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Characterizations of circle homeomorphisms of different regularities in the universal Teichmüller space

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Abstract. In this survey, we first give a summary of characterizations of circle homeomorphisms of different regularities (quasisymmetric, symmetric, or $C^{1+\alpha}$) in terms of Beurling–Ahlfors extension, Douady–Earle extension, and Thurston's earthquake representation of an orientation-preserving circle homeomorphism. Then we provide a brief account of characterizations of the elements of the tangent spaces of these sub-Teichmüller spaces at the base point in the universal Teichmüller space.

We also investigate the regularity of the Beurling–Ahlfors extension $BA(h)$ of a $C^{1+\text{Zygmund}}$ orientation-preserving diffeomorphism h of the real line, and show that the Beltrami coefficient $\mu(BA(h))(x+iy)$ vanishes as $O(y)$ uniformly on x near the boundary of the upper half plane if and only if h is $C^{1+\text{Lipschitz}}$. Finally, we show this criterion is indeed true when h is started with any homeomorphism of the real line that is a lifting map of an orientation-preserving circle homeomorphism.

1. Introduction

The 1992–93 academic year was the fourth year of my PhD study at Graduate Center of CUNY. I recall on a sunny day in Fall 1992 and in his office facing Bryant Park at the corner of 6th Avenue and 42nd Street in New York City, Dennis gave me a lecture of one hour on why a $C^{1+\text{Zygmund}}$ diffeomorphism (see Definition 9) of the circle with an irrational rotational number cannot have wandering intervals, and hence it is ergodic. At the end, Dennis also explained why this smooth condition is not weaker than the Denjoy's $C^{1+\text{bounded variation}}$ criterion (also see Definition 9) for the ergodicity of the map ([7]). That conversation and his work on the renormalization operator in [44] not only motivated our joint result in [34], but also have kept me interested in exploring different methods to develop Teichmüller theory of circle diffeomorphisms of different types.

In the following, I first give a very brief introduction to the universal Teichmüller space. Then I describe what will be covered in this survey.

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Let $\mathbb{D}$ be the unit disk on the complex plane $\mathbb{C}$ centered at the origin. Two quasiconformal homeomorphisms F_1 and F_2 of $\mathbb{D}$ are said to be equivalent if F_2 is equal to $A \circ F_1$ for some conformal homeomorphism A of $\mathbb{D}$. The *universal Teichmüller space* $T(\mathbb{D})$ is defined to be the collection of the equivalence classes of the quasiconformal homeomorphisms of $\mathbb{D}$ under this equivalence relation. Through Bers' embedding, this space is equipped with a complex Banach manifold structure, which is an infinitely-dimensional space. Furthermore, the Teichmüller space of any hyperbolic Riemann surface is a sub Banach manifold of $T(\mathbb{D})$, which is the reason for $T(\mathbb{D})$ to be called the universal Teichmüller space.

Given a holomorphic curve γ in $T(\mathbb{D})$ through the base point (represented by the identity map), γ is expressed by a curve f^t of quasiconformal mappings depending holomorphically on t, $|t| < 1$, and with $f^0 = \text{id}$. Denote by

$$V(z) = \frac{df^t}{dt}|_{t=0}(z), \quad z \in \mathbb{D}. \tag{1.1}$$

Then V is a (complex) tangent vector field on $\mathbb{D}$ with $\|\bar{\partial} V\|_\infty < \infty$ ([3]). The collection of all such tangent vector fields on $\mathbb{D}$ constitute the tangent vector space of $T(\mathbb{D})$ at the base point.

There is a real model to characterize the elements of $T(\mathbb{D})$ and correspondingly there is a real model to describe the points of the tangent space of $T(\mathbb{D})$ at the base point. Let $\mathbb{S}^1$ be the boundary of $\mathbb{D}$, which is the unit circle on $\mathbb{C}$ centered at the origin. An orientation-preserving homeomorphism f of $\mathbb{S}^1$ is said to be *quasisymmetric* if there exists a constant $M > 0$ such that for any $s \in \mathbb{R}$ and $0 < t < \frac{1}{2}$,

$$\frac{1}{M} \leq \frac{|f(e^{2\pi i(s+t)}) - f(e^{2\pi i s})|}{|f(e^{2\pi i s}) - f(e^{2\pi i(s-t)})|} \leq M. \tag{1.2}$$

Let $QS(\mathbb{S}^1)$ be the collection of all quasisymmetric homeomorphisms f of $\mathbb{S}^1$ and let $\text{Möb}(\mathbb{S}^1)$ be the collection of the Möbius transformations preserving $\mathbb{D}$. Two elements f_1 and f_2 of $QS(\mathbb{S}^1)$ are equivalent if there exists an element $g \in \text{Möb}(\mathbb{S}^1)$ such that $f_2 = g \circ f_1$. The Beurling–Ahlfors extension ([4]) was the first method to affirm that there is a one-to-one correspondence between the universal Teichmüller space $T(\mathbb{D})$ and the quotient space $\text{Möb}(\mathbb{S}^1) \setminus \text{QS}(\mathbb{S}^1)$.

A continuous tangent vector field V along the unit circle $\mathbb{S}^1$ is said to be *Zygmund bounded* ([47]) if there exists a positive constant C such that

$$|\frac{V(e^{2\pi i(s+t)}) - 2V(e^{2\pi i s}) + V(e^{2\pi i(s-t)})}{t}| \leq C \tag{1.3}$$

for all $s \in \mathbb{R}$ and $0 < t < \frac{1}{2}$. Aligned with the real model of the universal Teichmüller space $T(\mathbb{D})$, there is a real model for the tangent space of $T(\mathbb{D})$ at the base point ([42]), which is comprised of the Zygmund bounded tangent vector fields V along $\mathbb{S}^1$ vanishing at the three points -1, $-i$ and 1, and we denote by $\Lambda(\mathbb{S}^1)$.

For a textbook on the universal Teichmüller space, we refer to [2, 20, 35].

Based on what I have understood or studied, I give in this survey a summary of the characterizations of circle homeomorphisms of different regularities (quasisymmetric, symmetric, or $C^{1+\alpha}$) in terms of Beurling–Ahlfors extension, Douady–Earle extension, and Thurston's earthquake representation of an orientation-preserving circle homeomorphism, and a brief account of the characterizations of the tangent vector spaces of these spaces at the base point through infinitesimal Beurling–Ahlfors extension, infinitesimal conformally natural extension, and infinitesimal earthquake representation of a continuous vector field along $\mathbb{S}^1$.

There are two ways to characterize a $C^{1+\alpha}$ circle diffeomorphism f in terms of Beurling–Ahlfors extension, one uses the Beurling–Ahlfors extension of a lifting map of f to the real line $\mathbb{R}$ and the other considers this type of extension of the conjugacy of f by a Möbius transformation mapping $\mathbb{S}^1$ to $\mathbb{R}$. The former is quite straightforward, but the latter is not obvious and has not been found in the literature. So we include the work for the latter in this survey. Furthermore, we investigate the regularity of the Beurling–Ahlfors extension $BA(h)$ of a $C^{1+\text{Zygmund}}$ orientation-preserving diffeomorphism h of the real line, and show that the Beltrami coefficient $\mu(BA(h))(x+iy)$ vanishes as $O(y)$ uniformly on x near the boundary of the upper half plane if and only if h is $C^{1+\text{Lipschitz}}$. Finally, we show this criterion is indeed true when h is started with any homeomorphism of the real line that is a lifting map of an orientation-preserving circle homeomorphism.

2. Beurling–Ahlfors extension and infinitesimal Beurling–Ahlfors extension

2.1. Background for Beurling–Ahlfors extension

The Beurling–Ahlfors extension ([4]) of an orientation-preserving homeomorphism h of the real line $\mathbb{R}$ to the upper half plane $\mathbb{U}$, denoted by $BA(h)$, is defined as follows. Given a point $z = x + iy \in \mathbb{U}$, $BA(h)(z) = u(x, y) + iv(x, y)$ is defined by

$$u(x,y) = \frac{1}{2y}\int_{x-y}^{x+y} h(t)\,dt \quad \text{and} \quad v(x,y) = \frac{1}{y}\left(\int_{x}^{x+y} h(t)\,dt - \int_{x-y}^{x} h(t)\,dt\right). \tag{2.1}$$

The Beurling–Ahlfors extension $BA(h)$ provides the first method to affirm the real model for $T(\mathbb{D})$, which has become a classical technique tool in Teichmüller theory. A neat presentation of this topic is given in [2, Chapter IV], which we will not repeat in this survey. But note that the expression of v given in (2.1) is twice of the expression of v given in [2, Chapter IV]. This modification enables BA to extend the identity map on $\mathbb{R}$ to the identity map on $\mathbb{U}$. It is straightforward to check that the results stated in [2, Chapter IV] continue to be valid for the Beurling–Ahlfors extension defined by (2.1). For example,

(1) BA is compatible with affine maps in the sense that given any two affine mappings A_1, A_2,

$$BA(A_1 \circ h \circ A_2) = A_1 \circ BA(h) \circ A_2;$$

(2) $BA(h)$ is a diffeomorphism of the upper half plane $\mathbb{U}$ if h is quasisymmetric;

(3) $BA(h)$ is quasiconformal if h is quasisymmetric. Here one may use symmetric triples on $\mathbb{R}$ to define f to be quasisymmetric; that is, an orientation-preserving homeomorphism h of $\mathbb{R}$ is quasisymmetric if there exists a constant $M > 0$ such that for any $x, t \in \mathbb{R}$ with $t > 0$,

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M. \tag{2.2}$$

In the following subsection, we show how to apply the Beurling–Ahlfors extension in two different ways to confirm the real models for the universal Teichmüller space $T(\mathbb{D})$ and two sub-Teichmüller spaces of $T(\mathbb{D})$.

2.2. Characterizations of circle homeomorphisms of different regularities through Beurling–Ahlfors extension

Let f be an orientation-preserving circle homeomorphism of the unit circle $\mathbb{S}^1$. There are two different ways to apply the Beurling–Ahlfors extension of a homeomorphism of a real line $\mathbb{R}$ to an extension of f to the closed unit disk $\overline{\mathbb{D}}$. We describe them one by one. The first one is to use a lifting map $\tilde{f}$ of f to $\mathbb{R}$, from which one can easily see that a regularity on f transfers to the same regularity on $\tilde{f}$; and vice versa. In the following, we let $\tilde{f}$ be the lifting map of f fixing three points 0, 1 and ∞. Clearly, $\tilde{f}(x+1) = \tilde{f}(x) + 1$ for any $x \in \mathbb{R}$. From the first property of the Beurling–Ahlfors extension mentioned in the previous subsection, one can see that $BA(\tilde{f})(z+1) = BA(\tilde{f})(z) + 1$ for any $z \in \mathbb{U}$. Then $BA(\tilde{f})$ projects down to an extension map of f to $\overline{\mathbb{D}}$ under the projection

$$\pi : \mathbb{U} \to \mathbb{D} : z \mapsto e^{2\pi i z},$$

which we denote by $\pi \circ BA(\tilde{f}) \circ \pi^{-1}$.

