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Analysis on quasidisks: A unified approach through transmission and jump problems

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Abstract. We give an exposition of results from a crossroad between geometric function theory, harmonic analysis, boundary value problems, and approximation theory, which characterize quasicircles. We will specifically expose the interplay between the jump decomposition, singular integral operators, approximation by Faber series, and the Grunsky inequalities. Our unified point of view is made possible by the concept of transmission.

1. Introduction

A quasiconformal map in the plane is a homeomorphism between planar domains which maps small circles to small ellipses of bounded eccentricity. A quasicircle is by definition the image of the circle $\mathbb{S}^1$ under a quasiconformal map and a quasidisk is the interior of a quasicircle. In geometric function theory quasicircles play a fundamental role in the description of the universal Teichmüller space. They also play an important role in complex dynamical systems. The reader is referred to the book by F. Gehring and K. Hag [25] for a nice introduction to various ramifications of this topic.

It is a familiar fact in the field that quasicircles have an unusually large number of characterizations which are not obviously equivalent, and indeed are qualitatively quite different. See e.g. [25, Chapters 8,9] for some of the classical and also some less well-known ones. It is somewhat astonishing that these continue to be found. In this paper, we will focus on the relatively recent ones due to A. Çavuş [16], Y. Y. Napalkov and R. S. Yulmukhametov [41], Y. Shen [60], and the authors [53, 55]. Indeed, our purpose here is to highlight a characterization based on an interplay between geometric function theory, harmonic analysis, boundary value problems and approximation theory. This point of view was investigated by the authors in a series of papers, and in these works, it emerged that the key to a unified approach is the method of transmission of harmonic functions (or forms).

The goal of this paper is to give an essentially self-contained exposition of this circle of ideas and the method of transmission, not least because of its potential applications

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outside geometric function theory. In doing so we have also refined and improved many of our theorems in previous papers.

To define the notion of transmission, let Γ be a Jordan curve separating the Riemann sphere $\overline{\mathbb{C}}$ into two components Ω_1 and Ω_2. Given a harmonic function h on Ω_1 which extends continuously to Γ, there is a harmonic function on Ω_2 with the same continuous extension on Γ. We call the new function the transmission of h. We generalize the concept of transmission to Dirichlet bounded harmonic functions. For such harmonic functions, the transmission exists and is bounded with respect to the Dirichlet semi-norm if and only if the curve Γ is a quasicircle.

Returning to the problem of characterization of quasicircles, it came to light that in the setting of Dirichlet bounded harmonic functions, a number of perfect equivalences arise. To begin with, given a Jordan curve Γ as above, the following three statements are equivalent.

(1) Γ is a quasicircle.

(2) There is a transmission from the Dirichlet space of harmonic functions on Ω_1 to the Dirichlet space of harmonic functions on Ω_2, which is bounded with respect to Dirichlet energy and agrees with transmission of continuous functions.

(3) The linear operator taking the boundary values of a Dirichlet bounded harmonic function to its Plemelj–Sokhotski decomposition is a bounded isomorphism.

We also show that the jump formula holds up to constants for quasicircles. These results are due to the authors [53, 55].

The first three equivalent statements also are closely related to approximability by Faber series, the Faber and Grunsky operators, and the Schiffer operator. We thus have the following further equivalent statements. Attributions in brackets refers to the first proofs of the equivalence with (1), unless clarified below.

(4) The Faber operator corresponding to Ω_2 is an isomorphism (authors [55]).

(5) The sequential Faber operator is an isomorphism (Çavuş [16], Shen [60]).

(6) Every element of the holomorphic Dirichlet space of Ω_2 is uniquely approximable by a Faber series (Çavuş [16], Shen [60]).

(7) The Schiffer operator is an isomorphism (Napalkov and Yulmukhametov [41]).

The implications (1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6) are due to Çavuş [16], and later independently by Shen [60], while the reverse implications are due to Shen.

The characterization (7) is due to Napalkov and Yulmukhametov [41]. It was proven independently by the authors [55], using our characterization (4). Unfortunately we were not aware of the results of Napalkov and Yulmukhametov. We made use instead of the result of Shen. In some sense, transmission provides a bridge between the result (7) of Napalkov and Yulmukhametov [41] and (5) of Çavuş [16] and Shen [60], by making it possible to replace the sequential Faber operator with the Faber operator for non-rectifiable curves. For the special case of rectifiable Jordan curves, the equivalence of (1) and (2) is due to H. Y. Wei, M. L. Wang, and Y. Hu [69].

Finally, all of these results are closely connected to the classical result that Γ is a quasicircle if and only if:

(8) The norm of the Grunsky operator is strictly less than one.

The implication (1) $\Rightarrow$ (8) is due to R. Kühnau [33] and (8) $\Rightarrow$ (1) is due to C. Pommerenke [43]. In the literature, all proofs of the implication (k) $\Rightarrow$ (1) for $k = 4, \ldots, 7$ (including those due to the authors) rely on the result Pommerenke's converse to the strict Grunsky inequalities (8) $\Rightarrow$ (1). We give alternate proofs in this paper which use transmission only. Our present proofs in the reverse direction (that (1) implies (2) through (7)) also differ from previous ones given by the authors. We first applied this alternate approach in [56] in the case of Jordan curves on Riemann surfaces.

In order to define the transmission in a sensible way, some notion of boundary values is necessary. To this end we also include an exposition of a conformally invariant notion of non-tangential boundary value, which we call conformally non-tangential (CNT for short). This was developed by the authors for Jordan curves in Riemann surfaces [57]. The existence of such boundary values for the Dirichlet space of a simply connected domain is an automatic consequence of a well-known result of A. Beurling. On the other hand, it is not true in general that the boundary values of a harmonic function in one connected component of the complement of Γ are boundary values of a harmonic function in the other component. Even potential-theoretically negligible sets are not obviously the same: for example, sets of harmonic measure zero with respect to one side are not necessarily harmonic measure zero with respect to the other, even for quasicircles. We resolve these issues and give a general framework for the application of the CNT boundary values to sewing and transmission. Aside from the bounded transmission theorem mentioned above, the most important of these results are:

(i) for quasicircles, the potential-theoretically negligible sets on the boundary of Ω_1 are also negligible for Ω_2;

(ii) the operator (what we call the bounce operator) taking a Dirichlet-bounded harmonic function on a doubly-connected region in Ω_1, one of whose boundaries is Γ, to the harmonic function on Ω_1 with the same boundary values, is bounded for any Jordan curve;

(iii) limiting integrals taken over level curves of Green's function are the same for any two Dirichlet bounded harmonic function in a collar near Γ which have the same CNT boundary values (the anchor lemma).

The precise statements are given in Theorems 2.20, 4.6, and Theorem 4.10, respectively.

To conclude, we strive in this paper to show the clarifying power of the transmission theorem for understanding approximation by Faber series, the Grunsky operator, the Plemelj–Sokhotski jump theorem, and Schiffer operators. The results should have many applications in the investigation of the behaviour of function spaces, boundary value problems, and related operators under sewing. The results here are also the basis for a scattering theory of harmonic functions and one-forms for general Riemann surfaces [58].

The paper is organized as follows. In Section 2, we state necessary definitions and results regarding conformally non-tangential boundary values of Dirichlet bounded functions. After preliminaries on the Dirichlet and Bergman space, and quasisymmetric mappings in Sections 2.1 and 2.2, we define certain potential-theoretically negligible sets on a Jordan curve Γ with respect to the enclosed domain in Section 2.3, which we call null sets, and derive their basic properties. A particularly crucial fact is that, in the case that the Jordan curve is a quasicircle, sets that are null with respect to one of the regions enclosed by Γ are also null with respect to the other. In fact, for quasicircles not containing ∞, null sets in Γ are precisely Borel sets of capacity zero. After reviewing some basic results on boundary values of the Dirichlet space of the disk in Section 2.4, we give the definition and basic properties of CNT boundary values in Section 2.5.

Section 3 contains the first of the main results, namely (1) $\Leftrightarrow$ (2): a Dirichlet-bounded transmission exists on Dirichlet space if and only if Γ is a quasicircle. Section 3.1 reviews some known theorems which characterize quasisymmetries in terms of their action on the homogeneous Sobolev space $H^{1/2}$, and a reformulation in terms of CNT boundary values up to null sets. This refinement is necessary because sets of harmonic measure zero on a quasicircle with respect to one side of a curve – which are the images of sets of Lebesgue measure zero on the circle under a conformal map – need not be of harmonic measure zero with respect to the other side of the curve. Thus, null sets are necessary. Section 3.2 contains the transmission result.

In Section 4, we establish several useful results regarding boundary values and integrals. We prove that the so-called bounce operator described in the introduction is bounded. We also prove the "anchor lemma", which shows that certain limiting integrals taken over curves approaching the non-rectifiable Jordan curve depend only on the CNT boundary values. Finally, we give a few useful dense subsets of Dirichlet spaces on simply- and multiply-connected domains. These ultimately rely on density of polynomials.

Section 5 contains the main results on Plemelj–Sokhotski jump isomorphism and Schiffer isomorphism, that is (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (7). Section 5.1 defines the Schiffer operator and proves basic analytic results, Möbius invariance, and an identity of Schiffer. Section 5.3 defines a Cauchy integral operator adapted to non-rectifiable curves using limits of integrals over curves approaching the boundary. We show that for quasicircles the value of this operator is the same up to constants for curves approaching Γ over either side, in a certain sense involving transmission. We also prove basic identities relating the Cauchy integral operator to the Schiffer operators, and the Möbius invariance of the operator. Section 5.3 contains the main results which show that the Plemelj–Sokhotski jump decomposition exists when Γ is a quasicircle, and in a certain sense this decomposition is an isomorphism if and only if Γ is a quasicircle. We also give a new proof of Napalkov and Yulmukhametov's result that the Schiffer operator is surjective (and hence an isomorphism) if and only if Γ is a quasicircle.

In Section 6 we prove that the Faber operator is an isomorphism if and only if Γ is a quasicircle, as well as the existence and uniqueness of Faber series; that is, (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). We also give a brief review of the equivalence with strict Grunsky inequalities.

Finally, we would like to mention that, we make no claims to completeness in terms of the inclusion of all the related literature, and indeed we merely indicate the tip of the literary iceberg.

2. Function spaces and boundary values

2.1. Dirichlet and Bergman spaces

We denote the complex plane by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}}$. We define

$$\mathbb{D}^+ = \{z \in \mathbb{C} : |z| < 1\}$$

and also

$$\mathbb{D}^- = \{z \in \mathbb{C} \,:\, |z| > 1\} \cup \{\infty\}.$$

The circle $\partial\mathbb{D}^+ = \{z \in \mathbb{C} : |z| = 1\}$ is denoted $\mathbb{S}^1$ and a standard annulus is defined by

$$\mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < 1\}.$$

In what follows, we shall sometimes write $a \lesssim b$ as shorthand for $a \le Cb$, where C is a constant. In general, the value of C in the estimates may differ from line to line, but in each instance could be estimated if necessary. Also if $a \lesssim b$ and $b \lesssim a$ then we write $a \approx b$.

In this paper, a conformal map is always assumed to be one-to-one (not just locally one-to-one). That is, a conformal map is a biholomorphism onto its image.

The Riemann sphere $\overline{\mathbb{C}}$ is endowed with the standard complex structure given by the charts

$$\begin{aligned} \psi_0 &: \mathbb{C} \to \mathbb{C}, \quad \psi_0(z) = z, \\ \psi_\infty &: \overline{\mathbb{C}}\backslash\{0\} \to \mathbb{C}, \quad \psi_\infty(z) = 1/z,\ z \neq \infty, \quad \psi_\infty(\infty) = 0, \end{aligned}$$

and holomorphicity or harmonicity is defined with respect to these charts. That is, let Ω be an open connected set in $\overline{\mathbb{C}}$. A function h is holomorphic on Ω if (1) it is holomorphic on $\overline{\mathbb{C}}\backslash\{\infty\}$ and (2) if $\infty \in \Omega$, then $g(z) = f(1/z)$ is holomorphic in a neighbourhood of 0. Anti-holomorphic and harmonic functions on Ω are defined similarly.

We will also consider smooth one-forms on subsets of $\overline{\mathbb{C}}$, where these are defined in the usual way in terms of the Riemann surface structure of $\overline{\mathbb{C}}$. Any one-form α is given in local coordinates by $h_1(z)\,dz + h_2(z)\,d\bar{z}$ for smooth functions $h_1(z)$ and $h_2(z)$. A one-form α on Ω is said to be holomorphic if it can be expressed in local coordinates z as $h(z)\,dz$ where $h(z)$ is holomorphic. That is, $\alpha = a(z)\,dz$ on $\Omega\backslash\{\infty\}$, and if $\infty \in \Omega$, then $b(w) = -a(1/w)/w^2$ is holomorphic in an open set containing 0 (so that in a chart at ∞, we may write $\alpha = b(w)\,dw$). A one-form is anti-holomorphic if it is the complex conjugate of a holomorphic one-form.

We also define the $*$-operator as follows. If $\alpha = h_1(z)\,dz + h_2(z)\,d\bar{z}$ in local coordinates we define

$$*\alpha = *(h_1(z)\,dz + h_2(z)\,d\bar{z}) = -ih_1\,dz + ih_2\,d\bar{z}.$$

It is easily checked that this is well-defined with respect to the change of coordinates $z = \psi_0 \circ \psi_\infty^{-1}(w) = 1/w$.

Define

$$\|\alpha\|_\Omega^2 = \frac{1}{2\pi}\iint_\Omega \alpha \wedge *\overline{\alpha}, \tag{2.1}$$

which might of course diverge. Since any smooth one-form α on Ω can be written (uniquely) in z coordinates as

$$\alpha = h_1(z)\,dz + h_2(z)\,d\bar{z} \tag{2.2}$$

for smooth functions h_1 and h_2, then if in (2.2) z is the parameter in $\mathbb{C}$ (that is, in ψ_0 coordinates), then (2.1) can be written as

$$\|\alpha\|_\Omega^2 = \frac{1}{\pi}\iint_{\Omega\setminus\{\infty\}} \left(|h_1(z)|^2 + |h_2(z)|^2\right) dA, \tag{2.3}$$

where $dA = (d\bar{z} \wedge dz)/2i$ is the Euclidean area element in $\mathbb{C}$. This is justified as follows: when $\infty \in \Omega$, if $\|\alpha\|_\Omega^2 < \infty$ then there is an R such that

$$\iint_{|z|>R} \left(|h_1(z)|^2 + |h_2(z)|^2\right) dA < \infty.$$

Thus, the point at ∞ can be removed from the domain of integration without changing the convergence properties or value of the integral.

Definition 2.1. A smooth one-form α is said to be harmonic if $d\alpha = 0$ and $d * \alpha = 0$; equivalently, for any point $p \in \Omega$, $\alpha = dh$ for some harmonic function h on some open neighbourhood of p. Note that if $\infty \in \Omega$, this restricts the behaviour of α at ∞ since $h(1/z)$ must be harmonic at 0.

We then define the space of L^2 harmonic one-forms $\mathcal{A}_{\mathrm{harm}}(\Omega)$ to consist of those harmonic one-forms α on Ω such that $\|\alpha\|_\Omega < \infty$. This is a Hilbert space with inner product

$$(\alpha, \beta) = \frac{1}{2\pi}\iint_\Omega \alpha \wedge *\overline{\beta}, \tag{2.4}$$

which is also consistent with (2.1). The Bergman space of one-forms is

$$\mathcal{A}(\Omega) = \{\alpha \in \mathcal{A}_{\mathrm{harm}}(\Omega) : \alpha \text{ is holomorphic}\},$$

and for $\alpha = h_1(z)\,dz$, $\beta = h_2(z)\,dz \in \mathcal{A}(\Omega)$, we have

$$(\alpha, \beta) = \frac{1}{\pi}\iint_{\Omega\setminus\{\infty\}} h_1(z)\,\overline{h_2(z)}\,dA.$$

The anti-holomorphic Bergman space $\overline{\mathcal{A}(\Omega)}$ consists of complex conjugates of elements of $\mathcal{A}(\Omega)$.

Observe that $\mathcal{A}(\Omega)$ and $\overline{\mathcal{A}(\Omega)}$ are orthogonal with respect to the inner product. We then obtain the decomposition

$$\mathcal{A}_{\text{harm}}(\Omega) = \mathcal{A}(\Omega) \oplus \overline{\mathcal{A}(\Omega)},$$

which induces the projection operators

$$\begin{aligned} \mathbf{P}(\Omega)&: \mathcal{A}_{\text{harm}}(\Omega) \to \mathcal{A}(\Omega), \\ \overline{\mathbf{P}(\Omega)}&: \mathcal{A}_{\text{harm}}(\Omega) \to \overline{\mathcal{A}(\Omega)}. \end{aligned} \tag{2.5}$$

Definition 2.2. For an open connected set Ω and a smooth function $h\colon \Omega \to \mathbb{C}$ we define the Dirichlet energy of h by

$$D_\Omega(h) = \|dh\|_\Omega^2. \tag{2.6}$$

The harmonic Dirichlet space $\mathcal{D}_{\text{harm}}(\Omega)$ consists of those harmonic functions h on Ω such that $D_\Omega(h) < \infty$.

If z is the coordinate in $\mathbb{C}$ then (2.6) can be written as

$$D_\Omega(h) = \frac{1}{\pi} \iint_{\Omega\setminus\{\infty\}} \left(\left|\frac{\partial h}{\partial z}\right|^2 + \left|\frac{\partial h}{\partial \bar{z}}\right|^2 \right) dA. \tag{2.7}$$

The holomorphic Dirichlet space $\mathcal{D}(\Omega)$ is the set of holomorphic functions in $\mathcal{D}_{\text{harm}}(\Omega)$, and the anti-holomorphic Dirichlet space $\overline{\mathcal{D}(\Omega)}$ is given by the set of complex conjugates of elements of $\mathcal{D}(\Omega)$. The Dirichlet energy on $\mathcal{D}(\Omega)$ restricts to

$$D_\Omega(h) = \frac{1}{\pi} \iint_{\Omega\setminus\{\infty\}} |h'(z)|^2 \, dA,$$

and similarly for $\overline{\mathcal{D}(\Omega)}$. Observe that $\mathcal{D}_{\text{harm}}(\Omega)$ does not decompose into a sum of elements of $\mathcal{D}(\Omega)$ and $\overline{\mathcal{D}(\Omega)}$ unless Ω is simply connected (and even in that case, the decomposition is not unique because constants belong to both spaces).

Note also that the Dirichlet energy is not a norm, since $D(c) = 0$ for constants c. If we define the homogeneous Dirichlet spaces

$$\dot{\mathcal{D}}_{\text{harm}}(\Omega) = \mathcal{D}_{\text{harm}}(\Omega)/\sim,$$

where we say that $h_1 \sim h_2$ if $h_1 - h_2$ is constant, then this is a normed space, with norm

$$\|h\|_{\dot{\mathcal{D}}_{\text{harm}}(\Omega)} = (D_\Omega(h))^{1/2}.$$

For simply-connected domains $d\colon \dot{\mathcal{D}}_{\text{harm}}(\Omega) \to \mathcal{A}_{\text{harm}}(\Omega)$ is an isometry.

In general, if we mod out additive constants, we will denote the corresponding space with a dot. If an operator $\mathbf{L}\colon \mathcal{D}_{\text{harm}}(\Omega_1) \to \mathcal{D}_{\text{harm}}(\Omega_1)$ passes down to a well-defined operator from $\dot{\mathcal{D}}_{\text{harm}}(\Omega_1)$ to $\dot{\mathcal{D}}_{\text{harm}}(\Omega_2)$, we will denote it by $\dot{\mathbf{L}}$, and similarly for any other dotted spaces.

To equip the harmonic Dirichlet space with a norm, one could proceed as follows. Given $h \in \mathcal{D}_{\text{harm}}(\Omega)$, for fixed $p \in \Omega$ we define the pointed Dirichlet norm

$$\|h\|^2_{\mathcal{D}_{\text{harm}}(\Omega,p)} = \|dh\|^2_{\Omega} + |h(p)|^2. \tag{2.8}$$

One immediately sees that the pointed Dirichlet norm is conformally invariant in the sense that if f is a holomorphic bijection onto its image and $f(p) = q$ then

$$\|h \circ f\|_{\mathcal{D}_{\text{harm}}(\Omega,p)} = \|h\|_{\mathcal{D}_{\text{harm}}(f(\Omega),q)}.$$

Furthermore, an elementary argument shows that for any $p, q \in \mathbb{D}$

$$\|h\|_{\mathcal{D}_{\text{harm}}(\mathbb{D}^+,p)} \approx \|h\|_{\mathcal{D}_{\text{harm}}(\mathbb{D}^+,q)}.$$

For this reason, we will drop the point in the notation for the norm and simply write $\|\cdot\|_{\mathcal{D}_{\text{harm}}(\Omega)}$. From these two facts, it is easily deduced that (1) composition by a bijective Möbius transformation $T\colon \mathbb{D} \to \mathbb{D}$ is a bounded isomorphism of the pointed Dirichlet space for any fixed p, and (2) if $f\colon \Omega \to \Omega'$ is holomorphic, then

$$\begin{aligned} \mathbf{C}_f\colon \mathcal{D}_{\text{harm}}(\Omega') &\to \mathcal{D}_{\text{harm}}(\Omega), \\ h &\mapsto h \circ f \end{aligned}$$

is bounded, and if f is furthermore a bijection, then $\mathbf{C}_f$ is a bounded isomorphism.

If one restricts to normalized functions $h(p) = 0$ for some $p \in \Omega$, then we use the notation $\mathcal{D}_p(\Omega)$; in this case the $D_\Omega(h)^{1/2}$ is a genuine norm.

Finally, for a domain G in the plane, with boundary Γ, we also define the Sobolev spaces $H^1(G)$ and $H^{1/2}(\Gamma)$.

Definition 2.3. The Sobolev space $H^1(G)$ consists of functions in $L^2(G)$ such that

$$\|h\|_{H^1(G)} := \left(D_G(h) + \|h\|^2_{L^2(G)}\right)^{1/2} < \infty. \tag{2.9}$$

For any $p \in \mathbb{D}^+$ and $h \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$, it is known that

$$\|h\|_{H^1(\mathbb{D}^+)} \approx \|h\|_{\mathcal{D}_{\text{harm}}(\mathbb{D}^+,p)}, \tag{2.10}$$

with the norm of the right-hand side defined as in (2.8). One can also show that the result remains true for domains with sufficiently regular boundary. However, note that this is not true for general Jordan domains.

If the boundary Γ is regular enough then one can also take the restriction (trace) of an $H^1(G)$-function to Γ, which yields a function $h|_\Gamma \in H^{1/2}(\Gamma)$ where $H^{1/2}(\Gamma)$ is the space of functions in $L^2(\Gamma)$ for which

$$\|f\|_{H^{1/2}(\Gamma)} := \left(\int_\Gamma \int_\Gamma \frac{|f(z) - f(\zeta)|^2}{|z - \zeta|^2}\,|dz|\,|d\zeta| + \|f\|^2_{L^2(\Gamma)}\right)^{1/2} < \infty, \tag{2.11}$$

see [65, Chapter 4] for all the details regarding Sobolev spaces. We will reserve the use of Sobolev spaces for $\mathbb{D}^+$, so that $\Gamma = \mathbb{S}^1$ and the above boundary trace is defined.

The homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{S}^1)$ is defined as the space of measurable functions $f\colon \mathbb{S}^1 \to \mathbb{R}$ such that the seminorm

$$\|f\|_{\dot{H}^{1/2}(\mathbb{S}^1)} = \left(\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|f(z) - f(\zeta)|^2}{|z-\zeta|^2} \, |dz| \, |d\zeta| \right)^{1/2},$$

is finite.

By a Jordan curve Γ in $\overline{\mathbb{C}}$, we mean the image of $\mathbb{S}^1$ under a continuous map into $\overline{\mathbb{C}}$ which is a homeomorphism onto its image. Equivalently, it is the image of a Jordan curve in the plane under a Möbius transformation. Now given a Jordan domain $\Omega \subset \overline{\mathbb{C}}$, let $g_\Omega(z,w)$ denote Green's function of Ω, that is, the harmonic function in z on $\Omega\backslash\{w\}$ such that $g_\Omega(z,w) + \log|z-w|$ is harmonic near w and whose limit as $z \to z_0$ is zero for any point z_0 on the boundary of Ω.

Definition 2.4. Let Γ be a Jordan curve bounding a Jordan domain Ω in $\overline{\mathbb{C}}$. A collar neighbourhood of Γ in Ω is a set of the form

$$A_{p,r} = f(\mathbb{A}_r),$$

where $f\colon \mathbb{D}^+ \to \Omega$ is a biholomorphism such that $f(0) = p$.

Setting $A = A_{p,r}$, we define a norm on $\mathcal{D}_{\mathrm{harm}}(A)$ as follows. Let g_Ω be Green's function of Ω with singularity at $f(0)$. Let n be the outward unit normal, and ds denote the integral with respect to arc length, and set

$$\widehat{h}(0) = -\lim_{r \nearrow 1} \frac{1}{2\pi} \int_{f(|z|=r)} h(\zeta) \frac{\partial g_\Omega}{\partial n}(\zeta) \, ds. \tag{2.12}$$

We can also write the integrand as $h(\zeta) * dg_\Omega(\zeta)$. Then we define

$$\|h\|^2_{\mathcal{D}_{\mathrm{harm}}(A)} = \|dh\|^2_A + |\widehat{h}(0)|^2. \tag{2.13}$$

Note that the norm depends on the choice of curve Γ towards which we take the limit. This is a norm, since it is easily seen that

$$\|h+g\|_{\mathcal{D}_{\mathrm{harm}}(A)} \le \|h\|_{\mathcal{D}_{\mathrm{harm}}(A)} + \|g\|_{\mathcal{D}_{\mathrm{harm}}(A)},$$
$$\|c\,h\|_{\mathcal{D}_{\mathrm{harm}}(A)} = |c|\|h\|_{\mathcal{D}_{\mathrm{harm}}(A)}$$

for any constant c, and if $\|h\|_{\mathcal{D}_{\mathrm{harm}}(A)} = 0$ then $dh = 0$ and $\widehat{h}(0) = 0$, from which it follows that $h = 0$. Moreover, one has that

$$\|h\|^2_{\mathcal{D}_{\mathrm{harm}}(A)} = \|h \circ f\|^2_{\mathcal{D}_{\mathrm{harm}}(\mathbb{A}_r)} \approx \|h \circ f\|^2_{H^1(\mathbb{A}_r)} \approx \|h\|^2_{H^1(A)}. \tag{2.14}$$

Occasionally, we will say that a linear map $\mathbf{L}$ of Dirichlet spaces is "bounded with respect to Dirichlet energy", which means that we have an estimate of the form $D(\mathbf{L}f) \le CD(f)$; this is of course not a norm estimate. When we say that a linear map is bounded, we always mean that it is bounded with respect to the corresponding norm.

