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Generalised pairs in birational geometry

Caucher Birkar

Abstract. In this note we introduce generalised pairs from the perspective of the evolution of the notion of space in birational algebraic geometry. We describe some applications of generalised pairs in recent years and then mention a few open problems.

Unless stated otherwise we work over an algebraically closed field k of characteristic zero. Divisors are considered with rational coefficients; for simplicity we avoid real coefficients. Although we will make some historical remarks but this text is not meant to tell the history of birational geometry. This note is an expanded version of a talk given at the conference "Geometry at large" in Fuerteventura, Spain, in December 2018.

1. Varieties

Geometry is the study of "shapes", that is, "spaces". The notion of space varies in different types of geometries and it also changes historically while the subject evolves over time. The spaces of interest in classical algebraic geometry are algebraic varieties X defined in affine or projective spaces by vanishing of polynomial equations over an algebraically closed field k. It is not easy to classify such spaces even in dimension one due to the huge number of possibilities. It is then natural to restrict to smaller classes of varieties such as smooth ones which leads one to birational geometry.

Birational geometry aims to classify algebraic varieties up to birational transformations. This involves finding special representatives in each birational class and then classifying such representatives, say by constructing their moduli spaces. To start with, taking compactification and then normalisation one can assume X to be projective and normal. However, being normal is not a very special property so it is not easy to work in such generality. We can resolve the singularities and assume X to be smooth. This enables us to use many tools available for smooth varieties, e.g. the Riemann–Roch theorem in low dimension has a particularly simple expression.

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In dimension one each birational class contains a unique smooth projective model so smoothness is a satisfactory condition to work with. One then can focus on parametrising curves of fixed genus as did Riemann in the 19th century.

In dimension two one can still stick with the category of smooth varieties. Indeed running a classical minimal model program on X by successively blowing down -1-curves, smoothness is preserved. The program ends with either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a curve or a smooth projective variety Y with K_Y being nef. Here nefness means that the intersection number $K_Y \cdot C$ is non-negative for every curve $C \subset Y$. However, we are still led to consider some kind of singularities. Indeed if K_Y is big, then some positive multiple of it is base point free hence defines a birational contraction $Y \to Z$ where Z is a projective surface with Du Val singularities. The two-dimensional birational geometry was developed by the Italian school of algebraic geometry in late 19th century and early 20th century, in particular, by Castelnuovo and Enriques.

It is clear that the primary spaces to study in dimension one and two are smooth projective varieties. Many of the deep statements in classical algebraic geometry are stated for such varieties. However, later when people studied moduli spaces, say of curves of fixed genus, in greater detail they were led to consider more general spaces such as curves with nodal singularities which may not be irreducible [18]. Constructing moduli spaces for surfaces involves considering semi-lc surfaces which are analogues of curves with nodal curves (cf. [1, 42]).

Until the end of the 1960s there were some progress in higher dimensional birational geometry, e.g. work of Fano, resolution of singularities of 3-folds by Zariski, and resolution of singularities in arbitrary dimension by Hironaka. It was only in the 1970s that the subject really took off. For example, Iskovskikh and Manin [35] proved that smooth quartic 3-folds are not rational using earlier work of Fano. On the other hand, Iitaka proposed a program for birational classification of varieties (even open varieties) according to their Kodaira dimension (cf. [33, 34]) which as we will see led to the development of pairs. And the techniques developed in Mori's solution to the Hartshorne conjecture [54] in the late 1970s paved the way for taking some of the first steps toward a minimal model program in higher dimension [55]. In yet another direction Reid's study of singularities [58] helped to identify the right classes of singularities for this program to work. Moreover, Kawamata and Shokurov [37, 61] made extensive use of pairs and vanishing theorems [36] to establish foundational results such as the base point free theorem guaranteeing existence of relevant contractions generalising contraction of -1-curves on surfaces. With Mori's proof of existence of flips [56], (and the more general version by Shokurov [62]), and Miyaoka and Kawamata's proof of abundance [38, 51, 52] (and the more general version by Keel, Matsuki, McKernan [41]), and further input from Kollár and many others birational geometry for 3-folds was well-developed by the early 1990s. See [40, 44] for introductions to the subject.

As it is evident from the above discussion, it did not take long before people understood that the category of smooth varieties is not large enough for birational geometry in dimensions ≥ 3. One needs to consider singularities albeit of mild kinds. There are

several types of singularities such as Kawamata log terminal and log canonical which behave well (these are defined in Section 3). These singularities not only allow proofs to go through but their study reveals fascinating facts which are often reflections of global phenomena. Thus, the primary spaces in higher dimensional birational geometry were no longer smooth ones but those with mild singularities.

On the other hand, one can study the birational geometry of varieties X defined over arbitrary fields k. In characteristic zero many statements can be reduced to similar statements after passing to the algebraic closure of k, e.g. running a minimal model program. But the finer classification is not that simple, for example, there can be many non-isomorphic smooth projective varieties over k which all become isomorphic after passing to the algebraic closure. See for example [43] for birational geometry of varieties of dimension 3 over $\mathbb{R}$. In positive characteristic the story is more complicated because of non-separable field extensions.

