

Locally conformally symplectic and Kähler geometry

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Abstract. The goal of this note is to give an introduction to locally conformally symplectic and Kähler geometry. In particular, the first two sections aim to provide the reader with enough mathematical background to appreciate these geometric structures. The standard reference for locally conformally Kähler geometry is the book *Locally conformal Kähler geometry* by Sorin Dragomir and Liviu Ornea; many progresses in this area, however, were accomplished after its publication, hence are not covered there. On the other hand, there is no comprehensive reference for locally conformally symplectic geometry and many recent advances lie scattered in the literature. While the tone of this note is rather expository, I propose a (hopefully) exhaustive bibliography, to which the reader is referred for both the precise statements and the techniques used. Section 3 would like to demonstrate how these geometries can be used to give precise mathematical formulations to ideas deeply rooted in classical and modern Physics.

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1. Symplectic and locally conformally symplectic geometry

A *symplectic manifold* is a smooth manifold M^{2n} with a 2-form $\omega \in \Omega^2(M)$ which is *non-degenerate*, i.e. $\omega_p^n \neq 0$ for every $p \in M$, and *closed*, i.e. $d\omega = 0$. The non-degeneracy condition can be rephrased by saying that ω provides an isomorphism of vector bundles ${}^b: TM \rightarrow T^*M$, $X \mapsto X^b = \iota_X \omega$.

The word *symplectic* was coined by Hermann Weyl in 1939: he replaced the old terminology *complex group* with *symplectic group* to indicate the Lie group of matrices

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preserving the bilinear skew-symmetric form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ on \mathbb{R}^{2n} , see [160, p. 165]. The etymology is from the Greek *σμπλεκτικός*, which funnily enough means *complex*.

In the process of getting acquainted with symplectic geometry, something that one experiences quite early is, paraphrasing Mikhaïl Gromov, a *curious mixture of “hard” and “soft”*, see [72] as well as [107, p. 81]. This applies both to the mathematical aspects and to the techniques employed in symplectic geometry. An indication of softness in symplectic geometry is certainly *Darboux theorem*, asserting that, locally, two symplectic manifolds can not be distinguished from one another¹; see [16, Section 8.43] or [107, Theorem 3.15] for a modern proof. Thus, symplectic geometry is somehow a global thing. Of the two conditions ensuring that a 2-form on an even-dimensional manifold is symplectic, however, only one is of global nature, namely closedness. Closedness imposes strong cohomological restrictions on the existence of a symplectic structure on an even-dimensional compact manifold²: for instance, all Betti numbers of even degree must be non-zero. The general problem of determining which compact manifolds admit a symplectic structure is far from being solved, see [134].

The first true mathematical exposition of what a symplectic manifold is appeared in a paper of Hwa-Chung Lee in 1941, see [95]. Lee considers the general setting of an even-dimensional manifold M^{2n} endowed with a non-degenerate 2-form ω . He studies first the *flat* case, in which $d\omega = 0$, that is, what is nowadays known as symplectic. Then, he discusses the problem of two 2-forms ω and ω' which are *conformal* to one another: on an open set $U \subset M$ with local coordinates (x_1, \dots, x_{2n}) , write

$$\omega = \sum_{i < j} \omega_{ij}(x) dx_i \wedge dx_j \quad \text{and} \quad \omega' = \sum_{i < j} \omega'_{ij}(x) dx_i \wedge dx_j;$$

ω and ω' are *locally conformal* if there exists $\varphi \in \mathcal{C}^\infty(U)$, nowhere vanishing, with $\omega'_{ij} = \varphi \omega_{ij}$. Lee then finds necessary and sufficient conditions for a given $\omega \in \Omega^2(M)$ to be (locally) conformal to a flat, i.e. closed, one: for $n \geq 3$ this happens³ if and only if there exists a 1-form ϑ such that $d\omega = \vartheta \wedge \omega$. It is interesting to notice that the mathematical birthplace of *both* symplectic and locally conformally symplectic geometry is the very same paper of Lee.

The development of symplectic geometry after 1941 has been tremendous, kept up first by the French school (Charles Ehresmann, Paulette Libermann, André Lichnerowicz, Georges Reeb) in the 1950's, then by the Russian school, with the central figure of Vladimir Arnol'd, and by the American school (Dusa McDuff, Victor Guillemin, Alan Weinstein); a special place is occupied by Gromov⁴. This is however not the right place to extol the ubiquity of symplectic geometry in modern Mathematics; I refer the reader to the nice surveys [17, 64], and [106].

¹This is very different from the Riemannian case, where curvature provides a local invariant.

²The same is not true for open manifolds: as proved by Gromov [70, 71], any open manifold with a non-degenerate 2-form admits a symplectic structure.

³The case $n = 1$ is trivial: as remarked by Lee, every ω is in this case conformal to a flat one, due to dimension reasons. The case $n = 2$ is only slightly different; see the discussion below.

⁴This list of quoted mathematicians is of course far from being complete.

The fate of locally conformally symplectic geometry, on the contrary, was very different. Except for works of Libermann in 1955 [98] and Jean Lefebvre in 1966 [96] and 1969 [97], the subject remained in hibernation until two seminal papers of Izu Vaisman were published: *On locally conformal almost Kähler manifolds*, in 1976, see [146], and *Locally conformal symplectic manifolds*, in 1984, see [151].

In [146], Vaisman defines a *locally conformally symplectic manifold*⁵ as a manifold M^{2n} , $n \geq 1$, endowed with a non-degenerate 2-form $\omega \in \Omega^2(M)$ such that every point $p \in M$ has an open neighborhood U such that

$$d(e^\sigma \omega|_U) = 0, \tag{1}$$

where $\sigma \in \mathcal{C}^\infty(U)$ is a smooth function. If (1) holds for $U = M$, then (M, ω) is *globally conformally symplectic*; if it holds for σ a constant function, (M, ω) is clearly a *symplectic* manifold. The work of Lee shows that the above definition is equivalent to the following one: a manifold M^{2n} , $n \geq 1$, endowed with a non-degenerate 2-form $\omega \in \Omega^2(M)$, is locally conformally symplectic manifold if there exists a globally defined 1-form $\vartheta \in \Omega^1(M)$ such that

$$d\omega = \vartheta \wedge \omega \quad \text{and} \quad d\vartheta = 0. \tag{2}$$

The 1-form ϑ was baptized the *Lee form* by Vaisman. If $n = 1$ one has $d\omega = 0 = \vartheta \wedge \omega$ for any choice of ϑ . For $n \geq 2$, ϑ is completely determined by ω ; moreover, as remarked by Libermann in [98], the second condition in (2) follows from the first one if $n \geq 3$. (ω, ϑ) is called a locally conformally symplectic structure on M . According to this alternative definition, a locally conformally symplectic manifold is globally conformally symplectic if ϑ is exact and symplectic if $\vartheta = 0$.

