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Happy Birthday, Dennis!

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Abstract. These are thoughts on mathematics, intuition, inspiration, imagination, feeling, perception, and the pleasure with which it can fill us, written as a reflection on the work of Dennis Sullivan on the occasion of his 80th birthday.

Just imagine, Dennis, how much fun it is to be writing birthday greetings to you. Hey, it gets me to remember the span of years that we've known each other,

- from the 1970s: the early days at the IHES where we each saw our sons merrily careening around the Résidence de l'Ormaille on trikes (and bikes);
- and moving through, to the past few weeks where we had some email exchange with Mike Freedman about the existence or nonexistence of quasi-conformal or bi-Lipschitz versions of that mathematical gem: the Bing involution (see [2]) of S^3.

In those early days at the IHES, besides your important localizing and rationalizing homotopy theory in the lecture halls, and besides our conversations about 3-manifolds with our close friend Po (Valentin Poenaru), you were quite adept at calling up teams for volleyball in the tiny volleyball court at the Résidence – the net of that court needing constant tugging to be tightened – and with some of us – not you! – in need of much more training to be able to keep up a reasonable game.

The first thing I had learned about you, possibly even before meeting you, was that you took delight in every aspect of mathematics, new or old. I heard that "every time you thought about, say, the Pythagorean theorem, your eyes would light up." This sentiment, of course, is something we should all teach: that 'novelty' might well be a *sufficient* cause for the pleasure of a mathematical idea, but is hardly a *necessary* one. The imagination can also be (re-)inspired by the familiar as it is inspired by the new.

As for imagination and the pleasure you take in it, I immediately think of these opening lines in your papers (my italicization):

> *Imagine in the hyperbolic space* $\mathbf{H}^{d+1}$ *an infinite completely symmetrical array of points* [5]

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and

> *One* imagines *trying to push the input circles through levels of a harmonic function on the surface* [4].

There are moments in our mathematical thinking where it is less *how to prove something* that is the issue, but more: how to practice expanding our intuitions so as to be able to encompass the issue; it's a call to …"Imagine …" (e.g., "in the hyperbolic space $\mathbf{H}^{d+1}$ …").

For me, this call to stretch my imagination, sharpen my geometric intuition, happens whenever I try to think of R. H. Bing's theorem that the double of the solid Alexander horned sphere is homeomorphic to S^3; or equivalently, but taking S^3 as its starting point, when I try to think of the 'Bing involution' of S^3 that has a fixed point set that decomposes S^3 into two solid Alexander horned spheres (see also [1]). That topological object was one of my early fascinations.

There is something about the pliability of the substance of three-dimensional topology, enlisting an often surprising pliability of the imagination that makes it such a joy. Not only with the grand beacon conjectures of our subject. But even with easy theorems. Dennis, I bet you take as much delight in the following theorem – elementary though it is – as I do: given any two curves in a shell ($S^2 \times [0, 1]$) such that the two endpoints of each of the curves lie in the different components of the boundary, there exists a homeomorphism of the shell onto itself bringing the one curve to the other. Well, we could actually try to prove this thing – or we could just *name it*. If you call it the *lightbulb hanging by a wire from the ceiling theorem* you get both its title, and, pretty much, its proof.

One of the vantage points toward mathematics that you like is what you call *visceral mathematics*, where "visceral" means

> *having to do with the response of the body as opposed to the intellect, as in the distinction between feeling and thinking.*

You explained that visceral can mean *feeling* structure. And this reminds me of the ideas of Bernard Teissier who focusses exactly on the "body-aspect" (and perceptual-aspect) of geometry (see [6]), e.g., for the line, he makes a clear distinction between what he calls "the visual line" and "the vestibular line." The visual aspect is that the perception (and therefore the understanding) of a line is that it "corresponds to the detection of a curve of curvature zero" insisting on the fact that the "transfer of structure" between the vestibular line and the visual line (the Poincaré–Berthoz isomorphism) is the source of *most* of the meaning for us of the mathematical line.

Teissier quotes Poincaré who introduced a muscle-relation to understanding: "our perception of a point in space is our perception of the gesture we have to make to seize an object placed there." Teissier writes "Our vestibular system detects all changes of orientation and accelerations." He also brings in the engagement of our "biological time clock" in the formation of certain mathematical ideas.

Of course the “feel” of structure requires close understanding of the texture of structure; as in your 1979 result ([3]) that all topological manifolds (except possibly: surprise! in dimension 4) have a unique bi-Lipschitz structure.

This result of yours is part of a broader theme: you said that you were “trying since grad school to answer the question ‘what is a manifold?’ ” and especially in the context of special coordinates. For example, you said:

> *Even more attractive to me are the generalized versions of the Schottky uniformizations of Riemann surfaces given by quadratic differentials and solving for coordinates by the Schwarzian derivative equation.*

Your deep engagement uniting the close grain of mathematics with its grand general structure has inspired generations of mathematicians, and has also inspired me.

Happy Birthday,
Barry

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