

Area in real K3-surfaces

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Abstract. For a real K3-surface X , one can introduce areas of connected components of the real point set $\mathbb{R}X$ of X using a holomorphic symplectic form of X . These areas are defined up to simultaneous multiplication by a positive real number, so the areas of different components can be compared. In particular, it turns out that the area of a non-spherical component of $\mathbb{R}X$ is always greater than the area of any spherical component.

In this paper we explore further comparative restrictions on the area for real K3-surfaces admitting a suitable polarization of degree $2g - 2$ (where g is a positive integer) and such that $\mathbb{R}X$ has one non-spherical component and at least g spherical components. For this purpose we introduce and study the notion of simple Harnack curves in real K3-surfaces, generalizing planar simple Harnack curves from [8].

1. Introduction

A K3-surface X is a smooth simply connected complex surface admitting a holomorphic symplectic form, that is, a holomorphic 2-form Ω such that $\Omega \wedge \bar{\Omega}$ is a volume form. A K3-surface X is called *real* if it comes with an anti-holomorphic involution $\sigma: X \rightarrow X$. The fixed point set of σ is denoted with $\mathbb{R}X$ and called the *real locus* of X . If non-empty, $\mathbb{R}X$ is an orientable surface. All K3-surfaces are diffeomorphic, but their real loci may have different topological types, see [2, 10] for details.

There are finitely many possibilities for the topological type of $\mathbb{R}X$. For example, the surface $\mathbb{R}X$ may be diffeomorphic to the disjoint union of two tori. In this case, we call X a *hyperbolic* real K3-surface. The two components of the real locus of a hyperbolic K3-surface are homologous. If X is not hyperbolic, then $\mathbb{R}X$ has at most one non-spherical component. Denote by a the number of connected components of $\mathbb{R}X$, and denote by b the half of the first Betti number of $\mathbb{R}X$. As a corollary of the Smith–Thom inequality [13, 14], one obtains $a + b \leq 12$. There are further restrictions on a and b , namely we have

$$\begin{aligned} a - b &\equiv 0 \pmod{8} && \text{if } a + b = 12, \\ \text{and} &&& \\ a - b &\equiv \pm 1 \pmod{8} && \text{if } a + b = 11, \end{aligned}$$

according to the congruences on the Euler characteristic of the real locus of maximal (in the sense of the Smith–Thom inequality) and submaximal real algebraic surfaces (see [5, 7, 11]). The deformation classification of real K3-surfaces is, essentially, due to Nikulin [10].

Multiplying the holomorphic 2-form Ω by a non-zero complex number we may assume that the closed real 2-form $\alpha = \operatorname{Re} \Omega$ is invariant with respect to the involution σ while $\beta = \operatorname{Im} \Omega$ is anti-invariant. The form α is non-vanishing, and thus defines an orientation on $\mathbb{R}X$. Hence, α may be viewed as an area form on $\mathbb{R}X$, well-defined up to a real multiple. Thus, we may compare total areas of different components of $\mathbb{R}X$. If $K \subset \mathbb{R}X$ is a component, then we denote with $\operatorname{Area}(K) > 0$ the absolute value of $\int_K \Omega = \int_K \alpha$. For instance, if $\mathbb{R}X$ is hyperbolic, i.e., consists of two components T_1 and T_2 , each diffeomorphic to the 2-torus, then

$$\operatorname{Area}(T_1) = \operatorname{Area}(T_2)$$

since in this case the components T_1 and T_2 are homologous.

Suppose that there exists a smooth real curve $C \subset X$, that is, a smooth curve invariant with respect to the involution σ . All (not necessarily smooth) real curves in X linearly equivalent to C form a linear system. By the adjunction formula, all such curves constitute the real projective space $\mathbb{R}P^g$, where g is the genus of C . Such linear system is called *polarization* if $g > 0$ (we extend the standard terminology to the case $g = 1$). Accordingly, we say that the real K3-surface X is genus g *polarized* if such a linear system is fixed. The square of the homology class $[C] \in H_2(X)$ is equal to $2g - 2$ by the adjunction formula, so we also say that such a polarization is of *degree* $2g - 2$.

The real locus $\mathbb{R}C = C \cap \mathbb{R}X$ is a smooth 1-dimensional manifold and therefore diffeomorphic to the disjoint union of l circles. By the Harnack inequality [6], we have

$$l \leq g + 1.$$

Clearly, the homology class $[\mathbb{R}C] \in H_1(\mathbb{R}X; \mathbb{Z}_2)$ does not depend on the choice of the curve C in the polarization. We say that the polarization is *non-contractible* if $[\mathbb{R}C] \neq 0$. In such case, $\mathbb{R}X$ must contain a non-spherical component, which we denote with $N \subset \mathbb{R}X$. Unless X is hyperbolic, all other components of $\mathbb{R}X$ are spheres. We denote them with Σ_j , $j = 1, \dots, a - 1$.

The principal result of this paper is the following theorem.

Theorem 1. *Suppose that X is a real K3-surface admitting a non-contractible genus $g > 0$ polarization and such that $\mathbb{R}X$ has $a - 1 \geq g$ spherical components. Then, we have*

$$\operatorname{Area}(N) > \sum_{j=1}^{a-g} \operatorname{Area}(\Sigma_j),$$

where N is the non-spherical component of $\mathbb{R}X$ and Σ_j , $j = 1, \dots, a - 1$, are its spherical components.

2. Area inequalities from linear algebra

Consider a real K3-surface X (in this section, we do not assume that X is algebraic). Denote with $[A] \in H_2(X; \mathbb{R})$ the homology class dual to the real 2-form α . By the Hodge–Riemann relations we have

$$[A].[A] > 0.$$

Proposition 2.1. *If $N \subset \mathbb{R}X$ is a component of genus $b > 1$, and $\Sigma_1, \dots, \Sigma_k \subset \mathbb{R}X$ are the spherical components, then*

$$\text{Area}^2(N) \geq (b - 1) \left(\sum_{i=1}^k \text{Area}^2(\Sigma_i) + 2[A].[A] \right).$$

Proof. We have the decomposition

$$H_2(X; \mathbb{R}) = H_2^+(X; \mathbb{R}) \oplus H_2^-(X; \mathbb{R}),$$

where $H_2^+(X; \mathbb{R})$ stands for the σ -invariant part of the vector space $H_2(X; \mathbb{R})$ and $H_2^-(X; \mathbb{R})$ for its anti-invariant part. The decomposition is orthogonal with respect to the intersection form on $H_2(X; \mathbb{R})$. We have $[A] \in H_2^+(X; \mathbb{R})$, while the class dual to β belongs to $H_2^-(X; \mathbb{R})$. In addition, a Kähler form on X can be chosen in such a way that the class of this form belongs to $H_2^-(X; \mathbb{R})$. Thus, the intersection form restricted to $H_2^+(X; \mathbb{R})$ has one positive square.

