EMS Surv. Math. Sci. 3 (2016), 107–130
DOI 10.4171/EMSS/15

EMS Surveys in
Mathematical Sciences

Topological monsters in elliptic equations and spectral theory

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Abstract. In this paper we will discuss the appearance of complicated geometric and topological structures through the level sets of solutions to a wide range of elliptic equations. Applications to the analysis of nodal sets of eigenfunctions of Schrödinger operators will be discussed too.

Mathematics Subject Classification (*2010*). 35B05, 35J15, 58J50.

Keywords. Level sets, elliptic equations, nodal sets, eigenfunctions, harmonic functions.

Contents

*The authors are supported by the ERC Starting Grants 633152 (A.E.) and 335079 (D.P.-S.). This work is supported in part by the ICMAT–Severo Ochoa grant SEV-2015-0554.

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1. Introduction: level sets and critical points of solutions to elliptic PDEs

The study of level sets of solutions to an elliptic PDE is a major topic in partial differential equations and geometric analysis, with connections that range from geometric measure theory to purely topological aspects.

The origin of this kind of questions goes back to electrostatics and potential theory. There the key object is the electric potential, which is a scalar function satisfying the Poisson equation

$$\Delta u = f$$

in a domain $\Omega \subseteq \mathbb{R}^3$, where the function f describes the electric charge density and the level sets $u^{-1}(c)$ are called equipotential surfaces. If the domain Ω is not the whole space, boundary conditions need to be imposed (typically, in electrostatics one prescribes the value of the potential u on $\partial\Omega$ and suitable decay conditions at infinity). An analogous problem arises in the Newtonian theory of gravity, with u playing the role of the gravitational potential and the function f (which would now be nonpositive) describing the mass density.

It is remarkable that the study of the geometric and topological properties of the level sets of the electrostatic potential was actually pioneered by Faraday and Maxwell in the XIX century. The relevance for this task of Faraday's lines of force, which are simply the integral curves of the gradient vector field $-\nabla u$, was unveiled in Maxwell's celebrated treatise [40], where they led to the first hints of what is nowadays known as Morse theory.

In this survey we will be concerned with the geometry and topology of solutions to PDEs, focusing on solutions to linear elliptic PDEs on $\mathbb{R}^n$ and on eigenfunctions of the Laplacian (possibly plus a potential) on a bounded domain of $\mathbb{R}^n$ or on a closed Riemannian manifold. As we will see, in both cases, we will be interested in the "shape" of the level sets rather than only in their "size".

This survey is organized as follows. In Section 2 we will start by studying the level sets of a harmonic function, which is a classical topic where many fundamental questions remain open. This provides a gentle introduction to the subject. In Section 3 we will consider level sets for nonlinear equations close to the linear regime, illustrating the ideas with the Allen–Cahn equation. In Section 4 we will consider the closely related topic of critical points of solutions, that is, the study of the locus where the gradient of the function vanishes. This will be discussed in the context of Green's functions for Riemannian manifolds. Finally, nodal sets of eigenfunctions will be considered in Sections 5 and 6, first in the context of low-energy eigenfunctions of the Laplacian and then for high-energy eigenfunctions of other Schrödinger operators.

2. A basic example: harmonic functions on $\mathbb{R}^n$

2.1. Harmonic functions in the plane. To present the questions that will be discussed in this survey, let us start with the most elementary context that one can consider: a harmonic function u in the Euclidean space $\mathbb{R}^n$:

$$\Delta u = 0\,.$$

For simplicity, let us start with the two-dimensional case, that is, harmonic functions on the plane, and restrict our attention to *regular* level sets, that is, level sets on which the gradient ∇u does not vanish. It is an easy consequence of the implicit function theorem that all the connected components of a regular level set of a function are differentiable submanifolds without boundary.

A very easy observation is that the assumption that u is harmonic implies that a regular level curve of u cannot have any compact connected components. Indeed, otherwise there must be a closed curve γ (smooth and without any self-intersections) in the plane such that u takes some constant value c on γ. Since γ is the boundary of a bounded domain Ω (a topological disk, in fact), it follows from the maximum principle that $u \equiv c$ in Ω, and in fact $u \equiv c$ in the whole plane because harmonic functions are analytic. This contradicts the assumption that γ is a connected component of $u^{-1}(c)$, so the claim follows.

Hence we conclude that the connected components of any regular level set of a harmonic function in the plane must be open, i.e., diffeomorphic to the real line. Quite remarkably, one can also understand the topology of level sets that are not regular [5,34,42], essentially because the critical points of a harmonic function on the plane are necessarily isolated and because the critical level curve in a neighborhood of a critical point is homeomorphic to a $2m$-pointed star (that is, to the set $\{z \in \mathbb{C} : z^{2m} \in [0,1)\}$ for some integer $m \geqslant 2$).

Let us emphasize at this point that, in contrast to what the above easy arguments might suggest, our understanding of the level curves of harmonic functions on the plane is far from complete. Modulo homeomorphism, the natural question is to classify the (possibly singular) foliations that are defined by the level sets of a harmonic function on the plane, and in this respect we are not significantly better off than in the early 1950s, when the efforts of Morse, Kaplan and Boothby managed to shed some light on the singular foliations defined by the level sets of a harmonic function.

Let us briefly elaborate on this issue. Firstly, the aforementioned authors showed that the assumption that the function is harmonic on the whole plane (as opposed to only in a subset of it) implies that if the harmonic function does not have any critical points, then either the foliation it defines is homeomorphic to the trivial foliation of the plane by parallel straight lines or else it has infinitely many Reeb components. This result is in fact a consequence of Picard's theorem for holomorphic functions, and has an analogous but more sophisticated statement for general harmonic functions on the plane. It is not hard to see that there are infinitely many homeomorphism classes of foliations defined by harmonic functions on the plane without critical points; for example, it suffices to consider the real part of the functions e^z, e^{e^z}, $e^{e^{e^z}}$, ... A complete characterization (modulo homeomorphism) of the foliations that can arise from the level sets of a harmonic function on the plane is still wide open.

2.2. Harmonic functions in $\mathbb{R}^n$ with $n \geqslant 3$. The extension of these results to higher dimensions is a long-standing open problem, collected e.g. in the problem collections

[8, Problem 3.20] and [37, Problem R.7]. The reason for which little progress has been made concerning the topology of level sets of harmonic functions in higher dimensions is twofold. From an analytic point of view, all existing results for harmonic functions in $\mathbb{R}^2$ have been obtained using complex analytic techniques, which are no longer available in dimension $n \geqslant 3$. Topologically, the situation is also much simpler in two dimensions because any connected component of a regular level set of a harmonic function u on the plane is homeomorphic to a line, while in higher dimensions there are infinitely many topological types of noncompact level sets and it is not obvious which ones can actually appear in harmonic functions.

In fact, other than the obvious obstruction that all the connected components of a regular level set must be noncompact (by the same argument as above) and the analysis of explicit examples of harmonic functions, until very recently there were no results on the admissible topological types of the level sets of a harmonic function. For instance, in dimension $n = 3$ it is easy to construct explicit harmonic functions having level sets with a connected component homeomorphic to elementary surfaces such as the plane or the cylinder, but it is an entirely different matter to ascertain whether there can be more complicated surfaces, as in the following:

Question 1. *Is there a harmonic function u in $\mathbb{R}^3$ such that $u^{-1}(0)$ has a connected component homeomorphic to the genus g torus with N ends? (cf. Figure 1).*

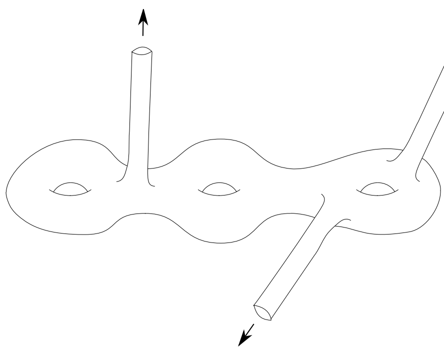

Figure 1. A torus of genus $g = 3$ and $N = 3$ ends.