2.2. Quasisymmetries and quasiconformal maps

In this section we review definitions and results about quasisymmetries and quasiconformal maps.

Definition 2.5. Let A and B be open connected subsets of the complex plane. An orientation-preserving homeomorphism $\Phi\colon A \to B$ is a k-*quasiconformal mapping* if:

(1) for every rectangle $[a,b] \times [c,d] \subset A$, $\Phi(x,\cdot)$ is absolutely continuous on $[c,d]$ for almost every $x \in [a,b]$;

(2) for every rectangle $[a,b] \times [c,d] \subset A$, $\Phi(\cdot,y)$ is absolutely continuous on $[a,b]$ for almost every $y \in [c,d]$;

(3) there is a $k \in (0,1)$ such that $|\Phi_{\bar{z}}| \leq k|\Phi_z|$ almost everywhere in A.

We say that a map is quasiconformal if it is k-quasiconformal for some $k \in (0,1)$.

Define

$$\begin{aligned} \iota\colon \mathbb{C}\backslash\{0\} &\to \mathbb{C}\backslash\{0\}, \\ z &\mapsto 1/z. \end{aligned}$$

Let A and B be open connected subsets of $\overline{\mathbb{C}}$. We say that a homeomorphism $\Phi\colon A \to B$ is a k-quasiconformal mapping if

$$\Phi, \quad \iota \circ \Phi, \quad \Phi \circ \iota, \quad \text{and} \quad \iota \circ \Phi \circ \iota$$

are all k-quasiconformal on their maximal domains of definition; as above, we say that Φ is quasiconformal if it is k-quasiconformal for some k. If $A, B \subsetneq \overline{\mathbb{C}}$ then Φ is quasiconformal if, given Möbius transformations S and T such that $S(A) \subset \mathbb{C}$ and $T(B) \subset \mathbb{C}$, $T \circ \Phi \circ S^{-1}$ is quasiconformal from $S(A)$ to $T(B)$.

Similarly, for open connected sets $A, B \subset \overline{\mathbb{C}}$ we say that a map $f\colon A \to B$ is conformal if

$$f, \quad \iota \circ f, \quad f \circ \iota, \quad \text{and} \quad \iota \circ f \circ \iota$$

are all conformal on their maximal domains of definition. Conformal maps are 0-quasiconformal onto their image, and it can be shown that 0-quasiconformal maps are conformal (see e.g. [2]). Furthermore, if $\Phi\colon A \to B$ is quasiconformal and $g\colon A' \to A$ is conformal then $\Phi \circ g\colon A' \to B$ is quasiconformal, and if $f\colon B \to B'$ is conformal, then $f \circ \Phi$ is quasiconformal. If C is any open connected subset of A, then the restriction of Φ to C is a quasiconformal map onto $\Phi(C)$.

Remark 2.6. Any quasiconformal map $\Phi\colon \mathbb{C} \to \mathbb{C}$ extends to a quasiconformal map from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$, which takes ∞ to ∞ [36, Theorem I.8.1].

Definition 2.7. An orientation-preserving homeomorphism h of $\mathbb{S}^1$ is called a *quasisymmetric mapping*, iff there is a constant $k > 0$, such that for every α, and every β not equal to a multiple of 2π, the inequality

$$\frac{1}{k} \leq \left| \frac{h(e^{i(\alpha+\beta)}) - h(e^{i\alpha})}{h(e^{i\alpha}) - h(e^{i(\alpha-\beta)})} \right| \leq k$$

holds.

Let $\mathrm{QS}(\mathbb{S}^1)$ denote the set of quasisymmetric maps from $\mathbb{S}^1$ to $\mathbb{S}^1$. Then $\mathrm{QS}(\mathbb{S}^1)$ consists precisely of boundary values of quasiconformal maps, as the following two theorems show.

Theorem 2.8. *Let $\Phi: \mathbb{D}^+ \to \mathbb{D}^+$ be a quasiconformal map. Then Φ has a continuous extension to $\mathbb{S}^1 \cup \mathbb{D}^+$, and the restriction of this extension to $\mathbb{S}^1$ is a quasisymmetry. Conversely, if $\phi: \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetry, then ϕ is the restriction to $\mathbb{S}^1$ of the continuous extension of a quasiconformal map $\Phi: \mathbb{D}^+ \to \mathbb{D}^+$. In the above, one may replace $\mathbb{D}^+$ everywhere by $\mathbb{D}^-$ and the result still holds.*

Proof. By reduction of the problem to the upper-half plane using the conformal equivalence of the unit disk and the former, this is just Ahlfors–Beurling's result in [13]. ∎

Definition 2.9. A Jordan curve Γ in $\overline{\mathbb{C}}$ is a *quasicircle* if and only if it is the image of $\mathbb{S}^1$ under a quasiconformal map $\Phi: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. We say that a Jordan domain is a quasidisk if its boundary is a quasicircle.

Example 2.1. It is known that quasicircles can arise as the Julia sets of rational maps. Another well-known example is von Koch's snowflake. See [25] for more information regarding quasidisks and quasicircles.

Quasidisks have the following important property [25, Corollary 2.1.5].

Theorem 2.10. *Let Ω be a quasidisk. If $f: \mathbb{D}^\pm \to \Omega$ is a biholomorphism, then f extends to a quasiconformal map of $\overline{\mathbb{C}}$.*

One of the main tools in this paper is the conformal welding theorem.

Theorem 2.11 (Conformal welding theorem). *For any quasisymmetry $\phi: \mathbb{S}^1 \to \mathbb{S}^1$, there are conformal maps $f: \mathbb{D}^+ \to \mathbb{C}$ and $g: \mathbb{D}^- \to \overline{\mathbb{C}}$, with the following properties:*

(1) *f and g are quasiconformally extendible to $\overline{\mathbb{C}}$ (so that, in particular, $\Omega^+ = f(\mathbb{D}^+)$ and $\Omega^- = g(\mathbb{D}^-)$ are quasidisks);*

(2) *$\partial f(\mathbb{D}^+) = \partial g(\mathbb{D}^-)$, where ∂ denotes the boundary; and*

(3) *$\phi = (g^{-1} \circ f)|_{\mathbb{S}^1}$.*

If we specify the normalization $f(0) = 0$, $g(\infty) = \infty$, and $g'(\infty) = 1$, then f and g are uniquely determined.

The normalization above can be replaced with any three normalizations in the interior of the domains of f or g if desired.

2.3. Null sets

In this section, we define null sets, which are potential-theoretically negligible sets on the boundary of a Jordan domain. That is, in specifying a harmonic function of bounded Dirichlet energy on a Jordan domain by its boundary values, changes to (or non-existence of) the boundary values on null sets have no effect. We will see that in the special case that the Jordan domain is bounded by a quasicircle, null sets are those sets of logarithmic capacity zero.

We first recall the definition of logarithmic capacity [3, 47]; we follow [47].

Definition 2.12. Let μ be a finite Borel measure in $\mathbb{C}$ with compact support. The potential of μ is the function

$$p_\mu(z) = \iint_{\mathbb{C}} \log|z-w|\, d\mu(w).$$

The energy of μ is then defined to be

$$I(\mu) = \iint p_\mu(z)\, d\mu(z).$$

The equilibrium measure of a compact set K is the measure ν such that

$$I(\nu) = \sup_{\mu \in \mathcal{P}(K)} I(\mu),$$

where $\mathcal{P}(K)$ is the set of Borel probability measures on K. Every compact set possesses an equilibrium measure [47, Theorem 3.3.2]. Now the logarithmic capacity of a set $E \subseteq \mathbb{C}$ is defined as

$$c(E) = \sup_{\substack{\mu \in \mathcal{P}(K) \\ K \subseteq E \text{ compact}}} e^{I(\mu)}.$$

For compact sets K, we have

$$c(K) = e^{I(\nu)},$$

where ν is the equilibrium measure of K.

We say a property holds *quasi-everywhere* if it holds except possibly on a set of logarithmic capacity zero.

Remark 2.13. If a property holds quasi-everywhere, it holds almost everywhere, but the converse is not true.

In what follows we will often drop the word “logarithmic” and use simply the word “capacity”.

The outer logarithmic capacity of a set $E \subseteq \mathbb{C}$ [21] is defined as

$$c^*(E) = \inf_{\substack{E \subseteq U \subseteq \mathbb{C} \\ U \text{ open}}} c(U).$$

By Choquet’s theorem [17, 47], for any bounded Borel set E, $c(E) = c^*(E)$.

Lemma 2.14. *Every bounded set of outer capacity zero in $\mathbb{C}$ is contained in a Borel set of capacity zero.*

Proof. Let F be a set of outer capacity zero. Thus, there are open sets U_n, $n \in \mathbb{N}$, containing F such that $c(U_n) < 1/n$. We can choose these sets such that $U_{n+1} \subseteq U_n$ for all n, by replacing U_n with $U_n' = \cap_{k=1}^n U_k$ if necessary, and observing that by [47, Theorem 5.1.2 (a)], $c(U_n') \leq c(U_n) < 1/n$ since $U_n' \subseteq U_n$.

The set $V = \cap_{n=1}^{\infty} U_n$ is a Borel set containing F. Since $V \subseteq U_n$ for all $n \in \mathbb{N}$, again applying [47, Theorem 5.1.2 (a)] we see that $c(V) < 1/n$ for all $n \in \mathbb{C}$, so $c(V) = 0$. ∎

Quasiconformal maps preserve compact sets of logarithmic capacity zero. We are grateful to M. Younsi for suggesting the proof below.

Lemma 2.15. *Let $K \subseteq \mathbb{C}$ be compact. Let U be an open set containing K and $f: U \to V$ be a homeomorphism onto the open set $V \subset \mathbb{C}$, which is Hölder continuous of exponent $\alpha > 0$. If K has capacity zero, then $f(K)$ also has capacity zero.*

Proof. Let μ be a probability measure with support in $f(K)$. If we define the Borel probability measure $\nu = f^*(\mu)$ by $\nu(A) = \mu(f(A))$, then

$$
\begin{aligned}
I(\mu) &= \iint_V \iint_V \log|z-w|\, d\mu(z)\, d\mu(w) = \iint_U \iint_U \log|f(z)-f(w)|\, d\nu(z)\, d\nu(w) \\
&= \iint_U \iint_U \log \frac{|f(z)-f(w)|}{|z-w|^\alpha}\, d\nu(z)\, d\nu(w) + \alpha I(\nu).
\end{aligned}
$$

Since the capacity of K is zero, $I(\nu) = -\infty$. Moreover, the Hölder-continuity of f means that $|f(z) - f(w)| \leq M|z-w|^\alpha$, which yields $I(\mu) = -\infty$. Now since μ was arbitrary, $f(K)$ has capacity zero. ∎

From this, we obtain the following lemma.

Lemma 2.16. *Let $E \subseteq \mathbb{C}$ be a bounded Borel set. Let $f: \mathbb{C} \to \mathbb{C}$ be a homeomorphism which is Hölder continuous of exponent $\alpha > 0$. If E has capacity zero, then $f(E)$ has capacity zero.*

Proof. By [47, Theorem 5.1.2(b)], we have

$$
c(f(E)) = \sup_{\substack{K \subseteq f(E) \\ K \text{ compact}}} c(K) \tag{2.15}
$$

(indeed, this follows directly from the definition of capacity). Thus, if $f(E)$ does not have capacity zero, there is a compact set $K \subseteq f(E)$ such that $c(K) > 0$. Since f is a homeomorphism, $f^{-1}(K)$ is a compact subset of E, so by the previous lemma $c(f^{-1}(K)) > 0$. Applying (2.15) again with E in place of $f(E)$ we see that $c(E) > 0$, a contradiction. ∎

In particular, quasiconformal maps preserve bounded Borel sets of capacity zero, since they are uniformly Hölder on every compact subset [36, p. 71].

Corollary 2.17. *Let $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a quasisymmetry. Then $I \subseteq \mathbb{S}^1$ is a Borel set of logarithmic capacity zero if and only if $\phi(I)$ is a Borel set of logarithmic capacity zero.*

Proof. By the Beurling–Ahlfors extension theorem (Theorem 2.8), ϕ has a quasiconformal extension $\Psi\colon \mathbb{D} \to \mathbb{D}$. In fact, this extends to a quasiconformal map of the plane via

$$\Phi(z) = \begin{cases} \Psi(z) & z \in \mathrm{cl}\,\mathbb{D}, \\ 1/\overline{\Psi(1/\bar{z})} & z \in \mathbb{C}\backslash \mathrm{cl}\,\mathbb{D}. \end{cases}$$

Since a quasiconformal map is uniformly Hölder-continuous on every compact subset, the claim follows from Lemma 2.16. See also [13] for further details. ■

Remark 2.18. In [6], N. Arcozzi and R. Rochberg gave a combinatorial proof that if $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetry and I is a closed subset of $\mathbb{S}^1$, then there is a constant $K > 0$ depending only on ϕ such that $\frac{1}{K}c(I) \leq c(\phi(I)) \leq Kc(I)$. This of course implies Corollary 2.17.

We now define null sets. Note that in the sphere, the boundary of a domain is taken with respect to the sphere topology. So it might include ∞.

Definition 2.19. Let Ω be a Jordan domain in $\overline{\mathbb{C}}$ with boundary Γ. Let $I \subset \Gamma$. We say that I is null with respect to Ω if I is a Borel set, and there is a biholomorphism $f\colon \mathbb{D}^+ \to \Omega$ such that $f^{-1}(I)$ has logarithmic capacity zero.

The meaning of $f^{-1}(I)$ requires an application of Carathéodory's theorem, which says that since Ω is a Jordan domain, any biholomorphism f has a continuous extension which takes $\mathbb{S}^1$ homeomorphically to Γ. This is true even if Γ contains the point at ∞, as can be seen by composing f by a Möbius transformation taking Γ onto a bounded curve and applying Carathéodory's theorem there, and then using the fact that T is a homeomorphism of the sphere. Thus, $f^{-1}(I)$ is defined using the extension of f. Note that I is a Borel set if and only if $f^{-1}(I)$ is Borel.

If there is one biholomorphism f such that $f^{-1}(I)$ has capacity zero, then $g^{-1}(I)$ has capacity zero for all biholomorphisms $g\colon \mathbb{D}^+ \to \Omega$. This is because the Möbius transformation $T = g^{-1} \circ f$ preserves Borel sets of capacity zero in $\mathbb{S}^1$, for example by Corollary 2.17. Also, it is easily seen that one may replace $\mathbb{D}^+$ with $\mathbb{D}^-$ in the above definition.

If Γ is a Jordan curve, bordering domains Ω_1 and Ω_2, then I might be null with respect to Ω_1 but not with respect to Ω_2, or vice versa. However, for quasicircles, the concept of null set is independent of the choice of "side" of the curve. This is a key fact.

Theorem 2.20. *Let Γ be a quasicircle in $\overline{\mathbb{C}}$, and let Ω_1 and Ω_2 be the connected components of $\overline{\mathbb{C}}\backslash\Gamma$. Then $I \subset \Gamma$ is null with respect to Ω_1 if and only if it is null with respect to Ω_2.*

Proof. Choose a Möbius transformation T so that $T(\Gamma)$ is a quasicircle in $\mathbb{C}$ (and in particular bounded). Clearly $T(I)$ is null in $T(\Gamma)$ with respect to $T(\Omega_i)$ if and only if it is null in Γ with respect to Ω_i, for $i = 1, 2$. Thus, it suffices to prove the claim for a quasicircle Γ in $\mathbb{C}$.

Let Ω^+ and Ω^- be the bounded and unbounded components of the complement of Γ respectively. Let $f_\pm : \mathbb{D}^\pm \to \Omega^\pm$ be conformal maps. These have quasiconformal extensions to $\mathbb{C}$. Thus, $\phi = f_-^{-1} \circ f_+$ has a quasiconformal extension to $\mathbb{C}$, and in particular is a quasisymmetry.

By definition I is null with respect to Ω^+ if and only if $f_+^{-1}(I)$ is a Borel set of logarithmic capacity zero in $\mathbb{S}^1$. By Corollary 2.17, this holds if and only if $f_-^{-1}(I) = \phi(f_+^{-1}(I))$ is a Borel set of logarithmic capacity zero in $\mathbb{S}^1$, that is if and only if I is null with respect to Ω^-. ∎

We are grateful to M. Younsi for pointing out that the converse to Theorem 2.20 is false by a result of J. Becker and C. Pommerenke [9]. They show that there is a non-quasiconformal Jordan curve such that the corresponding Riemann maps onto its complements are Hölder continuous.

Remark 2.21. The proof can be modified to show that if Ω is a Jordan domain bounded by a quasicircle, and $I \subseteq \Gamma$ is null with respect to Ω, then there is a Möbius transformation T such that $T(I)$ is a bounded Borel set of capacity zero (in fact, for any Möbius transformation such that $T(I)$ is bounded, it is a set of capacity zero). Similarly, E. Villamor [67, Theorem 3] showed that if $g: \mathbb{D}^- \to \mathbb{C}$ is a one-to-one holomorphic κ-quasiconformally extendible map satisfying $g(z) = z + \cdots$ near ∞, then there is a κ depending only on the quasiconformal constant such that for any closed $I \subset \mathbb{S}^1$, we have $c(I)^{1+\kappa} \leq c(g(A)) \leq c(I)^{1-\kappa}$. In light of what we have discussed, this implies Theorem 2.20.

2.4. Dirichlet space of the disk and boundary values

If we write $f(z) = \sum_{n=0}^{\infty} a_n z^n$ as a power series and set $z = re^{i\theta}$, one can use polar coordinates to see that

$$D_{\mathbb{D}^+}(f) = \sum_{n=1}^{\infty} n|a_n|^2. \tag{2.16}$$

Another important fact about the Dirichlet space is that if $f \in \mathcal{D}(\mathbb{D}^+)$ then f has radial boundary values, i.e. for almost every $z \in \mathbb{S}^1$, the limit $\lim_{r \to 1^-} f(rz) =: \tilde{f}(z)$ exists; see e.g. [21]. Moreover, by a result of J. Douglas [19], one has that

$$D_{\mathbb{D}^+}(f) = \int_0^{2\pi} \int_0^{2\pi} \frac{|\tilde{f}(z) - \tilde{f}(\zeta)|^2}{|z - \zeta|^2} \, |dz| \, |d\zeta|. \tag{2.17}$$

Now for $\zeta \in \mathbb{S}^1$, let

$$K(\zeta) = \frac{1}{|1 - \zeta|^{1/2}}, \tag{2.18}$$

and define the convolution of two functions f, g defined on the unit circle via

$$(f * g)(z) := \int_0^{2\pi} f(z\overline{\zeta})\, g(\zeta)\, |d\zeta|. \tag{2.19}$$

If $z \in \mathbb{D}^+$ and $\zeta \in \mathbb{S}^1$, then

$$P_z(\zeta) = \frac{1 - |z|^2}{|z - \zeta|^2}, \tag{2.20}$$

denotes the Poisson kernel of the disk, and we set

$$P(u)(z) = \int_0^{2\pi} P_z(\zeta) u(\zeta)\, |d\zeta|. \tag{2.21}$$

Regarding boundary values of harmonic functions with bounded Dirichlet energy, we will use the following result.

Theorem 2.22. *Let $f = P(K * \varphi)$ for some $\varphi \in L^2(\mathbb{S}^1)$. For fixed $\theta \in [0, 2\pi)$, consider the following four limits:*

(1) $\lim_{r \to 1^-} f(rz)$ *(the radial limit of f);*

(2) $\lim_{N \to \infty} \sum_{n=-N}^{N} \widehat{(K * \varphi)}(n) e^{in\theta}$ *(the limit of the partial sums of the Fourier series for $K * \varphi$);*

(3) $\lim_{h \to 0^+} \frac{1}{2h} \int_{\theta - h}^{\theta + h} (K * \varphi)(e^{it})\, dt$ *(the boundary trace of f).*

If one of them exists and is finite, they all do and they are equal. The equivalence of (1) and (2) is Abel's theorem and a result of E. Landau [34, pp. 65–66]. The equivalence of (2) and (3) is from Beurling in [12].

The boundary behaviour of elements of Dirichlet space is better than this result indicates in two ways. Firstly, the limit exists not just radially but non-tangentially. Secondly, the limit exists not just almost everywhere, but up to a set of outer capacity zero.

We now define non-tangential limit. A non-tangential wedge in $\mathbb{D}$ with vertex at $p \in \mathbb{S}^1$ is a set of the form

$$W(p, M) = \{z \in \mathbb{D} : |p - z| < M(1 - |z|)\} \tag{2.22}$$

for $M \in (1, \infty)$.

Definition 2.23. We say that a function $h\colon \mathbb{D} \to \mathbb{C}$ has a non-tangential limit of ζ at p in $\mathbb{S}^1$ if

$$\lim_{\substack{z \to p \\ z \in W(p,M)}} h(z) = \zeta$$

for all $M \in (1, \infty)$.

Equivalently, in the above definition one may replace non-tangential wedges with Stolz angles

$$\Delta(p, \alpha, \rho) = \{z : |\arg(1 - \overline{p}z)| < \alpha \text{ and } |z - p| < \rho\},$$

where $\alpha \in (0, \pi/2)$ and $\rho \in (0, 2\cos\alpha)$.

The following theorem of Beurling [21, Theorem 3.2.1] improves our understanding of the boundary behaviour, as promised.

Theorem 2.24. *Let $h \in \mathcal{D}_{\text{harm}}(\mathbb{D})$. Then there is a set $I \subseteq \mathbb{S}^1$ of outer capacity zero such that the non-tangential limit of h exists on $\mathbb{S}^1 \backslash I$.*

Remark 2.25. By Lemma 2.14, we may take I to be a Borel set of capacity zero.

Since a wedge at p contains a radial segment terminating at $p \in \mathbb{S}^1$, it is immediate that if the non-tangential limit exists, then the radial limit exists and equals the non-tangential limit. Using Theorems 2.24 and 2.22, one then has the following theorem.

Theorem 2.26. *Let $h \in \mathcal{D}_{\text{harm}}(\mathbb{D})$. Let H be the non-tangential boundary values of h. The Fourier series of H converges, except possibly on a set of outer capacity zero, to H.*

Finally, we have the following.

Theorem 2.27. *Let $h_1, h_2 \in \mathcal{D}_{\text{harm}}(\mathbb{D})$. If the non-tangential limits of h_1 and h_2 are equal except on a Borel set of capacity zero, then $h_1 = h_2$.*

To see this, it is enough to see that if the non-tangential limit of $h \in \mathcal{D}_{\text{harm}}(\mathbb{D})$ is zero, then h is zero. This follows essentially from the equality of the radial and non-tangential limits and (2.17).

2.5. Conformally non-tangential boundary values

We now extend the notion of non-tangential limits to arbitrary Jordan domains. This extension is an immediate consequence of the Riemann mapping theorem, and is uniquely determined by the requirement that the definition be conformally invariant. Although this extension is by itself trivial, substantial results arise when one considers boundary values from two sides of the curve, as we will see in Section 3.

Definition 2.28. Let Ω be a Jordan domain in $\overline{\mathbb{C}}$ with boundary Γ. Let $h\colon \Omega \to \mathbb{C}$ be a function. We say that the conformally non-tangential (CNT) limit of h is ζ at $p \in \Gamma$ if, for a biholomorphism $f\colon \mathbb{D}^+ \to \Omega$, the non-tangential limit of $h \circ f$ is ζ at $f^{-1}(p)$.

The existence of the limit does not depend on the choice of biholomorphism, as the following lemma shows.

Lemma 2.29. *Let $h\colon \mathbb{D} \to \mathbb{C}$, and let $T\colon \mathbb{D} \to \mathbb{D}$ be a disk automorphism. Then h has a non-tangential limit at $p \in \partial\mathbb{D}$ if and only if $h \circ T$ has a non-tangential limit at $T^{-1}(p)$, and these are equal.*

Proof. The claim follows from the easily verified fact that every Stolz angle at p is contained in the image under T of a Stolz angle at $T^{-1}(p)$, and every Stolz angle at $T^{-1}(p)$ is contained in the image under T^{-1} of a Stolz angle at p. ∎

If $h \circ f$ has non-tangential limit ζ at $f^{-1}(p)$ for a biholomorphism $f: \mathbb{D}^+ \to \Omega$, then $h \circ g$ has non-tangential limit ζ at $g^{-1}(p)$ for any biholomorphism $g: \mathbb{D}^+ \to \Omega$, by applying the lemma above to $T = g^{-1} \circ f$.

Remark 2.30. This notion of CNT limit is conformally invariant, in the following sense. If Ω_1 and Ω_2 are Jordan domains and $f: \Omega_1 \to \Omega_2$ is a biholomorphism, then the CNT limit of $h: \Omega_2 \to \mathbb{C}$ exists and equals ζ at $p \in \partial\Omega_2$ if and only if the CNT limit of $h \circ f$ exists and equals ζ at $f^{-1}(p) \in \Omega_1$. The only role that the regularity of the boundary curves plays in the definition, is that we use Carathéodory's theorem implicitly to uniquely associate points on the boundary of $\partial\Omega_1$ with points on $\partial\Omega_2$. Therefore, the boundary is required to be a Jordan curve. However, even this condition can be removed, by replacing the boundary of the domain in $\overline{\mathbb{C}}$ with the ideal boundary [58].

Remark 2.31. An obviously equivalent definition is as follows. The CNT limit of $h: \Omega \to \mathbb{C}$ is ζ at $p \in \partial\Omega$ if, given a conformal map $f: \mathbb{D}^+ \to \Omega$, defining

$$V(p, M) = f(W(f^{-1}(p), M)),$$

one has that

$$\lim_{\substack{z \to p \\ z \in V(p,M)}} h(z) = \zeta.$$

Note that, treating the ideal boundary of Ω as a border of Ω [4] (which can be done since Ω is biholomorphic to a disk), the angle of the wedge $V(p, M)$ has a sensible geometric meaning. That is, let ϕ be a border chart taking a neighbourhood U of p in Ω to a half-disk which takes a segment of the ideal boundary containing p to a segment of the real axis. In this neighbourhood, $\phi(V(p, M) \cap U)$ is a wedge in the ordinary sense. The boundary of $\phi(V(p, M) \cap U)$ meets the real axis at two angles which are independent of the choice of chart.

Using CNT limits, we can formulate a conformally invariant version of Beurling's theorem on non-tangential limits.

