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Flow box decomposition for gradients of univariate polynomials, billiards on the Riemann sphere, tree-like configurations of vanishing cycles for A_n curve singularities and geometric cluster monodromy

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Abstract. The base space of the universal unfolding of A_n-curve singularities is equipped with a stratification such that the geometric monodromy group is generated by wall-crossing mapping classes.

To Dennis, with admiration

1. Introduction

In his expository address "Inside and Outside Manifolds", Vancouver ICM 1974, Dennis Sullivan [33] suggests to study geometrical objects inside one manifold rather than classifying manifolds. We dream of studying isolated complex hypersurface singularities via such geometrical objects as proposed by Sullivan in the fibres of its universal unfolding as introduced by René Thom [34]. More precisely, let the 0-fibre of the polynomial mapping

$$f\colon \mathbb{C}^{n+1} \to \mathbb{C}$$

have an isolated singularity at the origin, and let

$$f_\lambda = f + \lambda_1 g_1 + \cdots + \lambda_\mu g_\mu$$

be its universal unfolding with parameters $\lambda \in \mathbb{C}^\mu$. Here, the polynomial mappings g_k, $1 \le k \le \mu$, induce a linear basis in the quotient vector space $\mathbb{C}[\![X_0, \ldots, X_n]\!]/J(f)$ of the local $\mathbb{C}$-algebra $\mathbb{C}[\![X_0, \ldots, X_n]\!]$ by the Jacobian ideal

$$J(f) = (\partial f(X)/\partial X_0, \ldots, \partial f(X)/\partial X_n).$$

The fibres $F_\lambda = \{f_\lambda(p) = 0 \mid p \in \mathbb{C}^{n+1}\}$ for $\lambda \in \mathbb{C}^\mu$ are hypersurfaces in the space $\mathbb{C}^{n+1}$ with its canonical symplectic structure ω and complex structure J. So, the fibres equipped with the induced structures ω_λ, J_λ become a family of Kähler manifolds.

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Besides the inside structures ω_λ, J_λ we propose as main inside geometrical object a real valued function $N^f_\lambda : F_\lambda \to \mathbb{R}$, which is defined as follows:

$$N^f_\lambda(p) = 0 \quad \text{if } (Df_\lambda)_p = 0,$$

i.e., p is a singular point of F_λ, and $N^f_\lambda(p) = |df_\lambda(u)|$, where $u \in T_p\mathbb{C}^{n+1}$ is chosen such that $\omega(u, Ju) = 1$ and $\omega(u, v) = 0$ for all $v \in T_p F_\lambda$. So, $N^f_\lambda(p)$ is the norm

$$\|(Df_\lambda)_p\| = \max |df_\lambda(u)|, \quad u \in T_p\mathbb{C}^{n+1},\ \omega(u, Ju) = 1$$

of the differential $(Df_\lambda)_p$ at $p \in F_\lambda$ of the defining function f_λ of the hypersurface F_λ. More simply, in coordinates we have

$$N^f_\lambda(p)^2 = \left|\frac{\partial f_\lambda}{\partial x_0}(p)\right|^2 + \cdots + \left|\frac{\partial f_\lambda}{\partial x_n}(p)\right|^2.$$

Let $N^f : \mathbb{C}^{n+1} \to \mathbb{R}$ be defined by $N^f(q) = \|df_q\|$. So, $N^f(p) = N^f_\lambda(p)$, $\lambda = 0$, holds for p on the central fibre $\{f(p) = 0\}$. The non-negative function N^f has an isolated zero at $0 \in \mathbb{C}^{n+1}$ by assumption. Following the seminal methods of Milnor [23], we use the curve selection lemma [13]. The function

$$(N^f)^2 = \left|\frac{\partial f}{\partial x_0}\right|^2 + \cdots + \left|\frac{\partial f}{\partial x_n}\right|^2$$

being real algebraic, the curve selection lemma implies that the differential $d(N^f)^2$ has at $0 \in \mathbb{C}^{n+1}$ an isolated zero too. It follows for $\varepsilon > 0$, and small enough, that the connected component $B^f_{2\varepsilon}$ of 0 of the set

$$\{q \in \mathbb{C}^{n+1} \mid N^f(q) \leq 2\varepsilon\}$$

is a differentiable ball and neighbourhood of 0 in $\mathbb{C}^{n+1}$ (see [18]). Further, it follows using the curve selection lemma again and after making ε eventually smaller, that the fibre F_0 intersects transversely in $B^f_{2\varepsilon}$ the t-levels, $0 < t \leq 2\varepsilon$, of the function N^f . By openness of the tranversality conditions, there exists $\delta > 0$ such that firstly the fibres F_λ intersect transversely in $B^f_{2\varepsilon}$ for $\|\lambda\| \leq \delta$ the t-levels, $\varepsilon/2 \leq t \leq 2\varepsilon$, of N^f_λ and secondly that $N^f_\lambda \leq N^f + \varepsilon/2$ holds on $B^f_{2\varepsilon}$.

For the sequel we fix ε and δ with the above properties. Moreover, for $\lambda \leq \delta$ we denote by F^f_λ the subset

$$\{p \in F_\lambda \cap B^f_{2\varepsilon} \mid N^f_\lambda(p) \leq \varepsilon\}.$$

Now we have constructed a family

$$\left(F^f_\lambda, \omega_\lambda, J_\lambda, N^f_\lambda\right)_{\lambda \in U}, \quad U = \{\lambda \mid \|\lambda\| < \delta\},$$

of Kähler manifolds with an extra inside geometric object N^f_λ.

The gradient of $N_0^f : F_0^f \to \mathbb{R}$ provides a smooth retraction of F_0^f to $\{0\}$, the levels of the function N_λ^f define a foliation with singularities on F_λ^f. Our project, still a dream, is to stratify the base space U of the unfolding by cells, such that the monodromy is given by wall crossing data. In the present situation, one could ask for instance that above each stratum the datum $(F_\lambda^f, N_\lambda^f)$ be globally trivial as a family of differential manifolds equipped with a singular foliation.

A seminal example of an inside geometric object is the spine $\Sigma^f = \Sigma_{a_0 a_1 \cdots a_n}$ introduced by Frédéric Pham in the fibre

$$X^f = \{z \in \mathbb{C}^{n+1} \mid f(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 1\}$$

(see [25]). The spine $\Sigma_{a_0 a_1 \cdots a_n}$ is a realisation of the simplicial complex consisting of the join-product $\mu_{a_0} * \mu_{a_1} * \cdots * \mu_{a_n}$ of the finite sets μ_{a_i} of a_i-roots of unity in the ith factor. The homological monodromy is induced by the cyclic permutations of roots of unity; see [23].

The embedding in X^f is such that all n-simplices are Lagrangian submanifolds, so Σ^f is a Lagrangian spine for the Stein manifold X^f. More concretely, the n-simplices are labelled by systems $(\rho_0, \ldots, \rho_n)$ of roots of unity, $\rho_i^{a_i} = 1$, and points on it are non-negative real linear combinations

$$b_0\rho_0 + \cdots + b_n\rho_n \quad \text{with } b_0^{a_0} + \cdots + b_n^{a_n} = 1.$$

The spine Σ^f is related to the real function N^f restricted to X^f: in case $a_i \geq 3$, the function N^f has exactly one critical point on each simplex $\sigma \subset \Sigma^f$ of index $n - \dim(\sigma)$. This is water on the mill of our dream. Moreover, more water comes from the construction of the geometric monodromy as a tête-à-tête twist with centre the spine Σ^f. The tête-à-tête property is realized by walking along broken geodesics on the spine over the distance π with respect to the right angled corresponding spherical n-simplex

$$\{(b_0^{a_0/2}, \ldots, b_n^{a_n/2})\}$$

in the unit sphere $S^n \subset \mathbb{R}^{n+1}$. If a walk hits the boundary of a n-simplex in the spine, one continues on the next n-simplex with respect to the cyclic order of roots of unity: if one hits a k-simplex, α, $0 \leq k < n$, in its simplicial interior, the next n-simplex is obtained by changing to the next root of unity all vertices that are not in the k-simplex α. The tête-à-tête property is like the spherical cut locus property, saying that all such walks that start at a point $p \in \Sigma^f$ meet again at their endpoints in a point $p' \in \Sigma^f$; see [4, p. 2652].