The answer to this question turns out to be positive. More generally, one can prove the following theorem [23]. Recall that a hypersurface of $\mathbb{R}^n$ is called *algebraic* if it can be written as the zero set of a polynomial on $\mathbb{R}^n$ whose gradient does not vanish at any point of the hypersurface. Throughout the paper, by a *hypersurface* we simply mean a codimension-1 embedded submanifold without boundary of $\mathbb{R}^n$.

Theorem 2.1. *Let Σ be a smooth noncompact algebraic hypersurface of $\mathbb{R}^n$. Then there is a harmonic function in $\mathbb{R}^n$ whose zero set has a connected component diffeomorphic to Σ.*

Remark 2.2. Given any analytic hypersurface Σ of $\mathbb{R}^n$, it is apparent that one has $\Sigma = w^{-1}(0)$ for some function satisfying $\Delta w = 0$ in some neighborhood of Σ (it is enough to take w as the local solution, given by the Cauchy–Kowalewski theorem, to the problem $\Delta w = 0, w|_\Sigma = 0, \partial_\nu w = 1$). The key point in Theorem 2.1 is that w is harmonic in the whole space, which in particular makes it essential to allow the hypersurfaces to be realized only modulo diffeomorphism: for example, it is known [28] that the curves $x_2 = x_1^s$, which are all diffeomorphic for any integer $s \geqslant 1$, can be a connected component of the zero level set of a harmonic function in $\mathbb{R}^2$ if and only if $s = 1$ or $s = 2$.

Remark 2.3. Since the genus g torus with N ends can be algebraically embedded in $\mathbb{R}^3$, this furnishes a positive answer to Question 1. It should be noticed that Theorem 2.1 does not apply to singular real algebraic varieties of codimension 1. In fact, the statement is not true in this case, a counterexample is the set $\{(x_1^2 + x_2^2 + x_3^2 - 1)x_3 = 0\}$ in $\mathbb{R}^3$, which cannot be homeomorphic to a union of connected components of the zero set of a harmonic function.

Sketch of the proof. To explain the gist of our approach we shall here consider the concrete problem of showing that there exists a harmonic function in $\mathbb{R}^3$ whose zero set has a connected component diffeomorphic to the torus of genus g with N ends, as displayed in Figure 1. The basic strategy to construct a harmonic function such that the above surface, which we shall call Σ, is diffeomorphic to a connected component of its zero set is the following. We will start with a function v that is harmonic in a neighborhood of the surface and has a level set diffeomorphic to Σ. We will then approximate this local solution by a global harmonic function u. Hence, the key for the success of this strategy is to be able to ensure that the level set of the local harmonic function is "robust" in the sense that it is preserved, up to a diffeomorphism, under the perturbation corresponding to the above global approximation.

Let us elaborate on this point to show the kind of difficulties that arise when carrying out this program. We have seen that the starting point of the strategy must be a topological stability theorem for the level sets of the local harmonic function v. If these level sets were compact, this could be accomplished using Thom's celebrated isotopy theorem [1, Theorem 20.2]; however, the fact that the level sets of a harmonic function are all noncompact makes the problem much subtler. Moreover, this stability result must be finely tailored to provide sufficient control of the deformation at infinity, for otherwise this part would be a bottleneck in the proof of the theorem. Therefore, one of the crucial parts of the proof is to establish a fine C^1 noncompact stability result under small perturbations, which holds provided that the function v satisfies a suitable *gradient* condition (namely, that $|\nabla v|$ is bounded away from zero) in a *saturated* neighborhood of the surface (roughly speaking, in $v^{-1}((-\epsilon, \epsilon))$ in order to control the zero set of v) and a C^2 bound.

In view of this result, the local harmonic function v must be constructed in such a way that the above stability conditions hold. These conditions rule out, for example, an approach based on the Cauchy–Kowalewski theorem, of the kind that we discussed in Remark 2.2, as it does not yield enough information on the domain of definition of the solution. Instead, we base our construction on the use of Green's functions. We take an unbounded domain Ω whose boundary is diffeomorphic to the surface Σ and consider its Dirichlet Green's function $\mathcal{G}_\Omega(x, y)$. If y is a point of this domain, $\mathcal{G}_\Omega(\cdot, y)$ defines a (local) harmonic function in a half-neighborhood of the boundary, which is the zero level set of this function and diffeomorphic to the surface Σ. However, the noncompact stability theorem cannot be applied to $\mathcal{G}_\Omega(\cdot, y)$ because this function does not satisfy the aforementioned saturation and gradient conditions (this is obvious because the gradient of the Green's function tends to zero at infinity).

In order to circumvent this difficulty, we start by deforming the surface with a diffeomorphism so that it has a controlled geometry at infinity (i.e., in this particular case, that the N ends of the torus are straight cylinders, as in Figure 1). We will call Σ' this deformation of the initial surface, and denote by Ω the inner domain bounded by it. We then insert a straight half-line in each end of the domain and, denoting by μ_i the length measure on the i^{th} half-line, consider the function

$$v(x) := \sum_{i=1}^{N} \int \mathcal{G}_\Omega(x, y)\, d\mu_i(y)\,.$$

It can be shown that this integral converges, thereby defining a function harmonic in the domain minus the half-lines whose zero set is the surface Σ' and which does satisfy the boundedness, saturation and gradient conditions of the C^1 stability theorem in a half-neighborhood of the surface. Roughly speaking, the proof of this fact is based on estimates that exploit the asymptotic Euclidean symmetries of the construction and the exponential decay of the Green's function of the domain. As an aside, notice that to resort to this construction one needs to prove the noncompact stability theorem also for functions defined only in a half-neighborhood of the surface under consideration.

The last ingredient is an approximation theorem that allows us to approximate the local harmonic function v by a global harmonic function in a certain neighborhood of the surface Σ. This approximation must be small in the C^1 norm in order to apply the stability theorem. If Σ' were compact, the usual approximation theorems for elliptic PDEs [6] would provide an adequate global approximation result. We recall that these theorems, which generalize the classical theorem of Runge in complex analysis, ensure that a function satisfying the Laplace equation in a compact subset (with a certain topological property) of $\mathbb{R}^n$ can be approximated in any C^k norm by a global harmonic function. As Σ' is noncompact, however, the situation is considerably more involved. The proof of our global approximation theorem relies on an iterative procedure (which does *not* apply to arbitrary elliptic PDEs) that is built over an appropriate exhaustion by compact sets and combines suitable Green's function estimates and a balayage-of-poles argument. □

Of course, one can also consider hypersurfaces of infinite topology, which cannot be embedded in $\mathbb{R}^n$ as the zero set of a polynomial and, consequently, are not covered by Theorem 2.1. The simplest case would be the following:

Question 2. *Is there a harmonic function u in $\mathbb{R}^3$ such that $u^{-1}(0)$ has a connected component homeomorphic to the torus of infinite genus or to the infinite jungle gym? (cf. Figure 2).*

It turns out that the answer to Question 2 is positive too. It follows from a more technical theorem that can be informally stated as follows. The definition of the tentacled hypersurfaces below and the precise statement of the theorem (which applies not just to one, but to finite unions of disjoint tentacled hypersurfaces) can be found in [23, Theorem 1.1].

The intuition behind the definition of "tentacled hypersurfaces", of which genus g tori with N straight cylindrical ends or periodic structures such as the jungle gym or the infinite torus drawn in Figure 2 are prime examples, is clear from the sketch of the proof of Theorem 2.1: they are hypersurfaces whose very rigid geometry at infinity provides good lower bounds on the gradient of the Green's function in certain regions.

