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Editors' introduction

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We started the present project of a collection of articles dedicated to Dennis Sullivan in February 2021, on the occasion of his 80th birthday. We wanted this collection to be a sign of the admiration that we, together with all the authors that would participate in the project, feel for Dennis's personality and work.

About his personality, we would like to stress the fact that he has always been available for sharing with colleagues—young and old—his ideas, his curiosity and his interest, even in the smallest details of mathematics, bringing into focus the beauty he finds there. Barry Mazur reports in a paper published in the present collection, addressing Sullivan: "I heard that every time you thought about, say, the Pythagorean theorem, your eyes would light up." We also would like to mention all the minds he has shaped, directly and indirectly, as well as the immense activity he has always generated around him. The seminars he organized and which he is still organizing, like the lectures he gave and he is still giving, and the papers, books and notes he wrote and is still writing, on subjects like geometric topology, manifolds, singularities, Kleinian groups, foliations and laminations, hyperbolic geometry, discrete groups, dynamics, homotopic algebra, theoretical physics, fluid mechanics, and other topics, will remain a source of inspiration for generations of mathematicians.

A certain number of reports have been published about Sullivan's works, in particular after the various prizes he received (the most recent one being the Abel prize, in 2022), but none of these reports can really do justice to Sullivan's titanic work; to know it, one has to read his papers, and there are still important parts of his work that remain poorly known. In particular, there are two forgotten (or even unnoticed) results from the 1970s, which are very important to Sullivan (we learned this from him), that we would like to highlight now. This will also give us the occasion of mentioning works of two of his current PhD students.

The first result was obtained around 1970–71, when Sullivan discovered that the automorphism group of a simply connected finite homotopy type of a topological space, together with its action on the structure set $S(X)$ of closed simply connected PL manifolds in the Poincaré duality homotopy type X, should be "arithmetic", namely, that it consists of the integral points of a $\mathbb{Q}$-algebraic group acting on $S(X)$ which, if non-empty, is an arithmetic torsor of the said arithmetic group.

Sullivan gave two proofs of the first part, with the second proof extending to the torsor. The first proof used Daniel Kan's idea behind Quillen's Lie models (1967). The fact that $\mathrm{Aut}(X)$ is arithmetic was discovered independently by Clarence Wilkinson. The second proof, also by Sullivan, uses the differential forms rational homotopy theory, and it was published in the paper *Infinitesimal computations in topology* (Publ. Math. IHÉS, 1977). Mentioning this "arithmetic" result is important, because this question and the related structure remain virtually unknown. Sullivan has a PhD student currently working on this, Runjie Hu. Recently, Alexander Berglund found new results related to this $\mathbb{Q}$-algebraic structure.

Even though forgotten or unnoticed in the 1970s, this arithmeticity (in turn motivated by "Galois Symmetry" from the talk Sullivan gave at the 1970 Nice ICM) was the reason behind Sullivan's work on rational homotopy theory with differential forms, discussed in the paper *Infinitesimal computations in topology*. Thus, although this paper is very well known, its motivation and central result were alas unnoticed until recently.

The second result, obtained in 1966–67, was announced in the BAMS article by Sullivan of July 1967 titled *On the Hauptvermutung for manifolds*. The title was based on an unexpected and difficult result about the structure set for PL manifolds, which was called the Characteristic Variety Theorem. It solved the obstruction theory from Sullivan's thesis. This was obtained by defining finitely many *a priori* numerical invariants in $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ associated with a finite system of closed manifolds or their $\mathbb{Z}/n\mathbb{Z}$ analogues mapping to the homotopy type (the characteristic variety). This was done by applying transversality to any homotopy equivalence between two manifolds in the homotopy type of X and computing the signatures, etc. of the various components of the characteristic variety and its transversal pre-image.

All this was the result of a determined effort. During the year 1966, the parts of this characteristic variety emerged slowly. Each time a part was achieved, one could use Novikov's technique from 1964 to prove the *a priori* invariants vanishing if the homotopy equivalence were a homeomorphism (except possibly for an element of order two in the integral fourth cohomology of the homotopy type). Thus, by obstruction theory, the homeomorphism was homotopic to a PL homeomorphism (for simply connected PL manifolds of dimension at least 5, if the fourth integral cohomology had no two torsion, 1966).

The Characteristic Variety Theorem used advanced algebraic and geometric topology including localization and manifolds with singularities. Geometrically focused topologists were relieved when in 1969 the corollary about the Hauptvermutung could essentially be achieved and extended, based on Kirby–Siebenmann's new theory on the category of homeomorphisms and topological manifolds. (Here, the terms "essentially" and "extended" are used because Kirby and Siebenmann produced one obstruction in the third cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients, which was the only obstruction to deforming a homeomorphism between PL manifolds through homeomorphisms, to a PL homeomorphism; the integral Bockstein of their obstruction is the order-two obstruction revealed by the characteristic variety). This is the "essentially" part. If one is content to have a homotopy to a PL homeomorphism in the simply connected case, it is enough to know that the integral Bockstein vanishes. The simply connectedness hypothesis is dropped in their result, this being the "extended" part.

The effect of this development is that the Characteristic Variety Theorem (solving the obstruction theory for classifying PL manifolds) was no longer needed for the Hauptvermutung. However, this applies as well to topological manifolds. In this form it will appear in the developing thesis of Runjie Hu which completes later developments (1970s and 1980s) due to Norman Levitt, Andrew Ranicki, Frank Quinn, Brumfiel–Morgan and Milgram–Madsen.

Another PhD student of Sullivan, Jiahao Hu (PhD in 2023) has extended the Characteristic Variety Theorem to real K-theory, giving numerical invariants that determine the isomorphism class of stable real vector bundles over a finite homotopy type. This result was unanticipated since 1966 because of difficulties related to two torsion in $KO(X)$. Jiahao overcomes this using quaternions.