Given a locally conformally symplectic manifold (M, ω) , the *conformal class* of ω is

$$\{\omega' \in \Omega^2(M) \mid \omega' = e^f \omega \text{ for } f \in \mathcal{C}^\infty(M)\}.$$

If ϑ is the Lee form of (M, ω) and $\omega' = e^f \omega$, then the Lee form of (M, ω') is $\vartheta' = \vartheta + df$, hence the cohomology class of ϑ in $H_{dR}^1(M)$ is an invariant of the conformal class.

Formula (1) implies, in particular, that at a local scale a symplectic manifold can not be distinguished from a locally conformally symplectic manifold. Thus not only all symplectic manifolds locally look alike, in view of Darboux theorem, but potentially there may exist manifolds which locally look like symplectic manifolds and however fail to do so globally! Locally conformally symplectic structures exist on open manifolds, as proved by Rui Loja Fernandes and Pedro Frejlich using an h -principle; see [53], in particular the Acknowledgements. It was proved very recently by Yakov Eliashberg and Emmy Murphy using again h -principle that a *closed* almost complex manifold (M, J) with a non zero cohomology class $\mu \in H_{dR}^1(M)$ admits a locally conformally symplectic structure; see [51, Theorem 1.8] for the precise statement. In [6, 27] explicit examples of compact locally conformally symplectic manifolds which do not admit any symplectic structure are provided.

⁵Vaisman uses locally *conformal* symplectic, while I stick with the terminology locally *conformally* symplectic in this note. Some recent papers use *conformal symplectic*, see [41] and [51].

For this reason, I prefer to consider locally conformally symplectic manifolds as something different from symplectic manifolds. Concretely, this means that our locally conformally symplectic structures will always be assumed to have a Lee form ϑ which is not exact.

In his 1976 paper Vaisman proves a few results about locally conformally symplectic manifolds but turns quickly his attention to the metric case, in the wake of Alfred Gray's work on almost Hermitian structures. It is in his 1984 article that he extensively studies the non-metric case. Motivated by the metric case, which I will discuss in Section 2, Vaisman distinguishes between locally conformally symplectic structures *of the first kind* and *of the second kind*. A locally conformally symplectic structure (ω, ϑ) on M is of the first kind if there exists a vector field $U \in \mathfrak{X}(M)$ such that

$$\mathcal{L}_U \omega = 0 \quad \text{and} \quad \vartheta(U) = 1.$$

Otherwise, it is of the second kind. One can show that the above conditions characterize U uniquely; it is the *Lee field* of the locally conformally symplectic structure. A sophisticated way to rephrase this goes as follows: define

$$\mathfrak{X}(M, \omega) = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0\}; \quad (3)$$

then $\mathfrak{X}(M, \omega) \subset \mathfrak{X}(M)$ is a subalgebra. If $X \in \mathfrak{X}(M, \omega)$ then $\mathcal{L}_X \vartheta = 0$, hence $\vartheta(X)$ is a constant function on M . The *Lee homomorphism* is $\ell: \mathfrak{X}(M, \omega) \rightarrow \mathbb{R}$, $\ell(X) = \vartheta(X)$ and is a morphism of Lie algebras. Thus (ω, ϑ) is of the first kind if and only if the Lee homomorphism is non zero, hence surjective; of the second kind otherwise. In particular, the Lee form of a locally conformally symplectic structure of the first kind is nowhere zero. I should remark here that in the conformal class of a locally conformally symplectic structure of the first kind there exist always locally conformally symplectic structures of the second kind. To see this, it is enough to choose a function f such that $df_p = -\vartheta_p$ for some $p \in M$; then the Lee form of $e^f \omega$ has a zero at p . In particular, being of the first kind is not a conformal notion. Notice that (3) defines an automorphism of a given element in the conformal class of a locally conformally symplectic structure. If one wants to deal with the whole conformal class, then the object to be considered is the subalgebra

$$\widehat{\mathfrak{X}}(M, \omega) = \{X \in \mathfrak{X}(M) \mid \exists f_X \in \mathcal{C}^\infty(M) \mid \mathcal{L}_X \omega = f_X \omega\};$$

here f_X should be nowhere 0. In this case as well one sees that the *extended Lee homomorphism* $\widehat{\ell}: \widehat{\mathfrak{X}}(M, \omega) \rightarrow \mathbb{R}$, $\widehat{\ell}(X) = \vartheta(X) + f_X$ is a morphism of Lie algebras (see [21]). The Lee homomorphism and its extended version have been investigated broadly, see for instance [21, 77], and [151].

Another way to tell locally conformally symplectic structures apart is according to the Morse–Novikov class of the 2-form ω . Given a 1-form ϑ on a manifold M , one can define a differential operator $d_\vartheta: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ by setting $d_\vartheta \sigma = d\sigma - \vartheta \wedge \sigma$. If ϑ is closed, then $d_\vartheta^2 = 0$ and the *Morse–Novikov*⁶ cohomology of $(\Omega^*(M), d_\vartheta)$ is

$$H_\vartheta^k(M) = \frac{\ker\{d_\vartheta: \Omega^k(M) \rightarrow \Omega^{k+1}(M)\}}{d_\vartheta(\Omega^{k-1}(M))}.$$

⁶The Morse–Novikov cohomology has more than two fathers. In the context of locally conformally symplectic geometry, for instance, it was first considered by Guédira and Lichnerowicz in [74]. It was also considered by Witten in his celebrated paper [161].

If M is compact, these cohomology spaces are always finite-dimensional, and $H_{\vartheta}^*(M) \cong H_{dR}^*(M)$ if ϑ is exact. Further, as noticed in [19], the Euler–Poincaré characteristic of the Morse–Novikov cohomology of a compact, orientable manifold equals that of the de Rham cohomology, hence it is topological. In general, however, Morse–Novikov cohomology behaves very differently from de Rham cohomology: indeed, if ϑ is not exact and M is connected then $H_{\vartheta}^0(M) = 0$, see [74]; if, in addition, M^n is compact and orientable, then a Poincaré duality holds, that is, $H_{\vartheta}^i(M) \cong H_{\vartheta}^{n-i}(M)^*$, hence $H_{\vartheta}^n(M) = 0$, see [76]. In [149] Vaisman proved that $H_{\vartheta}^*(M)$ is isomorphic to the cohomology of M with coefficients in the sheaf of smooth functions $f \in \mathcal{C}^\infty(M)$ which satisfy $d_{\vartheta}f = 0$. It was proved in [46] that if M carries a Riemannian metric for which ϑ is parallel, then $H_{\vartheta}^*(M) = 0$. Aside from these general results, the computation of Morse–Novikov cohomology is in general very difficult. For a nilmanifold or a completely solvable solvmanifold⁷ the computation of the Morse–Novikov cohomology can be performed algebraically; see [5, 8, 108], and [110]. For more details on the Morse–Novikov cohomology, I refer the reader to [24, 46, 52, 76], and [77].

