BC-system, absolute cyclotomy and the quantized calculus

Alain Connes and Caterina Consani

Abstract. We give a short survey on several developments on the BC-system, the adele class space of the rationals, and on the understanding of the "zeta sector" of the latter space as the Scaling Site. The new result that we present concerns the description of the BC-system as the universal Witt ring (i.e., K-theory of endomorphisms) of the "algebraic closure" of the absolute base $\mathbb{S}$. In this way we attain a conceptual meaning of the BC dynamical system at the most basic algebraic level. Furthermore, we define an invariant of Schwartz kernels in one dimension and relate the Fourier transform (in one dimension) to its role over the algebraic closure of $\mathbb{S}$. We implement this invariant to prove that, when applied to the quantized differential of a function, it provides its Schwarzian derivative. Finally, we survey the roles of the quantized calculus in relation to Weil's positivity, and that of spectral triples in relation to the zeros of the Riemann zeta function.

Dedicated to Dennis Sullivan on his 80th birthday

> *. . . La vraie jeunesse ne s'use pas.*
> *On a beau l'appeler souvenir,*
> *On a beau dire qu'elle disparaît,*
> *On a beau dire . . . que tout s'en va,*
> *Tout ce qui est vrai reste là.*
>
> – J. Prévert

1. Introduction

We dedicate this paper to Dennis Sullivan, whose genuine love for understanding mathematics and his generosity in communicating new ideas has always been an inspirational example to us. We take this opportunity to write an overview on the actual state of our enduring interest in the Riemann zeta function. The Riemann zeta function has many intriguing manifestations in science: our own interest was triggered with the discovery of the surprising relation that this function has with noncommutative geometry.

In quantum statistical mechanics in the first place, the zeta function appears as the partition function of a dynamical system determined by the analysis of the Hecke algebra of the affine group of rational numbers [3]. This result leads to the noncommutative space of adele classes of the rationals [5]. The impact in number theory is the spectral realization of the zeros of the Riemann zeta function and the geometric understanding of the Riemann–Weil explicit formula as a trace formula [5, 30]. Moreover, the quantized calculus in noncommutative geometry, jointly with the theory of prolate spheroidal functions in analysis provides, by means of a semilocal trace formula computation, a conceptual reason for the positivity of Weil's functional [19, 23].

The understanding of the adele class space of the rationals from a more classical geometric standpoint is provided by the theory of Grothendieck toposes: indeed, in many cases the space of the points of these toposes is noncommutative. The "zeta sector" of the adele class space of the rationals is described precisely by the set of points of the Scaling Site [14]. This result led us to a parallel and independent investigation of the algebraic landscape of semirings of characteristic one, where each integer acts by an endomorphism, thus generalizing the Frobenius operator on geometries in finite characteristic, and where categorically, one only adds one more (prime) "field": the Boolean semifield $\mathbb{B}$ (see [9]).

However, and in spite of its elementary definition, the Boolean $\mathbb{B}$ cannot qualify as realizing the original dream of J. Tits in his search for a basic algebraic structure rooting fundamental examples of combinatorial geometries, neither as an algebraic incarnation of Waldhausen's "initial ring". The weakness of the classical algebraic approach is in part due to the inherent deficiency of set-theory when compared to the more flexible categorical counterpart. By following the simple idea that an abelian group A is entirely encoded by the covariant functor HA that assigns to a pointed set X the pointed set of A-valued "divisors" on X, one quickly realizes that the functorial viewpoint is a very natural and versatile generalization of the original set-theoretical notion of abelian group.

This idea has led us to develop algebraic geometry over a "base" $\mathbb{S}$ that is the spherical counterpart of the multiplicative monoid $\mathbb{F}_1 = \{0, 1\}$ and the categorical backbone of the sphere spectrum in homotopy theory [27]. Rings over this base are the Γ-rings of G. Segal and the simplest of them is $\mathbb{S}$, namely the identity endofunctor on the category of pointed sets. Ordinary rings become Γ-rings through the Eilenberg–MacLane functor H. In [12] we gave an arithmetic application of these ideas by extending, at archimedean infinity, the structure sheaf of the algebraic spectrum of the ring of integers as a sheaf of Γ-rings, and more precisely as a subsheaf of the constant sheaf $H\mathbb{Q}$.

One instance of the relevance of the stalk at the archimedean place of this compactification (the Arakelov one point compactification) is exemplified through its relation with the Gromov norm [17]. The choice of this new base has also the further advantage to provide the right framework for Hochschild and cyclic homologies, since simplicial Γ-sets (i.e., Γ-spaces) are well understood and homotopy theory over them becomes, by means of the Dold–Kan correspondence, homological algebra taking place over $\mathbb{S}$ [20].

In Section 2, we give a short survey of all these developments centered on the key role played by the BC-system and the adele class space, and on the understanding of

the "zeta sector" of the latter space as the Scaling Site. The new result that we present in this context is introduced in Section 3, where we describe the BC-system as the Witt ring (i.e., K-theory of endomorphisms) of the "algebraic closure" of $\mathbb{S}$. In this way, we obtain a conceptual meaning of the BC dynamical system no longer from analysis (in terms of quantum statistical mechanics) but at the most basic algebraic level. The algebraic closure $\overline{\mathbb{S}}$ is defined by adjoining to $\mathbb{S}$ all (abstract) roots of unity, and the relation between its Witt ring and the BC-system suggests to perform the following two steps:

(i)   Determine the extension of scalars $\overline{\mathrm{Spec}\,\mathbb{Z}} \times_{\mathbb{S}} \overline{\mathbb{S}}$.

(ii) Define an appropriate De Rham–Witt complex for $\overline{\mathrm{Spec}\,\mathbb{Z}} \times_{\mathbb{S}} \overline{\mathbb{S}}$.

An educated guess on $\overline{\mathrm{Spec}\,\mathbb{Z}} \times_{\mathbb{S}} \overline{\mathbb{S}}$ suggests that this space ought to involve algebraically the cyclotomic extension of the field of rational numbers. The De Rham–Witt complex of $\overline{\mathrm{Spec}\,\mathbb{Z}} \times_{\mathbb{S}} \overline{\mathbb{S}}$ should mainly provide a strengthening of the link between two worlds: on one side (say, on the left) the classical world of Arakelov geometry now enriched over Γ-rings, while on the other side (the right) the analytic framework of noncommutative geometry stemming from the BC-system and directly related to the understanding of the zeros of the Riemann zeta function. These two worlds are, a priori, quite different in nature. Homological algebra over Γ-rings is, through the Dold–Kan correspondence, naturally encoded by the homotopy theory of Γ-spaces, so that the world on the left is that of homotopy theory, spectra, animas, . . .The world on the right side instead, is that of analysis, Hilbert space operators, the quantum, . . .