2. Schemes

With Grothendieck's revolution of algebraic geometry attention shifted from studying varieties to schemes in the 1960s, to some extent. Birational geometry of arithmetic surfaces, that is, regular schemes of dimension two was worked out [47,59]. This is important for arithmetic geometry. e.g. for constructing proper regular models of curves defined over number fields and eventually to construct Néron models of elliptic curves [65].

On the other hand, one may attempt to study birational geometry of schemes defined over a field; we mean allowing non-reduced structures. There does not seem to be much research in this direction except some cases in dimension one. For example, Mumford's canonical curves appear while studying minimal smooth projective surfaces Y of Kodaira dimension one [57]. An effective divisor X on Y such that $K_Y \cdot X = 0$ and $X \cdot C = 0$ for every component C of X, is a curve of canonical type. By adjunction $K_X \sim 0$ so we may consider X as a Calabi–Yau scheme of dimension one. Studying such X, especially when X is connected and the g.c.d. of its coefficients is 1, gives non-trivial information about Y.

From the 1970s focus was on the birational geometry of good old smooth varieties but one important point of view of Grothendieck remained which emphasized on working in the relative setting, that is, studying varieties with a morphism to another variety. This has had an important influence on the development of birational geometry. For example a Kawamata log terminal singularity can be considered as a local analogue of a Fano variety.

3. Pairs

In the 1970s Iitaka initiated a program to study the birational classification of open varieties (that is, non-compact varieties) according to their Kodaira dimension [33, 34]. This involved compactifying the variety by adding a boundary divisor. This approach then

evolved into the theory of pairs. Pairs are roughly speaking algebraic varieties together with an extra structure given by a certain type of divisor (see below for precise definition). They appear naturally even if one is only interested in studying projective varieties. We give a series of motivating examples before defining pairs rigorously.

Much of the machinery of birational geometry of the last four decades has been developed for pairs. The main concept of space is then that of a pair.

3.1. Open varieties

Suppose U is a smooth variety. Then the geometry of U is often best understood after taking a compactification. Suppose X is a smooth projective variety containing U as an open subset. Such a compactification is not unique except in dimension one. We can choose X so that $B := X \setminus U$ is a divisor with simple normal crossing singularities. Now we consider (X, B) as a pair. The geometry of (X, B) reveals so much about the geometry of U.

3.2. Adjunction

Suppose X is a smooth variety and B is a smooth prime divisor on X. Then the well-known adjunction formula says that $K_B = (K_X + B)|_B$. Generalisations of the adjunction formula play a central role in birational geometry. Very often one derives non-trivial statements about the geometry of (X, B) from that of B allowing proofs by induction on dimension. Thus, one is led to study pairs such as (X, B).

3.3. Canonical bundle formula

Suppose X is a smooth projective variety and $f\colon X \to Z$ is a contraction, that is, a projective morphism with connected fibres. Suppose in addition that $K_X \sim f^*L$ for some $\mathbb{Q}$-divisor L. Then the canonical bundle formula says that we can write

$$K_X \sim f^*(K_Z + B + M),$$

where the discriminant divisor B is a $\mathbb{Q}$-divisor uniquely determined and the moduli divisor M is a $\mathbb{Q}$-divisor determined up to $\mathbb{Q}$-linear equivalence [2,39]. The classical example is that of Kodaira's canonical bundle formula in which X is a surface and f is an elliptic fibration. The higher dimensional version in a more general context is discussed below. The canonical bundle formula allows one to investigate the geometry of X from that of $(Z, B + M)$. In the setting above, it is possible to choose M so that we can consider $(Z, B + M)$ as a pair [2]. We will see later that for many applications it is actually more appropriate to consider it as a generalised pair because it is important to keep track of the coefficients of M.

3.4. Quotient varieties

Suppose that X is a smooth variety and G is a finite group acting on X. Let Y be the quotient of X by G and $\pi\colon X \to Y$ be the quotient map. Then using Hurwitz formula

one can write $K_X = \pi^*(K_Y + B_Y)$ for some divisor B_Y whose coefficients are of the form $1 - \frac{1}{n}$ for certain natural numbers n. It is then natural to study Y and B_Y together rather than just Y, that is, to consider the pair (Y, B_Y).

3.5. Definition of pairs

We now define pairs and their singularities rigorously. A *pair* (X, B) consists of a normal quasi-projective variety X and a $\mathbb{Q}$-divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. We call B a *boundary* divisor if its coefficients are in $[0, 1]$. Let $\phi: W \to X$ be a log resolution of (X, B). We can write

$$K_W + B_W = \phi^*(K_X + B),$$

where B_W is uniquely determined. Here we choose K_W so that $\phi_* K_W = K_X$. We say (X, B) is *log canonical* (resp. *Kawamata log terminal*) (resp. *ϵ-log canonical*) if every coefficient of B_W is ≤ 1 (resp. < 1) (resp. $\leq 1 - \epsilon$). If D is a component of B_W with coefficient 1, then its image on X is called a *log canonical centre*.