Theorem 2.32. *Let Ω be a Jordan domain with boundary Γ. For $h \in \mathcal{D}_{\mathrm{harm}}(\Omega)$, the* CNT *boundary values of h exist at every point in Γ except possibly on a null set $I \subset \Gamma$ with respect to Ω. If h_1 and h_2 are* CNT *boundary values of some element of H_1 and H_2 in $\mathcal{D}_{\mathrm{harm}}(\Omega)$, respectively, and $h_1 = h_2$ except possibly on a null set, then $H_1 = H_2$.*

This follows directly from Theorem 2.24, Theorem 2.27, Lemma 2.14, and the conformal invariance of CNT limits (Remark 2.30).

In [53, 55] we used limits along hyperbolic geodesics (equivalently, orthogonal curves to level curves of Green's function) in place of CNT limits, following H. Osborn [42]. If the CNT limit exists, then the radial limit exists. Besides being a stronger property, the CNT limits we later defined [56, 57] seem to be more convenient.

We now define a particular class of boundary values. Let Ω be a Jordan domain in $\overline{\mathbb{C}}$ with boundary Γ. We say that two functions h_1 and h_2 on Γ are equivalent if $h_1 = h_2$

except possibly on a null set I with respect to Ω. Denote the set of such functions up to equivalence by $\mathcal{B}(\Gamma, \Omega)$. We say that $h_1 = h_2$ if they are equivalent.

Definition 2.33. The Osborn space of Γ with respect to Ω, denoted $\mathcal{H}(\Gamma, \Omega)$, is the set of functions $h \in \mathcal{B}(\Gamma, \Omega)$ which arise as boundary values of elements of $\mathcal{D}_{\text{harm}}(\Omega)$.

We then define the trace operator

$$\mathbf{b}_{\Omega,\Gamma} \colon \mathcal{D}_{\text{harm}}(\Omega) \to \mathcal{H}(\Gamma, \Omega)$$

and the extension operator

$$\mathbf{e}_{\Gamma,\Omega} \colon \mathcal{H}(\Gamma, \Omega) \to \mathcal{D}_{\text{harm}}(\Omega)$$

accordingly.

Example 2.2. In the case that $\Gamma = \mathbb{S}^1$ and $\Omega = \mathbb{D}^+$, these maps have simple expressions in terms of the Fourier series:

$$\mathbf{b}_{\mathbb{D}^+,\mathbb{S}^1}\left(\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\overline{z}^n\right) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$
$$\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\overline{z}^n.$$

Similar expressions can be obtained for $\mathbb{D}^-$.

We shall also need a result about the agreement of Sobolev and Osborn spaces.

Theorem 2.34. *Given a function $f \in H^{1/2}(\mathbb{S}^1)$ there exits a unique harmonic function $F \in \mathcal{D}_{\text{harm}}(\mathbb{D})$ whose* CNT *boundary values agree almost everywhere with values of f on $\mathbb{S}^1$.*

Proof. By the existence and uniqueness of the solution to the Dirichlet problem (see e.g. [65, Proposition 4.5, p. 334]), f has a unique harmonic extension $F \in H^1(\mathbb{D})$, and the CNT boundary values of F are equal to f almost everywhere. ∎

In particular, this shows that every $H^{1/2}(\mathbb{S}^1)$ (which is defined up to a measure zero set) has a unique extension to an element of $\mathcal{H}(\mathbb{S}^1, \mathbb{D}^+)$ (which is defined up to a set of capacity zero).

Finally, we pose the following question. Given a Jordan curve Γ in $\overline{\mathbb{C}}$, and let Ω_1 and Ω_2 be the components of its complement. For which Jordan curves Γ is $\mathcal{H}(\Gamma, \Omega_1) = \mathcal{H}(\Gamma, \Omega_2)$? We answer this question in the next section.

3. Transmission of harmonic functions in quasicircles

3.1. Vodop'yanov–Nag–Sullivan theorem

First we recall a result due to K. Vodop'yanov [68] regarding the boundedness of composition operators on fractional Sobolev spaces which will be useful in proving a characterization result for quasisymmetric homeomorphisms of $\mathbb{S}^1$. However, the original result is formulated for Sobolev spaces on the real line. To this end, one defines the homogeneous Sobolev (or Besov) space $\dot{H}^{1/2}(\mathbb{R})$ as the closure of $\mathcal{C}_c^\infty(\mathbb{R})$ (smooth compactly supported functions) in the seminorm

$$\|f\|_{\dot{H}^{1/2}(\mathbb{R})} = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy\right)^{1/2}. \tag{3.1}$$

Theorem 3.1. *The composition map $\mathbf{C}_\phi(h) := h\circ\phi$ is bounded from $\dot{H}^{1/2}(\mathbb{R})$ to $\dot{H}^{1/2}(\mathbb{R})$ if and only if ϕ is a quasisymmetric homeomorphism of $\mathbb{R}$ to $\mathbb{R}$.*

See [68, Theorem 2.2].

Remark 3.2. As is customary in Sobolev space theory, the constructions of compositions, traces and so on, are done using dense subsets of Sobolev spaces, e.g. the set of smooth compactly supported functions, where for example the composition $\mathbf{C}_\phi(h)$ is well-defined (i.e. for $h\in\mathcal{C}_c^\infty(\mathbb{R})$). Thereafter, one seeks boundedness estimates with bounds that are independent of h and extends the results by density to the desired Sobolev space.

In [39], S. Nag and D. Sullivan showed that quasisymmetries of $\mathbb{S}^1$ are characterized by the fact that they are bounded maps of the Sobolev space $H^{1/2}(\mathbb{S}^1)/\mathbb{R}$ and in doing so reproved Theorem 3.1. In what follows we give a presentation of their result adding also some more references for the sake of completeness.

Theorem 3.3. *Let $\phi\colon\mathbb{S}^1\to\mathbb{S}^1$ be a homeomorphism. Then the following are equivalent.*

(1) *ϕ is a quasisymmetry;*

(2) *ϕ has the following three properties:*

  (a) *ϕ takes Borel sets of capacity zero to Borel sets of capacity zero;*

  (b) *for every $h\in\mathcal{H}(\mathbb{S}^1)$, $\mathbf{C}_\phi(h)\in\mathcal{H}(\mathbb{S}^1)$;*

  (c) *the map $h\mapsto h\circ\phi$ obtained in (b) is bounded with respect to Dirichlet energy, i.e. there is a C such that*

$$D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(h\circ\phi)) \le C D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}h). \tag{3.2}$$

Proof. That (1) implies 2 (a) is Corollary 2.17.

That (1) implies 2 (b) are equivalent can be shown by transferring the problem to the real line. As a consequence of a much more general result for divergence-type elliptic operators due to A. Barton and S. Mayboroda [8, Theorem 7.18], if $\mathbb{H}$ denotes the upper half-plane, then there exists a solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ on } \mathbb{H}, \\ u|_{\partial\mathbb{H}} = f \in \dot{H}^{1/2}(\mathbb{R}), \end{cases} \tag{3.3}$$

which is unique (up to additive constants) and the estimate

$$\|u\|_{\dot{H}^1(\mathbb{H})} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{3.4}$$

holds.

Now the fact that for every $h \in \mathcal{H}(\mathbb{R})$, the composition $\mathbf{C}_\phi h \in \mathcal{H}(\mathbb{R})$, is then a consequence of Theorem 3.1.

That (1) implies 2(c) can be shown as follows. Let $H \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ be the function whose CNT boundary values equal h quasi-everywhere. Let $\Phi\colon \mathbb{D}^+ \to \mathbb{D}^+$ be a quasiconformal map whose boundary values equal ϕ (which exists by the aforementioned Beurling–Ahlfors extension theorem). By quasi-invariance of Dirichlet energy (see e.g. [1]), we have

$$D_{\mathbb{D}^+}(\mathbf{C}_\Phi H) \leq C D_{\mathbb{D}^+}(H) = C D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+} h),$$

where C is of course independent of H. Let $F := \mathbf{C}_\Phi H - \mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{C}_\phi h) \in H^1(\mathbb{D}^+)$. Then using $F|_{\mathbb{S}^1} = 0$, the harmonicity of $\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{C}_\phi h)$ and the Sobolev space divergence theorem (see e.g. [22, Theorem 4.3.1, p. 133]), one can show that

$$\int_{\mathbb{D}^+} \partial(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{C}_\phi h))\, \overline{\partial F}\, dA = 0.$$

This yields that

$$\begin{aligned} D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{C}_\phi h)) &\leq D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{C}_\phi h)) + D_{\mathbb{D}^+}(F) \\ &= D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{C}_\phi h)) + 2\int_{\mathbb{D}^+} \partial(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{C}_\phi h))\, \overline{\partial F}\, dA + D_{\mathbb{D}^+}(F) \\ &= D_{\mathbb{D}^+}(\mathbf{C}_\Phi H). \end{aligned} \tag{3.5}$$

Now assume (2) holds. If (3.2) is valid for any homeomorphism ϕ, then transference of Douglas's result to the real line in equation (2.17) yields that $\|\mathbf{C}_\phi u\|_{\dot{H}^{1/2}(\mathbb{R})} \lesssim \|u\|_{\dot{H}^{1/2}(\mathbb{R})}$, which by Theorem 3.1 yields that ϕ is a quasisymmetric homeomorphism of the real line. This completes the proof. ∎

In the remainder of the paper, we will say that an operator between Dirichlet spaces is bounded with respect to Dirichlet energy if it satisfies an estimate of the form given by equation (3.2).

Conditions (2) (a) and (2) (b) of Theorem 3.3 are not easy to verify, but of course the direction (2) → (1) can be stated in the following way. Let $\dot{\mathcal{C}}(\mathbb{S}^1)$ denote the space of continuous functions on $\mathbb{S}^1$ modulo additive constants.

Theorem 3.4. *Let $\phi: \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. Assume that there is a dense set $\mathcal{L} \subseteq \dot{H}^{1/2}(\mathbb{S}^1)$ such that $\mathcal{L} \subseteq \dot{\mathcal{C}}(\mathbb{S}^1)$ and there is an M such that*

$$\|\mathbf{C}_\phi h\|_{\dot{H}^{1/2}(\mathbb{S}^1)} \leq M \|h\|_{\dot{H}^{1/2}(\mathbb{S}^1)}.$$

Then ϕ is a quasisymmetry.

Proof. The proof of this result is embedded in the proof of Theorem 3.1. See also [39, Corollary 3.2] and [15, Theorem 1.3]. ∎

3.2. Transmission (overfare)

We are now able to prove the transmission theorem in the simplest case.

Theorem 3.5. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$, and let Ω_1 and Ω_2 be the components of the complement. The statements* (1) *and* (2) *below are equivalent.*

(1) *Γ is a quasicircle.*

(2) (a) *If $I \subseteq \Gamma$ is null with respect to Ω_1 then it is null with respect to Ω_2,*

(b) *$\mathcal{H}(\Gamma, \Omega_1) \subseteq \mathcal{H}(\Gamma, \Omega_2)$, and*

(c) *the map $\mathbf{e}_{\Gamma,\Omega_2}\mathbf{b}_{\Gamma,\Omega_1}: \mathcal{D}_{\text{harm}}(\Omega_1) \to \mathcal{D}_{\text{harm}}(\Omega_2)$ is bounded* with respect to Dirichlet energy.

Needless to say, the roles of 1 and 2 can be switched, so that if transmission from Ω_1 to Ω_2 is possible, it is then possible from Ω_2 to Ω_1.

Proof. We show that (2) implies (1). The truth of either (1) or (2) is unaffected by applying a global Möbius transformation, so we can assume that Γ is bounded. Let $\Omega^{\pm}$ be the connected components of the complement in $\overline{\mathbb{C}}$; assume for definiteness that $\Omega_1 = \Omega^+$ (this can be arranged by composing by $1/z$).

Now let $f_{\pm}: \mathbb{D}^{\pm} \to \Omega^{\pm}$ be conformal maps. By Carathéodory's theorem, $f_{\pm}$ each extend to homeomorphisms from $\mathbb{S}^1$ to Γ; denote the extensions also by $f_{\pm}$. The function

$$\phi = f_+^{-1} \circ f_-|_{\mathbb{S}^1}: \mathbb{S}^1 \to \mathbb{S}^1$$

is thus a homeomorphism. We will show that ϕ is a quasisymmetry. Once this is shown, it follows from the conformal welding theorem that Γ is a quasicircle.

To do this, we show that ϕ has properties 2 (a), 2 (b), and 2 (c) of Theorem 3.3. Let I be a Borel set of capacity zero in $\mathbb{S}^1$. Then $f_-(I)$ is by definition null with respect to Ω^-. So by 2 (a) of the present theorem, $f_-(I)$ is null with respect to Ω^+. By definition $\phi(I) = f_+^{-1}(f_-(I))$ is a Borel set of capacity zero in $\mathbb{S}^1$. This shows that ϕ has the property 2 (a) of Theorem 3.3.

Given $h \in \mathcal{H}(\mathbb{S}^1)$, there is an $H \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ with CNT boundary values equal to h except possibly on a null set I. Also, $H \circ f_+^{-1} \in \mathcal{D}_{\text{harm}}(\Omega^+)$. By definition, $H \circ f_+^{-1}$ has

CNT boundary values except on the null set $f_+(I)$. By assumption 2 (b) of the present theorem, there is a function

$$u = \mathbf{e}_{\Gamma,\Omega^-}\mathbf{b}_{\Gamma,\Omega^+}(H \circ f_+^{-1}) \in \mathcal{D}_{\mathrm{harm}}(\Omega^-)$$

whose CNT boundary values agree with those of $H \circ f_+^{-1}$ except on a null set K containing $f_+(I)$. Set $I' = f_+^{-1}(K)$, which is a null set containing I.

By definition, $u \circ f_- \in \mathcal{D}_{\mathrm{harm}}(\mathbb{D}^-)$ has CNT boundary values except on the null set $f_-^{-1}(K) = \phi^{-1}(I')$, which contains $\phi^{-1}(I)$. These CNT boundary values agree with

$$h \circ f_+^{-1} \circ f_- = h \circ \phi$$

except on $\phi^{-1}(I')$. Thus, the function $u \circ f_-(1/\bar{z})$ has CNT boundary values equal to h except on $\phi^{-1}(I')$. That is,

$$u \circ f_-(1/\bar{z}) = \mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(h \circ \phi),$$

which shows that $\mathbf{C}_\phi h \in \mathcal{H}(\mathbb{S}^1)$. Since h is arbitrary, this shows that property 2(b) of Theorem 3.3 holds.

To show that 2 (c) of Theorem 3.3 holds, by 2 (c) of the present theorem and conformal invariance of the Dirichlet norm, there is a constant $C > 0$ such that

$$D_{\Omega^+}(\mathbf{e}_{\Gamma,\Omega^+}\mathbf{b}_{\Gamma,\Omega^-}v) \leq C D_{\Omega^-}(v)$$

for all $v \in \mathcal{D}_{\mathrm{harm}}(\Omega^-)$. Then for arbitrary $h \in \mathcal{H}(\mathbb{S}^1)$, using the notation above we have

$$\begin{aligned} D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(h \circ \phi)) &= D_{\mathbb{D}^+}(u \circ f_-(1/\bar{z})) = D_{\mathbb{D}^-}(u \circ f_-) \\ &= D_{\Omega^-}(u) \leq C D_{\Omega^+}(H \circ f_+^{-1}) \\ &= C D_{\mathbb{D}^+}(H) = C D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}h). \end{aligned}$$

Thus, ϕ is a quasisymmetry, completing the proof that (2) implies (1).

So we need only show that (1) implies (2). Again, we can assume that Γ is bounded and denote the bounded and unbounded components of the complement by Ω^+ and Ω^- respectively. Let $f_\pm \colon \mathbb{D}^\pm \to \Omega^\pm$ be conformal maps, which have quasiconformal extensions to $\mathbb{C}$. Thus, $\phi = f_+^{-1} \circ f_-$ is a quasisymmetry of $\mathbb{S}^1$, and properties (2) (a)–(c) of Theorem 3.3 hold.

Given a Borel set $I \subseteq \Gamma$ which is null with respect to Ω^+, by definition $f_+^{-1}(I)$ is a Borel set of capacity zero in $\mathbb{S}^1$. Thus, since ϕ^{-1} is a quasisymmetry, by Theorem 3.3, $\phi^{-1}(f_+^{-1}(I)) = f_-^{-1}(I)$ is a Borel set of capacity zero. Thus, by definition I is null with respect to Ω^-. This shows that 2 (a) of the present theorem holds.

Denoting $R(z) = 1/\bar{z}$, a proof similar to that given above for the reverse implication shows that (2) (b) of the present theorem holds with the extension to Ω^- given by

$$\mathbf{e}_{\Gamma,\Omega^-}\mathbf{b}_{\Gamma,\Omega^+}H = \left[\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{b}_{\mathbb{S}^1,\mathbb{D}^+}(H \circ f_+) \circ \phi)\right] \circ R \circ f_-^{-1},$$

where $H \in \mathcal{D}_{\mathrm{harm}}(\Omega^+)$.

To show that (2) (c) of the present theorem holds, let C be the constant in Theorem 3.3 part (2) (c). Then we have

$$\begin{aligned} D_{\Omega^-}(\mathbf{e}_{\Gamma,\Omega^-}\mathbf{b}_{\Gamma,\Omega^+}H) &= D_{\mathbb{D}^+}(\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^+}(\mathbf{b}_{\mathbb{S}^1,\mathbb{D}^+}(H\circ f_+)\circ\phi)) \\ &\leq C D_{\mathbb{D}^+}(H\circ f_+) = C D_{\Omega^+}(H), \end{aligned}$$

which completes the proof. ■

Again, conditions (2) (a) and (2) (b) are difficult to verify in practice. So we give a more practical version of the (2) → (1) implication. First, we observe that harmonic functions which extend continuously to the boundary have a transmission. That is, let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into two components Ω_1 and Ω_2, and denote the set of functions continuous on the closure of Ω_j by $\mathcal{C}(\mathrm{cl}\Omega_j)$ and the set of functions in $\mathcal{C}(\mathrm{cl}\Omega_j)$ which are additionally harmonic in Ω_j by $\mathcal{C}_{\mathrm{harm}}(\Omega_j)$. Then by the existence and uniqueness of solutions to the Dirichlet problem, given any $h_1 \in \mathcal{C}_{\mathrm{harm}}(\Omega_1)$ there is an $h_2 \in \mathcal{C}_{\mathrm{harm}}(\Omega_2)$ whose boundary values agree with those of h_1 everywhere. We thus have the well-defined maps

$$\begin{aligned} \mathcal{O}_{\Omega_1,\Omega_2} &: \mathcal{C}_{\mathrm{harm}}(\Omega_1) \to \mathcal{C}_{\mathrm{harm}}(\Omega_2), \\ \mathcal{O}_{\Omega_2,\Omega_1} &: \mathcal{C}_{\mathrm{harm}}(\Omega_2) \to \mathcal{C}_{\mathrm{harm}}(\Omega_1). \end{aligned}$$

It follows immediately from the definition of CNT boundary values that if h extends continuously to a boundary point $p \in \Gamma$ then the CNT boundary value exists and equals its CNT limit. This motivates the definition of a transmission operator $\mathbf{O}_{\Omega_1,\Omega_2}$ by restricting $\mathcal{O}_{\Omega_1,\Omega_2}$ to $\mathcal{D}_{\mathrm{harm}(\Omega_1)} \cap \mathcal{C}_{\mathrm{harm}(\Omega_1)}$, which we shall define momentarily. Before doing that we gather our observations in the following theorem.

Theorem 3.6. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$, and let Ω_1 and Ω_2 be the connected components of the complement of Γ. If there is a dense set $\mathcal{L} \subseteq \mathcal{D}_{\mathrm{harm}}(\Omega_2)$, such that $\mathcal{L} \subset \mathcal{C}(\mathrm{cl}\,\Omega_2)$, and the continuous transmission on $\mathcal{L}$ is bounded with respect to Dirichlet energy on $\mathcal{L}$, then Γ is a quasicircle.*

Here, by dense set, we mean that for any $h \in \mathcal{D}_{\mathrm{harm}}(\Omega_2)$, for all $\varepsilon > 0$ there is an element $u \in \mathcal{L}$ such that $D_{\Omega_2}(u-h) < \varepsilon$. Equivalently, $\dot{\mathcal{L}}$ is dense in $\dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_2)$.

Proof. Let $f : \mathbb{D}^+ \to \Omega_1$ and $g : \mathbb{D}^- \to \Omega_2$ be biholomorphisms. Then by Carathéodory's theorem $\phi := g^{-1} \circ f$ is a well-defined homeomorphism of $\mathbb{S}^1$.

Given a $\mathcal{L}$ satisfying the hypotheses, observe that $\mathbf{C}_g\mathcal{L}$ is dense in $\mathcal{D}_{\mathrm{harm}}(\Omega_2)$ by conformal invariance of Dirichlet energy, and by Carathéodory's theorem $\mathbf{C}_g\mathcal{L} \subset \mathcal{C}(\mathbb{S}^1)$. Also $\mathbf{b}_{\mathbb{D}^-,\mathbb{S}^1}\mathbf{C}_g\mathcal{L}$ is dense. Now for $h \in \mathbf{b}_{\mathbb{D}^-,\mathbb{S}^1}\mathbf{C}_g\mathcal{L}$, define

$$\hat{\mathbf{C}}_\phi h = \mathbf{b}_{\mathbb{D}^+,\mathbb{S}^1}\mathbf{C}_f\mathcal{O}_{\Omega_2,\Omega_1}\mathbf{C}_{g^{-1}}\mathbf{e}_{\mathbb{S}^1,\mathbb{D}^-}h$$

and note that

$$\hat{\mathbf{C}}_\phi h = \mathbf{C}_\phi h.$$

By conformal invariance of the Dirichlet spaces and the hypothesis, this is a bounded operator on $\dot{H}^{1/2}(\mathbb{S}^1)$. Thus, applying Theorem 3.4 we see that ϕ is a quasisymmetry which in turn yields that Γ is a quasicircle. ∎

Remark 3.7. We have shown that a bounded transmission exists for quasicircles separating a compact Riemann surface into two components in [57]. The results of that paper develop a foundation for applying quasisymmetric sewing techniques to boundary value problems for general Riemann surfaces, and ultimately to a "scattering theory" viewpoint of Teichmüller theory [58].

Theorem 3.5 shows that if Γ is a quasicircle and Ω_1, Ω_2 are the connected components of the complement, then $\mathcal{H}(\Gamma, \Omega_1) = \mathcal{H}(\Gamma, \Omega_2)$. We thus define

$$\mathcal{H}(\Gamma) = \mathcal{H}(\Gamma, \Omega_1) = \mathcal{H}(\Gamma, \Omega_2)$$

in this special case. Now we are ready to define the transmission operators.

Definition 3.8. We have well-defined maps

$$\begin{aligned} \mathbf{O}_{\Omega_1,\Omega_2} &= \mathbf{e}_{\Gamma,\Omega_2}\mathbf{b}_{\Gamma,\Omega_1} \colon \mathcal{D}_{\mathrm{harm}}(\Omega_1) \to \mathcal{D}_{\mathrm{harm}}(\Omega_2), \\ \mathbf{O}_{\Omega_2,\Omega_1} &= \mathbf{e}_{\Gamma,\Omega_1}\mathbf{b}_{\Gamma,\Omega_2} \colon \mathcal{D}_{\mathrm{harm}}(\Omega_2) \to \mathcal{D}_{\mathrm{harm}}(\Omega_1), \end{aligned}$$

which are bounded with respect to Dirichlet energy.

We will also use the simplified notation

$$\mathbf{O}_{1,2} = \mathbf{O}_{\Omega_1,\Omega_2}, \quad \mathbf{O}_{2,1} = \mathbf{O}_{\Omega_2,\Omega_1},$$

wherever it can be done without ambiguity.

Remark 3.9. The symbol "**O**" stands for old English "oferferian" meaning "to transmit", which could be rendered as "overfare" in modern English.

The overfare operators are inverses of each other by definition:

$$\begin{aligned} \mathrm{Id}_{\mathcal{D}_{\mathrm{harm}}(\Omega_1)} &= \mathbf{O}_{1,2}\mathbf{O}_{2,1}, \\ \mathrm{Id}_{\mathcal{D}_{\mathrm{harm}}(\Omega_2)} &= \mathbf{O}_{2,1}\mathbf{O}_{1,2}, \end{aligned}$$

where Id stands of course for the identity on the space indicated by the subscript.

Example 3.1. The overfare operators have a simple form in the case that $\Gamma = \mathbb{S}^1$:

$$[\mathbf{O}_{\mathbb{D}^+,\mathbb{D}^-}h^+](z) = h^+(1/\bar{z}), \quad [\mathbf{O}_{\mathbb{D}^-,\mathbb{D}^+}h^-](z) = h^-(1/\bar{z})$$

for $h^\pm \in \mathcal{D}_{\mathrm{harm}}(\mathbb{D}^\pm)$.

The formulation of CNT boundary values and limits is entirely conformally invariant. However, in the context of transmission, the existence of the overfare depends on the relative geometry of the domain Ω and the sphere. That is, it depends on the regularity of the boundary. It is remarkable that complete symmetry between the boundary value problems for the inside and outside domains occurs precisely for quasicircles.

Finally, we record the following result.

Corollary 3.10. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ and Ω_1 and Ω_2 be the connected components of the complement, and assume that $f\colon \mathbb{D}^+ \to \Omega_1$ and $g\colon \mathbb{D}^- \to \Omega_2$ are biholomorphisms. If there is a dense set $\mathcal{L} \subseteq \mathcal{D}_{\mathrm{harm}}(\Omega_2)$ such that $\mathcal{L} \subset \mathcal{C}(\mathrm{cl}\,\Omega_2)$, on which $\mathbf{C}_f \mathcal{O}_{2,1}$ is bounded with respect to Dirichlet energy, then Γ is a quasicircle. Conversely, if Γ is a quasicircle, then $\mathbf{C}_f \mathbf{O}_{2,1}$ is bounded with respect to Dirichlet energy.*

Proof. If Γ is a quasicircle, then $\mathbf{O}_{2,1}$ is bounded by Theorem 3.5, and $\mathbf{C}_f$ is an isometry.