The purpose of the present paper is to bring this dream to more reality for the A_n singularities $f = y^2 + x^{n+1}$. Unfortunately at this point of our study, a modification of the functions N_λ^f was necessary. The unfolding

$$f_\lambda = y^2 + x^{n+1} + \lambda_n x^{n-1} + \cdots + \lambda_2 x + \lambda_1 = y^2 + P(x)$$

preserves the hyperelliptic symmetry $(+y, x) \to (-y, x)$. The modified function will be the modulus of the univariate monic, Tchirnhausen reduced, polynomial

$$N_\lambda^f(p) = \frac{1}{4}\left|\frac{\partial}{\partial y} f_\lambda(p)\right|^2 = |f_\lambda(p) - y(p)^2| = |P(x(p))|$$

for $p \in \mathbb{C}^2$ on the fibre $f_\lambda(p) = 0$. Our previous work on A_n singularities [2] gets upgraded from integral homological monodromy to geometric monodromy. Properties concerning monic univariate polynomials and billiards on the Riemann sphere are studied on the road.

The geometric picture of a univariate monic polynomial P, as introduced in [3, 8], is enriched with flow lines of the vector field

$$X_P = -\operatorname{grad}\big(\log(|P|)\big);$$

see Section 2.

The probability of hitting of a root r of P when starting at or near infinity is computed. A simultaneous flow box decomposition for the vector fields X_P and

$$Y_P = i \operatorname{grad}\big(\log(|P|)\big)$$

is constructed; see Sections 3 and 4.

A combinatorial description of the Morse function $|P|$ on the complement of the roots for generic polynomials P is studied. A question related to Lagrangian Skeleta and Arboreal Singularities (see [14, 15, 24]), by email and his Zoom talk in Moscow, of Roger Casals was our motivation to use this description and to discover rooted planar trees attached to generic monic polynomials. These trees can be realized up to isotopy by slalom polynomials and give tree-like configurations of vanishing cycles for the curve A_n singularities defined by $y^2 + x^{n+1}$. Perhaps a more geometric understanding of quiver and mutation theory for the A_n singularity comes in reach. Combinatorial and conformal problems appear; see Sections 5–9.

Our main ingredient is the pair of orthogonal foliations on $\mathbb{C}^*$ given by concentric circles and real rays. The inverse image of this pair of foliations by a polynomial is still orthogonal, but with equiangular singularities. This observation is not at all new. It was used in [5–7, 16, 17, 22, 29–31] leading to results partially overlapping or close to ours.

2. Flow box decomposition

Let $P(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0$ be a complex monic polynomial of degree $d > 0$. In a previous paper [3], we have introduced the picture $\operatorname{Pic}(P)$ of P which is a special graph in the Gaussian plane $\mathbb{C}$. More precisely, the picture is the union of the zero set of the real part, drawn blue, and the zero set of the imaginary part of P, drawn green. See Figure 1 for the Lehmer polynomial.

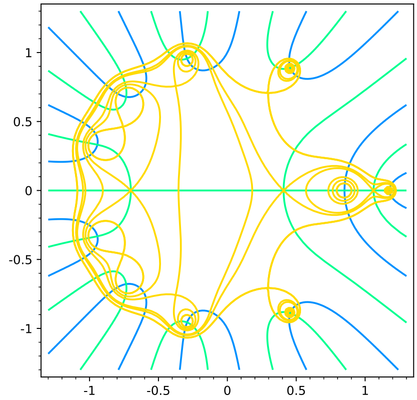

Figure 1. Flow box decomposition for the Lehmer polynomial $P = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$. The blue and green part is the picture $\mathrm{Pic}(P)$ of P.

The main result of [3] is the construction of a cell decomposition of the space of complex monic polynomials of degree d. More precisely, the equivalence relation $P \sim Q$ on monic polynomials of degree d defined by requiring that the pictures $\mathrm{Pic}(P)$ and $\mathrm{Pic}(Q)$ are isotopic relative to the asymptotic $4d$ rays induces a stratification Σ_d on the manifold Pol_d of monic polynomials of degree d by smooth real semi-algebraic contractible strata. The number of top-dimensional strata is $\frac{1}{3d+1}\binom{4d}{d}$, which is a Fuss-Catalan number. Moreover, the restriction of this stratification to the complement of the discriminant locus Δ_d is invariant by the action of the braid group on d strands. We propose, following [33], the stratification Σ_d as an example of an inside geometrical object in the manifold pair $(\mathrm{Pol}_d, \Delta_d)$ with braid group action.

Let (S, J) be a surface and let X be a vector field with isolated singularities on S. A weak flow box for X is a subset R that is the image of an embedding

$$\phi\colon [0, 1] \times [0, 1] \to S$$

such that the images $\phi(]0, 1[\times \{s\})$, $s \in [0, 1]$, are oriented re-parametrizations of trajectories of X. So, the image $\phi(]0, 1[\times [0, 1])$ contains no singularities of X. If, moreover,

$$\phi \circ \theta, \quad \theta(t, s) = (s, t)$$

is a parametrization for a weak flow box for the vector field $Y = J(X) = iX$, then we call R a conformal flow box for X or the pair (X, Y).

The weak flow box decomposition for the field $X_P = -\mathrm{grad}(\log(|P|))$ of this paper is given by the complementary regions of the graph $\mathrm{FlowBox}(P)$, which is the union of the preceding blue and green graphs together with a yellow one. The yellow one consists of the critical levels of the function $|P|$. More precisely, the zeros of P which we think as infinitesimal circles of radius 0, for each zero s of P' the level $|P(z)| = |P(s)|$, and an infinitesimal circle at ∞ in the compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$. One could avoid the use of circles of radius 0 by doing a real oriented blow-up of the zeros of P and $\infty \in \mathbb{C} \cup \{\infty\}$. The space $\mathbb{C} \cup \{\infty\}$ transforms to a surface with yellow boundary.

The mapping $P : \mathbb{C} \to \mathbb{C}$ being holomorphic, constraints for the planar graphs follow. The three graphs of the different colours intersect in such a way that the sectors that appear at an intersection point are equiangular. In particular, this means that the intersection is orthogonal precisely if only two smooth branches meet. Moreover, we remark that the smooth edges of the union of the green and the blue graph are trajectories of the field X_P.

See Figure 1 for the Lehmer polynomial. The infinitesimal circles are not drawn! The flow boxes are the connected components of the complement of $\mathrm{FlowBox}(P)$ in $\mathbb{C} \cup \{\infty\}$.

Each connected component of the complement of $\mathrm{FlowBox}(P)$ is the interior of a polygon. We define a side of such a polygon a monochromatic connected component of its boundary and call the polygon a combinatorial quadrilateral with two opposite yellow sides. Moreover, each quadrilateral has one green and one blue side. The yellow sides are eventually infinitesimal (except if one does above real blow-ups) and correspond to different levels of $|P|$. The vector field X_P pushes the yellow sides from high level to low level and is tangent to the sides of blue and green colour. Combinatorially one observes, also using [3], the following theorem.

Theorem 2.1. *The complement of* $\mathrm{FlowBox}(P)$ *in* $\mathbb{C} \cup \{\infty\}$ *is a union of combinatorial quadrilateral with two opposite yellow sides, one blue side and one green side, that are gradient trajectories.* ∎

Gradient lines are in general not real semi-algebraic. The blue and green graphs are real semi-algebraic since they are levels of the real or imaginary parts of P. The yellow one is also real semi-algebraic, so we have the following.

Theorem 2.2. *The graph* $\mathrm{FlowBox}(P)$ *is real semi-algebraic. Its complexity (say the number of vertices plus the number of edges) is bounded up to a factor* (≤ 16) *by the square of the degree* d *of* P. *The number of flow boxes of* $\mathrm{FlowBox}(P)$ *is bounded by* $4d^2$. *This bound is reached for generic* P, *i.e., in the present context polynomials* P *such that* $|P|$ *takes* $d-1$ *distinct positive values on the zeros of* P'. ∎

One can reduce the number of boxes drastically, by joining boxes that share a smooth yellow edge, that is perpendicular to a green and a blue edge at its ends.

For a general polynomial P of degree d one expects $d-1$ (visible) yellow levels in $\mathrm{FlowBox}(P)$. Only one yellow level appears for polynomials $P = z^n - z$, $n \geq 2$. As an example, for $P = z^4 - z$ with 4 roots one yellow level and 18 boxes appear. After joining flow boxes according to the above rule 12 flow boxes remain. The polynomial $P = z^4 - z - 3i$ of degree 4 is generic enough: the real function $|P|$ takes 3 distinct positive values on the zeros of P'. The bound $4d^2 = 64$ is reached; see Figure 2.