We now mention a few remembrances from the two seminars that Sullivan conducted at the Institut des Hautes Études Scientifiques in Bures-sur-Yvette, for more than 20 years, and in New York, for almost five decades now.

From the IHÉS seminar, we remember the famous year 1982. It was the time when Dennis started splitting his time between Paris and New York, where he was named the Albert Einstein Chair, after which he started the Einstein Chair Seminar.

In that year, Dennis returned to Paris after having spent some time in the USA in discussions with Bill Thurston. Dennis had been impressed by the works and ideas of the latter since the first time he met him, in December 1971, in Berkeley, where he was visiting from MIT to give a series of talks on differential forms and the homotopy theory of manifolds, and where Thurston was a graduate student; see the *Third story* in Sullivan's recollections that appeared in the present journal, in the collective article *W. P. Thurston and French mathematics* (2019).

During the stay in New York that Sullivan made on that year, Sullivan was the first to know about Thurston's characterization of postcritically finite rational maps of the sphere. In the same year, he organized a *Seminar on Conformal and Hyperbolic Geometry* in Bures-sur-Yvette. This was on Thursdays, all day. In the audience were Misha Gromov, usually in the first row and smoking, René Thom, Larry Siebenmann, John Hubbard, Nico Kuiper, Michel Herman, David Ruelle, and many others. During the first weeks, the talks were on Kleinian groups, but after a certain time, Dennis started talking about iterations of rational maps and the analogies between this theory and that of Kleinian groups. He then came with the idea of the Dictionary, pointing out the many analogies between the two theories: limit sets/Julia sets, discontinuity domain/Fatou set, etc. He then gave the proof of an enhanced version of Ahlfors' finiteness theorem and asked what there should be on the other side of the dictionary. One week later, he came with the statement of the No-Wandering Domain Theorem and an outline of the proof, which he completed within the next two weeks, just in time to finish it on the last day of the seminar.

We have included in the present collection a set of notes on the iteration on complex analytic rational maps, written by Sullivan, which he distributed in that same year, at his Bures seminar.

Let us now cross the ocean and see what was happening at essentially the same time in the USA, at another seminar conducted by Dennis.

The Albert Einstein Chairs were created by the State of New York in 1964 as an attempt to attract outstanding scholars. There were five of them, and five others named Albert Schweitzer Chairs in humanities. While the Schweitzer Chairs were enmeshed in controversy (see the article *Trouble bedeviling Schweitzer Chairs* in The New York Times, Sunday, December 13, 1970), the five Einstein Chairs were filled with relative ease and little publicity. In 1981, upon the retirement of Elliot T. Montroll, a mathematical physicist of the University of Rochester, Dennis P. Sullivan was appointed Albert Einstein Professor of Science at Queens College and the Graduate School of the City University of New York.

Upon commencing his position at the City University of New York, Dennis started a weekly seminar that attracted many of the best minds from around the world, bringing an atmosphere of excitement and energy to the Graduate Center that persists to this day. While the topics were wide-ranging, the Einstein Chair Seminar had periods with overarching themes.

The first decade of the Einstein Chair Seminar was devoted to the deep interrelation between structures on hyperbolic manifolds and holomorphic dynamics that grew out of Sullivan's seminal work and his collaboration with Thurston. His dictionary between dynamical properties of Kleinian groups and dynamics of rational maps has become a major tool in the study of holomorphic dynamics. His deep understanding of the deformation spaces of Riemann surfaces through Teichmüller theory led to his proof of the No-Wandering Domain Theorem for rational maps, and hence to a complete classification of the components on which the dynamics of a rational map is tame. Furthermore, Sullivan outlined a framework to develop a conceptual proof for Feigenbaum's rescaling rigidity on the period-doubling bifurcations.

The next two decades had derived algebraic topology, string topology, and fluid dynamics. The Einstein Chair Seminar was heavily influenced by Chas and Sullivan's discovery of natural operations on the homology of the free loop space of manifolds, combining the composition of loops with the intersection product of chains in an intricate fashion. Recent works have confirmed hypothetical statements that the string topology operations are sensitive to the structure of manifolds beyond their homotopy type.

The fourth decade included three-dimensional manifolds, finite combinatorial models for three-dimensional fluids, perturbation theory, and more. Featured speakers came from a variety of disciplines, pure and applied mathematics, physics and engineering, each of whom have made significant contributions. These talks led to deep discussions on the nature of turbulent flows and their relation to quantum field theory, and more.

We are now in the fifth decade of the Einstein Chair Seminar, and the seminar continues as ever to attract outstanding scholars and generate in-depth discussions.

While in this day and age it may be prosaic to say that these seminars are recorded and archived, that was not the case in 1981 when Sullivan set out to record them for current and future generations. At the time it was innovative to devote resources and energy to such an endeavor, but that is not surprising given Sullivan's emphasis on mathematical communication and revolutionary contributions to mathematics.

Let us turn to the papers included in the present collection. These papers reflect several of Sullivan's interests, namely in topics like singularity theory, circle homeomorphisms, random walks, dynamics, quasiconformal mappings, Galois theory, noncommutative geometry, homotopy theory, fluid dynamics, rational homotopy, solenoidal manifolds, hyperbolic geometry and Fuchsian groups.

In the quick overview that we give of these papers, we shall mention at the same time some related works of Sullivan.

The opening paper, by Barry Mazur, titled *Happy Birthday, Dennis!*, is a collection of memories of moments Barry spent with Dennis, together with thoughts on mathematics, intuition, inspiration, imagination, feeling, perception, and the pleasure with which mathematics can fill us, written as a reflection on the work of Dennis on the occasion of his 80th birthday.

