In the following four sections, we study large-time behaviors of aforementioned four synchronization models.

3. The Winfree model

In this section, we present a large-time dynamics of the Winfree model with all-to-all coupling $c_{kj} = \frac{1}{N}$:

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^{N} S(\theta_j)I(\theta_k), \quad j = 1, \dots, N. \tag{3.1}$$

As discussed in Section 2.1, system (3.1) does not admit any conservation laws except the number of oscillators, so this lack of conservation laws makes analysis for large-time dynamics difficult, however, it makes the model describe more diverse asymptotic patterns. For the description of the large-time dynamics of (3.1), we first introduce several jargons to be used in the sequel. To analyze the large-time patterns, we use a scalar quantity "rotation number" which plays a key role in the classifications of large-time patterns.

Definition 3.1. Let $\theta = \theta(t) \in \mathbb{R}$ be a phase of an oscillator. If the limit $\lim_{t\to\infty} \frac{\theta(t)}{t}$ exists, then we call this limit as a rotation number of the oscillator and denote it by ρ.

Remark 3.2. Note that for the free flow (2.3),

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \lim_{t\to\infty} \frac{\theta^0 + t\Omega}{t} = \Omega.$$

Then, several asymptotic states can be characterized by rotation numbers.

Definition 3.3 ([44,45,49]). Let $\Theta := (\theta_1, \dots, \theta_N)$ be a state of the ensemble of Winfree oscillators whose dynamics are governed by (3.1).

(1) The configuration Θ tends to "*complete oscillator death (COD)*" if the rotation numbers of all oscillators are zero, i.e.,

$$|\{i : \rho_i = 0\}| = N,$$

where $|A|$ is the size of set A.

(2) The configuration Θ tends to "*partial oscillator death (POD)*" if the rotation numbers of at least two oscillators are zero, i.e.,

$$2 \le |\{i : \rho_i = 0\}| < N.$$

Note that for POD, not all rotation numbers are zero; if all oscillator rotation numbers are zero, COD is achieved.

(3) The configuration Θ tends to "*complete phase-locked state (CPLS)*" if the rotation numbers of all oscillators are equal and nonzero, i.e., there exists a nonzero number ρ such that

$$|\{i : \rho_i = \rho\}| = N.$$

(4) The configuration Θ tends to "*partially phase-locked state (PPLS)*" if there exist at least two oscillators whose rotation numbers are the same, i.e., there exists a nonzero constant ρ such that

$$2 \leq |\{i \ : \ \rho_i = \rho\}| < N.$$

Note that for PPLS, not all rotation numbers are the same; if all oscillator rotation numbers are equal to ρ, CPLS is achieved.

Remark 3.4. Let $\mathcal{S}$ be a phase-locked sub-ensemble of oscillators $\{\theta_1, \ldots, \theta_N\}$, say $\mathcal{S} = \{\theta_1, \ldots, \theta_n\}$ for $n \leq N$. Suppose that one of the oscillators in $\mathcal{S}$, say θ_1, has a well-defined nonzero rotation number $\exists\, \rho_1 := \lim_{t\to\infty} \frac{\theta_1(t)}{t} \neq 0$. Then, the phase-locking relation in Definition 3.3 implies that the rest of the oscillators in $\mathcal{S}$ has a rotation number and equal to ρ_1:

$$\exists\, \rho_i := \lim_{t\to\infty} \frac{\theta_i(t)}{t} = \rho_1, \quad 2 \leq i \leq n.$$

A natural question will characterize sufficient frameworks leading to the above four asymptotic states. In the following two subsections, we present sufficient frameworks leading to three asymptotic states, COD, POD and PPLS.

3.1. Emergence of COD and POD.

In this subsection, we present a sufficient framework leading to the COD and POD. Let S and I be the sensitivity and influence functions satisfying the following structural conditions:

($\mathcal{B}1$) The sensitivity function S is a 2π-periodic, analytic, and odd function, and the influence function I is a 2π-periodic, analytic, and even function:

$$\begin{aligned} &S(\theta + 2\pi) = S(\theta), \quad S(-\theta) = -S(\theta), \quad \theta \in \mathbb{R}, \\ &I(\theta + 2\pi) = I(\theta), \quad I(-\theta) = I(\theta). \end{aligned} \tag{3.2}$$

($\mathcal{B}2$) The sensitivity and influence functions satisfy some geometric conditions: there exist positive constants θ_* and θ^*, satisfying

$$0 < \theta_* < \theta^* < 2\pi,$$

such that,

$$\begin{aligned} &S \leq 0 \text{ on } [0, \theta^*] \quad \text{and} \quad S' \leq 0,\ S'' \geq 0 \text{ on } [0, \theta_*], \\ &I \geq 0,\ I' \leq 0 \text{ on } [0, \theta^*], \quad \text{and} \quad I'' \leq 0 \text{ on } [0, \theta_*], \\ &(SI)' < 0 \text{ on } (0, \theta_*), \quad (SI)' > 0 \text{ on } (\theta_*, \theta^*), \end{aligned} \tag{3.3}$$

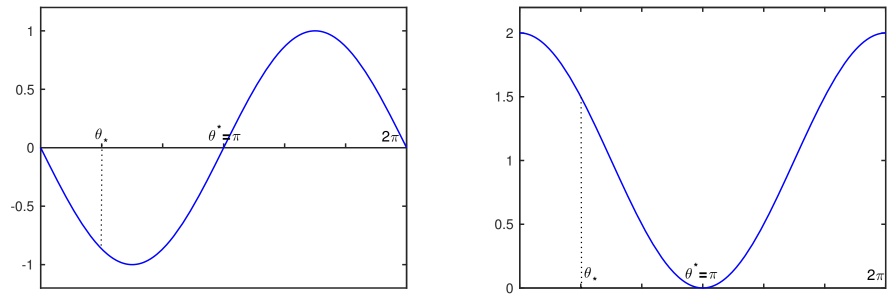
where S' denotes the θ-derivative of S (see Figure 2 for schematic graph of S and I).

Remark 3.5. The above structural conditions for S and I are motivated in the analysis of Theorem 3.8 and an explicit example [7, 70, 77, 78]:

$$(S(\theta), I(\theta)) = (-\sin\theta, 1 + \cos\theta). \tag{3.4}$$

In this case,

$$\theta_* = \frac{\pi}{3}, \quad \theta^* = \pi.$$



(a) $S(\theta) = -\sin\theta$ (b) $I(\theta) = 1 + \cos\theta$

Figure 2. Diagrams for $S(\theta)$ and $I(\theta)$

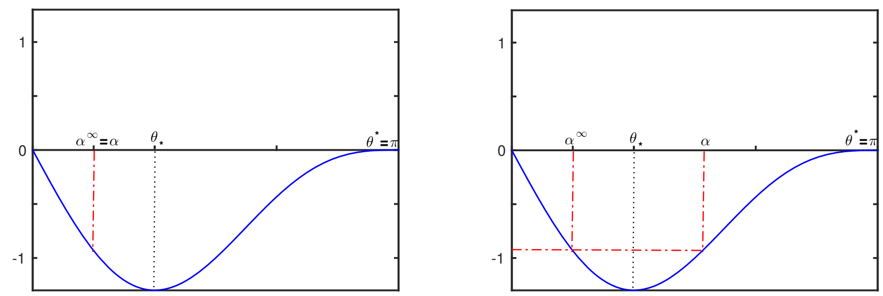
In order to state our result on the emergence of COD, we introduce some notation as follows. For a given $\alpha \in (0, \theta^*)$, consider the following equation on the interval $[0, \theta_*]$:

$$(SI)(x) = (SI)(\alpha), \quad x \in [0, \theta_*]. \tag{3.5}$$

Note that the conditions (3.2) and (3.3) yield the following geometric shape of the coupling function SI (see Figure 3):

$$\begin{aligned} &(SI)(0) = 0, \quad \theta_* = \operatorname*{argmin}_{0 \le \theta \le \theta^*}(SI)(\theta), \\ &(SI)(\theta) < 0 \text{ on } \theta \in (0, \theta^*), \quad (SI)(\theta^*) \le 0. \end{aligned} \tag{3.6}$$

Thus, the equation (3.5) has a unique solution α^∞ guaranteed by the relation (3.6). Moreover, for $\alpha \in (0, \theta_*]$, $\alpha^\infty = \alpha$.



(a) Position of α^∞ for $\alpha \in (0, \theta_*)$ (b) Position of α^∞ for $\alpha \in (\theta_*, \theta^*)$

Figure 3. Determination of α^∞

Theorem 3.6 ([49]). *Suppose that the following conditions hold.*

(1) *The coupling strength is sufficiently large such that*

$$\alpha \in (0,\pi), \quad K > K_e(\alpha^\infty) =: -\frac{\Omega^\infty}{S(\alpha^\infty)I(\alpha^\infty)}.$$

(2) *Initial data Θ^0 satisfy*

$$\Theta^0 \in \overline{\mathcal{R}}(\alpha),$$

where the set $\mathcal{R}(\alpha)$ is given as follows.

$$\mathcal{R}(\alpha) := \{\Theta = (\theta_1,\dots,\theta_N) \in \mathbb{R}^N \mid \theta_i \in (-\alpha,\alpha),\ j = 1,\dots,N\}.$$

Then, $\Theta(t)$ converges to a unique equilibrium state Θ^∞, i.e., the rotation numbers of all oscillators are zero:

$$\rho_j = 0, \quad 1 \le j \le N.$$

Remark 3.7. In [45], the following locally coupled Winfree model

$$\dot\theta_j = \Omega_j + K\sum_{k=1}^N c_{kj} S(\theta_j)I(\theta_k), \quad t>0, \ \ j = 1,\dots,N$$

has been studied, and it is shown that the partial oscillator death state emerges for

$$K \ge \max_{1\le j\le n}\left[\frac{|\Omega_j|}{\sum_{k=1}^n c_{kj}}\frac{1}{|SI(\alpha^\infty)|}\right]$$

for some class of initial configurations. In particular, when $n = N$, the complete oscillator death occurs.

3.2. Emergence of PPLS.

In the subsection, we are interested in the emergence of PPLS for (3.1) with the special pair (3.4):

$$\dot\theta_j = \Omega_j - \frac{K}{N}\sin\theta_j \sum_{k=1}^N \left(1+\cos\theta_j\right), \quad t>0, \quad 1\le i\le N. \tag{3.7}$$

For given initial configuration Θ^0, even if PPLS state emerges asymptotically, it will be very difficult to estimate the size of PPLS a priori. Hence, we instead choose inverse direction, namely, we first choose an sub-ensemble with size $n \le N$ and look for sufficient conditions which drive this specific sub-ensemble to an PPLS state asymptotically.

3.2.1. Identical oscillators. In this part, we consider identical oscillators, i.e.,

$$\Omega_j = \Omega > 0, \quad 1 \le j \le N.$$

In the sequel, we discuss a sufficient framework to guarantee the phase-locking of sub-ensemble $\mathcal{S} = \{\theta_1, \dots, \theta_n\}$ (see Figure 4). For this, we set relative sizes for the sub-ensemble and coupling strength as follows.

$$\gamma := \frac{n}{N}, \qquad \mu := \frac{K}{\Omega}.$$

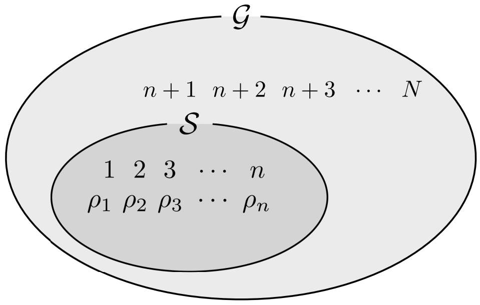


Figure 4. Schematic diagram for $\mathcal{S}$ and $\mathcal{G}$ extracted from [45]

We are now ready to state our framework for the emergence of partial locking.

($\mathcal{F}_A 1$) Sub-ensemble $\mathcal{S}$ is a majority group in total configuration:

$$0.56 \approx \frac{4}{4+\pi} < \gamma \le 1.$$

($\mathcal{F}_A 2$) The relative size μ of coupling strength compared to the common positive natural frequency Ω is moderately small:

$$0 < \mu < \left[\gamma\left(\frac{1}{2} + \frac{\pi}{8}\right) - \frac{1}{2}\right] =: \mu^\infty.$$

($\mathcal{F}_A 3$) Initial phases for sub-ensemble $\mathcal{S}$ is confined: for $\alpha \in (0, \alpha_\infty)$,

$$\max_{1 \le k,l \le n} |\theta_k^0 - \theta_l^0| \le 2\alpha \exp\left[-\frac{\mu}{1-2\mu}\left[\gamma\left(-2 + 2\alpha \pm \frac{\pi}{2}\right) + 4\right]\right].$$

where $\alpha_\infty := 1 + \frac{\pi}{4} - \frac{1}{\gamma}(1 + 2\mu)$.

Note that the upper bound for μ appearing in $0 < \mu < \frac{1}{2}\big[\gamma(1 + \frac{\pi}{4}) - 1\big]$ implies $K < \frac{\Omega}{2}$, and when the ensemble $\mathcal{S}$ is the same as the total ensemble $\mathcal{G}$, the above framework implies

$$\alpha_\infty = \frac{\pi}{4} - 2\mu, \quad \lim_{\mu \to \mu^\infty} \lim_{\alpha \to \alpha_\infty} \frac{\mu}{1-2\mu}\left[\gamma\left(-2 + 2\alpha \pm \frac{\pi}{2}\right) + 4\right] = \frac{\pi}{4}.$$

We are now ready to state our first result.