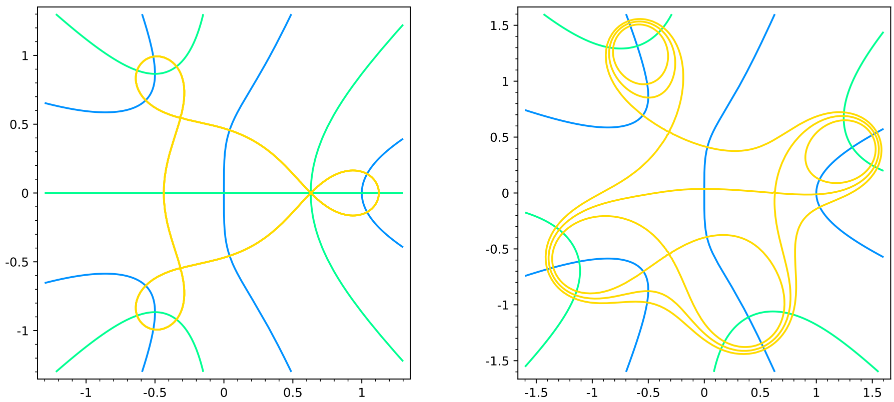

Figure 2. Flow box decomposition for $P = z^4 - z$ and $P = z^4 - z - 3i$.

3. Flow box decomposition with separatrices

In order to have a conformal combinatorial flow box decomposition, one needs to add the flow lines of X_P through the singular points of the yellow part of $\mathrm{FlowBox}(P)$. Such a singular point is by construction a zero s of the derivative P' of P with $P(s) \neq 0$. Define $R_{P(s)} \subset \mathbb{C}$ to be the ray

$$R_{P(s)} = \big\{x + yi \mid x\mathrm{Im}(P(s)) - y\mathrm{Re}(P(s)) = 0\big\}.$$

The flow lines through s are a subset in the inverse image $P^{-1}(R_{P(s)})$. Unfortunately, our pictures show the whole inverse image $P^{-1}(R_{P(s)})$. Those flow lines that run to a root of P, and not having a root of P' in their closure should have been deleted (see Figure 7).

Define the graph $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ by adding to $\mathrm{FlowBox}(P)$, in red, the flow lines belonging to $P^{-1}(R_{P(s)})$ that have a root s of P' with $P(s) \neq 0$ in their closure. The complement of $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ in $\mathbb{C} \cup \{\infty\}$ is a union of open flow boxes for the field X_P. Again X_P pushes yellow sides to yellow sides and the green, blue or red sides are flow lines. All points of the same open flow box flow by X_P to the same root of P. More generally, the graph $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ defines a stratification of $\mathbb{C}$ by contractible open strata, such that all points of the same stratum flow by X_P to the same root of P or of P'.

Moreover, the graph $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ is real semi-algebraic. The graph

$$\mathrm{FlowBox}^{\mathrm{sep}}(P) \cap \mathbb{C}$$

has no terminal vertices, i.e., vertices of valency one, and its non-bounded edges are near ∞ asymptotic to real radial rays in $\mathbb{C}$ through 0.

Maximal subintervals of the smooth part of a level of the function $|P|$ are flow lines of the field $Y_P = -iX_P$; see Figure 3. We obtain the following theorem.

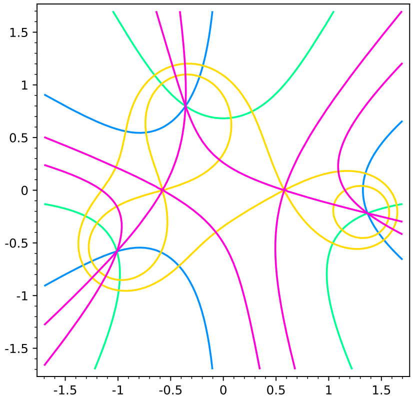

Figure 3. $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ for $P = z^3 - z - 1 + i$ with two red separatrices $C(s)$ trough the zeros of P' located at the double points of the yellow levels. In this picture twelve red flow lines having zeros of P in their closure should be deleted.

Theorem 3.1. *The real semi-algebraic graph* $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ *gives a conformal flow box decomposition for the field* $X_P = -\operatorname{grad}(\log(|P|))$ *that determines the combinatorics of its flow lines. The decomposition* $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ *is also a conformal flow box decomposition for the field* Y_P. ■

Let $\mu = \frac{|d\theta|}{2\pi}$ be the rotationally invariant probability measure on circles centred at $0 \in \mathbb{C}$. The pull back measure $\nu = P^*\mu$ defines a probability measure on the level curves of $|P|$ that is invariant by the flow of X_P. In particular, to each yellow side of a flow box of $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ is assigned a probability such that opposite sides of a flow box share equal probabilities. Also to each zero r of P is assigned a probability $\nu_P(r)$.

Let R_{ρ_1,ρ_2} be the ring domain

$$\left\{z \in \mathbb{C} \mid \rho_1 \leq |z| \leq \rho_2\right\}$$

with its Euclidean measure $|dx \wedge dy|$ and let $\nu_{P,\rho_1,\rho_2}(r)$ be the probability that a point from R_{ρ_1,ρ_2} flows by X_P to the root r of P. Flow lines of X_P are asymptotic to radial ones near ∞, which implies

$$\lim_{\rho_1,\rho_2 \to \infty} \nu_{P,\rho_1,\rho_2}(r) = \nu_P(r).$$

Let P be a monic polynomial of degree $d \geq 1$ and let $P_a = x^d - a(P - x^d)$, $a \in [0,1]$, be the family of monic polynomials that connects the polynomial P to the polynomial x^d. Note $P_1 = x^d$ and $P_0 = P$. First choose ρ_0 big enough such that every circle S_ρ, $\rho \geq \rho_0$ intersects all flow lines of all polynomials P_a transversely. Let θ be a flow line of X_{P_1} that starts at a point of S_{ρ_o}.

We want to find and study the flow line for X_P that is at infinity asymptotic to the ray θ. Consider the image $R = P_1(\theta)$ which is a ray. The inverse image $P_a^{-1}(R)$ is a disjoint family of d flow lines for X_{P_a}. So there exists a unique continuous family of flow lines (θ_a), where θ_a is a flow line for X_{P_a} and $\theta_1 = \theta$ holds. The flow line θ_0 of X_P is asymptotic to θ.

Let a_ρ, b_ρ be the intersection points of θ and θ_0 with S_ρ. One checks that $|a_\rho - b_\rho|$, $\rho \geq \rho_0$, is bounded. So, the probability measure of the circular segment $[a_\rho, b_\rho]$ decays like $O(\frac{1}{\rho})$. For a balanced polynomial P, i.e., having roots that sum up to 0, this decay is stronger, in fact $o(\frac{1}{\rho})$. Hence, we have the following theorem.

Theorem 3.2. *The probability $\nu_{P,\rho,\rho+1}(r)$ that a flow line starting at a random point $z \in R_{\rho,\rho+1}$ ends at the root r of P satisfies $|\nu_{P,\rho,\rho+1}(r) - \nu_P(r)| \leq O(\frac{1}{\rho})$ for $\rho \to \infty$.* ■

Note thatthe value of the probability $\nu_P(r)$ depends a priori on the combinatorics of $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ and the slopes of the rays $R_{P(s)}$. But, surprisingly, we have the following.

Theorem 3.3. *Let P be a polynomial of degree d with d distinct roots. The probability $\nu_P(r)$ is independent from the chosen root r of P, so $\nu_P(r) = \frac{1}{d}$. For polynomials P of degree d with a root r of multiplicity $m_P(r) > 1$ the probability $\nu_P(r)$ is $\nu_P(r) = \frac{m_P(r)}{d}$.*

Proof. Call in this proof a polynomial P of degree d P'-generic if it has d roots and if at each zero s of P' exactly two flow lines $F_1(s)$, $F_2(s)$ of X_P exist, that start at infinity and have s in their closure and do not contain a root of P. So P' has $d-1$ zeros and the rays $R_{P(s)}$ are pairwise distinct.

Firstly, we assume that P belongs to the dense set of P'-generic polynomials with d distinct roots. Let $C(s)$ for a zero s of P' be the union

$$F_1(s) \cup F_2(s) \cup \{s\}.$$

The curve $C(s)$ is near infinity asymptotic to radial rays. The image $P(C(s))$ is the half ray $[P(s), \infty] \subset R_{P(s)}$ that does not contain the origin. There exists a parametrization

$$t \in \mathbb{R} \mapsto \gamma_s(t) \in C(s)$$

such that $P(\gamma_s(t)) = P(\gamma_s(-t))$ holds. It follows that all $d-1$ path integrals

$$\int_{C(s)} \frac{dP}{P}$$

vanish.