One fundamental relation between these two worlds is the assembly map [1] which associates to K-homology classes of the universal proper quotient, classes in the K-theory of the reduced C^*-algebra of a group. This creates a bridge, of index theoretic nature, between the world of homotopy theory and the world of analysis where K-theory of C^*-algebras plays a key role. More precisely, the assembly map relates together two ways of effecting the quotient of a space by a group action. On the left world one sees a homotopy quotient as a special case of a homotopy colimit, while on the right world one effects a cross product which is a special case of a general principle in noncommutative geometry of encoding difficult quotients (such as leaf spaces of foliations) by noncommutative algebras.

It is also worth noticing that aside from quotient spaces, these tricky spaces also appear naturally as sets of points of a topos. For example, to a small category $\mathcal{C}$ one may associate the presheaf topos $\hat{\mathcal{C}}$ of contravariant functors from $\mathcal{C}$ to the category of sets. In general, the nature of the space of points of the topos $\hat{\mathcal{C}}$ is as delicate as that of a quotient space, and one may either use as a substitute the classifying space $B\mathcal{C}$ in the left world or view such spaces as noncommutative spaces, if one prefers to work in the right world.

In Section 4, we describe the role of the one-dimensional quantized calculus in relation to Weil's positivity. A central role is played both by the unitary obtained by composing Fourier transform with inversion, and by its quantized logarithmic derivative. An elementary lemma only meaningful in quantized calculus (called the *main lemma* in this paper) gives the conceptual reason to expect Weil's positivity. The fact that the hypothesis of

this lemma is only verified up to an infinitesimal prevents one concluding immediately that positivity holds. In [23] we showed that positivity can still be obtained, for a single archimedean place, by treating separately this infinitesimal. Moreover, in the semilocal case (i.e., when finitely many places, including the archimedean one, are involved), the same infinitesimal property continues to hold [19], and this fact opens the way for a strategy toward RH.

In Section 4.1, we define an invariant of Schwartz kernels in one dimension and relate the Fourier transform (in one dimension) to its role over $\overline{\mathbb{Z}}$ (see Section 3.3). Then, we implement this invariant to prove that, when applied to the quantized differential of a function, it delivers its Schwarzian derivative. This shows in particular that, as emphasized in the fax of D. Sullivan reported in Figure 1 (beginning of Section 4), the quantized differential calculus encodes in a subtle manner the conformal structure also in dimension one, where the Riemannian point of view gives no clue.

In Section 4.2, we state the main lemma in quantized calculus that yields Weil's positivity as a consequence of the triangular property of the quantized differential, and in Section 4.3 we discuss the triptych formed by Fourier, zeta and Poisson. The quantized calculus is then applied in the semilocal framework (Section 4.4) and provides, through the semilocal trace formula, both the operator theoretic formalism for the explicit formulas of Riemann–Weil and a conceptual reason for Weil's positivity.

We discuss the radical of Weil's quadratic form in Section 4.5, and the "almost radical" of its restriction to an interval $[\lambda^{-1}, \lambda]$ in Section 4.6. We then use spectral triples (through Dirac operators) to detect the zeros of the Riemann zeta function up to imaginary part $2\pi\lambda^2$. This provides the operator theoretic replacement for the Riemann–Siegel formula in analytic number theory and the approximation to the sought for cohomology discovered in [21].

2. The BC-system and its role

The origin of the relation between noncommutative geometry and the Riemann zeta function is a fundamental interplay between the mechanism of symmetry breaking in physics and the theory of ambiguity of E. Galois. In physics, the choice of an extremal equilibrium state at zero temperature breaks the symmetry of a system. On the Galois side the choice of such a state selects a group isomorphism of the abstract group $\mathbb{Q}/\mathbb{Z}$ with the group of roots of unity in $\mathbb{C}$. The link is established explicitly by implementing the formalism of quantum statistical mechanics [4] that encodes a quantum statistical system by a pair $(\mathcal{A}, \sigma_t)$ of a C^*-algebra $\mathcal{A}$ and a 1-parameter group of automorphisms $\sigma \colon \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$.

The main tool is the KMS condition that analytically encapsulates the relation existing in quantum mechanics between the Heisenberg time evolution of observables

$$\sigma_t(A) = \exp(itH) A \exp(-itH),$$

where H is the Hamiltonian of the system, and an equilibrium state ϕ at inverse temperature $\beta = \frac{1}{kT}$, whose evaluation on an observable A is

$$\phi(A) := Z^{-1} \operatorname{Tr}(A \exp(-\beta H)),$$

where $Z = \operatorname{Tr}(\exp(-\beta H))$. The precise mathematical encoding of this relation was obtained by Haag, Hugenholtz and Winnink [28], starting from earlier work of Kubo, Martin and Schwinger. A way to understand the KMS_β condition is provided by the equality

$$\big(\varphi(x\sigma_t(y))\big)_{t=i\beta} = \varphi(yx)$$

whose heuristic meaning is that σ_t at $t = i\beta$ compensates for the lack of tracial property of the state φ by allowing one to replace $\varphi(yx)$ with $\varphi(x\sigma_t(y))$ at $t = i\beta$. The states fulfilling the KMS_β condition form a (possibly empty) convex compact simplex.

The specific system that exhibits the interplay between the phenomenon of symmetry breaking in physics and the theory of ambiguity of E. Galois is the BC-system [3]. It is defined using the affine group

$$P^+(\mathbb{Q}) := \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Q}, a > 0 \right\}.$$

The subgroup $P^+(\mathbb{Z})$ of integral translations obtained by requiring that $a, b \in \mathbb{Z}$ is almost normal in $P^+(\mathbb{Q})$, and this fact allows one to define a Hecke algebra $\mathcal{A}$ in place of the convolution algebra of the quotient $P^+(\mathbb{Q})/P^+(\mathbb{Z})$. The action of $\mathcal{A}$ in the Hilbert space $\ell^2(P^+(\mathbb{Q})/P^+(\mathbb{Z}))$ plays the role of the regular representation. The significant fact here is that this representation determines a factor of type III, thus naturally endowed with a one parameter group of automorphisms σ_t of $\mathcal{A}$ (the time evolution). The pair $(\mathcal{A}, \sigma_t)$ constitutes the BC-system. Its first properties are as follows:

- The system exhibits a phase transition with spontaneous symmetry breaking. The KMS_β state is unique for $\beta \leq 1$. For $\beta > 1$ the extremal KMS_β states are parametrized by the points of the zero-dimensional Shimura variety $\mathrm{Sh}(\mathrm{GL}_1, \{\pm 1\})$.