The significance of Morse–Novikov cohomology in the context of locally conformally symplectic geometry stems from (2): if (M, ω, ϑ) is a locally conformally symplectic manifold then $d\vartheta = 0$ and $d_{\vartheta}\omega = d\omega - \vartheta \wedge \omega = 0$, hence the 2-form ω defines a cohomology class $[\omega]_{\vartheta} \in H_{\vartheta}^2(M)$. The locally conformally symplectic structure is *exact* if $[\omega]_{\vartheta} = 0$, *non exact* otherwise. It is easy to see that a locally conformally symplectic structure of the first kind is exact: by defining $\eta = -\iota_U\omega$, where U is the Lee field, one has $\omega = d\eta - \vartheta \wedge \eta$. The converse does not hold: in fact, being exact is an invariant of the conformal class of a locally conformally symplectic structure, while being of the first kind is not, as I remarked above. The locally conformally symplectic structures constructed by Eliashberg and Murphy in [51] are exact. The importance of Morse–Novikov cohomology in the context of locally conformally symplectic geometry is highlighted, for instance, by the recent research papers [8, 93], and [126].

Locally conformally symplectic structures of the first kind are strictly related to contact structures. A (*co-orientable*) *contact structure* on an odd-dimensional manifold P^{2n+1} ($n \geq 1$) consists of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$ at every point. It follows readily that the distribution $\xi = \text{Ker}\alpha$ is maximally non-integrable. Let (P, α) be a contact manifold and consider a *strict contactomorphism*, that is, a diffeomorphism $\varphi: P \rightarrow P$ satisfying $\varphi^*\alpha = \alpha$. Then, as observed for instance by Augustin Banyaga in [23], the *mapping torus*⁸ P_{φ} admits a locally conformally symplectic structure of the first kind. In the same paper, Banyaga proves a sort of converse to this result: namely, if a compact manifold M is endowed with a locally conformally symplectic structure of the first kind, then there exist a compact contact manifold (P, α) and a diffeomorphism $\varphi: P \rightarrow P$ such that M is diffeomorphic to the mapping torus P_{φ} . Banyaga’s result, however, does

⁷A *nilmanifold* is the quotient of a connected, simply connected nilpotent Lie group by a lattice. More generally, a *solvmanifold* is a compact quotient of a connected, simply connected solvable Lie group by a lattice. A solvmanifold is *completely solvable* if the adjoint representation on the corresponding Lie algebra has only real eigenvalues.

⁸Given a topological space X and a homeomorphism $\varphi: X \rightarrow X$, the mapping torus or suspension X_{φ} is the quotient space of $X \times \mathbb{R}$ by the \mathbb{Z} -action generated by $1 \cdot (x, t) = (\varphi(x), t + 1)$. The projection $\pi: X_{\varphi} \rightarrow S^1$, $[(x, t)] \mapsto [t]$ is a fiber bundle with fiber X . If M is a smooth manifold and φ is a diffeomorphism, then M_{φ} is a smooth manifold and $M \rightarrow M_{\varphi} \rightarrow S^1$ is a smooth fiber bundle.

not claim that the original locally conformally symplectic structure on M is related to the mapping torus construction. A similar result, in which the given locally conformally symplectic structure is preserved, is proved in [28].

Locally conformally symplectic structures of the second kind are much less understood. Concerning, in particular, non exact structures, Banyaga [24] proved that there exist two families of locally conformally symplectic structures on the 4-dimensional solvmanifold constructed in [45] and that they are non exact. These are the first acknowledged examples of this type of locally conformally symplectic structures. In [104, Appendix A] it was shown that the locally conformally symplectic structures of the Oeljeklaus–Toma manifolds constructed in [114] are not exact. In [6] the properties of non exact locally conformally symplectic structures extensively are investigated, producing many new examples.

It is interesting to notice that contact and locally conformally symplectic structures come together also in the context of Jacobi structures. According to [74], indeed, a *transitive* Jacobi manifold is a contact manifold if the dimension is odd and a locally conformally symplectic manifold if it is even.

I conclude this section with a collection of results in locally conformally symplectic geometry.

The problem of *reduction* in locally conformally symplectic geometry was tackled in [77, 78, 104], and [113]. Related to this, the study of group actions on locally conformally symplectic manifolds was addressed in [78] and, recently, a convexity result for the image of the momentum mapping of twisted Hamiltonian torus actions was obtained in [32]. Such actions, and their connection with the existence of locally conformally symplectic structures on the total space of fiber bundles with locally conformally symplectic fiber, have also been considered in [125]. In [104] the authors use reduction to show that Hopf manifolds (see Section 2) are universal models for compact, exact locally conformally symplectic manifolds; this is analogous to Tischler’s result on universal models for symplectic manifolds, see [141]. In [19] a *Moser trick* for locally conformally symplectic forms is proved. The blow-up of a locally conformally symplectic manifold at a point or along a compact symplectic submanifold, i.e. a submanifold such that the locally conformally symplectic form restricts to a *closed* form, was constructed in [42] and [163]. A *Lagrangian submanifold* of a locally conformally symplectic manifold $(M^{2n}, \omega, \vartheta)$ is a submanifold $\iota_L: L^n \rightarrow M$ such that $\iota_L^* \omega = 0$; this notion is of conformal nature. A result on neighbourhoods of Lagrangian submanifolds in locally conformally symplectic manifolds was obtained in [127], analogous to the known result of Weinstein in the symplectic case, [158]. The problem of displacing a Lagrangian submanifold in a locally conformally symplectic manifold is tackled in the paper [41], which also contains some interesting observations on the issues that appear when one tries to apply Floer’s machinery or results such as Gromov compactness to the locally conformally symplectic situation. Such issues depend, essentially, on the fact that ω is not closed, hence no bound *à la Gromov* on the energy of a J -holomorphic map is possible. The paper [136] suggests some ideas on how to control the failure of Gromov compactness. The properties of the group of diffeomorphisms preserving the conformal class of a locally conformally symplectic structure are studied in [77]; see also [21, 22], and [97]. Finally, for a description of locally conformally symplectic structures in the language of Lie algebroids as well as some generalizations I refer the reader to the papers [84] and [85].

2. Kähler and locally conformally Kähler geometry

A Kähler manifold is a complex manifold which admits a compatible Riemannian metric such that the complex structure is parallel with respect to the Levi-Civita connection. The Riemannian metric and the complex structure provide a non-degenerate 2-form which is also parallel, in particular closed. Thus Kähler geometry lies at the intersection between *complex*, *Riemannian* and *symplectic* geometry. The combination of three geometries produces a class of manifolds which possess distinctive properties within each of the three geometries.

As complex manifolds, Kähler manifolds can, to a certain extent, be studied with methods of complex algebraic geometry; indeed, the main source of examples of compact Kähler manifolds is provided by projective varieties, i.e. zero loci of homogeneous polynomials in $\mathbb{C}P^N$. Informally, the extent to which the class of compact Kähler is larger than the class of projective varieties is the content of the *Kodaira problem*:

Can every compact Kähler manifold be deformed to a projective manifold?