The article by Hyungryul Baik and Inhyeok Choi, titled *Random walks on mapping class groups*, is a survey on random walks in the setting of mapping class groups acting on Teichmüller spaces equipped with the Teichmüller metric.

This topic has been extensively studied by Kaimanovich and Masur, who made connections between mapping class group actions, the theory of discrete groups acting on negatively curved spaces, and the theory of lattices in Lie groups acting on homogeneous space. Baik and Choi, in their survey, discuss the relations between random walks on Teichmüller spaces compactified by the space of projective equivalence classes of measured foliations, and the Patterson–Sullivan theory of conformal measures. This theory was started by S.-J. Patterson, who, in his paper *The limit set of a Fuchsian group* (1976), associated to each point x in the hyperbolic plane a quasi-invariant measure ν_x on the boundary of the space. The construction was later extended by Sullivan to any dimension, in his paper *On the density at infinity of a discrete group of hyperbolic motions* (1979) in which he established deep connections between this theory, harmonic analysis, and the ergodic theory of geodesic flows.

The paper *Characterizations of circle homeomorphisms of different regularities in the universal Teichmüller space*, by Jun Hu, is also concerned with Teichmüller theory, more specifically, with the study of circle homeomorphisms in relation with the universal Teichmüller space. The author surveys several characterizations of various properties of circle homeomorphisms (quasisymmetric, symmetric, $C^{1+\alpha}$, etc.) in terms of Beurling–Ahlfors and Douady–Earle extensions and of Thurston's earthquake representations of such homeomorphisms. He also obtains a new result on the regularity of the Beurling–Ahlfors extension $BA(h)$ of a $C^{1+\text{Zygmund}}$ orientation-preserving diffeomorphism h of the real line, showing that the associated Beltrami coefficient $\mu(BA(h))(x + iy)$ vanishes as $O(y)$ uniformly on x near the boundary of the upper half-plane if and only if h is $C^{1+\text{Lipschitz}}$. Let us note that the author was Sullivan's PhD student, and that he has collaborated with him on circle homeomorphisms (cf. the paper by Hu and Sullivan, *Topological conjugacy of circle diffeomorphisms*, 1997).

The article by Richard Canary, titled *Hitchin representations of Fuchsian groups*, is a survey of the theory of representations of fundamental groups of closed surfaces into $\mathsf{PSL}(d, \mathbb{R})$, initiated by Hitchin using Higgs bundles, with a focus on the dynamical and geometric properties of these representations. In analogy with the theory of augmented Teichmüller space, in which one attaches to this space the strata at infinity consisting of Teichmüller spaces of finite area hyperbolic surfaces with cusps, the author is interested in the study of an "augmented Hitchin component" arising as the metric completion of the Hitchin component with respect to a pressure metric. Pressure metrics form a class of metrics originally obtained using the thermodynamic formalism introduced by Bowen and Ruelle in their study of Anosov flows and Anosov diffeomorphisms, in the 1970s, at IHÉS where Sullivan was also working. Recent works on the augmented Hitchin component are due to Loftin and Zhang who studied topological aspects of this space in the case $d = 3$, obtaining local parametrizations of neighborhoods of the strata at infinity. There are also works by Canary, Zhang and Zimmer, in which these authors developed a theory of Hitchin representations for geometrically finite Fuchsian groups, and by Bray, Kao and Martone which led to a construction of pressure metrics on components of Hitchin representations of Fuchsian lattices arising as strata at infinity of an augmented Hitchin component. Canary, in his paper, points out new relations with Sullivan's work, namely, through a recent work of Tholozan, in which the latter describes an embedding of the Hitchin component in the Teichmüller space of foliated complex structures on the unit tangent bundle of a surface, with ideas originating in Sullivan's paper *Linking the universalities of Milnor–Thurston, Feigenbaum and Ahlfors–Bers* (1993). Using this work, the pull-back of a Weil–Petersson-type metric on Teichmüller space appears as the so-called simple root pressure metric.

The article by Norbert A'Campo is titled *Flow box decomposition for gradients of univariate polynomials, billiards on the Riemann sphere, tree-like configurations of vanishing cycles for A_n curve singularities and geometric cluster monodromy*. In this article, the author studies isolated complex hypersurface singularities from a dynamical point of view, a point of view which he attributes to Sullivan. He starts by recalling the latter's expository talk at the 1974 Vancouver ICM, titled *Inside and outside manifolds*, in which he promotes the study of geometrical objects inside one manifold, in parallel to the question of classifying manifolds, which, by that time, had already evolved quite far. Sullivan declared that the qualitative study of dynamical systems is "one organizing center" for the study of individual manifolds. In the present paper, A'Campo's goal is to understand isolated complex hypersurface singularities in the sense proposed by Sullivan, in the fibres of the universal unfolding of a singularity, an object introduced and described by René Thom in his book *Stabilité structurelle et morphogénèse*.

About Thom, it is worth mentioning here a sentence by Sullivan (email to Athanase Papadopoulos, April 28, 2019): "My first math hero was René Thom. While visiting IHÉS in 73–74, as *professeur associé* at Orsay, I was offered a professorship at IHÉS. I was honored to accept and to become his 'colleague'." About heroes, Sullivan continues: "My second math hero was the Mozart-like figure Bill Thurston." We already talked about Sullivan's esteem for Thurston at the beginning of this introduction.

Let us now move to another paper in this collection in which ideas of Thom and Thurston both play a significant role. This is the paper by Michael Freedman titled *Controlled Mather–Thurston theorems*. Cobordism theory, for whose development Thom was awarded the Fields medal in 1958, appears at the center of this paper, where Freedman obtains substantial generalizations of theorems of Milnor–Wood for flat circle bundles over surfaces, and of Mather–Thurston on the existence, in the case of C^0-foliations, of cobordisms between the given manifold bundle and one admitting a flat connection with "controlled" topology. He proves, under suitable conditions, the existence of cobordisms that are semi-simple. With this, Freedman gives a very novel perspective on old deep theorems, and he provides techniques that are potentially useful for the study of other problems in foliation theory. He declares that his paper is inspired by his will to lay the mathematical foundation of a program in physics.