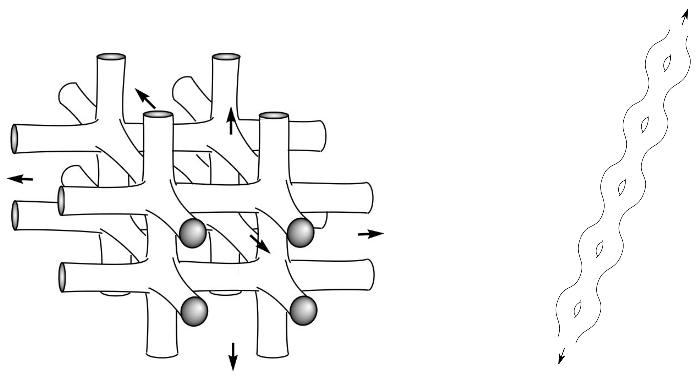

Figure 2. From the left to the right, an infinite jungle gym and a torus of infinite genus.

Theorem 2.4 (Informal statement). *Let Σ be a (connected) hypersurface of $\mathbb{R}^n$ which either only has "straight ends" (i.e., outside a compact set, it is equivalent modulo rigid motions to the union of finitely many "straight tubes" of the form $(0,\infty) \times \partial K$, with K a compact subset of $\mathbb{R}^{n-1}$) or is invariant under a discrete isometry subgroup G and only has "straight ends" modulo G. (These hypersurfaces are called "tentacled", as their ends, modulo a discrete isometry subgroup, are given by finitely many straight tentacles). Then there is a smooth diffeomorphism Φ of $\mathbb{R}^n$, arbitrarily close to the identity in the C^1 norm, so that $\Phi(\Sigma)$ is a connected component of the zero set of a harmonic function in $\mathbb{R}^n$.*

2.3. What if one considers more than one harmonic function? After discussing the level sets of a harmonic function, we will next consider the transverse intersection of the zero sets of $m \geqslant 2$ harmonic functions. Unlike a single level set, these intersections can be compact. Here one would be interested in sets diffeomorphic to pathological objects as in the following:

Question 3. *Let Σ be an exotic sphere of dimension 7, so that Σ is a smooth manifold homeomorphic to the standard 7-sphere $\mathbb{S}^7$ but not diffeomorphic to it. It is known that Σ can be smoothly embedded in $\mathbb{R}^{14}$. Are there seven harmonic functions $u_1, \dots, u_7$ on $\mathbb{R}^{14}$ such that the exotic sphere Σ is diffeomorphic to a component of the joint level set $u_1^{-1}(0) \cap \cdots \cap u_7^{-1}(0)$?*

The following theorem asserts that the answer to Question 3 is positive by providing a sufficient condition for the existence of m harmonic functions whose zero sets intersect transversally at a codimension-m submanifold of certain topology. In the statement, the only issue that needs some explanation is that the submanifold Σ is assumed to have trivial normal bundle (i.e., that it is has a tubular neighborhood diffeomorphic to Σ cross an m-dimensional ball). Indeed, it is well known that this is a topological obstruction for the submanifold to be the transverse intersection of m functions.

Theorem 2.5. *Given an integer $m \geqslant 2$, let Σ be a compact codimension-m embedded submanifold of $\mathbb{R}^n$ with trivial normal bundle. Then there are m harmonic functions $u_1, \dots, u_m$ on $\mathbb{R}^n$ and a smooth diffeomorphism Φ of $\mathbb{R}^n$, arbitrarily close to the identity in any C^p norm, so that $\Phi(\Sigma)$ is a a connected component of the intersection of their level sets $u_1^{-1}(0) \cap \cdots \cap u_m^{-1}(0)$.*

3. The linear regime of nonlinear equations: Allen–Cahn and the Helmholtz equation

While in the previous section we decided to restrict our attention to harmonic functions on $\mathbb{R}^n$ to keep statements as simple as possible, it is clear that the results can be extended to significantly wider classes of equations. Specifically, in Reference [23] it is discussed to which extent these results remain valid for more general second-order linear elliptic PDEs. Our concern in this section will be to show how these methods can be used to study nonlinear equations, as long as one stays in the linear regime.

To illustrate the idea, we will consider the paradigmatic case of the Allen–Cahn equation in $\mathbb{R}^n$:

$$\Delta u + u - u^3 = 0\,.$$

In this context, the basic question that we will discuss in this section is the following:

Question 4. *Let Σ be a compact hypersurface of $\mathbb{R}^n$. Is there a solution to the Allen–Cahn equation on $\mathbb{R}^n$ with (a connected component of) a level set diffeomorphic to Σ?*

Before getting into the heart of the matter, it is worth recalling that the study of the level sets of solutions to the Allen–Cahn equation has attracted much attention, especially due to De Giorgi's 1978 conjecture that all the level sets of any solution to the Allen–Cahn equation in $\mathbb{R}^n$ that is monotone in one direction must be hyperplanes for $n \leqslant 8$. This is a natural counterpart of the Bernstein problem for minimal hypersurfaces, which asserts that any minimal graph in $\mathbb{R}^n$ must be a hyperplane provided that $n \leqslant 8$. Ghoussoub–Ghi and Ambrosio–Cabré proved De Giorgi's conjecture for $n = 2, 3$ [3, 31], and the work of Savin [43] showed that it is also true for $4 \leqslant n \leqslant 8$ under a natural technical assumption. Del Pino, Kowalczyk and Wei [13] employed the Bombieri–De Giorgi–Giusti hypersurface to show that the statement of De Giorgi's conjeture does not hold for $n \geqslant 9$.

The analysis and possible classification of bounded entire solutions to the Allen–Cahn equation is an important open problem where the Morse index of the solution u (that is, the maximal dimension of a vector space $V \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \left(|\nabla v|^2 - v^2 + 3u^2 v^2\right) dx < 0$$

for all nonzero $v \in V$) plays a key role. Generally speaking, it is expected [14, 15] that the condition that the Morse index of the solution be finite should play a similar role as the finite total curvature assumption in the study of minimal hypersurfaces in Euclidean spaces. In particular, it is well known that there are many infinite-index solutions to the Allen–Cahn equation [7,11], and this abundance of solutions should translate into a wealth of possible level sets.

We shall next show how one can use ideas related to the matters discussed in Section 2 to explore the flexibility of bounded entire solutions to the Allen–Cahn equation of infinite index, proving, in particular, a positive answer to Question 4 [20]:

Theorem 3.1. *Let Σ be any compact orientable hypersurface without boundary of $\mathbb{R}^n$, with $n \geqslant 3$. Then there is an entire solution u of the Allen–Cahn equation in $\mathbb{R}^n$, which falls off at infinity as $|u(x)| < C(1+|x|)^{(1-n)/2}$ if $n \geqslant 4$ and is in $L^4(\mathbb{R}^3)$ if $n = 3$, such that its zero level set $u^{-1}(0)$ has a connected component diffeomorphic to Σ.*

Sketch of the proof. For concreteness, let us assume that $n \geqslant 4$. The result hinges on two key lemmas. The first one concerns the linearization of the Allen–Cahn equation at 0, that is, the Helmholtz equation. Specifically, using the same notation as in Theorem 3.1 without any further notice, one has the following:

Lemma 3.2. *There is a function w satisfying the Helmholtz equation*

$$\Delta w + w = 0$$

in $\mathbb{R}^n$ and a diffeomorphism Ψ of $\mathbb{R}^n$ such that $\Psi(\Sigma)$ is a structurally stable connected component of the zero set $w^{-1}(0)$. Furthermore, w falls off at infinity as $|D^j w(x)| < C_j(1+|x|)^{\frac{1-n}{2}}$ for any j.

Let us recall that a (connected component of a) level set $\Sigma' \subset w^{-1}(0)$ of a function w is *structurally stable* if there is some $\epsilon > 0$ such that if w' is a function with $\|w - w'\|_{C^1} < \epsilon$, then the level set $w'^{-1}(0)$ has a connected component diffeomorphic to Σ'. (Here one could have taken any other C^k norm and the diffeomorphism can be assumed to be close to the identity also in C^k, but we will not need this additional information.)