Conversely, assume that there is a dense subset $\mathcal{L}$ with the stated properties. Then $\mathbf{C}_g \mathcal{L}$ is dense in $\mathcal{D}_{\mathrm{harm}}(\mathbb{D}^-)$ since $\mathbf{C}_g$ preserves the Dirichlet energy, and furthermore $\mathbf{C}_g \mathcal{L} \subset \mathcal{C}(\mathrm{cl}\,\mathbb{D}^-)$. By assumption $\mathbf{O}_{\mathbb{D}^+,\mathbb{D}^-} \mathbf{C}_f \mathcal{O}_{2,1} \mathbf{C}_{g^{-1}}$ is bounded on $\mathcal{D}_{\mathrm{harm}}(\mathbb{D}^-)$. Hence, $\mathbf{C}_{g^{-1}\circ f}$ is bounded on $\dot{H}^{1/2}(\mathbb{S}^1)$, and therefore by Theorem 3.4 $\phi = g^{-1} \circ f$ is a quasisymmetry. Thus, Γ is a quasicircle. ∎

All of the previous theorems involves boundedness with respect to Dirichlet energy, which is not a norm. Transmission is bounded if one mods out the constants. We define

$$\dot{\mathcal{H}}(\Gamma) = \mathcal{H}(\Gamma)/\sim$$

in the same way as for the Dirichlet space. For quasicircles we then have a well-defined bounded map

$$\dot{\mathbf{O}}_{1,2} = \dot{\mathbf{b}}_{\Gamma,\Omega_1} \dot{\mathbf{e}}_{\Gamma,\Omega_2}\colon \mathcal{D}_{\mathrm{harm}}(\Omega_1) \to \mathcal{D}_{\mathrm{harm}}(\Omega_2)$$

and for general Jordan curves a continuous transmission

$$\dot{\mathcal{O}}_{1,2}\colon \dot{\mathcal{C}}(\Omega_1) \to \dot{\mathcal{C}}(\Omega_2),$$

where the dotted maps are all defined in the obvious way. We also immediately have the following reformulation of the results above.

Theorem 3.11. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$. There is a bounded map*

$$\dot{\mathbf{O}}_{1,2}\colon \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_1) \to \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_2),$$

which agrees with $\dot{\mathcal{O}}_{1,2}$ on a dense subset of $\dot{\mathcal{C}}_{\mathrm{harm}}(\Omega_1)$ if and only if Γ is a quasicircle.

As in Theorem 3.6, it is enough that the map agrees with $\dot{\mathcal{O}}_{1,2}$ on a dense subset of $\dot{\mathcal{C}}_{\mathrm{harm}}(\Omega_1)$. We also observe that there is a transmission on Bergman space. Namely, if Γ is a quasicircle we define

$$\mathbf{O}'_{1,2}\colon \mathcal{A}_{\mathrm{harm}}(\Omega_1) \to \mathcal{A}_{\mathrm{harm}}(\Omega_2)$$

to be the unique operator satisfying

$$\mathbf{O}'_{1,2} d = d\mathbf{O}_{1,2}, \tag{3.6}$$

and similarly for $\mathbf{O}'_{2,1}$. It is easily checked that this is well-defined using the fact that the transmission of a constant is (the same) constant. Similarly, for arbitrary Jordan curves Γ, continuous transmission induces the transmission on harmonic one-forms

$$\mathcal{O}'_{1,2} \colon d\mathcal{C}_{\mathrm{harm}}(\Omega_1) \to d\mathcal{C}_{\mathrm{harm}}(\Omega_2).$$

Boundedness of $\mathbf{O}'$ is obviously equivalent to boundedness of $\dot{\mathbf{O}}$ since $d \colon \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_k) \to \mathcal{A}_{\mathrm{harm}}(\Omega_k)$ is a well-defined isometry.

The question of boundedness of $\mathbf{O}_{1,2}$ on inhomogeneous spaces (such as the pointed Dirichlet space) is rather problematic. Indeed, let $f \colon \mathbb{D}^+ \to \Omega_1$ and $g \colon \mathbb{D}^- \to \Omega_2$ be the corresponding welding maps; if we like, we can choose the points in the pointed norms to be 0, $f(0)$, $g(\infty)$, and ∞ in $\mathbb{D}^+$, Ω_1, Ω_2, and $\mathbb{D}^-$, respectively. If transmission were bounded on the pointed Dirichlet space, then

$$\mathbf{C}_{f^{-1}} \mathbf{O}_{2,1} \mathbf{C}_g$$

would be bounded with respect to the pointed norms on $\mathcal{D}_{\mathrm{harm}}(\mathbb{D}^{\pm})$, and thus with respect to $H^1(\mathbb{D}^{\pm})$. Thus, for a general quasisymmetry $\phi = g^{-1} \circ f$, we would obtain a bounded map

$$\mathbf{C}_\phi \colon H^{1/2}(\mathbb{S}^1) \to H^{1/2}(\mathbb{S}^1).$$

Conversely, if the above map is bounded, then $\mathbf{O}_{2,1} = \mathbf{C}_f \mathbf{C}_\phi \mathbf{C}_{g^{-1}}$ is bounded. However, this boundedness of $\mathbf{C}_\phi$ is currently unknown. In [52], the authors made an attempt to prove this, based on an extension result of Z. Ibragimov, see [28, Theorem 3.1 (5)], which enables one to extend the quasisymmetry ϕ to a quasi-isometry Φ on $\mathbb{D}$. But the problem arises in connection to showing that composition maps with such quasi-isometries are bounded on the Sobolev space $H^1(\mathbb{D})$, which to our knowledge, is also unknown. Regarding optimal results concerning boundedness of $\mathbf{C}_\phi$ on a wide range of Sobolev (and in fact even Besov) spaces on the real line, the reader is referred to G. Bourdaud [14].

We end this section with the following interesting question.

Open problem. Characterize the class of quasicircles for which transmission is norm-bounded on Dirichlet space.

4. The bounce operator and density theorems

The bounce operator. In what follows we will introduce a particular operator

$$\mathcal{D}_{\mathrm{harm}}(A) \to \mathcal{D}_{\mathrm{harm}}(\Omega),$$

which is defined for any collar neighbourhood A of the boundary Γ of a Jordan domain, which shall enable us to approximate functions in $\mathcal{D}_{\mathrm{harm}}(\Omega)$ with respect to the Dirichlet semi-norm. In connection to the interpretation of the collar neighbourhood, note that in [56, 57] we used the term collar neighbourhood referred for more general domains, but here in what follows, the special case of Definition 2.4 is enough for our purposes. We will show that functions in the Dirichlet space of a collar neighbourhood of Γ have CNT boundary values. To prove this, we need a lemma.

Lemma 4.1. *Let*

$$\begin{aligned} A &= \{z \in \mathbb{C} : r < |z| < R\}, \\ B_1 &= \{z \in \mathbb{C} : |z| < R\}, \\ B_2 &= \{z \in \mathbb{C} : r < |z|\} \cup \{\infty\}. \end{aligned}$$

For any $h \in \mathcal{D}_{\mathrm{harm}}(A)$, there is a constant $c \in \mathbb{C}$ and functions $h_i \in \mathcal{D}_{\mathrm{harm}}(B_i)$ for $i = 1, 2$, such that

$$h = h_1 + h_2 + c \log\left(|z|/R\right)$$

for all $z \in A$. If h is real, it is possible to choose h_1, h_2, and c real.

Proof. We prove the claim for h real; the general case follows by separating h into real and imaginary parts.

Choose $s \in (r, R)$ and let γ be the curve $|z| = s$ traced once counterclockwise. Set

$$c = \frac{1}{2\pi} \int_\gamma *dh.$$

Since

$$\int_\gamma *d \log\left(|z|\right) = 2\pi$$

we then have that

$$\int_\gamma *d\left(h - c \log\left(|z|\right)\right) = 0.$$

Set $H = h - c \log |z|$. Since $*dH$ is exact, H has a single-valued harmonic anti-derivative G in A, which is the harmonic conjugate of H. Thus, $F = H + iG$ is a holomorphic function in A. Now define

$$F_1(z) = \lim_{s \nearrow R} \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{\zeta - z}\, d\zeta, \quad z \in B_1,$$

and define F_2 by

$$F_2(z) = \lim_{s \searrow r} \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{\zeta - z}\, d\zeta, \quad z \in B_2 \backslash \{\infty\}$$

and $F_2(\infty) = 0$. Observe that F_1 is holomorphic on B_1 and F_2 is holomorphic on B_2. Furthermore, for $z \in A$ clearly $F(z) = F_1(z) - F_2(z)$. Now, setting $h_1 = \mathrm{Re}(F_1)$ and

$h_2 = \mathrm{Re}(F_2)$ we obtain the desired decomposition, where h_1, h_2, and c are real. It remains to show that $h_i \in \mathcal{D}_{\mathrm{harm}}(B_i)$ for $i = 1, 2$.

To show that $h_1 \in \mathcal{D}_{\mathrm{harm}}(B_1)$, it is enough to show that there is an annulus

$$A' = \{z \in \mathbb{C} : r' < |z| < R\}$$

for $r' \in (r, R)$ such that h_1 is in $\mathcal{D}_{\mathrm{harm}}(A')$, since h_1 is holomorphic on an open neighbourhood of the closure of $|z| < r'$.

Given any such $r' \in (r, R)$, the closure of A' is in B_2, and thus the restriction of h_2 to A' is in $\mathcal{D}_{\mathrm{harm}}(A')$. Furthermore, the restriction of h to A' is in $\mathcal{D}_{\mathrm{harm}}(A')$, and a direct computation shows that $\log(|z|)$ is in $\mathcal{D}_{\mathrm{harm}}(A)$, and in particular in $\mathcal{D}_{\mathrm{harm}}(A')$. Since $h_1 = h - h_2 - c\log(|z|)$, this proves that $h_1 \in \mathcal{D}_{\mathrm{harm}}(A')$, and hence in $\mathcal{D}_{\mathrm{harm}}(B_1)$.

The same argument shows that $h_2 \in \mathcal{D}_{\mathrm{harm}}(B_2)$. ■

Remark 4.2. It is easy to adapt this argument to any doubly-connected domain bordered by non-intersecting Jordan curves, even on Riemann surfaces [57]. It can be shown that the decomposition is unique, up to the additive constant which can be transferred between h_1 and h_2.

Theorem 4.3. *Let Γ be a Jordan curve bounding a Jordan domain Ω in $\overline{\mathbb{C}}$. Let A be a collar neighbourhood of Γ in Ω. If $h \in \mathcal{D}_{\mathrm{harm}}(A)$ then h has* CNT *boundary values except possibly on a null set with respect to Ω. Furthermore, there is an $H \in \mathcal{D}_{\mathrm{harm}}(\Omega)$ whose* CNT *boundary values agree with those of h except possibly on a null set.*

Proof. By definition of collar neighbourhood, for some $p \in \Omega$ and $r \in (0, 1)$, we have $A = A_{p,r} = f(\mathbb{A}_r)$, where $\mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < 1\}$. By conformal invariance of Dirichlet spaces and CNT boundary values, it suffices to show this for $\Gamma = \mathbb{S}^1$, $\Omega = \mathbb{D}^+$, and $A = \mathbb{A}_r$.

Let $h \in \mathcal{D}_{\mathrm{harm}}(\mathbb{A}_r)$. By Lemma 4.1, $h = h_1 + h_2 + c\log|z|$ for some functions $h_i \in \mathcal{D}_{\mathrm{harm}}(B_i)$, $i = 1, 2$ where $B_1 = \mathbb{D}^+$ and $B_2 = \{z : |z| > r\} \cup \{\infty\}$. Now $c\log|z|$ extends continuously to 0 on $\mathbb{S}^1$, and thus the non-tangential boundary values exist and are zero everywhere on $\mathbb{S}^1$. Since $h_1 \in \mathcal{D}(\mathbb{D}^+)$, it has non-tangential boundary values except possibly on a null set by a direct application of Beurling's theorem (Theorem 2.24). Now h_2 is continuous on an annular neighbourhood of $\mathbb{S}^1$ and thus the non-tangential boundary values exist with respect to $\mathbb{D}^+$ everywhere. Thus, the non-tangential boundary values of h exist except possibly on a null set.

Furthermore, $u(z) = h_2(1/\overline{z}) \in \mathcal{D}_{\mathrm{harm}}(\mathbb{D}^+)$ is continuous on an open neighbourhood of $\mathbb{D}^+$, and its non-tangential boundary values exist everywhere with respect to $\mathbb{D}^+$ and equal those of h_2 with respect to $\mathbb{D}^+$. Thus, the function $H = h_1 + u$ is in $\mathcal{D}_{\mathrm{harm}}(\mathbb{D}^+)$ has non-tangential boundary values equal to h except possibly on a null set. ■

Remark 4.4. The proof actually shows a slightly stronger statement: there is an $H \in \mathcal{D}_{\mathrm{harm}}(\Omega)$ whose CNT boundary values exist and equal those of h, precisely where those of h exist.

Definition 4.5. Since the function $H \in \mathcal{D}_{\text{harm}}(\Omega)$ is uniquely determined by its CNT boundary values on Γ, Theorem 4.3 induces a well-defined operator

$$\mathbf{G}_{A,\Omega}\colon \mathcal{D}_{\text{harm}}(A) \to \mathcal{D}_{\text{harm}}(\Omega)$$

for any collar neighbourhood A of the boundary Γ of a Jordan domain. We call this the "bounce" operator.

It follows immediately from the conformal invariance of the Dirichlet space and CNT limits that the bounce operator is conformally invariant. That is, if $f\colon \Omega' \to \Omega$ is a biholomorphism and A' is the domain such that $f(A') = A$, then

$$\mathbf{G}_{A',\Omega'}(h \circ f) = (\mathbf{G}_{A,\Omega}\, h) \circ f. \tag{4.1}$$

We now prove boundedness of the bounce operator.

Theorem 4.6. *Let Ω be a Jordan domain in $\overline{\mathbb{C}}$ bounded by a Jordan curve Γ. For any collar neighbourhood A of Γ in Ω, $\mathbf{G}_{A,\Omega}$ is bounded with respect to the Dirichlet energy; equivalently, $\dot{\mathbf{G}}_{A,\Omega}$ is bounded. Furthermore, $\mathbf{G}_{A,\Omega}$ is bounded.*

Proof. By conformal invariance of the Dirichlet norm, Dirichlet energy, and CNT limits, it suffices to prove this for $A = \mathbb{A}_r = \{z : r < |z| < 1\}$ and $\Omega = \mathbb{D}^+$.

Let $h \in \mathcal{D}_{\text{harm}}(A)$. Then by [38, Proposition 1.25.2], h is in $H^1(A)$. By Theorem 2.34, $\mathbf{G}_{A,\mathbb{D}^+}h$ is the unique Sobolev extension of the Sobolev trace of h in $H^{1/2}(\mathbb{S}^1)$. Furthermore, by the result on the unique Sobolev extension, see e.g. [65, Proposition 4.5, p. 334], and the fact that $\mathbb{S}^1 \subsetneq \partial A$, we have

$$\|h|_\Gamma\|_{H^{1/2}(\mathbb{S}^1)} \leq \|h|_{\partial\Omega}\|_{H^{1/2}(\partial A)} \lesssim \|h\|_{H^1(A)}.$$

Also, by the existence of the unique solution to the Dirichlet problem with boundary data in Sobolev spaces (see e.g. [65, Proposition 1.7, p. 360]), the harmonic Sobolev extension H of $h|_{\mathbb{S}^1}$ satisfies

$$\|H\|_{H^1(\mathbb{D}^+)} \lesssim \|h|_{\mathbb{S}^1}\|_{H^{1/2}(\mathbb{S}^1)}.$$

This together with the estimate for $\|h|_\Gamma\|_{H^{1/2}(\mathbb{S}^1)}$ above yields that

$$\|\mathbf{G}_{A,\mathbb{D}^+}h\|_{H^1(\mathbb{D}^+)} \lesssim \|h\|_{H^1(A)}. \tag{4.2}$$

From this, (2.10) and (2.14), one can easily deduce the boundedness of $\mathbf{G}_{A,\Omega}$. Now if one applies (4.2) to the harmonic function $h - h_A$, where h_A is the average of h given by $\frac{1}{|A|}\int_A h$, then one has that

$$\|\mathbf{G}_{A,\mathbb{D}^+}h - \mathbf{G}_{A,\mathbb{D}^+}h_A\|_{H^1(\mathbb{D}^+)} \lesssim \|h - h_A\|_{H^1(A)}.$$

Moreover, we know that

$$\begin{aligned} D_{\mathbb{D}^+}(\mathbf{G}_{A,\mathbb{D}^+}h)^{1/2} &= D_{\mathbb{D}^+}(\mathbf{G}_{A,\mathbb{D}^+}h - \mathbf{G}_{A,\mathbb{D}^+}h_A)^{1/2} \\ &\leq \|\mathbf{G}_{A,\mathbb{D}^+}h - \mathbf{G}_{A,\mathbb{D}^+}h_A\|_{H^1(\mathbb{D}^+)} \end{aligned}$$

and that

$$\begin{aligned}\|h-h_A\|_{H^1(A)} &= D_A(h-h_A)^{1/2} + \|h-h_A\|_{L^2(A)} \\ &\le D_A(h)^{1/2} + CD_A(h)^{1/2} \lesssim D_A(h)^{1/2},\end{aligned}$$

where the inequality $\|h-h_A\|_{L^2(\Omega)} \lesssim D_A(h)^{1/2}$ is the well-known Poincaré–Wirtinger inequality. Thus, $D_{\mathbb{D}^+}(\mathbf{G}_{A,\mathbb{D}^+}h) \lesssim D_A(h)$, as desired. ■

Theorem 4.7. *Let Ω be a Jordan domain in $\overline{\mathbb{C}}$ bounded by Γ and let A be a collar neighbourhood of Γ in Ω. The set $\dot{\mathbf{G}}_{A,\Omega}(\mathcal{D}(A))$ is dense in $\dot{\mathcal{D}}_{\text{harm}}(\Omega)$, and $\mathbf{G}_{A,\Omega}(\mathcal{D}(A))$ is dense in $\mathcal{D}_{\text{harm}}(\Omega)$ with respect to the pointed norm.*

Proof. By conformal invariance of the Dirichlet semi-norm and (4.1), we may assume that $A = \mathbb{A}_r$ and $\Omega = \mathbb{D}^+$ as above.

First, observe that the polynomials $\mathbb{C}[z, z^{-1}]$ are contained in $\mathcal{D}(\mathbb{A}_r)$. But for any integer $n > 0$, we have

$$\mathbf{G}_{\mathbb{A}_r,\mathbb{D}^+} z^n = z^n \quad \text{and} \quad \mathbf{G}_{\mathbb{A}_r,\mathbb{D}^+} z^{-n} = \overline{z}^n,$$

so $\mathbf{G}_{\mathbb{A}_r,\mathbb{D}^+}\mathbb{C}[z, z^{-1}] = \mathbb{C}[z, \overline{z}]$. Since $\mathbb{C}[z, \overline{z}]$ is a dense subset of $\mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ this proves the claim. ■

In the next theorem, the norm on $\mathcal{D}(A)$ is taken to be the restriction of the norm on $\mathcal{D}_{\text{harm}}(A)$, choosing Γ to be the outer curve (see (2.12) and (2.13)).

Theorem 4.8. *Let A be any domain in $\overline{\mathbb{C}}$ bounded by two non-intersecting Jordan curves, such that 0 and ∞ are in distinct components of the complement of the closure of A. Then Laurent polynomials $\mathbb{C}[z, z^{-1}]$ are dense in $\mathcal{D}(A)$ and $\dot{\mathbb{C}}[z, z^{-1}]$ is dense in $\dot{\mathcal{D}}(A)$.*

Proof. Without loss of generality assume that the component of the complement of Γ_2 containing A also contains ∞, and let B_2 denote this component. Let B_1 then denote the component of the complement of Γ_1 containing A; it must also contain 0. We have that $A = B_1 \cap B_2$.

Now let $f_i \colon \mathbb{D}^+ \to B_i$ be biholomorphisms for $i = 1, 2$. Let $\gamma_i^r = f_i(|z| = r)$ for $r \in (0, 1)$, endowed with positive orientations with respect to 0. For any $h \in \mathcal{D}(A)$, setting

$$h_i(z) = \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{\gamma_i^r} \frac{h(\zeta)}{\zeta - z}\, d\zeta \quad z \in B_i,\ i = 1, 2$$

we have that h_i are holomorphic on B_i, and $h = h_1 - h_2$.

We will show that h_i are in $\mathcal{D}(B_i)$ for $i = 1, 2$. Let C_1 denote the open domain in B_1 bounded by Γ_1 and $f_1(\gamma_2^s)$ for s chosen close enough to 1 that it is entirely in A. This can be done, because the function $z \mapsto |f_1^{-1}(z)|$ is continuous on B_1, and is strictly less than one on B_1. This function has a maximum $R < 1$ on Γ_2 since Γ_2 is compact, so we can choose $s \in (R, 1)$. To show that $h_1 \in \mathcal{D}(B_1)$ it suffices to show that $h_1 \in \mathcal{D}(C_1)$, since h_1 is holomorphic on an open neighbourhood of $f(|z| \leq s)$. Now $h \in \mathcal{D}(C_1)$ since $C_1 \subseteq A$, and $h_2 \in \mathcal{D}(C_1)$, since the closure of C_1 is contained in B_2. Since $h_1 = h + h_2$, this proves the claim. A similar argument shows that $h_2 \in \mathcal{D}(B_2)$.

Now we prove the density claims. It clearly suffices to prove the first claim. For the purposes of this proof, by a "homogeneous" polynomial p in $\mathbb{C}[z]$ or $\mathbb{C}[1/z]$ we mean one satisfying $\hat{p}(0) = 0$. This can be arranged by adding a constant; in the former case, these are just the polynomials with zero constant term.

Fix $h \in \mathcal{D}_{\text{harm}}(A)$ and let the decomposition $h = h_1 - h_2$ be as above. Now B_1 is a Jordan domain, and hence a Carathéodory domain, so polynomial one-forms $\mathbb{C}[z]\,dz$ are dense in $\mathcal{A}(B_1)$ [37, Vol. 3, Section 16, Theorem 3.20]. Similarly, $\mathbb{C}[z]\,dz$ is dense in $\mathcal{A}(1/B_2)$, so $z^{-2}\mathbb{C}[1/z]\,dz$ is dense in $\mathcal{A}(B_2)$. Therefore, given any $\varepsilon > 0$ there exists $p_1 \in \mathbb{C}[z]$ and $p_2 \in \mathbb{C}[1/z]$ such that

$$\|dh_i - dp_i\|_A \leq \|dh_i - dp_i\|_{B_i} < \varepsilon/2.$$

We may choose these polynomials to be homogeneous, without altering the estimate. Set $c = \hat{h}(0)$ and let $p = p_1 - p_2 + c$. Using $h = h_1 - h_2$ we see that

$$\|h - p\|_{\mathcal{D}_{\text{harm}}(A)} = \left(\|dh_1 - dh_2 - dp_1 + dp_2\|_A^2 + |c - \hat{h}(0)|^2\right)^{1/2} < \varepsilon.$$

This proves the claim. ∎

Corollary 4.9. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ and let Ω_1 and Ω_2 be the connected components of the complement. Let A_1 and A_2 be collar neighbourhoods of Γ in Ω_1 and Ω_2 respectively, and let $U = A_1 \cup A_2 \cup \Gamma$. Let $\mathbf{R}_i : \mathcal{D}(U) \to \mathcal{D}(A_i)$ denote restriction from U to A_i for $i = 1, 2$. Then for $i = 1, 2$ $\mathbf{R}_i(\mathcal{D}(U))$ is dense in $\mathcal{D}(A_i)$ with respect to the norms and $\dot{\mathbf{R}}_i(\mathcal{D}(U))$ is dense in $\dot{\mathcal{D}}(A_i)$.*

Proof. Observe that U is open, so the statement of the theorem makes sense.

Now A_1 and A_2 are each bounded by two non-intersecting Jordan curves in $\overline{\mathbb{C}}$. By applying a Möbius transformation and conformal invariance of the Dirichlet spaces and Dirichlet semi-norm, we can assume that ∞ and 0 are each contained in the interior of one of the connected components of the complement of U, and not both in the same one. In that case, the same holds for A_1 and A_2. Thus, $\mathbb{C}[z, z^{-1}]$ is dense in $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ by Theorem 4.8. Since $\mathbb{C}[z, z^{-1}] \subseteq \mathcal{D}(U)$, the theorem is proven. ∎

It is an indispensable fact that the limiting integral of harmonic functions against L^2 one-forms is unaffected by application of the bounce operator.

Lemma 4.10 (Anchor lemma). *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ bounding a Jordan domain Ω. Let A be a collar neighbourhood of Γ in Ω and let $\Gamma_\varepsilon = f(|z| = e^{-\varepsilon})$ for a biholomorphism $f : \mathbb{D}^+ \to \Omega$ and $\varepsilon > 0$. For any $h \in \mathcal{D}_{\text{harm}}(A)$ and $\alpha \in \mathcal{A}(A)$, we have*

$$\lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} \alpha(w) h(w) = \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} \alpha(w) \mathbf{G}_{A,\Omega} h(w). \tag{4.3}$$

In particular, if h has CNT *boundary values equal to zero except possibly on a null set, then for any* $\alpha \in \mathcal{A}(A)$*, we have*

$$\lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} \alpha(w) h(w) = 0.$$

Proof. We assume that Γ_ε are positively oriented with respect to 0. The fact that the left integral in (4.3) is finite follows from the fact that α and dh are L^2 on A, since fixing ε_0 such that Γ_{ε_0} is in A, we have by Stokes' theorem that

$$\lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} \alpha(w) h(w) = \int_{\Gamma_{\varepsilon_0}} \alpha(w) h(w) + \iint_{A'} \alpha \wedge dh,$$

where $A' \subset A$ is the region bounded by Γ_{ε_0} and Γ.

Setting $\tilde{\alpha}(w) = \alpha(f(w)) f'(w)$ and $\tilde{h}(w) = h(f(w))$, and denoting the circle $|z| = e^{-\varepsilon}$ traced counterclockwise by C_ε, we have

$$\int_{\Gamma_\varepsilon} \alpha(w) h(w) = \int_{C_\varepsilon} \tilde{\alpha}(w) \tilde{h}(w),$$

so it suffices to prove the claim for $A = \mathbb{A} = \{z : e^{-\varepsilon_0} < |z| < 1\}$, $\Omega = \mathbb{D}^+$, and $\Gamma_\varepsilon = C_\varepsilon$.