3.6. Examples

(1) The simplest kind of pairs are the log smooth ones. A *log smooth* pair is a pair (X, B), where X is smooth and Supp B has simple normal crossing singularities. Such pairs are log canonical exactly when the coefficients of B are at most 1. If the coefficients of B are less than 1, then the pair is Kawamata log terminal.

(2) Let $X = \mathbb{P}^2$. When B is a nodal curve, then (X, B) is log canonical. But when B is a cuspidal curve, then (X, B) is not log canonical.

(3) Let X be the cone over a smooth rational curve. Then, $(X, 0)$ is Kawamata log terminal. In contrast if Y is the cone over an elliptic curve, then $(Y, 0)$ is log canonical but not Kawamata log terminal.

(4) Let X be a toric variety and B be the sum of the torus invariant divisors. Then (X, B) is log canonical.

(5) Let X be the variety in $\mathbb{A}^4$ defined by the equation $xy - zw = 0$. Then $(X, 0)$ is Kawamata log terminal.

3.7. Using pairs

We illustrate the power of pairs by an example which frequently comes up in inductive statements. Suppose (X, B) is a projective Kawamata log terminal pair and S is a normal prime divisor on X. Now suppose L is a Cartier divisor such that

$$A := L - (K_X + B + S)$$

is ample. Consider the exact sequence

$$H^0(X, L) \to H^0(S, L|_S) \to H^1(X, L - S).$$

By assumption,

$$L - S = K_X + B + A.$$

Since A is ample and (X, B) is Kawamata log terminal, $H^1(L - S) = 0$ by the Kawamata–Viehweg vanishing theorem (cf. [36]). Therefore, every section of $L|_S$ can be lifted to a section of L. This is very useful for example in situations where we want to show the linear system $|L|$ is non-empty or that it is base point free.

3.8. Progress in the last two decades

There has been huge progress in birational geometry of pairs in arbitrary dimension, in the last two decades which builds upon the machinery developed in the preceding decades. This is due to work of many people. To name few specific examples, existence of flips [10, 28, 64], existence of minimal models for varieties of log general type and finite generation of log canonical rings [10], ACC for log canonical thresholds and boundedness results for varieties of log general type [29–31] (ACC on smooth varieties was proved in [17]), boundedness of complements and Fano varieties [8, 9], existence of moduli spaces for stable pairs [42], existence of moduli spaces for polarised Calabi–Yau and Fano pairs [4] have all been established.

3.9. Semi-log canonical pairs

A semi-log canonical pair is roughly a pair in which the underlying space may not be irreducible or may have components with self-intersections but the intersections should be nice similar to those on nodal curves. Semi-log canonical pairs are important for constructing compact moduli spaces [42] as they appear as limits of usual pairs in families.

4. Generalised pairs

In recent years a new concept of space has evolved, that is, the generalised pairs. A generalised pair is roughly a pair together with a birational model polarised with some divisor having some positivity property. They were first defined and studied in [14] in their general form. However, some special cases were already investigated in [13] and they also appeared implicitly in the earlier works [7, 12]. Generalised pairs have found applications in various contexts which we will discuss in the next section. We begin with considering some motivating examples of such pairs and then give their precise definition afterwards.

4.1. Polarised varieties

Consider a projective variety X and an ample divisor M on it. We say X is polarised by M. For example, M can be a very ample divisor determining an embedding of X into some projective space. Influenced by Grothendieck's theory of Hilbert schemes and Mumford's geometric invariant theory, polarised varieties play a central role in modern moduli theory

as one often needs some kind of positivity in order to achieve "stability" hence be able to construct moduli spaces. For example, one may consider varieties X polarised by K_X.

For applications it is important to allow more general polarisations, that is, when M is not necessarily ample but only nef. Indeed, [13] studies the birational geometry of varieties and pairs polarised by nef divisors which paved the way for the development of generalised pairs. Here M is only a divisor class and not necessarily effective so we cannot consider (X, M) as a usual pair.

For moduli of log canonically polarised pairs of general type, see [42], and for moduli of polarised Calabi–Yau and Fano pairs, see [4].

4.2. Generalised polarised varieties

Suppose X is a projective variety and $X \dashrightarrow X'$ is a birational map to another projective variety. Suppose X' is polarised by a nef divisor M'. We can consider X as a generalised polarised variety, that is, a variety with some polarised birational model. Sometimes it is important to understand how X and X' are related. For example, when X' is a minimal model of X, then the canonical divisor $K_{X'}$ polarises X' so we can consider X as a generalised polarised variety. This can be used to prove results about termination of flips (see the next section).

4.3. Canonical bundle formula and subadjunction

Suppose (X,B) is a projective pair with Kawamata log terminal singularities and $f:X\to Z$ is a contraction. Suppose in addition that $K_X + B \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-divisor L. Then the canonical bundle formula says that we can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z),$$

where the discriminant divisor B_Z is a $\mathbb{Q}$-divisor uniquely determined and the moduli divisor M_Z is a $\mathbb{Q}$-divisor determined up to $\mathbb{Q}$-linear equivalence [39]. There is a resolution $Z' \to Z$ and a nef divisor $M_{Z'}$ on Z' whose pushdown to Z is M_Z (see [2]). We can then consider $(Z, B_Z + M_Z)$ as a generalised pair which happens to be generalised Kawamata log terminal according to the definitions below.