- The symmetries of the system are given by the group $\mathrm{GL}_1(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^*$ of invertible elements of the profinite completion of the integers. The zero-temperature KMS states evaluated on a natural arithmetic subalgebra of the algebra of observables of the system take values that are algebraic numbers and generate the maximal abelian extension $\mathbb{Q}^{\mathrm{cycl}}$ of $\mathbb{Q}$.

- The class field theory isomorphism intertwines the action of the symmetries and the Galois action on the values of states, thus providing a quantum statistical mechanical reinterpretation of the explicit class field theory of $\mathbb{Q}$.

- The partition function $Z(\beta)$ of the system is the Riemann zeta function evaluated at $\beta \in \mathbb{R}$.

The last property establishes the link between noncommutative geometry and the Riemann zeta function. The algebra of the BC-system describes the quotient space $\mathbb{Q}^\times \backslash \mathbb{A}_f$ of finite

adeles of $\mathbb{Q}$ acted upon by the multiplicative group $\mathbb{Q}^\times$. When passing to the dual system using the dynamics, and combining the dual action of $\mathbb{R}_+^*$ together with the symmetries

$$\mathrm{GL}_1(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^*$$

of the system, one obtains the action of the idele class group on the adele class space $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}$. This is the space that provides a geometric interpretation of the Riemann–Weil explicit formulas [5].

This latter result was the starting point of a "longue marche" pursuing the study of the geometry of the adele class space. This space provides the spectral realization of the zeros of L-functions with Grössencharacter, where the Riemann zeta function is associated to the trivial character and whose related space is the "zeta-sector"

$$X = \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*.$$

This zeta-sector provides a Hasse–Weil formula for the Riemann zeta function using the action of $\mathbb{R}_+^*$ on X (see [7, 8]). In view of this result it is clear that X may play the role of the space of the points of the curve for function fields. The geometric structure of X came with the discovery of the "Arithmetic Site" [11, 13]: this is the presheaf topos $\widehat{\mathbb{N}^\times}$ dual to the multiplicative monoid of positive integers, endowed with the structure sheaf provided by the only semifield F whose multiplicative group is infinite cyclic. The geometry of the Arithmetic Site is tropical and of characteristic one (the addition is unipotent: $1 + 1 = 1$). The structure sheaf of this topos is obtained by implementing the action of the semigroup $\mathbb{N}^\times$ on the semifield F by power maps $x \mapsto x^n$.

It is a general fact that in characteristic one the power maps define injective endomorphisms of a semifield and that there exists only one semifield which is finite and not a field, namely the Boolean semifield $\mathbb{B} := \{0, 1\}$. The Arithmetic Site is defined over $\mathbb{B}$ (because F is of characteristic one) and a key result is that the "zeta-sector" X gets canonically identified with the set of points of the Arithmetic Site defined over the semifield $\mathbb{R}_+^{\max}$ of tropical real numbers. This semifield appears both in tropical geometry and also in semiclassical analysis as a limit of deformations of real numbers.

One extremely convincing result of the dequantization program [29] is that the Fourier transform becomes the Legendre transform when taken to the classical limit. The semifield $\mathbb{R}_+^{\max}$ is an infinite extension of $\mathbb{B}$ and its absolute Galois group is determined by the power maps

$$\mathrm{Aut}_\mathbb{B}(\mathbb{R}_+^{\max}) = \{\mathrm{Fr}_\lambda \mid \lambda \in \mathbb{R}_+^*\}, \quad \mathrm{Fr}_\lambda(x) := x^\lambda. \tag{1}$$

This group acts on the points of the Arithmetic Site defined over $\mathbb{R}_+^{\max}$ and, under the canonical identification of these points with the "zeta-sector" $X = \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*$, this action corresponds to the action of the idele class group. In spite of the fact that the Arithmetic Site is an object of countable nature (the semigroup $\mathbb{N}^\times$ and the semifield F are countable) and hence there is no non-trivial action of $\mathbb{R}_+^*$ on the topos, $\mathbb{R}_+^*$ acts meaningfully using the theory of correspondences [11, 13].

The extension of scalars of the Arithmetic Site to $\mathbb{R}_+^{\max}$ determines the Scaling Site [14] namely the Grothendieck topos $[0, \infty) \rtimes \mathbb{N}^\times$ (where $\mathbb{N}^\times$ acts by multiplication) endowed with the structure sheaf of continuous convex functions with integral slopes. The set of points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$ identifies canonically with the "zeta sector" X. The restriction of the structure sheaf of the Scaling Site to the periodic orbits in X determines, for each prime p, the quotient $\mathbb{R}_+^* / p^{\mathbb{Z}}$ which appears in X as the counterpart of the prime-point p of Spec $\mathbb{Z}$. The emerging tropical structure describes an analogue of an elliptic curve and it also exhibits a few totally new features. For instance, the divisor degree on these curves is a real number and the Riemann–Roch formula is real valued. Such real valued indices are ubiquitous in the noncommutative geometry of foliations and the tropical geometry of the Scaling Site can be lifted in complex geometry [16].

In order to extend the geometric positivity argument used by Mattuck and Tate, and Grothendieck for function fields, to the field of rational numbers and on the above geometric space one needs to show a Riemann–Roch formula holding on the square of the Scaling Site. In this respect, the case of periodic orbits is far too simplified since for curves one can bypass the construction of a cohomology theory for divisors beyond H^0 using Serre duality as a definition of H^1. For surfaces, and in particular for the square of the Scaling Site, this trick handles only H^2 leaving H^1 still out of reach. One is thus faced with the problem of developing a good cohomology theory in characteristic one.

Motivated by this application, we developed a general theory of homological algebra for the (non-abelian) category of $\mathbb{B}$-modules [15], however the lack of the additive inverse makes the elimination of certain technical difficulties apparently quite hard. While trying to by-pass this issue, we were led to investigate a more fundamental base for algebraic manipulations, which is, as explained in the introduction, independent of the choice of a characteristic. The main reason for our turn of interests toward this new base is that it is the most natural one for Hochschild and cyclic homology theories. In the next section we show that the fundamental basis $\mathbb{S}$ provides the conceptual interpretation of the BC-system as the Witt construction over the algebraic closure of $\mathbb{S}$.