We first show (4.3) for $\alpha(w) = w^n dw$ for some integer n. By Lemma 4.1 we can write $h = h_1 + h_2 + c \log |z|$, where $h_1 \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ and $h_2 \in \mathcal{D}_{\text{harm}}(B)$, where $B = \{z : |z| > e^{-\varepsilon_0}\} \cup \{\infty\}$. Now α and h_2 extend continuously to $\mathbb{S}^1$; thus so does $\mathbf{G}_{\mathbb{A},\mathbb{D}^+} h_2$, and so trivially

$$\lim_{\varepsilon \searrow 0} \int_{C_\varepsilon} \alpha(w) h_2(w) = \lim_{\varepsilon \searrow 0} \int_{C_\varepsilon} \alpha(w) \mathbf{G}_{\mathbb{A},\mathbb{D}^+} h_2(w).$$

Similarly,

$$\lim_{\varepsilon \searrow 0} \int_{C_\varepsilon} \alpha(w) \log |w| = \lim_{\varepsilon \searrow 0} \int_{C_\varepsilon} \alpha(w) \mathbf{G}_{\mathbb{A},\mathbb{D}^+} \log |w|;$$

in fact, both sides are zero. Finally, since $\mathbf{G}_{\mathbb{A},\mathbb{D}^+} h_1 = h_1$, the claim follows.

Thus, the claim holds for any $\alpha(w) = p(w)\, dw$ for $p(w) \in \mathbb{C}[z, 1/z]$. Now the set of such α is dense in $\mathcal{A}(\mathbb{A})$. This follows from the density of $\mathbb{C}[z, 1/z]$ in $\mathcal{D}(\mathbb{A})$ (which is a special case of Theorem 4.8), and the fact that for some constant k, $\alpha - k/z$ is exact. So the proof of the claim will be complete if it can be shown that for h fixed,

$$\alpha \mapsto \lim_{\varepsilon \searrow 0} \int_{C_\varepsilon} \alpha(w) h(w)$$

is a continuous functional on $\mathcal{A}(\mathbb{A})$.

With ε_0 and A' the region bounded by Γ_{ε_0} and $\mathbb{S}^1$, let $M = \sup_{w \in \Gamma_{\varepsilon_0}} |h(w)|$. Since Γ_{ε_0} is a compact subset of $\mathbb{A}$, by a standard result for Bergman spaces there is a constant C independent of $\alpha(w) = a(w) dw$ such that

$$\sup_{w \in \Gamma_{\varepsilon_0}} |a(w)| \lesssim \|\alpha\|_{A(\mathbb{A})}.$$

Therefore, Stokes' theorem and Cauchy–Schwarz's inequality yield that

$$\begin{aligned}\Big|\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon}\alpha(w)h(w)\Big| &= \Big|\int_{\Gamma_{\varepsilon_0}}\alpha(w)h(w)+\iint_{A'}\alpha\wedge dh\Big|\\ &\leq 2\pi e^{-\varepsilon_0}M\sup_{w\in\Gamma_{\varepsilon_0}}|\alpha(w)|+\|\alpha\|_{\mathcal{A}(A')}\|h\|_{\mathcal{D}_{\mathrm{harm}}(A')}\\ &\lesssim \big(2\pi e^{-\varepsilon_0}M+\|h\|_{\mathcal{D}_{\mathrm{harm}}(A')}\big)\|\alpha\|_{\mathcal{A}(\mathbb{A})},\end{aligned}$$

which completes the proof of (4.3).

The proof of the second claim follows immediately from the observation that if h has CNT boundary values zero except possibly on a null set, then $\mathbf{G}_{A,\mathbb{D}^+}h=0$. ■

5. Schiffer and Cauchy operators

5.1. Schiffer operators

We will define certain operators on the Bergman space of anti-holomorphic one-forms which we call "Schiffer operators". We require an identity to facilitate the definition.

Now let g_Ω be the Green function of Ω from Section 2.1. Schiffer considered the following kernel function:

$$L_\Omega(z,w)\,dz\,dw=\frac{1}{\pi i}\frac{\partial^2 g_\Omega}{\partial z\partial w}(z,w)\,dz\,dw.$$

Note that L_Ω is a meromorphic function in z on Ω with a pole of order two at $z=w$ and no other poles. In fact, it is symmetric, so it is also holomorphic in w except for a pole at $w=z$.

Example 5.1. For $\Omega=\mathbb{D}^+$, one has

$$g_{\mathbb{D}^+}(z,w)=-\log\frac{|z-w|}{|1-\bar{w}z|},$$

and so

$$L_{\mathbb{D}^+}(z,w)=\frac{-1}{2\pi i}\frac{1}{(w-z)^2}.$$

Moreover, by the conformal invariance of the Green function, if φ is a conformal map of Ω onto a domain Ω', then

$$L_{\Omega'}(\varphi(z),\varphi(w))\,\varphi'(z)\varphi'(w)=L_\Omega(z,w),\tag{5.1}$$

which using Riemann's mapping theorem can be used to calculate the Schiffer kernels of simply connected domains in the plane.

Theorem 5.1. *Let* Γ *be a Jordan curve, and let* Ω *be one of the connected components of the complement of* Γ *in* $\overline{\mathbb{C}}$. *Let* $g_\Omega(z,w)$ *denote Green's function of* Ω. *Then for any one-form* $\overline{\alpha} = \overline{h(z)}\, d\overline{z} \in \overline{\mathcal{A}(\Omega)}$, *we have*

$$\left(\iint_\Omega L_\Omega(z,w)\, \overline{h(w)}\, d\overline{w} \wedge dw \right) \cdot dz = 0 \tag{5.2}$$

as a principal value integral.

Proof. Let $f : \mathbb{D}^+ \to \Omega$ be a biholomorphism, chosen so that $f(0) = z$ and η be such that $f(\eta) = w$. Let Γ_ε be the image of the circle with centre at the origin and radius $e^{-\varepsilon}$ under the biholomorphic map f, with positive orientation with respect to w. By Stokes' theorem, if we denote by C_r the circle of radius r centred at w with winding number one with respect to w, then

$$\begin{aligned}
&\iint_\Omega L_\Omega(z,w)\overline{h(w)}\, d\overline{w} \wedge dw \cdot dz \\
&\quad = \lim_{r \searrow 0} \iint_{\Omega \setminus B(w;r)} L_\Omega(z,w)\overline{h(w)}\, d\overline{w} \wedge dw \cdot dz \\
&\quad = \lim_{\varepsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\varepsilon} \frac{\partial g_\Omega}{\partial z}(z,w)\overline{h(w)}\, d\overline{w}\, dz - \lim_{r \searrow 0} \frac{1}{\pi i} \int_{C_r} \frac{\partial g_\Omega}{\partial z}(z,w)\overline{h(w)}\, d\overline{w}\, dz.
\end{aligned}$$

Note that all integrals take place over the w variable while z is fixed. To say that the output of the integral on the left-hand side is zero as a form, is equivalent to demanding that for fixed z the coefficient of dz of the output is zero. Therefore, it is enough to show that

$$\lim_{\varepsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\varepsilon} \frac{\partial g_\Omega}{\partial z}(z,w)\overline{h(w)}\, d\overline{w} = 0, \tag{5.3}$$

$$\lim_{r \searrow 0} \frac{1}{\pi i} \int_{C_r} \frac{\partial g_\Omega}{\partial z}(z,w)\overline{h(w)}\, d\overline{w} = 0. \tag{5.4}$$

Now Green's function of the disk is given by

$$g_{\mathbb{D}^+}(\zeta,\eta) = -\log\left| \frac{\zeta - \eta}{1 - \overline{\eta}\zeta} \right|,$$

so by conformal invariance of Green's function $g_\Omega(f(\zeta), f(\eta)) = g_{\mathbb{D}^+}(\zeta,\eta)$, we have that

$$\frac{\partial g_\Omega}{\partial z}(z, f(\eta)) = \frac{1}{f'(0)} \frac{\partial}{\partial \zeta}\bigg|_{\zeta=0} g_{\mathbb{D}^+}(\zeta,\eta) = \frac{1}{2f'(0)}\left(\frac{1}{\eta} - \overline{\eta}\right). \tag{5.5}$$

Now let $\mathbb{A}_r = \{z : r < |z| < 1\}$ for any $r \in (0,1)$ and set $A = f(\mathbb{A}_r)$. One can see explicitly from (5.5) that for fixed z, the function

$$K(\eta) := \frac{\partial g_\Omega}{\partial z}(z, f(\eta))$$

is in $\mathcal{D}_{\mathrm{harm}}(\mathbb{A}_r)$, so by conformal invariance of the Dirichlet space

$$k(w) = K(f^{-1}(w)) = \frac{\partial g_\Omega}{\partial z}(z, w)$$

is in $\mathcal{D}_{\mathrm{harm}}(A)$. Thus, we can apply the Anchor lemma (Lemma 4.10) to $\overline{k}$ and $\alpha(w) = h(w)\, dw$ to conclude that the integral in (5.3) is zero. On the other hand, for w in an open neighbourhood of z, by (5.5) (or directly from the definition of Green's function) we can write

$$\frac{\partial g_\Omega}{\partial z}(z, w) = \frac{1}{2(w - z)} + H(w),$$

where $H(w)$ is harmonic in w. Inserting this into the left-hand side of (5.4), we obtain that the integral is indeed zero. ■

We may now define the Schiffer operator.

Definition 5.2. Let Γ be a Jordan curve in $\overline{\mathbb{C}}$, and let Ω_1 and Ω_2 denote the connected components of the complement of Γ. For $\overline{h(w)}d\overline{w} \in \overline{\mathcal{A}(\Omega_1)}$, we define for $j = 1, 2$:

$$\mathbf{T}_{\Omega_1,\Omega_j}\overline{h(w)}d\overline{w} = \frac{1}{\pi} \iint_{\Omega_1} \frac{\overline{h(w)}}{(w - z)^2} \frac{d\overline{w} \wedge dw}{2i} \cdot dz, \quad z \in \Omega_j. \tag{5.6}$$

Note that the output is a one-form on Ω_j. In the case that $j = 1$, we interpret (5.6) as a principal value integral. We will see that this is in general a bounded map into $A(\Omega_j)$, and in that role we refer to $\mathbf{T}_{\Omega_1,\Omega_j}$ as Schiffer operators.

As in the case of the overfare operator $\mathbf{O}$, we will use the notation $\mathbf{T}_{j,k}$ in place of $\mathbf{T}_{\Omega_k,\Omega_k}$ wherever possible.

The operators $\mathbf{T}_{j,k}$ were first defined by Schiffer [49]. Schiffer investigated these operators extensively with others; see e.g. Bergman and Schiffer [11], Schiffer and Spencer [51], and Schiffer [18].

The terminology surrounding the Schiffer operators is not entirely consistent. As a Calderón–Zygmund singular integral operator acting on functions in the plane, the Schiffer operator is bounded on L^2 (more generally on L^p, $1 < p < \infty$). The integral operator on general functions in $L^2(\mathbb{C})$ is called the Beurling transform. It is also sometimes called the Hilbert transform, a term used more widely (in the harmonic analysis and integral equations context) for a principal value integral along the real line (the explicit formula is (5.22) ahead, if one chooses there $\Gamma = \mathbb{R}$). Napalkov and Yulmukhametov use the term Hilbert transform to refer specifically to $\mathbf{T}_{1,2}$. Of course these integral operators are all closely related. We reserve the term "Schiffer operator" for the restriction of the singular integral operator to anti-holomorphic functions on a subset of $\overline{\mathbb{C}}$.

The "nesting" – that the kernel function is derived from the Green's function of a larger domain than the domain of integration – is a central feature of the Schiffer operators, which he explored at length in [18]. The Schiffer kernel is closely related to the so-called fundamental differential. On general domains and Riemann surfaces, there is also a related operator derived from integrating against the Bergman kernel obtained from a larger domain. (This does not appear in the present paper, because the Bergman kernel of the sphere is zero). Adding to the terminological confusion described above, some authors refer to the fundamental bi-differential as the Bergman kernel. On the double of a Riemann surface, certain identities relate the Schiffer and Bergman kernels [51].

As a first step, we establish the existence of this integral. Assume for the moment that Ω_1 is bounded, that is, $\infty \in \Omega_2$. For fixed $z \in \Omega_j$, the integrand $1/(w-z)^2$ is obviously in $L^2(\Omega_1)$, so this is immediate. If $z \in \Omega_1$, for a biholomorphism $f : \mathbb{D}^+ \to \Omega_1$, let Γ_ε be the image of the curve $|z| = e^{-\varepsilon}$ under f with the same orientation, and let C_r be the circle centred on z traced counterclockwise. Then

$$\begin{aligned} &\frac{1}{\pi} \iint_{\Omega_1} \frac{\overline{h(w)}}{(w-z)^2} \frac{d\bar{w} \wedge dw}{2i} \cdot dz \\ &\quad = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{\overline{h(w)}\, d\bar{w}}{(w-z)}\, dz - \lim_{r \searrow 0} \frac{1}{2\pi i} \int_{C_r} \frac{\overline{h(w)}\, d\bar{w}}{(w-z)}\, dz. \end{aligned} \tag{5.7}$$

Let $\mathbb{A}_s = \{z : s < |z| < 1\}$ where s is fixed so that z is not in the closure of $B_s = f(\mathbb{A}_s)$. The first limit exists, by the fact that $h(w)\, dw$ and $dw/(w-z)^2$ are in $\mathcal{A}(B_s)$ and

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{\overline{h(w)}\, d\bar{w}}{(w-z)}\, dz \\ &\quad = \frac{1}{2\pi i} \int_{\Gamma_{-\log s}} \frac{\overline{h(w)}\, d\bar{w}}{(w-z)}\, dz + \frac{1}{\pi} \iint_{B_s} \frac{\overline{h(w)}}{(w-z)^2} \frac{d\bar{w} \wedge dw}{2i} \cdot dz. \end{aligned}$$

The limit of the second term in (5.7) can be shown to be zero by an explicit computation.

Theorem 5.1 can now be applied to de-singularize the kernel function. For $\alpha(w) = h(w)\, dw \in \mathcal{A}(\Omega_1)$, we have

$$\mathbf{T}_{\Omega_1,\Omega_1} \bar{\alpha}(z) = \iint_{\Omega_1} \left(\frac{1}{2\pi i} \frac{1}{(w-z)^2} - L_{\Omega_1}(z,w) \right) \overline{h(w)}\, d\bar{w} \wedge dw \cdot dz \tag{5.8}$$

since this new term does not have an effect on the existence or value of the integral.

We can deal with the general case that Ω_1 might be unbounded by establishing the invariance of the integrals under Möbius transformations, which is interesting on its own. To this end, define the pull-back of $\bar{\alpha}$ under $w = M(z)$ by

$$M^* \bar{\alpha}(z) = \overline{h(M(z)) M'(z)}\, d\bar{z},$$

and similarly define the pull-back of $\beta(w) = g(w)\, dw$ by

$$M^* \beta(z) = g(M(z)) M'(z)\, dz.$$

Theorem 5.3. *If $M : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a Möbius transformation taking Ω_j bijectively to $\widetilde{\Omega}_j$ for $j = 1, 2$, then for all $\bar{\alpha} = \overline{h(w)}\, d\bar{w} \in \overline{\mathcal{A}(\widetilde{\Omega}_1)}$, we have*

$$[\mathbf{T}_{\Omega_1,\Omega_j}\, M^* \bar{\alpha}] = M^* [\mathbf{T}_{\widetilde{\Omega}_1,\widetilde{\Omega}_j}\, \bar{\alpha}]. \tag{5.9}$$

Proof. Assume first that $j = 2$. Setting $w = M(z)$, $\eta = M(\zeta)$, $\overline{\alpha(\eta)} = \overline{h(\eta)}\, d\overline{\eta}$ and using the identity

$$\frac{M'(\zeta)M'(z)}{(M(\zeta) - M(z))^2} = \frac{1}{(\zeta - z)^2}, \tag{5.10}$$

which holds for arbitrary Möbius transformations, yields that

$$\begin{aligned} M^*[\mathbf{T}_{\tilde{\Omega}_1,\tilde{\Omega}_2}\overline{\alpha}](z) &= \frac{1}{\pi}\iint_{\tilde{\Omega}_1} \frac{\overline{h(\eta)}}{(\eta - M(z))^2} \frac{d\overline{\eta} \wedge d\eta}{2i} \cdot M'(z)\, dz \\ &= \frac{1}{\pi}\iint_{\Omega_1} \frac{\overline{h(M(\zeta))}}{(M(\zeta) - M(z))^2} \overline{M'(\zeta)} M'(\zeta) \frac{d\overline{\zeta} \wedge d\zeta}{2i} \cdot M'(z)\, dz \\ &= \frac{1}{\pi}\iint_{\Omega_1} \frac{\overline{h(M(\zeta))M'(\zeta)}}{(\zeta - z)^2} \frac{d\overline{\zeta} \wedge d\zeta}{2i} \cdot dz \\ &= [\mathbf{T}_{\Omega_1,\Omega_2} M^*\overline{\alpha}](z). \end{aligned}$$

In the case that $j = 1$, we use (5.1) with $\varphi = M$. When combined with (5.10), the argument above may be repeated using the expression (5.8). ■

Note that the de-singularization of the integral allowed the application of change of variables in the proof above. As a consequence, we see that the Möbius transformation preserves the original principal value integral. This can also be shown directly.

Remark 5.4. If one views the Schiffer operators as acting on a Bergman space of functions $\overline{h(z)}$ rather than on the L^2 space of one-forms $\overline{h(z)}\, d\overline{z}$, their Möbius invariance is obscured.

As promised, Theorem 5.3 implies the existence of the integrals defining the Schiffer operator, since we may apply a Möbius transformation to reduce the general case to the case that Ω_1 is bounded, which we dealt with above.

Remark 5.5. If $z \in \Omega_j$, the meaning of this one-form at $z = \infty$ is obtained by applying the change of coordinates $z = 1/\zeta$, $dz = -d\zeta/\zeta^2$ to express it in coordinates at ∞:

$$\frac{1}{\pi}\iint_{\Omega_1} \frac{\overline{h(w)}}{(1 - w\zeta)^2} \frac{d\overline{w} \wedge dw}{2i} \cdot d\zeta, \quad \zeta \in 1/\Omega_j.$$

Alternatively, one may transform both the input and output simultaneously using Theorem 5.3 with $M(z) = 1/z$.

Finally, we have the following.

Theorem 5.6. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$, and let Ω_1 and Ω_2 be the components of the complement of Γ in $\overline{\mathbb{C}}$. The Schiffer operators $\mathbf{T}_{\Omega_1,\Omega_j}$ are bounded from $\overline{\mathcal{A}(\Omega_1)}$ to $\mathcal{A}(\Omega_j)$ for $j = 1, 2$.*

Proof. By Theorem 5.3, we may assume that $\infty \in \Omega_2$, so that Ω_1 is bounded. The integrands in the definitions of $\mathbf{T}_{\Omega_1,\Omega_2}$ and $\mathbf{T}_{\Omega_1,\Omega_1}$ given in (5.6), and (5.8) respectively are non-singular and holomorphic in z for each $w \in \Omega_2$ and $w \in \Omega_1$ (in each case), and furthermore both integrals are locally bounded in z. Therefore, the holomorphicity of $\mathbf{T}_{\Omega_1,\Omega_j}$ follows by moving the $\overline{\partial}$ inside (5.6) and (5.8), and using the holomorphicity of the integrands in each case.

The L^p-boundedness of these operators for $1 < p < \infty$, considered as singular integral operators, is a consequence of the boundedness of singular integral operators of Calderón–Zygmund type, see e.g. [35, p. 26]. ∎

5.2. Cauchy operator

As usual, consider a Jordan curve Γ in $\overline{\mathbb{C}}$. For now we assume that ∞ is not in Γ. Let Ω_1 and Ω_2 denote the components of the complement.

Definition 5.7. For $h \in \mathcal{D}(\Omega_1)$ we will consider a kind of Cauchy integral obtained as follows. Let $f : \mathbb{D} \to \Omega_1$ be a biholomorphism. If we let Γ_ε be the image of the closed curve $|z| = e^{-\varepsilon}$ under f, with the same orientation, and $q \notin \Gamma$, then we define

$$\begin{aligned} \mathbf{J}^q_{\Omega_1} h(z) &= \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} h(w) \left(\frac{1}{w-z} - \frac{1}{w-q} \right) dw \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} h(w) \frac{z-q}{(w-z)(w-q)}\, dw \qquad z \in \overline{\mathbb{C}} \backslash \Gamma. \end{aligned} \tag{5.11}$$

The term involving q amounts to an arbitrary choice of normalization. In the case that $q = \infty$, this reduces to

$$\mathbf{J}^q_{\Omega_1} h(z) = \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} \frac{h(w)}{w-z}\, dw \quad z \in \overline{\mathbb{C}} \backslash \Gamma.$$

This is almost a Cauchy integral, of course. We will motivate the definition of $\mathbf{J}^q_{\Omega_1}$ after first establishing some of its properties.

First, we observe that $\mathbf{J}^q$ is Möbius invariant in a certain sense. The invariance follows from the identity

$$\frac{M'(w)(M(z) - M(q))}{(M(w) - M(z))(M(w) - M(q))} = \frac{z-q}{(w-z)(w-q)}, \tag{5.12}$$

which holds for any Möbius transformation M. Observe that the usual normalization $q = \infty$ obscures the Möbius invariance of the Cauchy kernel. Using this identity, together with a change of variables and conformal invariance of the Dirichlet space, we obtain the following.

Theorem 5.8. *Let Γ be a curve in $\overline{\mathbb{C}}$ and Ω_1, Ω_2 be the connected components of the complement. Let M be a Möbius transformation. Then for any $h \in \mathcal{D}(M(\Omega_1))$, we have*

$$[\mathbf{J}^{M(q)}_{M(\Omega_1)} h] \circ M = \mathbf{J}^q_{\Omega_1} (h \circ M).$$

Observe that Theorem 5.8 extends the definition of the integral to the case that $\infty \in \Gamma$. For the moment, this says only that if the limit exists on one side, then it exists on both, and the two sides are equal. We will show that the limit exists whenever $h \in \mathcal{D}_{\text{harm}}(\Omega_1)$.

In the remainder of the section we will: (1) provide identities relating $\mathbf{J}^q$ to the Schiffer operators; (2) show that the output is in $\mathcal{D}_{\text{harm}}(\Omega_1 \sqcup \Omega_2)$; and (3) show that for quasicircles, the limiting integral is, up to constants, independent of which side of Γ you choose to take the limit in.

Let ∂ and $\overline{\partial}$ denote the Wirtinger operators on the Riemann sphere.

Theorem 5.9. *Let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into connected components Ω_1 and Ω_2. Assume that $q \notin \Gamma$. Then*

$$\partial \mathbf{J}^q_{\Omega_1} h(z) = \mathbf{T}_{\Omega_1,\Omega_2} \overline{\partial} h(z), \qquad z \in \Omega_2, \tag{5.13}$$

$$\partial \mathbf{J}^q_{\Omega_1} h(z) = \partial h(z) + \mathbf{T}_{\Omega_1,\Omega_1} \overline{\partial} h(z), \qquad z \in \Omega_1, \tag{5.14}$$

$$\overline{\partial} \mathbf{J}^q_{\Omega_1} h(z) = 0, \qquad z \in \Omega_1 \cup \Omega_2. \tag{5.15}$$

Proof. If $q \in \Sigma_2$, then the first claim follows by applying Stokes' theorem and bringing ∂ under the integral sign, as does the third in the case that $z \in \Omega_2$. Denote the circle $|z - q| = r$ traced counter-clockwise by C_r. Using the fact that

$$\lim_{r \searrow 0} \frac{1}{\pi i} \int_{C_r} \frac{\partial g_{\Omega_1}}{\partial w}(w; z) h(w) = h(q),$$

by Stokes' theorem, we have for $q \in \Omega_1$ and $z \in \Omega_2$ that

$$\begin{aligned} \mathbf{J}^q_{\Omega_1} h(z) &= \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} \left[\frac{1}{2\pi i} \left(\frac{1}{w - z} - \frac{1}{w - q} \right) - \frac{1}{\pi i} \frac{\partial g_{\Omega_1}}{\partial w}(w, q) \right] h(w)\, dw + h(q) \\ &= \iint_{\Omega_1} \left[\frac{1}{2\pi i} \left(\frac{1}{w - z} - \frac{1}{w - q} \right) - \frac{1}{\pi i} \frac{\partial g_{\Omega_1}}{\partial w}(w, q) \right] \overline{\partial} h(w)\, d\overline{w} \wedge dw + h(q). \end{aligned} \tag{5.16}$$

Noting that the integrand is non-singular, applying ∂_z to both sides using Theorem 5.1) proves the first claim. Applying $\overline{\partial}_z$ proves the third claim in the case that $q \in \Omega_1$ and $z \in \Omega_2$.

If $z \in \Omega_1$ and $q \in \Omega_2$, we have similarly that

$$\mathbf{J}^q_{\Omega_1} h(z) = \iint_{\Omega_1} \left[\frac{1}{2\pi i} \left(\frac{1}{w - z} - \frac{1}{w - q} \right) + \frac{1}{\pi i} \frac{\partial g_{\Omega_1}}{\partial w}(w, z) \right] \overline{\partial} h(w)\, d\overline{w} \wedge dw + h(z). \tag{5.17}$$

Applying $\overline{\partial}_z$ completes the proof of the third claim, and applying ∂_z using (5.8) proves the second claim in the case that $q \in \Omega_2$. To prove the second claim in the case that $q \in \Omega_1$, we add a further term to (5.17) which removes the singularity at q as in (5.16) and apply ∂_z. ■

For $j = 1,2$ we denote

$$\mathbf{J}^q_{\Omega_1,\Omega_j} h = \mathbf{J}^q_{\Omega_1} h\big|_{\Omega_j}.$$

Theorem 5.9 immediately implies that these are bounded with respect to the Dirichlet energy.

Corollary 5.10. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ and choose $q \notin \Gamma$. Then*

$$\mathbf{J}^q_{\Omega_1} : \mathcal{D}_{\text{harm}}(\Omega_1) \to \mathcal{D}_{\text{harm}}(\Omega_1 \cup \Omega_2)_q,$$
$$\mathbf{J}^q_{\Omega_1,\Omega_j} : \overline{\mathcal{D}(\Omega_1)} \to \mathcal{D}(\Omega_j), \quad j = 1,2.$$

If $q \in \Omega_j$, then the image of $\mathbf{J}^q_{\Omega_1,\Omega_j}$ is $\mathcal{D}_q(\Omega_j)$. Furthermore, each of the operators above are norm-bounded and bounded with respect to Dirichlet energy.

Proof. The characterization of the images and boundedness with respect to the Dirichlet energy follow immediately from Theorems 5.6 and 5.9, together with the fact that the integral kernel of $\mathbf{J}^q_{\Omega_1}$ vanishes for $z = q$.

It remains to show that $\mathbf{J}^q$ is norm-bounded. Assume first that $q \in \Omega_2$. If we use q as the fixed point in the definition of the norm on Ω_2, we then have that

$$\|\mathbf{J}^q_{1,2} h\|_{\Omega_2} = D_{\Omega_2}(\mathbf{J}^q_{1,2} h)^{1/2}$$

since $\mathbf{J}^q_{1,2} h(q) = 0$. Thus, $\mathbf{J}^q_{1,2}$ is norm-bounded.