The article by Alain Connes and Caterina Consani titled *BC-system, absolute cyclotomy and the quantized calculus*, is concerned with the interplay between noncommutative geometry, the Riemann zeta function, symmetry breaking in physics, and Galois theory. The main object of study in this article is the BC dynamical system, of which the authors give a description at the most basic algebraic level, as the universal Witt ring, that is, the K-theory of endomorphisms, of the algebraic closure of the absolute base $\mathbb{S}$, obtained by adjoining to $\mathbb{S}$ all abstract roots of unity. A section of this paper is dedicated to the recent results of the authors on the analytic approach to noncommutative geometry, based on the quantized calculus and the notion of spectral triple, through Dirac operators, which give information on the zeros of Riemann's zeta function.

Connes and Consani recall a letter, sent by Sullivan to Alain Connes on September 27, 1993, that sets the bases of the interactions between noncommutative geometry and the geometry of manifolds. The letter (which was actually sent by fax) is reproduced in the paper, and it gave rise to a joint paper by Connes, Sullivan and Teleman, titled *Quasiconformal mappings, operators on Hilbert space, and local formulae for characteristic classes* (1994). Let us take this opportunity to recall that Sullivan was the first to introduce Galois theory in the study of the geometry of manifolds, introducing there the concepts of localization, rationalization, profinite and p-adic homotopy theory, etc.; cf. his 1970 ICM talk, titled *Galois symmetry in manifold theory at the primes*. These ideas are also at the basis of his algebraico-geometrical proof of the Adams conjecture on the relation between vector bundles and spherical fibrations (1967).

The paper by James Davis and Carmen Rovi, titled *Chain duality for categories over complexes*, contains a proof of the fact that the additive category of chain complexes parametrized by a finite simplicial complex is a category with chain duality. This result was stated without proof by Ranicki in his book *Algebraic L-theory and topological manifolds* (1992), and it is fundamental in the latter's algebraic formulation of the Sullivan–

Wall surgery exact sequence and his interpretation of the surgery obstruction map as the passage from local to global Poincaré duality. In their paper, Davis and Rovi give a full proof of this result. At the same time, they provide a new, conceptual, and geometric treatment of the theory of chain duality on K-based chain complexes. The results of this paper are closely related to Sullivan's works on the classification of manifolds, which he started in his PhD thesis defended in 1966 and titled *Triangulating homotopy equivalences*. In this thesis, Sullivan formulated an obstruction theory for the deformation of a homotopy equivalence between two manifolds to be a piecewise linear homeomorphism. This gave in particular further cases where the Hauptvermutung is true. We already talked about this at the beginning of the present introduction. The thesis was very shortly followed by a University of Warwick preprint titled *Triangulating and smoothing homotopy equivalences and homeomorphisms* (Geometric topology seminar Notes), written in 1967, in which Sullivan presented the notion of surgery exact sequence for PL manifolds. The paper was published several years later, in *The Hauptvermutung book* (1996).

The topic of the paper *Singularity formation in the incompressible Euler equation in finite and infinite time* by Theodore Drivas and Tarek Elgindi is fluid dynamics, one of Sullivan's favorite topics. The emphasis is on infinite-dimensional dynamical systems. The authors survey several classical and recent results on the incompressible Euler equations that govern perfect fluid motion and they also prove several new results on this topic. From the geometric viewpoint, a fluid flow is seen as a geodesic motion on the group of volume-preserving diffeomorphisms. The notion of wellposedness plays an important role in this domain. The authors discuss wellposedness results for the Eulerian velocity field belonging to various function spaces. They examine 2D fluid motion, especially from the point of view of their long-time persistent behavior. They examine mechanisms for finite-time singularity formation and they review recent advances on blowup of classical solutions for 3D Euler equations. They consider the formation of singularities in 3D and they discuss a class of global solutions in any dimension (with constant pressure) and use them to give an example of a finite time blowup from smooth initial data for the Euler equations in infinite spatial dimensions. All along their paper, the authors discuss several open problems and conjectures on this topic. A number of papers of Sullivan related to fluid dynamics are quoted, in particular, *A finite time blowup result for quadratic ODE's* (2011), *Algebra, topology and algebraic topology of 3D ideal fluids* (2011), *3D incompressible fluids: combinatorial models, eigenspace models, and a conjecture about well-posedness of the 3D zero viscosity limit* (2014), *A discourse on the measurable Riemann mapping theorem and incompressible fluid motion* (2015), and *Lattice hydrodynamics* (2020).

The paper *Rational homotopy via Sullivan models and enriched Lie algebras* by Yves Félix and Steve Halperin is a survey on the theory of minimal models and rational homotopy. Rational homotopy theory originated in the late sixties and the early seventies with the simultaneous but distinct approaches of Quillen, Sullivan and Bousfield–Kan. In each of these approaches, one associates to a path-connected space X an algebraic object which is used to construct a *rational completion* of this space, $X \to X_{\mathbb{Q}}$. In their paper, Félix and Halperin concentrate on the use of Lie algebras encoding homotopy groups in the setting of Sullivan's $\mathbb{Q}$-completion. They present a conjecture of Avramov–Félix saying that the homotopy Lie algebra of a finite cell complex, in the infinite-dimensional case, contains a free graded Lie algebra on two generators. They present several related examples in detail, including hyperbolic Riemann surfaces, classifying spaces of right-angled Artin groups, arrangements of hyperplanes in complex Euclidean space, and configuration spaces of points in a Euclidean space. The authors mention several consequences of Sullivan's minimal model, including the existence of infinitely many geometrically distinct closed geodesics on most closed manifolds obtained by Vigué-Poirrier and Sullivan in the paper *The homology theory of the closed geodesic problem* (1976), and also later by Grove and Halperin in their paper *Contributions of rational homotopy theory to global problems in geometry* (1982). Another consequence of Sullivan's minimal model that the authors present is the construction of Sullivan models for the space of sections of a fibration by Haefliger, in his paper *Rational homotopy of the space of sections of a nilpotent bundle* (1982) and by Buijs, Félix and Murillo, in their paper *Lie models for the components of sections of a nilpotent fibration* (2009). Félix and Halperin also discuss the problem of the completion of the universal enveloping algebra. Besides being a survey, the paper contains new results by the authors, and a list of open problems.