3. The conceptual meaning of the BC-system

The convolution algebra of the quotient $P^+(\mathbb{Q})/P^+(\mathbb{Z})$ has an integral model [10, Section 3], given by the Hecke algebra

$$\mathcal{H}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}.$$

The ring endomorphisms $\sigma_n(e(r)) = e(nr)$, $n \in \mathbb{N}$ act on the canonical generators of the group ring $e(r) \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, $r \in \mathbb{Q}/\mathbb{Z}$. There are natural quasi-inverse linear maps

$$\tilde{\rho}_n \colon \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \tilde{\rho}_n(e(\gamma)) = \sum_{n\gamma'=\gamma} e(\gamma').$$

These two operators are used both in the definition of the crossed product $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ and in the presentation of the algebra.

There is a striking analogy between the algebraic rules fulfilled by the pair $\{\sigma_n, \widetilde{\rho}_n\}$ and the relations fulfilled, in the global Witt construction, by the Frobenius and Verschiebung operators. The invariant part of the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ for the action of the group

$$\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}^*$$

is described in terms of Almkvist's ring of endomorphisms $\mathbb{W}_0(\mathbb{S})$ as follows.

Theorem 3.1 ([20, Theorem 2.3]). *The ring $\mathbb{W}_0(\mathbb{S})$ is canonically isomorphic to the invariant part of the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ for the action of the group $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}^*$.*

In this section, we extend Theorem 3.1 by showing a natural isomorphism of the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with the ring $\mathbb{W}_0(\overline{\mathbb{S}})$, where $\overline{\mathbb{S}}$ denotes the monoid $\mathbb{S}$-algebra $\mathbb{S}[M]$ of the multiplicative pointed monoid $M = (\mathbb{Q}/\mathbb{Z})_+$, with elements the base point $* = 0$ and the $e(r)$'s for $r \in \mathbb{Q}/\mathbb{Z}$. The multiplication in M is defined by

$$e(r)e(s) = e(r+s) \quad \forall\, r, s \in \mathbb{Q}/\mathbb{Z}.$$

The functor $\overline{\mathbb{S}} \colon \Gamma^{\mathrm{op}} \to \mathfrak{Sets}_*$, is defined by $\overline{\mathbb{S}}[X] = X \wedge M$, where the monoid structure in M yields the algebra structure $\overline{\mathbb{S}}[X] \wedge \overline{\mathbb{S}}[Y] \to \overline{\mathbb{S}}[X \wedge Y]$.

3.1. Endomorphisms and matrices

In [20] we considered the class of $\mathbb{S}$-modules of the form $\mathbb{S}[F] = \mathbb{S} \wedge F$, where F is a finite object of the category $\mathfrak{Sets}_*$ of pointed sets. As a functor $\mathbb{S}[F] \colon \Gamma^{\mathrm{op}} \to \mathfrak{Sets}_*$ associates to a finite pointed set X the smash product $\mathbb{S}[F](X) := F \wedge X$ and to a map of finite pointed sets $g \colon X \to Y$ the map $\mathbb{S}[F](g) := \mathrm{Id} \wedge g$. An endomorphism of $\mathbb{S}[F]$ is a natural transformation.

Lemma 3.2. *Let F, F' be two finite objects in $\mathfrak{Sets}_*$. The map*

$$\mathrm{Hom}_{\mathbb{S}}(\mathbb{S}[F], \mathbb{S}[F']) \to \mathrm{Hom}_{\mathfrak{Sets}_*}(F, F'), \quad \phi \mapsto \phi(1_+),$$

where $\phi(1_+)$ denotes the restriction of ϕ to $1_+ = \{0, 1\}$ is a bijection of sets. The inverse map is

$$\mathrm{Hom}_{\mathfrak{Sets}_*}(F, F') \to \mathrm{Hom}_{\mathbb{S}}(\mathbb{S}[F], \mathbb{S}[F']), \quad \psi \mapsto \widetilde{\psi} = \mathrm{Id} \wedge \psi,$$

where $\widetilde{\psi}(X) = \mathrm{Id}_X \wedge \psi \colon X \wedge F \to X \wedge F'$.

Proof. Let $\phi \in \mathrm{Hom}_{\mathbb{S}}(\mathbb{S}[F], \mathbb{S}[F'])$ and X a finite pointed set. An element $y \in \mathbb{S}[F](X) = F \wedge X$, $y \neq *$, is determined by a pair $y = (f, x) \in F \times X$, and there exists a (unique) map of pointed sets $g \colon 1_+ \to X$ with $g(1) = x$. By the naturality of the transformation ϕ, one has

$$\phi \circ \mathbb{S}[F](g) = \mathbb{S}[F](g) \circ \phi.$$

This shows that ϕ is uniquely determined by its restriction $\phi(1_+)$ on $\mathbb{S}[F](1_+) = F$, with $\phi(1_+) \in \mathrm{Hom}_{\mathfrak{Sets}_*}(F, F')$. Conversely, given $\psi \in \mathrm{Hom}_{\mathfrak{Sets}_*}(F, F')$ one associates to it

the natural transformation $\widetilde{\psi}\colon \mathbb{S}[F] \to \mathbb{S}[F']$ that maps a finite pointed set X to the map $\mathrm{Id}_X \wedge \psi\colon \mathbb{S}[F] \to \mathbb{S}[F']$. It is immediate to verify that the two maps are inverse of each other.   ∎

In the following part we shall consider endomorphisms of $\bar{\mathbb{Z}}$-modules of the form $\bar{\mathbb{Z}}[F] = \bar{\mathbb{Z}} \wedge F$, with F a finite pointed set. For $n \in \mathbb{N}$, $n_+ := \{0, 1, \ldots, n-1, n\}$.

Definition 3.3. Let $\mathrm{Mat}_n^R(\bar{\mathbb{Z}})$ be the multiplicative pointed monoid of $n \times n$ matrices with entries in the multiplicative monoid $\bar{\mathbb{Z}}(1_+) = M = (\mathbb{Q}/\mathbb{Z})_+$, which have only one non-zero (i.e., not equal to the base point $*$) entry in each column.