Consider the subspace $V \subset H_2^+(X; \mathbb{R})$ generated by $[A], [N], [\Sigma_1], \dots, [\Sigma_k]$, where $N, \Sigma_1, \dots, \Sigma_k$ are oriented by α . The determinant of the intersection matrix of these vectors is

$$\mathcal{D} = \begin{vmatrix} [A].[A] & \text{Area}(N) & \text{Area}(\Sigma_1) & \text{Area}(\Sigma_2) & \dots & \text{Area}(\Sigma_k) \\ \text{Area}(N) & 2(b - 1) & 0 & 0 & \dots & 0 \\ \text{Area}(\Sigma_1) & 0 & -2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \text{Area}(\Sigma_k) & 0 & 0 & \dots & 0 & -2 \end{vmatrix},$$

since the self-intersection of a component of $\mathbb{R}X$ in X is minus the Euler characteristic of the component. We get

$$(-1)^{k+1} \mathcal{D} = 2^k \left(\text{Area}^2(N) - (b - 1) \sum_{i=1}^k \text{Area}^2(\Sigma_i) - 2(b - 1)[A].[A] \right) \geq 0,$$

since a diagonalization of the intersection form on $V \subset H_2^+(X; \mathbb{R})$ contains exactly one positive square. ■

Corollary 2.2. *If $N \subset \mathbb{R}X$ is a component of genus $b > 1$, and $\Sigma_1, \dots, \Sigma_k \subset \mathbb{R}X$ are spherical components, where $k \leq b - 1$, then*

$$\text{Area}(N) > \sum_{i=1}^k \text{Area}(\Sigma_i).$$

Proof. The statement is an immediate corollary of Proposition 2.1 and the inequality

$$n(x_1^2 + \dots + x_n^2) \geq (x_1 + \dots + x_n)^2$$

valid for any integer $n \geq 1$ and any real numbers x_1, \dots, x_n . ■

Remark 2.3. In the first version of our paper, the main result was a weaker version of Theorem 1, namely, only the inequality $\text{Area}(N) > \text{Area}(S_1)$ under similar hypotheses. In view of Corollary 2.2 this inequality is only non-trivial if the genus of N is 1. However, the referee of our paper suggested an elegant simple proof of this inequality based on so-called *Donaldson's trick* (which consists in changing the complex structure on a K3-surface so that an anti-holomorphic involution becomes holomorphic; see [3] or [2]) and applicable even for non-projective real K3-surfaces with a torus component. We are strongly indebted to the referee for this remark which has pushed us to find a stronger version of Theorem 1 considered in this paper. This stronger version still comes as an application of simple Harnack curves in a K3-surface that are studied in this paper.

The referee's suggestion mentioned in the previous remark can be generalized in the following way strengthening Corollary 2.2.

Proposition 2.4. *If $N \subset \mathbb{R}X$ is a component of genus $b \geq 1$, and $\Sigma_1, \dots, \Sigma_k \subset \mathbb{R}X$ are spherical components, where $k \leq b$, then*

$$\text{Area}(N) > \sum_{i=1}^k \text{Area}(\Sigma_i).$$

Proof. Let ω be a σ -anti-invariant Kähler form on X such that

$$[\omega]^2 = [\alpha]^2 = [\beta^2] > 0.$$

Consider the hyperkähler rotation that cyclically exchange the triple α, β, ω (in such a way that α becomes a new Kähler form), and denote the resulting K3-surface with X' . Since the new holomorphic 2-form on X' is σ -anti-invariant, the involution σ is holomorphic on X' and $\mathbb{R}X$ becomes a holomorphic curve in X' (for details, see [3] or [2]).

The component N is a holomorphic curve in X' and defines a polarisation of genus b of X' . Choose a point $p_i \in \Sigma_i$ for every $i = 1, \dots, k$. Since $k \leq b$, there exists a holomorphic curve N' from the polarisation such that N' passes through $\{p_i\}_{i=1}^k$. For any $i = 1, \dots, k$, the class $[N'] = [N] \in H_2(X)$ is orthogonal to $[\Sigma_i]$. Thus, Σ_i is an irreducible component of N' . It remains to notice that $\int_{C'} \alpha > 0$ for any holomorphic curve $C' \subset X'$. ■

3. Curves in X and their deformations

We take a closer look at the polarization (i.e., the linear system $|C| \approx \mathbb{P}^g$) defined by a smooth real curve $C \subset X$. Denote by $\mathcal{M} \subset |C|$ the space of all smooth curves linearly equivalent to C . It is well known that \mathcal{M} is a smooth manifold of dimension g . The tangent space $T_C \mathcal{M}$ consists of holomorphic normal vector fields to C in X . The non-degenerate holomorphic 2-form Ω provides an identification between $T_C \mathcal{M}$ and the space of holomorphic 1-forms on C (through plugging into Ω the normal vector field corresponding to a vector from $T_C \mathcal{M}$).

Let $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ be the universal covering consisting of pairs $\tilde{C}' = (C', \gamma)$, where $C' \in \mathcal{M}$ and γ is a homotopy class of a path connecting C and C' in \mathcal{M} . For $Z \in H_1(C)$ we define the map $I_Z: \tilde{\mathcal{M}} \rightarrow \mathbb{C}$ by

$$I_Z(\tilde{C}') = \int_{Z_\gamma} \Omega. \tag{1}$$

Here Z_γ is the surface spanned by a cycle from Z under the monodromy from γ . Since the closed 2-form Ω vanishes on any holomorphic curve (including C and C'), the value $I_Z(\tilde{C}') \in \mathbb{C}$ is well-defined.

Let $a_1, \dots, a_g \in H_1(C)$ be a maximal collection of a -cycles, i.e., a collection of linearly independent primitive elements with trivial pairwise intersection numbers.

Lemma 3.1. *The map*

$$I = (I_{a_1}, \dots, I_{a_g}): \tilde{\mathcal{M}} \rightarrow \mathbb{C}^g \tag{2}$$

is a local diffeomorphism.

Proof. Since I is a map between manifolds of the same dimension, it suffices to prove that its differential is injective. The kernel of dI at $\tilde{C}' \in \tilde{\mathcal{M}}$ consists of holomorphic forms on $C' \in \mathcal{M}$ with zero periods along a_1, \dots, a_g . By the Riemann theorem, any such holomorphic form on C' must vanish. ■

Remark 3.2. Lemma 3.1 is a shadow of the so-called Beauville–Mukai integrable system (see [1]) on the universal Jacobian over $\tilde{\mathcal{M}}$. The maps I_Z are the integrals of this system.

The system (a_1, \dots, a_g) of a -cycles can be represented with a system a of g pairwise disjoint simple loops on C . Their complement in C is a sphere with $2g$ holes. Let $\tilde{\mathcal{M}}^a$ be the space consisting of pairs $(C', [\gamma])$, such that $C' \in |C|$, is (at worst) a nodal curve, and $[\gamma]$ is a homotopy class of a path

$$\gamma: [0, 1] \rightarrow |C|$$

such that $\gamma(0) = C$, $\gamma(1) = C'$, and for all $t \in [0, 1]$ the curve $\gamma(t)$ is at worst a nodal curve whose vanishing cycles under the monodromy are represented by simple loops on C disjoint with the family a . Note that the forgetting map

$$\tilde{\mathcal{M}}^a \rightarrow |C|, \quad (C', [\gamma]) \mapsto C'$$

is a local diffeomorphism. The definition of the map (2) naturally extends to the map

$$I^a: \tilde{\mathcal{M}}^a \rightarrow \mathbb{C}^g. \tag{3}$$

Lemma 3.1 extends to the following proposition.