It is worth emphasizing that, if we compare Lemma 3.2 with the results for harmonic functions that we described in Section 2, the key difference is that the solution w of the Helmholtz equation in $\mathbb{R}^n$ falls off at infinity (by the Liouville theorem, a nontrivial harmonic function in $\mathbb{R}^n$ simply cannot do so). In fact, the decay as $|x|^{\frac{1-n}{2}}$ is known to be sharp for solutions to the Helmholtz equation.

To pass from the Helmholtz equation to the Allen–Cahn equation one resorts to an iterative procedure. To set the iteration, let us recall that a fundamental solution of the Helmholtz equation is given by

$$G(x) := \frac{2^{1-\frac{n}{2}}\pi}{|\mathbb{S}^{n-1}|\,\Gamma(\frac{n}{2}-1)}\,|x|^{1-\frac{n}{2}}\,Y_{\frac{n}{2}-1}(|x|)\,, \tag{3.1}$$

where $Y_{\frac{n}{2}-1}$ denotes the Bessel function of the second kind of order $\frac{n}{2}-1$, $|\mathbb{S}^{n-1}|$ is the area of the unit $(n-1)$-sphere and Γ is the Gamma function. This means that for any smooth function f that decays fast enough at infinity, the convolution $G * f$ satisfies the equation

$$\Delta(G * f) + G * f = f\,.$$

Using the function G, the iterative scheme can then be assumed to be

$$u_0 := \delta w\,,$$
$$u_{j+1} := \delta w + G * (u_j^3)\,, \tag{3.2}$$

where w is the above solution to the Helmholtz equation and δ is some small positive constant that will be fixed later. One can prove that, if δ is small enough, then u_j converges as $j \to \infty$ to some function u in the weighted space $C^k_{\frac{n-1}{2}}(\mathbb{R}^n)$ of functions whose norm

$$\|u\|_{k,\frac{n-1}{2}} := \max_{|\alpha|\leqslant k} \sup_{x\in\mathbb{R}^n} \left|(1+|x|)^{\frac{n-1}{2}}\, \partial^\alpha u(x)\right|$$

is finite. Here $k \geqslant 2$ is any integer. The function u must then satisfy

$$u = \delta w + G * (u^3)\,,$$

so in particular we infer that

$$\Delta u + u - u^3 = 0$$

by applying the Helmholtz operator to the above identity.

To prove the convergence of the iteration, one needs to use the sharp falloff of the function w at infinity and the following lemma, which provides a weighted estimate for convolutions with the fundamental solution:

Lemma 3.3. *For any $v \in C^k_{\frac{n-1}{2}}(\mathbb{R}^n)$, one has the estimate*

$$\|G * (v^3)\|_{k,\frac{n-1}{2}} \leqslant C\, \|v\|^3_{k,\frac{n-1}{2}}\,.$$

Once that one has established the converge of the iteration, the rest of the argument is easy. Indeed, it readily follows that the solution to the Allen–Cahn equation must satisfy the bounds

$$\|u\|_{k,\frac{n-1}{2}} \leqslant C\delta$$

and

$$\left\| w - \frac{u}{\delta} \right\|_{k,\frac{d-1}{2}} = \frac{1}{\delta}\, \|\delta w - u\|_{k,\frac{d-1}{2}} = \frac{1}{\delta}\, \|G * (u^3)\|_{k,\frac{d-1}{2}} \leqslant \frac{C}{\delta}\, \|u\|^3_{k,\frac{d-1}{2}} \leqslant C\delta^2\,.$$

Since the zero set of u and u/δ obviously coincides, a moment's thought shows that the fact that the zero set $w^{-1}(0)$ has a structurally stable connected component diffeomorphic to Σ implies that, for small enough δ (specifically, for $C\delta^2 < \epsilon$, with ϵ the constant introduced right after Lemma 3.2) the zero set $u^{-1}(0)$ must also have a component diffeomorphic to Σ. The theorem then follows. □

Remark 3.4. In fact, it follows from the proof that the connected component of $u^{-1}(0)$ can be assumed to be an *ϵ-rescaling* of Σ, that is, the image of Σ under a diffeomorphism of $\mathbb{R}^n$ that can be written as $\Phi = \Phi_1 \circ \Phi_2$, where Φ_2 is a rescaling and $\|\Phi_1 - \mathrm{id}\|_{C^1(\mathbb{R}^n)} < \epsilon$ (and here we could have taken any other fixed C^k norm instead).

4. From level sets to critical points: appearance in Green's functions

The study of the topology of the level sets of a function u is intimately connected with that of its critical points, that is, with the locus where its gradient ∇u vanishes. Indeed, it is well known that the critical values of u (i.e., the values that u takes at its critical points) are those at which the compact regular level sets $u^{-1}(c)$ can change its topology. Essentially, the reason for this is that if the level set $u^{-1}(c)$ is compact for all $c_1 \leqslant c \leqslant c_2$ and u does not have any critical values in the interval $[c_1, c_2]$, the vector field $X := \nabla u/|\nabla u|^2$ is well defined in this set and satisfies

$$X \cdot \nabla u = 1\,,$$

so its time t flow maps $u^{-1}(c_1)$ into $u^{-1}(c_1 + t)$ for all $0 < t < c_2 - c_1$.

To illustrate the study of the critical points of solutions to a PDE, we will consider the case of the Green's function $\mathcal{G}(x, y)$ of the Laplacian Δ_g of a noncompact n-dimensional manifold M endowed with a Riemannian metric g. The Green's function then satisfies the equation

$$\Delta_g \mathcal{G}(\cdot, y) = -\delta_y \tag{4.1}$$

for each $y \in M$ and is symmetric and smooth on $M \times M$ outside the diagonal. We will find it notationally convenient to fix a point $y \in M$, once and for all, and consider a Green's function $G := \mathcal{G}(\cdot, y)$ with pole y, which is then harmonic in $M\backslash\{y\}$. Let us recall that a noncompact Riemannian manifold does not need to admit a positive Green's function (that is the case of the Euclidean plane, for instance), but if there is a positive Green's function, there is a unique minimal one.

An important difficulty in the study of critical points lies in the fact that Green's function estimates are not sufficiently fine to elucidate whether the gradient of G vanishes in a certain region. Moreover, it is well known that the codimension of the critical set of G is at least 2 [32], which introduces additional complications in the analysis. For these reasons, the results on critical points of Green's functions are surprisingly scarce.

We shall next see how an approach based on a combination of techniques from differential topology and second-order elliptic PDEs can be employed to show that there are Green's functions with an arbitrarily high number of critical points provided that $n \geqslant 3$. It is worth emphasizing that the way that we manage to control the critical points is precisely by proving the existence of level sets of complicated topology and showing that, in a precise sense, one can generically assume that the critical points of the Green's function are nondegenerate in a certain region of space. For simplicity, let us state our results in the case where M is diffeomorphic to $\mathbb{R}^n$ [22]:

Theorem 4.1. *Given any integer N, there is a Riemannian metric g on $\mathbb{R}^n$, with $n \geqslant 3$, whose minimal Green's function has at least N nondegenerate critical points.*

Sketch of the proof. The key step of the proof is to show that one can choose a metric on $\mathbb{R}^n$ whose minimal Green's function approximates the (Euclidean) Dirichlet Green's function of any fixed bounded domain. To state the lemma, given a bounded domain Ω in $\mathbb{R}^n$ containing the point y, let us denote by G_Ω its Dirichlet Green's function with a pole at the point y, which satisfies the equation

$$\Delta G_\Omega = -\delta_y \quad \text{in } \Omega\,, \qquad G_\Omega|_{\partial\Omega} = 0$$

with Δ the ordinary Laplacian of $\mathbb{R}^n$.