We consider in this survey the following three types of regularities imposed on f: quasisymmetry, symmetry, and $C^{1+\alpha}(0 < \alpha < 1)$. The quasisymmetry of f is defined by (1.2) and the symmetry of f is defined by further requiring that

$$\frac{|f(e^{2\pi i(s+t)}) - f(e^{2\pi i s})|}{|f(e^{2\pi i s}) - f(e^{2\pi i(s-t)})|} \text{ converges to 0 uniformly on } s \text{ as } t \to 0. \tag{2.3}$$

By $f \in C^{1+\alpha}$ for some $0 < \alpha < 1$ we mean that f is a diffeomorphism and f' is α-Hölder continuous. Applying the properties of Beurling–Ahlfors extension summarized in [2, Chapter IV], one can characterize each type of f in terms of a condition on the Beltrami coefficient of $BA(\tilde{f})$. We summarize the results in Table 1, but we skip detailed

Properties	Main references
$\pi \circ BA(\widetilde{f}) \circ \pi^{-1}$ is quasiconformal iff f is quasisymmetric	[4]
$\pi \circ BA(\widetilde{f}) \circ \pi^{-1}$ is asymptotically conformal iff f is symmetric	[16] or [21]
$\mu_{\pi \circ BA(\widetilde{f}) \circ \pi^{-1}}(z) = O((1-\lvert z\rvert)^{\alpha})$ iff f is a $C^{1+\alpha}$ diffeomorphism for each $0 < \alpha < 1$	mean value theorem

Table 1. Relationships among various regularities of f and $\pi \circ BA(\widetilde{f}) \circ \pi^{-1}$.

proofs since they follow from the work of [2, Chapter IV] in a relatively straightforward way.

In this paper, we are more focused on another way, more natural in Teichmüller theory, to apply the Beurling–Ahlfors extension to characterize a circle homeomorphism in each of the three types. Let us first introduce a conformal invariant to characterize the quasisymmetry or symmetry of an orientation-preserving circle homeomorphism f. Given a quadruple $Q = \{a, b, c, d\}$ consisting of four points a, b, c and d on the unit circle $\mathbb{S}^1$ arranged in counterclockwise order, we denote the cross ratio $cr(Q)$ of Q by

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

Definition 1 (Quasisymmetry by cross ratio distortion). The cross ratio distortion norm of f is defined as

$$\|f\|_{cr} = \sup_{cr(Q)=1} |\ln cr(f(Q))|, \tag{2.4}$$

where $f(Q)$ be the image quadruple $\{f(a), f(b), f(c), f(d)\}$. We say f is quasisymmetric if $\|h\|_{cr}$ is finite.

Clearly, the cross ratio distortion norm $\|f\|_{cr}$ of f is invariant under pre- or post-composition by a conformal homeomorphism preserving $\mathbb{D}$. Therefore, we may assume that f fixes three points -1, i and $-i$ on the circle $\mathbb{S}^1$. This definition of quasisymmetry is equivalent to the one using the condition (1.2) on the symmetric triples on $\mathbb{S}^1$ ([12]).

Similarly, the symmetry of f can be defined by imposing an asymptotic condition on the cross ratio distortion of f as follows. For any quadruple $Q = \{a, b, c, d\}$ on $\mathbb{S}^1$, the minimal scale $s(Q)$ is defined as

$$s(Q) = \min\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

A sequence of quadruples

$$\{Q_n = \{a_n, b_n, c_n, d_n\}\}_{n=1}^{\infty}$$

is said to be *degenerating* if $cr(Q_n) = 1$ for each n and $\lim_{n\to+\infty} s(Q_n) = 0$.

Definition 2 (Symmetry by cross ratio distortion). A quasisymmetric homeomorphism f of $\mathbb{S}^1$ is said to be symmetric if

$$\sup_{\{Q_n\}} \limsup_{n\to\infty} |cr(h(Q_n))| = 0, \tag{2.5}$$

where the supremum is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.

See [12] for the equivalence of this definition for symmetry with the one using the asymptotic vanishing condition of the ratio distortion (1.2) of f on symmetric triples.

Now we are ready to introduce the second way to apply Beurling–Ahlfors extension to construct an extension of f to $\bar{\mathbb{D}}$. After post composition by a Möbius transformation, we may assume that f fixes i, -1 and $-i$. We have this assumption for f through the remaining part of this subsection. Let $w(z) = \frac{1}{i}\frac{z+i}{z-i}$. Then w maps the circle $\mathbb{S}^1$ onto the real line $\mathbb{R}$ and the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{U}$ with 0 mapped to i. Note also that w fixes -1 and 1. Now let $h = w \circ f \circ w^{-1}$. By associating with any symmetric triple $\{x-t, x, x+t\}$ to a quadruple $\{x-t, x, x+t, \infty\}$, one can see that $\|f\|_{cr} < \infty$ implies that h is a quasisymmetric homeomorphism of $\mathbb{R}$ defined by (2.2). Therefore, the Beurling–Ahlfors extension $BA(h)$ is a quasiconformal homeomorphism of the upper plane $\mathbb{U}$ (see [4]). It follows that $w^{-1} \circ BA(h) \circ w$ is a quasiconformal extension of f to the closure of the unit disk $\mathbb{D}$, which we call the Beurling–Ahlfors extension of f and denote by $BA(f)$; that is,

$$BA(f) = w^{-1} \circ BA(w \circ f \circ w^{-1}) \circ w. \tag{2.6}$$

In summary, $BA(f)$ is quasiconformal if f is quasisymmetric (using Definition 1).

Now let us show why $BA(f)$ is asymptotically conformal if f is symmetric (by Definition 2). At first, $h = w \circ f \circ w^{-1}$ is a symmetric homeomorphism of the real line $\mathbb{R}$ in the sense that

$$\frac{h(x+t)-h(x)}{h(x)-h(x-t)} \text{ converges to 0 uniformly on } x \in R \text{ as } t \to 0. \tag{2.7}$$

With the above condition on h, one can see that the Beurling–Ahlfors extension $BA(h)$ of h has its Beltrami coefficient $\mu_{BA(h)}(x+iy)$ converge to 0 uniformly on $x \in \mathbb{R}$ as $y \to 0^+$. But in order to show that the Beltrami coefficient $\mu_{BA(f)}(z)$ converges to 0 uniformly as $|z| \to 1$, one needs to obtain first that

$$\mu_{BA(h)}(x+iy) = o\left(\frac{1}{\sqrt{x^2+y^2}}\right) \quad \text{as } \sqrt{x^2+y^2} \to \infty.$$

This goal can be achieved by using the condition that $|\frac{h(-x)}{h(x)}|$ converges to 1 as x goes to ∞. We obtain the following theorem.

Theorem 1. *If f is an orientation-preserving symmetric homeomorphism of $\mathbb{S}^1$ fixing three points i, -1 and $-i$, then the Beurling–Ahlfors extension $BA(f)$ of f defined by (2.6) is asymptotically conformal.*

In the remaining part of this subsection, we show the following theorem.

Theorem 2. *If f is an orientation-preserving $C^{1+\alpha}$ diffeomorphism of $\mathbb{S}^1$ fixing three points i, -1 and $-i$, where $0<\alpha<1$, then the Beurling–Ahlfors extension $BA(f)$ of f defined by (2.6) has its Beltrami coefficient $\mu_{BA(f)}$ satisfying*

$$\mu_{BA(f)}(z) = O\big((1-|z|)^{\alpha}\big). \tag{2.8}$$

Proof. Let $w(z)$ be the Möbius transformation defined as the above. Given a point $z \in \mathbb{D}$, let $w = w(z) = x + iy$, where $y > 0$. We use the same notation that $h = w \circ f \circ w^{-1}$. Since f is a $C^{1+\alpha}$ diffeomorphism of $\mathbb{S}^1$, where $0<\alpha<1$, it follows that h is a $C^{1+\alpha}$ diffeomorphism of $\mathbb{R}$ and furthermore for $x \in \mathbb{R}$,

$$h'(x) = 1 + O\Big(\frac{1}{|x|^{\alpha}}\Big). \tag{2.9}$$

Denote by $BA(h)(w) = BA(h)(x+iy) = u(x,y) + iv(x,y)$. Then the Beltrami coefficient $\mu_{BA(h)}(w)$ is expressed by

$$\mu_{BA(h)}(w) = \mu_{BA(h)}(x+iy) = \frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)}, \tag{2.10}$$

where

$$\begin{aligned}
u_x &= \frac{1}{2y}\big(h(x+y) - h(x-y)\big),\\
u_y &= \frac{1}{y}\Big[-\frac{1}{2y}\int_{x-y}^{x+y} h(t)\,dt + \frac{1}{2}\big((h(x+y) + h(x-y)\big)\Big],\\
v_x &= \frac{1}{y}\big[h(x+y) - 2h(x) + h(x-y)\big],\\
v_y &= -\frac{1}{y^2}\Big(\int_{x}^{x+y} h(t)\,dt - \int_{x-y}^{x} h(t)\,dt\Big) + \frac{1}{y}\big((h(x+y) - h(x-y)\big).
\end{aligned}$$

Since h is a $C^{1+\alpha}$ diffeomorphism of $\mathbb{R}$, it follows from the mean value theorem that

$$u_x = v_y = f'(x) + O(y^{\alpha}) \quad \text{and} \quad u_y = v_x = O(y^{\alpha}).$$

Thus, when x is bounded and y is small,

$$\mu_{BA(h)}(w) = O(y^{\alpha}).$$

It remains to verify that

$$\mu_{BA(h)}(w) = O\Big(\frac{1}{|w|^{\alpha}}\Big)$$

when $|w|$ is large, which can be achieved by considering two cases.