Now assume (Y, Δ) is a projective log canonical pair and V is a minimal log canonical centre. Here minimality is among log canonical centres with respect to inclusion. Assume (Y, Θ) is Kawamata log terminal for some Θ. Assume V is normal (this actually holds automatically). Then we can write

$$(K_Y + \Delta)|_V \sim_{\mathbb{Q}} K_V + C + N,$$

where C is a divisor with coefficients in $[0, 1]$ and N is a divisor class which is the pushdown of some nef divisor. What happens is that from the setup we can find a Kawamata log terminal pair (S, Γ) and a contraction $g\colon S \to V$ such that

$$K_S + \Gamma \sim_{\mathbb{Q}} g^*(K_Y + \Delta)|_V.$$

Thus, we can use the previous paragraph to decompose $(K_Y + \Delta)|_V$ as $K_V + C + N$. In particular, $(V, C + N)$ is a generalised pair. This construction is called *subadjunction* [39] – in practice though people often try to perturb $C + N$ to get a boundary divisor.

4.4. Definition of generalised pairs

A projective *generalised pair* consists of

- a normal projective variety X,
- a $\mathbb{Q}$-divisor $B \geq 0$ on X, and
- a birational contraction $\phi\colon X' \to X$ and a nef $\mathbb{Q}$-divisor M' on X',

such that $K_X + B + M$ is $\mathbb{Q}$-Cartier where $M := \phi_* M'$.

Actually we specify X', M' only up to birational transformations, that is, if we replace X' with a resolution and replace M' with its pullback, then the pair would be the same. In other words, we view M' as a so-called b-divisor which is determined by its trace on the model X'.

We can similarly define generalised pairs in the relative setting, that is, when we are given a projective morphism $X \to Z$ (but X may not be projective) and then we only assume M' to be nef over Z.

Now we define generalised singularities for a generalised pair $(X, B + M)$. Replacing X' we can assume ϕ is a log resolution of (X, B). We can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

for some uniquely determined B'. We say $(X, B + M)$ is *generalised log canonical* (resp. *generalised Kawamata log terminal*) (resp. *generalised ϵ-log canonical*) if each coefficient of B' is ≤ 1 (resp. < 1) (resp. $\leq 1 - \epsilon$).

4.5. Examples

We present a series of examples of generalised pairs.

(1) The most obvious way to construct a generalised pair is to take a projective pair (X, B) and a nef $\mathbb{Q}$-divisor M to get $(X, B + M)$. Then $(X, B + M)$ is generalized log canonical (resp. generalized Kawamata log terminal) iff (X, B) is log canonical (resp. Kawamata log terminal). In this example $M' = M$ does not contribute to the singularities even if its coefficients are large. In contrast, the larger the coefficients of B are, the worse the singularities of $(X, B + M)$ are.

(2) In general, M' does contribute to singularities. For example, assume $X = \mathbb{P}^2$ and that ϕ is the blowup of a point x. Let E' be the exceptional divisor, L a line passing through x and L' the birational transform of L.

If $B = 0$ and $M' = 2L'$, then we can calculate $B' = E'$ hence $(X, B + M)$ is generalized log canonical but not generalized Kawamata log terminal. However, if $B = L$ and $M' = 2L'$, then $(X, B + M)$ is not generalized log canonical because in this case $B' = L' + 2E'$.

(3) Suppose that X is a smooth projective variety such that $-K_X$ is nef. Letting $M := -K_X$ we can consider (X, M) as a generalised Calabi–Yau pair as $K_X + M = 0$. This is very useful for tackling problems about X, see Sections 5.8 and 5.10 below.

(4) Let X be a smooth projective variety and D be a divisor with $h^0(X, D) \neq 0$. We can find a resolution of singularities $\phi\colon X' \to X$ such that the movable part M' of the linear system $|\phi^* D|$ is base point free, hence in particular nef. Writing C' for the fixed part of $|\phi^* D|$ we get $M' + C' \sim \phi^* D$. Let M, C be the pushdowns of M', C' respectively. Then $(X, C + M)$ is a generalised pair with nef part M' which remembers the movable and fixed parts of $|\phi^* D|$. Its singularities reflect the singularities of (X, L) where L is a general member of $|D|$, e.g. $(X, C + M)$ is generalised log canonical iff (X, L) is log canonical. Moreover, for any non-negative rational number t, the singularities of the generalised pair $(X, tC + tM)$ with nef part tM' reflect the singularities of the pair (X, tL).

More generally suppose $W \subseteq |D|$ is a linear system. Again pick a resolution $X' \to X$ so that the movable part M' of the linear system $\phi^* W$ is base point free, hence in particular nef. Writing C' for the fixed part of $\phi^* W$ we get $M' + C' \sim \phi^* D$. Let M, C be the pushdowns of M', C' respectively. Then $(X, C + M)$ is a generalised pair with nef part M' which remembers the movable and fixed parts of W. Similarly, for a non-negative rational number t, $(X, tC + tM)$ is a generalised pair with nef part tM' whose behaviour reflects that of (X, tL) where L is a general member of W. Generalised pairs of this kind have appeared in the context of birational rigidity and non-rationality of Fano varieties and the Sarkisov program going back to work of Iskovskikh–Manin which in turn is based on ideas going back as far as Fano and Noether (see [15] for a survey on these topics, especially Section 0.2).