Given $\mu = (\mu_{ij}) \in \mathrm{Mat}_n^R(\bar{\mathbb{Z}})$ one defines a map of pointed sets by setting

$$\rho(\mu)\colon M \wedge n_+ \to M \wedge n_+, \quad \rho(\mu)(\alpha, j) := \begin{cases} * & \text{if } \mu_{ij} = * \ \forall i, \\ (\mu_{ij}\alpha, i) & \text{if } \mu_{ij} \neq *. \end{cases} \tag{2}$$

Note that for $\mu \in \mathrm{Mat}_n^R(\bar{\mathbb{Z}})$, there exists, for a given j, at most one $i \in \{1, \ldots, n\}$ with $\mu_{ij} \neq *$.

Proposition 3.4. *With the notations of Lemma* 3.2, *the map*

$$\widetilde{\rho}\colon \mathrm{Mat}_n^R(\bar{\mathbb{Z}}) \to \mathrm{End}_{\bar{\mathbb{Z}}}(\bar{\mathbb{Z}}[n_+]) \quad \widetilde{\rho}(\mu) := \widetilde{\rho(\mu)}$$

is an isomorphism of multiplicative pointed monoids.

Proof. With $\mu = (\mu_{ij}) \in \mathrm{Mat}_n^R(\bar{\mathbb{Z}})$, $\widetilde{\rho(\mu)}$ defines a natural transformation

$$\widetilde{\rho(\mu)}\colon \bar{\mathbb{Z}}[F] \to \bar{\mathbb{Z}}[F], \quad \widetilde{\rho(\mu)}(X) = \mathrm{Id}_X \wedge \rho(\mu)\colon X \wedge M \wedge n_+ \to X \wedge M \wedge n_+$$

that commutes with the action of M. Thus, it determines an endomorphism

$$\widetilde{\rho(\mu)} \in \mathrm{End}_{\bar{\mathbb{Z}}}(\bar{\mathbb{Z}}[F]).$$

Let $\mu, \mu' \in \mathrm{Mat}_n^R(\bar{\mathbb{Z}})$: their product is given by

$$(\mu\mu')_{ik} = \begin{cases} \mu_{ij}\mu'_{jk} & \text{if } \exists \, j \text{ such that } \mu_{ij} \neq * \text{ and } \mu'_{jk} \neq *, \\ * & \text{otherwise.} \end{cases}$$

By applying (2), one gets
$$\rho(\mu\mu') = \rho(\mu) \circ \rho(\mu'),$$

since $\rho(\mu) \circ \rho(\mu')(\alpha, k) \neq *$ if and only if there exist j with $\mu'_{jk} \neq *$ and i with $\mu_{ij} \neq *$. In that case, one has

$$\rho(\mu) \circ \rho(\mu')(\alpha, k) = (\mu_{ij}\mu'_{jk}\alpha, i) = ((\mu\mu')_{ik}\alpha, i).$$

This shows that $\widetilde{\rho}$ is a multiplicative map. It is injective by construction. Next we show that it is also surjective. Let $\phi \in \mathrm{End}_{\overline{\mathbb{Z}}}(\overline{\mathbb{Z}}[F])$. Then by Lemma 3.2, $\phi = \widetilde{\psi}$, where ψ is the restriction $\phi(1_+)$. This restriction commutes with the action of M on

$$(\overline{\mathbb{Z}}[F])(1_+) = F \wedge M,$$

and thus it is given by a matrix $\rho(\mu)$ acting as in (2). ∎

For a given $\mathbb{S}$-algebra A, we denote by $\mathbb{Mat}_n^R(A)$ the $\mathbb{S}$-algebra of matrices over A defined in [27, Section 2.1.4.1, Example 2.1.4.3 (6)]. Note that, up to transposition, there are two equivalent definitions for such matrices: we let $\mathbb{Mat}_n^L(A)$ be the functor ($\mathbb{S}$-algebra) from finite pointed sets to pointed sets that maps a finite pointed set X to the set of $n \times n$ matrices of elements of $A(X)$ with only one non-zero entry in each row. Similarly, $\mathbb{Mat}_n^R(A)$ is the functor mapping a finite set X to the set of $n \times n$ matrices of elements of $A(X)$ with only one non-zero entry in each column.

Next proposition shows that one can define a bimodule $\mathbb{Mat}_n(A)$ over these two $\mathbb{S}$-algebras as the functor from finite pointed sets to pointed sets mapping X to the set of $n \times n$ matrices of elements of $A(X)$ with no restriction on the matrix entries. The proposition is in fact a special case of the composition law for $\mathbb{S}$-algebras viewed as endofunctors.

Proposition 3.5. *Let A be an $\mathbb{S}$-algebra. The following facts hold:*

(i) *The action of $\mathbb{Mat}_n^L(A)$ on $\mathbb{Mat}_n(A)$ by left multiplication*

$$\mathbb{Mat}_n^L(A)(X) \times \mathbb{Mat}_n(A)(Y) \to \mathbb{Mat}_n(A)(X \wedge Y)$$

turns $\mathbb{Mat}_n(A)$ into a left module over $\mathbb{Mat}_n^L(A)$.

(ii) *The action of $\mathbb{Mat}_n^R(A)$ on $\mathbb{Mat}_n(A)$ by right multiplication*

$$\mathbb{Mat}_n(A)(X) \times \mathbb{Mat}_n(A)^R(Y) \to \mathbb{Mat}_n(A)(X \wedge Y)$$

turns $\mathbb{Mat}_n(A)$ into a right module over $\mathbb{Mat}_n^R(A)$.

Proof. The proof is the same as the one in [27]: one simply needs to check that the product in the $\mathbb{S}$-algebra A determines a well-defined product of matrices. To this end, the point is that the sum involved in determining the matrix element at position (i, j) is obtained from a row by column product of two matrices that only contain one non-zero term. This fact holds as long as either the rows of one matrix or the columns of the other one contain only one non-zero element: this is the case in (i) and (ii). ∎

For $A = \overline{\mathbb{Z}}$ and $X = 1_+$, there is an isomorphism of pointed monoids[1]

$$\mathbb{Mat}_n^R(\overline{\mathbb{Z}})(1_+) = \mathrm{Mat}_n^R(\overline{\mathbb{Z}}).$$

---

[1]Note that the $\mathbb{S}$-algebra $\mathbb{Mat}_n^R(\overline{\mathbb{Z}})$ is not the same as the spherical algebra of the monoid $\mathrm{Mat}_n^R(\overline{\mathbb{Z}})$.