The $d-1$ curves $C(s)$ are disjoint and show near infinity $2(d-1)$ different asymptotic directions. The curves $C(s)$ divide the Gaussian plane in d regions. Let R be such a region. The Cauchy integral

$$I_R = \frac{1}{2\pi i} \int_{\partial R} \frac{dP}{P}$$

does not vanish, since the arcs at infinity contribute d times their angular measure $\nu = \frac{1}{2\pi i}\frac{dz}{z}$, and since the contribution along the curves $C(s)$ vanishes. The integral counts the number of roots of P in R, so $I_R \geq 1$. From $\sum_R I_R = d$ follows $I_R = 1$ for each of the d regions. Hence, the total angular measure at infinity of a region is

$$I_R/d = \frac{1}{d}.$$

The flow lines of X_P do not cross the curves $C(s)$. Only $2d - 2$ flow lines end at a zero of P'. It follows that for a root r of P lying in a region R the equality

$$\nu_P(r) = I_R/d = \frac{1}{d}$$

holds.

Secondly, the numbers $\nu_P(r)$ for P'-generic polynomials depend continuously upon the coefficients of P. Hence, by approximation of a general polynomial P with roots of multiplicity one by P'-generic polynomials one obtains the equality $\nu_P(r) = \frac{1}{d}$.

The last statement $\nu_P(r) = \frac{m_P(r)}{d}$ is obtained by additivity. A small deformation will separate a root of multiplicity m in m roots that lie close to each other. So, the polynomial P is deformed to a polynomial Q with distinct roots. A further small deformation separates, if necessary, the roots of Q'. Each root of multiplicity $m, m > 1$, will correspond to a subtree with $m - 1$ edges in the tree E_Q. The corresponding separatrices separate the regions of the m roots. Now by additivity of the Cauchy integral the result follows. ■

4. Billiards on the Riemann sphere

Let S be a finite subset of $\mathbb{P}^1(\mathbb{C})$ with $d + 1 \geq 2$ elements and let $t \in \mathbb{P}^1(\mathbb{C})$, $t \notin S$, be any point. For $p \in S$ let S_p be the degree 0 divisor

$$S_p = -d\ p + \sum_{q \in S, q \neq p} q$$

and let $f_{S,p}$ the meromorphic function with $(f_{S,p}) = S_p$ and $f_{S,p}(t) = 1$. The restriction of $f_{S,p}$ to $\mathbb{P}^1(\mathbb{C}) \setminus \{p\}$ is a polynomial with d distinct roots at $S \setminus \{p\}$. The foliation on $\mathbb{P}^1(\mathbb{C}) \setminus S$ given by the levels of $|f_{S,p}|$ does not depend upon the choice of the point t.

The following defines a billiard B_S by its trajectories. A trajectory of B_S is a C^1 smooth curve $\gamma\colon]-\infty, +\infty[\to \mathbb{P}^1(\mathbb{C})$ with $\gamma(0) \in S$, $\dot{\gamma}(t) \neq 0$, that hits infinitely often the set S at times

$$\cdots < t_{-2} < t_{-1} < 0 = t_0 < t_1 < t_2 < \cdots$$

such that the restriction of γ to $]t_k, t_{k+1}[$, $k = \ldots -2, -1, 0, 1, 2, \ldots$, is orthogonal to the foliation given by the levels of the function $|f_{S,\gamma(t_k)}|$. The cushions of this billiard are the disjoint union of $d + 1$ circles, each one being the circle of non-zero tangent vectors to X

at $s \in S$ up to positive stretching. The rules of reflection are unusual: if one hits a cushion at θ, one continues at $\theta + \pi$.

Equivalently, one can construct the surface with boundary that underlies the billiard B_S doing a real-oriented blow-up of $\mathbb{P}^1(\mathbb{C})$ centred at all points of S. In this description the cushions are the "exceptional divisors" with points that represent non-zero tangent vectors at points $p \in S$ up to positive stretching. We denote by E_p the exceptional divisor above the point p. Again, instead of bouncing back, a trajectory continues through the antipodal point. In fact, one has a whole circle of billiards $B_S(\alpha)$: one continues at $\theta + \alpha$. The billiard $B_S(0)$ works with reflections and $B_S = B_S(\pi)$.

Up to re-parametrization, a trajectory γ is determined by its initial velocity

$$\dot{\gamma}(0) = -\operatorname{grad}(|f_{S,p}|), \quad p \in S,$$

at $q \notin S$. Not every non-zero tangent vector v to $\mathbb{P}^1(\mathbb{C})$ at $q \notin S$ determines a trajectory as above: the vector v determines a continuous, piecewise smooth, curve $\gamma\colon [0, T] \to X$ that hits the set S only $k, k \geq 1$, times at

$$t_0 < t_1 < t_2 < \cdots < t_{k-1}$$

and is as before, putting $t_k = T$, on intervals $]t_h, t_{h+1}[$ orthogonal to the foliation of the levels of $|f_{S,\gamma(t_h)}|$, $0 \leq h < k$, and moreover terminates at T in a zero of the derivative of the function $f_{S,\gamma(t_{k-1})}$. Also, a trajectory may not hit the set S, if it runs to a critical point of $|f_{S,p}|$. Again such a trajectory is determined by its initial velocity v and we call it a stopped trajectory. Similarly, there are stopped trajectories in backward time. The set of initial velocities of stopped forward or backward trajectories is countable, hence of measure 0.

Let $W(S)$ be the set of words on the alphabet S. A trajectory γ that visits S at times

$$\cdots < t_{-2} < t_{-1} < 0 = t_0 < t_1 < t_2 < \cdots$$

produces the infinite word $w_\gamma = (\gamma(t_i))_{i \in \mathbb{Z}}$. A stopped trajectory produces a word that is finite in forward or backward time.

The grammar of those words has a first obvious rule that is very non-Dutch: repetitions of letters $\cdot\cdot\, pp \,\cdot\cdot$ are not allowed. From Theorem 3.3 it follows that the probability of finding $\cdot\cdot\, pq \,\cdot\cdot$, $p \neq q$, in average equals $1/(-1 + \#S)$.

Let $W^\infty(S)$ be the set of words of infinite trajectories. The natural probability measure on it is generated by assigning to the cylinder

$$C(i, p) = \left\{ w : \mathbb{Z} \to\in W^\infty(S) \mid w(i) = p \right\}$$

the measure $1/\#S$. The shift operator $T\colon W^\infty(S) \to W^\infty(S)$ is an interesting measure preserving dynamical system.

To a monic polynomial P of degree $d > 0$ with distinct roots one associates the divisor (P), the set $S_P = \operatorname{supp}((P)) \subset \mathbb{P}^1(\mathbb{C})$ with $d + 1 \geq 2$ elements and the corresponding billiard B_{S_P} and dynamical system $W^\infty(S_P)$. How much information about the polynomial P can be retrieved from $W^\infty(S_P)$?

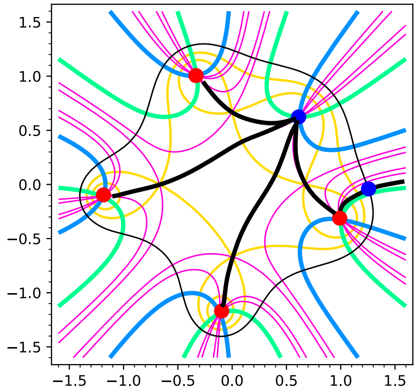

Figure 4. Rooted tree Γ_P, in black, for $P = z^5 - z + i + 1$. The subtree $E_P \subset F_P$ having as vertices the roots of P is the extended Dynkin diagram $\widetilde{D}_4$.

Let $\mu(w)$ for a finite word $(w(i))_{0 \leq i \leq k}$ be the measure of the set of infinite words W with $W(i) = w(i)$, $i = 0, \dots, k$. How many finite words w are necessary for a given monic polynomial P, such that the measures $\mu(w)$ determine P up to substitutions of z by $z - t$?

The group $\mathrm{PGL}(2, \mathbb{C})$ acts 3-transitively on $\mathbb{P}^1(\mathbb{C})$, so the systems $W^\infty(S_P)$ for polynomials of degree 1, or of degree 2, are isomorphic. A first question is how to read off the cross-ratio of the support of the divisor (P) for a polynomial of degree 3?

5. The rooted planar tree Γ_P

First we repeat and strengthen a little the definition of *generic polynomials* that was used in the proof of Theorem 3.3.

A monic polynomial P of degree $n + 1$ is called *very generic* if the following three open conditions are fulfilled:

(1) P has $n + 1$ distinct roots, P' has n distinct roots,

(2) at each root s of P' exactly two flow lines $F_1(s)$, $F_2(s)$ of X_P exist, that start at infinity and have s in their closure and so do not contain a root of P,

(3) the positive part of the real axis is asymptotic at infinity to one of the flow lines that flows to a root of P.

Let P be a very generic polynomial of degree $n + 1$. The (mountain) pass path at a root s of P' is the closure $E_P(s)$ of the union $E_1(s) \cup E_2(s)$, where $E_1(s)$, $E_2(s)$ are flow lines that have s in their closure and run down to roots of P.