Case 1: Assume that $\frac{|x|}{y} \geq 2$. Using the estimate (2.9) for h' and applying the mean value theorem, we obtain

$$u_x = v_y = 1 + O\big((|x| - y)^{-\alpha}\big) \quad \text{and} \quad u_y = v_x = O\big((|x| - y)^{-\alpha}\big),$$

which implies that

$$\mu_{BA(h)}(w) = O\big((|x| - y)^{-\alpha}\big).$$

Let $s = \frac{|x|}{y}$. Then $s \geq 2$. Now we rewrite

$$\frac{|x| - y}{\sqrt{x^2 + y^2}} = \frac{s - 1}{\sqrt{s^2 + 1}}.$$

Since $\frac{\sqrt{5}}{5} \leq \frac{s-1}{\sqrt{s^2+1}} \leq 1$ when $s \geq 2$, it follows that $\frac{\sqrt{5}}{5} \leq \frac{|x|-y}{\sqrt{x^2+y^2}} \leq 1$. Therefore,

$$\mu_{BA(h)}(w) = O(|w|^{-\alpha}).$$

Case 2: Assume that $\frac{|x|}{y} \leq 2$. We continue to use the estimate (2.9) for h', but we use an integral of h' to estimate $h(x + y) - h(0)$ and $h(x - y) - h(0)$. So we obtain

$$\begin{aligned} h(x + y) - h(0) &= \int_0^{x+y} h'(t)\,dt = \int_0^{x+y} [1 + O(|t|^{-\alpha})]\,dt \\ &= x + y + O(|x - y|^{1-\alpha} + |x + y|^{1-\alpha}) \\ &= x + y + O(|x - y|^{1-\alpha}) + O(|x + y|^{1-\alpha}). \end{aligned}$$

Similarly,

$$h(x - y) - h(0) = x - y + O(|x - y|^{1-\alpha}) + O(|x + y|^{1-\alpha}).$$

Thus,

$$\begin{aligned} u_x = v_y &= 1 + \frac{1}{y}\big[O\big((|x - y|)^{1-\alpha}\big) + O\big((|x + y|)^{1-\alpha}\big)\big] \\ &= 1 + \frac{1}{y^\alpha}\Big[O\Big(\Big(\Big|\frac{x}{y} - 1\Big|\Big)^{1-\alpha}\Big) + O\Big(\Big(\Big|\frac{x}{y} + 1\Big|\Big)^{1-\alpha}\Big)\Big] \end{aligned}$$

and

$$u_y = v_x = \frac{1}{y^\alpha}\Big[O\Big(\Big(\Big|\frac{x}{y} - 1\Big|\Big)^{1-\alpha}\Big) + O\Big(\Big(\Big|\frac{x}{y} + 1\Big|\Big)^{1-\alpha}\Big)\Big].$$

Clearly,

$$O\Big(\Big(\Big|\frac{x}{y} - 1\Big|\Big)^{1-\alpha}\Big) + O\Big(\Big(\Big|\frac{x}{y} + 1\Big|\Big)^{1-\alpha}\Big) = O\Big(\Big(\Big|\frac{|x|}{y} - 1\Big|\Big)^{1-\alpha}\Big) + O\Big(\Big(\frac{|x|}{y} + 1\Big)^{1-\alpha}\Big),$$

which shows that this expression is less than or equal to $O(1) + O(3^{1-\alpha})$ by using $\frac{|x|}{y} \leq 2$. Therefore, we obtain

$$u_x = v_y = 1 + O\Big(\frac{1}{y^\alpha}\Big) \quad \text{and} \quad u_y = v_x = O\Big(\frac{1}{y^\alpha}\Big).$$

Properties	Main references
$BA(f)$ is quasiconformal iff f is quasisymmetric	[4]
$BA(f)$ is asymptotically conformal iff f is symmetric	Theorem 1
$\mu_{BA(f)}(z) = O((1-\lvert z\rvert)^\alpha)$ iff f is a $C^{1+\alpha}$ diffeomorphism for each $0<\alpha<1$	Theorem 2

Table 2. Relationships among various regularities of f and $BA(f)$.

We also know $|w| = \sqrt{x^2+y^2} \le \sqrt{5}y$ since $\frac{|x|}{y} \le 2$. Then $\frac{1}{y} \le \frac{\sqrt{5}}{|w|}$. So we obtain

$$u_x = v_y = 1 + O\Big(\frac{1}{|w|^\alpha}\Big) \quad \text{and} \quad u_y = v_x = O\Big(\frac{1}{|w|^\alpha}\Big).$$

Thus,

$$\mu_{BA(h)}(w) = O(|w|^{-\alpha}).$$

The proof of this theorem is complete. ■

Finally, we summarize the results on the relationships among various regularities of f and $BA(f)$ in Table 2.

2.3. Characterizations of vector fields of different regularities through infinitesimal Beurling–Ahlfors extension

As we have mentioned in the introduction, the real model $\Lambda(\mathbb{S}^1)$ for the tangent space of $T(\mathbb{D})$ at the base point ([42]) consists of all Zygmund bounded tangent vector fields V along $\mathbb{S}^1$ vanishing at the three points -1, $-i$ and 1.

From the expressions of $u(x,y)$ and $v(x,y)$ given by (2.6), one can see that the Beurling–Ahlfors extension is a linear operator that extends a continuous map from $\mathbb{R}$ to $\mathbb{R}$ to a continuous map from $\overline{\mathbb{U}}$ to $\overline{\mathbb{U}}$, which is differentiable on $\mathbb{U}$. Thus, the Beurling–Ahlfors extension is a linear operator on the space of continuous vector fields on $\mathbb{R}$. This implies that the infinitesimal Beurling–Ahlfors extension of a continuous vector field along $\mathbb{R}$ is defined in the same way as the Beurling–Ahlfors extension of a homeomorphism of $\mathbb{R}$.

As in the previous subsection, we are more interested in using $\mathbb{D}$ as the hyperbolic plane and viewing $\mathbb{S}^1$ as the boundary of the hyperbolic plane. Analogously, there is a conformal invariant to characterize a Zygmund bounded tangent vector field V along $\mathbb{S}^1$, which is defined as follows.

Definition 3. The cross ratio distortion norm of V is defined as

$$\|V\|_{cr} = \sup_{cr(Q)=1} |V[Q]| < +\infty,$$

where

$$V[Q] = \frac{V(b)-V(a)}{b-a} - \frac{V(c)-V(b)}{c-b} + \frac{V(d)-V(c)}{d-c} - \frac{V(a)-V(d)}{a-d}.$$

It is shown in [25] that

(i) V is Zygmund bounded if and only if $\|V\|_{cr} < +\infty$, and

(ii) $\|V\|_{cr}$ is a conformal invariant in the sense that for any Möbius transformation g from $\mathbb{D}$ to $\mathbb{D}$ or $\mathbb{U}$,

$$\|g_* V\|_{cr} = \|V\|_{cr},$$

where $g_* V$ is the pushforward of V by g; that is

$$g_* V = \frac{V \circ g^{-1}}{(g^{-1})'} = g' \circ g^{-1} \cdot V \circ g^{-1}.$$

By $V \in \Lambda_0(\mathbb{S}^1)$ we mean that the ratio in the expression (1.3) approaches 0 uniformly on s as $t \to 0$, which is equivalent to the following definition.

Definition 4. A Zygmund bounded vector field V along $\mathbb{S}^1$ belongs to $\Lambda_0(\mathbb{S}^1)$ if

$$\sup_{\{Q_n\}} \limsup_{n\to\infty} |V[Q_n]| = 0,$$

where the supremum is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.

We continue to use the map $w(z) = \frac{1}{i}\frac{z+i}{z-i}$. According to [42], $\|\bar\partial BA(w_* V)\|_\infty$ is finite if V is Zygmund bounded, and furthermore if $V \in \Lambda_0(\mathbb{S}^1)$, then $\bar\partial BA(w_* V)(x+iy)$ vanishes uniformly on x as $y \to 0$ and vanishes as $|x+iy| \to \infty$.

Now we define $BA(V) = (w^{-1})_*(BA(w_* V))$. Then

$$\bar\partial(BA(V))(z) = \bar\partial\left(\frac{BA(w_* V)(w(z))}{w'(z)}\right) = (\bar\partial BA(w_* V))(w(z))\frac{\overline{w'(z)}}{w'(z)}.$$

Thus,

$$\|\bar\partial(BA(V))\|_\infty = \|\bar\partial BA(w_* V)\|_\infty,$$

and $\bar\partial(BA(V))(z)$ vanishes uniformly as z approaches the boundary of $\mathbb{D}$ if $V \in \Lambda_0(\mathbb{S}^1)$.

In analogy with Theorem 2, one can obtain the following theorem.

Theorem 3. *If V is a $C^{1+\alpha}$ smooth vector field along $\mathbb{S}^1$ for some $0 < \alpha < 1$, then*

$$\bar\partial BA(V)(z) = O\big((1-|z|)^\alpha\big).$$

We summarize the characterizations of three types of V in terms of $\bar\partial(BA(V))$ in Table 3.

Properties	Main references
$\|\bar{\partial} BA(V)\|_\infty < \infty$ iff V is Zygmund bounded	[42]
$\bar{\partial} BA(V)$ vanishes uniformly near $\mathbb{S}^1$ iff $V \in \Lambda_0(\mathbb{S}^1)$	[42]
$\bar{\partial} BA(V)(z) = O((1-\|z\|)^\alpha)$ iff V is a $C^{1+\alpha}$ smooth for each $0 < \alpha < 1$	Theorem 3

Table 3. Relationships among various regularities of V and $BA(V)$.

3. Douady–Earle extension and infinitesimal conformally natural extension

3.1. Background for Douady–Earle extension

We first introduce the conformal barycenter of a probability measure μ supported on $\mathbb{S}^1$. A point w of $\mathbb{D}$ is called a *conformal barycenter* of μ, denoted by $B(\mu)$, if

$$\int_{\mathbb{S}^1} \frac{\zeta - w}{1 - \bar{w}\zeta}\, d\mu(\zeta) = 0.$$

We call μ an *admissible* measure if it has no atom of measure $\geq \frac{1}{2}$. Douady and Earle ([8]) pointed out that $B(\mu)$ exists uniquely for any admissible measure μ, although they proved the existence and uniqueness of $B(\mu)$ for the case when μ has no atom. In the following, we outline the proof, given in [31], of the existence and uniqueness of $B(\mu)$ under the admissible condition on μ.

To see the existence and uniqueness of $B(\mu)$, the following smooth vector field ξ_μ on $\mathbb{D}$ is considered in [8]:

$$\xi_\mu(w) = (1 - |w|^2) \int_{\mathbb{S}^1} \frac{\zeta - w}{1 - \bar{w}\zeta} d\mu(\zeta), \quad w \in \mathbb{D}. \tag{3.1}$$

It is found in [28] that the vector field $\xi_\mu(w)$ becomes relatively easy to handle if the scalar factor $1 - |w|^2$ is dropped. So let

$$\tilde{\xi}_\mu(w) = \frac{1}{1 - |w|^2} \xi_\mu(w). \tag{3.2}$$

The work to reach the existence and uniqueness of $B(\mu)$ is comprised of two steps. One can see first that if μ is admissible, then $\xi_\mu(w)$ (and hence $\tilde{\xi}_\mu(w)$) points inside the circle centered at the origin and passing through w when w is sufficiently close to the boundary $\mathbb{S}^1$. (In fact, if μ has no atom, then $\tilde{\xi}_\mu(w)$ extends to a continuous vector field on the closed disk $\bar{\mathbb{D}}$ and points inside at every point on $\mathbb{S}^1$.) This implies the existence of a singular point of the vector field, which is a conformal barycenter of μ. Secondly, in order to have the uniqueness of $B(\mu)$, it suffices to know that the Jacobian of the vector field is positive at every point of $\mathbb{D}$. Now one can find that it is quite easy to achieve this property by using $\widetilde{\xi}_\mu(w)$ since it satisfies

$$\widetilde{\xi}_\mu(w) = \widetilde{\xi}_{(g_w)_*(\mu)}(0), \tag{3.3}$$

where $g_w(\zeta) = \frac{\zeta - w}{1 - \bar{w}\zeta}$ and $(g_w)_*(\mu)$ is the pushforward measure of μ under the map g_w (see the following expression (3.7)), and hence the values of $\widetilde{\xi}_\mu$ at two points z and w in $\mathbb{D}$ are related by

$$\widetilde{\xi}_\mu(z) = \widetilde{\xi}_{(g_w)_*(\mu)}(g_w(z)). \tag{3.4}$$

Then the Jacobian of $\widetilde{\xi}_\mu$ at w is the product of the Jacobian of $\widetilde{\xi}_{(g_w)_*(\mu)}$ at 0 and the Jacobian of g_w at w. Therefore, it is sufficient to show that the Jacobian of $\widetilde{\xi}_\mu$ is positive at 0, which is

$$\mathrm{Jac}(\widetilde{\xi}_\mu)(0) = \frac{1}{2} \int\int_{\mathbb{S}^1 \times \mathbb{S}^1} |\eta^2 - \zeta^2|^2 \, d\mu(\zeta) \, d\mu(\eta) > 0. \tag{3.5}$$

Now by the Poincaré–Hopf index theorem, we know that the singular point of ξ_μ in $\mathbb{D}$ is unique. The details for the first step can be found in [31].