The two papers *Solenoidal manifolds, laminations, profinite completions* by Alberto Verjovsky and *A note on laminations with symmetric leaves*, by Michael Kapovich, are concerned with the theory of solenoids. These are objects whose study was very much promoted by Sullivan, who used them as a tool in his work on the Ehrenpreis conjecture, a conjecture due to Leon Ehrenpreis (the date he formulated it is not clear) and proved by Vlad Markovich and Jeremy Kahn in 2011, stating that for any closed Riemann surfaces of genus ≥ 2 and for $\varepsilon > 0$, there are finite degree covers of the two surfaces that are $(1 + \varepsilon)$-quasi-isometric.

Solenoids are spaces laminated by n-dimensional leaves carrying a Cantor set transverse structure. Such a structure already appears in works of Bing, McCord, Rogers, Schori and Tollefson in the 1960s. By a theorem of A. Clark and S. Hurder, topologically homogeneous compact solenoidal manifolds are obtained as inverse limits of increasing towers of finite regular covering spaces of compact manifolds with infinite and residually finite fundamental groups, and they behave like laminated versions of compact manifolds. A version of the solenoid was used in dynamical systems theory, as a one-dimensional expanding attractor, by Robert Williams. Solenoids also play a role in Smale's Axiom A diffeomorphisms and in holomorphic dynamics. From the topological viewpoint, they also naturally appear as Pontryagin duals of discrete locally compact Hausdorff abelian groups. Among Sullivan's works related to solenoids, we mention the papers *Solenoidal manifolds* (2014) and *Linking the universalities of Milnor–Thurston, Feigenbaum and Ahlfors–Bers* (1993).

The paper by Verjovsky combines solenoids, laminations and profinite topology. It is mainly a survey of n-dimensional solenoidal manifolds for $n = 1, 2$ and 3, but it also contains new material on this subject. One of them is a discussion of laminated spaces with exotic differentiable structures along the leaves. Other objects that play an important role in this paper are Cantor groups, that is, topological groups homeomorphic to the Cantor set, and in particular, profinite groups, which are profinite completions of residually finite groups.

The paper *A note on laminations with symmetric leaves* by Michael Kapovich is concerned with solenoidal manifolds from the geometrical, dynamical and analytical points of view. The main result proved in this paper is a theorem which gives a description of n-dimensional homogeneous solenoidal laminations whose leaves are all isometric to a symmetric space of noncompact type. This result says that a solenoidal manifold is homeomorphic to the quotient of a symmetric space by a discrete, torsion-free cocompact group of isometries, provided the dimension is not 4. In the case of dimension 4, the manifold is homotopy equivalent to such a quotient. The proof uses fundamental ideas in geometric group theory and conformal topology.

The next paper, *The Sullivan dictionary and Bowen–Series maps*, by Mahan Mj and Sabyasachi Mukherjee, is concerned with two important themes from Sullivan's work on holomorphic dynamics, namely, the dictionary he established between Kleinian groups and the dynamics of rational maps of the sphere, and his work on the so-called Patterson–Sullivan measure. The authors survey this dictionary, with a stress on a new component on which they have made recent progress, namely, a combination theorem making a bridge between Fuchsian groups and complex polynomials, combining them into a single dynamical system on the Riemann sphere. They call this process *mating a Fuchsian group with a polynomial*. This study is based on an item in Sullivan's dictionary that highlights the similarity between, on the one hand, Bers's Simultaneous Uniformization Theorem for Kleinian groups and, on the other hand, polynomial mating in rational dynamics, introduced by Douady and Hubbard. A key ingredient in this study is the use of the so-called class of Bowen–Series maps, which are examples of piecewise Fuchsian Markov maps of the circle that are orbit equivalent to finitely generated Fuchsian groups and which first appeared in the 1979 works of Bowen and Series. The authors introduce maps which they call *higher Bowen–Series maps*, which in some precise sense are piecewise Bowen–Series maps, and which give rise to combination theorems and to dynamically natural homeomorphisms between limit sets and Julia sets. The new material in this paper includes a recent work of the authors, *Combining rational maps and Kleinian groups via orbit equivalence*. The paper ends with a list of open questions.

The paper *The diagonal of cellular spaces and effective algebro-homotopical constructions* by Anibal Medina-Mardones is a survey of certain homotopy coherent enhancements of the coalgebra structure of the chains of a cellular complex defined by an approximation of the diagonal of the complex. It is known that over the rational numbers, C_∞-coalgebra structures on the chain control the $\mathbb{Q}$-complete homotopy theory of a space, whereas over the integers, E_∞-coalgebra structures provide an appropriate setting to model the full homotopy type of the complex. The survey focuses on the constructive nature of the C_∞ and E_∞ extensions of the diagonal map over the rationals and integers respectively. Such structures carry geometric and combinatorial information, and the author mentions applications in several fields, from deformation theory to higher category theory and condensed matter physics. As a motivation, he mentions Sullivan's local inductive construction of a C_∞-coalgebra structure on the chains of cell complexes whose closed cells have the $\mathbb{Q}$-homology of a point, carried in his *Appendix A in "infinity structure of Poincaré duality spaces"* (2007), and the following challenge he posed in his paper with Ruth Lawrence, *A formula for topology/deformations and its significance* (2014): "Study this free differential Lie algebra attached to a cell complex, determine its topological and geometric meaning as an intrinsic object. Give closed form formulae for the differential and for the induced maps associated to subdivisions." Another related paper by Sullivan is his *Infinitesimal computations in topology* (1977).