Proposition 3.3. *The map I^a is a local diffeomorphism.*

Proof. For a nodal curve $C' \in |C|$ the holomorphic 2-form gives an identification between $T_{C'}|C|$ and the space of meromorphic forms on the normalization of C' with at worst simple poles over the nodes such that the residues at the two preimages of the same node are opposite. The space of such forms is g -dimensional. The kernel of dI^a over C' consists of the forms with zero periods over $a_j, j = 1, \dots, g$, and thus trivial. ■

4. Simple Harnack curves and their degenerations

Simple Harnack curves in toric surfaces were introduced and studied in [8]. A toric surface $Y \supset (\mathbb{C}^\times)^2$ may be considered as a *log K3-surface*, or a K3-surface relative to its toric divisor $D = Y \setminus (\mathbb{C}^\times)^2$. Indeed, D is the pole divisor for the meromorphic extension of the holomorphic form $dz_1 \wedge dz_2$ on $(\mathbb{C}^\times)^2$. In this section, we define and study counterparts of these curves in (closed) K3-surfaces.

Recall that a smooth (and irreducible over \mathbb{C}) real curve C is called an *M-curve* (or a *maximal curve*), if the number of its real components is equal to one plus its genus (i.e., if it has the maximal number of real components allowed by the Harnack inequality). An M-curve C is *dividing*, i.e., $C \setminus \mathbb{R}C$ consists of two components interchanged by the involution of complex conjugation.

An orientation on the real locus $\mathbb{R}C$ of a dividing curve C is called the *complex orientation* if it comes as the boundary orientation of a component of $C \setminus \mathbb{R}C$, see [12]. Clearly there are two complex orientations on $\mathbb{R}C$ and they are opposite.

Definition 4.1. A smooth real M-curve $C \subset X$ is called *simple Harnack* if for any component $K \subset \mathbb{R}X \setminus \mathbb{R}C$ and any two distinct components $L, L' \subset \mathbb{R}C$ adjacent to K a complex orientation of L and L' can be extended to an orientation of K .

Note that this definition allows for two types of components $K \subset \mathbb{R}X \setminus \mathbb{R}C$. Either we have $\partial K = \partial \bar{K}$ for the closure $\bar{K} \subset \mathbb{R}X$, or the closure \bar{K} is the entire connected component $\Sigma \subset \mathbb{R}X$. In the first case, the complex orientations of the components of $\mathbb{R}C$ must alternate as in Figure 1.

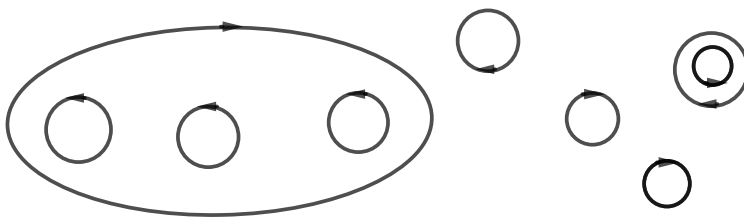


Figure 1. Orientations imposed by Definition 4.1.

In the second case, the component $\Sigma \subset \mathbb{R}X$ contains a single component $L \subset \mathbb{R}X$ and $[L] \neq 0 \in H_1(\mathbb{R}X; \mathbb{Z}_2)$.

Definition 4.2. A component $L \subset \mathbb{R}C$ is said to be *modifiable* or an *m-component* if $[L] \neq 0 \in H_1(\mathbb{R}X; \mathbb{Z}_2)$ and the component of $\mathbb{R}X$ containing L does not contain any other component of $\mathbb{R}C$.

Unless X is a hyperbolic K3-surface, $\mathbb{R}X$ has not more than one non-spherical component and thus $\mathbb{R}C$ may have not more than one *m-component*.

Remark 4.3. Simple Harnack curves in toric surfaces from [8] can be defined through a relative version of Definition 4.1. Namely, a real M-curve C in a real toric surface Y is simple Harnack if $C \setminus (\mathbb{C}^\times)^2 = \mathbb{R}C \setminus (\mathbb{C}^\times)^2$ (i.e., all intersection points of C and the toric divisor are real) and the orientation of $\partial K \subset (\mathbb{R}C \cap (\mathbb{C}^\times)^2)$ induced from $K \subset (\mathbb{R}^\times)^2 \setminus \mathbb{R}C$ agrees with an orientation of a component of $C \setminus \mathbb{R}C$.

Remark 4.4. It seems that Definition 4.1 also might be meaningful for the case when X is a surface different from a K3-surface. In a more general setting we add an assumption that each component of $\mathbb{R}X \setminus \mathbb{R}C$ is orientable (the condition that holds automatically in the case of K3-surfaces thanks to the non-vanishing 2-form Ω).

We say that a curve $C_0 \subset X$ is a *degeneration* of simple Harnack curves if there exists a continuous family $C_t \subset X, t \in [0, 1]$, such that C_t is a simple Harnack curve for every $t \in (0, 1]$. Clearly, C_0 is a real curve which may develop some singularities. Also, the degeneration C_0 does not have to be irreducible, or even reduced. It consists of several components while some of these components may be taken with multiplicity greater than 1 (multiple components). We refer to components of C_0 whose multiplicity is equal to 1 as simple components of C_0 .

Proposition 4.5. *Let $C_0 \subset X$ be a degeneration of simple Harnack curves. Then, a singular point of C_0 either belongs to a multiple component, or is an ordinary double point, i.e. a node.*

Furthermore, if a node of C_0 is non-real, then it corresponds to a transverse intersection point of two different simple components of C_0 . If a node p of C_0 is real, then p is either a solitary node (given in local analytic coordinates by $x^2 + y^2 = 0$), or corresponds to a transverse intersection of two different real simple components of C_0 . In the latter case, C_0 is a union of two real curves intersecting only at p ; the two real branches of $\mathbb{R}C_0$ at p come from the same connected component of $\mathbb{R}C_t, t > 0$, under degeneration.

Corollary 4.6. *If $C_0 \subset X$ is a reduced irreducible degeneration of simple Harnack curves, then all singular points of C_0 are solitary nodes.*

Proof of Proposition 4.5. Away from multiple components, each singular point $p \in C_0$ is isolated and (as a hypersurface singularity) can be described through vanishing cycles on a curve $C_t, t > 0$, which is a simple Harnack curve. Each vanishing cycle $Z_t \subset C_t$ corresponds to a critical point of a morsification of p (i.e., a holomorphic function with

non-degenerate critical points which approximates the local equation of C_0 near p). Definition and properties of vanishing cycles can be found in [9]. To find an appropriate collection of conjugation-invariant vanishing cycles we follow the procedure below.