Theorem 3.8 ([45]). *Suppose that the framework* $(\mathcal{F}_A 1)$–$(\mathcal{F}_A 3)$ *hold, and let* Θ *be a global solution to* (3.7)*. Then, there exist positive constants* λ *and* C *such that*

$$\max_{1\le k,l\le n} |\theta_k(t)-\theta_l(t)| \le Ce^{-\lambda t}, \quad t\ge 0.$$

Remark 3.9. Our result says that once ρ_i exists for some $i \in \{1,\dots,n\}$, then all rotation numbers for oscillators in $\mathcal{S}$ are equal:

$$\rho_1 = \cdots = \rho_n.$$

3.2.2. Nonidentical oscillators. In this part, we consider a sub-ensemble $\mathcal{S}$ consisting of nonidentical oscillators with slightly different natural frequencies. When we turn off coupling $K=0$ in (3.7), it is easy to see that for $\Omega_k \neq \Omega_j$,

$$\rho_k = \Omega_k, \quad \rho_j = \Omega_j.$$

Thus, nonidentical oscillators cannot be locked in the absence of couplings, in contrast when the coupling is too strong as in [49], the rotation numbers of oscillators will be zero, i.e., complete oscillator death appears. Hence, to get a partial locking of nonidentical oscillators, we must have a reasonable range of coupling strength. Before, we state the emergence of partial locking, we choose a maximal frequency fluctuation Δ_Ω^∞ with respect to a base natural frequency Ω to satisfy

$$\Delta_\Omega^\infty < \Omega - 6K, \quad \frac{\Delta_\Omega^\infty}{3K}\left[\exp\left(\frac{6\pi K}{|\Omega| - \Delta_\Omega^\infty - 2K}\right) - 1\right] \le \min\left\{\frac{1}{2}\alpha,\ \alpha\Big(1 - e^{-\frac{\sin\alpha}{\alpha}|\bar{\Lambda}|}\Big)\right\},$$

where $\bar{\Lambda}$ is given by the following relation:

$$\bar{\Lambda} := \frac{\gamma\mu}{1+2\mu}\Big(2 - 2\alpha + \frac{\pi}{2}\Big) - \frac{\mu}{1-2\mu}\Big[\gamma\Big(-2 + 2\alpha - \frac{\pi}{2}\Big) + 4\Big].$$

Note that $\Delta_\Omega^\infty = 0$ satisfies the above relations. Therefore, by the continuity of the relations, there exists a positive constant Δ_Ω^∞ satisfying the above relations.

We are now ready to state our second framework on the emergence of partial locking.

$(\mathcal{F}_B 1)$ Sub-ensemble $\mathcal{S}$ is a majority group:

$$0.56 \approx \frac{4}{4+\pi} < \gamma \le 1.$$

$(\mathcal{F}_B 2)$ The relative size μ_K of coupling strength compared to the common positive natural frequency Ω is moderately small:

$$0 < \mu < \min\left\{\gamma\left(\frac{1}{2} + \frac{\pi}{8}\right) - \frac{1}{2}, \frac{1}{6}\right\}.$$

($\mathcal{F}_B3$) Initial phases Θ_n^0 for sub-ensemble is confined:

$$\alpha \in (0, \alpha_\infty), \qquad \max_{1 \le k,l \le n} |\theta_k^0 - \theta_l^0| \le \alpha.$$

where $\alpha_\infty := 1 + \frac{\pi}{4} - \frac{1}{\gamma}(1 + 2\mu)$.

($\mathcal{F}_B4$) The natural frequencies Ω_i are small perturbations of the common frequency Ω:

$$\Omega > 0, \quad \Omega_i \in \Big[\Omega - \Delta_\Omega^\infty, \Omega + \Delta_\Omega^\infty\Big], \quad i = 1, \dots, n.$$

Finally, we state our second theorem on the existence of partial synchronization.

Theorem 3.10 ([45]). *Suppose that the framework ($\mathcal{F}_B1$)–($\mathcal{F}_B4$) hold, and let Θ be a global solution to (3.7). Then, the diameter of the sub-ensemble $\mathcal{S}$ is uniformly bounded:*

$$\sup_{0 \le t < \infty} \max_{1 \le k,l \le n} |\theta_k(t) - \theta_l(t)| \le 3\alpha.$$

3.2.3. Existence of periodic locked orbit. In this part, we briefly summarize the result from [72]. Suppose that the natural frequencies Ω_i are randomly chosen from the interval I:

$$\Omega_i \in I := (1 - \gamma, 1 + \gamma) \quad \text{for } \gamma \in [0, 1), \quad i = 1, \dots, N.$$

Let K_* be the locking bifurcation critical parameter for $\gamma = 0$ and $N = 1$ determined by

$$K_* = \sup\{K > 0 : 1 - K(1 + \cos\theta)\sin\theta > 0, \forall \theta \in \mathbb{R}\} = \frac{4}{3\sqrt{3}}.$$

Define an open set U by

$$U := \{(\gamma, K) \in (0, 1) \times (0, K_*) : 0 < \gamma < K\big(D(K)\big)^2\},$$

where $D(K) := L_*(1 - \frac{K}{K_*})^2$ with a constant $L_* = \frac{1}{280}\exp\big(-\frac{\pi K_*}{4(1-K_*)} - \frac{\pi K_*}{2}\big)$.

For $(\gamma, K) \in U$ and $N \ge 1$, define an open set $C_{\gamma,K}^N$ by

$$C_{\gamma,K}^N := \Big\{\Theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N : \max_{i,j} |\theta_i - \theta_j| < \Gamma_{\gamma,K}\Big(\frac{1}{N}\sum_{k=1}^N \theta_k\Big)\Big\},$$

where the dispersion curve $\Gamma_{\gamma,K} : \mathbb{R} \to (0, 1)$ is a 2π-periodic function satisfying

$$\max_{s \in [0, 2\pi]} \Gamma_{\gamma,K}(s) < D(K).$$

The main results from [72] are the existence of a positively invariant set and complete phase-locked states for each $(\gamma, K) \in U$.

Theorem 3.11 ([72]). *Let $(\gamma, K) \in U$ be an arbitrary point. Then, the following assertions hold.*

(1) *The set $C^N_{\gamma,K}$ is positively invariant under the Winfree flow Φ^t.*

(2) *There exists an initial configuration $\Theta_0 \in C^N_{\gamma,K}$ and a constant rotation vector $\mathbf{J}_{\gamma,K} = (\rho_{1,\gamma,K}, \ldots, \rho_{N,\gamma,K})$ satisfying*

$$\Phi^t_i(\Theta_0) = \mathbf{J}_{\gamma,K} t + \Psi^N_i(t) \quad \text{for } i = 1, \ldots, N, \ t \geq 0,$$

where $\Psi^N_i : \mathbb{R}^+ \to \mathbb{R}$ is $\frac{2\pi}{\rho_{\gamma,K}}$-periodic, C^∞ and uniformly bounded with respect to N.

4. The Kuramoto model

In this section, we summarize results on the emergent properties of the Kuramoto model such as the existence of phase-locked states introduced in Definition 2.3, well-posedness of the kinetic Kuramoto model, namely the Kuramoto–Sakaguchi equation. Compared to the Winfree model, the Kuramoto model has a balance of total phase. This excludes the existence of complete oscillator death studied in the previous section. Due to the balance law (i) in Proposition 2.4, from now on, without loss of generality we may assume that the total sum of natural frequency is zero. Otherwise, we can choose a rotating frame with speed $\Omega_c := \frac{1}{N}\sum_{k=1}^N \Omega_k$, and we also assume that connection topology is all-to-all:

$$\sum_{k=1}^N \Omega_k = 0, \quad c_{kj} = \frac{1}{N}. \tag{4.1}$$

Then, under this condition (4.1), the Kuramoto model (2.12) becomes

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^N \sin(\theta_k - \theta_j). \tag{4.2}$$

Since the R.H.S. of (4.2) is 2π-periodic, system (4.2) is a dynamical system on tori, but tori can be embedded to $\mathbb{R}^N$. Hence in the sequel, we regards (4.2) as a dynamical system on $\mathbb{R}^N$.

4.1. Order parameters. In this subsection, we introduce real order parameters measuring the degree of synchronizability. Recall that for the Winfree model, rotation number ρ plays a key role in the analyzing the asymptotic dynamics of the Winfree model. For a phase configuration $\Theta = \Theta(t)$ governed by (4.2), the Kuramoto order parameters R and ϕ are defined by the following relation:

$$Re^{\mathrm{i}\phi} := \frac{1}{N}\sum_{k=1}^N e^{\mathrm{i}\theta_k}, \tag{4.3}$$

and will play a key role in the synchronization estimates. Once the R.H.S. of (4.3) is not zero, then R is well defined and ϕ is also well defined modulo 2π. In the visualization of oscillators as point rotors, the R.H.S. of (4.3) corresponds to the centroid of N-rotor positions on the unit circle (a convex combination of N-points in a unit disc), hence the centroid should be inside the unit disk. Thus, the modulus R is always bounded by 1, and is invariant under uniform rotation. The state $R = 1$ corresponds to the state in which all phases are the same, i.e., phase synchronization:

$$R = 1 \quad \Longleftrightarrow \quad \theta_i \equiv \alpha \pmod{2\pi}, \quad i = 1, \dots, N, \quad \text{for some } \alpha \in \mathbb{R},$$

and $R = 0$ corresponds to the splay-state which is uniformly distributed on the unit circle.

We next rewrite the Kuramoto model (4.2) in terms of mean-field quantity R and ϕ. For this, we divide (4.3) by $e^{\mathrm{i}\theta_j}$ to obtain:

$$Re^{\mathrm{i}(\phi-\theta_j)} = \frac{1}{N}\sum_{k=1}^{N} e^{\mathrm{i}(\theta_k-\theta_j)},$$

and compare the imaginary part of the above relation to find

$$R\sin(\phi - \theta_j) = \frac{1}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_j). \tag{4.4}$$

By comparing the second relation in (4.4) and the coupling terms in (2.12), it is easy to see that the Kuramoto system (2.12) can be rewritten in a mean-field form:

$$\dot{\theta}_j = \Omega_j - KR\sin(\theta_j - \phi), \quad t > 0. \tag{4.5}$$

The equation (4.5) looks like decoupled, but the mean-field R and ϕ are functions of other θ_k's. Hence, it is not a decoupled one.

4.2. Existence of phase-locked states. In this subsection, we briefly provide an existence of phase-locked states by using time-asymptotic approach. The existence of phase-locked states is closely related to the so called "*complete synchronization problem*":

> Find conditions on the coupling strength K and a natural frequency vector $(\Omega_1, \dots, \Omega_N)$ to guarantee the complete synchronization, i.e., for any solution to (4.2) issued from a generic initial configuration Θ^0,
>
> $$\lim_{t\to\infty} \max_{1\le k,j\le N} |\dot{\theta}_k(t) - \dot{\theta}_j(t)| = 0.$$

Of course, in the course of verification of the complete synchronization problem, we will explicitly characterize the genericity of initial configurations (see Theorem 4.7).

Note that under the zero sum condition (4.1), the phase-locked state $\Theta^\infty = (\theta_1^\infty, \dots, \theta_N^\infty)$ corresponds to the equilibrium of (4.2):

$$0 = \Omega_j + \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k^\infty - \theta_j^\infty), \quad j = 1, \dots, N. \tag{4.6}$$

Note that the relations in (4.6) yield the following two simple observations:

- Once $\Theta^\infty = (\theta_1^\infty, \dots, \theta_N^\infty)$ is a phase-locked state, then its translation $\Theta^\infty + \alpha(1, \dots, 1)$ is also a phase-locked state.
- Since the totality of nonlinear term satisfies

$$\Big|\frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k^\infty - \theta_j^\infty)\Big| \le K,$$

  If $K < |\Omega_j|$ for some j, then system (4.6) do not have any solution. Thus, in order to guarantee the existence of phase-locked state, the coupling strength K should be large.

The system of transcendental equations (4.6) is known to be solvable for $N = 2$, and it is not known whether it is explicitly solvable or not for $N \ge 3$, which is related to the integrability of the Kuramoto model (see [84,85]). We do not try to solve the system (4.6), but instead, we will construct the phase-locked states using time-asymptotic approach. More precisely, we will show that phase-locked states emerge from any generic initial configurations along the Kuramoto flow (4.2). We now recall the concept of complete synchronization as follows.

Definition 4.1. Let $\Theta = (\theta_1, \dots, \theta_N)$ be a solution to (4.2). Then, the Kuramoto model (4.2) has a complete synchronization asymptotically if the relative frequencies tend to zero as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} |\dot\theta_k(t) - \dot\theta_j(t)| = 0, \quad 1 \le k, j \le N.$$

As a warming-up for the complete synchronization, we consider a simple example.

Example 4.2. Consider a two-oscillator system:

$$\dot\theta_1 = \Omega_1 + \frac{K}{2}\sin(\theta_2 - \theta_1), \quad t > 0,$$
$$\dot\theta_2 = \Omega_2 + \frac{K}{2}\sin(\theta_1 - \theta_2).$$

We now consider the differences θ and Ω:

$$\theta := \theta_2 - \theta_1, \quad \Omega := \Omega_2 - \Omega_1.$$

The difference θ satisfies the Adler equation:

$$\dot\theta = \Omega - K\sin\theta, \quad t > 0.$$

Then, it is easy to show that for $K > |\Omega|$, complete synchronization occurs for any initial configuration:

$$\lim_{t\to\infty} \|\dot\theta_1(t) - \dot\theta_2(t)\|_\infty = 0.$$

We first recall previous mathematical works on the Kuramoto model. Ermentrout [34] found a critical coupling at which all oscillators become phase-locked, independent of their number, and the linear and nonlinear stabilities of this phase-locked state have been studied in several papers [2, 25, 27, 32, 40, 53, 67, 68, 81] using tools such as Lyapunov functionals, spectral graph theory, and control theory. In particular, the works in [25, 32, 53] use the phase diameter $D(\Theta) := \max_{1\le j,k\le N} |\theta_j - \theta_k|$ as a Lyapunov functional, and study its temporal evolution via Gronwall's inequality for initial configurations whose phase diameter is at most $\pi + \varepsilon$ with $\varepsilon \ll 1$. More precisely, the first author and his collaborators [39] extended the previous work of Choi et al [25] to allow initial configurations whose diameter is slightly larger than π for sufficiently large coupling strength. In fact, they showed that sufficiently large coupling can push initial configurations into configurations confined in the half circle so that they can use the result in [25]. For this, they used the dynamics of order parameters r and ϕ (see [39] for details). We now recall a most recent result on complete synchronization from [25, 39]. By a slight modification of the arguments in [25, 39], we obtain the following estimate on the emergence of phase-locked states.