Given a point $z \in \mathbb{D}$, let η_z be the normalized harmonic measure on $\mathbb{S}^1$ as viewed from z; that is, for any Borel set $E \subset \mathbb{S}^1$,

$$\eta_z(E) = \frac{1}{2\pi} \int_E |dg_z(\zeta)| = \frac{1}{2\pi} \int_E \frac{1 - |z|^2}{|z - \zeta|^2} \, |d\zeta|. \tag{3.6}$$

If I is an arc on $\mathbb{S}^1$ and $\theta_z(I)$ is the radian of the angle of I as viewed from z (in the hyperbolic metric on $\mathbb{D}$), then $\eta_z(I)$ is the ratio of $\theta_z(I)$ to 2π. Let f be an orientation-preserving homeomorphism on the unit circle $\mathbb{S}^1$ and $f_*(\eta_z)$ be the pushforward of η_z under f. That is, for any Borel set $E \subset \mathbb{S}^1$,

$$f_*(\eta_z)(E) = \eta_z(f^{-1}(E)). \tag{3.7}$$

Now we denote the conformal barycenter of the measure $f_*(\eta_z)$ by w; that is, $w = B(f_*(\eta_z))$. Then w and z satisfy the following equation:

$$\frac{1}{2\pi} \int \frac{\zeta - w}{1 - \bar{w}\zeta} \, df_*(\eta_z)(\zeta) = 0. \tag{3.8}$$

The *Douady–Earle extension* $DE(f)$ of f is defined as: $DE(f)(z) = B(f_*(\eta_z))$ for each $z \in \mathbb{D}$ and $DE(f)(z) = f(z)$ for each $z \in \mathbb{S}^1$.

In summary, given a point $z \in \mathbb{D}$, $DE(f)(z)$ is defined to be the unique point $w \in \mathbb{D}$ such that

$$F(z, w) = 0,$$

where

$$F(z, w) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{f(\xi) - w}{1 - \bar{w} f(\xi)} \cdot \frac{1 - |z|^2}{|z - \xi|^2} \, |d\xi|. \tag{3.9}$$

The extension $DE(f)$ has the following two important features:

(1) $DE(f)$ is an orientation-preserving homeomorphism of the closed disk $\overline{\mathbb{D}}$ and a real analytic diffeomorphism of $\mathbb{D}$.

(2) $DE(f)$ is *conformally natural* in the sense that for any two orientation-preserving conformal homeomorphisms A and B of $\mathbb{D}$,

$$DE(f)(A \circ f \circ B) = A \circ DE(f) \circ B.$$

For their proofs, we refer to [8]. Similar work has been developed in [1] for conformal barycentric extensions of monotone circle maps (not necessarily continuous) and in [28] for circle endomorphisms. Finally, conformally natural extensions are introduced and studied in [31] for arbitrary continuous maps from the circle to itself. The continuity of such an extension is proved by two different methods in [31] and [27].

3.2. Characterizations of circle diffeomorphisms of different regularities through Douady–Earle extension

In the previous subsection, we have provided a brief account on how the Douady–Earle extension is defined and the basic properties such as conformal naturality and being a diffeomorphism of $\mathbb{D}$. The most important feature of Douady–Earle extension is its conformal naturality. As a compensation to this valuable property, the Douady–Earle extension is defined by a quite implicit process through the equation $F(z, w) = 0$, where $w = DE(f)(z)$ and $F(z, w)$ is given by the expression (3.9). Thus, one has to apply the implicit function theorem to estimate the Jacobian, Beltrami coefficient or maximal dilatation of $DE(f)$ at any point $z \in \mathbb{D}$. This is a complicated process. In order to use relatively simple expressions for partial derivatives of F, we may pre-compose f by a conformal homeomorphism A of $\mathbb{D}$ mapping 0 to z and post-compose f by a conformal homeomorphism B mapping w to 0. Then by the conformal naturality,

$$DE(B \circ f \circ A)(0) = B(DE(f)(A(0))) = B(DE(f)(z)) = B(w) = 0.$$

Instead of considering the Beltrami coefficient of $DE(f)$ at z, one works with the Beltrami coefficient of $DE(B \circ f \circ A)$ as 0.

We say that the Douady–Earle extension $DE(f)$ is *normalized* if $DE(f)(0) = 0$. Let $DE(f)$ be such a normalized extension. In the following, we give the expressions for the Jacobian and the Beltrami coefficient of $DE(f)$ at 0. Let

$$c_1 = \frac{\partial F}{\partial z}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \overline{\xi} f(\xi)\,|d\xi|, \quad c_{-1} = \frac{\partial F}{\partial \overline{z}}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \xi f(\xi)\,|d\xi| \tag{3.10}$$

and

$$d_1 = \frac{\partial F}{\partial w}(0,0) = -1, \quad d_{-1} = \frac{\partial F}{\partial \overline{w}}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(\xi)^2\,|d\xi|. \tag{3.11}$$

Then

$$\frac{\partial w}{\partial \overline{z}}(0) = -\frac{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial \overline{z}}(0,0) - \frac{\partial F}{\partial \overline{w}}(0,0)\overline{\frac{\partial F}{\partial z}(0,0)}}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \overline{w}}(0,0)|^2} \tag{3.12}$$

and

$$\frac{\partial w}{\partial z}(0) = -\frac{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial z}(0,0) - \frac{\partial F}{\partial \overline{w}}(0,0)\overline{\frac{\partial F}{\partial \overline{z}}(0,0)}}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \overline{w}}(0,0)|^2}. \tag{3.13}$$

The Jacobian of $DE(f)$ at 0 is equal to

$$\left|\frac{\partial w}{\partial z}(0)\right|^2 - \left|\frac{\partial w}{\partial \overline{z}}(0)\right|^2 = \frac{|\frac{\partial F}{\partial z}(0,0)|^2 - |\frac{\partial F}{\partial \overline{z}}(0,0)|^2}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \overline{w}}(0,0)|^2} = \frac{|c_1|^2 - |c_{-1}|^2}{|d_1|^2 - |d_{-1}|^2}. \tag{3.14}$$

Let us present here how it is shown in [8] that the Jacobian is positive. Take $h\colon \mathbb{R} \to \mathbb{R}$ be a lifting of f to the real line $\mathbb{R}$; that is, $f(e^{is}) = e^{ih(s)}$ for any $s \in \mathbb{R}$, where $h(s + 2\pi) = h(s) + 2\pi$. Then

$$\begin{aligned}
|d_{-1}|^2 &= \Big(\frac{1}{2\pi}\Big)^2 \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} f(z)^2\overline{f(\xi)}^2\,|dz|\,|d\xi| \\
&= \Big(\frac{1}{2\pi}\Big)^2 \int_0^{2\pi}\int_0^{2\pi} e^{2ih(s)}e^{-2ih(t)}\,ds\,dt \\
&= \Big(\frac{1}{2\pi}\Big)^2 \int_0^{2\pi}\int_0^{2\pi} \cos 2(h(s) - h(t))\,ds\,dt,
\end{aligned}$$

and hence one can rewrite

$$|d_1|^2 - |d_{-1}|^2 = 2\Big(\frac{1}{2\pi}\Big)^2 \int_0^{2\pi}\int_0^{2\pi} \sin^2(h(s) - h(t))\,ds\,dt. \tag{3.15}$$

It follows that $|d_1|^2 - |d_{-1}|^2 > 0$.

In [8], Douady and Earle expressed $|c_1|^2 - |c_{-1}|^2$ as

$$|c_1|^2 - |c_{-1}|^2 = \Big(\frac{1}{2\pi}\Big)^2 \int_{s=0}^{\pi} \sin s \int_{t=0}^{2\pi} H(t,s)\,dt\,ds \tag{3.16}$$

with

$$\begin{aligned}
H(t,s) &= \sin(h(t+s) - h(t)) + \sin(h(t+2\pi) - h(t+s+\pi)) \\
&\quad + \sin(h(t+\pi+s) - h(t+\pi)) + \sin(h(t+\pi) - h(t+s)).
\end{aligned}$$

Let

$$\begin{aligned}
\alpha_1 &= h(t+s) - h(t), \quad \alpha_2 = h(t+2\pi) - h(t+s+\pi), \\
\alpha_3 &= h(t+\pi+s) - h(t+\pi), \quad \alpha_4 = h(t+\pi) - h(t+s).
\end{aligned}$$

Clearly, all α_j's are non-negative and their sum is

$$\sum \alpha_j = h(t+2\pi) - h(t) = 2\pi.$$

Properties	References
$DE(g_2 \circ f \circ g_1) = g_2 \circ DE(f) \circ g_1$	[8]
$DE(f)$ is quasiconformal if f admits a quasiconformal extension	[8]
$DE(f)$ is quasiconformal iff f is quasisymmetric	[8] and [4] or [29]
$\ln K(DE(f)) \leq C\|f\|_{cr}$ for a universal constant C	[29]
$DE(f)$ is asymptotically conformal if f admits an asymptotically conformal extension	[10]
$DE(f)$ is asymptotically conformal iff f is symmetric	[10] and [21] or [30]
$\mu_{DE(f)}(z) = O((1-\|z\|)^\alpha)$ iff f is a $C^{1+\alpha}$ diffeomorphism for each $0<\alpha<1$	[37]

Table 4. Relationships among various regularities of f and $DE(f)$.