By composing with a Möbius transformation and applying Theorem 5.8, we can assume that $q = \infty$ and $0 \in \Omega_1$. Since we have already shown boundedness with respect to Dirichlet energy, it remains only to show that $|\mathbf{J}^q_{1,1} h(0)|$ is controlled by the norm of h. Now let $f : \mathbb{D}^+ \to \Omega_1$ be a conformal map such that $f(0) = 0$. Using $g_{\Omega_1}(z,0) = -\log|f^{-1}(z)|$ and arguing as in (5.16) and (5.17), we see that

$$\begin{aligned}
\mathbf{J}^q_{\Omega_1} h(0) &= \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon} \left[\frac{1}{2\pi i}\frac{1}{w} + \frac{1}{\pi i}\frac{\partial g_{\Omega_1}}{\partial w}(w,0)\right] h(w)\, dw + h(0) \\
&= \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \left[\frac{1}{w} - \frac{(f^{-1})'(w)}{f^{-1}(w)}\right] h(w)\, dw + h(0) \\
&= \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{|z|=r} \left[\frac{f'(\zeta)}{f(\zeta)} - \frac{1}{\zeta}\right] h(f(\zeta))\, d\zeta + h(0) \\
&= \iint_{\mathbb{D}^+} \left[\frac{f'(\zeta)}{f(\zeta)} - \frac{1}{\zeta}\right] \overline{\partial}(h \circ f)(\zeta)\, \frac{d\bar{\zeta} \wedge d\zeta}{2i} + h(0). \qquad (5.18)
\end{aligned}$$

It is clear that if we choose the point in the definition of the norm on $\mathcal{D}_{\text{harm}}(\Omega_1)$ to be 0, then it suffices to estimate the first integral. Since f is one-to-one, the integrand is non-singular. Now let $R = \inf_{|\zeta|=r} |f(\zeta)|$. Since $R > 0$, we have

$$\iint_{r<|\zeta|<1} \left|\frac{f'(\zeta)}{f(\zeta)}\right|^2 dA_\zeta = \iint_{f(\mathbb{A}_r)} \left|\frac{1}{\eta}\right|^2 dA_\eta < \infty.$$

Obviously,

$$\iint_{\mathbb{A}_r} \left|\frac{1}{\zeta}\right|^2 dA_\zeta < \infty,$$

so by Minkowski's inequality applied to the domain $\mathbb{A}_r$, we have

$$\iint_{\mathbb{A}_r} \left|\frac{f'(\zeta)}{f(\zeta)} - \frac{1}{\zeta}\right|^2 dA_\zeta < \infty,$$

and therefore we obtain that $\| \log(f(z)/z)\|_{\Omega_1} < \infty$. Inserting this in (5.18) and applying Cauchy–Schwarz's inequality, we obtain

$$|\mathbf{J}^q_{1,1} h(0)| \le |h(0)| + D_{\Omega_1}(\log(f(z)/z))^{1/2} D_{\Omega_1}(h \circ f)^{1/2},$$

which (using the conformal invariance of the Dirichlet energy) proves the boundedness of $\mathbf{J}^q_{1,1}$.

In the case that $q \in \Omega_1$, once again one trivially obtains that $\mathbf{J}^q_{1,1}$ is norm-bounded by choosing q to be point in the norm on $\mathcal{D}_{\mathrm{harm}}(\Omega_1)$. Then $\mathbf{J}^q_{1,2}$ is shown to be bounded by choosing the point in the norm on $\mathcal{D}_{\mathrm{harm}}(\Omega_2)$ to be ∞, and estimating

$$\mathbf{J}^q_{1,2} h(\infty) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{-h(w)}{w-q}\, dw$$

as above. ■

Observing that $\mathbf{J}^q_{\Omega_1} c$ is constant in Ω_1 and Ω_2 for any constant c, we also obtain well-defined operators

$$\dot{\mathbf{J}}_{\Omega_1} \colon \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_1) \to \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_1 \cup \Omega_2),$$
$$\dot{\mathbf{J}}_{\Omega_1,\Omega_k} \colon \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_1) \to \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_k), \quad k = 1, 2.$$

These are obviously bounded by Corollary 5.10, and it is easily verified that these are independent of q.

As in the case of the overfare and Schiffer operators, we will use the notation $\mathbf{J}^q_k$ in place of $\mathbf{J}^q_{\Omega_k}$ and $\mathbf{J}^q_{j,k}$ in place of $\mathbf{J}^q_{\Omega_j,\Omega_k}$ wherever possible.

The operator $\mathbf{J}^q_1$ is motivated as follows. Setting aside the normalization at q, we would like to define the Cauchy integral

$$\frac{1}{2\pi i} \int_\Gamma \frac{h(w)}{w-z}\, dw$$

of a function $h \in \mathcal{H}(\Gamma, \Omega_1)$, but there are two obstacles: the curve Γ is not rectifiable, and functions in h are not particularly regular. This problem is solved by considering instead $\mathbf{J}^q_{\Omega_1} \mathbf{e}_{\Gamma,\Omega_1} h$.

The question immediately arises: if one considers instead $\mathbf{J}^q_{\Omega_2} \mathbf{e}_{\Gamma,\Omega_2} h$, is the result the same? Of course, this requires that $\mathcal{H}(\Gamma, \Omega_1) \subseteq \mathcal{H}(\Gamma, \Omega_2)$ at least, which we know is true for quasicircles. In fact, up to constants, it is indeed sufficient that Γ is a quasicircle.

Theorem 5.11. *Let Γ be a quasicircle in $\overline{\mathbb{C}}$, and let Ω_1 and Ω_2 be the connected components of the complement. Fix $q \notin \Gamma$, then for any $h \in \mathcal{D}_{\text{harm}}(\Omega_1)$, we have*

$$\dot{\mathbf{J}}_1 h = -\dot{\mathbf{J}}_2 \dot{\mathbf{O}}_{1,2} h.$$

The same results hold with the roles of Ω_1 and Ω_2 switched.

Proof. Let B_1 and B_2 be collar neighbourhoods of Γ in Ω_1 and Ω_2, respectively. Let $U = B_1 \cup B_2 \cup \Gamma$. This is an open set bordered by two analytic curves. By Corollary 4.9, the class $\dot{\mathbf{R}}_1 \mathcal{D}(U)$ of elements of $\mathcal{D}(U)$ is dense in $\dot{\mathcal{D}}(B_1)$. Furthermore, by Theorem 4.7, $\dot{\mathbf{G}}_{A_1,\Omega_1} \mathcal{D}(B_1)$ is dense in $\dot{\mathcal{D}}_{\text{harm}}(\Omega_1)$. Thus, since $\dot{\mathbf{G}}_{B_1,\Omega_1}$ is bounded by Theorem 4.6, $\dot{\mathbf{G}}_{B_1,\Omega_1} \dot{\mathbf{R}}_1 \mathcal{D}(U)$ is dense in $\dot{\mathcal{D}}_{\text{harm}}(\Omega_1)$. By Theorem 3.11, $\dot{\mathbf{O}}_{1,2}$ is bounded, so it is enough to prove the theorem for such functions.

Let $h \in \mathcal{D}(U)$. Since h extends continuously to Γ, the CNT boundary values of $\mathbf{R}_1 h$ and $\mathbf{R}_2 h$ with respect to Ω_1 and Ω_2 both equal the continuous extension, and hence each other. Thus,

$$\mathbf{O}_{1,2} \mathbf{G}_{B_1,\Omega_1} \mathbf{R}_1 h = \mathbf{G}_{B_2,\Omega_2} \mathbf{R}_2 h \tag{5.19}$$

from which we conclude that the formula holds in the homogeneous sense; that is, with dots.

Fix $z \in \overline{\mathbb{C}} \backslash \Gamma$. Since B_i are collar domains, by definition there are biholomorphisms $f_i : \mathbb{D}^+ \to \Omega_i$ so that $f_i(\mathbb{A}_{r_i}) = B_i$ for annuli $\mathbb{A}_{r_i} = \{z : r_i < |z| < 1\}$ for $i = 1, 2$. Let Γ^i_ε denote the limiting curves $f_i(|z| = e^{-\varepsilon})$ with orientations induced by f_i. By Carathéodory's theorem, the maps f_i extend homeomorphically to maps from $\mathbb{S}^1$ to Γ, so for any fixed ε, the curves Γ^i_ε are each homotopic to Γ, and hence to each other. Thus, since z and q are eventually not in the domain bounded by Γ^1_ε and Γ^2_ε, we have

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma^1_\varepsilon} h(w) \left(\frac{1}{w-z} - \frac{1}{w-q} \right) dw \\ &\quad = - \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma^2_\varepsilon} h(w) \left(\frac{1}{w-z} - \frac{1}{w-q} \right) dw, \end{aligned} \tag{5.20}$$

where the negative sign arises from the change of orientation between the integrals.

Finally, applying the anchor lemma 4.10 for fixed z with

$$\alpha(w) = \left(\frac{1}{w-z} - \frac{1}{w-q} \right) dw,$$

we have for $i = 1, 2$ that

$$\mathbf{J}^q_i \mathbf{G}_{B_i,\Omega_i} \mathbf{R}_i h(z) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma^i_\varepsilon} h(w) \left(\frac{1}{w-z} - \frac{1}{w-q} \right) dw. \tag{5.21}$$

Here, we may have to shrink the domain B_j so that neither z nor q are in the closure, to ensure that $\alpha \in L^2(B_j)$. This does not affect the validity of the argument, since given nested collar neighbourhoods $B'_j \subset B_j$, by definition

$$\mathbf{G}_{B'_j,\Omega_j} h|_{B'_j} = \mathbf{G}_{B_j,\Omega_j} h|_{B_j}.$$

Thus, combining (5.19), (5.20), and (5.21), we have

$$\mathbf{J}_1^q \mathbf{G}_{B_1,\Omega_1} \mathbf{R}_1 h = -\mathbf{J}_2^q \mathbf{O}_{1,2} \mathbf{G}_{B_1,\Omega_1} \mathbf{R}_1 h.$$

The equation also holds in the homogeneous sense. Using density and boundedness completes the proof. ■

Remark 5.12. The negative sign is an artifact of the change of orientation induced by the switch from the domain Ω_1 to Ω_2. In previous publications [53, 56] we chose the orientations in such a way that the sign did not change.

Remark 5.13. Our proof of Theorem 5.11 given in [53] contains a gap, which is not hard to fill in a couple of ways, but only if one mods out by constants. Here it is filled by the proof of the anchor lemma, which we stated and proved for the first time in [56].

Finally, we record the following obvious fact.

Theorem 5.14. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ and let Ω_1 and Ω_2 be the components of the complement. Fix $q \notin \Gamma$, and let $h \in \mathcal{D}(\Omega_j)$.*

If $q \in \Omega_j$, then

$$\mathbf{J}_j^q h(z) = \begin{cases} h(z) - h(q) & z \in \Omega_j, \\ -h(q) & z \notin \Omega_j \cup \Gamma, \end{cases}$$

whereas if $q \notin \Omega_j$, then

$$\mathbf{J}_j^q h(z) = \begin{cases} h(z) & z \in \Omega_j, \\ 0 & z \notin \Omega_j \cup \Gamma. \end{cases}$$

Proof. This follows from the ordinary Cauchy integral formula. ■

5.3. The Schiffer isomorphisms and the Plemelj–Sokhotski isomorphisms

The classical Plemelj–Sokhotski jump formula can be expressed using a principle value integral on the curve. That is, if u is a smooth function and Γ is a smooth Jordan curve in $\mathbb{C}$, for $z_0 \in \Gamma$, define

$$\mathcal{H}u(z_0) = \text{P.V.} \frac{1}{2\pi i} \int_\Gamma \frac{u(\zeta)}{\zeta - z_0}\, d\zeta. \tag{5.22}$$

Of course, one could weaken the analytic assumptions. We have [10]

$$\lim_{z \to z_0^\pm} \frac{1}{2\pi i} \int_\Gamma \frac{u(\zeta)}{\zeta - z}\, d\zeta = \pm \frac{1}{2} u(z_0) + \mathcal{H}u(z_0), \tag{5.23}$$

where $\lim_{z \to z_0^\pm}$ respectively denotes the limits taken in the bounded and unbounded components Ω_+ and Ω_- of the complement of Γ.

Thus, defining the functions

$$h_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{u(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Omega_k,\ k = 1, 2,$$

for any point $z_0 \in \Gamma$, it follows that

$$\lim_{z \to z_0} h_2(z) - \lim_{z \to z_0} h_1(z) = u(z_0),$$

which is referred to as the Plemelj–Sokhotski jump formula. When the map that takes u to (h_1, h_2) is an isomorphism, we will refer to it as the Plemelj–Sokhotski isomorphism.

Of course, this decomposition has been studied extensively, for various curves Γ and various boundary values. Since quasicircles are not in general rectifiable, this hinders the formulation of the jump decomposition. B. Kats studied Riemann–Hilbert problems (i.e. more general versions of the Plemelj–Sokhotski jump problems) on non-rectifiable curves, see e.g. [32] for the case of Hölder continuous boundary values, and the survey article [31] and references therein. Also, the jump decomposition was shown to hold for a range of Besov spaces of boundary values by the authors, for d-regular quasidisks [54], which are not necessarily rectifiable.

In this paper, we do not integrate over the curve directly, but use limiting curves as in previous sections. Thus, rectifiability is not necessary. We show that the map $u \mapsto (h_1, h_2)$ is an isomorphism precisely for quasicircles, and that the jump formula holds up to constants. Here, we circumvent the need for a principal value integral along the curve. Nevertheless, we have the following problem.

Open problem. Can a meaningful principal value integral $\mathcal{H}u$ be defined when Γ is a quasicircle and $u \in \mathcal{H}(\Gamma)$, and a corresponding formula (5.23) found?

We will show the result of Napalkov and Yulmukhametov, that $\dot{\mathbf{J}}_{1,2}$, $\mathbf{J}^q_{1,2}$, and the Schiffer operator $\mathbf{T}_{1,2}$ are isomorphisms precisely for quasicircles. To do this, we first require a lemma. This lemma says that the jump formula holds for functions which are boundary values of holomorphic maps in a collar neighbourhood of Γ.

Lemma 5.15. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ and let Ω_1 and Ω_2 be the connected components of the complement. Let B be a collar neighbourhood of Γ in Ω_1. Assume that q is not in the closure of B. If $H \in \mathcal{D}(B)$, then $\mathbf{J}^q_{1,2}\mathbf{G}_{B,\Omega_1} H$ extends to a holomorphic function H_2 on $\Omega_2 \cup \Gamma \cup B$, which satisfies*

$$H_2(z) = \mathbf{J}^q_{1,1}\mathbf{G}_{B,\Omega_1} H(z) - H(z), \quad z \in B.$$

Furthermore, $\mathbf{J}^q_{1,2}\mathbf{G}_{B,\Omega_1} H$ has a transmission in $\mathcal{D}_{\mathrm{harm}}(\Omega_1)$, given explicitly by

$$\mathbf{O}_{2,1}\mathbf{J}^q_{1,2}\mathbf{G}_{B,\Omega_1} H = \mathbf{G}_{B,\Omega_1} H_2 = \mathbf{J}^q_{1,1}\mathbf{G}_{B,\Omega_1} H - \mathbf{G}_{B,\Omega_1} H.$$

Recall that $\mathbf{O}_{2,1}$ is the solution of the Dirichlet problem on Ω_1 with continuous boundary values $H_2|_\Gamma$.

Proof. Assume that $q \in \Omega_2$; by Theorem 5.8 and conformal invariance of the bounce operator and CNT boundary values, we can assume that $q = \infty$. The first claim is just the ordinary Cauchy integral formula combined with the anchor lemma. Let $f : \mathbb{D}^+ \to \Omega_1$ be the biholomorphism such that $f(\mathbb{A}) = B$ for an annulus $\mathbb{A} = \{z : r < |z| < 1\}$, and let Γ_1^ε be the corresponding images under f of circles $|z| = e^{-\varepsilon}$ as usual, with orientation induced by f. Let γ be the analytic curve which is the inner boundary of B; that is, the image of $|z| = r$ under f.

Define

$$H_2(z) = \frac{1}{2\pi i} \int_\gamma \frac{H(w)}{w - z}\, dw,$$

which is holomorphic in the open set $\Omega_2 \cup \Gamma \cup B$. By the Anchor lemma (Lemma 4.10) and the fact that H is holomorphic, for all $z \in \Omega_2$, we have

$$\mathbf{J}^q_{1,2}\mathbf{G}_{B,\Omega_1} H(z) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma^1_\varepsilon} \frac{H(w)}{w - z}\, dw = H_2(z). \tag{5.24}$$

By the ordinary Cauchy integral formula, for all $z \in B$, we have

$$H(z) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i} \int_{\Gamma^1_\varepsilon} \frac{H(w)}{w - z}\, dw - H_2(z).$$

Applying the Anchor lemma (Lemma 4.10) again, we see that

$$H(z) = \mathbf{J}^q_{1,1}\mathbf{G}_{B,\Omega_1} H(z) - H_2(z) \tag{5.25}$$

for all $z \in B$.

We now prove the second claim. Since H_2 extends continuously to Γ, its CNT boundary values with respect to Ω_1 equal its CNT boundary values with respect to Ω_2, which are equal to those of $\mathbf{J}^q_{1,2}\mathbf{G}_{B,\Omega_1} H$ by (5.25). Of course the CNT boundary values are all continuous extensions. Thus,

$$\mathbf{G}_{B,\Omega_1} H_2 = \mathbf{O}_{2,1}\mathbf{J}^q_{1,2}\mathbf{G}_{B,\Omega_1} H. \tag{5.26}$$

To see that $\mathbf{G}_{B,\Omega_1} H_2 \in \mathcal{D}_{\text{harm}}(\Omega_2)$, let $B_1 = f(\mathbb{A}')$ be a collar neighbourhood of Γ in Ω_1 where $\mathbb{A}'$ is chosen so that its inner boundary is contained in $\mathbb{A}$. Since H_2 is holomorphic on an open neighbourhood of the closure of B_1, its restriction to B_1 is in $\mathcal{D}(B_1)$. Since $\mathbf{G}_{B,\Omega_1}$ is bounded by Theorem 4.6, the transmission $\mathbf{G}_{B,\Omega_1} H_2 = \mathbf{G}_{B_1,\Omega_1} H_2$ (where H_2 is restricted to B_1) is in $\mathcal{D}_{\text{harm}}(\Omega_1)$ as claimed.

Finally, applying $\mathbf{G}_{B,\Omega_1}$ to both sides of (5.25), which leaves the first term of the right-hand side unchanged, and using (5.26), we obtain

$$\mathbf{G}_{B,\Omega_1} H = \mathbf{J}^q_{1,1}\mathbf{G}_{B,\Omega_1} H(z) - \mathbf{O}_{2,1}\mathbf{J}^q_{1,2}\mathbf{G}_{B,\Omega_1} H$$

on Ω_1. This completes the proof of the case that $q \in \Omega_2$.

When $q \in \Omega_1$, the proof above can be repeated with the Cauchy kernel replaced by $1/(w - z) - 1/(w - q)$ in each step, with no other changes. ∎

Theorem 5.16 (Transmitted jump formula for quasicircles). *Let* Γ *be a quasicircle in* $\overline{\mathbb{C}}$ *and let* Ω_1 *and* Ω_2 *be the connected components of the complement. For all* $h \in \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_1)$*, we have*

$$h = \dot{\mathbf{J}}_{1,1} h - \dot{\mathbf{O}}_{2,1} \dot{\mathbf{J}}_{1,2} h.$$

Proof. By Lemma 5.15, the claim holds for all h of the form $\mathbf{G}_{B,\Omega_1} H$ for $H \in \mathcal{D}(B)$. By Theorem 4.7, $\dot{\mathbf{G}}_{B,\Omega_1} \mathcal{D}(B)$ is dense in $\dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_1)$. Thus, the theorem follows from the fact that $\dot{\mathbf{J}}$ is bounded by Corollary 5.10. ■

Remark 5.17. This can be thought of as the classical jump formula modulo constants expressed in terms of the transmission.

Lemma 5.15 generates a large class of functions in the Dirichlet space for which there is a continuous transmission, which will be useful in proving injectivity of the jump and Schiffer operators ahead. Recall that the overfare operator for one-forms $\mathbf{O}'$ used below was defined by equation (3.6).

Lemma 5.18. *Let* Γ *be a Jordan curve separating* $\overline{\mathbb{C}}$ *into components* Ω_1 *and* Ω_2*.*

(1) *For all* $\overline{h} \in \mathbf{G}_{B,\Omega_1} \mathcal{D}(B) \cap \overline{\mathcal{D}(\Omega_1)}$*,* $\mathbf{J}^q_{1,2} \overline{h}$ *has a continuous transmission in* $\mathcal{D}_{\mathrm{harm}}(\Omega_1)$ *given by*

$$\mathcal{O}_{2,1} \mathbf{J}^q_{1,2} \overline{h} = \mathbf{J}^q_{1,1} \overline{h} - \overline{h}.$$

(2) *For all* $\overline{\alpha} \in \overline{\partial}[\mathbf{G}_{B,\Omega_1} \mathcal{D}(B) \cap \overline{\mathcal{D}(\Omega_1)}]$*,* $\mathbf{T}_{1,2} \overline{h}$ *has a continuous transmission in* $\mathcal{A}_{\mathrm{harm}}(\Omega_1)$ *given by*

$$\mathcal{O}'_{2,1} \mathbf{T}_{1,2} \overline{\alpha} = -\overline{\alpha} + \mathbf{T}_{1,1} \overline{\alpha}.$$

Proof. The first claim follows directly from Lemma 5.15. Now let $\overline{\alpha} = \overline{\partial} \overline{h}$. Applying now Theorem 5.9 to the right-hand side of (1), we see that

$$\mathbf{P}(\Omega_1) \mathcal{O}'_{2,1} \mathbf{T}_{1,2} \overline{\alpha} = \mathbf{T}_{1,1} \overline{\alpha},$$
$$\overline{\mathbf{P}(\Omega_1)} \mathcal{O}'_{2,1} \mathbf{T}_{1,2} \overline{\alpha} = -\overline{\alpha}.$$

Applying ∂ to the left-hand side of (1) and using Theorem 5.9 again proves the claim. ■

We can now prove that $\mathbf{T}_{1,2}$ is one-to-one.

Theorem 5.19. *Let* Γ *be a Jordan curve separating* $\overline{\mathbb{C}}$ *into components* Ω_1 *and* Ω_2*.*

(1) $\mathbf{T}_{1,2}$ *is injective (Napalkov and Yulmukhametov* [41]*).*

(2) *For any collar neighbourhood* B *of* Γ *in* Ω_1*,* $\mathbf{T}_{1,2}$ *restricted to*

$$\overline{\partial}[\mathbf{G}_{B,\Omega_1} \mathcal{D}(B) \cap \overline{\mathcal{D}(\Omega_1)}]$$

has left inverse $-\overline{\mathbf{P}(\Omega_1)} \mathcal{O}'_{2,1}$*, where* $\overline{\mathbf{P}(\Omega_1)}$ *is the projection defined in* (2.5)*.*

Proof. The second claim follows immediately from Lemma 5.18 (2).

Let $B = f(\mathbb{A})$ be a collar neighbourhood of Γ in Ω_1 induced by some biholomorphism $f : \mathbb{D}^+ \to \Omega_1$ and annulus $\mathbb{A}$. Now for any $n > 0$, by conformal invariance of the bounce operator (4.1), we have

$$\mathbf{G}_{B,\Omega_1}\mathbf{C}_{f^{-1}}w^{-n} = \mathbf{C}_{f^{-1}}\mathbf{G}_{\mathbb{A},\mathbb{D}^+}w^{-n} = \mathbf{C}_{f^{-1}}\overline{w}^n,$$

and so

$$\overline{\partial}\mathbf{C}_{f^{-1}}\mathbb{C}[\overline{z}] \subseteq \overline{\partial}[\mathbf{G}_{B,\Omega_1}\mathcal{D}(B) \cap \overline{\mathcal{D}(\Omega_1)}].$$

Furthermore, $\overline{\partial}\mathcal{C}_{f^{-1}}\mathbb{C}[\overline{z}]$ is dense in $\overline{A(\Omega_1)}$.

By the second claim, for any $\overline{\alpha} \in \overline{\partial}\mathbf{C}_{f^{-1}}\mathbb{C}[\overline{z}]$ if $\mathbf{T}_{1,2}\overline{\alpha} = 0$, then $\overline{\alpha} = 0$. On the other hand, for $z \in \Omega_2$ fixed, $dw/(w-z)^2 \in \mathcal{A}(\Omega_1)$, and for any $\overline{\alpha} \in \overline{\mathcal{A}(\Omega_1)}$, we have

$$\mathbf{T}_{1,2}\overline{\alpha}(z) = \left(\alpha(w), \frac{dw}{(w-z)^2}\right).$$

This proves the first claim since, by Lemma 1 in [41], the system

$$\left\{w \in \Omega_1;\ \frac{1}{(w-z)^2},\ z \in \Omega_2\right\}$$

is complete in $\mathcal{A}(\Omega_1)$, and therefore if $(\alpha(w), dw/(w-z)^2) = 0$, then $\alpha = 0$. ■

This implies that the Cauchy-type operator $\mathbf{J}^q_{1,2}$ is injective. It is convenient to record the two cases $q \in \Omega_1, \Omega_2$.

Corollary 5.20. *Let Γ be a Jordan curve Γ in $\overline{\mathbb{C}}$.*

(1) (a) $\dot{\mathbf{J}}_{1,2}$ *is injective.*

(b) *For any collar neighbourhood B of Γ in Ω_1, on $\overline{\dot{\mathcal{D}}(\Omega_1)} \cap \dot{\mathbf{G}}_{B,\Omega_1}\mathcal{D}(B)$, the left inverse is given by $-\overline{\dot{\mathbf{P}}(\Omega_1)}\dot{\mathcal{O}}_{2,1}$.*

(2) *Fix $q \in \Omega_2$. For any $p \in \Omega_1$, $\mathbf{J}^q_{1,2}$ is injective from $\overline{\mathcal{D}_p(\Omega_1)}$ to $\mathcal{D}_q(\Omega_2)$.*

(3) *Fix $q \in \Omega_1$. Then $\mathbf{J}^q_{1,2}$ is injective from $\overline{\mathcal{D}(\Omega_1)}$ to $\mathcal{D}(\Omega_2)$.*

Proof. (1) (a) follows immediately from Theorems 5.9 and 5.19. To see (1) (b), observe that Lemma 5.15 tells us, for any $\overline{h} \in \mathbf{G}_{B,\Omega_1}\overline{\mathcal{D}(\Omega_1)}$, that

$$-\mathcal{O}_{2,1}\mathbf{J}^q_{1,2}\overline{h} = -\mathbf{J}^q_{1,1}\overline{h} + \overline{h}.$$

Thus, the same formula holds for the homogeneous spaces, and the claim follows.