Theorem 4.3 ([25, 39]). *Suppose that the coupling strength K satisfies*

$$K > D(\Omega) := \max_{1\le j,k\le N} |\Omega_k - \Omega_j|,$$

and let $\Theta = \Theta(t)$ be a solution to (4.2) such that there exists a positive time $T \in (0, \infty)$ such that

$$0 < D(\Theta(T)) < \pi - \arcsin\left(\frac{D(\Omega)}{K}\right).$$

Then, there exist positive constants $C_0(T)$ and Λ such that

$$D(\dot{\Theta}(t)) \le C_0 \exp(-\Lambda(t - T)), \quad \textit{as } t \to \infty.$$

Remark 4.4. (i) For identical oscillators $D(\Omega) = 0$, complete synchronization has been shown in [29] for an arbitrary initial configuration with $D(\Theta^0) < 2\pi$. Of course, the synchronization estimate given in [29] does not yield the detailed relaxation process toward a phase-locked state as it is.

(ii) The result in [25] corresponds to the case $T = 0$, i.e., $D(\Theta^0) < \pi$, and the result in [39] deals with the case $D(\Theta^0) < \pi + \varepsilon$, $\varepsilon \ll 1$ and $K \gg D(\Omega)$.

(iii) We refer to a survey paper [30] on mathematical results for the complete synchronization with $D(\Theta^0) < \pi$.

For some years, the complete synchronization problem has been restricted to initial configurations with $D(\Theta^0) < \pi$. We next briefly discuss how this restrictions on initial configuration has been removed by generic conditions in the following discussions based on the material from [40].

Proposition 4.5 ([40]). *Let Θ be a solution to (4.2) with $\Omega_j = 0$, $\forall\, j$ and initial data Θ^0:*

$$R^0 > 0, \quad \theta_j^0 \ne \theta_k^0, \quad 1 \le j, k \le N.$$

Then, there exists at most one j_0 such that

$$\lim_{t\to\infty} |\theta_{j_0}(t) - \phi(t)| = \pi, \quad \lim_{t\to\infty} |\theta_k(t) - \phi(t)| = 0, \quad k \neq j_0.$$

We now return to the existence of phase-locked states. As mentioned before, the existence of phase-locked states is closely related to the "*complete synchronization problem*". In the following, we will briefly present a recent resolution of the complete synchronization problem in [40]. The proof is based on the following four ingredients.

First ingredient (scaling). We divide (4.2) by K and we set

$$\tau := Kt, \quad \tilde{\Omega}_i := \frac{\Omega_i}{K}$$

to obtain the rescaled Kuramoto model:

$$\frac{d\theta_j}{d\tau} = \tilde{\Omega}_j + \frac{1}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_j).$$

Thus, we have

$$K \gg 1, \quad \Omega_j \,:\, \text{fixed} \quad \Longleftrightarrow \quad K = 1, \quad |\tilde{\Omega}_j| \ll 1.$$

Second ingredient (gradient flow). The Kuramoto model (4.2) is a gradient flow with analytical potential (Proposition 2.5) so, the complete synchronization holds if and only if the fluctuations

$$\hat{\theta}_j := \theta_j - \theta_c, \quad \theta_c := \frac{1}{N}\sum_{k=1}^{N} \theta_k$$

are uniformly bounded (Theorem 2.7)

Third ingredient (asymptotic dynamics of identical oscillators). Consider the Kuramoto model for identical oscillators:

$$\dot{\theta}_j = \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_j). \tag{4.7}$$

For a solution Θ issued from initial data Θ^0 satisfying

$$R^0 > 0, \quad \theta_j^0 \neq \theta_k^0, \quad 1 \le j,k \le N,$$

asymptotically, there are two possible scenarios, complete phase synchronization and bi-polar state (Proposition 4.5).

Fourth ingredient (variation between identical and non-identical oscillators). The Kuramoto models (4.2) and (4.7) for identical and nonidentical oscillators can be made as close as possible in finite-time interval, as long as the natural frequencies are small enough. Let Θ^I and Θ^{NI} be solutions to

$$\dot{\theta}_j^I = \frac{K}{N}\sum_{k=1}^N \sin(\theta_k^I - \theta_j^I),$$
$$\dot{\theta}_j^{NI} = \Omega_j + \frac{K}{N}\sum_{k=1}^N \sin(\theta_k^{NI} - \theta_j^{NI}), \quad \sum_{k=1}^N \Omega_k = 0,$$

subject to the same initial data:

$$\theta_j^I(0) = \theta_j^{NI}(0) = \theta_j^0, \quad 1 \le j \le N.$$

Then, we have

$$\|\Theta^{NI}(t) - \Theta^I(t)\|_\infty \le \frac{\|\Omega\|_\infty}{2K}(e^{2Kt} - 1), \quad t > 0. \tag{4.8}$$

Here, $\|\Omega\|_\infty$ is defined as the ℓ^∞-norm of Ω_i:

$$\|\Omega\|_\infty := \max_{1\le j\le N} |\Omega_j|.$$

Fifth ingredient (formation of a strong black-hole). Note that we are viewing the Kuramoto model (4.2) as a dynamical system on $\mathbb{R}^N$, i.e., each phase θ_j will be treated as a real number.

Consider a real line and imagine an infinite chain $\mathcal{C}(t)$ of open intervals

$$\mathcal{C}(t) := \bigcup_{k=-\infty}^{\infty} \Big(\gamma(\phi(t)) + 2k\pi\Big), \quad \gamma(\phi(t)) := \Big(\phi(t) - \frac{\ell}{2}, \phi(t) + \frac{\ell}{2}\Big),$$

where $\phi(t)$ is the mean-field phase defined in Section 4.1. Suppose that more than half of oscillators in a given total ensemble lie on some small interval $\gamma(\phi)$ at time t_0. Then, we can classify oscillators as confined oscillators (which lie on the interval $\gamma(\phi)$) and drifting oscillators (which does not lie on $\gamma(\phi)$) at time t_0. Then, we will choose a large coupling strength K so that confining oscillators still lie on the time-varying interval $\gamma(\phi(t))$ and drifting oscillators drift to immediate neighboring intervals $\gamma(\phi(t)) - 2\pi$, $\gamma(\phi(t))$ or $\gamma(\phi(t)) + 2\pi$ and then confine there afterwards. This leads to the uniform boundedness of phase fluctuations. This can be formalized in the following proposition.

Proposition 4.6 ([40]). *Suppose that the initial configuration Θ_0 satisfies*

$$\theta_{j0} \in [-\pi, \pi), \quad 1 \le j \le N,$$

and let n_0, ℓ, and K satisfy

$$n_0 \in \mathbb{Z}_+ \cap \Big(\frac{N}{2}, N\Big], \quad \ell \in \Big(0, 2\cos^{-1}\frac{N-n_0}{n_0}\Big),$$
$$\max_{1\le j,k\le n_0} |\theta_{j0} - \theta_{k0}| < \ell, \quad K > \frac{D(\Omega)}{\frac{n_0}{N}\sin\ell - \frac{2(N-n_0)}{N}\sin\frac{\ell}{2}}. \tag{4.9}$$

Let Θ be a global solution to system (4.2) with zero sum of natural frequencies. Then, we have

$$\sup_{0 \le t < \infty} D(\Theta(t)) \le 4\pi + \ell, \quad \lim_{t \to \infty} \|\dot{\Theta}(t)\|_\infty = 0.$$

Finally, we have the existence of phase-locked states for generic initial configurations in a large coupling strength regime.

Theorem 4.7 ([40]). *Suppose that the initial configuration Θ_0 and natural frequencies Ω_i satisfy the zero sum condition and*

$$R^0 > 0, \quad \theta_j^0 \ne \theta_k^0, \quad 1 \le j \ne k \le N, \quad \max_{1 \le j \le N} |\Omega_j| < \infty.$$

Then there exists a large coupling strength $K_\infty > 0$ such that if $K \ge K_\infty$ there exists a phase-locked state Θ^∞ such that the solution with initial data Θ_0 satisfies

$$\lim_{t \to \infty} \|\Theta(t) - \Theta^\infty\|_\infty = 0,$$

where the norm $\|\cdot\|_\infty$ is the standard ℓ^∞-norm in $\mathbb{R}^N$.

Proof. We briefly sketch the proof. The details can be found in the original paper [40]. Consider an initial data Θ^0 and coupling strength K:

$$R^0 > 0, \quad \theta_j^0 \ne \theta_k^0, \quad 1 \le j, k \le N, \qquad \|\Omega\|_\infty \le L < \infty, \quad K = 1.$$

The proof consists of four steps.

Step A. We use the first and third ingredients to solve the Kuramoto model with $K = 1$, $\Omega_i = 0$ and see that that from a given initial configuration Θ_0:

$$\exists\, T_* \in (0, \infty) \quad \text{such that } \theta_1^I(T_*), \dots, \theta_{N-1}^I(T_*) \in \gamma(\phi(T_*)), \quad |\gamma(\phi(T_*))| < \frac{\pi}{2},$$

where $|\gamma(\phi(T_*))|$ denote length of the open interval $\gamma(\phi(T_*))$.

Step B. We use the fourth ingredient and the estimate (4.8) for $\|\Theta^{NI} - \Theta^I\|_\infty$ to see that there exists $L_0 \ll 1$ such that if $|\Omega_j| \le L_0 \ll 1$, then

$$\theta_j^{NI}(T_*) \in \tilde{\gamma}, \quad |\tilde{\gamma}| < \pi, j = 1, \dots, N-1,$$

where $\tilde{\gamma}$ is some open interval.

Step C. We restart the Kuramoto flow with initial configuration $\Theta(T_*)$ as a new initial data and apply Proposition 4.6, and get the uniform boundedness for our flow $\Theta(t)$.

Step D. The gradient flow structure (second ingredient) of the Kuramoto model implies the convergence toward the phase-locked state. □

Remark 4.8. (i) The complete synchronization problem has also been treated in diverse settings, e.g., inertia effect [26], frustration [42, 43, 61], and hierarchical leadership [46].

(ii) A lower bound for the coupling strength stated in theorem is not optimal. In relation to the optimal coupling strength for the complete synchronization, we refer to [82, 83] for all-to-all and bipartite graphs.

4.3. Finiteness of phase-locked states. In previous subsection, we discussed the existence of phase-locked states as asymptotic states from generic initial configurations along the Kuramoto flow. However, the asymptotic approach does not give much information on the structure of phase-locked states. Once existence is guaranteed, then next question to address will be

"How many phase-locked states are there?"

For an ensemble of identical Kuramoto oscillators with $N \geq 4$, it is easy to construct an infinite number of phase-locked states whose totality is a measure zero in configuration space. In the following example, we construct a continuum of phase-locked states with zero order parameter.

Example 4.9 ([50]). For $\mu \in [0, 2\pi]$ and $N \geq 4$, we define a state Θ^μ: $k = 1, \dots, N$,

$$\theta_k^\mu := \begin{cases} \frac{2k\pi}{N-2} + \mu, & k = 1, \dots, N-2, \\ 0, & k = N-1, \\ \pi, & k = N. \end{cases}$$

Then, it is easy to see that Θ^μ satisfies (4.6). Moreover, for $\mu_1 \neq \mu_2 \in [0, 2\pi]$, we have

$$\theta_k^{\mu_1} - \theta_k^{\mu_2} = \begin{cases} \mu_1 - \mu_2, & k = 1, \dots, N-2, \\ 0, & k = N-1, N \end{cases}$$

Thus, distinct μ's clearly yield non-equivalent phase locked states, and these states all have a Kuramoto order parameter of 0.

$$R = \frac{1}{N} \Big| \sum_{k=1}^{N-2} e^{\mathrm{i} \frac{2k\pi}{N-2}} + e^{\mathrm{i}0} + e^{\mathrm{i}\pi} \Big| = 0.$$

Thus, alternative form (4.5) of the Kuramoto model implies

$$\dot{\theta}_j = \Omega, \quad j = 1, \dots, N.$$

Thus, Θ^μ is a phase-locked state.

For a given natural frequency vector $(\Omega_1, \dots, \Omega_N)$ and parameters K and N, we denote by $\mathcal{P} = \mathcal{P}(\{\Omega_i\}, K, N)$ a set of all phase-locked states(solutions) for system (4.2) up to phase-shift.

Proposition 4.10 ([50]). *For identical Kuramoto oscillators with $K > 0$, every phase-locked state with a positive order parameter is a permutation of*

$$(0, \dots, 0, \pi, \dots, \pi),$$

where the number of 0's is greater than $N/2$, and the π's do not necessarily exist, i.e., phase-locked state with $R > 0$ is a bi-polar state or completely synchronization state. Thus, we have

$$|\mathcal{P}| = \begin{cases} 2^{N-1}, & N \text{ odd}, \\ 2^{N-1} - \frac{1}{2}\binom{N}{N/2}, & N \text{ even}, \end{cases}$$

where $|A|$ is the cardinality of the set A.

We next list the results on the sufficient and necessary condition for the nonemptiness of the set $\mathcal{P}$ and its finiteness even for non-identical oscillators without the proofs.