In the paper *String topology in three flavours* by Florian Naef, Manuel Rivera and Nathalie Wahl, the authors highlight major string topology operations that illustrate the richness of the subject. They restrict their attention to the original Chas–Sullivan loop product and its dual, the Goresky–Hingston coproduct. They describe these two operations geometrically and algebraically. The geometric construction uses Thom–Pontrjagin intersection theory while the algebraic construction is phrased in terms of Hochschild homology. The authors compare the two descriptions using methods from rational homotopy theory, emphasizing the role of configuration spaces. They illustrate the geometric aspect of string topology through computations of loop products and coproducts on lens spaces via intersections of geometric cycles. From the algebraic point of view, they describe the structure that these operations define together on the Tate–Hochschild complex. They show that the two operations are encoded in the data of a Manin triple, and they address the question of homotopy invariance for the product and coproduct, both algebraically and geometrically.

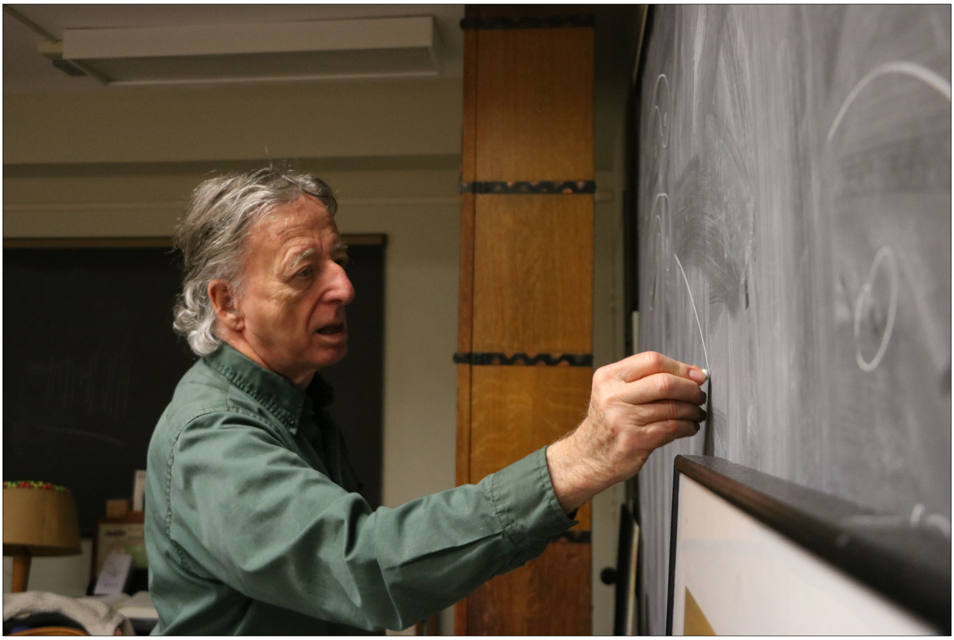
The paper *On the Whitehead nightmare and some related topics* is written by Valentin Poénaru. In this paper the author first surveys his work on what he calls the QSF (quasi-simply filtered group) property in geometric group theory, where he met infinitistic complications to which he gave the name Whitehead nightmare. He then sketches work in progress, joint with Louis Funar and Daniele Otera, whose goal is to show that all finitely presented groups can avoid this nightmare, and that all groups have the GSC (geometric simple connectivity) property, which is much stronger than QSF.

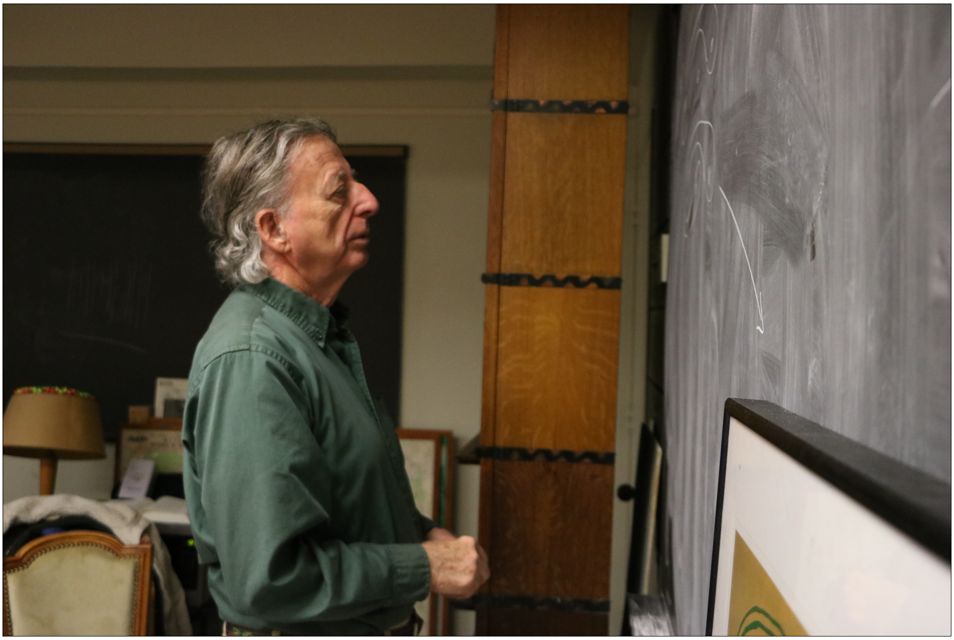
The paper *Formality and finiteness in rational homotopy theory*, by Alex Suciu, is a broad overview of the various geometrico-topological techniques of Sullivan's rational homotopy theory, with applications to the study of closed orientable 3-manifolds, smooth (quasi-)projective varieties, compact Kähler manifolds, compact Lie group actions, Sasakian manifolds and hyperplane arrangement complements. The author makes a particular stress on formality and finiteness properties of spaces, and how these properties are reflected in algebraic models for these spaces, the Malcev Lie algebra completions of their fundamental groups and the geometry of some associated varieties, such as resonance and characteristic varieties. The theory is illustrated by enlightening examples from complex algebraic geometry, compact Lie group actions and three-dimensional manifolds. Some papers of Sullivan that are connected with these theories are *Genetics of homotopy theory and the Adams conjecture* (1974), *On the intersection ring of compact three* *manifolds* (1975), *Infinitesimal computations in topology* (1977), and the influential set of notes *Geometric topology: localization, periodicity and Galois symmetry*. These notes, written in 1970 and published in 2005, were widely circulated at the time they were written and they had a major influence on the development of both algebraic and geometric topology. In those notes appeared also the formulation of the *Sullivan conjecture* on the contractibility of the space of maps from the classifying space of a finite group to a finite-dimensional CW complex, proved later by Haynes Miller.