For simplicity we assumed X to be smooth but similar constructions apply for example if X has klt singularities and D is $\mathbb{Q}$-Cartier.

(5) Let X be a smooth projective variety such that $K_X \sim_{\mathbb{Q}} 0$ (that is, X is a Calabi–Yau variety). Let M be a nef divisor on X. A difficult conjecture predicts that M is numerically equivalent to a semi-ample divisor. Viewing (X, M) as a generalised pair, the question is whether $K_X + M$ is numerically semi-ample. See the last section for more general statements.

(6) Let X be a normal $\mathbb{Q}$-factorial variety and let $V = \sum b_i V_i$ be a formal linear combination of closed subvarieties where $V_i \neq X$ and b_i are non-negative rational numbers. One can consider (X, V) as some kind of pair generalising the traditional notion of a pair [19] (when all the V_i are prime divisors, then (X, V) is a pair in the usual sense). Take a log resolution $\phi\colon X' \to X$ such that the scheme-theoretic inverse images $V_i' := \phi^{-1} V_i$ are all Cartier divisors. Here we are assuming that the exceptional locus of ϕ union the support of all the V_i' is a divisor with simple normal crossing singularities (hence the name log resolution). Write $K_{X'} + E' = \phi^* K_X$ and then define the log discrepancy of a prime divisor D' on X' with respect to (X, V) to be $1 - \mu_{D'}(E' + \sum b_i V_i')$. We say (X, V) is log canonical (resp. Kawamata log terminal) if all the log discrepancies are ≥ 0 (resp. > 0).

The singularities of (X, V) can be interpreted in terms of generalised pairs. Indeed, for each i, the morphism $X' \to X$ factors through the blowup $Y_i \to X$ of X along V_i. The scheme-theoretic inverse image of V_i on Y_i is a Cartier divisor which is anti-ample over X, and V_i' is the pullback of this anti-ample divisor. Thus, $-V_i'$ is nef over X, for each i. Now let $M' := -\sum b_i V_i'$ which is nef over X, and let $B' := E' + \sum b_i V_i'$. Consider the generalised pair $(X, B + M)$ with nef part M' relatively over X where $B = \phi_* B'$ and $M = \phi_* M'$. Then we have

$$K_{X'} + B' + M' = K_{X'} + E' = \phi^* K_X = \phi^*(K_X + B + M)$$

as $B + M = 0$. Thus, the log discrepancies of (X, V) are the same as the generalised log discrepancies of $(X, B + M)$.

Note that even though $B + M = 0$ but $(X, B + M)$ as a generalised pair is not the same as the usual pair $(X, 0)$. Also note that even if X is projective, the generalised pair $(X, B + M)$ only makes sense relatively over X since M' obviously may not be nef globally. This suggests that $(X, B + M)$ is useful for studying local properties of (X, V).

4.6. Geometry of generalised pairs

In the next section we present some of the applications of generalised pairs in recent years. We want to emphasize that in addition to such applications studying the geometry of generalised pairs is interesting on its own. Many questions for varieties and usual pairs can be asked in the context of generalised pairs which leads to some deep problems. For example, can we always run a minimal model program for a generalised pair and get a minimal model or a Mori fibre space in the end? See the last section for some more problems.

Various geometric aspects of generalised pairs have been studied in recent years, for example, see [13,14,26,53] for the minimal model program and termination, [14] for birational boundedness of linear systems and ACC for generalised lc thresholds, [8,16,22,23] for boundedness of complements, [20] for boundedness of generalised pairs of general type, [21] for the canonical bundle formula, [5] for boundedness of Fano type generalised log Calabi–Yau fibrations, [4] for birational boundedness of linear systems and boundedness of polarised varieties, [11] for boundedness of rationally connected generalised log Calabi–Yau 3-folds, [45,46] for abundance, [49] for the Sarkisov program, [48] for accumulation points of generalised lc thresholds, [25] for invariance of plurigenera and boundedness, [3, 24] for connectedness properties of non-klt loci of generalised log Calabi–Yau pairs, etc.

5. Some applications of generalised pairs

In this section we discuss some applications of generalised pairs in recent years.

5.1. Effective Iitaka fibrations and pluricanonical systems of generalised pairs

Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$. The Kodaira dimension $\kappa(W)$ is the largest number κ among $\{-\infty, 0, 1, \dots, \dim X\}$ such that

$$\limsup_{m \in \mathbb{N}} \frac{h^0(X, mK_W)}{m^\kappa} > 0.$$

Then by a well-known construction of Iitaka, there is a birational morphism $V \to W$ from a smooth projective variety V, and a contraction $V \to X$ onto a projective variety X such that a (very) general fibre F of $V \to X$ is smooth with Kodaira dimension zero, and $\dim X$ is equal to the Kodaira dimension $\kappa(W)$. The map $W \dashrightarrow X$ is referred to as an *Iitaka fibration* of W, which is unique up to birational equivalence. For any sufficiently divisible natural number m, the pluricanonical system $|mK_W|$ defines an Iitaka fibration. The following hard conjecture predicts that we can choose m uniformly depending only on the dimension.