Theorem 4.11 ([41, 83]). *Let $\{\Omega_i\}$ and K be a set of natural frequencies and coupling strength satisfying*

$$\sum_{i=1}^{N} \Omega_i = 0, \qquad K > 0, \; K \geq \|\Omega\|_\infty := \max_{1 \leq i \leq N} |\Omega_i|.$$

Then, the following assertions hold.

(1) *The set $\mathcal{P}$ is not empty if and only if there exists $\beta \in \left[\frac{\|\Omega\|_\infty}{K}, 1\right]$ and $\Sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$ such that*

$$\beta = \frac{1}{N} \sum_{j=1}^{N} \sigma_j \sqrt{1 - \left(\frac{\Omega_j}{K\beta}\right)^2}. \tag{4.10}$$

(2) *Suppose that a pair (β, Σ) satisfy the relation (4.10). Then the corresponding phase-locked state Θ is the unique solution (up to global rotation), satisfying*

$$K\beta \sin(\phi(\Theta) - \theta_j) = -\Omega_j, \quad j = 1, \ldots, N,$$
$$\sigma_j \cos(\phi(\Theta) - \theta_j) \geq 0.$$

In this case, $\beta = R(\Theta)$ (order parameter corrresponding to phase Θ). Therefore, this property establishes a one to one relation between a pair (β, Σ) and a element of $\mathcal{P}$.

(3) *If*

$$\frac{\|\Omega\|_\infty}{K} \leq \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left(\frac{\Omega_j}{\|\Omega\|_\infty}\right)^2},$$

then, $\mathcal{P}$ is not empty.

(4) *If $K \gg 1$, then, we have*

$$2^{N-1} \leq |\mathcal{P}| \leq 2^N. \tag{4.11}$$

Remark 4.12. The first three statements and lower bound estimate in (4.11) has been obtained in [83], and the upper bound estimate in (4.11) has been obtained recently in [41].

In the next subsection, we finally discuss the structure of phase-locked states which can be confined a half circle geometrically. However, we do not have any information on the structure of general phase-locked states.

4.4. Structure of phase-locked states. In this subsection, we discuss the structure of phase-locked states with diameter less than π. As aforementioned, the existence of phase-locked states will be guaranteed by the time-asymptotic approach. So far, we do not know detailed structure of general phase-locked state. However, for phase-locked states confined in a half circle, we can see below how the phase-locked states look like. The phases in a phase-locked state are arranged into the order of natural frequencies in counter-clock wise. Moreover, phase-locked states confined in a half circle are unique up to rotation (see [25] for details).

Theorem 4.13 ([25]). *Let $\Theta = \Theta(t)$ be a global smooth solution to the system (2.12) satisfying*

$$\max_{j,k} |\Omega_j - \Omega_k| > 0, \quad 0 < D(\Theta^0) < \pi, \quad K > K_e = \frac{\max_{j,k} |\Omega_j - \Omega_k|}{\sin D(\Theta^0)}.$$

(1) *If $\Omega_i < \Omega_j$, then there exists a positive time t^*_{ij} such that*

$$\theta_i(t) < \theta_j(t) \quad \text{for } t \geq t^*_{ij}.$$

That is the oscillators with large natural frequencies will be advanced to the front starting from any initial configurations in counter-clock wise, and the ordering of θ will be well-defined since $0 < D(\Theta) < \pi$. Actually, we can precisely count the number of collision times.

(2) *Suppose (i, j) satisfies*

$$\theta_i(t_0) > \theta_j(t_0) \quad \text{and} \quad \Omega_i > \Omega_j \quad \text{for some } t_0 \in \mathbb{R}_+$$

Then i and j-oscillators will not meet after $t = t_0$, i.e.,

$$\theta_i(t) > \theta_j(t), \quad t > t_0.$$

We next show the lower and upper bounds for the transversal phase differences during the relaxation process toward the phase-locked state. For this, we explicitly provide the lower and upper bounds for the transversal phase differences $\theta_{ij} := \theta_i - \theta_j$, $i \neq j$. We set $\Omega_{ij} := \Omega_i - \Omega_j$.

We next introduce the dual angle $\mathcal{D}^\infty$ of initial phase diameter $D(\Theta^0)$ by

$$\mathcal{D}^\infty \in \left(0, \frac{\pi}{2}\right), \quad \sin \mathcal{D}^\infty = \sin D(\Theta^0).$$

For an initial diameter $D(\Theta^0) \in [0, \frac{\pi}{2}]$, the dual angle $\mathcal{D}^\infty$ coincides with $D(\Theta^0)$. We also set

$$L(N, \mathcal{D}^\infty) := 1 - \frac{N-2}{N}(1 - \cos \mathcal{D}^\infty),$$
$$U(N, D(\theta^0)) := 1 - \frac{N-2}{N}\left(1 - \frac{1}{\cos \frac{D(\theta^0)}{2}}\right).$$

Theorem 4.14 ([25]). *Let $\Theta = \Theta(t)$ be a global smooth solution to the system* (4.2) *satisfying*

$$\max_{j,k} |\Omega_j - \Omega_k| > 0, \quad 0 < D(\Theta^0) < \pi \quad \textit{and} \quad K > K_e.$$

Then, we have

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) \leq \lim_{t\to\infty} \theta_{ij}(t) \leq \mathcal{D}^\infty.$$

Moreover, if

$$\left|\frac{\Omega_{ij}}{KL}\right| \leq 1 \quad \textit{and} \quad \sin^{-1}\left(\frac{\Omega_{ij}}{KL}\right) \leq \mathcal{D}^\infty,$$

then we obtain

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) \leq \lim_{t\to\infty} \theta_{ij}(t) \leq \sin^{-1}\left(\frac{\Omega_{ij}}{KL}\right).$$

In the following subsection, as a last topic for the Kuramoto model, we discuss the well-posedness issues for the Kuramoto–Sakaguchi equation which can be obtained from the Kuramoto model as $N \to \infty$.

4.5. The Kuramoto–Sakaguchi equation. In this subsection, we consider the kinetic Kuramoto model which can be obtained as a mean-field limit from the Kuramoto model (4.2) and discuss the well-posedness of weak solutions. Let $f = f(\theta, \Omega, t)$ be a one-oscillator probability density function at phase θ, natural frequency Ω, at time t. Using the standard BBGKY Hiearchy arguments [55], it is easy to see that f satisfies the Kuramoto–Sakaguchi (K–S) equation [1, 59]:

$$\begin{aligned} &\partial_t f + \partial_\theta(\omega[f]f) = 0, \qquad (\theta,\Omega) \in \mathbb{T}\times\mathbb{R}, \ \ \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \ \ t > 0, \\ &\omega[f](\theta,\Omega,t) = \Omega - KL[f], \quad L[f] := \int_{\mathbb{T}} \sin(\theta-\theta_*)\rho(\theta_*,t)\,d\theta_*, \end{aligned} \tag{4.12}$$

subject to suitable initial data:

$$f^0(\theta,\Omega) = f^0(\theta+2\pi,\Omega) \geq 0, \quad \int_{\mathbb{T}} f^0(\theta,\Omega)\,d\theta = g(\Omega), \quad \iint_{\mathbb{T}\times\mathbb{R}} f^0\,d\Omega d\theta = 1, \tag{4.13}$$

where K and $g = g(\Omega)$ are the nonnegative coupling strength and the distribution function for natural frequencies, respectively, and

$$\rho(\theta,t) := \int_{\mathbb{R}} f(\theta,\Omega,t)\,d\Omega$$

is the local phase density.

Proposition 4.15 ([6]). *Let f be a C^1-solution to* (4.12)–(4.13)*. Then we have*

$$\int_{\mathbb{T}} f(\theta,\Omega,t)\,d\theta = g(\Omega), \quad \iint_{\mathbb{T}\times\mathbb{R}} f(\theta,\Omega,t)\,d\Omega d\theta = 1, \quad t \geq 0.$$

The existence of strong solutions for the Kuramoto–Sakaguchi–Fokker–Planck equation has been studied in [60] and recently Landau damping phenomenon to the K–S equation (4.12) and its noisy version has been extensively studied in literature [13,20,35,51,52]. More precisely, Kuramoto conjecture that below the critical coupling strength the incoherent solution is expected to be nonlinearly stable, in contrast above the critical coupling strength, it is expected to be nonlinearly unstable. The verification of this nonlinear phenomena rigorously has been done in aforementioned literature. We refer to Introduction of Chiba's paper [20] for interested readers.

4.5.1. Well-posedness of K–S equation. We next recall several concepts of measure-valued solutions and weak solutions to (4.12) following the presentation in [6,19,48,59]. Let $\mathcal{M}(\mathbb{T}\times\mathbb{R})$ be the set of nonnegative Radon measures on $\mathbb{T}\times\mathbb{R} = [0,2\pi]\times\mathbb{R}$, which can be understood as nonnegative bounded linear functionals on $C_0(\mathbb{T}\times\mathbb{R})$. For a Radon measure $\nu \in \mathcal{M}(\mathbb{T}\times\mathbb{R})$, we use a standard duality relation:

$$\langle \nu, h\rangle := \int_0^{2\pi}\int_{\mathbb{R}} h(\theta,\Omega)\nu(d\theta,d\Omega), \quad h \in C_0(\mathbb{T}\times\mathbb{R}).$$

The definitions of a measure valued solution and weak solution to the equation (4.12) are given as follows.

Definition 4.16 ([6,19,21,59]). (1) For $T\in(0,\infty)$, let $\mu \in L^\infty([0,T);\mathcal{M}(\mathbb{T}\times\mathbb{R}))$ be a measure valued solution to (4.12) with an initial Radon measure $\mu_0\in\mathcal{M}(\mathbb{T}\times\mathbb{R})$ if and only if μ satisfies the following conditions:

- μ is weakly continuous:

$$\langle \mu_t, h\rangle \text{ is continuous as a function of } t, \quad \forall\, h\in C_0(\mathbb{T}\times\mathbb{R}).$$

- μ satisfies the integral equation: $\forall\, h \in C_0^1(\mathbb{T}\times\mathbb{R}\times[0,T))$,

$$\langle \mu_t, h(\cdot,\cdot,t)\rangle - \langle \mu_0, h(\cdot,\cdot,0)\rangle = \int_0^t \langle \mu_s, \partial_s h + \omega\partial_\theta h\rangle\, ds,$$

  where $\omega = \omega(\theta,\Omega,\mu_s)$ is defined by

$$\omega(\theta,\Omega,\mu_s) := \Omega - K\langle \mu_s, \sin(\theta-\cdot)\rangle.$$

(2) For $T\in(0,\infty)$, let $f\in L^\infty(\mathbb{T}\times\mathbb{R}\times[0,T))$ be a L^∞-weak solution to (4.12) with initial data $f_0\in L^\infty(\mathbb{T}\times\mathbb{R})$ if and only if f satisfies the following conditions:

- The map $t\to f(\cdot,\cdot,t)\in L^1_{\text{loc}}(\mathbb{T}\times\mathbb{R})$ is weakly continuous as a function of t.
- f is a distributional solution to (4.12): for any test function $\phi\in C_c^\infty(\mathbb{T}\times\mathbb{R}\times\mathbb{R}_+)$,

$$\int_0^\infty\iint_{\mathbb{T}\times\mathbb{R}} (f\partial_t\phi + \omega[f]f\partial_\theta\phi)(\theta,\Omega,t)\, d\Omega d\theta dt = \iint_{\mathbb{T}\times\mathbb{R}} f^0(\theta,\Omega)\phi(\theta,\Omega,0)\, d\Omega d\theta.$$

Remark 4.17. (i) Let $f = f(\theta, \Omega, t)$ be a classical solution to the K–S equation (4.12). Then $\mu_t := f(\theta, \Omega, t) d\Omega d\theta$ is a measure valued solution.

(ii) Note that the empirical measure

$$\mu_t = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\theta_i(t), \Omega_i)}, \quad (\theta_i(t), \Omega_i) \text{ solution of (4.2)}$$

is a measure valued solution.

(iii) The existence of measure-valued solution for the K–S equation has been studied in [19, 21, 59] using the empirical measure and particle-in-cell method [71]. See [18] for the corresponding issues for other kinetic models.

4.5.2. Existence of BV weak solutions. In this part, we briefly summarize the well-posedness results for bounded variation (BV) weak solutions in [6] motivated from the relevant works [7, 28] in hyperbolic conservation laws with nonlocal flux. Note that the equation (4.12) can be written into a quasilinear form:

$$\partial_t f + \omega[f] \partial_\theta f + f \partial_\theta(\omega[f]) = 0\,,$$

that is

$$\partial_t f + (\Omega - KL[\rho]) \partial_\theta f = K f \partial_\theta L[\rho]$$

or equivalently, it can be rewritten as a characteristic system:

$$\dot{\theta} = \Omega - KL[\rho], \quad \dot{f} = K f \partial_\theta L[\rho]. \tag{4.14}$$

We next discuss the global existence of BV weak solutions to (4.12) equation only for identical oscillators. A corresponding theory for nonidentical oscillators is not complete yet (see [5]). Without loss of generality, we assume $g(\Omega) = \delta$ where δ is the Dirac delta located at $\Omega = 0$. As a consequence

$$f(\theta, \Omega, t) = \rho(\theta, t) \otimes \delta(\Omega)$$

and

$$\omega[f] f = (\Omega - KL[\rho])\, \rho \otimes \delta(\Omega) = -KL[\rho] \rho \otimes \delta(\Omega).$$

Therefore (4.12) reduces to an equation for $\rho(\theta, t)$:

$$\partial_t \rho - K \partial_\theta (L[\rho] \rho) = 0. \tag{4.15}$$

The initial datum ρ_0 is assumed to be in the set $\mathcal{X}$ defined by

$$\mathcal{X} = \left\{ \rho : \mathbb{T} \to \mathbb{R}, \ \ \rho \in BV(\mathbb{T})\,, \ \ \rho(\theta) \geq 0\,, \ \ \int_{\mathbb{T}} \rho(\theta)\, d\theta = 1 \right\}. \tag{4.16}$$

We next recall the concept of an entropy weak solution to the continuity equation (4.15) with non-local flux as follows.