Moreover, the set of matrices $\mathrm{Mat}_n(\bar{\mathbb{Z}}) = M_n(\mathbb{S}[M])$ (where $M = (\mathbb{Q}/\mathbb{Z})_+$) coincide with $\mathbb{M}\mathrm{at}_n(\bar{\mathbb{Z}})(1_+)$. By Proposition 3.5, they form a bimodule with right and left actions provided by $\mathrm{Mat}_n^R(\bar{\mathbb{Z}})$ acting on the right of $\mathrm{Mat}_n(\bar{\mathbb{Z}})$ by matrix multiplication and by

$$\mathrm{Mat}_n^L(\bar{\mathbb{Z}}) = \mathbb{M}\mathrm{at}_n^L(\bar{\mathbb{Z}})(1_+)$$

acting similarly on the left. The role of the bimodule $\mathrm{Mat}_n(\bar{\mathbb{Z}}) = \mathbb{M}\mathrm{at}_n(\bar{\mathbb{Z}})(1_+)$ is to encode similarities.

Given a field k and the associated $\mathbb{S}$-algebra Hk, morphisms of $\mathbb{S}$-algebras

$$\bar{\mathbb{Z}} = \mathbb{S}[M] \to Hk$$

correspond bijectively to (multiplicative) monoid homomorphisms $M \to k$ ([18, Proposition 2.2 (i)]). In particular, to an injective morphism $M \to k$ corresponds an extension of $\bar{\mathbb{Z}}$ by the field k. In view of this fact, we introduce the following definition.

Definition 3.6. An element $\alpha \in \mathrm{Mat}_n(\bar{\mathbb{Z}}) = M_n(\mathbb{S}[M])$ is invertible if and only if the matrix $\alpha \in M_n(k)$ is invertible in all field extensions k of $\bar{\mathbb{Z}}$.

Matrix similarity in $\mathrm{Mat}_n(\bar{\mathbb{Z}})$ is stable by taking powers, as illustrated by the following lemma.

Lemma 3.7. *Let $\alpha \in \mathrm{Mat}_n(\bar{\mathbb{Z}})$, $\mu \in \mathrm{Mat}_n^R(\bar{\mathbb{Z}})$, $\gamma \in \mathrm{Mat}_n^L(\bar{\mathbb{Z}})$, such that $\gamma\alpha = \alpha\mu$. Then one has $\gamma^k\alpha = \alpha\mu^k$ for all $k \in \mathbb{N}$.*

Proof. One has $\gamma^2\alpha = \gamma\alpha\mu = \alpha\mu^2$, and by induction on k one derives $\gamma^k\alpha = \alpha\mu^k$. ∎

In view of Proposition 3.4, it is equivalent to consider endomorphisms $T \in \mathrm{End}_{\bar{\mathbb{Z}}}(\bar{\mathbb{Z}}[F])$ (where F is a finite-pointed set) of $\bar{\mathbb{Z}}$-modules $E = \bar{\mathbb{Z}}[F]$, or matrices $\mu \in \mathrm{Mat}_*^R(\bar{\mathbb{Z}})$, where $*$ is the integer recording the cardinality of the complement of the base point in F. One defines the notion of invariant (of endomorphisms) as follows.

Definition 3.8. An invariant is a map

$$\chi : \mathrm{Mat}_*^R(\bar{\mathbb{Z}}) \to R$$

to a commutative ring R that satisfies the following conditions:

(i) $\chi(E, T) = \chi(T(E), T)$,

(ii) $\chi(E_1 \vee E_2, T_1 \vee T_2) = \chi(E_1, T_1) + \chi(E_2, T_2)$,
      $\chi(E_1 \wedge E_2, T_1 \wedge T_2) = \chi(E_1, T_1)\chi(E_2, T_2)$,
      where the smash product is taken over $\bar{\mathbb{Z}}$

(iii) χ is invariant under similarity, i.e., $\chi(\gamma) = \chi(\mu)$ if $\gamma\alpha = \alpha\mu$ for an invertible matrix $\alpha \in \mathrm{Mat}_*(\bar{\mathbb{Z}})$.

Condition (i) is the same as in [20, Definition 2.2], and has the role to mod out the zero endomorphisms. The second condition implements the ring structure. Finally, (iii) realizes invariance under similarity.

3.2. Construction of the universal invariant

We shall define the universal invariant of endomorphisms after applying the extension of scalars from $\bar{\mathbb{Z}}$ to the maximal cyclotomic extension of $\mathbb{Q}$. In that set-up Almkivst's original result applies and associates to (square) matrices a divisor with coefficients in the multiplicative group of the field. Our result states that the divisor has coefficients in the group of roots of unity.

Next proposition gives the construction of the invariant of endomorphisms. We keep the same notations of Section 3.1. In particular, $M = (\mathbb{Q}/\mathbb{Z})_+$ denotes the multiplicative, pointed monoid of abstract roots of unity.

Proposition 3.9. *Let $T \in \mathrm{Mat}_n^R(\bar{\mathbb{Z}})$ and $\kappa\colon M \hookrightarrow k$ be an injective morphism into an algebraically closed field extension of $\bar{\mathbb{Z}}$ of characteristic zero.*

(i) *The divisor D defined by Almkivst's invariant of $\kappa(T) \in \mathrm{Mat}_n(k)$ has coefficients in $\kappa(M^\times)$. The divisor $\tau(T) := \kappa^{-1}(D)$ with coefficients in $M^\times$ is independent of the choice of κ.*

(ii) *The map*

$$\tau\colon \mathrm{Mat}_n^R(\bar{\mathbb{Z}}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \tau(T) := \kappa^{-1}(D)$$

defines an invariant.

Proof. (i) Let $t = (t_{ij}) \in M_n(k)$ be a matrix whose non-zero entries are roots of unity and with at most one non-zero element t_{ij} in each column. We claim that the eigenvalues of t are either 0 or roots of unity. Let $E = k^n$ be the k-vector space on which t acts. The subspaces $E_j := t^j(E)$ form a decreasing filtration of E for which there exists a finite index ℓ such that $E_{\ell+1} = E_\ell$. The non-zero eigenvalues of t are the same as the eigenvalues of the restriction t_ℓ of t on E_ℓ. We verify that the endomorphism t_ℓ has finite order. Indeed, let

$$\phi\colon n_+ \to n_+, \quad \phi(j) = \begin{cases} i & \text{if } t_{ij} \neq 0, \\ * & \text{if } t_{ij} = 0 \ \forall i. \end{cases} \tag{3}$$

The range of ϕ^ℓ labels a basis of E_ℓ: in this basis the matrix of t_ℓ describes the permutation obtained by restricting ϕ, whose entries are in roots of unity. Such a matrix is periodic thus all of its eigenvalues are roots of unity. This shows that Almkivst's invariant of $\kappa(T) \in \mathrm{Mat}_n(k)$, i.e., a divisor D with coefficients in $k^\times$, has in fact coefficients in $\kappa(M^\times)$. Moreover, one also derives that the divisor $\tau(T) := \kappa^{-1}(D)$ with coefficients in $M^\times$ is independent of the choice of κ.