Conjecture 5.2 (Effective Iitaka fibration, cf. [27]). *Let W be a smooth projective variety of dimension d and Kodaira dimension $\kappa(W) \geq 0$. Then there is a natural number m_d depending only on d such that the pluricanonical system $|mK_W|$ defines an Iitaka fibration for any natural number m divisible by m_d.*

In [14] the conjecture is reduced to bounding certain invariants of the very general fibres of the Iitaka fibration. The point is that, perhaps after replacing W, X birationally, bounding such invariants implies that the Iitaka fibration induces a canonical bundle type formula giving $\mathbb{Q}$-divisors $B \geq 0$ and M on X where the coefficients of B are in a DCC set and M is nef with bounded Cartier index. Thus, it would be enough to prove the following theorem regarding generalised pairs. Here by DCC set we mean the set does not contain any infinite strictly decreasing sequence of numbers.

Theorem 5.3. *Let Λ be a DCC set of non-negative real numbers, and d, r be natural numbers. Then there is a natural number $m(\Lambda, d, r)$ depending only on Λ, d, r such that if*

(i) *(X, B) is a projective log canonical pair of dimension d,*

(ii) *the coefficients of B are in Λ,*

(iii) *rM is a nef Cartier divisor, and*

(iv) *$K_X + B + M$ is big,*

then the linear system $|m(K_X + B + M)|$ defines a birational map if $m \in \mathbb{N}$ is divisible by $m(\Lambda, d, r)$.

For usual pairs, that is when $M = 0$, the theorem was previously known [30, Theorem 1.3]. However, the general case proved in [14] uses very subtle properties of the theory of generalised pairs.

5.4. Boundedness of complements and of Fano varieties

Let (X, B) be a projective pair where B is a boundary. Let $T = \lfloor B \rfloor$ and $\Delta = B - T$. An *n-complement* of $K_X + B$ is of the form $K_X + B^+$ such that

- (X, B^+) is log canonical,
- $n(K_X + B^+) \sim 0$, and
- $nB^+ \geq nT + \lfloor (n+1)\Delta \rfloor$.

From the definition one sees that

$$-nK_X - nT - \lfloor (n+1)\Delta \rfloor \sim nB^+ - nT - \lfloor (n+1)\Delta \rfloor \geq 0,$$

so existence of an n-complement for $K_X + B$ implies that the linear system

$$|-nK_X - nT - \lfloor (n+1)\Delta \rfloor|$$

is non-empty. In particular, this means that we should be looking at varieties X with K_X "non-positive", e.g. Fano varieties. Complements were defined by Shokurov [62] in the context of construction of flips. The following was conjectured by him [63] and proved in [8].

Theorem 5.5. *Let d be a natural number and $\Re \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\Re$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is log canonical of dimension d,*
- *the coefficients of B are in $\Phi(\Re)$,*
- *X is of Fano type, and*
- *$-(K_X + B)$ is nef.*

Then there is an n-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$. Moreover, the complement is also an mn-complement for any $m \in \mathbb{N}$.

Here X being of Fano type means that (X, C) is klt and $-(K_X + C)$ is ample for some boundary C. And $\Phi(\Re)$ stands for the set

$$\left\{1 - \frac{r}{m} \mid r \in \Re,\ m \in \mathbb{N}\right\}.$$

A special case of the theorem is when $K_X + B \sim_{\mathbb{Q}} 0$ along a fibration $f\colon X \to T$. This is where generalised pairs come into the picture. Applying the canonical bundle formula we can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_T + B_T + M_T),$$

where B_T is the discriminant divisor and M_T is the moduli divisor. It turns out that the coefficients of B_T are in $\Phi(\mathfrak{S})$ for some fixed finite set $\mathfrak{S}$ of rational numbers, and that pM_T is integral for some bounded number $p \in \mathbb{N}$. Now we want to find a complement for $K_T + B_T + M_T$ and pull it back to X. As mentioned elsewhere in the text

$(T, B_T + M_T)$ is not a pair but it is a generalised pair. Thus, we actually need to construct complements in the more general setting of generalised pairs which can be defined similar to the case of usual pairs. Once we have a bounded complement for $K_T + B_T + M_T$ we pull it back to get a bounded complement for $K_X + B$.

Also see [60] for an extensive study of complements where generalised pairs are used.

The theory of complements is applied in [9] to prove the following statement which was known as the BAB conjecture. Thus, generalised pairs play an important (indirect) role in the proof of this theorem.

Theorem 5.6. *Let d be a natural number and ϵ be a positive real number. Then the projective varieties X such that*

- *(X, B) is ϵ-log canonical of dimension d for some boundary B, and*
- *$-(K_X + B)$ is nef and big,*

form a bounded family.

This theorem in turn has been applied to various problems in recent years.