To prove (2), by Theorem 5.19 (1) and Theorem 5.9, if $\mathbf{J}^q_{1,2}\overline{h} = 0$, then $\overline{h}$ is a constant c. If $\overline{h} \in \overline{\mathcal{D}_p(\Omega)}$, then $c = h(p) = 0$, so $h = 0$. On the other hand, if $q \in \Omega_1$, then $c = h(q) = 0$. This proves (3). ■

The left inverses in (2) and (3) are easily obtained by keeping track of constants appropriately.

Theorem 5.21 (Napalkov and Yulmukhametov [40, 41]). *Let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. If any of the following four conditions hold, then Γ is a quasicircle.*

(1) $\mathbf{T}_{1,2}$ *is surjective.*

(2) *The restriction of $\dot{\mathbf{J}}_{1,2}$ to $\overline{\dot{\mathcal{D}}(\Omega_1)}$ is surjective onto $\dot{\mathcal{D}}(\Omega_2)$.*

(3) *The restriction of $\mathbf{J}^q_{1,2}$ to $\overline{\mathcal{D}_p(\Omega_1)}$ is surjective onto $\mathcal{D}_q(\Omega_2)$ for some $q \in \Omega_2$ and $p \in \Omega_1$.*

(4) *The restriction of $\mathbf{J}^q_{1,2}$ to $\overline{\mathcal{D}(\Omega_1)}$ is surjective onto $\mathcal{D}(\Omega_2)$ for $q \in \Omega_1$.*

Proof. The first and second claim are equivalent by Theorem 5.9. If (3) or (4) holds, then (2) clearly holds. Thus, it suffices to show that (2) implies that Γ is a quasicircle.

Assume that (2) holds. Let

$$\mathbf{K}\colon \mathcal{D}(\Omega_2) \to \overline{\mathcal{D}(\Omega_1)}$$

be the right inverse of $\dot{\mathbf{J}}_{1,2}$ on $\overline{\dot{\mathcal{D}}(\Omega_1)}$. Choose a collar neighbourhood $B = f(\mathbb{A}_r)$ of Γ in Ω_1, where $f\colon \mathbb{D}^+ \to \Omega_1$ is a biholomorphism. As in the proof of Theorem 5.19, $\dot{\mathbf{G}}_{B,\Omega_1}\dot{\mathbf{C}}_{f^{-1}}\dot{\mathbb{C}}[1/z]$ is dense in $\overline{\dot{\mathcal{D}}(\Omega_1)}$, since

$$\dot{\mathbf{G}}_{B,\Omega_1}\dot{\mathbf{C}}_{f^{-1}}\dot{\mathbb{C}}[1/z] = \dot{\mathbf{C}}_{f^{-1}}\dot{\mathbf{G}}_{\mathbb{A},\mathbb{D}^+}\dot{\mathbb{C}}[1/z] = \dot{\mathbf{C}}_{f^{-1}}\dot{\mathbb{C}}[\overline{z}]$$

and polynomials are dense in $\dot{\mathcal{D}}(\mathbb{D}^+)$. Since $\dot{\mathbf{J}}_{1,2}$ is bounded and surjective, the set

$$\mathcal{L} = \dot{\mathbf{J}}_{1,2}\dot{\mathbf{G}}_{B,\Omega_1}\dot{\mathbf{C}}_{f^{-1}}\dot{\mathbb{C}}[1/z]$$

is dense in $\overline{\dot{\mathcal{D}}(\Omega_1)}$. Furthermore, Lemma 5.15 guarantees that $\mathcal{L} \subset \dot{\mathcal{C}}(\mathrm{cl}\Omega_2)$, and by Lemma 5.18 for every element $\overline{h} \in \mathcal{L}$, we have

$$\dot{\mathcal{O}}_{2,1}\overline{h} = \dot{\mathcal{O}}_{2,1}\dot{\mathbf{J}}_{1,2}\mathbf{K}\overline{h} = (\dot{\mathbf{J}}_{1,1}\mathbf{K} - \mathbf{K})\overline{h}.$$

We can also conjugate to get transmission of elements $h \in \overline{\mathcal{L}} \subset \dot{\mathcal{D}}(\Omega_1)$, that is

$$\dot{\mathcal{O}}_{2,1}h = \overline{(\dot{\mathbf{J}}_{1,1}\mathbf{K} - \mathbf{K})h}.$$

Since $\dot{\mathbf{J}}_{1,1}\mathbf{K} - \mathbf{K}$ is bounded, Theorem 3.11 applies, and we can conclude that Γ is a quasicircle. This proves (2). ■

Theorem 5.22 (Napalkov and Yulmukhametov [40, 41]). *Let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. The following are equivalent.*

(1) Γ *is a quasicircle.*

(2) $\dot{\mathbf{J}}_{1,2}$ *is a bounded isomorphism from $\overline{\dot{\mathcal{D}}(\Omega_1)}$ onto $\dot{\mathcal{D}}(\Omega_2)$.*

(3) $\mathbf{T}_{1,2}$ *is a bounded isomorphism.*

(4) *For* $q \in \Omega_2$ *and* $p \in \Omega_1$, $\mathbf{J}^q_{1,2}$ *is a bounded isomorphism from* $\overline{\mathcal{D}_p(\Omega_1)}$ *into* $\mathcal{D}_q(\Omega_2)$.

(5) *For* $q \in \Omega_1$, $\mathbf{J}^q_{1,2}$ *is a bounded isomorphism from* $\overline{\mathcal{D}(\Omega_1)}$ *into* $\mathcal{D}(\Omega_2)$.

If any of these hold, then the inverse of $\dot{\mathbf{J}}_{1,2}$ *is* $-\overline{\dot{\mathbf{P}}}(\Omega_1)\dot{\mathbf{O}}_{2,1}$, *and the inverse of* $\mathbf{T}_{1,2}$ *is* $-\overline{\mathbf{P}(\Omega_1)}\mathbf{O}'_{2,1}$.

Proof. If (2), (3), (4), or (5) holds, then by Theorem 5.21 Γ is a quasicircle.

Conversely, assume that Γ is a quasicircle. By Theorem 5.19 and Corollary 5.20, we have that the maps in (2)–(5) are injective.

By the inverse mapping theorem it is enough to show that the maps in (2)–(5) are surjective. For $q \in \Omega_2$, surjectivity of $\mathbf{J}^q_{1,2}$ follows immediately from surjectivity of $\dot{\mathbf{J}}_{1,2}$, as does surjectivity of $\mathbf{T}_{1,2}$ by applying Theorem 5.9. For $q \in \Omega_1$, surjectivity of $\mathbf{J}^q_{1,2}$ follows from that of $\dot{\mathbf{J}}_{1,2}$ after observing that in this case $\mathbf{J}^q_{1,2}c = -c$ for any constant c, and thus the constants can be adjusted as needed. Thus, it suffices to prove surjectivity of $\dot{\mathbf{J}}_{1,2}$.

Assume that $q \in \Omega_2$. To see that $\dot{\mathbf{J}}_{1,2}$ is surjective from $\overline{\dot{\mathcal{D}}(\Omega_1)}$ to $\dot{\mathcal{D}}(\Omega_2)$, let $h \in \dot{\mathcal{D}}(\Omega_2)$. Let $H = -\dot{\mathbf{O}}_{2,1}h$, where the bounded transmission $\dot{\mathbf{O}}_{2,1}$ exists by Theorem 3.5. Now $H = H_1 + \overline{H_2}$, where $H_1 \in \dot{\mathcal{D}}(\Omega_1)$ and $\overline{H_2} \in \overline{\dot{\mathcal{D}}(\Omega_1)}$. For all $z \in \Omega_2$, we have

$$\dot{\mathbf{J}}_{1,2}\overline{H_2}(z) = \dot{\mathbf{J}}_{1,2}H(z) = \dot{\mathbf{J}}_{2,2}h(z) = h(z),$$

where the first equality is by part one of Theorem 5.14 with $j = 2$ (one can choose there $q \in \Omega_2$), the second equality is by Theorem 5.11, and the third equality is by Theorem 5.14 part two with $j = 2$. This completes the proof. ■

Again, the inverse of $\mathbf{J}^q_{1,2}$ can be obtained by adjusting constants.

Remark 5.23. As far as we know, the fact that $\mathbf{T}_{1,2}$ is an isomorphism was not known to Schiffer even for stronger assumptions on the curve. We have generalized this and the jump isomorphism to various settings (taking into account topological obstacles); namely to compact Riemann surfaces separated by a quasicircle [56]; and with M. Shirazi, to compact Riemann surfaces with n quasicircles enclosing simply connected domains in [44]. The converse, that $\mathbf{T}_{1,2}$ is an isomorphism then Γ is a quasicircle, only exists in genus zero. It is an open question whether a suitably formulated converse holds in genus $g > 0$, though it seems plausible once the topological differences are taken into account.

We now prove that the Plemelj–Sokhotski jump decomposition is an isomorphism precisely for quasicircles. For $q \in \Omega_2$, define

$$\begin{aligned} \mathbf{M}^q(\Omega_1)\colon \mathcal{D}_{\text{harm}}(\Omega_1) &\to \mathcal{D}(\Omega_1) \oplus \mathcal{D}_q(\Omega_2), \\ h &\mapsto (\mathbf{J}^q_{1,1}h, \mathbf{J}^q_{1,2}h), \end{aligned}$$

and for $q \in \Omega_1$, define

$$\mathbf{M}^q(\Omega_1)\colon \mathcal{D}_{\text{harm}}(\Omega_1) \to \mathcal{D}_q(\Omega_1) \oplus \mathcal{D}(\Omega_2).$$

Similarly, we have the following operator on harmonic Bergman space:

$$\begin{aligned} \mathbf{M}'(\Omega_1)\colon \mathcal{A}_{\text{harm}}(\Omega_1) &\to \mathcal{A}(\Omega_1) \oplus \mathcal{A}(\Omega_2), \\ \alpha + \overline{\beta} &\mapsto (\alpha + \mathbf{T}_{1,1}\overline{\beta}, \mathbf{T}_{1,2}\overline{\beta}), \end{aligned} \tag{5.27}$$

where $\alpha \in A(\Omega_1)$ and $\overline{\beta} \in \overline{A(\Omega_1)}$.

With this notation, we have the following theorem.

Theorem 5.24. *Let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. The following are equivalent.*

(1) *Γ is a quasicircle.*

(2) *For any $q \in \overline{\mathbb{C}} \backslash \Gamma$, $\mathbf{M}^q(\Omega_1)$ is an isomorphism.*

(3) *$\mathbf{M}'(\Omega_1)$ is an isomorphism.*

(4) *$\dot{\mathbf{M}}(\Omega_1)$ is an isomorphism.*

It is enough that (2) *holds for a single q.*

Proof. We first prove that (1) is equivalent to (3). Assuming that Γ is a quasicircle, by Theorem 5.22, $\mathbf{T}_{1,2}$ is an isomorphism. Given $\tau = \alpha + \overline{\beta} \in \mathcal{A}_{\text{harm}}(\Omega_1)$, assume that $\mathbf{M}'(\Omega_1)\tau = 0$. Then $\mathbf{T}_{1,2}\overline{\beta} = 0$, so $\overline{\beta} = 0$. Since $\alpha = \alpha + \mathbf{T}_{1,1}\overline{\beta} = 0$, we see that $\tau = 0$, so $\mathbf{M}'(\Omega_1)$ is injective. Given any $(\tau, \sigma) \in \mathcal{A}(\Omega_1) \oplus \mathcal{A}(\Omega_2)$, choose $\overline{\beta}$ such that $\mathbf{T}_{1,2}\overline{\beta} = \sigma$. Setting $\alpha = \tau - \mathbf{T}_{1,1}\overline{\beta}$, we have

$$\mathbf{M}'(\Omega_1)(\alpha + \overline{\beta}) = (\tau, \sigma).$$

Conversely, if $\mathbf{M}'(\Omega_1)$ is an isomorphism, in particular $\mathbf{T}_{1,2}$ is surjective, so by Theorem 5.21 Γ is a quasicircle.

(4) is obviously equivalent to (3), after observing that $d\mathbf{M}^q(\Omega_1) = \mathbf{M}'(\Omega_1)d$ by Theorem 5.9. It remains to show that (2) and (1) are equivalent. If (2) holds, then $\mathbf{J}^q_{1,2}$ is an isomorphism and in particular surjective, so Γ is a quasicircle by Theorem 5.21. Assume that Γ is a quasicircle. We have already shown that $\dot{M}(\Omega_1)$ is an isomorphism, so in particular $\dot{\mathbf{M}}(\Omega_1)$ is surjective. The fact that $\mathbf{M}^q(\Omega_1)$ is surjective follows from observing, for any constant function c, that

$$\mathbf{M}^q(\Omega_1)c = \begin{cases} (0, -c) & q \in \Omega_1, \\ (c, 0) & q \in \Omega_2, \end{cases} \tag{5.28}$$

so the constant can be adjusted as needed. Also, if $\mathbf{M}^q(\Omega_1)h = 0$, then $\mathbf{M}'(\Omega_1)dh = 0$ Since we have already shown that $\mathbf{M}'(\Omega_1)$ is injective, we have that $h = c$ for some constant c. By (5.28), $h = 0$. ■

In the case that Γ is a quasicircle, we call $\mathbf{M}^q(\Omega_1)$ the Plemelj–Sokhotski jump isomorphism. This establishes that the jump decomposition holds up to constants on quasicircles, with data in $\dot{\mathcal{H}}(\Gamma)$.

Theorem 5.25. *Let Γ be a quasicircle separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. For any $u \in \dot{\mathcal{H}}(\Gamma)$, there exist $h_j \in \dot{\mathcal{D}}(\Omega_j)$ such that the* CNT *boundary values u_j of h_j satisfy*

$$u = u_1 - u_2$$

except possibly on a null set. The h_1 and h_2 are unique and are given explicitly by

$$(h_1, h_2) = \dot{\mathbf{M}}(\Omega_1)\,\dot{\mathbf{e}}_{\Gamma,\Omega_1}\,u.$$

Proof. Fix $q \in \overline{\mathbb{C}} \backslash \Gamma$. Given $u \in \mathcal{H}(\Gamma)$, denote $h = \mathbf{e}_{\Gamma,\Omega_1} u$, so that $(h_1, h_2) = \mathbf{M}^q h$. To show that $u_1 - u_2 = u$ it suffices to show that

$$h = h_1 - \dot{\mathbf{O}}_{2,1} h_2.$$

But this is precisely Theorem 5.16.

To see that the decomposition is unique, let H_j be another pair of functions such that $u = \dot{\mathbf{b}}_{\Gamma,\Omega_1} H_1 - \dot{\mathbf{b}}_{\Gamma,\Omega_2} H_2$. In that case, $h = H_1 - \dot{\mathbf{O}}_{2,1} H_2$, so

$$h_1 - H_1 + \dot{\mathbf{O}}_{2,1}(H_2 - h_2) = 0. \tag{5.29}$$

By Theorem 5.22, $\overline{\dot{\mathbf{P}}(\Omega_1)}\dot{\mathbf{O}}_{2,1}$ is one-to-one on $\mathcal{D}(\Omega_2)$. Applying this to (5.29), we obtain that $H_2 - h_2 = 0$. Inserting this back into (5.29) yields $h_1 - H_1 = 0$. ■

Interestingly, the jump formula holds for $u \in \mathcal{H}(\Gamma)$ for arbitrary quasicircles, for those u which are boundary values of the set $\mathbf{G}_{B,\Omega_1}\mathcal{D}(B)$, which is dense in $\mathcal{D}_{\text{harm}}(\Omega_1)$. Although $\mathbf{M}^q(\Omega_1)$ is bounded, this does not obviously imply that the jump formula holds on $\mathcal{H}(\Gamma)$ for arbitrary quasicircles. We thus have the following open question.

Open question. For which quasicircles Γ does the jump formula hold on $\mathcal{H}(\Gamma)$?

This is clearly closely related to the open problem on boundedness of transmission.

Remark 5.26. The operator $\overline{\mathbf{P}(\Omega_1)}\mathbf{O}_{2,1}$ also appears in conformal field theory (usually with stronger analytic assumptions). Theorem 6.5 generalizes to higher genus, that is, this operator is inverse to a kind of Faber operator. This fact can be exploited to give an explicit description of the determinant line bundle of this operator; see [44] for the case of genus g surfaces with one boundary curve. The general case of genus g with n boundary curves is work in progress with D. Radnell.

Remark 5.27. The connection of the jump problem to Schiffer operators $\mathbf{T}_{j,k}$, was explicit from the beginning in the work of Bergman and Schiffer [11]. The survey [50] focusses in particular on the real jump problem and its relation to boundary layer potentials. The connection to the complex jump theorem which we give here is more direct. The paper of H. Royden [48] connects the Schiffer kernel functions to the jump problem on Riemann surfaces. His results are phrased somewhat differently, in terms of topological conditions for extensions of holomorphic and harmonic extensions on domains; indeed, the

Plemelj–Sokhotski jump formula is not mentioned explicitly. However, it can be derived as a special case of his results, but with more restrictive analytic assumptions on the function on the curve; namely, that it extend holomorphically to a neighbourhood of the curve. Our paper [56] considers jump decompositions and Schiffer kernels on Riemann surfaces, in the setting of Dirichlet spaces and quasicircles, extending some of the results given here to higher genus.

6. Faber and Grunsky operators

6.1. The Faber operator and Faber series

The Faber operator, see e.g. [63], arises in the theory of approximation by Faber series in domains in the plane or sphere, and has its origin in the work of G. Faber [23]. The Faber operator is typically defined as follows. Let Γ be a rectifiable Jordan curve separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. Assume for the moment that Ω_1 is bounded, that is $\infty \in \Omega_2$, and that $0 \in \Omega_1$. Let $f\colon \mathbb{D}^+ \to \Omega_1$ be a conformal map, such that $f(0) = 0$. Let h be a holomorphic function on $\mathbb{D}^-$ and assume that $h \circ f^{-1}$ extends to an integrable function on Γ; that is, h has boundary values in some sense (e.g. non-tangential) and $h \circ f^{-1} \in L^1(\Gamma)$. Then the Faber operator is defined by

$$\mathcal{F} h(z) = -\frac{1}{2\pi i} \int_\Gamma \frac{h \circ f^{-1}(w)}{w - z}\, dw.$$

For various choices of the regularity of Γ and the space of holomorphic functions on $\mathbb{D}^+$, this is called the Faber operator. The Faber operator is closely related to the approximation by Faber polynomials of a holomorphic function on Ω_2 in general. The nth Faber polynomial corresponding to the domain Ω_2 is defined by

$$F_n(z) = \mathcal{F}((\cdot)^{-n})(z) = -\frac{1}{2\pi i} \int_\Gamma \frac{(f^{-1}(w))^{-n}}{w - z}\, dw.$$

The Faber operator produces a Faber series as follows. Let $h(z)$ be a holomorphic function in $\mathrm{cl}(\mathbb{D}^-)$ which vanishes at the origin. Assume that Γ is an analytic Jordan curve, so that we may focus on the heuristic idea. If $h(z) = h_1 z^{-1} + h_2 z^{-2} + \cdots$, then it is easily verified that the function $H(z) = \mathcal{F} h(z)$ is holomorphic on the closure of Ω_2 and vanishes at ∞. Furthermore, one has the polynomial series

$$H(z) = \mathcal{F} h(z) = \sum_{n=1}^{\infty} h_n \mathcal{F}((\cdot)^{-n}) = \sum_{n=1}^{\infty} h_n F_n(z)$$

called the Faber series of H. This series converges uniformly on the closure of Ω_2. One of the advantages of Faber series over power series, is that for sufficiently regular functions it converges uniformly on compact subsets of the domain. That is, unlike power series, they are adapted to the geometry of the domain.

If one refines the analytic setting, as we do below, then one can investigate different kinds of convergence of the series. Existence and uniqueness of a Faber series correspond to surjectivity and injectivity of the Faber operator respectively. Since h is defined on Ω_2 and f^{-1} on Ω_1, the composition $h \circ f^{-1}$ is not necessarily defined anywhere except on Γ. Thus, the boundary behaviour of h and f^{-1} play a central role in the study of the Faber operator.

There is a vast literature on the Faber operator, Faber series, and their approximation properties. Moreover, the analytic properties of the Faber operator as they relate to the regularity of the curve and the function space, and approximability in various senses by series of Faber polynomials has been extensively studied. See the books of J. M. Anderson [5] and P. K. Suetin [63] (note that the 1998 English translation of the 1984 original has an extensively updated bibliography). Some more recent papers are H. Y. Wei, M. L. Wang and Y. Hu [69], D. Gaier [24], Y. E. Yıldırır and R. Çetintaş [70].

Here we define a Faber operator with domain $\mathcal{D}(\mathbb{D}^-)$ for arbitrary Jordan curves using transmission on the circle. Since the boundary behaviour of a holomorphic function $h(z)$ on $\mathbb{D}^+$ is identical in every sense to that of $\mathbf{O}_{\mathbb{D}^-,\mathbb{D}^+}h(z) = h(1/\bar{z})$, we will replace the domain $\mathcal{D}(\mathbb{D}^-)$ of the operator by $\overline{\mathcal{D}(\mathbb{D}^+)}$.

Definition 6.1. For $q \in \Omega_2$, we define a Faber operator by setting

$$\mathbf{I}_f^q = -\mathbf{J}_{1,2}^q \mathbf{C}_{f^{-1}} \colon \overline{\mathcal{D}_0(\mathbb{D}^+)} \to \mathcal{D}_q(\Omega_2). \tag{6.1}$$

It follows immediately from Corollary 5.10 and conformal invariance of Dirichlet space that $\mathbf{I}_f^q$ is a bounded operator. The choice $q = \infty$ and $p = f(0)$ corresponds to the operator $\mathcal{F}$ described above. From here on, we refer to (6.1) as the Faber operator, and use the new notation to distinguish it from the heuristic discussion above.

Denote the set of polynomials vanishing at q by $\mathbb{C}_q[z]$. Assume that $f(0) = p \in \overline{\mathbb{C}}$, and let Γ' be a fixed simple closed analytic curve in Ω_1 with winding number zero with respect to p. By Lemma 4.10, for any $\bar{h} \in \mathbb{C}_0[\bar{z}]$, we have

$$\mathbf{I}_f^q \bar{h} = \mathbf{I}_f^q \mathbf{O}_{\mathbb{D}^-,\mathbb{D}^+} u = -\frac{1}{2\pi i} \int_{\Gamma'} u \circ f^{-1}(w) \left(\frac{1}{w - z} - \frac{1}{w - q} \right) dw,$$

where $u(z) = \bar{h}(1/\bar{z})$. It is easily shown that the output is a polynomial in $(z - p)^{-1}$. In particular, we define the Faber polynomials as follows.

Definition 6.2. Let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into Ω_1 and Ω_2. Assume that $q \in \Omega_2$ and let $p = f(0)$. Let $f \colon \mathbb{D}^+ \to \Omega_1$ be a conformal map. The nth Faber polynomial with respect to f is

$$\Phi_n(z) = \mathbf{I}_f^q(z^{-n}) \in \mathbb{C}_q[1/(z - p)].$$

If $q = \infty$ and $p = 0$, we have $\Phi_n(z) \in \mathbb{C}_\infty[1/z]$. It is easily checked that Φ_n has degree $-n$ in $(z - p)$.

Remark 6.3. For a bounded domain D bounded by a Jordan curve, polynomials are dense in $\mathcal{A}(D)$ [37], so polynomials vanishing at a fixed point q are dense in $\mathcal{D}_q(D)$. So for $p \in \Omega_1$, setting $M(z) = 1/(z-p)$ and $D = M(\Omega_1)$, we see that $\mathbb{C}_q[1/(z-p)]$ is dense in $\mathcal{D}_q(\Omega_1)$. Thus, since Φ_n has degree $-n$ in $(z-p)$ for each n, we see that the image of $\mathbf{I}_f^q$ is dense in $\mathcal{D}_q(\Omega_2)$.

Example 6.1. Consider the elliptic domain

$$\mathbb{E} = \left\{ x + iy \in \mathbb{C};\, \frac{x^2}{(5/4)^2} + \frac{y^2}{(3/4)^2} < 1 \right\}.$$

The function $g(z) = z + \frac{1}{4z}$ is a one-to-one, holomorphic mapping of $\mathbb{D}^-$ onto $\mathbb{C}\backslash\mathrm{cl}(\mathbb{E})$. Substituting $z + \frac{1}{4z}$ for $g(z)$ in $\Phi_n = \mathbf{I}_g^0(z^n)$, a rather lengthy calculation yields that the Faber polynomials $\{\Phi_n(z)\}_{n=0}^{\infty}$ associated with $\mathbb{E}$ are the monic Chebyshev polynomials, given by

$$\begin{aligned} &\Phi_0(z) = 1, \\ &\Phi_n(z) = 2^{-n}\left\{[z + \sqrt{z^2-1}]^n + [z - \sqrt{z^2-1}]^n\right\}, \quad n = 1, 2, \ldots. \end{aligned}$$

By a Faber series we mean a series of the form

$$\sum_{n=1}^{\infty} \lambda_n \Phi_n(z),$$

whether or not it converges in any sense. With notation as in Definition 6.2, we also define what we call the *sequential Faber operator*

$$\begin{aligned} \mathbf{I}_f^{\mathrm{seq}} \colon \ell^2 &\to \mathcal{D}(\Omega_2)_q, \\ (\lambda_1, \lambda_2, \ldots) &\mapsto \sum_{k=1}^{\infty} \frac{\lambda_k}{\sqrt{k}} \Phi_k. \end{aligned}$$

It commonly appears in this form in univalent function theory [20, 43].