Near a real singular point p of C_0 the family of curves C_t can be given as the zero set of a family of holomorphic functions $f_t: U \rightarrow \mathbb{C}$ on a small neighbourhood $p \in U \subset X$. Here U can be chosen to be σ -invariant with the contractible real part $\mathbb{R}U = U \cap \mathbb{R}X$, while f_t can be chosen to be real (i.e., $\text{conj} \circ f_t = f_t \circ \sigma$, where $\text{conj}: \mathbb{C} \rightarrow \mathbb{C}$ is the complex conjugation). Multiplying by (non-vanishing on U) holomorphic functions if needed, we may assume that $f_t, t > 0$, is a complex Morse function (i.e., its critical points are isolated and have non-degenerate Hessians). Similarly, we may also assume for f_t that the images of different critical points are different and that the image of a non-real critical point is not real.

The multiplication trick also allows us to assume that the restriction $f_t|_{\mathbb{R}U}$ is a generic Morse function, i.e., that the stable and unstable manifolds for different critical points are transverse. Suppose that there exist two critical points with positive critical values, with indices different by 1, and such that these points are connected with a gradient trajectory. Then, such a gradient trajectory must be unique. Indeed, since the index of one of the critical points must be one, there could be not more than two such trajectories. However, existence of two trajectories would imply a non-trivial mod 2 homology cycle in $\mathbb{R}U$ which is impossible. If two critical points with positive values are connected with a single trajectory, then these critical points are removable. Multiplying f_t by an appropriate non-vanishing real function we can make such a pair of critical points into a complex conjugate pair. Thus, inductively, we may assume that no pair of critical points with positive critical values can be connected with a gradient trajectory. (Note that the points of indices 0 and 2 cannot be connected in this way, since in the absence of trajectories to index 1 points it would imply an S^2 -component for $\mathbb{R}U$.)

The critical points of f_t for small $t > 0$ can be thought of as the result of perturbation of the singular point p for f_0 . In particular, the number of critical points of f_t coincides with the Milnor number μ_p of the singularity p . The set $\Pi_t \subset \mathbb{C}$ of critical values of f_t is conj -invariant and close to zero. Let us connect the points of Π_t with 0 by a conj -invariant collection Γ_p of μ_p smooth embedded paths (the paths connecting real points of Π_t to 0 may contain each other).

Let γ_t be one of these paths. Its inverse image $f_t^{-1}(\gamma_t) \subset U$ is a hypersurface. A σ -anti-invariant Kähler symplectic form on X has a 1-dimensional radical in the tangent space to $f_t^{-1}(\gamma_t)$. With its help a tangent vector field to γ_t , oriented towards the critical value, canonically lifts to a vector field in $f_t^{-1}(\gamma_t)$ vanishing at a critical point of f_t . We define the membrane $M_t \subset X$ as the stable manifold of this point. Since the critical points of f_t are Morse, and no trajectories over our paths may connect critical points, M_t is an embedded disk. The corresponding vanishing cycle $Z_t \subset C_t$ is the boundary of this disk M_t . We have $M_t \cap C_t = Z_t$, while the membrane M_t is never tangent to the curve C_t along Z_t . It is an embedded disk in X of self-intersection -1 (to define the

self-intersection of a membrane we use a normal vector field to Z_t in C_t as the boundary framing).

For non-real singular points $p \in C_0$ the construction of the cycles Z_t and the membranes M_t is similar but locally we do not have to worry about the complex conjugation invariance. Instead we use $\sigma(Z_t)$ and $\sigma(M_t)$ for the singular point $\sigma(p) \in C_0$.

For a given singular point $p \in C_0$, denote with $A_t \subset C_t$ the union of all vanishing cycles in C_t , and with $B_t \subset X$ the union of all their membranes. Both spaces A_t and B_t are connected. Their union over all singular points of C_0 is σ -invariant. The vanishing cycles from A_t intersect transversely. Two cycles are either disjoint or intersect in a single point. The dual graph of the vanishing cycles from A_t cannot have cycles, see [9].

Suppose that $p \notin \mathbb{R}C_0$. Then $A_t \subset C_t \setminus \mathbb{R}C_t$, but each component of $C_t \setminus \mathbb{R}C_t$ is of genus 0 since C_t is an M-curve. Thus, A_t consists of a single vanishing cycle, and p is a Morse point. Furthermore, $A_t \cup \sigma(A_t)$ separates C_t into several connected components, so p must be a transverse intersection point of distinct components from C_0 .

Suppose that $p \in \mathbb{R}C_0$. Then, the tree of vanishing cycles of A_t is σ -invariant, so it must have an invariant vertex or an invariant edge. However, an invariant edge would correspond to a transverse intersection of vanishing cycles. If these cycles are not real, then they intersect $\mathbb{R}C_t$ transversely in a single point which is impossible since $C_t \setminus \mathbb{R}C_t$ is disconnected. Thus, A_t possesses at least one σ -invariant vanishing cycle Z' whose membrane M' is also σ -invariant.

Suppose that σ acts on Z' non-trivially. Then $Z' \cap \mathbb{R}C_t$ consists of two points, while $\gamma = M' \cap \mathbb{R}X$ is a path connecting these points and transversal to $\mathbb{R}C_t$ at the endpoints. Let $M' \subset X$, $\partial M' \subset C_t$, be a small perturbation of the membrane M' such that $\partial M'$ and $\partial M'$ are disjoint. Let $\gamma' = M' \cap \mathbb{R}X$. The parity of the self-intersection of M' coincides with the intersection number of γ and γ' since all other points of $M' \cap M'$ come in pairs. This parity in its turn is determined by the displacement of $\partial\gamma' \subset \mathbb{R}C_t$ with respect to $\partial\gamma \subset \mathbb{R}C_t$. Let us enhance $\mathbb{R}C_t$ with the boundary orientation of one of the halves of $C_t \setminus \mathbb{R}C_t$. Since $\partial M' \cap \partial M' = \emptyset$, one of the points of $\partial\gamma$ must move in the direction of this orientation, while the other one moves contrary to this direction. Definition 4.1 implies that the intersection number of γ and γ' is even whenever γ connects two different components of $\mathbb{R}C_t$. However, this is incompatible with the odd self-intersection of M' .

If γ connects a component $L \subset \mathbb{R}C_t$ with itself, then $C_t \setminus Z'$ is disconnected. Thus, C_0 is a union of two real curves intersecting only at p . Furthermore, $A_t = Z'$, since otherwise there must be another cycle $Z' \subset A_t$ intersecting Z' transversally at a single point. In this case p is an ordinary double point with two real branches which is a transverse intersection point of two distinct components of C_0 .

Any other real vanishing cycle $Z' \subset C_t$ must be point-wise preserved by σ . Suppose that Z' intersects another cycle Z' in A_t . The cycle Z' cannot be point-wise preserved since it intersects Z' at a single point. Thus, Z' is imaginary. Then, since Z' and $\sigma(Z')$ are transverse and $C_t \setminus \mathbb{R}C_t$ is disconnected, $Z' \cap \sigma(Z')$ consists at least of two points which is impossible. Therefore, any solitary real singular point $p \in C_0$ has a unique vanishing cycle corresponding to an oval of $\mathbb{R}C_t$, which implies that p is a solitary node. ■

Remark 4.7. The proof of Proposition 4.5 is based on concordance, ensured by a real vanishing cycle, of complex orientations of a dividing real curve. This concordance is a well-known phenomenon in real algebraic geometry, responsible, in particular, for *Fielder’s orientation alternation*, see [4].