5.7. Termination of flips and existence of minimal models

In [7] existence of minimal models is linked with existence of weak forms of Zariski decompositions. A given divisor D on a projective variety has a weak Zariski decomposition if its pullback to some resolution of X can be written as $P + N$ where P is nef and N is effective. When (X, B) is a pair, we would be interested in weak Zariski decompositions of $K_X + B$. On the other hand, termination of flips is linked with log canonical thresholds in [6]. Extending these to the case of generalised pairs, termination of flips for generalised pairs with weak Zariski decompositions is derived from termination in lower dimension for generalised pairs, in [26,32]; it is also shown that existence of weak Zariski decompositions for pseudo-effective generalised pairs is equivalent to existence of minimal models for such pairs. In particular, termination of flips is established for pseudo-effective pairs of dimension four which at the moment does not follow from any other technique (this was first established in [53] for usual pairs).

5.8. Boundedness of certain rationally connected varieties

McKernan and Prokhorov [50] conjectured a more general form of BAB.

Conjecture 5.9. *Let d be a natural number and ϵ be a positive real number. Consider projective varieties X such that*

- *(X, B) is ϵ-log canonical of dimension d for some boundary B,*
- *$-(K_X + B)$ is nef, and*
- *X is rationally connected.*

Then the set of such X forms a bounded family.

The rational connectedness assumption cannot be removed: indeed it is well known that K3 surfaces do not form a bounded family; they satisfy the assumptions of the theorem with $d = 2$ and $\epsilon = 1$ and $B = 0$ except that they are not rationally connected.

The conjecture fits nicely into the framework of generalised pairs and related conjectures. Indeed letting $M := -(K_X + B)$ we get a generalised ϵ-log canonical pair $(X, B + M)$ with $K_X + B + M = 0$, hence a generalised Calabi–Yau pair. The advantage of this point of view is that running a minimal model program preserves the Calabi–Yau condition, hence one can get information by passing to Mori fibre spaces when $B + M \not\equiv 0$ (which is always the case on some birational model). Using this and the machinery developed in [5], a slightly weaker form of the conjecture is verified in dimension three in [11] where one replaces boundedness with boundedness up to isomorphism in codimension one.

5.10. Connected components of non-klt loci

Generalised pairs are used in [3, 24] to verify a conjecture of Hacon and Han on the connected components of non-klt loci of certain pairs. More precisely, it is proved that given a pair (X, B) and a projective morphism $X \to Z$ with connected fibres such that $-(K_X + B)$ is nef over Z, the non-klt locus of (X, B) has at most two connected components near each fibre of $X \to Z$. Similar to the previous subsection, letting $M := -(K_X + B)$ and considering $(X, B + M)$ as a generalised pair which is generalised log Calabi–Yau over Z, is the key to proving the above statement.

For similar reasons, it is expected that generalised pairs play a prominent role in tackling other problems about the class of pairs (X, B) with $-(K_X + B)$ nef over some base.

5.11. Varieties fibred over abelian varieties

In [12], generalised pairs, more precisely, polarised pairs, are used to investigate pairs that are relatively of general type over a variety of maximal Albanese dimension. Suppose that (X, B) is a projective Kawamata log terminal pair and $f : X \to Z$ is a surjective morphism where Z is a normal projective variety with maximal Albanese dimension, e.g. an abelian variety. It is shown that if $K_X + B$ is big over Z, then (X, B) has a good log minimal model. Moreover, if F is a general fibre of f, then

$$\kappa(K_X + B) \geq \kappa(K_F + B_F) + \kappa(Z) = \dim F + \kappa(Z),$$

where $K_F + B_F = (K_X + B)|_F$.

6. Problems

In this section we discuss some open problems regarding generalised pairs.

6.1. Existence of contractions and flips

Let $(X, B+M)$ be a projective generalised log canonical pair. Assume that R is a $K_X + B + M$-negative extremal ray. When $(X, B + M)$ is generalised Kawamata log terminal, existence of the contraction associated to R follows easily from the similar result for usual pairs. This is because we can easily find an ample $\mathbb{Q}$-divisor A and a Kawamata log terminal pair (X, Δ) such that

$$K_X + \Delta \sim_{\mathbb{Q}} K_X + B + M + A$$

and such that R is $(K_X + \Delta)$-negative. In particular, if R defines a flipping contraction, then its flip exists [10].

Now assume $(X, B + M)$ is not generalised Kawamata log terminal. If (X, C) is Kawamata log terminal for some C, then we can take an average and use the previous paragraph. More precisely, taking $t > 0$ to be a small rational number,

$$(X, tC + (1-t)B + (1-t)M)$$

is generalised Kawamata log terminal and

$$K_X + tC + (1-t)B + (1-t)M$$

intersects R negatively. Thus, in this case the contraction of R exists and in the flipping case its flip exists. If there is no C as above, e.g. when X itself has some non-Kawamata log terminal singularities, then the situation is more complicated and for now existence of contractions and flips in full generality is not proved yet. This is important for running minimal model program for generalised pairs.