Lemma 4.2. *Let Ω be a bounded domain of $\mathbb{R}^n$ with connected boundary. Given a compact subset S of $\Omega\backslash\{y\}$, for any $\delta > 0$ there is a complete metric g on $\mathbb{R}^n$ whose minimal Green's function G tends to zero at infinity and satisfies*

$$\|G - G_\Omega\|_{C^2(S)} < \delta\,.$$

To see that Lemma 4.2 leads to Theorem 4.1, one starts by proving that, by perturbing the domain Ω a little if necessary, one can assume that the Green's function G_Ω is Morse (that is, all its critical points are nondegenerate). This is not hard to believe but the proof is not immediate. Since the gradient of G_Ω points inwards (and in particular it does not vanish) on the boundary of Ω by Hopf's boundary point lemma, one can apply Morse theory for manifolds with boundary to the auxiliary C^2 function $f : \overline{\Omega} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -1/(G_\Omega(x) + 1)^2 & \text{if } x \neq y\,, \\ 0 & \text{if } x = y\,. \end{cases}$$

It then follows that f has at least $b_p(\overline{\Omega})$ critical points of index $n - p$, for all $0 \leqslant p \leqslant n$, where b_p denotes the p^{th} Betti number. It is a straightforward computation that all the critical points of f other than y are also critical points of G_Ω and they have the same Morse indices. Since any C^2-small perturbation of a Morse function is Morse and has the same number of critical points, to prove Theorem 4.1 it is then enough to apply the statement of Lemma 4.2, for instance, to a domain Ω in $\mathbb{R}^n$ whose boundary is the connected sum of N copies of $\mathbb{S}^1 \times \mathbb{S}^{n-2}$, for in this case its first Betti number is $b_1(\overline{\Omega}) = N$.

To conclude, let us sketch the proof of Lemma 4.2. For any positive integer j, let $\varphi_j : \mathbb{R}^n \to [1, \infty)$ be a smooth function such that $\varphi_j(x) = 1$ if $x \in \Omega$ and $\varphi_j(x) = j$ if $\operatorname{dist}(x, \Omega) > \frac{1}{j}$. Let us now define the conformally flat metrics $g_j := \varphi_j g_0$ on $\mathbb{R}^n$, where g_0 denotes the Euclidean metric. As the metric is flat outside a compact set, it is standard [35] that there is a unique minimal Green's function G_j with a pole at a point y. For simplicity of notation, let us choose the origin of coordinates so that $y = 0$.

The idea now is that, as the metric is very large outside Ω, the Green's function tends to zero outside Ω, so it tends to G_Ω as $j \to \infty$. The way we make this precise is via the variational formulation of the problem. For this, one writes G_j as a limit of the unique solution G_j^R to the boundary problem

$$\Delta_j G_j^R = 0 \quad \text{in } A^R := \big\{x \in \mathbb{R}^n : 1/R < |x| < R\big\}\,,$$
$$G_j^R\big|_{|x|=1/R} = \frac{R^{n-2}}{|\mathbb{S}^{n-1}|}\,, \qquad G_j^R\big|_{|x|=R} = 0\,.$$

on the annulus A^R centered at 0 and of inner radius $1/R$ and outer radius R. Here Δ_j refers to the Laplacian associated with the metric g_j. Indeed, one can prove that

$$G_j = \big(1 + o(1)\big)\, G_j^R + o(1)\,, \tag{4.2}$$

so that we clearly have uniform convergence in any compact set $K \subset \mathbb{R}^n\backslash\{0\}$:

$$\big\|G_j - G_j^R\big\|_{C^0(K)} = o(1)\,. \tag{4.3}$$

Here and in what follows, $o(1)$ denotes a quantity that tends to 0 as $R \to \infty$ uniformly in j (in the previous equation, this quantity obviously depends on the compact set K too).

Without getting into unnecessary details, the rough idea is to start by showing that the L^2 norm of the gradient of G_j^R becomes very small outside Ω for large j:

$$\int_{A^R\setminus\Omega} |\nabla G_j^R|^2 \, dx = o_R(1) \,. \tag{4.4}$$

Combining this with standard Schauder estimates, one sees that if Ω_ϵ is a thickening of the original domain Ω with a small width ϵ and $v_{\epsilon,j}^R$ is the product of G_j^R by a suitable cutoff function supported in Ω_ϵ and equal to 1 in Ω, then the energy of

$$\mathcal{E}_{\Omega_\epsilon}^R[v_{\epsilon,j}^R] := \int_{\Omega_\epsilon\cap A^R} |\nabla v_{\epsilon,j}^R|^2 \, dx$$

is very close to its minimum:

$$\mathcal{E}_{\Omega_\epsilon}^R[v_{\epsilon,j}^R] \leqslant \inf \mathcal{E}_{\Omega_\epsilon}^R + o_{\epsilon,R}(1) + \tilde{o}_R(1) \,, \tag{4.5}$$

where $\tilde{o}_R(1)$ denotes a quantity that goes to 0 as $\epsilon \to 0^+$ (*not* as $j \to \infty$) and is uniform in j but not necessarily in R.

As the Green's function G_{Ω_ϵ} is the unique minimizer of the above energy functional, one can ultimately infer that

$$\int |\nabla G_{\Omega_\epsilon}^R - \nabla v_{\epsilon,j}^R|^2 \, dx \leqslant o_{\epsilon,R}(1) + \tilde{o}_R(1) \,,$$

This L^2 estimate can be promoted to a pointwise bound. Since G_{Ω_ϵ} tends to G_Ω as $\epsilon \to 0^+$, with some work, on the set S one can take the limits $\epsilon \to 0^+$, $R \to \infty$ and $j \to \infty$ in the right order to obtain Lemma 4.2. ☐

It is remarkable that the picture is completely different in two dimensions, where the number of critical points of a "reasonable" Green's function admits a topological bound [22]. By "reasonable" we mean a Green's function that arises from an exhaustion procedure by compact sets, as in the proof of the existence of Green's functions due to Li and Tam in [38]. For simplicity, we will refer to these Green's functions as *Li–Tam*. Any Li–Tam Green's function coincides with the minimal one when the latter exists.

Theorem 4.3. *Let M be an open Riemannian surface of finite topological type. The number of critical points of any Li–Tam Green's function G on M is not larger than twice the genus of M plus the number of ends minus* 1*:*

$$\#\,\textit{critical points} \leqslant 2\,\textit{genus}(M) + \#\,\textit{ends} - 1 \,.$$

Sketch of the proof. In addition to the analytic proof of this result given in [22], a different, more visual proof of this result that hinges on the qualitative properties of the flow defined by the gradient of G can be found in [26]. In this second approach, the result follows from a more general heuristic principle, which informally asserts that M can be decomposed as the union of a topological disk D and a (possibly disconnected) noncompact graph $\mathcal{F}$, both of which consist of integral curves of ∇G. The disk is the union of the pole y and the points of M whose ω-limit along the flow of ∇G is y. The graph $\mathcal{F}$ consists of the critical points of G, their stable components, and certain integral curves of ∇G that escape to infinity. When suitably compactified, $\mathcal{F}$ is a connected graph that encodes the topology of the surface, the rank of the first homology group of $\mathcal{F}$ being twice the genus of M. The bounds for the number of critical points of the Green's function stem from the structure of the graph $\mathcal{F}$.

No matter which approach is taken, key ideas that underlie the proof are the behavior of the Green's function at each end and the fact that the critical points of the Green's function in two dimensions have a very rigid structure (in particular, their Hopf index is negative and determines the geometry of the critical point completely). □

5. Flexibility of the nodal sets of low-energy eigenfunctions

After analyzing problems on harmonic and Green's functions, we shall next outline how complicated geometric shapes can appear naturally in spectral theory.

For concreteness, let M be a compact Riemannian n-dimensional manifold with boundary and consider the sequence of its Dirichlet eigenfunctions, which satisfy the equation

$$\Delta u_k = -\lambda_k u_k \text{ in } M\,, \quad u_k|_{\partial M} = 0\,.$$

Here $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ are the Dirichlet eigenvalues of M. In this section we will be concerned with the geometry of the nodal sets $u_k^{-1}(0)$ of the eigenfunctions of the Laplacian, which is a classic topic in geometric analysis with a number of important open problems [45, 46].