The answer to this question is, perhaps surprisingly, no. This was proved by Claire Voisin in [155] and [156]; see also the survey [80]. A certain class of compact Kähler manifolds, namely *Hodge manifolds*, can be holomorphically embedded into a complex projective space: this is the content of Kodaira's embedding theorem, see [154, Theorem 7.11]. In this case the Kähler class is the pullback of the Fubini–Study class but the embedding is, in general, not isometric.

From the perspective of Riemannian geometry, the reduced holonomy of a compact Kähler manifold is contained in the unitary group $U(n)$, where n is half the dimension of the manifold. Manifolds with special holonomy turn out to have many applications in Physics, see for instance [83].

From the point of view of symplectic geometry, compact Kähler manifolds satisfy the *Hard Lefschetz property*, see [79], while symplectic manifolds need not, see [26]. The Lefschetz property implies the well-known fact that the Betti numbers of odd degree are even on a compact Kähler manifold (this follows also directly from Hodge theory). In a very actual research area such as homological mirror symmetry, the fact that a symplectic structure is part of a Kähler structure on a compact manifold sheds a great deal of light in the study of such duality; see by way of example [138]. I should also mention here that it was originally believed, and to some extent even erroneously proved, see [75], that every compact symplectic manifold admitted a Kähler metric. It was only in 1976 that Thurston provided the first example of a compact symplectic manifold with first Betti number equal to 3, hence no Kähler metric, see [140]. Since then, the quest for compact symplectic manifolds with no Kähler metrics has inspired beautiful Mathematics; see for instance the papers [55, 63, 99, 105] and the book [142].

Finally, concerning the topology of compact Kähler manifolds, I should point out that they are formal in the sense of Sullivan, see [47], and that their fundamental groups are constrained, see [4].

For many purposes it can be convenient to relax the strong integrability properties characterizing the three geometries that come together in a Kähler structure⁹. The right

⁹I am intentionally vague here. I refer the reader to [2] for a more convincing explanation of this claim.

framework to do this is that of *almost Hermitian structures*. An almost Hermitian structure on a manifold consists of a triple (g, J, ω) , where g is a Riemannian metric, J is an almost complex structure and ω is a 2-form, called the Kähler form, such that J is an isometry for g . Actually two of the three structures determine the third one through the equation

$$\omega(X, Y) = g(X, JY).$$

In their celebrated 1980 paper *The sixteen classes of almost Hermitian manifolds and their linear invariants* [69], Gray and Luis Hervella classified almost Hermitian structures in terms of the covariante derivative, with respect to the Levi-Civita connection, of the Kähler form. Kähler structures are recovered as those almost Hermitian structures whose Kähler form is parallel with respect to the Levi-Civita connection. This opens the doors to a whole series of almost Hermitian structures in which some of the integrability properties are not satisfied. Starting with this paper, the study of these structures was undertaken in a systematic way. Notice that some of them had already appeared in the literature. For instance, nearly Kähler structures were considered by Fukami and Ishihara in 1955 ([57], on the six sphere) and then studied extensively by Gray in [66, 67], and [68]. Locally conformally almost Kähler structures were discussed in Vaisman's 1976 paper [146]. To an almost Hermitian structure (g, J, ω) on M^{2n} with $n \geq 2$ one can associate the *Lee form* $\vartheta \in \Omega^1(M)$, defined as

$$\vartheta = -\frac{1}{n-1}J(d^*\omega). \quad (4)$$

An almost Hermitian structure (g, J, ω) is *locally conformally almost Kähler* if $d\omega = \vartheta \wedge \omega$ and $d\vartheta = 0$. If J is integrable, then the locally conformally almost Kähler is *locally conformally Kähler*. Thus a locally conformally Kähler structure is a Hermitian structure such that $d\omega = \vartheta \wedge \omega$ and $d\vartheta = 0$, with ϑ the Lee form.

The cases $n = 2$ and $n \geq 3$ are slightly different, as already anticipated in Section 1. In fact, if $n = 2$ there always exists a 1-form ϑ such that $d\omega = \vartheta \wedge \omega$, since the map $\Omega^1(M) \rightarrow \Omega^3(M)$, $\alpha \mapsto \alpha \wedge \omega$ is an isomorphism; ϑ , however, need not being closed; moreover, one easily checks that $d\omega = \vartheta \wedge \omega$ is equivalent to (4) for $n = 2$. If $n \geq 3$, on the other hand, the equation $d\omega = \vartheta \wedge \omega$, with ϑ defined by (4), need not hold on a generic Hermitian manifold; if it holds, however, ϑ is automatically closed, because the map $\Omega^2(M) \rightarrow \Omega^4(M)$, $\beta \mapsto \beta \wedge \omega$ is injective for $n \geq 3$. This discrepancy between the cases $n = 2$ and $n \geq 3$ also reflects on the fact that, in complex dimension 2, there exist only two “pure” classes in the Gray–Hervella classification. We shall also see that locally conformally Kähler in complex dimension 2 are strictly related to various geometric structures on 4-dimensional real manifolds.

Similarly to what happened for Kähler manifolds, locally conformally Kähler manifolds can be considered simultaneously as complex, Riemannian and locally conformally symplectic manifolds. As I mention above, a Kähler manifold M^{2n} can be characterized as a Riemannian manifold whose holonomy lies in $U(n)$. One could think of a conformal version of manifolds with special holonomy. For instance, locally conformally hyperkähler manifolds are studied in [50, Chapter 11]. References for locally conformally G_2 and $Spin(7)$ structures are [30, 43, 54], and [82].

As it happens in the locally conformally symplectic case, a locally conformally Kähler manifold is actually Kähler, in case $\vartheta = 0$, or *globally* conformal to a Kähler manifold, if ϑ is exact. In general, one can only argue that this conformal property holds locally. I prefer to consider locally conformally Kähler manifolds as a class which is distinct from that of Kähler manifolds; sometimes one uses the terminology *strictly* locally conformally Kähler to indicate locally conformally Kähler metrics which are not globally conformally Kähler. The reference for locally conformally Kähler geometry is the monograph [50] by Dragomir and Ornea; see also [115] and [121].

Of particular importance within locally conformally Kähler manifolds are *Vaisman manifolds*¹⁰; these are characterized by the property that the Lee form is parallel with respect to the Levi-Civita connection. I will implicitly assume that $\|\vartheta\| \neq 0$ on a compact Vaisman manifold, hence ϑ is nowhere zero and a Vaisman metric is never globally conformally Kähler on a compact manifold. Similarly to the Kähler case, the topology of a compact Vaisman manifold is constrained: of course, its Euler characteristic vanishes; moreover, its first Betti number is odd and there is a Hodge-type decomposition of the de Rham cohomology, see [87, 144], and [150].