6.2. Generalised minimal model program

Suppose that $(X, B + M)$ is a projective generalised log canonical pair. Assuming that existence of contractions and flips are established for such pairs, we can run the minimal model program on $K_X + B + M$ which, if terminates, produces a generalised Mori fibre space or a generalised minimal model. Termination of the program does not seem to follow from termination for usual pairs (although we can inductively treat the case when $K_X + B + M$ is pseudo-effective, as in [26]). For polarised varieties it was proved in [13] that termination of *some* choice of minimal model program can be guaranteed if termination holds for usual pairs which is only known up to dimension three. In low dimension the picture is clearer. Generalised termination holds trivially in dimension two. It is also known in dimension three [53] and in dimension four in the pseudo-effective case [26].

6.3. Generalised abundance

Although generalised pairs behave like usual pairs in many ways but there are some crucial differences. Assume $(X, B + M)$ is a projective generalised Kawamata log terminal pair with $K_X + B + M$ nef. In the case of usual pairs, that is, when $M = 0$, the abundance

conjecture says that $K_X + B$ is semi-ample which means that $m(K_X + B)$ is base point free for some natural number m. In general when $M \neq 0$, we cannot expect $K_X + B + M$ to be semi-ample as was already pointed out in [13]. Indeed this already fails in dimension one when X is an elliptic curve, $B = 0$, and M is a numerically trivial but non-torsion divisor. In dimension one at least numerical abundance holds, that is, $K_X + B + M \equiv D$ for some semi-ample divisor D. In dimension two even numerical abundance fails. The obstructions to numerical abundance seem to arise only when $K_X + B$ is not pseudo-effective (in which case X is uniruled so rational curves play a role); it is conjectured that if $K_X + B$ is pseudo-effective and if M is nef, then numerical abundance holds [45, 46] (these papers prove some results in this direction).

6.4. Boundedness of generalised pairs with fixed volume

Fix natural numbers d, p, a DCC set $\Phi \subset [0, 1]$ of rational numbers, and a positive rational number v. Consider projective generalised log canonical pairs $(X, B + M)$ of dimension d such that the coefficients of B are in Φ, the divisor pM' is Cartier where M' is the nef part of the pair $(X, B + M)$, and $K_X + B + M$ is ample with volume $(K_X + B + M)^d = v$. Then it is expected that such X form a bounded family. That is, one expects to find a natural number l depending only on d, p, Φ, v such that $l(K_X + B + M)$ is very ample hence defining an embedding of X into some fixed projective space $\mathbb{P}^n$ such the image of X under this embedding has bounded degree in $\mathbb{P}^n$. This statement is verified in dimension two in [20]. The case of usual pairs, that is, when $M' = 0$, was proved in [31] in any dimension.

This problem appears while studying projective pairs (Y, Δ) with semi-ample $K_Y + \Delta$ when we take $Y \to X$ to be the contraction defined by $K_Y + \Delta$ and let $(X, B + M)$ be given by the canonical bundle formula. Conjecturally, running the minimal model program on a log canonical pair ends with such a pair (Y, Δ) or a Mori fibre space. Thus, it is important to understand these pairs.

6.5. Classification of generalised pairs

One may ask to classify generalised pairs in given dimension. The extra information in generalised pairs as opposed to usual pairs makes the classification theory richer and more complex. For example let's look at the case of dimension one. Let $(X, B + M)$ be a projective generalised log canonical pair of dimension one. To classify such generalised pairs we need to fix some invariants. We can fix the degree of $K_X + B + M$ and also assume that lB, lM are both integral divisors for some fixed natural number l. In particular, the number of components of B is bounded. In fact, (X, B) varies in some bounded family of pairs. On the other hand, lM is a nef Cartier divisor whose degree takes only finitely many possibilities but $(X, \mathrm{Supp}(B + M))$ need not belong to a bounded family. Indeed even when $X = \mathbb{P}^1$ and $B = 0$, $\mathrm{Supp}\, M$ can have any number of components. One idea is to consider $(X, B + M)$ up to some kind of equivalence, e.g. up to isomorphism of (X, B) and up to linear equivalence of lM. For example if we fix X to be an elliptic curve and assume $B = 0$ and $M \equiv 0$ and then consider lM up to linear equivalence, then the classes

of such (X, M) are parametrised by the elements of $\mathrm{Pic}^0(X)$ (by sending (X, M) to lM). If we do not fix X, then we need a larger space to parametrise the classes of (X, M). It is not hard to imagine that going to higher dimension the classification problem gets quite complicated.

Another idea is to consider only the case $M \geq 0$. Indeed even in higher dimension this works in some interesting situations. For example, fix a natural number d and positive rational numbers c, v, and then consider projective semi-log canonical Calabi–Yau pairs (X, B) of dimension d and ample Weil divisors $M \geq 0$ such that $\frac{1}{c}B$ is a Weil divisor, the volume $\mathrm{vol}(M) = v$, and $(X, B + tM)$ is log canonical for some $t > 0$ (here t is not fixed). Then it is shown in [4, Theorem 1.10] that there is a projective coarse moduli space for such (X, B), M with the fixed data d, c, v. Note that M being ample, $(X, B + M)$ is a generalised pair.

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Caucher Birkar
DPMMS, Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WB, UK; c.birkar@dpmms.cam.ac.uk