5. Deformations of simple Harnack curves in K3-surfaces

Assume that the surface X is not hyperbolic. For a simple Harnack curve $C \subset X$ we choose an order on the components $L_0, \dots, L_g \subset \mathbb{R}C$, as well as their orientations compatible with a half of $C \setminus \mathbb{R}C$. Real curves in X linearly equivalent to C form the real part $\mathbb{R}|C| \approx \mathbb{R}\mathbb{P}^g$ of the projective space $|C| \approx \mathbb{P}^g$. The homology classes $a_j = [L_j] \in H_1(C)$, $j = 1, \dots, g$, form a maximal collection of a -cycles making the map (3) well-defined. We denote with $\widetilde{\mathbb{R}\mathcal{M}^a}$ the fixed locus of the involution induced by σ on $\widetilde{\mathcal{M}^a}$, the source of map (3).

Consider the subspace

$$\widetilde{\mathbb{R}\mathcal{M}_C^i} \subset \widetilde{\mathbb{R}\mathcal{M}^a}$$

consisting of pairs $(C', [\gamma])$, $\gamma: [0, 1] \rightarrow \mathbb{R}|C|$, $\gamma(0) = C$, $\gamma(1) = C'$, where, for any $t \in [0, 1]$, the real curve $\gamma(t)$ is at worst nodal and any non-singular real curve belonging to $\mathbb{R}|C|$ and sufficiently close to $\gamma(t)$ is simple Harnack.

Remark 5.1. If $(C', [\gamma]) \in \widetilde{\mathbb{R}\mathcal{M}_C^i}$, then, for any $t \in [0, 1]$, the curve $\gamma(t)$ does not have real solitary nodes.

Proposition 5.2. *Suppose that $L_j \subset \mathbb{R}C$ is not an m -component. Then, during the deformation γ , the component L_j remains non-singular, i.e., we may consistently distinguish a smooth real component in $\gamma(t)$, $t \in [0, 1]$, coinciding with L_j for $t = 0$.*

Proof. Let us, first, show the statement assuming that $\gamma(t)$ is a non-singular curve for any $t \in [0, 1]$. If $p \in \gamma(1)$ is a real singular point, then by Proposition 4.5, the point p corresponds to a transversal intersection of two different real irreducible components of the reducible curve $\gamma(1)$, and p is a unique intersection point of these components. Furthermore, the vanishing cycle of p connects a real component $L(1 - \varepsilon)$ of $\gamma(1 - \varepsilon)$, $\varepsilon > 0$, with itself. Thus,

$$[L(1 - \varepsilon)] \neq 0 \in H_1(\mathbb{R}X; \mathbb{Z}_2),$$

and Definition 4.1 implies that the non-spherical component $N \subset \mathbb{R}X$ containing $L(1 - \varepsilon)$ does not contain any other component of the real part of the curve $\gamma(1 - \varepsilon)$, that is, $L(1 - \varepsilon)$ is an m -component. This implies the statement of the proposition in the case considered. Moreover, g connected components of the real part of $\gamma(1 - \varepsilon)$ are contained in the union of spherical components of $\mathbb{R}X$.

Consider now the general case. For any $t \in [0, 1]$, the curve $\gamma(t)$ is a degeneration of simple Harnack curves. Thus, the particular case above implies that all singular points

of $\gamma(t)$ belong to N , and g connected components of the real part of $\gamma(t)$ are contained in the union of spherical components of $\mathbb{R}X$. This implies the statement. ■

We may reorder the components of $\mathbb{R}C$ so that all components except possibly for L_0 are not m -components. Thus, for any oriented component $L_j \subset \mathbb{R}C$, $j = 1, \dots, g$, the map (3) restricted to $\mathbb{R}\mathcal{M}_C^i$ induces the map

$$\mathbb{R}I_{L_j}: \widetilde{\mathbb{R}\mathcal{M}_C^i} \rightarrow \mathbb{R}. \tag{4}$$

We define

$$I_{\mathbb{R}C} = (\mathbb{R}I_{L_1}, \dots, \mathbb{R}I_{L_g}): \widetilde{\mathbb{R}\mathcal{M}_C^i} \rightarrow \mathbb{R}^g. \tag{5}$$

A component $L_j \subset \mathbb{R}C$, $j = 0, \dots, g$, is either non-contractible (i.e., $[L_j] \neq 0 \in H_1(\mathbb{R}X; \mathbb{Z}_2)$) or such that $\Sigma_j \setminus L_j = \Sigma_j^+ \cup \Sigma_j^-$ consists of two components. Here, Σ_j is the component of $\mathbb{R}X$ containing the component L_j . This component is oriented by the 2-form Ω . We denote with Σ_j^+ the component of $\Sigma_j \setminus L_j$ whose boundary orientation agrees with the chosen complex orientation of $\mathbb{R}C$ and with Σ_j^- the other one. Put

$$s_j^+ = \int_{\Sigma_j^+} \Omega = \text{Area}(\Sigma_j^+), \quad s_j^- = - \int_{\Sigma_j^-} \Omega = - \text{Area}(\Sigma_j^-).$$

Clearly, $s_j^+ - s_j^- = \text{Area}(\Sigma_j)$. If L_j is non-contractible, we put $s_j^+ = \infty, s_j^- = -\infty$. Let

$$\Delta = \left\{ (x_1, \dots, x_g) \mid s_0^- < - \sum_{j=1}^g x_j < s_0^+, s_j^- < x_j < s_j^+ \right\} \subset \mathbb{R}^g.$$

Proposition 5.3. *The map*

$$I_{\mathbb{R}C}: \widetilde{\mathbb{R}\mathcal{M}_C^i} \rightarrow \mathbb{R}^g$$

is a local diffeomorphism whose image is contained in Δ .

Proof. The map $I_{\mathbb{R}C}$ is a local diffeomorphism by Proposition 3.3. Note that for a holomorphic curve $C' \subset X$ the area of a membrane whose boundary is contained in $C \cup C'$ depends only on the class of the membrane in $H_2(X, C \cup C')$. In particular, to compute $I_{\mathbb{R}C}$ we may use the membranes contained in $\mathbb{R}X$. We have $s_j^- < \mathbb{R}I_{L_j}(C', \gamma) < s_j^+$ since the corresponding oval of $\mathbb{R}C'$ bounds two membranes of areas $s_j^+ - \mathbb{R}I_{L_j}(C', \gamma)$ and $\mathbb{R}I_{L_j}(C', \gamma) - s_j^-$, so these differences must be positive. If $[L_0] = 0 \in H_1(\mathbb{R}X; \mathbb{Z}_2)$, then the corresponding ovals of the real curves from the deformation γ cannot develop singularities by Proposition 5.2. Since $[L_0] + \dots + [L_g] = 0 \in H_1(C)$, the corresponding oval of C bounds the membranes of area

$$s_0^+ + \sum_{j=1}^g \mathbb{R}I_{L_j}(C', \gamma) \quad \text{and} \quad - \sum_{j=1}^g \mathbb{R}I_{L_j}(C', \gamma) - s_0^-,$$

so these quantities are also positive. ■

6. Proof of Theorem 1

For any algebraic curve D (not necessarily irreducible or reduced), the *multiplicity* of D is the minimum among the multiplicities of irreducible components of D .