An interesting example of a locally conformally Kähler manifold in complex dimension 2 is the Hopf surface. This is a compact complex surface whose universal cover is $\mathbb{C}^2 \setminus \{0\}$. As shown in [60], each primary¹¹ Hopf surface admits a locally conformally Kähler metric and some Hopf surfaces (those of class 1) admit Vaisman metrics (see also [128]). Since every primary Hopf surface is diffeomorphic to $S^3 \times S^1$, no Hopf surface admits Kähler metrics.

Notice that by a result of Franco Tricerri [143], a complex manifold has a locally conformally Kähler metric if and only if its blow-up at one point does (see also [124] and [157]). Thus, to study locally conformally Kähler metrics on compact complex surfaces it is sufficient to restrict to *minimal* surfaces. This means that one has the full power of the Kodaira-Enriques classification at hand. In [148, Theorem 2.1], Vaisman proved that a locally conformally Kähler manifold (M, J, g) is globally conformally Kähler if and only if (M, J) admits some Kähler metric. It is well known that a compact complex surface admits a Kähler metric if and only if its first Betti number is even. A posteriori, this motivates the following question of Vaisman [152, Remark 1]:

Does every compact complex surface with odd first Betti number admit a locally conformally Kähler metric?

Locally conformally Kähler, albeit non Vaisman, metrics on some Inoue surfaces have been constructed by Tricerri in [143]. Afterwards, in [31], Florin Belgun carried out a systematical analysis of locally conformally Kähler metrics on compact complex surfaces. His analysis showed that locally conformally Kähler structures are quite different from Kähler structures¹². For instance, a small deformation of the complex structure of a Kähler manifold remains Kähler (see [154, Theorem 9.23]). Belgun proved that this is

¹⁰Vaisman manifolds were first called *generalized Hopf manifolds* by Vaisman, see [150].

¹¹A Hopf surface is called *primary* if its fundamental group is isomorphic to \mathbb{Z} . Every Hopf surface is finitely covered by a primary one.

¹²In the short note [18], Aubin erroneously claimed that a compact locally conformally Kähler manifold is actually Kähler.

not the case for complex structures neither on locally conformally Kähler nor on Vaisman manifolds. Indeed, he described a certain Inoue surface which carries locally conformally Kähler metrics but admits deformations which do not carry any such metrics. The smooth manifold underlying these complex manifolds has first Betti number equal to 1, thus Belgun's example also answers Vaisman's question in the negative. See [56] for a report on Vaisman's question.

Apart from Hopf and Inoue surfaces, it was shown by Marco Brunella that Kato surfaces admit locally conformally Kähler metrics (see [37] and [38]). A Kato surface is a surface of class VII_0 with $b_2 > 0$ and which contains a global spherical shell¹³, see [111, 112], and [123]. The *global spherical shell conjecture* predicts that every class VII_0 surface with $b_2 > 0$ is a Kato surface. If the global spherical shell conjecture holds true, the remaining non-Kähler compact complex surfaces admit locally conformally Kähler metrics (see [49] and [139] for recent advances on this conjecture).

Building on the Kodaira-Enriques classification, the quest for suitable Hermitian metrics on compact complex non-Kähler surfaces has been a very active area of research. Perhaps the first instance of the intersection of this problem with the existence of locally conformally Kähler metrics occurs in a paper by Charles Boyer, see [35]. The author proved that if a compact complex surface M with $b_1(M)$ odd admits a Hermitian metric which is conformally anti-self-dual (that is, the self-dual part of the Weyl tensor vanishes), then the Hermitian metric is locally conformally Kähler; as remarked by Massimiliano Pontecorvo in [130, Proposition 1.5], $b_1(M) = 1$ in this case. In [131], Pontecorvo recovers Boyer's result using twistor methods. In [94], Claude LeBrun addresses the question of which compact complex surfaces admit a Hermitian metric which is Einstein. He proves that if the starting metric is not Kähler, it must be globally conformally Kähler, and that the compact complex surface is obtained from $\mathbb{C}P^2$ by blowing up 1, 2 or 3 points in general position. A *bi-Hermitian* structure on a compact complex manifold (M, J) (see [132] and [135]) consists of a pair $(J_+ = J, J_-)$ of complex structures on M , inducing the same orientation, both orthogonal with respect to a common Riemannian metric g . It is usual to assume that there exists $x \in M$ with $J_-(x) \neq \pm J_+(x)$. The study of bi-Hermitian structures on compact complex surfaces with odd first Betti number, and of their relations with locally conformally Kähler structures, was undertaken in [11, 12] and [15]; see [133] for an overview. Bi-Hermitian structures are also intimately related to generalized Kähler structures, see [73, Theorem 1.16].

A strict locally conformally Kähler manifold is never simply connected. A locally conformally Kähler manifold can be equivalently defined as a manifold admitting a Kähler covering whose deck group acts by conformal transformations (see [150]). As proved by Misha Verbitsky in [153], the Kähler metric on the universal covering of a Vaisman manifold admits a global Kähler potential. Since this property is stable under small deformations, a Vaisman structure deforms to a locally conformally Kähler one, not necessarily a Vaisman one. Motivated by this observations, Ornea and Verbitsky defined a class of locally conformally Kähler manifolds, which strictly contains Vaisman manifolds, namely *locally conformally Kähler manifolds with (proper) potential*, see [119] and [122].

¹³A *global spherical shell* S in a compact complex surface M is a real submanifold diffeomorphic to S^3 , such that $M \setminus S$ is connected and S has a neighbourhood which is biholomorphic to an annulus in \mathbb{C}^2 .

Nice results for such manifolds are available. For instance, it was proved in [119] that they admit an embedding into a Hopf manifold, provided the complex dimension is at least 3 (see also [118]). *Hopf manifolds* are generalizations of Hopf surfaces to arbitrary complex dimensions: they are defined as quotients of $\mathbb{C}^n \setminus \{0\}$ by a discrete subgroup of linear holomorphisms. A *primary* Hopf manifold is the quotient of $\mathbb{C}^n \setminus \{0\}$ by the action of the abelian group generated by complex numbers $\lambda_1, \dots, \lambda_n$, with $0 < |\lambda_1| \leq \dots \leq |\lambda_n| < 1$, where the action sends z_i to $\lambda_i z_i$, for $i = 1, \dots, n$ (see [86]). Compact Vaisman manifolds can be embedded into primary Hopf manifolds. In this sense, Vaisman manifolds and, more generally, locally conformally Kähler manifolds with proper potential are the analogue of Hodge manifolds in Kähler geometry. In locally conformally Kähler geometry, the statement corresponding to the Kodaira problem in Kähler geometry would be the following:

Can every compact locally conformally Kähler manifold be deformed to a Vaisman manifold?