By applying the summation formula in trigonometry, it is obtained in [1] that

$$\sum_{j=1}^{4} \sin \alpha_j = 4 \sin \frac{\alpha_1+\alpha_2}{2} \sin \frac{\alpha_1+\alpha_3}{2} \sin \frac{\alpha_2+\alpha_3}{2}. \tag{3.17}$$

Now we can see $|c_1|^2 - |c_{-1}|^2 > 0$ since $H(t,s) \geq 0$ for all t and s and is not identically equal to 0. Therefore, the Jacobian of $DE(f)$ at the origin is positive. By the conformal naturality, the Jacobian of $DE(f)$ is positive at every point $z \in \mathbb{D}$, which implies that $DE(f)$ is an orientation-preserving homeomorphism of $\mathbb{D}$ ([8]).

The Beltrami coefficient of $DE(f)$ at 0 is expressed by

$$\mu_{DE(f)}(0) = \frac{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial z}(0,0) - \frac{\partial F}{\partial \bar{w}}(0,0)\overline{\frac{\partial F}{\partial \bar{z}}(0,0)}}{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial \bar{z}}(0,0) - \frac{\partial F}{\partial \bar{w}}(0,0)\overline{\frac{\partial F}{\partial z}(0,0)}}. \tag{3.18}$$

This expression is used in [29,30,32,36,37] to estimate the Beltrami coefficient of $DE(f)$ in several different situations. Though the above expression of $\mu_{DE(f)}(0)$ and the expressions of involved partial derivatives of F at $(0,0)$, one may have some sense on the complexity in such processes. In the following, we summarize the features of the Douady–Earle extension $DE(f)$ that characterize a circle homeomorphism in three different types of regularities. We refer the proofs to the corresponding references.

Let f be an orientation preserving homeomorphism of $\mathbb{S}^1$ and let g_1 and g_2 be two Möbius transformations preserving $\mathbb{D}$. Denote by $\|f\|_{cr}$ the cross-ratio distortion of f, $K(DE(f))$ the maximal dilatation of $DE(f)$ on $\mathbb{D}$, and $\mu_{DE(f)}$ the Beltrami coefficient of $DE(f)$. Let $0<\alpha<1$. Relationships among the features of $DE(f)$ and the regularities of f in the three different types are given in Table 4.

Remark 1. Douady–Earle extension $DE(f)$ of an orientation-preserving homeomorphism f of $\mathbb{S}^1$ has also been applied to study the contraction property of other sub-Teichmüller spaces of the universal Teichmüller space, such as the universal Weil–Petersson Teichmüller space, the universal L^p ($p \geq 2$) Teichmüller space, and the so-called VMOA Teichmüller space (for examples, see [5, 14, 45]).

3.3. Background for infinitesimal conformally natural extension

Infinitesimal conformally natural extension introduces a linear operator that extends continuous tangent vector fields along the unit circle $\mathbb{S}^1$ to continuous tangent vector fields on the closed unit disk $\overline{\mathbb{D}}$ in a conformally natural way. Earle studied whether or not such a linear operator is unique in [9]. Later, an integral operator that extends continuous vector fields along $\mathbb{S}^1$ to continuous vector fields on $\overline{\mathbb{D}}$ was studied in [41]. A few years ago, I started a project to develop the properties of infinitesimal conformally natural extensions that are analogous to those already found for Douady–Earle extensions. My collaborator Jinhua Fan directed our attention to an integral operator studied in [41]. Results obtained in our collaboration on this project have just been published in [15], which is quite detailed. So I just introduce here a very brief background on this infinitesimal extension.

Let $C^0(\mathbb{S}^1, \mathbb{C})$ be the collection of all continuous maps from $\mathbb{S}^1$ to the complex plane $\mathbb{C}$. Given an element $V \in C^0(\mathbb{S}^1, \mathbb{C})$ and any $z \in \mathbb{D}$, $L_0(V)(z)$ is defined as

$$L_0(V)(z) = \frac{(1-|z|^2)^3}{2\pi i} \int_{\mathbb{S}^1} \frac{V(\zeta)}{(1-\overline{z}\zeta)^3(\zeta - z)}\, d\zeta. \tag{3.19}$$

It is proved in [15] that the operator L_0 is conformally natural in the following sense:

(1) If $V \in C^0(\mathbb{S}^1, \mathbb{C})$ has a continuous extension H to $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$, then

$$L_0(V) = H.$$

(2) For any element g in Möb($\mathbb{S}^1$) and any $V \in C^0(\mathbb{S}^1, \mathbb{C})$,

$$L_0(g_* V) = g_*(L_0(V)). \tag{3.20}$$

As a consequence of the main result of [9], any linear operator from $C^0(\mathbb{S}^1, \mathbb{C})$ to $C^0(\mathbb{D}, \mathbb{C})$ satisfying the above condition (2) is equal to L_0 up to multiplication by a constant. Furthermore, if such a linear operator is required to either satisfy the above condition (1) or extend the elements of $C^0(\mathbb{S}^1, \mathbb{C})$ to the elements of $C^0(\overline{\mathbb{D}}, \mathbb{C})$ with the given boundary maps, then it is equal to L_0. In the following, we recapitulate how it is explained in [9] that this extension operator L_0 is viewed as an infinitesimal version of the Douady–Earle extension operator.

Let f_t be a smooth curve of diffeomorphisms of $\mathbb{S}^1$ with

$$f_t(\zeta) = \zeta + tV(\zeta) + o(t), \tag{3.21}$$

Properties	References
$L_0(g_*(V)) = g_*(L_0(V))$	[15]
Uniqueness of the operator L_0	[9]
$\|\bar{\partial} L_0(V)\|_\infty$ is finite iff V is Zygmund bounded	[41]
$\frac{1}{C}\|V\|_{cr} \leq \|\bar{\partial} L_0(V)\|_\infty \leq C\|V\|_{cr}$ for a universal constant C	[15]
$\bar{\partial} L_0(V)$ is uniformly vanishing near boundary iff V satisfies the little Zygmund bounded condition	[15]
$\bar{\partial} L_0(V)(z) = O((1-\|z\|)^\alpha)$ iff V is $C^{1+\alpha}$-smooth for each $0 < \alpha < 1$	[15]

Table 5. Relationships among various regularities of V and $L_0(V)$.

where $\zeta \in \mathbb{S}^1$, t is a real parameter, and V is a smooth tangent vector field along $\mathbb{S}^1$. It is proved in [9, Theorem 2] that

$$\frac{dDE(f_t)}{dt}|_{t=0} = L_0(V). \tag{3.22}$$

This means that acting on the smooth tangent vector fields V along $\mathbb{S}^1$, the operator L_0 is the derivative of the Douady–Earle extension operator at the identity map. Therefore, L_0 is called the infinitesimal version of the Douady–Earle extension operator. Here an interesting question arises. Suppose V is a continuous tangent vector field along $\mathbb{S}^1$ and a smooth curve f_t of homeomorphisms of $\mathbb{S}^1$ satisfies (3.21). What regularities (weaker than smoothness) on V are sufficient for the equality (3.22) to hold?

Let us finish this subsection by pointing out that a conformally natural extension operator L_0 defined for vector fields in higher-dimensional spaces is studied by McMullen in [38, Appendix B], where $L_0(V)$ is called the visual extension of a vector field V.

3.4. Characterizations of vector fields of different regularities through infinitesimal Douady–Earle extension

Let V be a continuous tangent vector field along the unit circle $\mathbb{S}^1$ and $L_0(V)$ be the infinitesimal conformally natural extension of V. Denote by $\|V\|_{cr}$ the cross-ratio distortion norm of V and by $\|\bar{\partial} L_0(V)\|_\infty$ the L^∞-norm of the $\bar{\partial}$-derivative of $L_0(V)$ on $\mathbb{D}$. The characterizations of the tangent vector fields V along $\mathbb{S}^1$ in the three different classes of regularities in terms of conditions on $\bar{\partial} L_0(V)$ are summarized in Table 5.

4. Thurston's earthquake map and infinitesimal earthquake representation

4.1. Background for Thurston's earthquake map

Consider the open unit disk $\mathbb{D}$ as a model for the hyperbolic plane. A *geodesic lamination* $\mathcal{L}$ in the hyperbolic plane $\mathbb{D}$ is a collection of geodesics which foliate a closed subset L of $\mathbb{D}$. The set L is called *the locus* of $\mathcal{L}$, the geodesics are called the *leaves* of $\mathcal{L}$, the connected components of $\mathbb{D} \setminus L$ are called the *gaps*, and the gaps and the leaves of $\mathcal{L}$ are called the *strata* of the lamination.

Definition 5. Let $\mathcal{L}$ be a geodesic lamination in $\mathbb{D}$. By an *$\mathcal{L}$-left earthquake map E* we mean that E is an injective and surjective (and often discontinuous) map from $\mathbb{D}$ to $\mathbb{D}$ satisfying:

(i) for any stratum A, the restriction of E on A is the restriction on A of a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{D}$;

(ii) for any two strata A and B, the *comparison map*

$$\mathrm{comp}(A, B) = (E|_A)^{-1} \circ E|_B : \mathbb{D} \to \mathbb{D}$$

is a hyperbolic transformation whose axis weakly separates A and B and which translates to the left as viewed from A. Here $E|_A$ and $E|_B$ denote the Möbius transformations representing E on A and B, and we say that a line l *weakly separates* two sets A and B if any path connecting a point $a \in A$ to a point $b \in B$ intersects l.

Thurston [46] showed that each left earthquake map $(E, \mathcal{L})$ extends uniquely to a map $\widetilde{E}$ defined on $\mathbb{D} \cup \mathbb{S}^1$. The extension is continuous at each point $x \in \mathbb{S}^1$, and the restriction of $\widetilde{E}$ to $\mathbb{S}^1$ is a homeomorphism. Conversely, every circle homeomorphism h can be realized in this way.

Theorem 4 (Thurston's earthquake theorem). *Let f be an orientation preserving homeomorphism of the unit circle $\mathbb{S}^1$. There exists a left earthquake map $(E, \mathcal{L})$ such that $\widetilde{E}|_{\mathbb{S}^1} = f$, and the lamination is uniquely determined by f. Moreover, f determines the isometries of E on all gaps, and for any leaf $l \in \mathcal{L}$, two possibly different isometries on l only differ by a hyperbolic isometry with axis l and translation length between 0 and the limit value of the translation lengths of the comparison maps for E on the two sides of l.*

We call $(E, \mathcal{L})$ a left earthquake representation of f. In brief, we call it an *earthquake map* of f and denote it by E_f.