In this section we will be interested in low-energy eigenfunctions. It is worth recalling that, while the behavior of λ_k as $k \to \infty$ is extremely rigid, as captured by Weyl's law, the low-energy behavior of the eigenvalues is quite flexible. A landmark in this direction is the proof that one can prescribe an arbitrarily high number of eigenvalues of the Laplacian, including multiplicities. More precisely [9], in the case where M does not have boundary, Colin de Verdière proved that given any finite sequence of positive real numbers $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$, there is a metric g on M having this sequence as its first N nontrivial eigenvalues. Our goal will be to show that the nodal sets of the first eigenfunctions are surprisingly flexible too.

Since the first Dirichlet eigenfunction does not vanish, we will be especially interested in the shape of the nodal set $u_2^{-1}(0)$. In this direction, a major open problem is Payne's conjecture, which asserts that when M is a bounded simply connected domain of the plane the nodal line of u_2 is an arc connecting two distinct points of the boundary. Payne's conjecture is known to hold for convex domains [2, 41], but the general case is still open.

Yau [46, Problem 45] raised the question of the validity of Payne's conjecture in Riemannian n-manifolds with boundary. In this direction, Freitas [29] showed that there is a metric on the two-dimensional ball for which Payne's conjecture does not hold, as the nodal set is a closed curve contained in the interior of the ball. It is possible to obtain a powerful higher-dimensional analog of this result [25]. For simplicity and in view of Payne's conjecture, we will state it in the case of the n-dimensional ball, although an analogous statement holds true in any compact manifold with boundary. As in the construction of Green's functions with many critical points, the proof is based on the construction of metrics that ensure that the eigenfunctions are small in certain sets.

Theorem 5.1. *Let Σ be a compact orientable hypersurface without boundary contained in the n-dimensional ball $\mathbb{B}^n$, with $n \geqslant 3$. Then there exists a Riemannian metric g on $\overline{\mathbb{B}^n}$ such that the nodal set of its second Dirichlet eigenfunction is Σ.*

Sketch of the proof. The gist of the proof is to choose the metric so that the low-energy eigenvalues are simple and the corresponding eigenfunctions are close, in a suitable sense, to functions whose nodal set is known explicitly.

To begin with, let us consider a small neighborhood $\Omega \subset \mathbb{B}^n$ of the orientable hypersurface Σ, which we can identify with $(-1, 1) \times \Sigma$. Let us take a metric g_0 on $\mathbb{B}^n$ whose restriction to Ω is

$$g_0|_\Omega = dx^2 + g_\Sigma \,,$$

where x is the natural coordinate in $(-1, 1)$ and g_Σ is a Riemannian metric on Σ. We can assume that the first nontrivial eigenvalue of the Laplacian on Σ defined by g_Σ is larger than $\pi^2/4$.

It is then clear that the first two Neumann eigenfunctions v_1, v_2 of the domain Ω can be written as

$$v_1 := \frac{1}{\sqrt{2|\Sigma|}} \,, \quad v_2 := \frac{1}{\sqrt{|\Sigma|}} \cos \frac{\pi(x+1)}{2} \,,$$

where $|\Sigma|$ stands for the area of Σ. Observe that, with this normalization,

$$\int_\Omega v_j v_k = \delta_{jk} \,.$$

Denoting by Δ_0 the Laplacian corresponding to the metric g_0, these eigenfunctions satisfy the equation

$$\Delta_0 v_k = -\mu_k v_k \text{ in } \Omega \,, \quad \partial_\nu v_k|_{\partial\Omega} = 0$$

with $\mu_1 := 0$, $\mu_2 := \pi^2/4$.

The key idea now is that for each $\epsilon > 0$ we can define a piecewise smooth metric g_ϵ on $\mathbb{B}^n$ by setting

$$g_\epsilon := \begin{cases} g_0 & \text{in } \overline{\Omega} \,, \\ \epsilon g_0 & \text{in } \mathbb{B}^n \backslash \overline{\Omega} \,. \end{cases}$$

For our present purposes it is convenient to work with this metric, which we will eventually approximate by a smooth one. To define the spectrum of this discontinuous metric one resorts to the quadratic form

$$\begin{aligned} Q_\epsilon(\varphi) &:= \int_{\mathbb{B}^n} |d\varphi|_\epsilon^2 \, dV_\epsilon \\ &= \int_\Omega |d\varphi|^2 + \epsilon^{\frac{n}{2}-1} \int_{\Omega^c} |d\varphi|^2 \end{aligned}$$

together with the natural L^2 norm corresponding to the metric g_ϵ:

$$\begin{aligned}\|\varphi\|_\epsilon^2 &:= \int_{\mathbb{B}^n} \varphi^2 \, dV_\epsilon \\ &= \int_\Omega \varphi^2 + \epsilon^{\frac{n}{2}} \int_{\Omega^c} \varphi^2\end{aligned}$$

Here the possibly disconnected set $\Omega^c := \mathbb{B}^n \backslash \overline{\Omega}$ stands for the interior of the complement of Ω, the subscripts ϵ refer to quantities computed with respect to the metric g_ϵ and we are omitting the subscripts (and indeed the measure in the integrals) when the quantities correspond to the reference metric g_0.

As is well known, the domain of the quadratic form Q_ϵ can be taken to be the Sobolev space $H_0^1(\mathbb{B}^n)$ of functions with square-integrable derivatives and zero trace to the boundary. By the min-max principle, the k^{th} Dirichlet eigenvalue $\lambda_{k,\epsilon}$ of this quadratic form is

$$\lambda_{k,\epsilon} = \inf_{W \in \mathcal{W}_k} \max_{\varphi \in W \backslash \{0\}} q_\epsilon(\varphi) \,, \tag{5.1}$$

where $\mathcal{W}_k$ stands for the set of k-dimensional linear subspaces of $H_0^1(\mathbb{B}^n)$ and

$$q_\epsilon(\varphi) := \frac{Q_\epsilon(\varphi)}{\|\varphi\|_\epsilon^2} \tag{5.2}$$

is the Rayleigh quotient. The k^{th} Dirichlet eigenfunction $u_{k,\epsilon}$ is then a minimizer of the above variational problem for $\lambda_{k,\epsilon}$, in the sense that any subspace that minimizes the variational problem can be written as $\text{span}\{u_{1,\epsilon}, \dots, u_{k,\epsilon}\}$.

An important property of the eigenvalues $\lambda_{k,\epsilon}$ is that they are almost upper bounded by the corresponding Neumann eigenvalues. More precisely, for $k = 1, 2$ (and actually for any k, although not uniformly) one has

$$\limsup_{\epsilon \to 0^+} \lambda_{k,\epsilon} \leqslant \mu_k \,. \tag{5.3}$$

The way that this relation is proved is by considering the harmonic extension $\psi_k \in H_0^1(\mathbb{B}^n)$ of the Neumann eigenfunction v_k, which given by

$$\psi_k := \begin{cases} v_k & \text{in } \overline{\Omega} \,, \\ \hat{v}_k & \text{in } \Omega^c \,, \end{cases}$$

with $\hat{v}_k$ defined as the solution to the boundary value problem

$$\Delta_0 \hat{v}_k = 0 \;\text{ in } \Omega^c \,, \quad \hat{v}_k|_{\partial\Omega} = v_k \,, \quad \hat{v}_k|_{\partial\mathbb{B}^n} = 0 \,.$$

Indeed, one can plug these functions in the Rayleigh quotient to find, after some work,

$$\lambda_{k,\epsilon} \leqslant \max_{\varphi \in \text{span}\{\psi_1, \dots, \psi_k\}} q_\epsilon(\varphi) = [1 + O(\epsilon^{\frac{n}{2}})] \, \mu_k + O(\epsilon^{\frac{n}{2} - 1}) \,,$$

which proves (5.3).