Definition 4.18 ([5, 6]). Let $T > 0$. A function $\rho : \mathbb{T} \times [0, T] \mapsto [0, \infty)$ is an entropy weak solution to (4.15) with initial data $\rho_0 \in L^\infty(\mathbb{T})$ if the following hold.

(1) The map $[0, T] \ni t \mapsto \rho(\cdot, t) \in L^\infty(\mathbb{T})$ is continuous, with $\rho(\cdot, 0) = \rho_0$.

(2) $\rho(\theta, t)$ satisfies, for all $\alpha \in \mathbb{R}$

$$\partial_t |\rho - \alpha| - K\partial_\theta [\ell(\theta, t)\, |\rho - \alpha|] - \operatorname{sgn}(\rho - \alpha) K\alpha(\partial_\theta \ell) = 0$$

in the sense of distributions, with

$$\ell(\theta, t) = L[\rho(\cdot, t)](\theta).$$

We next explain how to construct approximate solutions to (4.15) consisting of functions which are piecewise constant in phase, but not constant in time. The jump discontinuities are located on a finite number of curves.

Suppose that the 2π–periodic initial datum ρ_0 is piecewise constant with a finite number of jump discontinuities. More precisely, for $N \in \mathbb{N}$, let $\theta_{01} < \theta_{02} < \cdots < \theta_{0N}$ be the locations of the discontinuities in the interval $[0, 2\pi)$. By periodicity, one has that

$$\rho_0(\theta_{01}-) = \rho_0(\theta_{0N}+).$$

Recalling (4.14), we consider a system of characteristic equations:

$$\begin{cases} \dot{\theta}_i = -KL[\rho](\theta_i), \quad i = 1, \dots, N, \\ \dot{\rho}_i = K\rho_i \dfrac{L[\rho](\theta_{i+1}) - L[\rho](\theta_i)}{\theta_{i+1} - \theta_i}, \end{cases} \tag{4.17}$$

subject to initial data

$$(\rho_{0i}, \theta_{0i}) = (\rho_0(\theta_{0i}+), \theta_{0i}), \quad i = 1, \dots, N, \tag{4.18}$$

where we set $\theta_{N+1} = \theta_1 + 2\pi$, $\theta_0 = \theta_N - 2\pi$.

Note that the system (4.17) and (4.18) can be written as an autonomous system in terms of the variables $U = (\theta_1, \dots, \theta_N, \rho_1, \dots, \rho_N)$, as $\dot{U} = F(U)$, with F Lipschitz continuous. Thus, the local existence and uniqueness follow from the standard Cauchy–Lipschitz theory, and one can show that the approximate solutions $(\rho_i(t), \theta_i(t))$ satisfy the following estimates: for some positive I_0,

(i) $\rho_i(0)\, e^{-KI_0 t} \le \rho_i(t) \le \rho_i(0)\, e^{KI_0 t}$,

(ii) $\mathrm{TV}\rho(\cdot, t) \le e^{KI_0 t} \cdot \mathrm{TV}\rho(\cdot, 0) + 2\big[e^{KI_0 t} - 1\big]$.

The global existence of BV solutions is established in the following theorem.

Theorem 4.19 ([6]). *Let $T > 0$ and consider the Cauchy problem (4.15) for $0 \le t < T$ with initial data in $\mathcal{X}$, defined in (4.16).*

(i) *For $\rho_0 \in \mathcal{X}$, there exists an entropic weak solution $\rho = \rho(\theta, t)$ of (4.15) in the sense of Definition 4.18. Moreover $\rho(\cdot, t) \in \mathcal{X}$ for every $t \in [0, T)$.*

(ii) *Let $\rho_1(\cdot,t)$ and $\rho_2(\cdot,t)$ be entropy weak solutions of* (4.15)*, corresponding to initial data $\rho_{0,1}$ and $\rho_{0,2} \in \mathcal{X}$, respectively. Assume moreover that*

$$\mathrm{TV}\left(\rho_1(\cdot,t)\right), \mathrm{TV}\left(\rho_2(\cdot,t)\right) \le C_0 \qquad \text{for all } t \in [0,T).$$

Then one has

$$\|\rho_1(\cdot,t) - \rho_2(\cdot,t)\|_{L^1(\mathbb{T})} \le \mathrm{e}^{Ct}\|\rho_{0,1} - \rho_{0,2}\|_{L^1(\mathbb{T})}, \quad C := K\left(1 + (2\pi+1)C_0\right).$$

5. The Lohe model

In this section, we study emergent properties of the Lohe model introduced in Section 2.3. So far, most analytical studies on the synchronization were focused on classical oscillators [1]. Even for quantum mechanical phenomena appearing in the Josephson junction array for superconductors, classical models such as Kuramoto type models have been widely used in literature instead of quantum models [56]. Synchronization of quantum mechanical systems is an emerging research area that has been studied mostly numerically and experimentally for possible applications in the control and stability of quantum devices and quantum computation [63, 64]. In the sequel, our main interest lies on the matrix-valued ODE model introduced by Max Lohe as a non-abelian generalization of the Kuramoto model.

Let $U_j = U_j(t)$ be an $d \times d$ unitary matrix encoding the quantum information of j-th quantum oscillator. The Lohe model reads as follows.

$$\mathrm{i}\dot{U}_j U_j^* = H_j + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(U_k U_j^* - U_j U_k^*\right), \quad 1 \le j \le N. \tag{5.1}$$

We have already seen that for $d = 1$, system (5.1) reduces to the Kuramoto model. This is why (5.1) is called a non-abelian generalization of the Kuramoto model. For $d = 2$, the unitary matrix U_j can be written as a linear combination of Pauli matrices σ_k, $k = 1, 2, 3$ and the identity matrix I_2 up to a phase factor θ_j:

$$U_j = e^{-\mathrm{i}\theta_j}\begin{pmatrix} x_j^4 + \mathrm{i}x_j^3 & x_j^2 + \mathrm{i}x_j^1 \\ -x_j^2 + \mathrm{i}x_j^1 & x_j^4 - \mathrm{i}x_j^3 \end{pmatrix}, \quad H_j = \begin{pmatrix} \omega_j^3 + \Omega_j & \omega_j^1 - \mathrm{i}\omega_j^2 \\ \omega_j^1 + \mathrm{i}\omega_j^2 & -\omega_j^3 + \Omega_j \end{pmatrix}.$$

Then, the dynamics of U_j is completely determined by the state $(\theta_j, x_j) \in \mathbb{R} \times \mathbb{R}^4$ governed by the coupled ODE system:

$$\begin{aligned} \dot{\theta}_j &= \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j)\langle x_j, x_k\rangle, \quad t > 0, \\ \dot{x}_j &= A_j x_j + \frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k - \theta_j)(x_k - \langle x_j, x_k\rangle x_j), \end{aligned} \tag{5.2}$$

where A_j and Ω_j are 4×4 skew-symmetric real matrix and real number, respectivley:

$$A_j := \begin{pmatrix} 0 & -\omega_j^3 & \omega_j^2 & -\omega_j^1 \\ \omega_j^3 & 0 & -\omega_j^1 & -\omega_j^2 \\ -\omega_j^2 & \omega_j^1 & 0 & -\omega_j^3 \\ \omega_j^1 & \omega_j^2 & \omega_j^3 & 0 \end{pmatrix}, \quad \Omega_j \in \mathbb{R},$$

and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^4$.

5.1. Two complex dimensions. In this subsection, we study the emergent dynamics of system (5.2) which summarize the results of [23]. We first note that system (5.2) admits a conservation law.

Proposition 5.1. *Let $\mathcal{X} = (x_1, \dots, x_N)$ be any solution to system (5.2) with initial data $\|x_j^0\| = 1$ for all j. Then, the modulus of x_j is invariant along the flow (5.2).*

$$\|x_j(t)\| = \|x_j^0\|, \quad t \geq 0.$$

where $\|\cdot\|$ is ℓ^2-norm in $\mathbb{R}^4$.

Before we move on further, we recall two definitions of synchronizations for (5.2).

Definition 5.2. Let $(\Theta, \mathcal{X})$ be a solution to system (5.2).

(1) The position ensemble $\mathcal{X}(t)$ approaches *position synchronization* asymptotically if

$$\lim_{t \to \infty} \max_{1 \leq k, j \leq N} \big(\|x_k(t) - x_j(t)\| + |\theta_k(t) - \theta_j(t)| \big) = 0.$$

(2) The whole ensemble $(\Theta, \mathcal{X})$ exhibits practical synchronization asymptotically if

$$\lim_{K \to \infty} \limsup_{t \to \infty} \max_{1 \leq k, j \leq N} \big(|\theta_k(t) - \theta_j(t)| + \|x_k(t) - x_j(t)\| \big) = 0.$$

Remark 5.3. Note that asymptotic complete position synchronization implies practical synchronization.

A main machinery to analyze emergent dynamic is a Lyapunov functional approach. For this, we set Lyapunov functionals for system (5.2) as follows.

$$D(\Theta) := \max_{1 \leq j,k \leq N} |\theta_k - \theta_j|, \qquad D(\mathcal{X}) := \max_{1 \leq j,k \leq N} \|x_k - x_j\|,$$
$$D(\Omega) := \max_{1 \leq j,k \leq N} |\Omega_k - \Omega_j|, \qquad D(A) := \max_{1 \leq j,k \leq N} \|A_k - A_j\|,$$

where the matrix norm $\|A\|$ is ℓ^2-norm by embedding 4×4 matrix into $\mathbb{R}^{16}$. Then, by rather tedious and lengthy calculations, we can derive a coupled system of first-order differential inequalities for $D(\Theta)$ and $D(\mathcal{X})$ under some a priori condition.

Proposition 5.4 ([23]). *Let T be any positive number in $(0, \infty]$, and let $(\Theta, \mathcal{X})$ be any solution to (5.2) satisfying*

$$D(\Theta(t)) < \frac{\pi}{6}, \quad \|x_j(0)\| = 1, \quad t \in [0, T), \quad j = 1, \dots, N.$$

Then, we have

$$\begin{aligned} \frac{dD(\Theta)}{dt} &\le D(\Omega) - \frac{\sqrt{3}K}{\pi} D(\Theta) + 2KD(\mathcal{X})D(\Theta), \quad t > 0, \\ \frac{dD(\mathcal{X})}{dt} &\le 4D(A) - \frac{\sqrt{3}K}{\pi} D(\mathcal{X}) + 2K(D(\mathcal{X}))^2. \end{aligned}$$

5.1.1. Identical oscillators. In this part, we consider system (5.2) for identical oscillators.

$$\begin{aligned} \dot{\theta}_j &= \Omega + \frac{K}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j) \langle x_j, x_k \rangle, \quad t > 0, \\ \dot{x}_j &= Ax_j + \frac{K}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta_j)(x_k - \langle x_j, x_k \rangle x_j), \end{aligned} \tag{5.3}$$

As a direct application of Proposition 5.4, we obtain the complete position synchronization to (5.2) for identical oscillators.

Theorem 5.5 ([23]). *Let $(\Theta, \mathcal{X})$ be the solution to (5.2) satisfying the following conditions:*

$$\begin{aligned} &D(\Omega) = 0, \qquad D(A) = 0, \qquad K > 0, \\ &D(\Theta_0) < \frac{\pi}{6}, \quad D(\mathcal{X}_0) < \frac{\sqrt{3}}{2\pi}, \quad \|x_{j0}\| = 1, \ j = 1, \dots, N. \end{aligned}$$

Then, we have

$$D(\Theta(t)) \le D(\Theta_0)e^{-KC_0 t}, \quad D(\mathcal{X}(t)) \le D(\mathcal{X}_0)e^{-KC_0 t}, \quad t \ge 0,$$

where C_0 is a positive constant defined by

$$C_0 := \frac{\sqrt{3}}{\pi} - \frac{D(\mathcal{X}_0)}{2}.$$

Proof. We briefly sketch the proof here, but details can be found in [23]. Note that $D(\Theta)$ and $D(\mathcal{X})$ satisfy

$$\begin{aligned} \frac{dD(\Theta)}{dt} &\le -\frac{\sqrt{3}K}{\pi} D(\Theta) + 2KD(\mathcal{X})D(\Theta), \quad t > 0, \\ \frac{dD(\mathcal{X})}{dt} &\le -\frac{\sqrt{3}K}{\pi} D(\mathcal{X}) + 2K(D(\mathcal{X}))^2. \end{aligned} \tag{5.4}$$

By using a continuous induction argument and (5.4), we show that $D(\Theta)$ and $D(\mathcal{X})$ are non-increasing in t:

$$D(\Theta(t)) \le D(\Theta_0) < \frac{\pi}{6}, \quad D(\mathcal{X}(t)) \le D(\mathcal{X}_0) < \frac{\sqrt{3}}{2\pi}, \quad t \in [0, \infty). \tag{5.5}$$

Then, we again use (5.5) in (5.4) to obtain Gronwall's inequalities:

$$\begin{aligned} \frac{dD(\Theta)}{dt} &\le -K\left[\frac{\sqrt{3}}{\pi} - \frac{D(\mathcal{X}_0)}{2}\right] D(\Theta), \quad t > 0, \\ \frac{dD(\mathcal{X})}{dt} &\le -K\left[\frac{\sqrt{3}}{\pi} - \frac{D(\mathcal{X}_0)}{2}\right] D(\mathcal{X}). \end{aligned}$$

This yields the desired estimates. □

For identical oscillators, we can show that scattering type results hold. For this, consider a free flow associated with (5.3):

$$\dot{\theta}_j = \Omega, \quad \dot{x}_j = Ax_j, \quad t > 0, \ j = 1, \dots, N. \tag{5.6}$$

We next define solution operators U_Ω and U_A corresponding to free flow (5.6):

$$U_\Omega(t)\theta_{j0} := \theta_{j0} + t\Omega, \quad U_A(t)x_{j0} = e^{tA}x_{j0}, \quad 1 \le j \le N.$$

Then, it is easy to see that the solution operator U_A is ℓ^2-isometry.

$$\|U_A(t)x_{j0}\| = \|x_{j0}\|, \quad t \ge 0.$$

We integrate (5.3) to obtain

$$\begin{aligned} \theta_j(t) &= \theta_{j0} + t\Omega + \frac{K}{N}\sum_{k=1}^{N}\int_0^t \sin(\theta_k(s) - \theta_j(s))\langle x_j(s), x_k(s)\rangle \, ds, \\ x_j(t) &= e^{tA}x_{j0} + e^{tA}\frac{K}{N}\sum_{k=1}^{N}\int_0^t e^{-sA}\cos(\theta_k(s) - \theta_j(s)) \\ &\qquad \times \big[x_k(s) - \langle x_j(s), x_k(s)\rangle x_j(s)\big] \, ds. \end{aligned}$$

We now define asymptotic states θ_j^+ and x_j^+:

$$\begin{aligned} \theta_j^+ &= \theta_{j0} + \frac{K}{N}\sum_{k=1}^{N}\int_0^\infty \sin(\theta_k(s) - \theta_j(s))\langle x_j(s), x_k(s)\rangle \, ds, \\ x_j^+ &= x_{j0} + \frac{K}{N}\sum_{k=1}^{N}\int_0^\infty e^{-sA}\cos(\theta_k(s) - \theta_j(s))[x_k(s) - \langle x_j(s), x_k(s)\rangle x_j(s)] \, ds. \end{aligned}$$

Then, the following scattering type results hold.