Each earthquake map naturally introduces a transversal shearing measure supported on the lamination associated with the earthquake map, which quantifies the amount of shearing along the geodesics in the lamination. Given an earthquake map $(E, \mathcal{L})$ and two geodesic lines l_* and l^* in $\mathcal{L}$, let β be a closed geodesic segment which is transversal to both l_* and l^* and intersects them at its endpoints. Note that each geodesic line in $\mathcal{L}$

either intersects β once or not at all. Thus β is transversal to the lamination $\mathcal{L}$. The amount $\nu(\beta)$ of *relative transversal shearing* of the earthquake map $(E, \mathcal{L})$ along β is defined as follows. Let $P = \{I_i\}_{i=1}^n$ be a partition of β into small geodesic segments, and T_i the comparison map between the strata containing the endpoints of the segment I_i. The *translation length*, denoted by $\tau(T_i)$, of each hyperbolic Möbius transformation $T_i\colon \mathbb{H} \to \mathbb{H}$ can be defined as the logarithm of the derivative of T_i at its expanding fixed point. Let $\nu(P) = \sum_{i=1}^n \tau(T_i)$. Now we define

$$\nu(\beta) = \inf_P \nu(P). \tag{4.1}$$

Note that ν is an intermediate quantity that can be conveniently defined for any closed geodesic segment β transversal to the lamination $\mathcal{L}$, but it does not suffice to define a Borel measure on $\mathcal{L}$ yet. In the following, we first use ν to define the earthquake measure $\sigma(\beta)$ of β; then we show how $\nu(\beta)$ can be well approximated by the sum of the translation lengths of comparison maps; and finally we show that σ and ν have norms commensurable to each other.

Definition 6 (Earthquake measure induced by earthquake map). The *earthquake measure* $\sigma(\beta)$ of β, induced by the earthquake map $(E, \mathcal{L})$, is defined to be

$$\sigma(\beta) = \inf_{\beta'} \nu(\beta'), \tag{4.2}$$

where β' is a closed geodesic segment containing β in its interior. Then σ naturally extends to a Borel measure on the space $\mathbb{X}$ with support consisting of all pairs of the endpoints of the leaves in $\mathcal{L}$, where $\mathbb{X}$ denotes the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\text{the diagonal}\}$ factorized by the equivalence relation $(a, b) \sim (b, a)$. We call σ the earthquake measure induced by $(E, \mathcal{L})$ and denote it by $\sigma(E)$.

As pointed out by Thurston in [46], $\nu(\beta)$ can be well approximated by the sum of the translation lengths of comparison maps. In fact, the notion of earthquake measure can be introduced more generally as follows.

Definition 7 (Earthquake measure). Let $\mathbb{X}$ denote the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\text{the diagonal}\}$ factorized by the equivalence relation $(a, b) \sim (b, a)$. Let $\mathcal{L}$ be a lamination on the hyperbolic plane. By *an earthquake measure* supported on $\mathcal{L}$ we mean a Borel measure on $\mathbb{X}$ supported on the set of the pairs consisting of two endpoints of geodesics in $\mathcal{L}$.

A nice example that helps to understand an earthquake map $(E, \mathcal{L})$ and its induced earthquake measure $\sigma(E)$ is the so-called finite earthquake theorem.

Theorem 5 (Finite earthquake theorem). *Assume that $S = \{p_1, \ldots, p_{n+3}\}$ and $S' = \{p'_1, \ldots, p'_{n+3}\}$ are two sets consisting of same number of points on $\mathbb{S}^1$ arranged in counter-clockwise cyclic order, and f is a bijection from S to S' with $f(p_j) = p'_j$. Then there exists a unique allowable lamination $\mathcal{L}$ consisting of finitely many non-intersecting hyperbolic geodesics connecting points in S and a unique measure σ supported on $\mathcal{L}$ such* *that, up to post-composition by a conformal homeomorphism of* $\mathbb{D}$*, the left earthquake map* E_σ *maps* p_j *to* p'_j*. The measure* σ *and its corresponding lamination are uniquely determined by the locations of the points of* S *and* S'*.*

There are at least two methods to prove the finite earthquake theorem. A proof by induction is developed in [20], in which a recursive process is developed to find the lamination and the weights that are assigned to its lines. Another straightforward proof is given in [26] by applying Thurston's original idea in [46] to search for extremal left homeomorphisms in the coset $\mathrm{PSL}(2,\mathbb{R})\circ f$, where $\mathrm{PSL}(2,\mathbb{R})$ represents the group of all hyperbolic isometries of the hyperbolic plane.

The following concept of an earthquake measure plays an analogous role for earthquake maps as the L^∞-norm of a Beltrami coefficient does for quasiconformal mappings.

Definition 8. The Thurston norm of an earthquake measure $(\sigma, \mathcal{L})$ is defined to be

$$\|\sigma\|_{Th} = \sup_{l(\beta)\leq 1} \sigma(\beta) = \sup_{l(\beta)=1} \sigma(\beta),$$

where β is a closed geodesic segment transversal to the lamination $\mathcal{L}$ and $l(\beta)$ denotes the hyperbolic length of β. If $\|\sigma\|_{Th}$ is finite, then we say that $(\sigma, \mathcal{L})$ is Thurston bounded.

Theorem 6 (Thurston). *If an earthquake measure* $(\sigma, \mathcal{L})$ *is Thurston bounded, then there exists an earthquake map* $(E, \mathcal{L})$ *such that* σ *is the induced earthquake measure by* E*. Moreover, up to post-composition by a Möbius transformation,* σ *determines the isometries of* E *on all gaps, and for any leaf* $l \in \mathcal{L}$*, two possibly different isometries on* l *only differ by a hyperbolic isometry with axis* l *and translation length between* 0 *and the measure* $\sigma(l)$ *of* l*.*

Remark 2. In [33], a sufficient condition is introduced for a type of Thurston unbounded earthquake measures to be induced by earthquake maps, which is an analogue of David's theorem ([6]) for solutions of Beltrami differential equation ([3]) in earthquake theory.

The first reference for background on earthquake maps is the paper [46] by Thurston. For a self-contained introduction on earthquake maps and interplay between earthquake maps and earthquake measures through approximations by finite earthquake maps, we refer to [19] or [26].

4.2. Characterizations of circle homeomorphisms of different regularities through earthquake measure

Let f be an orientation preserving homeomorphism of $\mathbb{S}^1$ and let g_1 and g_2 be two Möbius transformations preserving $\mathbb{D}$. Denote by $\|f\|_{cr}$ the cross-ratio distortion of f, $\|\sigma(E_f)\|_{Th}$ the Thurston norm of the earthquake measure $\sigma(E_f)$ induced by an earthquake map E_f of f. Let D be a disk on the hyperbolic plane of diameter 1 in the hyperbolic metric and $\delta(D)$ be the Euclidean distance from D to the boundary circle of the hyper-

Properties	References
$E_{g_2 \circ f \circ g_1} = g_2 \circ E_f \circ g_1$	[46]
$\sigma(E_f)$ is Thurston bounded iff f is quasisymmetric	[46]
$\frac{1}{C}\|f\|_{cr} \leq \|\sigma(E_f)\|_{Th} \leq C\|f\|_{cr}$ for a universal constant $C > 0$	[23]
$\sigma(E_f)$ is asymptotically vanishing iff f is symmetric	[19] or [23]
$\sigma(E_f) = O(\delta(D)^\alpha)$ iff f is a $C^{1+\alpha}$ diffeomorphism for each $0 < \alpha < 1$	[19] or [23]

Table 6. Relationships among various regularities of f and $\sigma(E_f)$.

bolic plane. Finally, we let $\sigma(D)$ denote the measure of the leaves in the lamination of E_f that intersect D.

Relationships among various regularities of f and $\sigma(E_f)$ have been studied in [19, 23, 25, 46], and other papers. The characterization of the Teichmüller topology on $T(D)$ in terms of a topology on the space $\mathcal{ML}_b(\mathbb{D})$ of Thurston bounded measured geodesic laminations on $\mathbb{D}$ is developed in [40]. Furthermore, the Teichmüller topology on the asymptotic Teichmüller space $AT(\mathbb{D}) = T(\mathbb{D})/T_0(\mathbb{D})$ is characterized in [13] by a topology on a quotient space $\mathcal{AML}_b(\mathbb{D})$ of $\mathcal{ML}_b(\mathbb{D})$ under an equivalence relation. The relevant results to this paper are summarized in Table 6.

4.3. Background for infinitesimal earthquake representation

Let $(\sigma, \mathcal{L})$ be a Thurston bounded earthquake measure. For each $t \geq 0$, there exists an earthquake map E_t inducing $t\sigma$. Take a stratum S and normalize E_t by requiring E_t to be the identity map on S. Then the restriction f_t of E_t on the boundary circle gives a continuous curve of homeomorphisms of the circle, which is called an earthquake curve determined by σ or $t\sigma$. The curve $f_t(x), t \geq 0$, is differentiable in t for each point x on the boundary circle and satisfies a non-autonomous ordinary differential equation ([19]). The holomorphic dependence of $f_t(x)$ on t for each fixed x is developed in [43], and complex earthquakes and applications to Teichmüller theory and to the deformation theory of the unit disk can be found in [39] and [11]. In this survey we focus on the real theory of earthquakes. The following theorem is proved in [19].

Theorem 7 ([19]). *Let $(\sigma, \mathcal{L})$ be a Thurston bounded earthquake measure and f_t, $t \geq 0$, be an earthquake curve determined by σ. Then for each $x \in \mathbb{S}^1$, $f_t(x)$ is differentiable in t for all $t \geq 0$. Moreover, let us take the upper-half plane model of the hyperbolic plane and assume that the upper-half imaginary axis is contained in a stratum determined by $\mathcal{L}$, then*

$$\frac{d}{dt} f_t(x) = V_t(f_t(x)), \tag{4.3}$$

where V_t is the vector field given by

$$V_t(x) = \int\int E_{f_t(a)f_t(b)}(x)\, d\sigma(a,b) + \text{a quadratic polynomial}, \tag{4.4}$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ *if* $x \in [a,b]$, *and otherwise* $E_{ab}(x) = 0$; *and in particular,*

$$V_0(x) = \int\int E_{ab}(x)\, d\sigma(a,b) + \text{a quadratic polynomial}. \tag{4.5}$$

Furthermore, if the upper-half imaginary axis happens to be contained in the stratum which normalizes the earthquake curve f_t, *then*

$$V_t(x) = \int\int E_{f_t(a)f_t(b)}(x)\, d\sigma(a,b); \tag{4.6}$$

and in particular,

$$V_0(x) = \int\int E_{ab}(x)\, d\sigma(a,b). \tag{4.7}$$

The above expression (4.7) reflects the idea for infinitesimal earthquake theory, which is developed in [17].

Theorem 8 ([17]). *For any Zygmund bounded function* $V:\mathbb{S}^1 \to \mathbb{C}$, *there exists a Thurston bounded earthquake measure* $(\sigma, \mathcal{L})$ *such that*

$$V(x) = \pi \int\int_{\mathcal{L}} E_{ab}(x)\, d\sigma(a,b) \text{ modulo a quadratic polynomial}; \tag{4.8}$$

and furthermore, if two functions V *differ by a quadratic polynomial then the corresponding measures* σ *are the same.*

We simply call $(\sigma, \mathcal{L})$ the infinitesimal earthquake representation of V and denote it by $\sigma(V)$.