One can also prove a similar upper bound

$$\mu_k \leqslant \liminf_{\epsilon\to 0^+} \lambda_{k,\epsilon}\,,$$

thereby proving that the eigenvalues $\lambda_{k,\epsilon}$ (with $k = 1, 2$) converge to the eigenvalues μ_k of the Neumann problem. Since this convergence is proved essentially by testing the Rayleigh quotient on the space generated by the Neumann eigenfunctions v_k, it stands to reason that the eigenfunction $u_{k,\epsilon}$ should also converge to the Neumann eigenfunctions v_k, up to a multiplicative constant, in some suitable sense. This intuition is indeed true and, not surprisingly, the convergence turns out to happen in the H^1 norm associated with the quadratic form. Therefore there is some constant β_k such that

$$\lim_{\epsilon\to 0^+} \|u_{k,\epsilon} - \beta_k v_k\|_{H^1(\Omega)} = 0\,. \tag{5.4}$$

For simplicity of notation, one can redefine the normalization constants of the eigenfunctions $u_{k,\epsilon}$ if necessary to take $\beta_k = 1$ for $k = 1, 2$.

Let us take a sequence of smooth metrics g_j that converge pointwise to g_ϵ. By choosing ϵ small enough and setting $\tilde{g} := g_j$ for a large enough j, it is not hard to prove that the second Dirichlet eigenfunction $\tilde{u}_2$ of the metric $\tilde{g}$ satisfies

$$\lim_{\epsilon\to 0^+} \|\tilde{u}_2 - v_2\|_{H^1(\Omega)} < \delta\,,$$

where δ is an arbitrarily small number. Notice that, if K is any compact subset of Ω that contains the hypersurface Σ, standard elliptic estimates allow us to promote the H^1 convergence to a $C^1(K)$ convergence:

$$\lim_{\epsilon\to 0^+} \|\tilde{u}_2 - v_2\|_{C^1(K)} < C\delta\,.$$

Since the nodal set of the Neumann eigenfunction v_2 is obviously given by Σ and ∇v_2 does not vanish on this hypersurface, Thom's isotopy theorem [1, Section 20.2] ensures that the nodal set $\tilde{u}_2^{-1}(0)$ has a connected component given by $\Phi(\Sigma)$ with Φ a diffeomorphism, provided that δ is small enough. Moreover, by combining Courant's nodal domain theorem with a a result of Uhlenbeck [44] about the non-genericity of critical nodal sets one can ensure that $\tilde{u}_2^{-1}(0)$ does not have any other components. The theorem follows by taking the pulled back metric $g := \Phi^*(\tilde{g})$. □

The strategy of the proof of Theorem 5.1 is quite versatile and can be used to derive a number of related results, for manifolds with or without boundary. In fact, a nontrivial modification of this strategy can be used to prove the following result [27], which we state in the case of the first (nontrivial) eigenfunction of a closed Riemannian manifold and is valid even for metrics within a conformal class and with fixed volume. It should be noticed that these requirements are known to severely restrict the flexibility that one enjoys in prescribing the low-energy eigenvalues of the metric; in particular, it has been proved [17, 30, 39] that the supremum of the first nontrivial eigenvalue λ_1 corresponding to g is finite as g ranges over the set of metrics conformal to g_0 and with the same volume.

Concerning the statement of the theorem, we recall that a hypersurface Σ of a manifold M is *separating* if its complement $M\backslash\Sigma$ is the union of two disjoint open sets.

Theorem 5.2. *Let M be a compact n-dimensional manifold without boundary ($n \geqslant 3$) endowed with a Riemannian metric g_0 and let Σ be a separating hypersurface. Then there is a metric g in M conformally equivalent to g_0 and with the same volume such that its first nontrivial eigenvalue λ_1 is simple and the nodal set of the corresponding eigenfunction u_1 is diffeomorphic to Σ. Furthermore, the diffeomorphism can be taken arbitrarily close to the identity in the C^0 norm.*

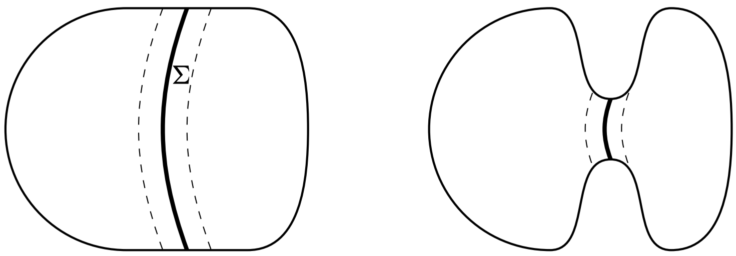

Figure 3. A separating hypersurface Σ and the η-neighbourhood Ω_η. Making the metric small in Ω_η turns M into a dumbbell.

Sketch of the proof. The proof is based on the explicit construction of a conformal factor which is of order ϵ in a neighborhood Ω_η of Σ of width η. Geometrically, this ensures that the manifold, endowed with the rescaled metric, has the structure of a dumbbell, as depicted in Figure 1. The basic idea behind the proof of the theorem is to exploit this dumbbell structure through a fine analysis of the first eigenfunction (see Figure 3). More precisely, we show that as ϵ tends to zero the first eigenfunction approximates a harmonic function in Ω_η with constant boundary values. In turn, the zero set of this harmonic function can then be controlled provided that η is small. □

6. Topological complexity in high-energy eigenfunctions

In this section we will continue to explore the emergence of complicated geometries in spectral theory. In contrast to the previous section, which considered the first eigenfunctions of the Laplacian, here we will be interested in high-energy eigenfunctions. As we explained in the previous section, high energy eigenfunctions exhibit a much more rigid behavior than low energy ones; in particular, the asymptotic behavior of the eigenvalues is controlled by Weyl's law and, at least if the metric is analytic [16], the nodal set of an eigenfunction u_λ with eigenvalue λ has an area of order $\sqrt{\lambda}$, in the sense that

$$C_1\sqrt{\lambda} < |u_\lambda^{-1}(0)| < C_2\sqrt{\lambda}\,.$$

A well known conjecture of Yau asserts that this bound should also hold for smooth metrics.

To motivate our analysis of high energy eigenfunctions, we will discuss a conjecture of Berry [4] concerning the nodal set of eigenfunctions of Schrödinger operators in $\mathbb{R}^3$.

In his paper, Berry constructs eigenfunctions of the hydrogen atom in $\mathbb{R}^3$ whose nodal set contains a certain torus knot as a connected component. He then raises the question as to whether there exist eigenfunctions of a quantum system whose nodal set has components with higher order linking, as in the case of the Borromean rings, see Fig. 4. It is worth mentioning [4, 36] that a physical motivation to study the nodal set of a quantum system is that it is the locus of destructive interference of the wave function. It is related to the existence of singularities (often called dislocations) of the phase $\mathrm{Im}(\log \psi)$ and of vortices in the current field $\mathrm{Im}(\overline{\psi}\, \nabla \psi)$. The existence of knotted structures of this type in physical models, especially in optics and in fluid mechanics, has recently attracted considerable attention, both from the theoretical [21, 24] and experimental [12, 33] viewpoints.

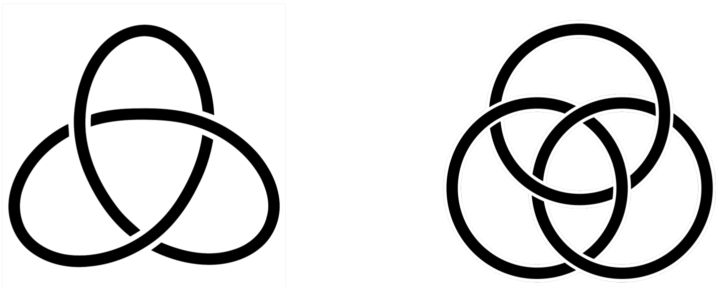

Figure 4. Berry's conjecture involves showing that there are high-energy eigenfunctions ψ of the harmonic oscillator realizing an arbitrary link (e.g. the trefoil knot or the Borromean rings depicted above) in their nodal set $\psi^{-1}(0)$.