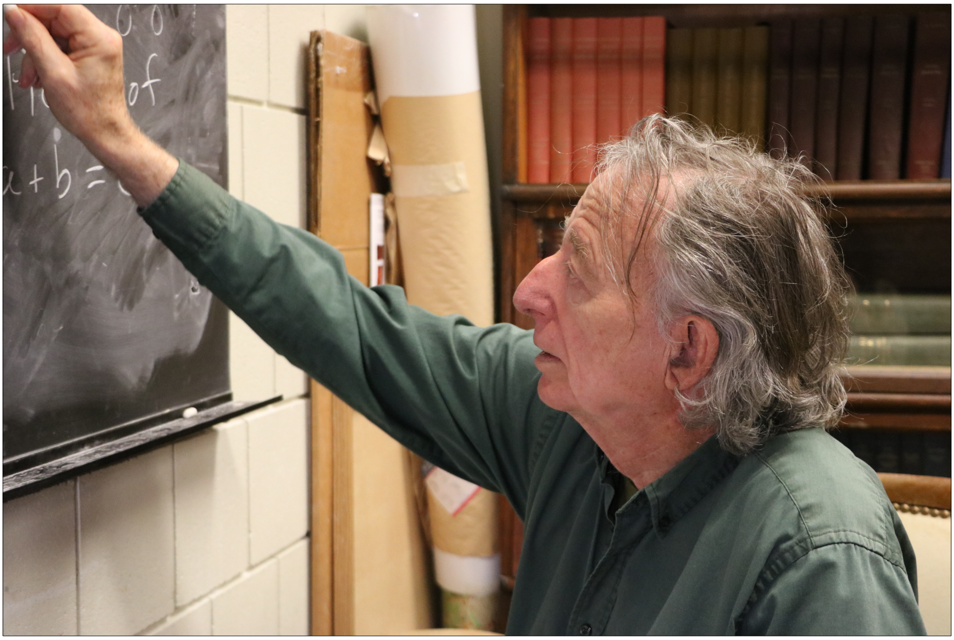
Finally, there is a musical offering. The paper by Bob Penner is titled *Music of moduli spaces*. In this paper, a musical instrument is constructed. The author calls it a *plastic hormonica*, and it is based on the Farey tessellation of the hyperbolic plane, decorated by its standard osculating horocycles centered at the rationals. There are several types of ways to play the instrument, for instance, tapping or holding points of another tessellation τ with the same decorating horocycles. The sounds produced depend on the lambda length of $e \in \tau$ with its decoration. One may also apply flips that are equivariant by a Fuchsian group preserving τ, etc. and there are also chords that arise from Markoff triples. The musical instrument is dedicated by Bob Penner to Dennis Sullivan, who pioneered and helped popularize the hyperbolic geometry underlying its construction.

This collection of articles dedicated to Sullivan reflects his wide interests, and at the same time, the richness of some of the developments that took place in topology, geometry and dynamics in the last sixty years, on topics whose foundations he led.


DISCRETE MATHEMATICS







The photos were taken by Dennis Sullivan's daughter, Clara Sullivan, in December 2022, and they are published here with her permission.

This is a set of notes on the iteration on complex analytic rational maps, written by Dennis Sullivan, which he distributed at the seminar he led at Bures-Sur-Yvette in 1982. They are included here with Sullivan's permission.

1

On the iteration of complex analytic maps

by Dennis Sullivan

If f is a complex analytic self-mapping of the Riemann sphere, Fatou (1918) defined the domain (of equicontinuity) where points have neighborhoods so that the restrictions of $f, f\circ f, f\circ f\circ f, \ldots$ form an equicontinuous family. Julia (1918) in effect defined the complement J_f of the domain of equicontinuity F_f as the closure of the expanding periodic points. In this paper we add certain techniques to the general study in the memoires of Fatou and Julia, using the analogy with the modern study of discrete groups of hyperbolic motions.

A component Ω of the domain of equicontinuity is called cyclic (of order k) if for some $k>0$ $f^k\Omega = \Omega$.

Theorem 1. Every component of the domain of equicontinuity has an image under f^n, some $n>0$, which is cyclic. (There are no wandering components.)