The strategy, used in [17], to prove Theorem 8 is to develop first a finite version of this theorem and then take the limit of a sequence of finite approximations. Included in [18], one may find an example that shows how to find the leaves and weights of $(\sigma, \mathcal{L})$ used in the integral expression (4.8) for a vector field V defined on a finite subset A of $\mathbb{S}^1$.

4.4. Characterizations of vector fields of different regularities through infinitesimal earthquake representation

Let V be a continuous vector field along the unit circle $\mathbb{S}^1$ and $(\sigma(V), \mathcal{L}(V))$ be the infinitesimal earthquake representation of V. Denote by $\|V\|_{cr}$ the cross-ratio distortion norm of V and $\|\sigma(V)\|_{Th}$ the Thurston norm of $\sigma(V)$. Let D be a disk on the hyperbolic plane of diameter 1 in the hyperbolic metric and $\delta(D)$ be the Euclidean distance from D to the boundary circle of the hyperbolic plane. Finally, we let $\sigma(D)$ denote the measure of the leaves in the lamination of $\mathcal{L}(V)$ that intersect D. The characterizations of vector fields V along $\mathbb{S}^1$ with different types of regularities in terms of $\sigma(V)$ are summarized in Table 7.

Properties	References
$\sigma(g_*(V)) = g_*(\sigma(V))$	[20]
Uniqueness of $(\sigma(V), \mathcal{L}(V))$	[17]
$\sigma(V)$ is Thurston bounded iff V is Zygmund bounded	[17]
$\frac{1}{C}\|V\|_{cr} \leq \|\sigma(V)\|_{Th} \leq C\|V\|_{cr}$ for a universal constant $C > 0$	[24] or [25]
$\sigma(V)$ is uniformly vanishing near boundary iff V satisfies the little Zygmund bounded condition	[19]
$\sigma(D) = O(\delta(D)^{\alpha})$ iff V is $C^{1+\alpha}$-smooth for each $0 < \alpha < 1$	[19]

Table 7. Relationships among various regularities of V and $\sigma(V)$.

5. Open problems on characterizing a $C^{1+\text{Zygmund}}$ circle homeomorphism through three extensions

Let us first define three types of diffeomorphisms.

Definition 9. Let f be an orientation-preserving diffeomorphism of the unit circle $\mathbb{S}^1$ or the real line $\mathbb{R}$. We define three types of smoothness on f as follows:

(1) $f \in C^{1+\text{Zygmund}}$ if f' is Zygmund bounded in the sense of (1.3);

(2) $f \in C^{1+\text{bounded variation}}$ if f' is of bounded variation;

(3) $f \in C^{1+\text{Lipschitz}}$ if f' satisfies the Lipschitz condition.

In this last section, we first investigate the regularity, near the boundary of the upper half plane, of the Beurling–Ahlfors extension $BA(h)$ of a $C^{1+\text{Zygmund}}$ orientation-preserving diffeomorphism h of the real line. We first show the following theorem.

Theorem 9. *Assume that h is a $C^{1+\text{Zygmund}}$ orientation-preserving diffeomorphism of the real line $\mathbb{R}$ with $\frac{1}{M} < h'(x) < M$ for all $x \in \mathbb{R}$ and some positive constant M. Then the Beltrami coefficient $\mu(BA(h))(x+iy)$ vanishes as $O(y)$ uniformly on x near the boundary of the upper half plane if and only if h is $C^{1+\text{Lipschitz}}$.*

Then we show that the criterion given in Theorem 9 can also be achieved when h is started with any homeomorphism of the real line that is a lifting map of an orientation-preserving circle homeomorphism. See Theorem 10. Finally, we raise three open questions on how to characterize a $C^{1+\text{Zygmund}}$ circle diffeomorphism in terms of vanishing condition on the Beltrami coefficient of the Beurling–Ahlfors or Douady–Earle extension or on the earthquake measure of Thurston's earthquake representation.

Before we prove Theorem 9, let us recall a proposition given by Sullivan.

Proposition 1 ([44, p. 427]). *If a real-valued function ϕ on an interval J is Zygmund bounded, then the average of ϕ on J differs from the average of the values of ϕ at the endpoints of J by $O(|J|)$, where $|J|$ represents the length of J.*

Proof of Theorem 9. Given a point $z = x + iy \in \mathbb{U}$, $BA(h)(z) = u(x, y) + iv(x, y)$ is defined by (2.6). Then the Beltrami coefficient $\mu_{BA(h)}(z)$ of $BA(h)(z)$ is expressed by

$$\mu_{BA(h)}(z) = \frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)}, \tag{5.1}$$

where

$$u_x = \frac{1}{2y}\big(h(x + y) - h(x - y)\big), \tag{5.2}$$

$$u_y = \frac{1}{y}\left[-\frac{1}{2y}\int_{x-y}^{x+y} h(t)\,dt + \frac{1}{2}\big(h(x + y) + h(x - y)\big)\right], \tag{5.3}$$

$$v_x = \frac{1}{y}\big[h(x + y) - 2h(x) + h(x - y)\big], \tag{5.4}$$

$$v_y = -\frac{1}{y^2}\left(\int_{x}^{x+y} h(t)\,dt - \int_{x-y}^{x} h(t)\,dt\right) + \frac{1}{y}\big(h(x + y) - h(x - y)\big). \tag{5.5}$$

Since h' is Zygmund bounded, it follows from Proposition 1 that

$$\frac{h(x + t) - h(x)}{t} - \frac{h'(x + t) + h'(x)}{2} = O(t).$$

Then

$$h(x + t) = h(x) + \frac{h'(x + t) + h'(x)}{2}t + O(t^2).$$

Thus,

$$\begin{aligned}\int_0^y h(x + t)\,dt &= h(x)y + \int_0^y \frac{h'(x + t) + h'(x)}{2}t\,dt + O(y^3)\\ &= h(x)y + \frac{h'(x)}{4}y^2 + \frac{1}{2}\int_0^y h'(x + t)t\,dt + O(y^3).\end{aligned}$$

Clearly,

$$\begin{aligned}\int_0^y h'(x + t)t\,dt &= \int_0^y t\,dh(x + t) = th(x + t)|_{t=0}^{y} - \int_0^y h(x + t)\,dt\\ &= yh(x + y) - \int_0^y h(x + t)\,dt.\end{aligned}$$

Thus,

$$\int_0^y h(x + t)\,dt = h(x)y + \frac{h'(x)}{4}y^2 + \frac{1}{2}yh(x + y) - \frac{1}{2}\int_0^y h(x + t)\,dt + O(y^3).$$

Therefore,

$$\int_0^y h(x+t)\,dt = \frac{2}{3}h(x)y + \frac{h'(x)}{6}y^2 + \frac{1}{3}yh(x+y) + O(y^3). \tag{5.6}$$

Similarly, we obtain

$$\int_0^y h(x-t)\,dt = \frac{2}{3}h(x)y - \frac{h'(x)}{6}y^2 + \frac{1}{3}yh(x-y) + O(y^3). \tag{5.7}$$

From (5.6) and (5.7), we obtain

$$\begin{aligned}\int_{x-y}^{x+y} h(t)\,dt &= \int_0^y h(x+t)\,dt + \int_0^y h(x-t)\,dt \\ &= \frac{4}{3}h(x)y + \frac{1}{3}y\big(h(x-y)+h(x+y)\big) + O(y^3)\end{aligned}$$

and

$$\begin{aligned}\int_x^{x+y} h(t)\,dt - \int_{x-y}^{x} h(t)\,dt &= \int_0^y h(x+t)\,dt - \int_0^y h(x-t)\,dt \\ &= \frac{h'(x)}{3}y^2 + \frac{1}{3}y\big(h(x+y)-h(x-y)\big) + O(y^3).\end{aligned}$$

Then from the expressions of the partial derivatives of u and v in (5.3)–(5.5), we first obtain

$$u_y = \frac{2}{3y}\left(\frac{h(x+y)+h(x-y)}{2} - h(x)\right) + O(y)$$

and

$$v_y = -\frac{h'(x)}{3} + \frac{2}{3y}\Big(h(x+y)-h(x-y)\Big) + O(y).$$

Thus,

$$\begin{aligned}u_x - v_y &= \frac{h'(x)}{3} - \frac{1}{6y}\big(h(x+y)-h(x-y)\big) + O(y) \\ &= \frac{h'(x)}{3} - \frac{1}{6y}\int_{-y}^{y} h'(x+t)\,dt + O(y) \\ &= \frac{h'(x)}{3} - \frac{1}{6y}\int_0^y h'(x+t)+h'(x-t)\,dt + O(y) \\ &= -\frac{1}{3y}\int_0^y \left[\frac{h'(x+t)+h'(x-t)}{2} - h'(x)\right]dt + O(y) = O(y), \\ v_x + u_y &= \frac{5}{3y}\left(\frac{h(x+y)+h(x-y)}{2} - h(x)\right) + O(y),\end{aligned}$$

$$
\begin{aligned}
u_x + v_y &= -\frac{h'(x)}{3} + \frac{7}{6y}\big(h(x+y) - h(x-y)\big) + O(y) \\
&= -\frac{h'(x)}{3} + \frac{7}{3y}\int_0^y \frac{h'(x+t) + h'(x-t)}{2}\,dt + O(y) \\
&= 2h'(x) + \frac{7}{3y}\int_0^y \left[\frac{h'(x+t) + h'(x-t)}{2} - h'(x)\right]dt + O(y) \\
&= 2h'(x) + O(y), \\
v_x - u_y &= -\frac{1}{3y}\left(\frac{h(x+y) + h(x-y)}{2} - h(x)\right) + O(y).
\end{aligned}
$$

Now using the expression (5.1) for the Beltrami coefficient $\mu_{BA(h)}(z)$, we can see that $\mu_{BA(h)}(x+iy) = O(y)$ if and only if

$$
\frac{h(x+y) + h(x-y)}{2} - h(x) = O(y^2)
$$

uniformly on x. It suffices to prove that $\frac{h(x+y)+h(x-y)}{2} - h(x) = O(y^2)$ uniformly on x if and only if h' is Lipschitz. Clearly, if h' is Lipschitz, then $\frac{h(x+y)+h(x-y)}{2} - h(x) = O(y^2)$ uniformly on x. It remains to show that $\frac{h(x+y)+h(x-y)}{2} - h(x) = O(y^2)$ uniformly on x implies h' is Lipschitz.

Since h' is Zygmund bounded, it follows from Proposition 1 that

$$
\frac{1}{y}\int_x^y h'(t)\,dt - \frac{h'(x) + h'(x+y)}{2} = O(y),
$$

which means

$$
h(x+y) - h(x) = \frac{h'(x) + h'(x+y)}{2} + O(y).
$$

Similarly,

$$
h(x) - h(x-y) = \frac{h'(x-y) + h'(x)}{2} + O(y).
$$

Then

$$
h(x+y) + h(x-y) - 2h(x) = \frac{h'(x+y) - h'(x-y)}{2} + O(y).
$$

Thus,

$$
\frac{h'(x+y) - h'(x-y)}{2} = O(y) + O(y^2),
$$

which implies that h' is Lipschitz. We complete the proof. ∎

Corollary 1. *Let f be a $C^{1+\text{Zygmund}}$ orientation-preserving diffeomorphism of the unit circle $\mathbb{S}^1$ and let $\tilde{f}$ be a lifting map of f to the universal covering space $\mathbb{R}$. Then the Beltrami coefficient $\mu(BA(\tilde{f}))(x+iy)$ vanishes as $O(y)$ uniformly on x near the boundary of the upper half plane if and only if f is $C^{1+\text{Lipschitz}}$.*

Now we are ready to prove the following statement.