Theorem 6.4. *Let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. Let $q \in \Omega_2$ and fix a conformal map $f \colon \mathbb{D}^+ \to \Omega_1$. The following are equivalent.*

(1) *Γ is a quasicircle.*

(2) *The Faber operator $\mathbf{I}_f^q$ is a bounded isomorphism.*

(3) *The sequential Faber operator is a bounded isomorphism (Shen* [60]*).*

(4) *Every element of $\mathcal{D}_q(\Omega_2)$ is approximable in norm by a unique Faber series $\sum_{n=1}^{\infty} h_n \Phi_n$ satisfying $(h_1, h_2/\sqrt{2}, h_3/\sqrt{3}, \ldots) \in \ell^2$ (Shen* [60]*).*

If any of conditions (2)–(4) *hold for a single q and single choice of $f \colon \mathbb{D}^+ \to \Omega_1$, then they hold for every $q \in \Omega_2$ and every choice of f.*

Proof. The equivalence of (1) and (2) follows immediately from Theorem 5.22 together with the fact that $\mathbf{C}_{f^{-1}}\colon \overline{\mathcal{D}_0(\mathbb{D}^+)} \to \overline{\mathcal{D}_p(\Omega_1)}$ is a bounded isomorphism, where $p = f(0)$. The equivalence of (2) and (3) follows from the fact that

$$(\lambda_1, \lambda_2, \lambda_3, \ldots) \mapsto \sum_{k=1}^{\infty} \frac{\lambda_k}{\sqrt{k}} \bar{z}^k \tag{6.2}$$

is a bounded isomorphism from ℓ^2 to $\overline{\mathcal{D}_0(\mathbb{D}^+)}$.

To show that (2) and (4) are equivalent, first observe that for any Jordan curve $\mathbf{I}_f^q$ is injective, since $\mathbf{C}_{f^{-1}}\colon \overline{\mathbf{D}_0(\mathbb{D}^+)} \to \overline{\mathbf{D}_p(\Omega_1)}$ is an isomorphism and $\mathbf{J}_{1,2}^q$ is injective by Corollary 5.20 (1). Now assume that (2) holds. Given any $H(z) \in \mathcal{D}_q(\Omega_2)$, let $H = \mathbf{I}_f^q \bar{h}$. This function h has a power series expression

$$\bar{h}(z) = h_1 \bar{z} + h_2 \bar{z}^2 + \cdots,$$

which converges in $\overline{\mathcal{D}_0(\mathbb{D}^+)}$ to $\bar{h}$. Since $\mathbf{I}_f^q$ is bounded, applying it to both sides we see that

$$H(z) = \sum_{n=1}^{\infty} h_n \Phi_n(z)$$

is convergent in the norm. Uniqueness follows from injectivity of $\mathbf{I}_f^q$.

To see that (4) implies (2), observe that (4) implies that the sequential Faber operator is surjective. Since (6.2) is an isomorphism, $\mathbf{I}_f^q$ is surjective, and hence an isomorphism. ∎

The result that (1) implies (3) and (4) was obtained earlier by Çavuş [16]. Note that Çavuş and Shen work with the conformal map on the outside of the disk. Another difference of convention is that they phrase their results in terms of Bergman spaces, which is equivalent under the isometry $h \mapsto h'$ between Dirichlet and Bergman spaces. We altered both conventions in order to harmonize our presentation.

Wei, Wang, and Hu [69] showed that for a rectifiable Jordan curve, the Faber operator is an isomorphism if and only if the curve is a rectifiable quasicircle. Our results rely on transmission and the limiting integral, which enable us to remove the assumption of rectifiability. A key result in our approach is the equality of limiting integrals up to constants from either side (Theorem 5.11), which was originally proven in [53].

The proofs given here that (2)–(4) imply (1) are also new in that they do not use Pommerenke's result that the strict Grunsky inequalities imply that Γ is a quasicircle (see Theorem 6.13 ahead).

The inverse can be given explicitly. Denote by $\overline{\mathbf{P}_0(\mathbb{D}^+)}$ the projection onto $\overline{\mathcal{D}_0(\mathbb{D}^+)}$.

Theorem 6.5. *Let Γ be a quasicircle separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. Let $q \in \Omega_2$, and fix $f\colon \mathbb{D}^+ \to \Omega_1$. The inverse of $\mathbf{I}_f^q$ is $\overline{\mathbf{P}_0(\mathbb{D}^+)} \mathbf{C}_f \mathbf{O}_{2,1}$.*

Proof. Let $p = f(0)$. Observe that $\overline{\mathbf{P}_0(\mathbb{D}^+)}\mathbf{C}_f = \mathbf{C}_f\overline{\mathbf{P}_p(\Omega_1)}$. Thus, by Theorem 5.22, we have

$$-\overline{\mathbf{P}_0(\mathbb{D}^+)}\mathbf{C}_f\mathbf{O}_{2,1}\mathbf{J}^q_{1,2}\mathbf{C}_{f^{-1}}\overline{h} = -\mathbf{C}_f\overline{\mathbf{P}_p(\Omega_1)}\mathbf{O}_{2,1}\mathbf{J}^q_{1,2}\mathbf{C}_{f^{-1}}\overline{h} = \overline{h}$$

for all $\overline{h} \in \overline{\mathcal{D}_0(\mathbb{D}^+)}$. So this is a left inverse, which must also be the right inverse by Theorem 6.4. ∎

Remark 6.6. In Theorem 6.4, the result of Shen that (3) ⇒ (1) is stronger in comparison to our result (2) ⇒ (1); on the other hand, (1) ⇒ (2) is stronger than (1) ⇒ (3). Also, Shen's result is still stronger in that it does not require assuming that the domain is a Jordan curve. It is not immediately clear what the meaning of transmission would be when the complement of $f(\mathbb{D}^+)$ is not the closure of a Jordan domain. We did not pursue this issue. It seems to be of interest, in light of the fact that Faber polynomials have meaning for degenerate domains, among other reasons.

6.2. Grunsky inequalities

The Grunsky inequalities originally stem from H. Grunsky's studies in the context of univalent function theory [26]. The operator (or matrix) involved in those studies, which is referred to as Grunsky operator, have grown to become a powerful tool in many areas of mathematics.

The Grunsky operator (in various shapes) has been explored by many authors, for example A. Baranov and H. Hedenmalm [7] and G. Jones [30]. L. A. Takhtajan and L.-P. Teo showed that it provides an analogue of the classical period mapping of compact surfaces for the universal Teichmüller space [64]. See also V. L. Vasyunin and N. K. Nikol'skiĭ for an exposition of its appearance in de Branges' work on complementary spaces [66]. There are many interesting results relating the analytic properties of the Grunsky matrix $\mathbf{Gr}_f$ to the geometric or analytic properties of the conformal map f and/or its image $f(\mathbb{D}^+)$; see for example Jones [30], Shen [59], or Takhtajan and Teo [64].

The treatment as an integral operator goes at least as far back as Bergman and Schiffer's classic paper [11], as described in the explanation following Theorem 6.17 ahead.

We shall first define the Grunsky operators acting on polynomials, and later extend them by Theorem 6.9 to Dirichlet spaces. In Theorem 6.11 we will define the Grunsky operators in the more general setting of quasicircles.

Definition 6.7. Given a Jordan curve Γ separating $\overline{\mathbb{C}}$ into Ω_1 and Ω_2 as above, let $f: \mathbb{D}^+ \to \Omega_1$ be a conformal map with $f(0) = p$ and fix $q \in \Omega_2$. The Grunsky operator on polynomials is defined by

$$\mathbf{Gr}_f = \mathbf{P}_0(\mathbb{D}^+)\,\mathbf{C}_f\,\mathbf{O'}_{2,1}\,\mathbf{I}^q_f : \mathbb{C}_0[\overline{z}] \to \mathcal{D}_0(\mathbb{D}^+). \qquad (6.3)$$

As we saw in the previous section, $\mathbf{I}_f^q$ takes polynomials to polynomials in $\mathbb{C}_q[1/(z-p)]$, which have continuous transmission. It follows from Lemma 5.15 that the output of $\mathbf{Gr}_f$ on polynomials is in $\mathcal{D}_0(\mathbb{D}^+)$. Since for any $\overline{h} \in \overline{\mathcal{D}_0(\mathbb{D}^+)}$, we have that

$$\mathbf{I}_f^q \overline{h} - \mathbf{I}_f^{q_1} \overline{h} = [\mathbf{J}_{1,2}^{q_1} - \mathbf{J}_{1,2}^{q}]\mathbf{C}_{f^{-1}} \overline{h}$$

is constant, and the transmission and pull-back of constants are also constant, the Grunsky operator is independent of q.

Remark 6.8. By the Anchor lemma (Lemma 4.10), this agrees with the classical definition of the Grunsky coefficients. We choose $q = \infty \in \Omega_1$ and $p = 0$ to be consistent with the usual conventions, though the reasoning works for arbitrary q and p. The classical definition (in fact, one of several) is that the Grunsky coefficients b_{nk} of a univalent map of the disk are given by

$$\Phi_n(f(z)) = z^{-n} + \sum_{k=1}^{\infty} b_{nk} z^k, \tag{6.4}$$

where Φ_n is the nth Faber polynomial. Recalling that $\mathbf{I}_f^0(z^{-n}) = \Phi_n$, the fact that $\Phi_n(f(z))$ has this form follows from a simple contour integration argument (or from Corollary 5.20). By the Anchor lemma (Lemma 4.10) applied to Φ_n, together with the fact that

$$\mathbf{P}_0(\mathbb{D}^+)\,\mathbf{C}_f = \mathbf{C}_f \mathbf{P}_0(\Omega^+),$$

we have

$$\mathbf{Gr}_f(\overline{z}^n) = \sum_{k=1}^{\infty} b_{nk} z^k.$$

Thus, we see that the Grunsky coefficients are just the coefficients of the matrix representation of $\mathbf{Gr}_f$.

The Grunsky operator and Faber polynomials are often formulated for conformal maps $g: \mathbb{D}^- \to \Omega_1$, of the form $g(z) = z + b_0 + b_1/z + \cdots$, where Ω_1 contains the point at ∞. Choosing $q = 0$, the Faber polynomials are then defined by $\Phi_n = \mathbf{I}_g^0(z^n)$, and the Grunsky coefficients by

$$\Phi_n(g(z)) = z^{-n} + \sum_{n=-\infty}^{-1} b_{nk} z^k.$$

The convention that g takes $\mathbb{D}^-$ onto a domain containing ∞ seems to provide an advantage in some proofs [20, 43], in that the area of the complement of $g(\mathbb{D}^-)$ is finite; this appears to be the motivation for the choice. However, the advantage is illusory: the important fact is that the functions to which the Grunsky operator is applied have finite Dirichlet energy. The following identity shows that either choice is as good as the other. Setting $f(z) = 1/g(1/z)$, then it is easily checked, for $n > 0$ and $m > 0$, that

$$\sqrt{nm}\, b_{-n,-m}(g) = \sqrt{nm}\, b_{nm}(f). \tag{6.5}$$

Another approach to the definition of the Grunsky coefficients is through generating functions, e.g.

$$\log \frac{g(z)-g(w)}{z-w} - \log g'(\infty) = \sum_{n=-\infty,m=-\infty}^{-1} b_{n,m} z^n w^m,$$

where $g'(\infty) = \lim_{z\to\infty} g(z)/z$ for suitably chosen branches of logarithm. One can recognize immediately the relation to the integral kernel in Theorem 6.17. This also visibly demonstrates (6.5). The Faber polynomials can also be defined using generating functions related to the integral kernel of the Faber operator, see e.g. [20, 29, 63].

We now extend $\mathbf{Gr}_f$ to the full Dirichlet space.

Theorem 6.9. *Let Γ be a Jordan curve separating $\overline{\mathbb{C}}$ into components Ω_1 and Ω_2. Fix $q \in \Omega_2$. Let $f : \mathbb{D}^+ \to \Omega_1$ be a conformal map. $\mathbf{Gr}_f$ extends to a bounded operator*

$$\mathbf{Gr}_f : \overline{\mathcal{D}_0(\mathbb{D}^+)} \to \mathcal{D}_0(\mathbb{D}^+)$$

of norm less than or equal to one. For all $\overline{h} \in \overline{\mathcal{D}_0(\mathbb{D}^+)}$, the extended operator satisfies

$$\|\mathbf{I}_f^q \overline{h}\|^2_{\mathcal{D}_q(\Omega_1)} \le \|\overline{h}\|^2_{\overline{\mathcal{D}_0(\mathbb{D}^+)}} - \|\mathbf{Gr}_f\, \overline{h}\|^2_{\mathcal{D}_0(\mathbb{D}^+)}. \tag{6.6}$$

If Γ has measure zero, then equality holds.

Proof. First observe that the Grunsky operator satisfies the following invariance property when restricted to polynomials. For any Möbius transformation M, we have

$$\mathbf{Gr}_{M\circ f} = \mathbf{Gr}_f. \tag{6.7}$$

This follows from the fact that $\mathbf{C}_M\, \mathbf{O}'_{M(\Omega_2),M(\Omega_1)} = \mathbf{O}'_{2,1}\, \mathbf{C}_M$ and, by Theorem 5.8,

$$\mathbf{C}_M \mathbf{J}^{M(q)}_{M(\Omega_1),M(\Omega_2)} = \mathbf{J}^q_{1,2}\, \mathbf{C}_M.$$

We will first show the inequality (6.6) for polynomials. By the above observation, it is enough to prove it when Ω_2 contains ∞, and $p = 0 \in \Omega_1$. We can also assume that $q = \infty$.

For $r \in (0,1)$, let C_r be the positively oriented curve $|z| = r$. For $\overline{h} = h_1\bar{z} + \cdots + h_m \bar{z}^m \in \mathbb{C}_0[\bar{z}]$, we set

$$H(w) = \mathbf{I}_f^q \overline{h} = \sum_{n=1}^{m} h_n \Phi_n.$$

Observe that since $H(w)$ vanishes at ∞, we have

$$\lim_{R\nearrow\infty} \int_{|z|=R} \overline{H(w)} H'(w)\, dw = 0.$$

Thus, by (6.4) and using the fact that $\overline{z} = r^2/z$ on C_r, we have

$$
\begin{aligned}
\|\mathbf{I}_f^q \overline{h}\|^2_{\mathcal{D}_q(\Omega_1)} &\leq - \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(C_r)} \overline{h}(w) H'(w)\, dw \\
&= - \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{C_r} \overline{H(f(z))} (H \circ f)'(z)\, dz \\
&= \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{C_r} \left(\sum_{n=1}^{m} \overline{h_n} \overline{z}^{-n} + \sum_{n=1}^{m} \sum_{k=1}^{\infty} \overline{b_{nk}} \overline{z}^{k} \right) \\
&\qquad \times \left(\sum_{n=1}^{m} n h_n z^{-n-1} - \sum_{n=1}^{m} \sum_{k=1}^{\infty} k b_{nk} z^{k-1} \right) dz \\
&= \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{C_r} \left(\sum_{n=1}^{m} \overline{h_n} r^2 z^{n} + \sum_{n=1}^{m} \sum_{k=1}^{\infty} \overline{b_{nk}} r^2 z^{-k} \right) \\
&\qquad \times \left(\sum_{n=1}^{m} n h_n z^{-n-1} - \sum_{n=1}^{m} \sum_{k=1}^{\infty} k b_{nk} z^{k-1} \right) dz \\
&= \|h\|^2_{\overline{\mathcal{D}_0(\mathbb{D}^+)}} - \left\| \sum_{k=1}^{\infty} \sum_{n=1}^{m} b_{nk} h_n \right\|^2_{\mathcal{D}_0(\mathbb{D}^+)}.
\end{aligned}
$$

If Γ has measure zero, equality holds. The theorem now follow from density of $\mathbb{C}_0[\overline{z}]$ in $\overline{\mathcal{D}_0(\mathbb{D}^+)}$. ■

From now on, $\mathbf{Gr}_f$ refers to this extended operator.

Corollary 6.10. *For any Möbius transformation M, $\mathbf{Gr}_{M \circ f} = \mathbf{Gr}_f$.*

Proof. This follows from (6.7), since the extended operator must also satisfy this identity. ■

Usually normalizations are imposed on the function classes, especially the derivative at the origin or at ∞. These normalizations obscure the Möbius invariance of various objects, such as the Grunsky operator and Cauchy integral operator, and furthermore limit the applicability of the stated theorems unnecessarily. So we have removed them as much as possible throughout the paper.

Theorem 6.11. *If Γ is a quasicircle dividing $\overline{\mathbb{C}}$ into Ω_+ and Ω_-, and $f : \mathbb{D}^+ \to \Omega_+$ is a biholomorphism, then*

$$\mathbf{Gr}_f = \mathbf{P}_0^h(\mathbb{D}^+)\, \mathbf{C}_f\, \mathbf{O}_{2,1}\, \mathbf{I}_f^q.$$

Proof. The expression is a bounded extension of (6.3) by Theorem 3.5. ■

Remark 6.12. As is well known, using the identity in Remark 6.8, one can show that only injectivity of f is necessary in order to define the bounded Grunsky operator on Dirichlet space. This is usually formulated as an extension to sequences in ℓ^2.

Theorem 6.13 (Kühnau [33], Pommerenke [43]). *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$. The following are equivalent.*

(1) *Γ is a quasicircle.*

(2) *The Grunsky operator has norm strictly less than one.*

(3) *The Grunsky operator has norm strictly less than one on polynomials.*

(4) *There is a κ such that or all $\overline{h} \in \overline{\mathcal{D}_0(\mathbb{D}^+)}$,*

$$\mathrm{Re}\langle \mathbf{O}_{\mathbb{D}^+,\mathbb{D}^-}\overline{h}, \mathbf{Gr}_f\,\overline{h}\rangle \leq \kappa \|h\|^2. \tag{6.8}$$

(5) *The inequality (6.8) holds for polynomials.*

Before the proof of the theorem, let us recall the classical Grunsky inequalities in geometric function theory and their connection to Theorem 6.13. Setting

$$\overline{h}(z) = \lambda_1 \overline{z} + \cdots \lambda_n \overline{z}^n,$$

(3) says that there is some $\kappa < 1$ such that for all choices of parameters $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we have

$$\sum_{k=1}^{n} k \left| \sum_{m=1}^{n} b_{mk}\lambda_k \right|^2 \leq \kappa \sum_{k=1}^{n} k|\lambda_k|^2. \tag{6.9}$$

Item (5) says that there is some $\kappa < 1$ such that for all such choices of parameters, we have

$$\left| \sum_{k=1}^{n} \sum_{m=1}^{n} b_{mk}\lambda_k\lambda_m \right| \leq \kappa \sum_{k=1}^{n} k|\lambda_k|^2. \tag{6.10}$$

Proof. Since the Grunsky operator has a bounded extension to the whole space by Theorem 6.9 and polynomials are dense in the Dirichlet space, claims (2) and (3) are equivalent. Similarly, claims (4) and (5) are equivalent. By the Cauchy–Schwarz inequality, applied to

$$\mathrm{Re}\langle \overline{h}, \mathbf{O}_{\mathbb{D}^-,\mathbb{D}^+}\mathbf{Gr}_f\overline{h}\rangle = \mathrm{Re}\langle \mathbf{O}_{\mathbb{D}^+,\mathbb{D}^-}\overline{h}, \mathbf{Gr}_f\,\overline{h}\rangle,$$

(2) implies (4), using the fact that $\mathbf{O}_{\mathbb{D}^-,\mathbb{D}^+}$ is norm-preserving. Thus, it is enough to show (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). If Γ is a quasicircle, then by Theorem 6.4 $\mathbf{I}_f^q$ is an isomorphism, so there is a $c > 0$ such that

$$\|\mathbf{I}_f^q\overline{h}\|_{\mathcal{D}_q(\Omega_1)} \geq c\|\overline{h}\|_{\overline{\mathcal{D}_0(\mathbb{D}^+)}}$$

for all $\overline{h}$. Inserting this in (6.6), we see that

$$\|\mathbf{Gr}_f\overline{h}\|_{\mathcal{D}_0(\mathbb{D}^+)} \leq \sqrt{1-c^2}\|\overline{h}\|_{\overline{\mathcal{D}_0(\mathbb{D}^+)}}.$$

(5) $\Rightarrow$ (1). This is [43, Theorem 9.12] applied to (6.8), applied to $g(z) = 1/f(1/z)$. The different convention for the mapping function does not alter the result; see (6.5) ahead. ∎

Remark 6.14. A simple functional analytic proof that any of (2)–(5) implies (1) can be given, if we assume in addition that Γ is a measure zero Jordan curve. Assuming that $\|\mathbf{Gr}_f\| \leq k < 1$ say, and applying the equality case of (6.6), we obtain

$$\|\mathbf{I}_f^q \overline{h}\|_{\mathcal{D}_q(\Omega_1)} \geq \sqrt{1-k^2}\|\overline{h}\|_{\overline{\mathcal{D}_0(\mathbb{D}^+)}}.$$

So the image of $\mathbf{I}_f^q$ is closed, and by Remark 6.3 it is $\mathcal{D}_q(\Omega_2)$. Hence, by the open mapping theorem $\mathbf{I}_f^q$ is a bounded isomorphism, and therefore Theorem 6.4 yields that Γ is a quasicircle. The proof that (2) implies (1) that we give here is adapted from Shen [60].

Remark 6.15. In order to define the Faber polynomials and the Grunsky coefficients b_{nk} in (6.6), it is only required that f is defined in a neighbourhood of 0 and has non-vanishing derivative there. It is classical that (6.9) and (6.8) each hold for $\kappa = 1$ if and only if f extends to a one-to-one holomorphic function on $\mathbb{D}^+$ [20]. Equation (6.9) (with $\kappa = 1$) is usually called the strong Grunsky inequalities, while (6.8) is usually called the weak Grunsky inequalities [20]. The computation in the proof of Theorem 6.9 is the usual proof of the strong Grunsky inequalities.

Remark 6.16. The proof of Theorem 6.9 is easily modified to show that for any one-to-one holomorphic function f on $\mathbb{D}$, the Grunsky operator (expressed as a function of the parameters $\alpha_k = \lambda_k/\sqrt{k}$) extends to a bounded operator on ℓ^2, see e.g. [20, 43].

The Grunsky inequalities have been generalized in many ways. J. A. Hummel [27] generalized the inequalities to pairs of non-overlapping maps. The authors have extended this to arbitrary numbers of non-overlapping maps in genus zero [46]; Grunsky inequalities were proven for the case of n non-overlapping maps into a compact surface of genus g by M. Shirazi [61, 62]. This has applications to Teichmüller theory and is related to generalizations of the classical period mapping to the infinite-dimensional Teichmüller space of bordered surfaces of arbitrary genus and number of boundary curves [45]. For an early overview of Faber–Grunsky matters with a tidy description of the algebraic identities and relations involved, see E. Jabotinsky [29]. The thesis of Shirazi [61] also contains a historical survey of the Faber and Grunsky operator.

Finally, we include an integral expression for the Grunsky operator, due to Bergman and Schiffer [11]. It is most conveniently expressed in terms of the Bergman space of one-forms.

If Ω is simply connected, then

$$d\colon \mathcal{D}(\Omega) \to A(\Omega)$$

is norm-preserving and, in fact, if Ω is simply connected it becomes an isometry when restricted to $\mathcal{D}(\Omega)_q$ for any $q \in \Omega$. We then define

$$\mathbf{Gr}'_f\colon \overline{\mathcal{A}(\mathbb{D}^+)} \to \mathcal{A}(\mathbb{D}^+)$$

by

$$\partial\, \mathbf{Gr}_f = \mathbf{Gr}'_f\, \overline{\partial}.$$

With this definition, we have the following theorem.

Theorem 6.17. *For any Jordan curve* Γ *and conformal map* $f\colon \mathbb{D}^+ \to \Omega_1$, *we have*

$$\begin{aligned}\mathbf{Gr}'_f\overline{\alpha} &= f^*\mathbf{T}_{1,1}(f^{-1})^*\overline{\alpha} \\ &= \iint_{\mathbb{D}^+} \frac{1}{2\pi i}\left(\frac{f'(w)f'(z)}{(f(w)-f(z))^2} - \frac{1}{(w-z)^2}\right)\overline{\alpha(w)} \wedge dw \cdot dz.\end{aligned}$$

Proof. Set $p = f(0)$. Assume for the moment that Γ is a quasicircle. Then, fixing some $q \in \Omega_2$, we have

$$\begin{aligned}\mathbf{Gr}'_f\overline{\alpha} &= -\partial\mathbf{P}_0^h(\mathbb{D}^+)\mathbf{C}_f\mathbf{O}_{2,1}\mathbf{J}_{1,2}^q\mathbf{C}_{f^{-1}}\overline{\partial}^{-1}\overline{\alpha} \\ &= -\partial\mathbf{C}_f\mathbf{P}_p^h(\Omega_1)\mathbf{O}_{2,1}\mathbf{J}_{1,2}^q\overline{\partial}^{-1}(f^{-1})^*\overline{\alpha} \\ &= -f^*\mathbf{P}(\Omega_1)\partial\mathbf{O}_{2,1}\mathbf{J}_{1,2}^q\overline{\partial}^{-1}(f^{-1})^*\overline{\alpha}.\end{aligned}$$

Here, it is understood that $\overline{\partial}^{-1}$ is a choice of inverse of $\overline{\partial}\colon \mathcal{D}_0(\mathbb{D}^+) \to \mathcal{A}(\mathbb{D}^+)$. Now applying Theorems 5.16 and 5.14, we see that

$$\begin{aligned}\mathbf{Gr}'_f\overline{\alpha} &= -f^*\mathbf{P}(\Omega_1)\partial(\mathbf{J}_{1,1}^q - \mathrm{Id})\overline{\partial}^{-1}(f^{-1})^*\overline{\alpha} \\ &= f^*\mathbf{P}(\Omega_1)\mathbf{T}_{1,1}(f^{-1})^*\overline{\alpha},\end{aligned}$$

which proves the first claim. The integral formula is obtained by substituting (5.8) into the right-hand side and changing variables, using the fact that

$$L_{\Omega_1}(w,z) = \frac{1}{2\pi i}\frac{(f^{-1})'(w)(f^{-1})'(z)}{(f^{-1}(w)-f^{-1}(z))^2}dw \cdot dz.$$

If Γ is a Jordan curve but not a quasicircle, we apply the same argument to polynomials $\overline{h} \in \mathbb{C}_0[\overline{z}]$, using Lemma 5.18 in place of Theorem 5.16. The result is then extended to $\overline{\mathcal{D}_0(\mathbb{D}^+)}$ using Theorem 6.9. ∎

Bergman and Schiffer directly defined an operator using this integral formula, and observed that it recovers the Grunsky operator when applied to polynomials, see [11, pp. 239–240]. In particular, their integral formulation agrees with the unique operator extension of the Grunsky matrix to ℓ^2, if we identify sequences with elements of $\overline{\mathcal{A}(\mathbb{D}^+)}$ as in Theorem 6.4.

Remark 6.18. It can be shown that the integral formula in Theorem 6.17 is a bounded operator for arbitrary one-to-one $f\colon \mathbb{D} \to \mathbb{C}$, and this agrees with the extension of the Grunsky operator of Theorem 6.9. Although Bergman and Schiffer [11] assume that the boundaries are analytic Jordan curves, they were certainly aware of this fact.

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