2

The method of proof combines ideas of orbit equivalence (to study the equivalence relations $\{x \simeq y$ if $\exists\, n \geq 0$ so that $f^n x = f^n y\}$ and $\{x \sim y$ if $\exists\, n, m \geq 0$ so that $f^n x = f^m y\}$.) and hyperbolic geometry (analytic self maps of general Riemann surfaces are distance decreasing for the hyperbolic metric. And discrete subgroups of $PSL(2, \mathbb{R})$ can be characterized by nature of elliptic elements) via the Riemann mapping theorem for measurable Riemannian metrics on $\overline{\mathbb{C}}$.

Theorem 2 There are only finitely many cycles of domains. There are Fatou domains which contain non-expanding / periodic points either in the interior or on the frontier to which all points of the domain tend. And there are Siegel-Arnold-Herman domains, disks or annuli where the k^{th} power of f is conjugate to an irrational rotation. There are no others.

3

<u>Theorem 3</u> There is a cyclic covering of the domain of equicontinuity* so that the quotient by the large orbits of f (the equivalence classes of $x \sim y \Leftrightarrow f^n x = f^m y$ some n, m) is a <u>Riemann surface</u>, S_f.

<u>Theorem 4</u> All the conformal structures on S_f compatible with the cyclic cover appear for rational maps quasi-conformally homeomorphic to f. Thus we have a finiteness theorem for S_f (the analogue of the Ahlfors finiteness theorem for finitely generated Kleinian groups.)

<u>Theorem 5</u> There is no measurable set $A \subset J_f$ in the Julia set which wanders ($A, fA, f^2A, \ldots$ are all disjoint) and which has positive Lebesgue 2-dimensional measure.

* minus the fixed point if any and all its inverse images under $f, \ldots$.

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A) Hyperbolic preliminaries

1) An analytic transformation of an open Riemann surface R covered by the disk is either an isometry or strictly distance decreasing for the unique conformally equivalent complete metric of curvature -1. By easy arguments one can then show

a) If f has a fixed point x then f is either a rotation of a disk or ~~[illegible]~~ for all y $f^n y \to x$ as $n \to \infty$. If f has no fixed point then for all y $f^n y \to \infty$.

b) If R is a domain on the sphere and f extends continuously to ∂R where it has only finitely many pts fixed (at most), then there is a unique fixed point p in $R \cup \partial R$ to which all orbits of f tend ($f^n y \to p$ for y in R $n \to +\infty$). (This uses a) and the fact the hyperbolic metric and the ~~[illegible]~~ spherical metric are related by factor tending to $+\infty$ at ∂R.)

2) (Siegel) A non-elementary subgroups of $PSL(2,R)$ is discrete iff it contains no irrational elliptic.

B) The proofs and the construction of the Riemann surface S_f

1) (Wandering domains) If $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{f} \cdots$ are all disjoint the relations $x \sim y$ and $x \simeq y$ are identical in Ω_1. We may discard finitely many to have no branched points.

a) If Ω_1 is a disk the f are all injective and $\Omega_1/\sim \; = \Omega_1/\simeq \; = \Omega_1$ is a Riemann surface with ideal boundary in Julia set.

b) If Ω_1 is an annulus we can show that f is eventually injective (see Appendix (wandering annulus)). so $\Omega_1/\sim$ is a Riemann surface with ideal boundary in the Julia set.

c) If Ω_1 has higher connectivity then $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{f} \cdots$ determines an increasing union of discrete subgroups of $PSL(2,\mathbb{R})$ whose union is discrete by A)2). Thus either the f are eventually injective and we have $\Omega_1/\sim$ is a Riemann surface with boundary in the Julia set or $\Omega_1/\sim$ is a Riemann surface with a non-finitely generated fundamental group.

<u>Proof of theorem 1</u> In all the cases a) b) c) we use the <u>measurable Riemann mapping theorem</u> (<u>Appendix - infinite parameters</u>) to construct an infinite dimension space of rational maps homeomorphic to f. (In a), b), and the first part of c) we use the fact that small conjugacies are unique on the Julia set <u>Appendix (conjugacy on Julia set)</u>. We also use the theory of prime ends to relate the frontier of a domain and the boundary of the standard disk.) (<u>Appendix - infinite parameters</u>)

2) (<u>Invariant domains</u>) Suppose $f: \Omega \to \Omega$. If there is a fixed point in Ω remove it and its full orbit from Ω (keep the name Ω). (This is a discrete set because the inverse orbit $\to$ Julia set). We say that (in a cyclic cover) <u>the full orbit of any critical points is now discrete</u> (Appendix (critical point)

Construct a function $m: \Omega \to \{1, 2, 3, \ldots\}$ satisfying $m(f(x)) = (\text{local degree of } f)(x) \cdot m(x)$ which is identically 1 outside the full orbits of critical points. Then (Ω, m) is a Riemann surface with branch points and $(\Omega, m) \xrightarrow{f} (\Omega, m)$ is a geometric covering.

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Then $(\Omega, m) \xrightarrow{f} (\Omega, m) \xrightarrow{f} (\Omega, m) \to \cdots$ determines an increasing sequence of discrete groups whose union is either elementary or discrete by A2). Leaving the elementary cases aside we have then a branched Riemann surface (in the cyclic cover) representing the $\approx$ equivalence classes on which f becomes an analytic isomorphism. Now f acts discontinuously here and we can form a ~~####~~ further quotient which is a Riemann surface and classifies the $\sim$ equivalence classes (in the cyclic cover).

This analysis proves Theorem 3. Theorem 4 uses this and the measurable Riemann mapping theorem. Theorem 5 uses the uniqueness of small conjugacies on the Julia set Appendix (conjugacies on the Julia set) and the measurable Riemann mapping theorem. Theorem 2 follows from A) 1).

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