This is certainly not true, in general; indeed, any Vaisman manifold has vanishing Euler characteristic; its blow-up at one point does have a locally conformally Kähler metric, but the Euler characteristic is positive, hence it can not carry any Vaisman metric¹⁴. However, every compact locally conformally Kähler manifold with potential can be deformed to a Vaisman one, as shown in [120, Theorem 2.1].

Recall that a compact locally conformally Kähler manifold is globally conformally Kähler if and only if the underlying complex manifold admits a Kähler metric. Related to the above question, I mention the following two conjectures (see [147] and [148]):

- A compact locally conformally Kähler manifold satisfying the topological conditions of a Kähler manifold admits some global Kähler metric.
- A compact locally but not globally conformally Kähler manifold has an odd-degree Betti number which is odd.

As I remarked above, the first Betti number of a compact Vaisman manifold is odd, hence the second conjecture holds for locally conformally Kähler manifolds with potential. A compact complex surface which admits a locally conformally Kähler but no Kähler metrics has odd first Betti number. In [114] Karl Oeljeklaus and Matei Toma disproved the second conjecture by constructing a compact complex 3-fold admitting locally conformally Kähler metrics with all odd-degree Betti numbers even. The so-called *Oeljeklaus–Toma manifolds* are generalizations to arbitrary complex dimensions of Inoue surfaces. They can also be described as solvmanifolds, see [88]; their de Rham and Morse–Novikov cohomologies have been computed in [81]. Oeljeklaus–Toma manifolds also give a further negative answer to the Kodaira problem in the locally conformally Kähler context.

The above mentioned result of LeBrun [94] implies that on a compact complex surface there exist no strictly locally conformally Kähler metrics which are Einstein. In complex dimension greater than 2, Andrzej Derdziński and Gideon Maschler [48] proved that in, the compact case, the only Kähler metrics which are conformal (but not homothetic) to an Einstein metric are those constructed by Lionel Bérard-Bergery in [33]. In [101], Farid Madani, Andrei Moroianu and Mihaela Pilca consider locally conformally Kähler metrics

¹⁴I would like to thank Liviu Ornea for pointing this fact out to me.

which are Einstein, showing that they are globally conformally Kähler with positive scalar curvature and that the only examples are those of LeBrun ($n = 2$) or Bérard-Bergery ($n \geq 3$). The authors also study the holonomy of proper¹⁵ locally conformally Kähler metrics. They show that if the holonomy is not generic, then either the metric is Vaisman or globally conformally Kähler; in the latter case, the reduced holonomy is $U(n)$ or $SO(2n - 1)$, being n the complex dimension of the manifold, and some classification results are given.

Although the complex structure is not parallel with respect to the Levi-Civita connection, it can be useful to have an auxiliary metric connection which does fulfill this property. To any Hermitian structure (g, J) on a manifold M one can associate a unique connection, called *Chern connection* ∇^C , which satisfies $\nabla^C g = 0 = \nabla^C J$ and whose torsion T is of type $(2, 0)$, that is,

$$T(JX, Y) = JT(X, Y) \quad \forall X, Y \in \mathfrak{X}(M).$$

The Chern connection coincides with the Levi-Civita connection if the Hermitian structure is Kähler. In [58] Gauduchon associated a 1-form $\tilde{\vartheta}$ to the Chern connection, the *torsion 1-form*, as follows:

$$\tilde{\vartheta}(X) = \text{trace}(Y \mapsto T(X, Y)).$$

One can show that $\tilde{\vartheta} = (n - 1)\vartheta$, hence the Lee form and the torsion 1-form are strictly related. Thus the Lee form of a locally conformally Kähler structure measures, in a certain sense, its lack of integrability, where integrability is the Kähler case.

A *Weyl structure* on a conformal manifold (M, c) is a torsion-free linear connection ∇^W , the Weyl connection, which preserves the conformal class c . This means that there exists a 1-form ϑ such that $\nabla^W g = g \otimes \vartheta$ for every $g \in c$. A conformal Hermitian manifold is a conformal manifold (M, c) with a complex structure J which is Hermitian for some, hence all, $g \in c$. If $\nabla^W J = 0$, then (M, c, J) is a *Kähler–Weyl manifold*. As pointed out by Kokarev in [89], locally conformally Kähler manifolds are examples of Kähler–Weyl manifolds; the Weyl connection is related to the Levi-Civita connection ∇ by the formula

$$\nabla_X^W Y = \nabla_X Y - \frac{1}{2}\vartheta(X)Y - \frac{1}{2}\vartheta(Y)X + \frac{1}{2}g(X, Y)U,$$

where $U = \vartheta^\sharp$ is the *Lee field*. This point of view on locally conformally Kähler geometry was adopted in [89] and [90], with applications to the topology of compact Vaisman¹⁶ manifolds, in particular their fundamental group.

Since the Lee form of a Vaisman structure is parallel, the results of [46] imply that the underlying locally conformally symplectic structure is exact. But more is true: if (g, J) is Vaisman then, up to a homothety, one can assume that $\|\vartheta\| = 1$ and one can show that the underlying locally conformally symplectic structure is of the first kind, see [19] and [50]; more precisely, one has $\mathcal{L}_U \omega = 0$ and $\omega = d\eta - \eta \wedge \vartheta$ for $\eta = -\iota_U \omega$ (see also [115, Section 9]).

¹⁵A locally conformally Kähler structure is *proper* if the Lee form does not vanish identically.

¹⁶Kokarev defined in [89] *pluricanonical* locally conformally Kähler metrics (actually Kähler–Weyl structures) as those for which $(\nabla\vartheta)^{1,1} = 0$. In [120] it was erroneously claimed that a locally conformally Kähler metric is pluricanonical if and only if it admits a potential. The mistake was clarified in [109] and [122], where it was proved that a compact pluricanonical locally conformally Kähler manifold is in fact Vaisman.

In Section 1 we discussed the relation between locally conformally symplectic structures of the first kind and contact structures. A similar relation exists between Vaisman and Sasakian structures. A Sasakian structure is a normal contact metric structure, see [34] and [36]. Indeed, the mapping torus of a Sasakian manifold and a Sasakian automorphism, that is, a diffeomorphism which respects the whole Sasakian structure, carries a natural Vaisman structure. In [117] the authors claimed that, in the compact case, the converse also holds; as explained in [122], however, the proof is flawed. Nevertheless, the result holds up to diffeomorphism: a compact Vaisman manifold is diffeomorphic to the mapping torus of a Sasakian manifold and a Sasakian automorphism. Morally, this discrepancy between the two directions is similar to what happens in the non-metric case. Based on this approach, a global splitting result for compact Vaisman manifolds was obtained in [29]. As in the non-metric case, let me notice the absence of structure results for compact locally conformally Kähler manifolds which are not Vaisman.

Analogous to the symplectic versus Kähler case, Ornea and Verbitsky formulated in [121] the following problem:

Construct a compact locally conformally symplectic manifold which admits no locally conformally Kähler metrics.