The union $E_P = \bigcup_{s, P'(s)=0} E_P(s)$ is a planar graph having the roots of P as vertices and n edges $E_P(s)$. This graph is a forest. Indeed, a cycle Z would bound a compact planar region on which the function $|P|$ is unbounded from above since at each root s of P' one of the unbounded flow lines $F_1(s)$, $F_2(s)$ lies in that region. In fact, this graph is a tree since it is a forest with $n+1$ vertices and n edges.

The tree E_P is called by Alexis Marin in [22] the Shub–Smale tree of the polynomial P. Marin answers a question of Smale by proving that all isotopy classes of finite planar trees occur. His proof is based on monodromy: the isotopy class of the tree E_P and the monodromy of the branched covering $P:\mathbb{C} \to \mathbb{C}$ determine each other.

Let $E_P(+\infty)$ be the flow line that at infinity is asymptotic to the positive part of the real axis. It has a root of P in its closure, since by assumption P is very generic. Define

$$\Gamma_P = E_P \cup E_P(+\infty).$$

It is a planar tree, rooted at $+\infty$ with $n+1$ vertices and a root $+\infty$ of valency 1; see Figure 4. The number of such planar rooted trees up to isotopy fixing $+\infty$ is the Catalan number

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

For a general monic polynomial P the definition of Γ_P is as follows. First consider all flow lines $\gamma: I \to \mathbb{C}$ of $-\operatorname{grad}(\log(|P|))$ that satisfy:

- P' and P do not vanish on the image of γ,
- γ is maximal, i.e., γ is not the restriction of a flow line defined on an interval $J \neq I$, $I \subset J$,
- the image of γ is bounded in $\mathbb{C}$, except if the image of γ is at $+\infty$ asymptotic to the positive real half-line.

Define Γ_P as the closure of the union of all flow lines satisfying the three properties above. Define E_P as the closure of the union of all bounded flow lines satisfying the three properties above.

Again, the subset Γ_P is a rooted planar tree. At a zero s of P' with $P(s) \neq 0$ of multiplicity m the germ of the graph Γ_P is homeomorphic to the germ at its centre of a star-shaped tree with $m+1$ legs, except if the flow line from $+\infty$ hits s too. Instead of being connected by edges, in the more general trees Γ_P the roots of P are connected by subtrees with $m+1$ terminal edges.

6. Unfolding of the A_n curve singularity

This section is inspired by the work of Roger Casals, as presented in his Zoom talk, broadcasted from California into the Moscow seminar of Sabir Gusein-Zade and received in Switzerland.

The space Pol_{n+1} of monic polynomials of degree $n+1$ can be considered as the universal parameter space for the unfolding of the plane curve singularity A_n.

Let ρ be a positive real number. For P very generic, i.e., P satisfying the above three open conditions, consider the Riemann surface

$$F_P = \{(y,x) \in \mathbb{C}^2 \mid y^2 + P(x) = 0,\ x \in D_P\},$$

where D_P is the subset

$$D_P = \{x \in \mathbb{C} \mid |P(x)| \leq \rho + \max_{s,P'(s)=0} |P(s)|\}$$

of the Gaussian plane. The projection $\pi\colon (y,x) \in F_P \mapsto x \in D_P$ is a branched covering of the disc D_P of degree 2 with branching points the roots of P. The actual positive value of ρ is of no importance. The value $\rho = 5$ was used in Figure 4.

The inverse image

$$\Delta_P = \pi^{-1}(E_P)$$

is a union of vanishing cycles. Indeed, $\pi^{-1}(E_P(s))$ is an embedded copy of the circle $v(s)$ in F_P, to which correspond two oriented integral cycles $\pm v(s)$ that add up to 0 in the homology of F_P. The cycles $\pm v(s), \pm v(s')$ intersect in one point with intersection number ± 1 if and only if the corresponding edges $E_P(s), E_P(s')$ are incident in a vertex of E_P. So the cycles are primitive elements in the integral homology of $H_1(F_P, \mathbb{Z})$. Moreover, the cycles $v(s)$ are quadratic vanishing cycles. Indeed, consider the family $P_{s,t}$ of monic polynomials having the same roots as P except for the two roots of P at the ends of the edge $E_P(s)$ and two roots of $P_{s,t}$ that travel with a parameter $t \in [0,1]$ from the ends of $E_P(s)$ and meet at the midpoint of $E_P(s)$. The corresponding path in the unfolding contracts the cycle $v(s)$ and ends at a smooth point of the discriminant.

The inverse image $\Delta_P \subset F_P$ is a generic configuration of embedded circles in the surface F_P on which the surface F_P retracts. In other words, Δ_P is a spine for the surface F_P.

The construction above can be adapted for generic polynomials P, i.e., polynomials such that P and P' have no common roots. Indeed, define a *cluster* in E_P to be a connected component of $E_P \setminus \{s \in E_P \mid P(s) = 0\}$. In the case of a very generic polynomial P the clusters of E_P are precisely the edges of E_P. The tree E_P for the Lehmer polynomial, which is not very generic, contains one cluster of multiplicity 2 with two distinct roots of P'; see Figure 5.

Let $C \subset E_P$ be a cluster. The frontier $\overline{C} \setminus C$ of C in E_P is a subset of roots of P. Observe that $m(C) = -1 + \#\overline{C} \setminus C$ equals the number of roots counted with multiplicity of P' in C. So, we call $m(C)$ the *multiplicity* of the cluster. The inverse image

$$v(C) = \pi^{-1}(\overline{C}) \subset F_P$$

is a graph which can be obtained as the union of $m(C)$ embedded circles in F_P. More precisely, the $m(C)+1$ terminal vertices $s_0, s_1, \dots, s_{m(C)+1}$ of $\overline{C}$ are cyclically ordered by the orientation of $\mathbb{C}$. The oriented shortest path in $\overline{C}$ from s_i to s_{i+1} lifts to an oriented cycle $v_i(C)$ on F_P, $i = 0, \dots, m(C)+1$. Those $m(C)+1$ cycles on F_P that pairwise

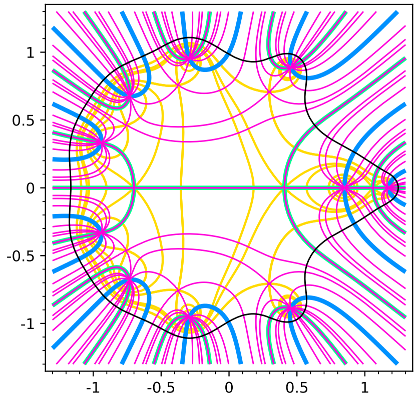

Figure 5. Region for the Lehmer polynomial P with black boundary. The dominating real zero, in fact Lehmer's number $\lambda = 1.17628\ldots$, of P is the limit point of the flow line that comes from $+\infty$. The tree E_P has 7 clusters of multiplicity 1 and one cluster of multiplicity 2. The cluster of multiplicity 2 has its frontier located at the real root $\frac{1}{\lambda}$ of P and the two roots of P with minimal real part.

intersect with intersection number ± 1 or 0 add up to 0 as a chain, so also in homology. More precisely, $v_i(\bar{C}), v_j(\bar{C})$, $j \neq i+1, i-1$, are disjoint or intersect along the inverse image by π of an internal edge of $\bar{C}$. In both cases, the homological intersection number vanishes. In fact, in the latter case both cycles become disjoint by an isotopy. If $m(C) > 1$, then $v_i(\bar{C}), v_{i+1}(\bar{C})$ intersect along the inverse image by π of the terminal edge running to s_{i+1} with homological intersection number $+1$. So the Dynkin diagram of the cycles belonging to a cluster C is the extended diagram $\widetilde{A}_{m(C)}$. The surface F_P retracts to the union of all $v(C)$, C being a cluster in E_P. The mapping cone $D(C)$ of the restriction

$$\pi : v(C) \to C$$

is a union of $m(C)$ disc that are attached to F_P. The union of the surface F_P and all $D(C)$, C cluster in E_P is a skeleton Sk_P, with Lagrangian pieces $D(C)$ and symplectic piece F_P, for the ambient space $\mathbb{C}^2$. This construction leads to a stratification of $\mathrm{Pol}_{n+1} \setminus \Delta_{n+1}$. First define an equivalence relation on the set Pol^*_{n+1} of generic polynomials: $P \sim Q$ if the planar rooted trees Γ_P, Γ_Q are equal up to an isotopy that fixes the $+\infty$ asymptotic direction. Top-dimensional strata are the equivalence classes of this relation. Closures of top-dimensional strata are cells with real semi-algebraic boundary. Define the lower-dimensional strata inductively as maximal non-singular connected components of highest dimension.