Proposition 5.6 ([23]). *Let* $(\Theta, \mathcal{X})$ *be the solution to* (5.2) *satisfying the following conditions:*

$$D(\Omega) = 0, \qquad D(A) = 0, \qquad K > 0,$$
$$D(\Theta_0) < \frac{\pi}{6}, \quad D(\mathcal{X}_0) < \frac{\sqrt{3}}{2\pi}, \quad \|x_{j0}\| = 1, \ j = 1, \ldots, N.$$

Then, there exists a positive constant C_1 *which depends on initial data such that*

$$\|U_\Omega(t)\theta_j^+ - \theta_j(t)\| \le C_1 e^{-KC_0 t}, \quad \|U_A(t)x_j^+ - x_j(t)\| \le e^{-\frac{Kt}{2}}, \quad \text{as } t \to \infty.$$

5.1.2. Nonidentical oscillators. Since the key differential inequalities for $D(\Theta)$ and $D(\mathcal{X})$ in Proposition 5.4 gives non-decaying upper bound, we cannot obtain the strong synchronization (position synchronization) as in Theorem 5.5 as it is. However, thanks to differential inequalities in Proposition (5.4), it is still possible to obtain a relaxed synchronization, namely practical synchronization in the sense of Definition 5.2. Without a proof, we state the main results as follows.

Proposition 5.7. *Let* $(\Theta, \mathcal{X})$ *be a solution to* (5.2) *satisfying the following set-up:*

(i) $\|x_{j0}\| = 1 > 0, \ j = 1, \ldots, N, \quad D(\Omega) < \frac{K}{8\sqrt{3}}, \quad D(A) < \frac{3K}{32\pi^2},$

(ii) $D(\Theta_0) < \frac{\pi}{12}, \quad D(\mathcal{X}_0) < \frac{\sqrt{3}}{4\pi}.$

Then, we have practical synchronization:

$$\lim_{K\to\infty} \limsup_{t\to\infty} D(\Theta(t)) = 0, \qquad \lim_{K\to\infty} \limsup_{t\to\infty} D(\mathcal{X}(t)) = 0.$$

Remark 5.8. In the course of the proof, we can derive upper bounds for $D(\Theta)$ and $D(\mathcal{X})$:

$$D(\Theta(t)) \le e^{-\frac{\sqrt{3}}{2\pi}Kt} D(\Theta_0) + \frac{2\pi D(\Omega)}{\sqrt{3}K}\left(1 - e^{-\frac{\sqrt{3}}{2\pi}Kt}\right),$$
$$D(\mathcal{X}(t)) \le e^{-\frac{\sqrt{3}}{2\pi}Kt} D(\mathcal{X}_0) + \frac{8\pi D(A)}{\sqrt{3}K}\left(1 - e^{-\frac{\sqrt{3}}{2\pi}Kt}\right).$$

5.2. General dimensions. In this subsection, we consider a general case with $d \ge 1$. Of course, for $d = 1, 2$, we have already studied synchronization estimates in Section 4 and Section 5.1. Thus, our main focus in this subsection is for high-dimensional case with $d \ge 3$. However, the machinery used in the synchronization estimates covers all dimensions.

5.2.1. Phase-locked states. For synchronization estimate, we first define what we mean by "*phase-locked states*" in an ensemble of matrices. Note that our oscillator in this situation is a matrix. Thus, suitable concept of phase-locked states should contain a special case $d = 1$ in the identification done in (2.22). We next introduce a concept of phase-locked states for the Lohe model. Recall that for the Kuramoto model in (4.2), the phase-locked state $\Theta = (\theta_1, \dots, \theta_N)$ is defined by the invariance property of phase differences in time (see Definition 2.3):

$$|\theta_i(t) - \theta_j(t)| = \text{constant}, \quad t \geq 0,\ 1 \leq i, j \leq N.$$

We now consider the product $U_i U_j^*$ and observe that for $d = 1$,

$$U_i U_j^* = e^{-\mathrm{i}(\theta_i - \theta_j)}.$$

Then, time-invariance of $\theta_i - \theta_j$ is equal to the time-invariance of $U_i U_j^*$. This simple observation leads to the suitable concept of phase-locked state for the matrix model.

Definition 5.9 ([50]). Let $\{U_i(t)\}$ be a solution to (5.1).

(1) $\{U_i(t)\}$ is a *phase-locked state* if and only if $U_i(t)U_j(t)^*$ is constant for all i, j, and $t \geq 0$.

(2) The Lohe flow $\{U_i(t)\}$ achieves *asymptotic phase-locking* if and only if the limit of $U_i U_j^*$ as $t \to \infty$ exists for all i, j.

Remark 5.10. From Proposition 2.9, the right multiplication of a phase-locked state by a unitary matrix is also a phase-locked state; likewise, a right multiplication of a Lohe flow achieving asymptotic phase-locking also achieves asymptotic phase-locking.

We now return to characterize phase-locked states defined in Definition 5.9. For the Kuramoto model, phase-locked state corresponds to the ensemble of phase oscillators moving with the average natural frequency $\Omega_c := \frac{1}{N}\sum_{k=1}^N \Omega_k$ so that trajectory of each oscillators follows the same linear motion on the unit circle:

$$\theta_j(t) = \theta_j(0) + t\Omega_c.$$

This one-dimensional concept of phase-locked state can be generalized to the matrix model as in the following proposition.

Proposition 5.11 ([63,64]). *The phase-locked states $\{U_j\}$ of (5.1) are of the form*

$$U_j = U_j^\infty e^{-\mathrm{i}\Lambda t},$$

where $U_j^\infty \in \mathcal{U}(d)$ and Λ is the constant $d \times d$ Hermitian matrix satisfying

$$U_j^\infty \Lambda U_j^{\infty *} = H_j - \frac{\mathrm{i}K}{2N}\sum_{k=1}^N \left(U_j^\infty U_k^{\infty *} - U_k^\infty U_j^{\infty *}\right).$$

5.2.2. Ensemble diameters. In this part, we will introduce a suitable Lyapunov functional and a Gronwall type inequality for it. For this, we will use a trace norm (or Frobenius norm) for a square matrix, which is nothing but ℓ^2 in $\mathbb{R}^{d^2}$. Let $M = (m_{ij})$ be a $d \times d$ complex matrix. Then, the trace norm of M is defined as follows:

$$\|M\| := \left[\mathrm{tr}(MM^*)\right]^{\frac{1}{2}} = \left[\sum_{1 \leq i,j \leq d} |m_{ij}|^2\right]^{\frac{1}{2}},$$

and we now introduce the ensemble diameter as a Lyapunov functional. For a finite collection of square matrices $\{U_j(t)\}_{j=1}^N$ and $\{H_j\}_{j=1}^N$, set

$$D(U) := \max_{1\le i,j\le N} \|U_i - U_j\|, \qquad D(H) := \max_{1\le i,j\le N} \|H_i - H_j\|. \tag{5.7}$$

Lemma 5.12 ([50]). *Let $\{U_i\}$ be a solution to (5.1). Then the ensemble diameter $D(U)$ in (5.7) satisfies Gronwall's differential inequality:*

$$\left|\frac{d}{dt}D(U) + KD(U)\right| \le D(H) + \frac{K}{2}D(U)^3 \qquad \textit{a.e. } t \in (0,\infty). \tag{5.8}$$

Identical oscillallators. We assume that the Hermitian matrices H_j satisfy

$$H_j = H_c, \quad \forall\ j = 1,\dots,N, \qquad \text{i.e., } D(H) = 0.$$

In this case, the matrix U_j satisfy

$$\mathrm{i}\dot{U}_j U_j^* = H_c - \frac{\mathrm{i}K}{2N}\sum_{k=1}^N \left(U_j U_k^* - U_k U_j^*\right), \quad j = 1,\dots,N. \tag{5.9}$$

By Gronwall's inequality in Lemma 5.12, we have an exponential synchronization of the Lohe model.

Theorem 5.13 ([50]). *Suppose that the Hermitian matrices $H = \{H_j\}_{j=1}^N$ and initial data $U^0 = \{U_j^0\}_{j=1}^N$ satisfy*

$$K > 0, \quad D(H) = 0, \quad D(U^0) < \sqrt{2}.$$

Then for any solution $\{U_j\}$ to (5.9), $D(U)$ approaches zero exponentially fast:

$$\sqrt{\frac{2D(U^0)^2}{(D(U^0)^2+2)e^{2Kt} - D(U^0)^2}} \le D(U(t)) \le \sqrt{\frac{2D(U^0)^2}{D(U^0)^2 + (2 - D(U^0)^2)e^{2Kt}}}.$$

Non-identical oscillallators. We next discuss a positively invariant set for the Lohe flow in (5.1). The Gronwall inequality in (5.8) of Lemma 5.12 motivates a cubic polynomial f:

$$f(x) := x - \frac{1}{2}x^3, \quad x \ge 0,$$

so that the upper bound in (5.8) becomes

$$\frac{d}{dt}D(U) \le D(H) - Kf(D(U)), \quad \text{a.e. } t > 0.$$

It is easy to verify that f satisfies the following:

(i) f has only two roots, 0 and $\sqrt{2}$, in the interval $[0,\infty)$.

(ii) $f \geq 0$ for $x \in \left[0, \sqrt{2}\right]$, $\quad f' \geq 0$ for $x \in \left[0, \sqrt{\frac{2}{3}}\right]$, $\quad f' \leq 0$ for $x \in \left[\sqrt{\frac{2}{3}}, \sqrt{2}\right]$.

(iii) $\displaystyle\arg\max_{x\geq 0} f(x) = \sqrt{\frac{2}{3}}, \quad \max_{x\geq 0} f(x) = \frac{2}{3}\sqrt{\frac{2}{3}}$.

Henceforth, K_e will denote a given sufficient positive coupling strength so that $\frac{D(H)}{K_e} < \frac{2}{3}\sqrt{\frac{2}{3}}$. With this K_e, the equation

$$f(x) = \frac{D(H)}{K_e}, \quad x \in [0, \sqrt{2}],$$

has a solution in $\left[0, \sqrt{\frac{2}{3}}\right]$, which we denote by α_1, and a solution in $\left[\sqrt{\frac{2}{3}}, \sqrt{2}\right]$, which we denote by α_2. We set

$$\mathcal{S}(\alpha_i) := \{U \;:\; D(U) < \alpha_i\}, \quad i = 1, 2.$$

Proposition 5.14 (Existence of a positively invariant set [50]). *Suppose that the coupling strength K and initial data U^0 satisfy*

$$K > K_e > \frac{3}{2}\sqrt{\frac{3}{2}}D(H), \quad D(U^0) \leq \alpha_2.$$

Then the following assertions hold:

(a) *As long as the diameter $D(U(t))$ remains in the interval $[\alpha_1, \alpha_2]$, it is strictly decreasing:*

$$\frac{d}{dt}D(U(t)) \leq c, \quad \text{a.e. } t > 0 \text{ for which } D(U(t)) \in [\alpha_1, \alpha_2],$$

where $c < 0$ is a negative constant determined by $D(H)$, K, and K_e.