(ii) The map τ fulfills the three conditions of Definition 3.8 since they hold true for Almkivst's invariant. In particular, the operations in condition (ii) correspond to direct sum and tensor product of modules. ∎

3.3. Completeness of the invariant τ

To prove that the above construction defines a universal invariant, one applies the same proof as in [20, Theorem 3.3] (in the case of endomorphisms of finite $\mathbb{S}$-modules), to show the injectivity of τ. The main fact to verify is that by implementing the (algebraic) Fourier transform one can diagonalize any matrix in $\mathrm{Mat}_n(\bar{\mathbb{Z}})$ corresponding to a permutation at the set level. We shall see (in the proof of Theorem 3.11) that for a cycle of such permutation one can choose a basis so that the matrix of such permutation is equivalent to the cyclic permutation matrix multiplied by a root of unity. Next proposition gives the algebraic relation between the endomorphisms determined by the cyclic permutation matrix $C(n)$ of order n

$$C(n)_{ij} := \begin{cases} 1 & \text{if } i = j + 1 \ (n), \\ 0 & \text{otherwise.} \end{cases} \qquad \forall\, i, j \in \{0, \dots, n-1\}$$

and the diagonal matrix $\Delta(n)$ whose entries are the full set of n-th roots of unity, i.e., $\Delta(n)_{jj} = e(j/n)$ for $j \in \{0, \dots, n-1\}$. The proposition shows that there is a non-trivial algebraic relation between $\Delta(n)$ and $C(n)$, and by Lemma 3.7 the same relation holds true when arbitrary powers of these two matrices are involved.

Proposition 3.10. *For $n \in \mathbb{N}$, let $\mu_n := \{e(a/n) \mid a \in \mathbb{Z}/n\mathbb{Z}\}$ be the group of n-th roots of unity.*

(i) *The matrix $V = (V_{ij}) \in \mathrm{Mat}_n(\mathbb{S}[\mu_n])$: $V_{ij} = e(ij/n)$ is the matrix of the Fourier transform on the cyclic group $\mathbb{Z}/n\mathbb{Z}$.*

(ii) *In any field extension of $\mathbb{S}[\mu_n]$ one has $n \neq 0$, and the inverse of V is, up to the overall factor n, the matrix $W = (W_{ij})$: $W_{ij} = e(-ij/n)$.*

(iii) *The following relations hold:*

$$\Delta(n)V = VC(n), \quad C(n)W = W\Delta(n). \tag{4}$$

Proof. (i) It suffices to recall that the Fourier transform on the cyclic group $\mathbb{Z}/n\mathbb{Z}$ is the transformation F of functions $f: \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ defined by

$$F(f)(a) = \sum \varepsilon(ab/n) f(b), \quad \varepsilon(x) := \exp(-2\pi i x) \quad \forall\, x \in \mathbb{R}.$$

(ii) Let k be a field extension of $\mathbb{S}[\mu_n]$, then k contains n distinct roots of unity of order n, thus the characteristic of k is prime to n. It follows that $n \neq 0$ in k and the inverse of V is $\frac{1}{n}W$.

(iii) One checks (4) by direct computation using the equality $e(x)e(y) = e(x+y)$. $\blacksquare$

We can now state and prove the main result of this section.

Theorem 3.11. *The ring $\mathbb{W}_0(\bar{\mathbb{Z}})$ is canonically isomorphic to the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. The invariant $\tau: \mathrm{Mat}_*^R(\bar{\mathbb{Z}}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ is universal and it extends the additive invariant of Theorem 3.1.*

Proof. Let $\chi \colon \mathrm{Mat}_*^R(\bar{\mathbb{Z}}) \to R$ be an invariant (thus fulfilling the conditions of Definition 3.8), then consider the map

$$\beta \colon \mathbb{Q}/\mathbb{Z} \to R, \quad \beta(r) := \chi([e(r)]) \quad \forall r \in \mathbb{Q}/\mathbb{Z},$$

where $[e(r)]$ is the endomorphism of the one dimensional module $\bar{\mathbb{Z}}$ given by multiplication by $e(r)$. By Definition 3.8 (ii), $\beta \colon \mathbb{Q}/\mathbb{Z} \to R^\times$ is a group homomorphism, and hence it extends to a ring homomorphism $\beta \colon \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to R$.

Next, we show that $\chi = \beta \circ \tau$. Let $T \in \mathrm{End}_{\bar{\mathbb{Z}}}(\bar{\mathbb{Z}}[F])$, we prove that

$$\chi(T) = \beta(\tau(T)).$$

We identify $F = n_+$ and let $\mu = (\mu_{ij})$ be the matrix with $\tilde{\rho}(\mu) = T$ (see Proposition 3.4). Let $\phi \colon n_+ \to n_+$ be the map as in (3). The ranges X_ℓ of the powers ϕ^ℓ form a decreasing sequence of subsets and we let ℓ be such that $X_\ell = X_{\ell+1}$. The matrix of the restriction of T to X_ℓ is the matrix of the permutation obtained by restricting ϕ, thus using Definition 3.8 (i) we can just consider the case where μ is the matrix of a permutation with entries in roots of unity. The required additivity in (ii) of Definition 3.8, allows one to assume that the permutation is a cyclic permutation. Then we observe that the matrix of a cyclic permutation ν with entries roots of unity is equivalent, using a diagonal matrix whose entries are ratios of entries of ν, to the matrix of a cyclic permutation of type $e(s)C(m)$, whose entries are all the same root of unity $e(s)$. It then follows from Proposition 3.10 and Definition 3.8 (iii) that

$$\chi(T) = \chi(e(s)\Delta(m)),$$

and finally, using again (ii) of Definition 3.8, one obtains $\chi(T) = \beta(\tau(T))$. ∎