Lemma 6.1. *Let $D \subset X$ be a real algebraic curve. Assume that D is either connected or consists of two connected non-real conjugated curves. Put $d = [D] \in H_2(X; \mathbb{Z})$, and denote by k the multiplicity of D . Then, the number of connected components of the real part $\mathbb{R}D$ of D is at most $2 + d^2/2k^2$.*

In particular, $2 + d^2/2k^2 \geq 0$ under the hypotheses of the lemma.

Proof. Assume, first, that D is irreducible over \mathbb{R} (but not necessarily reduced). Let D' be the reduced curve having the same set of points as D . The curve D' is real and $[D] = k[D']$. The required inequalities are equivalent for D and D' . If D' is irreducible over \mathbb{C} , the required inequality for D' is a corollary of the Harnack inequality and the fact that the number of solitary real points of D' is bounded from above by the difference between the arithmetic and geometric genera of D' . Suppose that D' has two irreducible components over \mathbb{C} (exchanged by the anti-holomorphic involution of X), and denote these components by D'_1 and D'_2 . Put $d'_1 = [D'_1] \in H_2(X; \mathbb{Z})$ and $d'_2 = [D'_2] \in H_2(X; \mathbb{Z})$. We have

$$\frac{[D']^2}{2} + 2 = \frac{(d'_1)^2}{2} + \frac{(d'_2)^2}{2} + 2 + d'_1 d'_2 \geq d'_1 d'_2,$$

since $(d'_i)^2 \geq -2$, $i = 1, 2$. The number of real points of D' is bounded from above by $d'_1 d'_2$. This implies the required inequality for D' , and thus, for D .

Assume now that $D = D_1 \cup D_2$, where D_1 and D_2 are two real curves without common components. Assume, in addition, that each of these two curves is either connected or consists of two connected non-real conjugated curves, and that the required inequality is true for D_1 and D_2 . Put $d_i = [D_i] \in H_2(X; \mathbb{Z})$, $i = 1, 2$, and denote by k_i the multiplicity of D_i , $i = 1, 2$. Denote by n the number of intersection points of D_1 and D_2 . Suppose that $k_1 \leq k_2$. In this case, $k = k_1$. We have

$$\frac{d^2}{2k^2} + 2 \geq \frac{d_1^2}{2k_1^2} + \frac{d_2^2}{2k_2^2} + \frac{d_1 d_2}{k_1} + 2 \geq \frac{d_1^2}{2k_1^2} + \frac{d_2^2}{2k_2^2} + \frac{nk_2}{k_1} + 2. \tag{6}$$

If $n \geq 2$, then

$$\frac{d_1^2}{2k_1^2} + \frac{d_2^2}{2k_2^2} + \frac{nk_2}{k_1} + 2 \geq \left(\frac{d_1^2}{2k_1^2} + 2 \right) + \left(\frac{d_2^2}{2k_2^2} + 2 \right),$$

and the number of connected components of $\mathbb{R}D$ is at most $d^2/2k^2 + 2$. If $n = 1$, then

$$\frac{d_1^2}{2k_1^2} + \frac{d_2^2}{2k_2^2} + \frac{nk_2}{k_1} + 2 \geq \left(\frac{d_1^2}{2k_1^2} + 2 \right) + \left(\frac{d_2^2}{2k_2^2} + 2 \right) - 1.$$

In this case, the only intersection point of D_1 and D_2 is real, and the number of connected components of $\mathbb{R}D$ is at most

$$\left(\frac{d_1^2}{2k_1^2} + 2\right) + \left(\frac{d_2^2}{2k_2^2} + 2\right) - 1 \leq \frac{d^2}{2k^2} + 2. \quad \blacksquare$$

We say that a real algebraic curve $D \subset X$ is *r-maximal* if D is either connected or consists of two connected non-real conjugated curves, and the number of connected components of the real part $\mathbb{R}D$ of D is equal to $d^2/2k^2 + 2$, where $d = [D] \in H_2(X; \mathbb{Z})$ and k is the multiplicity of D .

Lemma 6.2. *Let $D = D_1 \cup D_2 \subset X$ be an r-maximal real algebraic curve, where D_1 and D_2 are real curves such that each of them is either connected or consists of two connected non-real conjugated curves. Then, the curves D_1 and D_2 are r-maximal and have the same multiplicity.*

Proof. Put $d_i = [D_i] \in H_2(X; \mathbb{Z})$, $i = 1, 2$, and denote by k_i the multiplicity of D_i , $i = 1, 2$. Denote by n the number of intersection points of D_1 and D_2 . The *r*-maximality of D and the inequalities (6) imply that $k_1 = k_2$ and the curves D_1 and D_2 are *r*-maximal. \blacksquare

Recall that in the linear system of genus g polarizing the K3-surface X we may choose a curve passing through arbitrary g points. Let C be the real curve passing through g points on g distinct spherical components $\Sigma_1, \dots, \Sigma_g$ of $\mathbb{R}X$. Thus, $\mathbb{R}C$ contains at least g components L_1, \dots, L_g at these spherical components. Slightly perturbing the curve C if needed we may assume that C is smooth. Since the polarization is non-contractible, the real locus $\mathbb{R}C$ must also contain a non-contractible component $L_0 \subset \mathbb{R}C$ at the non-contractible component $N \subset \mathbb{R}X$. Thus, C is a simple Harnack curve.

Lemma 6.3. *Let $C' \in \mathbb{R}|C|$ be a connected curve intersecting each connected component $\Sigma_1, \dots, \Sigma_g$. If $g \geq 2$, the curve C' is reduced.*

Proof. Note that C' necessarily intersects N . In addition, $[C']^2 = [C]^2 = 2g - 2 > 0$ (since $g \geq 2$). Thus, C' is *r*-maximal and of multiplicity 1. Lemma 6.2 implies that all irreducible components of C' are of multiplicity 1, that is, C' is reduced. \blacksquare

Assume that $g \geq 2$. Choose a complex orientation of $\mathbb{R}C$. For every $j = 1, \dots, g$, the connected component Σ_j is oriented by the 2-form Ω and is divided by the oval L_j in two disks Σ_j^+ and Σ_j^- , where Σ_j^+ is the disk whose boundary orientation agrees with the chosen complex orientation of $\mathbb{R}C$. Denote by s_j^+ and $-s_j^-$ the areas of the disks Σ_j^+ and Σ_j^- .