A first answer to this question was provided by Bande and Kotschick in [20]. Different answers are contained in [27] and [28].

Related to this problem is a conjecture of Ugarte which aims to give a complete characterization of locally conformally Kähler structures on nilmanifolds. In [145, p. 200], he conjectured the following:

A compact nilmanifold of dimension $2n \geq 4$ admitting a locally conformally Kähler structure is the product of N with S^1 , where N is a quotient of $H(1, n)$.

Here $H(1, n)$ is the generalized Heisenberg group,

$$H(1, n) = \left\{ \begin{pmatrix} 1 & y_1 & y_2 & \dots & y_n & z \\ 0 & 1 & 0 & \dots & 0 & x_1 \\ \vdots & 0 & \ddots & \ddots & \vdots & x_2 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & & \ddots & 1 & x_n \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} \mid x_i, y_i, z \in \mathbb{R}, i = 1, \dots, n \right\}.$$

The conjecture holds in full generality in dimension 4 ([27]). In higher dimension it holds if one assumes that the complex structure of the locally conformally Kähler structure is invariant¹⁷ (see [137]) or if the locally conformally Kähler structure is Vaisman (see [25]).

We mention here the fact that compact Vaisman manifolds satisfy a Hard Lefschetz property (see [39]); this result builds on the Hard Lefschetz property for compact Sasakian manifolds proved in [40]. Again, the lack of structure theorems for general locally conformally Kähler manifolds reflects on the absence of a Hard Lefschetz property in the most general setting.

¹⁷This means that it comes from a left-invariant complex structure on the corresponding Lie group.

Compact Vaisman manifolds are, in general, non-formal. In 2001, Kotschick introduced the notion of *geometric formality*: a closed manifold is geometrically formal if it admits a Riemannian metric such that the product of two harmonic forms is harmonic (see [91]). Geometric formality implies formality in the sense of Sullivan, but the converse is not true, see for instance [92]. In [116], Ornea and Pilca showed that geometrically formal compact Vaisman manifolds obey to strong topological restrictions. It is not yet clear the extent to which a compact Vaisman manifold is non formal.

I end this section by quoting some other results about locally conformally Kähler manifolds.

Homogeneous locally conformally Kähler structures are in fact Vaisman, see [3] and [59]. Locally conformally Kähler structures on four-dimensional solvable Lie algebras have been classified in [7]. The papers [61] and [62] consider the problem of reduction in locally conformally Kähler geometry. In [62] the authors introduce the notions of *presentation* and *rank* of a locally conformally Kähler manifold. The rank of a locally conformally Kähler structure and his relation with other properties such as the existence of a potential have been further investigated in [129]. Toric locally conformally Kähler manifolds, and in particular Vaisman, are considered in [102]. In [10], the authors study isometric embeddings of locally conformally Kähler manifolds in some Hopf manifold, with special emphasis on the case of surfaces. An interesting contact point between locally conformally symplectic and Kähler geometry appears in the papers [13] and [14]. The authors consider locally conformally symplectic structures (ω, ϑ) on compact complex surfaces (M, J) such that ω tames J , i.e. the $(1, 1)$ -part of ω is positive definite. The Morse–Novikov cohomology of locally conformally Kähler surfaces has been investigated in [126]. Results on the deformations of Lee classes of locally conformally Kähler structures have been obtained in [65]. In [144], the author also addressed the question of which subset of $H_{dR}^1(M)$, being M a compact Vaisman manifold, is occupied by Lee classes of Vaisman metrics. In the more general context of Hermitian structures, metrics which are locally conformal to notable ones, for instance to balanced ones, have been studied in [9].

3. Classical mechanics

Now those Quantities which I consider as gradually and indefinitely increasing, I shall hereafter call *Fluents*, or *Flowing Quantities*, [. . .] And the Velocities by which every Fluent is increased by its generating Motion, (which I may call *Fluxions*, or simply Velocities or Celerities,) [. . .] The Relation of the Flowing Quantities to one another being given, to determine the Relation of their Fluxions.

A relation being proposed, including the Fluxions of Quantities, to find the Relations of those Quantities to one another.

— Sir Isaac Newton, *De methodis serierum et fluxionum*, 1671.¹⁸

¹⁸I am grateful to Prof. Antonio Giorgilli for having written amazing lecture notes for the Mathematical Physics courses he taught at University of Milano-Bicocca. As an undergrad I was lucky enough to attend a few of them, a very fruitful experience. His lecture notes contained, among other things, this reference to Newton's original work; see <http://www.mat.unimi.it/users/antonio/meccanica/meccanica.html>.

The three sentences of Isaac Newton define the objects of interest and summarize the goals of the study of dynamical systems. It was Newton who gave a mathematically precise definition of the three laws that govern classical mechanics, that is, the study of the movement of a body as a response to being exposed to a force. He developed a theory, called in his honor *Newtonian mechanics*, to state and solve the problems posed by classical mechanics, notably arising from planetary motions. In this formalism, the equations of motion of a physical system with n degrees of freedom are given as solutions of n differential equations involving velocities and their derivatives (that is, differential equations of order 2).

Analytical techniques in the study of the problems of classical mechanics, especially celestial mechanics, were brought in by Joseph-Louis Lagrange at the beginning of the 19th century, founding what is nowadays known as *Lagrangian formalism*; an important role in this formalism is played by the *principle of minimal action*. In particular, as recalled by Weinstein in [159], in his 1808 book *Mémoire sur la théorie des variations des éléments des planètes*, Lagrange uses explicitly a certain skew-symmetric 6×6 matrix. The appearance of geometric techniques in classical mechanics is due to William Rowan Hamilton, who rewrote Newton's equations as a set of $2n$ differential equations of order 1. In terms of position coordinates (q_1, \dots, q_n) and corresponding momenta (p^1, \dots, p^n) , the motion is governed by a function $H = H(q_1, \dots, q_n, p^1, \dots, p^n)$, the *Hamiltonian* of the system, through the equations

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p^i}, \\ \dot{p}^i = -\frac{\partial H}{\partial q_i}. \end{cases} \quad (5)$$

Nowadays¹⁹ it is known that, in the simplest case, the phase space of a Hamiltonian system is the cotangent bundle T^*Q of a manifold Q which parametrizes the positions q of the physical system; the corresponding momenta p live on the fibers of the cotangent bundle over a point $q \in Q$ and the Hamiltonian of the system is $H \in \mathcal{C}^\infty(T^*Q)$. T^*Q is in a natural way a symplectic manifold; the symplectic form on T^*Q is very easy to describe: if $\pi: T^*Q \rightarrow Q$ is the canonical projection, define the *Liouville* or *tautological* 1-form $\lambda_{\text{can}} \in \Omega^1(T^*Q)$ by $\lambda_{(q,p)}(v) = p(d\pi_{(q,p)}(v))$ for a tangent vector v at $T_{(q,p)}T^*Q$. Then $\omega_{\text{can}} = -d\lambda_{\text{can}}$ is a symplectic form on T^*Q ; in local coordinates, one has $\omega_{\text{can}} = \sum_{i=1}^n dq_i \wedge dp^i$. In the *Hamiltonian formalism*, the equations of motions (5) are given as integral curves of the *Hamiltonian vector field* X_H ; if $H: T^*Q \rightarrow \mathbb{R}$ is the Hamiltonian function of the system, then X_H is uniquely determined by the condition $dH = \omega_{\text{can}}(X_H, \cdot)$.