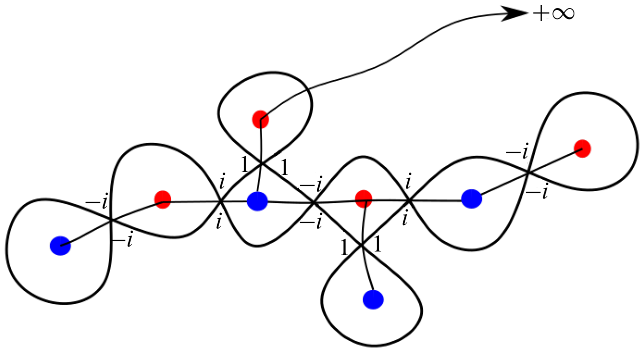

Figure 6. Rooted bi-coloured tree having one edge asymptotic to $+\infty$ with slalom curve and marked with critical values at double points.

7. Slalom polynomials

Our next goal is the construction of preferred representatives in the different strata of very generic polynomials. Shabat polynomials [10, 28], are generalizations of Chebyshev polynomials, a forerunner for Belyi maps and the Grothendieck interpretation, now called Dessins d'Enfants [19]. A Shabat polynomial $Sh(z)$ is a polynomial which has only 0 and 1 as critical values. The inverse image $Sh^{-1}([0,1]) \subset \mathbb{C}$ is a bi-coloured planar tree with vertices $Sh^{-1}(\{0,1\})$ coloured by the value of the Shabat polynomial Sh. The theorem of Shabat affirms that every isotopy class Γ of finite bi-coloured planar trees is obtained as inverse image $Sh^{-1}(\{0,1\})$ of a Shabat polynomial Sh. The polynomial is unique up to holomorphic automorphisms of source and target.

We start from the rooted planar tree Γ with $n+1$ edges in which one vertex is connected to $+\infty$ by an unbounded edge. Let $Sl(\Gamma)$ be the slalom curve of Γ. It is an immersed copy $Sl(\Gamma)\colon S^1 \to \mathbb{C}$ of the circle:

(1) with n transversal double points at the midpoints of the bounded edges of Γ,

(2) the curve $Sl(\Gamma)$ intersects Γ only and transversely twice at the midpoints of the bounded edges and once the unbounded edge such that each bounded complementary region contains one vertex.

A *slalom polynomial* $Sl\,P_\Gamma(z)$ for Γ is a generic monic polynomial P of degree $n+1$ with $|P(s)| = 1$ for each zero s of P' such that the coloured trees Γ_P and Γ are isotopic relative to $+\infty$.

A *slalom polynomial* is a generic monic polynomial P that is a slalom polynomial for the tree Γ_P.

Slalom polynomials P share properties with Shabat and Chebyshev polynomials: P' has $n = \text{degree}(P) - 1$ distinct roots and all critical values of P are of absolute value 1. So, the roots of P' are the double points of the curve $|P(z)| = 1$, which is a slalom curve

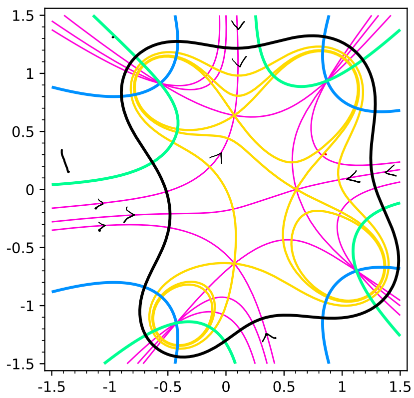

Figure 7. Region for $P = z^4 - z^3 + z^2 - z + 2 + I$
with black boundary. The lines $C(s)$ are marked with arrows.

for the tree Γ_P. The following is a strengthening of the above result of Marin for planar trees. Our proof is based on the Riemann uniformization theorem.

Theorem 7.1. *Given a rooted planar tree* Γ *with* $n+1$ *edges as above, there exists a slalom polynomial* P *for the isotopy class of* Γ.

Proof. There exists a continuous map $\phi\colon \mathbb{C} \to \mathbb{C}$ that is smooth except at the midpoints of Γ such that:

(1) the restriction of ϕ to the unbounded component of the complement of $Sl(\Gamma)$ is a degree $n+1$ covering map of the complement of the closed unit disc in $\mathbb{C}$,

(2) the restriction of ϕ to a bounded region of the complement of $S(\Gamma)$ is a diffeomorphism to the open unit disc in $\mathbb{C}$ that sends the vertex to 0 and half edges to radial rays,

(3) ϕ is holomorphic near ∞.

Let J be the unique conformal structure on $\mathbb{C}$ such that $\phi\colon (\mathbb{C}, J) \to \mathbb{C}$ is holomorphic. The conformal structure J extends to the one point compactification of $\mathbb{C}$ using properties (1) and (3) of ϕ, hence $\mathbb{C}$ and $(\mathbb{C}, J)$ are bi-holomorphic. Let $U\colon \mathbb{C} \to (\mathbb{C}, J)$ be a uniformization of the structure J. The composition

$$P = \phi \circ U\colon \mathbb{C} \to \mathbb{C}$$

is holomorphic and takes with finitely many exceptions every value $n+1$ times, so is a degree $n+1$ polynomial. The polynomial P has $n+1$ roots, the derivative has n roots at the points s with $U(s)$ being a midpoint of an edge. By stretching the uniformization map U one achieves that P is monic and generic.

The flow line from $+\infty$ ends at a root of P that we colour red and becomes the root of E_P. This red root can be changed by the transformation

$$P(z) \mapsto P_\lambda(z) = \lambda^{n+1} P\left(\frac{z}{\lambda}\right),$$

which preserves monic and balanced polynomials. The roots of P_λ are obtained by multiplying the roots of P by λ, so, $E_{P_\lambda} = \lambda E_P$ and this substitution turns the picture of E_P by the argument of λ and gives $2n$ combinatorial possibilities for attaching the unbounded edge of F_{P_λ} to a root of P_λ. Colour the vertices of the tree Γ in a bi-coloured way with colours red and blue (I am writing this in the days after 3 November 2020). Choose a disc D_R with centre 0 that contains Γ.

It follows that Γ and Γ_P are isotopic relative to $+\infty$, showing that all the $\mathrm{Cat}(n)$ combinatorial data consisting of a rooted planar tree as above are realized. The polynomial $P = P_\Gamma$ is a slalom polynomial for the tree Γ. ∎

As an example, see Figure 2. The polynomial $z^4 - z$ is the slalom polynomial of the tree D_4. The Chebyshev polynomials are up to a real translation precisely the slalom polynomials with the critical values among ± 1.

The slalom polynomial P_Γ is not unique up to holomorphic changes of coordinates in range and target. Uniqueness can be forced for trees with greatest valency $\leq k$ if one imposes moreover that the critical values are roots of unity of order depending on k; see Figure 6. For $k = 3$ order four works.

As in [3], one obtains a decomposition of the space of monic polynomials Pol_d of degree d and of the complement of the discriminant $\mathrm{Pol}_d \setminus \Delta$. We expect that this decomposition is a cell decomposition of $\mathrm{Pol}_d \setminus \Delta$, but state here only a weaker result concerning the top-dimensional strata.

Theorem 7.2. *The equivalence relation $\Gamma_P \sim \Gamma_Q$ induces a stratification with $\mathrm{Cat}(d-1)$ top-dimensional strata on the spaces Pol_d and $\mathrm{Pol}_d \setminus \Delta$. Moreover, the top-dimensional strata are cells and have a representative by a slalom polynomial.* ∎

Remark. The tree of vanishing cycles Δ_P that is constructed in the Milnor fibre of $y^2 + z^{\mu+1}$ is in general not a Dynkin diagram of a morsification, since no product of the positive Dehn twist of length μ computes the monodromy except if Δ_P is the diagram A_μ; see [1].

8. Slalom skeleton

Let SPol_d denote the subset of all polynomials with d distinct roots such that 1 is the only positive critical value of the function $|P|: \mathbb{C} \to \mathbb{R}_{\geq 0}$. So SPol_d is a subset of the complement of the discriminant $\Delta_d \subset \mathrm{Pol}_d$. The space SPol_d is semi-algebraic of real dimension d. The following result states that the subspace SPol_d is a deformation retract, i.e., a skeleton, of the space of monic polynomials of degree d that have only roots of multiplicity 1.

Theorem 8.1. *The space* $\mathrm{Pol}_d \setminus \Delta_d$ *retracts by deformation to* SPol_d.

Proof. The idea is to shrink the conformal height of all flow boxes in between two critical levels of $|P|$ by the factor $1-t$, $t \in [0,1]$, then to use Riemann's uniformization theorem to create a family of polynomials P_t with $P_0 = P$ and $P_1 \in \mathrm{SPol}_d$.