One can indeed solve these problems of Berry by showing [18] that any finite link (i.e., any finite collection of non-intersecting closed curves in space) can be realized as a collection of connected components of the nodal set of a high-energy eigenfunction of the harmonic oscillator. We recall that the eigenfunctions of the harmonic oscillator are the H^1 functions ψ satisfying the equation

$$-\Delta\psi + |x|^2\psi = \lambda\psi \tag{6.1}$$

in $\mathbb{R}^3$. It is well-known that the eigenvalues are of the form

$$\lambda = 2N + 3\,,$$

with N a nonnegative integer, and that the degeneracy of the corresponding eigenspace is $\frac{1}{2}(N+1)(N+2)$.

The result that ensures that there are high-energy eigenfunctions of the harmonic oscillator whose nodal set contains a knot of any fixed topology can then be stated as follows:

Theorem 6.1. *Let L be any finite link in $\mathbb{R}^3$. Then, for any large enough N there is a complex-valued eigenfunction ψ of the harmonic oscillator with eigenvalue $\lambda = 2N + 3$ whose nodal set $\psi^{-1}(0)$ has a subset of connected components diffeomorphic to L.*

Remark 6.2. It follows from the proof that the linked components of the nodal set that we construct are actually contained in a small ball whose radius is of order $\lambda^{-1/2}$.

Sketch of the proof. The key idea is that, as we implicitly saw in Section 3, one can prove that there are complex-valued solutions to the Helmholtz equation

$$\Delta\varphi + \varphi = 0$$

in $\mathbb{R}^3$ such that the link L is a union of connected components of the nodal set $\varphi^{-1}(0)$, up to a diffeomorphism. This is pertinent to the study of the eigenvalues of the harmonic oscillator because, in balls of radius $\lambda^{-1/2}$, the high-energy asymptotics of the eigenfunctions are determined by the Helmholtz equation. Heuristically, one can understand why this is true by introducing the rescaled variable $\tilde{x} := \lambda^{1/2} x$, in terms of which Eq. (6.1) is read as

$$\Delta_{\tilde{x}}\psi + \psi = \frac{|\tilde{x}|^2\,\psi}{\lambda^2}\,.$$

The way to make this precise is by computing the high-order asymptotics of the Laguerre polynomials, which govern the radial part of the eigenfunctions of the harmonic oscillator. Going over the fine details we will see that the accidental degeneracy of the eigenvalues of the harmonic oscillator is an essential ingredient of the proof too, essentially because it ensures the existence of families of isoenergetic eigenfunctions with a rich behavior in the angular variables.

Since the proof is not overly technical, we shall next give an indication of how the details can be carried out. Let us begin by fixing an orthogonal basis of eigenfunctions associated with the harmonic oscillator Hamiltonian, which we will write in spherical coordinates as

$$\psi_{klm} := e^{-\frac{r^2}{2}}\, r^l\, L_k^{l+\frac{1}{2}}(r^2)\, Y_{lm}(\theta,\phi)\,. \tag{6.2}$$

Here we are using the standard notation for the Laguerre polynomials and the spherical harmonics, the indices of the eigenfunctions range over the set

$$k \geqslant 0\,, \quad l \geqslant 0\,, \quad -l \leqslant m \leqslant l$$

and the eigenvalue corresponding to ψ_{klm} is

$$\lambda_{kl} := 4k + 2l + 3\,.$$

Notice that the eigenvalue is independent of m.

It is classical that the behavior of the eigenfunction ψ_{klm} for large values of k is

$$\psi_{klm}(x) = A_{kl}\left[j_l(\sqrt{\lambda_{kl}}\, r) + O(\tfrac{1}{k})\right] Y_{lm}(\theta,\phi)\,, \tag{6.3}$$

where j_l is the spherical Bessel function and A_{kl} are constants. A useful fact about this expansion is that it only involves Bessel functions, which are solutions to the Helmholtz equation. In fact, one can refine Lemma 3.2, which we used to study the Allen–Cahn equation, to construct global solutions to the Helmholtz equation in $\mathbb{R}^3$ whose zero set has a prescribed set of connected components and which are given by a finite series of spherical Bessel functions and spherical harmonics. An important additional technical property is that one can choose this global solution to be even:

Lemma 6.3. *There are finitely many complex numbers c_{lm} such that the complex-valued function*

$$\varphi := \sum_{l=0}^{l_0} \sum_{m=-l}^{l} c_{lm}\, j_l(r)\, Y_{lm}(\theta, \phi)$$

has the following properties:

(i) *The zero set $\varphi^{-1}(0)$ has a union of connected components diffeomorphic to L which is structurally stable under C^1-small perturbations of φ.*

(ii) *The function φ is even, so $c_{lm} = 0$ for all odd l.*

To exploit this fact, let us take a large integer $\widehat{k}$ that will be fixed later. For each even integer l smaller than $2\widehat{k}$ we set

$$\widehat{k}_l := \widehat{k} - \frac{l}{2}, \tag{6.4}$$

so that the eigenvalue

$$\lambda := \lambda_{\widehat{k}_l l} = 4\widehat{k} + 3 \tag{6.5}$$

does not depend on the choice of l. The desired eigenfunction ψ of the harmonic oscillator can then be derived from the function φ constructed in Lemma 6.3 by setting

$$\psi := \sum_{l=0}^{l_0} \sum_{m=-l}^{l} \frac{c_{lm}}{A_{\widehat{k}_l l}}\, \psi_{\widehat{k}_l lm}$$

for a large enough number $\widehat{k}$. Notice that, by construction, ψ is a smooth complex-valued eigenfunction with eigenvalue $\lambda = 4\widehat{k} + 3$, and that we have used that $c_{lm} = 0$ for odd l because the number $\widehat{k}_l$ is an integer only for even l.

Using the asymptotics (6.3) it is not hard to show that for any $\delta > 0$ one can choose $\widehat{k}$ large enough so that

$$\left\| \psi\left(\frac{\cdot}{\sqrt{\lambda}}\right) - \varphi(\cdot) \right\|_{C^1(B)} < \delta\,. \tag{6.6}$$

Lemma 6.3 then ensures that, if δ is small enough, the function $\psi(\cdot/\sqrt{\lambda})$ has a collection of connected components in its nodal set $\{\psi(\cdot/\sqrt{\lambda}) = 0\}$ diffeomorphic to L, so the theorem follows. □

It is worth emphasizing that the key point of the proof is not exactly that the eigenvalue is very high, but rather that it is very degenerate and that the corresponding eigenfunctions have a very rich behavior in the angular variables. Consequently, one can prove a similar result for highly excited states of the hydrogen atom [19], which was the original context that Berry considered. We recall that the eigenfunctions of the hydrogen atom satisfy the equation

$$\left(\Delta + \frac{2}{|x|} + \lambda\right)\psi = 0, \tag{6.7}$$

in $\mathbb{R}^3$.

The eigenvalues are given by

$$\lambda_n := -\frac{1}{n^2},$$

where n is a positive integer, and the corresponding eigenfunctions are given, with $0 \leqslant l \leqslant n-1$ and $-l \leqslant m \leqslant l$, by

$$\psi_{nlm} := e^{-r/n}\, r^l\, L^{2l+1}_{n-l-1}\left(\frac{2r}{n}\right) Y_{lm}(\theta, \phi),$$

where L^α_ν are the Laguerre polynomials.

The result for the hydrogen atom can be stated as follows:

Theorem 6.4. *Let L be any finite link in $\mathbb{R}^3$. Then, for any Coulomb eigenvalue with energy λ_n close enough to zero, there exist a complex-valued eigenfunction ψ of energy λ_n whose nodal set $\psi^{-1}(0)$ has a union of connected components diffeomorphic to L.*

A key difficulty here, which does not appear in the case of the harmonic oscillator, concerns estimates for the Green's function of the Coulomb Hamiltonian with zero energy, $\Delta + 2/|x|$.

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Received 26 November, 2015; revised 29 January, 2016