From the point of view of Hamiltonian formalism, the fact that non-degeneracy is a local condition implies that the definition of the Hamiltonian vector field is local. Following the illuminating introduction of Vaisman's paper [146], I propose to show that locally conformally symplectic manifolds provide an adequate and more general context for Hamiltonian mechanics. One can make the *Ansatz* that the dynamics on the phase

¹⁹One could name here many other scientists who contributed to elaborate thorough foundations for classical mechanics — I prefer to direct the reader to the much more complete references [1, 16, 44, 64] for further historical and mathematical background.

space consists of the orbits of a globally defined vector field X . Consider an open set $U_\alpha \subset T^*Q$ with local coordinates $(q_1^\alpha, \dots, q_n^\alpha, p_\alpha^1, \dots, p_\alpha^n)$. Then one obtains a local function $H_\alpha: U_\alpha \rightarrow \mathbb{R}$ such that the orbits of X are defined by a local version of Hamilton's equations,

$$\begin{cases} \dot{q}_i^\alpha = \frac{\partial H_\alpha}{\partial p_\alpha^i} \\ \dot{p}_\alpha^i = -\frac{\partial H_\alpha}{\partial q_i^\alpha} \end{cases} \quad (6)$$

Of course, X is the Hamiltonian vector field of the local Hamiltonian function H_α with respect to the local symplectic form $\omega_{\text{can}}^\alpha = \sum_{i=1}^n dq_i^\alpha \wedge dp_\alpha^i$. Suppose $\{U_\alpha\}_\alpha$ is an open covering of T^*Q . One then usually requires $\{\omega_{\text{can}}^\alpha\}$ and $\{H_\alpha\}$ to piece together to a global symplectic form ω_{can} and a global Hamiltonian H . However, following our *Ansatz*, in order to globalize this local assertion one only needs to prescribe the fact that the transition functions

$$q_i^\beta = q_i^\alpha(q_j^\alpha, p_\alpha^k) \quad \text{and} \quad p_\beta^i = p_\beta^i(q_j^\alpha, p_\alpha^k) \quad (7)$$

on $U_\alpha \cap U_\beta$ preserve (6). Of course, if (7) are *canonical* transformations of the phase space, then $\omega_{\text{can}}^\alpha = \omega_{\text{can}}^\beta$ and one is back to the symplectic context. However, allowing a *homothetical* change of coordinates, i.e. taking $H_\beta = \mu_{\beta\alpha} H_\alpha$ for a constant $\mu_{\beta\alpha} \neq 0$, then $\omega_{\text{can}}^\alpha = \mu_{\beta\alpha} \omega_{\text{can}}^\beta$. Thus our phase space consists of T^*Q with an open covering $\{U_\alpha\}$ and a symplectic form $\omega_{\text{can}}^\alpha$ on each U_α such that, on $U_\alpha \cap U_\beta \neq \emptyset$,

$$\omega_{\text{can}}^\alpha = \mu_{\beta\alpha} \omega_{\text{can}}^\beta. \quad (8)$$

Equation (8) implies that the collection $\{\mu_{\alpha\beta}\}$ satisfies the *cocycle condition* $\mu_{\beta\gamma} = \mu_{\beta\alpha} \mu_{\alpha\gamma}$, hence one obtains a real line bundle $L \rightarrow T^*Q$ with transition functions $\{\mu_{\alpha\beta}\}$. The *global* Hamiltonian is not anymore a smooth function on T^*Q but rather a smooth section of L . The cocycle condition can be rephrased by saying that

$$\mu_{\beta\alpha} = \frac{e^{\sigma_\alpha}}{e^{\sigma_\beta}}$$

for functions $\sigma_\alpha: U_\alpha \rightarrow \mathbb{R}$ (resp. $\sigma_\beta: U_\beta \rightarrow \mathbb{R}$). Now equation (8) shows that the collection of local 2-forms $\{e^{\sigma_\alpha} \omega_{\text{can}}^\alpha\}$ piece together to a global, non-degenerate 2-form ω on T^*Q . Clearly, the 1-forms $\{d\sigma_\alpha\}$ piece together to a 1-form ϑ and $d\omega = \vartheta \wedge \omega$. Thus, ϑ is the Lee form of the locally conformally symplectic structure (ω, ϑ) and $(T^*Q, \omega, \vartheta)$ is a locally conformally symplectic manifold.

As pointed out in [77], given any manifold Q with a closed 1-form $\bar{\vartheta}$, the cotangent bundle T^*Q admits a canonical exact locally conformally symplectic structure

$$(\omega, \vartheta) = (d_\vartheta(-\lambda_{\text{can}}), \vartheta),$$

where $\pi: T^*Q \rightarrow Q$ is the canonical projection and $\vartheta = \pi^* \bar{\vartheta}$.

I conclude this jaunt into classical mechanics by mentioning a couple of more papers where ideas of conformally symplectic geometry find applications to physical problems.

Let (M, ω) be a symplectic manifold, let $H \in \mathcal{C}^\infty(M)$ be a Hamiltonian function and X_H be the corresponding Hamiltonian vector field. If $f \in \mathcal{C}^\infty(M)$ is a function, then

the vector field $e^f X_H$ is *conformally Hamiltonian* with Hamiltonian H and conformal factor e^f . It clearly satisfies $e^f dH = \omega(e^f X_H, \cdot)$. Moreover, $e^f X_H$ is the Hamiltonian vector field of H for the 2-form $\omega' = e^{-f} \omega$, which is not closed anymore, but conformally closed with Lee form $-df$.

In [100] Maciejewski, Przybylska and Tsiganov consider conformally Hamiltonian vector fields in the theory of bi-Hamiltonian systems, in order to produce examples of completely integrable systems.

In [103] Marle used conformally Hamiltonian vector fields to study, in a new perspective, a certain diffeomorphism between the phase space of the Kepler problem and an open subset of the cotangent bundle of S^3 (resp. of a 2-sheeted hyperboloid, according to the energy of the motion).

In [162], Wojtkowski and Liverani apply the formalism of conformally symplectic geometry in order to model concrete physical situations such as the Gaussian isokinetic dynamics, also with collisions, and the Nosé–Hoover dynamics. More precisely, the authors show that such systems fall under the formalism of conformal Hamiltonian dynamics and explain how to easily deduce results about the symmetry of the Lyapunov spectrum.

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