Let us give more details. Define

$$a = \mathrm{Min}_{P'(z)=0} |P(z)|, \quad A = \mathrm{Max}_{P'(z)=0} |P(z)|.$$

Choose, for $0 \le t < 1$, the function $\chi_t \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ to be smooth with $\chi_t'(s) > 0$ and $\chi_t(s) = s, s < a,\ \chi_t(s) = s - (1-t)(A-a), A \le s$. Define

$$Q_t(z) = \frac{\chi_t |P(z)|}{|P(z)|} P(z).$$

There exists a complex structure J_t on $\mathbb{C}$ such that $Q_t \colon (\mathbb{C}, J_t) \to \mathbb{C}$ is holomorphic. There exists a unique uniformization $u_t \colon \mathbb{C} \to (\mathbb{C}, J_t)$ such that

$$\lim_{|z| \to \infty} \frac{u_t(z)}{z}$$

is real and positive, and moreover $P_t = Q_t \circ u_t$ is a monic polynomial with

$$\mathrm{Min}_{P_t'(z)=0} |P_t(z)| = 1.$$

By construction $P_0 = P$ and the trees Γ_{P_t} are isotopic. Moreover, the limiting polynomial P_1 is monic and slalom. Indeed slalom since

$$\mathrm{Min}_{P_1'(z)=0} |P_1(z)| = 1 = \mathrm{Max}_{P_1'(z)=0} |P_1(z)|. \qquad \blacksquare$$

9. Geometric cluster monodromy

Combinatorial or pictorial properties of a monic polynomial mapping $P \colon \mathbb{C} \to \mathbb{C}$ do not change by Tschirnhausen substitutions, i.e., by substituting $z - t$ for z. By Tschirnhausen substitutions, the space Pol_d retracts by deformation to the space Pol_d^* of balanced polynomials P having roots that sum up to 0.

Denote for a subset $X \subset \mathrm{Pol}_d$ by X^* the intersection of the set of polynomials X with the space of balanced polynomials. As before the space $\mathrm{Pol}_d^* \setminus \Delta_d^*$ retracts to the space

$$\mathrm{SPol}_d^* \setminus \Delta_d \cap \mathrm{SPol}_d^*.$$

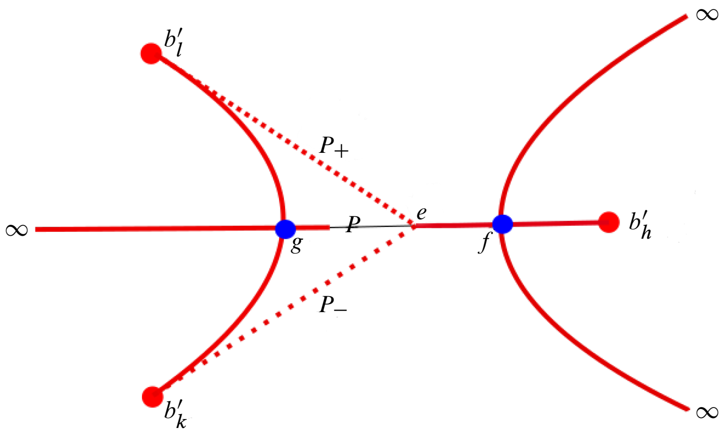

Figure 8. If the polynomial P lies on a generic cluster-branching stratum, three flow lines from ∞ run to the blue saddle points of $|P|$ that bound an edge e of E_P. The edge e disconnects E_P. By a small deformation the half edge f of e will join the root b'_k or b'_l leading to very generic polynomials P_+, P_-. The polynomial is P_+ if the half edge f connects to b'_l leaving the half edge g on its left. The polynomial is P_- if f connects to b'_k.

The family $y^2 + P(x) \mid P \in \mathrm{Pol}^*_{\mu+1}$ of functions on $\mathbb{C}^2$ is a model for the universal unfolding of the plane curve singularity A_μ. In this section, we relate the above stratification of the space of generic (balanced) polynomials $\mathrm{Pol}^*_{\mu+1} \setminus \Delta^*_{\mu+1}$ to the monodromy of the family of Riemann surfaces

$$F_P = \{y^2 + P(x) = 0 \mid P \in \mathrm{Pol}^*_{\mu+1} \setminus \Delta^*_{\mu+1}\}.$$

The open top-dimensional strata are the sets of very generic polynomials with given tree Γ_P. We need a description of the strata of real codimension one. For a polynomial P there are two elementary ways to stay generic, without staying very generic:

(1) the tree Γ_P has only one cluster C with $m(C) = 2$ and P' has μ distinct roots,

(2) all clusters in Γ_P are of multiplicity one, but the gradient line of X_P that starts at $+\infty$ ends at a root of P'.

So we have two types of walls of real codimension one in $\mathrm{Pol}^*_{\mu+1}$. The first we call *cluster-branching* (see Figure 8), the second one *root-branching*. The Lehmer polynomial (see Figure 5) lies on a cluster-branching wall. The polynomial $x^2 + 1$ lies on a root-branching wall.

A smooth oriented loop Λ in $\mathrm{Pol}^*_{\mu+1} \setminus \Delta$ is called transverse if it lies in the union of all strata of real codimension ≤ 1 and is transverse to the walls of cluster- and root-branching type. The following is a description of the monodromy in the universal unfolding induced by transverse loops. The description goes in several steps; see Figure 9.

Step 1. A very generic monic polynomial P of degree $\mu + 1$ gives us the tree Γ_P. Let $\widetilde{\Gamma}_P$ be a (narrow) closed regular tubular neighbourhood of Γ_P in $\mathbb{C}$. The flow line of X_P

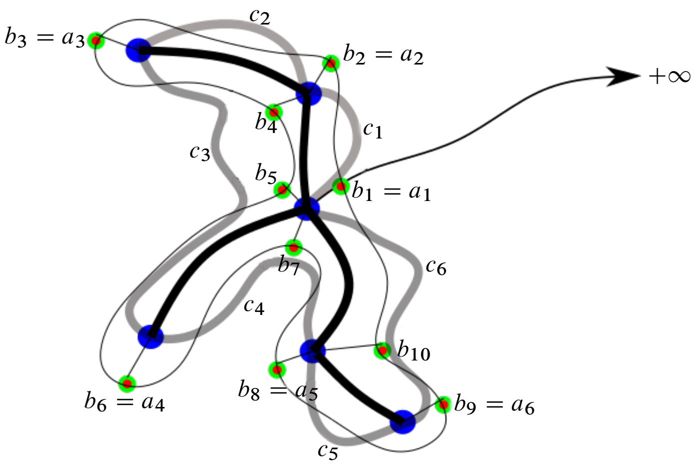

Figure 9. The tree Γ_P in black, 6 roots of $P(x)$ in blue, $\mu = 5$, images of cycles $c_1, c_2, \dots, c_5, c_6$ in grey, cycles $c_1, c_2, \dots, c_5$ form the Dynkin diagram A_5 in the fibre $y^2 + P(x) = 0$.

coming from $+\infty$ intersects the oriented boundary $\partial\widetilde{\Gamma}_P$ in a point b_1. Walk along $\partial\widetilde{\Gamma}_P$ and mark points $b_2, b_3, \dots$ on that boundary each time one passes near a root of P. We get a word of length 2μ. Let b'_k be the root near b_k and let r_k be a piece of the X_P-flow line from b_k to b'_k . Each root appears in the word $W_P = (b'_1, b'_2, \dots, b'_{2\mu})$ according to its valency in the tree E_P.

Step 2. Lift the path $(-r_1) * [b_1, b_2] * r_2$ to F_P. It is a cycle c_1 realized by a simple closed curve on F_P.

Let b_2^* be among $b_3, b_4, \dots,$ be the first point not close to b_1, b_2. Here *not close* means $(b_2^*)' \neq b_1, b_2$. Say, $b_1 = a_1, b_2 = a_2, b_2^* = a_3$.

Now start from a_3 along $\partial\widetilde{\Gamma}_P$ and stop at the first point a_4 among $b_4, b_5, \dots,$ that is not close to b_1, b_2, b_3.

Repeat μ times and get a sequence $a_1, a_2, \dots, a_{\mu+1}$, all points on $\partial\widetilde{\Gamma}_P$ close to roots $a'_1, a'_2, \dots, a'_{\mu+1}$ of P. Let r'_k be the piece of X_P-flow line from a_k to a'_k.

Lift the path $(-r'_k) * [a_k, a_{k+1}] * r'_{k+1}$ to the cycle c_k, again a simply closed curve on F_P. Also lift $(-r'_{\mu+1}) * [a_{\mu+1}, a_1] * r'_1$ to the cycle $c_{\mu+1}$.

The oriented curve $\partial\widetilde{\Gamma}_P$ lifts to an oriented curve c on F_P. Orient the cycles $c_1, \dots,$ $c_{\mu+1}$ such that the orientations agree along $c \cap c_k$.