(b) *The set $\mathcal{S}(\alpha_2)$ is a positively invariant set for the flow in* (5.1)*:*

$$U(t) \in \mathcal{S}(\alpha_2), \quad t > 0.$$

(c) *There exists a positive time $t_e \geq 0$ such that*

$$U(t) \in \mathcal{S}(\alpha_1), \quad t \geq t_e.$$

5.2.3. Stability with respect to initial data. In this part, we discuss a stability estimate with respect to initial data. Consider two Lohe flows $\{U_j\}$ and $\{\tilde{U}_j\}$ of (5.1) with common natural frequencies $H_j = \tilde{H}_j$ and different initial data $\{U_j^0\}$ and $\{\tilde{U}_j^0\}$, respectively. We introduce a metric $d(\cdot, \cdot)$ measuring the distance between $U_i U_j^*$ and $\tilde{U}_i \tilde{U}_j^*$:

$$d(U(t), \tilde{U}(t)) := \max_{1\leq i,j\leq N} \|U_i(t)U_j^*(t) - \tilde{U}_i(t)\tilde{U}_j^*(t)\|. \tag{5.10}$$

To motivate the meaning of the functional $d(U, \tilde{U})$, we consider a familiar case when $d = 1$. In this case, (5.10) implies that for $\theta_{ij} := \theta_i - \theta_j$,

$$d(U(t), \tilde{U}(t)) = \max_{1 \le i,j \le N} \left| 1 - e^{-\mathrm{i}(\tilde{\theta}_{ij} - \theta_{ij})} \right|.$$

Thus, it is easy to verify that

$$d(U(t), \tilde{U}(t)) = 0 \quad \Longleftrightarrow \quad \Theta \text{ and } \tilde{\Theta} \text{ are congruent up to constant shift.}$$

Hence, $d(U(t), \tilde{U}(t))$ measures the degree of maximal mismatch in configurations U and $\tilde{U}$. It is easy to verify that $d(\cdot, \cdot)$ can be controlled by the sum of ensemble diameters:

$$\begin{aligned} d(U(t), \tilde{U}(t)) &\le \max_{i,j} \|U_i(t) U_j^*(t) - I\| + \max_{i,j} \|I - \tilde{U}_i(t) \tilde{U}_j^*(t)\| \\ &= D(U(t)) + D(\tilde{U}(t)). \end{aligned}$$

We now present the orbital stability of the Lohe flow in (5.1).

Theorem 5.15 (Stability estimates [50]). *Suppose that the coupling strength K and initial data U^0 and $\tilde{U}^0$ satisfy*

$$K > K_e > \frac{54}{17} D(H) \approx 3.1765 D(H), \qquad U^0,\ \tilde{U}^0 \in \mathcal{S}(\alpha_1).$$

Then for the two Lohe flows $\{U_i\}$ and $\{\tilde{U}_i\}$, the following assertions hold:

(1) *The relative positions synchronize exponentially fast: $\alpha_1 < \frac{1}{3}$, and*

$$d(U^0, \tilde{U}^0) e^{-K(1+3\alpha_1)t} \le d(U(t), \tilde{U}(t)) \le d(U^0, \tilde{U}^0) e^{-K(1-3\alpha_1)t}, \quad t \ge 0.$$

(2) *The normalized velocities $\dot{U}_i U_i^*$ and $\dot{\tilde{U}}_i \tilde{U}_i^*$ synchronize:*

$$\left\| \dot{\tilde{U}}_i \tilde{U}_i^* - \dot{U}_i U_i^* \right\| \le K d(U^0, \tilde{U}^0) e^{-K(1-3\alpha_1)t}.$$

(3) *There exists a unitary matrix $L_\infty \in \mathcal{U}(d)$ independent of i such that*

$$\lim_{t \to \infty} U_i(t)^* \tilde{U}_i(t) = L_\infty \quad \textit{and} \quad \lim_{t \to \infty} \|\tilde{U}_i(t) - U_i(t) L_\infty\| = 0, \quad i = 1, \dots, N.$$

The convergence is exponential with rate bounded above by $-K(1 - 3\alpha_1)$.

5.2.4. Emergence of phase-locked states. In this part, we present the existence of phase-locked states defined in Definition 5.9 without the proof.

Theorem 5.16 ([50]). *Suppose that the coupling strength K and initial data U^0 satisfy*

$$K > \frac{54}{17} D(H), \quad D(U^0) \le \alpha_2.$$

Then for the Lohe flow $\{U_j\}$, the following assertions hold:

(1) $\{U_j\}$ *achieves asymptotic phase-locking:*

$$\lim_{t\to\infty} U_j U_k^*$$

converges exponentially fast, with rate bounded above by $-K(1-3\alpha_1)$.

(2) *There exists a phase-locked state* $\{V_j\}$ *and a unitary matrix* $L \in \mathcal{U}(d)$ *such that*

$$D(V) < \alpha_1 \quad \text{and} \quad \lim_{t\to\infty} \|U_j - V_j L\| = 0.$$

The convergence rate is exponentially fast, with rate bounded above by $-K(1-3\alpha_1)$.

(3) *Phase locked states in* $\overline{S(\alpha_2)}$ *are unique up to right-multiplication. That is, if* $\{V_j\}$ *and* $\{W_j\}$ *are two phase-locked states in* $\overline{S(\alpha_2)}$, *then there exists a unitary matrix* $L \in \mathcal{U}(d)$ *such that*

$$W_j = V_j L, \quad j = 1, \dots, N.$$

So far, we have discussed a synchronization of a quantum system with a finite state space. Next, we will discuss the modeling and analysis for a synchronization with an infinite state space.

6. The Schrödinger–Lohe model

In this section, we study the basic structure of the S–L model and its emergent properties. Next we recall the S–L system discussed in Section 2.3.2:

$$\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + V_j\psi_j + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(\psi_k - \frac{\langle\psi_j,\psi_k\rangle}{\langle\psi_j,\psi_j\rangle}\psi_j\right), \quad j = 1, \dots, N. \tag{6.1}$$

We first recall the definitions of several concepts on synchronizations for the S–L model.

Definition 6.1 ([22, 24]). Let $\Psi = (\psi_1, \dots, \psi_N)$ be a solution to the S–L model (6.1).

(1) The S–L model (6.1) exhibits complete wave function synchronization asymptotically if the following estimate holds:

$$\lim_{t\to\infty} \|\psi_i(t) - \psi_j(t)\| = d_{ij}, \quad 1 \le i, j \le N,$$

where d_{ij} is a nonnegative constant and $\|\cdot\|$ is the $L^2(\mathbb{R}^d)$ or $L^2(\mathbb{T}^d)$ norm.

(2) The S–L model (6.1) exhibits practical wave function synchronization asymptotically if the following estimate holds:

$$\lim_{K\to\infty} \limsup_{t\to\infty} \max_{1\le i,j\le N} \|\psi_i(t) - \psi_j(t)\| = 0.$$

Remark 6.2. If we assume the normalized oscillators $\|\psi_j(0)\| = 1, 1 \le j \le N$, it follows from the relation

$$\|\psi_i - \psi_j\|^2 = 2 - \int_{\mathbb{R}^d} (\psi_i \bar{\psi_j} + \psi_j \bar{\psi_i}) dx$$

that we have

$$\lim_{t\to\infty} \|\psi_i(t) - \psi_j(t)\| = d_{ij} \quad \iff \quad \lim_{t\to\infty} \mathrm{Re}(\langle \psi_i(t), \psi_j(t)\rangle) = 1 - \frac{(d_{ij})^2}{2}.$$

Hence, the quantity $\mathrm{Re}\langle \psi_i(t), \psi_j(t)\rangle$ does play a key role in the synchronization estimate (see Section 6.3).

Lemma 6.3. *Let $\Psi = (\psi_1, \ldots, \psi_N)$ be a global solution to the (6.1) satisfying the condition:*

$$\|\psi_j(0)\| = 1, \quad 1 \le j \le N.$$

Then, we have for any i, j,

$$\begin{aligned}
&\frac{d}{dt}\|\psi_i - \psi_j\|^2 \\
&\le \frac{2K}{N} \sum_{k=1}^{N} \Big[\big(\|\psi_i - \psi_k\| + \|\psi_j - \psi_k\| - 1\big) \cdot \|\psi_i - \psi_j\|^2 - |1 - \langle \psi_i, \psi_j\rangle|^2\Big] \\
&\qquad + 2\int_{\mathbb{R}^d} \mathrm{Im}[\psi_i \overline{\psi_j}](V_j - V_i) dx. \quad (6.2)
\end{aligned}$$

Remark 6.4. For identical potentials $V_j = V_i$, the differences $\psi_i - \psi_j$ satisfy

$$\frac{d}{dt}\|\psi_i - \psi_j\|^2 \le \frac{2K}{N} \sum_{k=1}^{N} \Big[\big(\|\psi_i - \psi_k\| + \|\psi_j - \psi_k\| - 1\big) \cdot \|\psi_i - \psi_j\|^2\Big]. \quad (6.3)$$

In the following three subsections, we will study the emergent properties of (6.1) for identical one-body potentials $V_j = V$ for all j and nonidentical potentials $V_j \ne V_k$ for some j and k, respectively, and discuss a finite-dimensional reduction of (6.1) in the study of synchronization estimates.

6.1. Identical potentials. We next discuss synchronous behavior of (6.1) for the identical potentials:

$$V_j = V, \quad j = 1, \ldots, N.$$

Then the S–L model becomes

$$\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + V\psi_j + \frac{\mathrm{i}K}{2N} \sum_{k=1}^{N} \Big(\psi_k - \frac{\langle \psi_k, \psi_j\rangle}{\langle \psi_j, \psi_j\rangle}\psi_j\Big), \quad j = 1, \ldots, N. \quad (6.4)$$

Given a set of N wave functions $\Psi = \{\psi_1, \ldots, \psi_N\}$, we set the diameter $D(\Psi)$:

$$D(\Psi) := \max_{1\le i,j\le N} \|\psi_i - \psi_j\|, \quad \|\psi\| := \left(\int_{\mathbb{R}^d} \psi\bar{\psi}\,dx\right)^{\frac{1}{2}}.$$

As in a finite-dimensional case discussed in Section 5.1.1, solutions to the S–L system (6.4) for identical potentials can be decomposed as a composition of the free solution operator and Lohe solution operator. More precisely, we consider two equations:

The free Schrödinger equation:

$$\partial_t \psi = \frac{\mathrm{i}}{2}\Delta\psi - \mathrm{i}V\psi, \quad x \in \mathbb{R}^d,\ t > 0, \tag{6.5}$$

and

The space-homogeneous Lohe system:

$$\frac{d\psi_j}{dt} = \frac{K}{2N}\sum_{k=1}^{N}\left(\psi_k - \frac{\langle\psi_k, \psi_j\rangle}{\langle\psi_j, \psi_j\rangle}\psi_j\right), \quad t > 0. \tag{6.6}$$

Let $S(t)$ and $L(t)$ be linear and nonlinear solution operators corresponding to (6.5) and (6.6), respectively. Then, it is easy to see that the solution operators S and L are $L^2(\mathbb{R}^d)$-isometry:

$$\|S(t)\psi\| = \|\psi\|, \quad \|L(t)\psi\| = \|\psi\|, \quad t \ge 0.$$

In the following proposition, we see that the solution operator to (6.4) can be decomposed of two solutions operator S and L.

Proposition 6.5 ([24]). *Let $\Psi = (\psi_1, \ldots, \psi_N)$ be a solution to system (6.4) with L^2 initial data Ψ^0. Then, the solution ψ_j can be decomposed as the successive composition of S and L operators to initial datum ψ_{j0}, i.e.,*

$$\psi_j(x,t) = S(t) \circ L(t)\psi_j^0(x), \quad x \in \mathbb{R}^d,\ t > 0.$$

Remark 6.6. Note that the operator $S(t)$ is linear. Decomposition of solution operator implies

$$\|\psi_i(t) - \psi_j(t)\| = \|S(-t)\psi_i(t) - S(-t)\psi_j(t)\| = \|L(t)\psi_i^0 - L(t)\psi_j^0\|.$$

Thus, quantum synchronization of the S–L model reduces to the synchronization of the Lohe system (6.6).

We now state the synchronization estimate of (6.1) for identical potentials.

Theorem 6.7 ([24]). *Suppose that the coupling strength and initial data satisfy*

$$K > 0, \quad \|\psi_j^0\| = 1,\ 1 \le j \le N, \quad D(\Psi^0) < \frac{1}{2}.$$

Then, for any solution $\Psi = (\psi_1, \ldots, \psi_N)$ to (6.4), the diameter $D(\Psi)$ satisfies

$$D(\Psi(t)) \le \frac{D(\Psi^0)}{D(\Psi^0) + (1 - 2D(\Psi^0))e^{Kt}}, \quad t \ge 0.$$

Proof. It follows from (6.3) that $D(\Psi)$ satisfies

$$\dot{D}(\Psi) \le \frac{K}{2}\Big(-D(\Psi) + 2D(\Psi)^2 \Big).$$

This yields

$$D(\Psi(t)) \le \frac{D(\Psi^0)}{D(\Psi^0) + (1 - 2D(\Psi^0))e^{Kt}}. \qquad \square$$

We next study the nonidentical potentials for the S–L model.

6.2. Nonidentical potentials. We now consider the dynamics of N S–L oscillators with normalized L^2 norm:

$$\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + V_j\psi_j + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\big(\psi_k - \langle \psi_j, \psi_k\rangle \psi_j\big), \quad \|\psi_j\| = 1, \quad x \in \mathbb{R}^d,\ t > 0. \tag{6.7}$$

Note that the estimate (6.2) can give only one-side differential inequality for $D(\Psi)$. To consider the nonidentical potentials, define $D(\mathcal{V}) = \max_{i,j} \|V_j - V_i\|_\infty$.

Consider the cubic equation:

$$f(x) := 2x^3 - x^2 + \frac{2D(\mathcal{V})}{K} = 0, \quad x \in [0,\infty), \quad K > 54D(\mathcal{V}). \tag{6.8}$$

Then, the equation (6.8) has a positive local maximum $\frac{2D(\mathcal{V})}{K}$ and a negative local minimum $-\frac{1}{27} + \frac{2D(\mathcal{V})}{K}$ at $x = 0$ and $\frac{1}{3}$, respectively (see Figure 5).