Theorem 10. *Let f be an orientation-preserving homeomorphism of the unit circle $\mathbb{S}^1$ and let $\widetilde{f}$ be a lifting map to the universal covering space $\mathbb{R}$. Then the Beltrami coefficient $\mu(BA(\widetilde{f}))(x+iy)$ vanishes as $O(y)$ uniformly on x near the boundary of the upper half plane if and only if $\widetilde{f}$ is $C^{1+\text{Lipschitz}}$.*

Proof. In order to use the same notation as in the proof of Theorem 9, we let $h = \widetilde{f}$. Then

$$h(x+1) = h(x) + 1$$

for any $x \in \mathbb{R}$. Applying the mean value theorem, it is straightforward to prove that if such a homeomorphism h of $\mathbb{R}$ is $C^{1+\text{Lipschitz}}$, then $\mu(BA(h)(x+iy)$ vanishes as $O(y)$ uniformly on x near the boundary of the upper half plane $\mathbb{U}$. It remains to prove the converse.

Assume that $\mu(BA(h)(x+iy)$ vanishes as $O(y)$ uniformly on x near the boundary of $\mathbb{U}$. Then

$$\mu(BA(h)(x+iy) = O(y^{\alpha})$$

for any $0 < \alpha < 1$ and any $0 < y < 1$, where the big O does not depend on x and α. By the third property listed in Table 1, we know that h is $C^{1+\alpha}$ and then f is a $C^{1+\alpha}$ diffeomorphism of $\mathbb{S}^1$. Thus, there exists a constant $M > 0$ such that

$$\frac{1}{M} \leq h'(x) \leq M$$

for any $x \in \mathbb{R}$. In the following, we first apply the condition $\mu(BA(h)(x+iy) = O(y)$ to show that h' is Zygmund bounded.

Denote by $BA(h)(x+iy) = u(x,y) + iv(x,y)$. Using the partial derivatives of u and v given by (5.2)–(5.5), we first obtain

$$u_x - v_y = \frac{1}{y^2}\int_0^y \big(h(x+t) - h(x-y+t)\big)\,dt - \frac{1}{2y}\big(h(x+y) - h(x-y)\big) \quad (5.8)$$

and

$$v_x + u_y = -\frac{1}{2y^2}\int_0^y \big(h(x+t) + h(x-y+t)\big)\,dt + \frac{1}{2y}\big(3h(x+y) - 4h(x) + 3h(x-y)\big). \quad (5.9)$$

We further apply the estimate $h(x+t) = h(x) + h'(x)t + O(|t|^{1+\alpha})$ with $0 < \alpha < 1$ to obtain

$$u_x + v_y = 2h'(x) + O(y^{\alpha}) \quad \text{and} \quad v_x - u_y = O(y^{\alpha}). \quad (5.10)$$

Now from the expression (5.1) for $\mu(BA(h)(z))$ and the estimate (5.10) for the denominator of $\mu(BA(h)(z))$, we can see that $\mu(BA(h)(z)) = O(y)$ implies that $u_x - v_y = O(y)$

and $v_x + u_y = O(y)$. From the expressions (5.8) and (5.9), we obtain

$$\frac{1}{y^2}\int_0^y \big(h(x+t)-h(x-y+t)\big)\,dt - \frac{1}{2y}\big(h(x+y)-h(x-y)\big) = O(y) \qquad (5.11)$$

and

$$-\frac{1}{y^2}\int_0^y \big(h(x+t)+h(x-y+t)\big)\,dt + \frac{2}{2y}\big(3h(x+y)-4h(x)+3h(x-y)\big) = O(y). \qquad (5.12)$$

The summation and subtraction of the above two expressions (5.11) and (5.12) show

$$-\frac{2}{y^2}\int_0^y h(x-y+t)\,dt + \frac{1}{2y}\big(5h(x+y)-8h(x)+7h(x-y)\big) = O(y) \qquad (5.13)$$

and

$$\frac{2}{y^2}\int_0^y h(x+t)\,dt - \frac{1}{2y}\big(7h(x+y)-8h(x)+5h(x-y)\big) = O(y). \qquad (5.14)$$

Replacing x by $x+y$ in the expression (5.13), we obtain

$$-\frac{2}{y^2}\int_0^y h(x+t)\,dt + \frac{1}{2y}\big(5h(x+2y)-8h(x+y)+7h(x)\big) = O(y). \qquad (5.15)$$

Adding (5.14) and (5.15), we see

$$\frac{1}{2y}\big(5h(x+2y)-15h(x+y)+15h(x)-5h(x-y)\big) = O(y),$$

that is,

$$h(x+2y)-3h(x+y)+3h(x)-h(x-y) = O(y^2). \qquad (5.16)$$

From the estimate (5.16), we conclude that h is $C^{1+\text{Zygmund}}$ from the following Proposition 2. Finally, the previous Theorem 9 implies that h is $C^{1+\text{Lipschitz}}$. We complete the proof. ∎

Proposition 2. *Let h be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then h is $C^{1+\text{Zygmund}}$ if and only if for any $x \in \mathbb{R}$ and any $t > 0$,*

$$h(x+2t)-3h(x+t)+3h(x)-h(x-t) = O(t^2). \qquad (5.17)$$

This is a known result. A sketch of the proof follows.

Proof. It is relatively straightforward to see that if h is $C^{1+\text{Zygmund}}$, then h satisfies the condition (5.17). In the following, we outline the proof of the converse.

We first explain why the condition (5.17) implies that h is differentiable. We rewrite the condition (5.17) as

$$\frac{h(a+2t)-h(a-t)}{3t} - \frac{h(a+t)-h(a)}{t} = O(t).$$

This means that the difference quotient on the middle third interval $[a, a+t]$ differs from the difference quotient on the interval $[a-t, a+2t]$ by $O(t)$. Repeating the estimate of the difference quotient on the middle third interval of the middle third interval inductively, one can see that the difference quotient on the smaller and smaller middle third interval has a limit as the middle third interval shrinks to a point b, which shows that h is differentiable at b. By setting up an arbitrary point x of $\mathbb{R}$ as the intersection of a sequence of the nested intervals with next one being the middle third of the one obtained already, we can see a strategy to show that h is differentiable at x. Furthermore, rewrite the condition (5.16) as

$$\left[\frac{h(x+2t)-h(x+t)}{t}-\frac{h(x+t)-h(x)}{t}\right] \\ -\left[\frac{h(x+t)-h(x)}{t}-\frac{h(x)-h(x-t)}{t}\right]=O(t). \tag{5.18}$$

Using the same method to prove that any Zygmund bounded function ϕ is α-Hölder continuous for any $0<\alpha<1$ (see [34]), one can obtain

$$\frac{h(x+t)-h(x)}{t}-\frac{h(x)-h(x-t)}{t}=O(t^{\alpha}). \tag{5.19}$$

Then one can use the condition (5.19) to show that h is differentiable and h' is α-Hölder continuous.

It remains to show that given a differentiable function h from $\mathbb{R}$ to itself, h' is Zygmund bounded if and only if the condition (5.17) holds.

Rewrite

$$\begin{aligned} &h(a+2t)-3h(a+t)+3h(a)-h(a-t)\\ &\quad=h(a+2t)-h(a+t)-2(h(a+t)-h(a))+h(a)-h(a-t)\\ &\quad=\int_{a+t}^{a+2t}h'(s)\,ds-2\int_{a}^{a+t}h'(s)\,ds+\int_{a-t}^{a}h'(s)\,ds\\ &\quad=\int_{0}^{t}\big[h'(a+t+s)-2h'(a+s)+h'(a-t+s)\big]\,ds\\ &\quad=t\big[h'(a+t+\xi_t(a))-2h'(a+\xi_t(a))+h'(a-t+\xi_t(a))\big], \end{aligned}$$

where $0\le\xi_t(a)\le t$. Thus, if h' is Zygmund bounded, then the condition (5.17) is satisfied. Conversely, if the condition (5.17) is satisfied, then we obtain

$$h'(a+t+\xi_t(a))-2h'(a+\xi_t(a))+h'(a-t+\xi_t(a))=O(t). \tag{5.20}$$

By fixing t, we only need to understand why $\xi_t(a)$ can chosen to be a continuous function of a. This is indeed true by letting $\xi_t(a)$ be the smallest input s on the interval $[0, t]$ at which the value of

$$h'(a+t+s)-2h'(a+s)+h'(a-t+s)$$

is equal to its average on $[0, t]$. Then the function $a \mapsto a + \xi_t(a)$ is continuous from $\mathbb{R}$ onto $\mathbb{R}$ since $0 \leq \xi_t(a) \leq t$. This implies that every real number x is a value of the function $a \mapsto a + \xi_t(a)$ at some real number a. Therefore, the condition (5.20) shows that h' is Zygmund bounded. We complete the proof. ∎

From the above Theorem 10, one can see that $\mu(BA(\tilde{f}))(x + iy)$ vanishes as $O(y)$ uniformly on x near the boundary of $\mathbb{U}$ is stronger than the condition that f is $C^{1+\text{Zygmund}}$. As we mentioned in the introduction, a $C^{1+\text{Zygmund}}$ circle diffeomorphism with an irrational rotation number is rigid in terms of topological conjugacy. So it is interesting to characterize a $C^{1+\text{Zygmund}}$ circle diffeomorphism f in term the Beltrami coefficient of the Beurling–Ahlfors extension of a lifting map $\tilde{f}$ of f or the Douady–Earle extension of f, or the earthquake measure of Thurston's earthquake representation of f.

Let f be an orientation-preserving homeomorphism of $\mathbb{S}^1$ and $\tilde{f}$ a lifting map to $\mathbb{R}$. We finish this paper by raising the following three questions.

Question 1. What type of vanishing condition on $\mu_{BA(\tilde{f})}$ near the boundary of $\mathbb{U}$ is necessary and sufficient for f to be $C^{1+\text{Zygmund}}$?

Question 2. What type of vanishing condition on $\mu_{DE(f)}$ near the boundary of $\mathbb{D}$ is necessary and sufficient for f to be $C^{1+\text{Zygmund}}$?

Question 3. What type of vanishing condition on $\sigma(E_f)$ near the boundary of $\mathbb{D}$ is necessary and sufficient for f to be $C^{1+\text{Zygmund}}$?

Some work related to Question 3 appears in [22].

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