Lemma 6.3 and Propositions 4.5, 5.3 imply that the inverse image of the line

$$\left\{ \left(\frac{s_1^+ + s_1^-}{2}, \dots, \frac{s_{g-1}^+ + s_{g-1}^-}{2}, u \right) \mid u \in \mathbb{R} \right\} \subset \mathbb{R}^g$$

under the map

$$I_{\mathbb{R}C} : \widetilde{\mathbb{R}\mathcal{M}_C^i} \rightarrow \mathbb{R}^g$$

is a segment $\mathcal{S} \subset \widetilde{\mathbb{R}\mathcal{M}}_C^i$ whose closure in $\widetilde{\mathbb{R}\mathcal{M}}^a$ has two extremal points corresponding to nodal curves C_+ and C_- ; each of the curves C_+ and C_- has a solitary double point in Σ_g . Furthermore, according to Proposition 5.2, the curves C_+ and C_- do not have other singular points in $\mathbb{R}X \setminus N$, and any curve corresponding to a point of \mathcal{S} does not have singular points in $\mathbb{R}X \setminus N$. Let D' and D'' be any two curves corresponding to distinct points of \mathcal{S} . The ovals of D' and D'' at the spherical components $\Sigma_1, \dots, \Sigma_{g-1}$ divide the corresponding spheres into disks of equal areas, so they intersect in at least $2(g - 1)$ points at these components. By the Bézout theorem,

$$D' \cap D'' \cap (N \cup \Sigma_g) = \emptyset.$$

Thus, $(C_+ \cap N) \cup (C_- \cap N)$ bounds a proper compact subsurface of N of area

$$s_g^+ - s_g^- = \text{Area}(\Sigma_g).$$

Similarly, for every connected component $\Sigma \subset \mathbb{R}X$ different from $\Sigma_1, \dots, \Sigma_{g-1}, N$, there is a proper compact subsurface $R \subset N$ whose area is equal to the area of Σ . The Bézout theorem implies that all these subsurfaces R are pairwise disjoint. This proves the statement of the theorem in the case $g \geq 2$.

Assume now that $g = 1$. In this case, the linear system $|C|$ is 1-dimensional, $\mathbb{R}|C| \simeq \mathbb{R}\mathbb{P}^1$, and the previous arguments can be easily adapted. Through any point of X one can trace a unique curve belonging to $|C|$. This defines a projection $\pi_{\mathbb{R}}: \mathbb{R}X \rightarrow \mathbb{R}\mathbb{P}^1$. Note that $\pi_{\mathbb{R}}(N)$ coincides with $\mathbb{R}\mathbb{P}^1$. The image under $\pi_{\mathbb{R}}$ of any spherical component Σ_j of $\mathbb{R}X$ is a closed segment, and all such segments are pairwise disjoint. Each segment $\pi_{\mathbb{R}}(\Sigma_j)$ gives rise to a proper compact subsurface in N (the intersection of N with the inverse image under $\pi_{\mathbb{R}}$ of the segment) whose area is equal to the area of Σ_j , and all these subsurfaces are pairwise disjoint. ■

7. Simple Harnack curves in K3 surfaces: further directions and questions

For the proof of Theorem 1 we have used simple Harnack curves $C \subset X$ of rather special type: each component of $\mathbb{R}X$ contained not more than one component of the curve $\mathbb{R}C$. Under this assumption a real curve is a simple Harnack whenever it is an M-curve.

We finish the paper by taking a look at more general simple Harnack curves. Namely, we assume that $C \subset X$ is a simple Harnack curve, and X is a real K3-surface which is not hyperbolic. Then the locus of the M-curve $\mathbb{R}C$ has not more than one m -component. We order the components $L_j, j = 0, \dots, g$ of $\mathbb{R}C$ so that all of them, except possibly L_0 , are not m -components. Thus the map $I_{\mathbb{R}C}$ from Proposition 5.3 is well-defined.

Note that for a degeneration $C_t, t \in [0, 1]$, of simple Harnack curves such that $C_1 = C$, any curve $C_t, t > 0$, is naturally identified with a point in $\mathbb{R}\mathcal{M}_C^i$.

Definition 7.1. A simple Harnack curve $C \subset X$ is called *minimal* if for any degeneration $C_t, t \in [0, 1]$, of simple Harnack curves such that $C_1 = C$ and $\lim_{t \rightarrow 0} I_{\mathbb{R}C}(C_t) \in \Delta$, the curve C_0 is reduced and irreducible.

Example 7.2. If X does not contain embedded curves of genus less than $g \geq 2$ (in particular, it does not contain (-2) -curves), then any simple Harnack curve of genus g is a minimal simple Harnack curve.

Proposition 7.3. *If C is a minimal simple Harnack curve, then the map*

$$I_{\mathbb{R}C}: \widetilde{\mathbb{R}\mathcal{M}_C^i} \rightarrow \Delta \tag{7}$$

from Proposition 5.3 is a diffeomorphism.

Proof. Since Δ is simply connected and $I_{\mathbb{R}C}$ is a local diffeomorphism by Proposition 5.3, it suffices to prove that (7) is proper. The space $\mathbb{R}\mathcal{M}_C^i$ is an open manifold covering a subset of $|C|$. By Proposition 4.5, for any degeneration $C_t, t \in [0, 1]$, of simple Harnack curves such that $C_1 = C$ and $\lim_{t \rightarrow 0} I_{\mathbb{R}C}(C_t) \in \Delta$, the limit curve C_0 is smooth. ■

We say that the components L_{j-}, L_{j_0}, L_{j+} nest if they are contractible (i.e., $[L_{j-}] = [L_{j_0}] = [L_{j+}] = 0 \in H_1(\mathbb{R}X; \mathbb{Z}_2)$), belong to the same component $\Sigma_j \subset \mathbb{R}X$, and one component of $\Sigma_j \setminus L_{j_0}$ contains L_{j-} while the other one contains L_{j+} , see Figure 2.

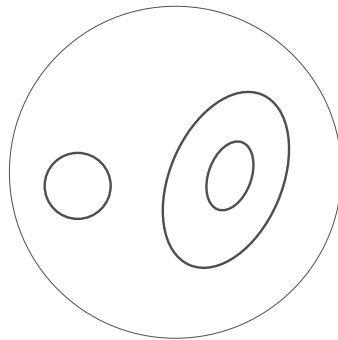


Figure 2. Three nesting ovals in a spherical component of $\mathbb{R}X$.

Proposition 7.4. *No three components of a minimal simple Harnack curve C can nest.*

Proof. Passing to different nesting components if needed we may assume that L_{j+} and L_{j_0} (resp. L_{j+} and L_{j_0}) are adjacent to the same component $K_{j+} \subset \mathbb{R}X \setminus \mathbb{R}C$ (resp. $K_{j-} \subset \mathbb{R}X \setminus \mathbb{R}C$). Then Definition 4.1 implies that the complex orientations of L_{j-}, L_{j_0}, L_{j+} alternate. Renumbering the components of $\mathbb{R}C$ if needed we may assume that $L_j^+ = L_g$ and $L_j^- = L_{g-1}$. Also, we may assume that the boundary orientation of ∂K_{j+} induced by Ω agree with the complex orientation of $\mathbb{R}C$.