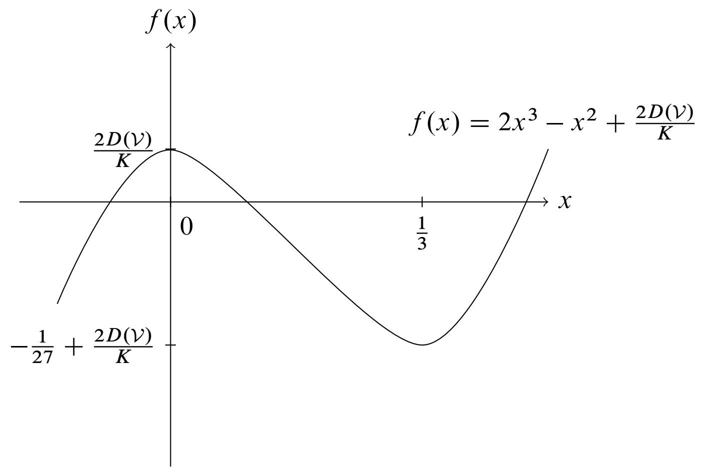


Figure 5. Schematic plot of $f(x)$

Moreover, (6.8) has two positive real roots $\alpha_1 < \alpha_2$:

$$0 < \alpha_1 < \frac{1}{3} < \alpha_2 < \frac{1}{2}, \quad \lim_{K\to\infty} \alpha_1 = 0, \quad \lim_{K\to\infty} \alpha_2 = \frac{1}{2}.$$

Theorem 6.8 ([22]). *Suppose that the following assumptions hold.*

(1) *The coupling strength is sufficiently large in the sense that*

$$K > 54 D(\mathcal{V}).$$

(2) *The initial data Ψ^0 satisfies the smallness assumption:*

$$\|\psi_i^0\| = 1, \quad j = 1, \dots, N, \quad D(\Psi^0) < \alpha_2.$$

Then, it achieves practical synchronization:

$$\lim_{K\to\infty} \limsup_{t\to\infty} D(\Psi(t)) = 0.$$

6.3. A finite-dimensional reduction. In this subsection, we derive a finite-dimensional dynamical system associated with the synchronization problem for (6.1) for congruent one-body potentials of the form

$$V_j(x) = V(x) + \Omega_j, \quad \Omega_j \in \mathbb{R}, \ x \in \mathbb{R}^d, \ 1 \le j \le N. \tag{6.9}$$

We basically summarize the results from [37]. Under the condition (6.9) and normalization condition $\|\psi_k\| = 1$, ψ_j satisfies

$$\mathrm{i}\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + (V(x) + \Omega_j)\psi_j + \frac{\mathrm{i}K}{2N}\sum_{k=1}^N \Big(\psi_k - \langle \psi_j, \psi_k\rangle \psi_j\Big). \tag{6.10}$$

It follows from Remark 6.4 that the quantity $\mathrm{Re}\langle \psi_i(t), \psi_j(t)\rangle$ does play a key role in the synchronization estimate. For this, we set

$$h_{ij}(t) := \int_{\mathbb{R}^d} \psi_i \bar{\psi_j}\, dx \in \mathbb{C}, \quad 1 \le i, j \le N.$$

Then, it is easy to see that we have, for $1 \le i, \ j \le N$,

$$h_{ij} = \bar{h}_{ji}, \quad h_{ii} = \|\psi_i\|^2 = 1, \quad |h_{ij}| = \left| \int_{\mathbb{R}^d} \psi_i \bar{\psi_j} dx \right| \le \|\psi_i\|\|\psi_j\| = 1.$$

Proposition 6.9 ([37]). *Let ψ_i be a solution to (6.10). Then h_{ij} satisfies the coupled system of ODEs:*

$$\begin{aligned} \frac{dh_{ij}}{dt} &= -\mathrm{i}\Omega_{ij}h_{ij} + \frac{K}{N}\left[\sum_{k=1}^N h_{ik} + \sum_{k=1}^N h_{kj}\right](1 - h_{ij}), \\ &= -\mathrm{i}\Omega_{ij}h_{ij} + \frac{K}{N}\left[2 + \sum_{k\neq i}^N h_{ik} + \sum_{k\neq j}^N h_{kj}\right](1 - h_{ij}), \quad 1 \le i, j \le N, \ t > 0, \end{aligned} \tag{6.11}$$

where $\Omega_{ij} := \Omega_i - \Omega_j$, and we used the fact that $h_{ii} = 1$.

We now set the real and imaginary parts of h_{ij}:

$$R_{ij} := \mathrm{Re}\, h_{ij}, \quad I_{ij} := \mathrm{Im}\, h_{ij}, \quad h_{ij} = R_{ij} + \mathrm{i} I_{ij}, \quad 1 \le i, j \le N,$$

and we set the synchronization matrices to be

$$\mathcal{R} := (R_{ij}) \in M_N(\mathbb{R}), \quad \mathcal{I} := (I_{ij}) \in M_N(\mathbb{R}).$$

Thus, it follows from Definition 6.1 that it is enough to show the convergence of the matrices $\mathcal{R}$ and $\mathcal{I}$ as $t \to \infty$. System (6.11) can be rewritten in terms of (R_{ij}, I_{ij}):

$$\begin{aligned} \frac{dR_{ij}}{dt} &= \Omega_{ij} I_{ij} + \frac{K}{N}\Big[\Big(2 + 2R_{ij} + \sum_{k \ne i,j}^{N} (R_{ik} + R_{kj})\Big)(1 - R_{ij}) \\ &\qquad + I_{ij}\Big(2I_{ij} + \sum_{k \ne i,j}^{N} (I_{ik} + I_{kj})\Big)\Big], \\ \frac{dI_{ij}}{dt} &= -\Omega_{ij} R_{ij} + \frac{K}{N}\Big[\Big(2I_{ij} + \sum_{k \ne i,j}^{N} (I_{ik} + I_{kj})\Big)(1 - R_{ij}) \\ &\qquad - I_{ij}\Big(2 + 2R_{ij} + \sum_{k \ne i,j}^{N} (R_{ik} + R_{kj})\Big)\Big]. \end{aligned} \tag{6.12}$$

Note that for the zero-coupling case $K = 0$, system (6.12) becomes

$$\frac{dh_{ij}}{dt} = -\mathrm{i}\Omega_{ij} h_{ij}.$$

This equation yields a closed orbit solution:

$$h_{ij}(t) = h_{ij}^0 e^{-\mathrm{i}\Omega_{ij} t}.$$

We next consider a situation where the coupling strength K is sufficiently large compared with Ω_{ij} that our situation is close to the situation in Section 4:

$$R_{ij} \approx 1, \quad I_{ij} \approx 0.$$

In this regime, system (6.12) can be approximated by a linearized system:

$$\begin{aligned} \frac{d\tilde{R}_{ij}}{dt} &= \Omega_{ij} \tilde{I}_{ij} + 2K(1 - \tilde{R}_{ij}), \quad t > 0, \\ \frac{d\tilde{I}_{ij}}{dt} &= -\Omega_{ij} \tilde{R}_{ij} - 2K \tilde{I}_{ij}. \end{aligned} \tag{6.13}$$

System (6.13) has a unique equilibrium

$$(\tilde{R}_{ij}^e, \tilde{I}_{ij}^e) = \Big(\frac{4K^2}{\Omega_{ij}^2 + 4K^2}, \ -\frac{2K\Omega_{ij}}{\Omega_{ij}^2 + 4K^2}\Big),$$

and we have

$$\tilde{R}_{ij}(t) = e^{-2Kt}\Big(C_1 \sin\Omega_{ij}t + C_2\cos\Omega_{ij}t\Big) + \frac{4K^2}{\Omega_{ij}^2 + 4K^2},$$
$$\tilde{I}_{ij}(t) = -\frac{\Omega_{ij}e^{-2Kt}}{2K}\Big(C_1 \sin\Omega_{ij}t + C_2\cos\Omega_{ij}t\Big) - \frac{2\Omega_{ij}K}{\Omega_{ij}^2 + 4K^2},$$

where C_1 and C_2 are constants. Although we do not have a rigorous analysis of the dynamic qualitative behavior of (6.12), we can see that, as K increases, (6.12) might exhibit a bifurcation phenomenon at some critical coupling strength. This bifurcation phenomenon can be seen from the explicit example of a two-oscillator system.

For a two-oscillator system, $h := h_{12}$ and $\omega := \Omega_{12}$ satisfy

$$\begin{aligned} &\frac{dh}{dt} = -\mathrm{i}\omega h + K(1-h^2) = -K\Big[\Big(h + \mathrm{i}\frac{\omega}{2K}\Big)^2 + \frac{\omega^2}{4K^2} - 1\Big], \quad t > 0, \\ &h(0) = h^0. \end{aligned} \tag{6.14}$$

Depending on the relative sizes of K and ω, we consider the following three cases:

$$K > \frac{\omega}{2}, \quad K = \frac{\omega}{2}, \quad K < \frac{\omega}{2}.$$

Case A: $K > \frac{\omega}{2}$. In this case, equation (6.14) has two equilibria, $h_{\infty,-}$ and $h_{\infty,+}$:

$$h_{\infty,-} := -\frac{1}{2}\sqrt{4 - \Big(\frac{\omega}{K}\Big)^2} - \mathrm{i}\frac{\omega}{2K} \quad \text{and} \quad h_{\infty,+} := \frac{1}{2}\sqrt{4 - \Big(\frac{\omega}{K}\Big)^2} - \mathrm{i}\frac{\omega}{2K}.$$

By direct calculation, the solution of (6.14) is given by the following explicit formula:

$$h(t) = \frac{h_{\infty,+}(h^0 - h_{\infty,-}) + h_{\infty,-}(h^0 - h_{\infty,+})e^{-\sqrt{4K^2-\omega^2}t}}{h^0 - h_{\infty,-} - (h^0 - h_{\infty,+})e^{-\sqrt{4K^2-\omega^2}t}}.$$

Thus, it is easy to see that for any initial data $h^0 \neq h_{\infty,-}$, we have

$$h(t) \to h_{\infty,+} \quad \text{as } t \to \infty.$$

Numerical simulation in Figure 6(a) shows that there are heteroclinic orbits connecting from $h_{\infty,-}$ to $h_{\infty,+}$.

Case B: $K = \frac{\omega}{2}$. In this case, the unique equilibrium is

$$h_\infty := -\mathrm{i},$$

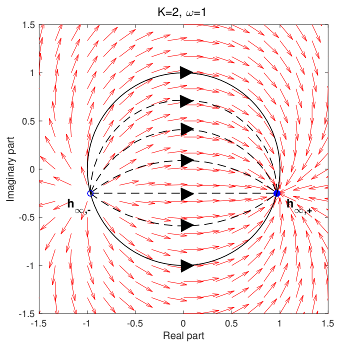
and the solution to (6.14) is given by the following formula.

$$h(t) = \frac{h^0 - \mathrm{i}(h^0 + \mathrm{i})Kt}{1 + (h^0 + \mathrm{i})Kt}.$$

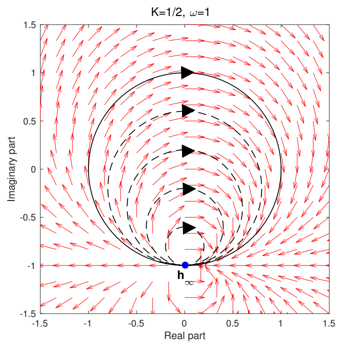
Thus, we have

$$h(t) \to h_\infty \quad \text{as} \quad t \to \pm\infty.$$

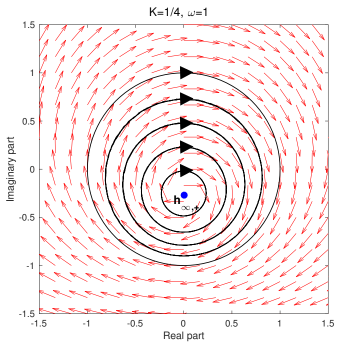
Numerical simulation in Figure 6(b) shows that there are homoclinic orbits connecting from h_∞ to h_∞.



(a) Dynamics for $K > \frac{\omega}{2}$



(b) Dynamics for $K < \frac{\omega}{2}$



(c) Dynamics for $K = \frac{\omega}{2}$

Figure 6. Phase portraits for three different cases

Case C: $K < \frac{\omega}{2}$. In this case, the equilibria of system (6.14) are as follows.

$$h_{\infty,-} = \mathrm{i}\left(-\frac{\omega}{2K} - \frac{1}{2}\sqrt{\left(\frac{\omega}{K}\right)^2 - 4}\right), \quad h_{\infty,+} = \mathrm{i}\left(-\frac{\omega}{2K} + \frac{1}{2}\sqrt{\left(\frac{\omega}{K}\right)^2 - 4}\right).$$

Because $|h_{\infty,-}| > 1$, only $h_{\infty,+}$ is admissible. By direct calculation, the solution of (6.14) is given as follows.

$$h(t) = \frac{h^0 \cos\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right) - \frac{K}{\sqrt{\omega^2-4K^2}}\left(\frac{\mathrm{i}\omega h^0}{K} - 2\right)\sin\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right)}{\cos\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right) + \frac{K}{\sqrt{\omega^2-4K^2}}\left(2h^0 + \frac{\mathrm{i}\omega}{K}\right)\sin\left(\frac{t\sqrt{\omega^2-4K^2}}{2}\right)}. \tag{6.15}$$

Note that (6.15) implies that h is a periodic orbit with period $\frac{4\pi}{\sqrt{\omega^2-4K^2}}$. Thus, we can see that a two-oscillator system has a bifurcation at $K = \frac{\omega}{2}$. Numerical simulation in Figure 6(c) shows that there are centers circling $h_{\infty,+}$.

7. Conclusion

In this survey paper, we have discussed asymptotic behaviors of several synchronization models; the Peskin model, the Winfree model, the Kuramoto model, the Lohe model and the Schrödinger–Lohe model from a unified framework. For each proposed model, we presented sufficient conditions leading to the collective behaviors of the synchronization models in terms of initial data and system parameters(the coupling strength, natural frequencies). There are many theoretical issues needed to be explored, e.g., existence of a critical coupling strength, structure of phase-locked states and stability of emergent asymptotic states, etc. The results presented in Section 5 and Section 6 are far from completeness. We have discussed only analogous issues as the classical models. However, we can expect more richer dynamic features for the quantum models, compared to the classical ones. We leave the investigation on the genuine features of the quantum models as interesting problems for a future direction.

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