If $\mu > 1$, the homological intersection numbers c_k, c_{k+1} equal $+1$ and all other equal 0. The cycles c_k, c_{k+1} intersect along a sub-interval, so by a small isotopy they will intersect transversally in one point.

If $\mu = 1$ the cycles c_1, c_2 intersect along two disjoint sub-intervals with local opposite intersection numbers. So the homological intersection number equals 0.

Step 3. Consider the system of oriented cycles $\widetilde{A}_\mu = c_1, c_2, \ldots, c_{\mu+1}$ as part of a marking M_P on F_P. The labelling is part of the marking. The complete marking M_P is obtained by adding the oriented lift c_∞ on F_P of the flow line that starts at $+\infty$. Orient the relative arc c_∞ such that the intersection number with any cycle in $\widetilde{A}_\mu$ is $+1$. The first μ cycles in $\widetilde{A}_\mu$ span freely the integral homology of F_P.

Observe that two such markings on an oriented surface are related by orientation-preserving diffeomorphism of F_P which is unique up to isotopy.

Step 4. If the loop Λ crosses at P a wall of cluster- or root-branching type connecting open strata, due to the above observation one gets a diffeomorphism from F_{P_-} to F_{P_+}, where both polynomials $P_-, P_+ \in \Lambda$ just before and after P are very generic. The monodromy induced by Λ is the composition of these wall crossing diffeomorphisms. The trees $\Gamma_{P_-}, \Gamma_{P_+}$ differ by elementary moves. In fact, by planar Whitehead moves. Also the labelling of the roots changes at a wall crossing. A similar change occurs at a root-branching. The flow line coming from $+\infty$ will have different roots in its closure.

The cycles $c_i, i = 1, \ldots, \mu$, are quadratic vanishing cycles that intersect geometrically to build the A_μ diagram inside the fibre F_P, so F_P retracts on their union. So the cycles c_i, $i, \ldots, \mu$, form a $\mathbb{Z}-$basis of the homology $H^1(F_P, \mathbb{Z})$. Since each cycle c_i is a sum of some cycles $\pm v(s)$, it follows that the cycles $v(s)$ (see above) span the first integral homology of F_P. Also it follows that in general the Milnor monodromy is a product of length $> \mu$ of Dehn twists with core the cycles $v(s)$.

The relations between the *wall crossing monodromy* as above, the *tête-à-tête monodromy* (see [4]), and the classical *Picard–Lefschetz monodromy of a morsification* need more investigation.

Recently, Pablo Portilla Cuadrado and Nick Salter have equipped the Milnor fibre of an isolated plane curve singularity with a preferred framing, relative to the boundary, as inside geometric object [26]. Very interestingly, the geometric monodromy group consists precisely of those mapping classes (relative to the boundary) that preserve this relative framing. Also the quadratic vanishing cycles are characterized in terms of this framing. The relative framing on the Milnor fibre depends upon a symplectic tubular neighbourhood together with a normal direction, as the above function N^f that we want to use depends upon a normal vector X with $i_X df = 1$. Again, more investigation is needed.

10. Problems

The signature of a monic polynomial P is defined as the relative isotopy class of its picture. Relative means here with respect to ends at infinity. In [3] the set Σ_d of realized signatures is combinatorially characterized as a set of isotopy classes of coloured graphs in the Gaussian plane. The characterisation is by six local properties and one global property. Moreover, the space of monic polynomials of a given signature is a contractible set of polynomials.

It would be interesting to give similar results for relative isotopy classes of flow box graphs of polynomials. One main problem is to characterize, if possible combinatorially, the flow box decompositions $\mathrm{FlowBox}(P)$ or $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ that can be realized.

In the decomposition $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ of a generic polynomial appear curved rectangles with four right angles and rectangles with three right and one acute angle $\frac{\pi}{4}$. Following Herbert Grötzsch [20], a curved rectangle with four right angles can be mapped conformally to a unique up to stretching Euclidean rectangle. Thinking of the yellow sides as base and roof, one gets a conformal invariant, namely the height $h = \frac{a}{b}$ of the Euclidean image. The Grötzsch map is explicit, the function $\log(P)$ maps each rectangle to an Euclidean rectangle in $\mathbb{C}$. For the rectangles in $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ the number b equals the measure of the yellow base, and, if the yellow sides are not infinitesimal, $a = \log(|P(s)|/|P(s')|)$, where s, s' are the zeros of P' on the level of the base and top.

Following the method of Grötzsch again, a rectangle with three right and one acute angle $\frac{\pi}{4}$ can be mapped conformally to a unique Khayyam–Saccheri hyperbolic quadrilateral with acute angle $\frac{\pi}{4}$. Again thinking the yellow sides as base and roof, one gets a conformal invariant $h = \frac{\sinh(a)}{\sinh(b)}$.

The problem is to study the conformal moduli h_P. Study means for instance to provide sharp estimates in terms of the coefficients of P.

The region D_P in the Gaussian plane that is bounded by the level $|P| = 5 + \max|P(s)|$, $P'(s) = 0$, is decomposed by $\mathrm{FlowBox}^{\mathrm{sep}}(P)$ in curved combinatorial rectangles; see Figures 4 and 7. The region D_P is homeomorphic to a disc and contains the yellow part of the flow box decomposition. The Euclidean length of flow lines of X_P that stay inside D_P is estimated in geometric terms from above by Damien Roy [27].

One can add to the rectangles in a consistent way diagonals and obtain a triangulation of D_P. Following the proof of Walter Brägger of the circle packing theorem of Andreev, each triangle is subdivided in 6 rectangular triangles and each one becomes the base of an ideal hyperbolic tetrahedron in hyperbolic three space [11]; see also [12]. The hyperbolic volumes of these tetrahedra provide more conformal invariants of Brägger type b_P that we inscribe to the sectors of the triangulation of $\mathrm{FlowBox}^{\mathrm{sep}}(P)$.

The next problem is to study and estimate the Brägger invariants b_P.

The most enriched data set for a polynomial P is $(\mathrm{FlowBox}^{\mathrm{sep}}(P), \nu_P, h_P, b_P)$ in which are inscribed the measures of the yellow edges together with all the numbers h_P and b_P.

The final problem, which is part of the initial motivation for this work, is to estimate in terms of the above data set the Mahler measure from below for the minimal polynomials P of Salem or Pisot numbers and to achieve progress on Lehmer's problem [21]. For a survey on Mahler's measure and the Lehmer problem, see [9,32]. In particular, how the dynamical system $W^\infty(S_P)$ for the Lehmer polynomial P differs from the systems $W^\infty(S_Q)$ for products Q of cyclotomic polynomials?

The unfolding space $\mathbb{C}^\mu$ with coordinates λ_k of an isolated singularity with real equation has a real basis by monomials such that the discriminant function Δ is with real coefficients. Now we can enrich the discriminant like in [3] by asking that the real functions $\mathrm{Re} P_\lambda$ and $\mathrm{Im} P_\lambda$ are regular above 0. Is the germ at 0 of the complement of the enriched discriminant a union of cells?

11. Sage + Pari

The pictures are drawn by using SAGEMATH and PARI in the following SAGE-cell.

```
import matplotlib;p=Graphics(); S=1.3
pari('f=z^10+z^9-z^7-z^6-z^5-z^4-z^3+z+1')
pari('g=deriv(f)'); pari('s=polroots(g)')
c=pari('vector(length(s),j,subst(f,z,s[j]))')
cf=pari('vector(poldegree(f)+1,n,polcoef(f,n-1,z))')
cc=c.sage(); cff=cf.sage()
M=max([abs(cc[n])^2 for n in range(0,len(cc))])
var('x''y',domain=RR)
ff=sum([cff[n]*(x+i*y)^n for n in range(0,len(cff))])
ff=expand(ff)
u=(ff+conjugate(ff))/2; v=-i*(ff-u)
w=u^2+v^2; w=expand(w)
p1=implicit_plot(u==0,(x,-S,S),(y,-S,S),
color=rainbow(7)[4])
p2=implicit_plot(v==0,(x,-S,S),(y,-S,S),
color=rainbow(7)[3])
bdry=implicit_plot(w==5+M,(x,-S,S),(y,-S,S),color='black')
pp=[p1,p2,bdry]
qq=[implicit_plot(w==abs(cc[n])^2,(x,-S,S),(y,-S,S),
color=rainbow(7)[1]) for n in range(0,len(cc))]
sep=[u*cc[n].imag()-v*cc[n].real()
for n in range(0,len(cc))]
ss=[implicit_plot(sep[n]==0,(x,-S,S),(y,-S,S),
color=rainbow(7)[6]) for n in range(0,len(cc))]
p=sum(pp+qq+ss); p.show()
```

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