Consider the inverse image of the interval

$$\{(0, \dots, 0, -u, u) \in \Delta \mid 0 \leq u < s_g^+\} \subset \Delta$$

under the diffeomorphism (7). It corresponds to the elements of $\widetilde{\mathbb{R}\mathcal{M}_C^i}$ such that the area of K_{j+} becomes smaller by u . For

$$u > \text{Area}(K_{j+}) + \text{Area}(K_{j-}) < s_j^+$$

we get self-contradicting conditions for the resulting smooth curve in Σ_j . ■

Proposition 7.5. *If a minimal simple Harnack curve C has an m -component, then each component $\Sigma \subset \mathbb{R}X$ contains not more than one component of $\mathbb{R}C$.*

Proof. Let L_0 be the m -component. Then, the component $N \subset \mathbb{R}X$ containing L_0 is disjoint from the other components of $\mathbb{R}C$ by the definition of the m -component. Since L_0 is not contractible, we have $s_0^- = -\infty, s_0^+ = +\infty$, thus Δ is a cube. Suppose that a component $K \subset \mathbb{R}X \setminus \mathbb{R}C$ is adjacent to L_1 and L_2 . Considering the inverse image of the line $\{(0, u, 0, \dots, 0) \mid u \in \mathbb{R}\} \subset \mathbb{R}^g$ under (7) we get a contradiction at the value $u = \pm \text{Area}(K)$ as in the proof of Proposition 7.4. ■

Proposition 7.6. *If a minimal simple Harnack curve C does not have an m -component, but has a non-contractible component L contained in the component $N \subset \mathbb{R}X$, then N contain another non-contractible component $L' \subset \mathbb{R}C$ homologous to L . Furthermore, in this case we have $\mathbb{R}C \cap N = L \cup L'$.*

Proof. If $(N \cap \mathbb{R}C) \setminus L$ consists of contractible components, then we have a contradiction with Definition 4.1. Thus, there must be another non-contractible component $L' \subset \mathbb{R}C$ on N . Without loss of generality, we may assume that $L = L_0, L' = L_1$. If there exists yet another component $L'' \subset \mathbb{R}C$ on N (contractible or not), then considering the inverse image of the ray $\{(u, 0, \dots, 0) \mid u \geq 0\} \subset \mathbb{R}^g$ under (7) we get a contradiction as in the proofs of Propositions 7.4 and 7.5. If L and L' are the only components on N , then they must be homologous by Definition 4.1. ■

The following example shows that a simple Harnack curve can have two homologous non-contractible components.

Example 7.7. Recall that a real K3-surface X polarized by genus 2 admits a double covering $\pi: X \rightarrow \mathbb{P}^2$ branching along a real curve $B \subset \mathbb{P}^2$ of degree 6. Denote with $\rho: X \rightarrow X$ the involution of deck transformation of π . Since π is a real map, the holomorphic involutions ρ and the anti-holomorphic involution σ commute. Therefore, the composition $\rho \circ \sigma$ is an antiholomorphic involution on X . Denote with $\mathbb{R}X'$ its fixed locus.

Let X be the K3-surface obtained as the double covering $\pi: X \rightarrow \mathbb{P}^2$ branched along a real sextic $B \subset \mathbb{P}^2$ whose real locus is depicted at Figure 3 (more precisely, the figure shows the isotopy type of the real locus and the position of ovals of the curve with respect to two auxiliary straight lines). The involution of complex conjugation on \mathbb{P}^2 can be lifted

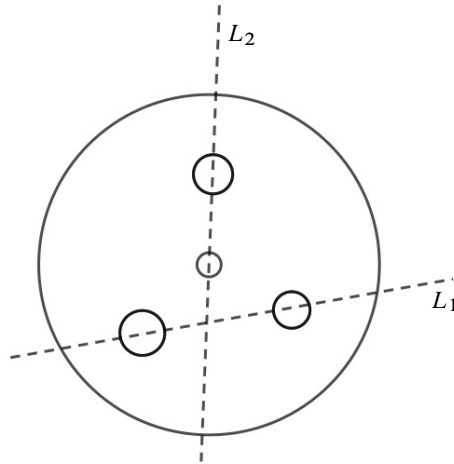


Figure 3. Two hyperplane sections in a genus 2 polarized K3-surface.

to X in two ways differing by the deck transformation ρ . We may assume that $\mathbb{R}X$ covers the non-orientable half of $\mathbb{R}\mathbb{P}^2 \setminus \mathbb{R}B$. Then, $\mathbb{R}X$ is homeomorphic to the disjoint union of a torus and four spheres, while $\mathbb{R}X'$ is homeomorphic to a surface of genus 4.

The equation $z^2 = f(x_0, x_1, x_2)$, where f is a homogeneous polynomial defining the curve B , gives an embedding of X into the weighted projective space $\mathbb{P}(1, 1, 1, 3)$. Accordingly, the inverse image $A_L = \pi^{-1}(L)$ of a real line $L \subset \mathbb{P}^2$ sits in the weighted projective plane $\mathbb{P}(1, 1, 3)$ embedded in $\mathbb{P}(1, 1, 1, 3)$. Suppose that L intersects B in 6 distinct real points. Then, $\mathbb{R}A_L = A_L \cap \mathbb{R}X$ is an M-curve. Its three ovals in the real part of $\mathbb{P}(1, 1, 3)$ have alternate complex orientations (with respect to the pencil of lines passing through the singular point of $\mathbb{P}(1, 1, 3)$) by Fiedler’s orientation alternation, cf. Remark 4.7. Furthermore, $\mathbb{R}A'_L = A_L \cap \mathbb{R}X'$ is also an M-curve, while complex orientations of $\mathbb{R}A_L$ and $\mathbb{R}A'_L$ agree over any point of $B \cap A_L$ after multiplication by i .

Consider the lines L_1 and L_2 from Figure 3. The intersection $A_{L_1} \cap A_{L_2}$ consists of two points from $\mathbb{R}X'$. The complement $(A_{L_1} \cup A_{L_2}) \setminus (\mathbb{R}A_{L_1} \cup \mathbb{R}A_{L_2})$ consists of two connected components each obtained as a bouquet of two planar domains. Accordingly, the real curve C obtained by a small perturbation of $A_{L_1} \cup A_{L_2}$ is an M-curve. Its real part $\mathbb{R}C = C \cap \mathbb{R}X$ has two non-contractible ovals at the torus component of $\mathbb{R}X$ and an oval at each spherical component of $\mathbb{R}X$. The complex orientation of $\mathbb{R}C$ is determined by the complex orientations of $\mathbb{R}A_{L_j}$, $j = 1, 2$, chosen so that the corresponding orientations at $\mathbb{R}A'_{L_j}$ are associated to the intersecting halves of $A_{L_j} \setminus \mathbb{R}A_{L_j}$. Thus, the complex orientations of the non-contractible ovals of $\mathbb{R}C$ are opposite, and C is a simple Harnack curve.

For other polarizations of X , existence of simple Harnack curves with two homologous non-contractible components remains unknown to us.

In the end, we formulate some open (to the best of our knowledge) questions concerning simple Harnack curves.

Question 7.8. Does there exist a non-minimal simple Harnack curve?

Question 7.9. Does there exist a K3-surface containing simple Harnack curves of arbitrary large genus?

Question 7.10. Which K3-surfaces contain simple Harnack curves?

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