3.4. Frobenius and Verschiebung

The Frobenius endomorphisms and the Verschiebung maps are operators in $\mathbb{W}_0(\mathbb{S})$ ([20]). The Frobenius ring endomorphisms F_n, $n \in \mathbb{N}$, are defined by the equality

$$F_n((E,T)) := (E, T^n)$$

(we refer to op. cit. for the notations). One easily checks that

$$\tau(F_n(x)) = \sigma_n(\tau(x)) \quad \forall\, n \in \mathbb{N}, \ x \in \mathbb{W}_0(\bar{\mathbb{Z}}), \tag{5}$$

where the group ring endomorphism σ_n is defined, for each $n \in \mathbb{N}$, by

$$\sigma_n \colon \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \sigma_n(e(\gamma)) = e(n\gamma). \tag{6}$$

The Verschiebung maps V_n replace a pair (E, T) by the endomorphism of the sum $\vee^n E$ that cyclically permutes the terms and uses $T \colon E \to E$ to turn back from the last term to

the first. The map V_n is additive by construction and when applied to a one dimensional $\bar{\mathbb{Z}}$-module $[e(a)]$ it gives the sum of the $[e(b)]$'s where $nb = a$. One thus obtains

$$\tau(V_n(x)) = \tilde{\rho}_n(\tau(x)) \quad \forall\, n \in \mathbb{N}, \ x \in \mathbb{W}_0(\bar{\mathbb{Z}}), \tag{7}$$

where $\tilde{\rho}_n, n \in \mathbb{N}$, is defined by

$$\tilde{\rho}_n \colon \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \tilde{\rho}_n(e(\gamma)) = \sum_{n\gamma'=\gamma} e(\gamma'). \tag{8}$$

Then, in analogy to and generalizing what is stated in [20, Section 2.2], one derives the following theorem.

Theorem 3.12. *The correspondences $\sigma_n \to F_n$, $\tilde{\rho}_n \to V_n$ determine a canonical isomorphism of the integral BC-system (i.e., the Hecke algebra $\mathcal{H}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$) with the Witt ring $\mathbb{W}_0(\bar{\mathbb{Z}})$ endowed with the Frobenius and Verschiebung maps.*

4. Analytic approach and noncommutative geometry

On September 27, 1993, Dennis Sullivan sent to the first author a fax, reproduced in Figure 1, that sets the scene of the interactions between noncommutative geometry and geometry of manifolds.[2] In [6], the account of the analytic approach based on [5] was reduced to the minimum. This section is dedicated to explain our recent results on this analytic approach. Two main tools in noncommutative geometry play a key role here, they are:

- the quantized calculus;
- the notion of spectral triple.

The quantized calculus is applied in the semilocal framework and it provides, through the semilocal trace formula, both the operator theoretic formalism for the explicit formulas of Riemann–Weil and a conceptual reason for Weil's positivity (Sections 4.2–4.4).

Spectral triples (through Dirac operators) together with the understanding of the radical of the Weil quadratic form restricted to an interval $[\lambda^{-1}, \lambda]$ using prolate functions, and the implementation of the map

$$\mathcal{E}(f)(x) := x^{1/2} \sum_1^\lambda f(nx),$$

allow one to detect the zeros of the Riemann zeta function up to imaginary part $2\pi\lambda^2$, thus providing the operator theoretic replacement for the Riemann–Siegel formula in analytic number theory (Sections 4.5 and 4.6).

---

[2]A joint paper (with N. Teleman) appeared in *Topology* in 1994 ([26]).

Both notions make essential use of operators in Hilbert space and of the following dictionary:

Real variable $f : X \to \mathbb{R}$	Self-adjoint operator H in Hilbert space
Range $f(X) \subset \mathbb{R}$ of the variable	Spectrum of the operator H
Composition $\phi \circ f$, ϕ measurable	Measurable functions $\phi(H)$ of self-adjoint operators
Bounded complex variable Z	Bounded operator A in Hilbert space
Infinitesimal variable dx	Compact operator T
Infinitesimal of order $\alpha > 0$	Characteristic values $\mu_n(T) = O(n^{-\alpha})$ for $n \to \infty$
Algebraic operations on functions	Algebra of operators in Hilbert space
Integral of function $\int f(x)\,dx$	$\rlap{/}{\int} T = $ coefficient of $\log(\Lambda)$ in $\mathrm{Tr}_\Lambda(T)$
Line element $ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$	$ds = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet :$ Fermion propagator D^{-1}
$d(a,b) = \mathrm{Inf} \int_\gamma \sqrt{g_{\mu\nu}\,dx^\mu\,dx^\nu}$	$d(\mu, \nu) = \mathrm{Sup}\,|\mu(A) - \nu(A)|,|\ \|[D, A]\| \le 1.$
Riemannian geometry (X, ds^2)	Spectral geometry $(\mathcal{A}, \mathcal{H}, D)$
Curvature invariants	Asymptotic expansion of spectral action
Gauge theory	Inner fluctuations of the metric
Weyl factor perturbation	$D \mapsto \rho D \rho$
Conformal Geometry	Fredholm module $(\mathcal{A}, \mathcal{H}, F)$, $F^2 = 1$.
Perturbation by Beltrami differential	$F \mapsto (aF + b)(bF + a)^{-1}, a = (1 - \mu^2)^{-1/2}, b = \mu a$
Distributional derivative	Quantized differential $dZ := [F, Z]$
Measure of conformal weight p	$f \mapsto \rlap{/}{\int} f(Z)|dZ|^p$

## 4.1. Schwartz kernels and Schwarzian derivative

Let V be a one-dimensional manifold and let $\mathcal{H} = L^2(V)$ be the Hilbert space of square integrable half densities: $\xi(x) = f(x)\,dx^{1/2} \in \mathcal{H}$. The Schwartz kernel of an operator $T : \mathcal{H} \to \mathcal{H}$ is of the form: $k_T(x, y)\,dx^{1/2}\,dy^{1/2}$. This means that, as a function of two variables, the kernel depends on choices of positive sections of the one-dimensional bundle of $1/2$-densities. By varying the choice of a section, i.e., by dividing it with a positive function $\rho(x)$, the kernel $k_T(x, y)$ gets modified accordingly to $\rho(x)\rho(y)k_T(x, y)$. The next lemma detects an invariant of the above change.

Lemma 4.1. *The differential form*

$$\omega = \partial_x \partial_y \log(k_T(x, y))\,dx\,dy \tag{9}$$

is independent of the choice of sections of the bundle of $1/2$-densities and defines an invariant $\omega(T)$ of the operator T.
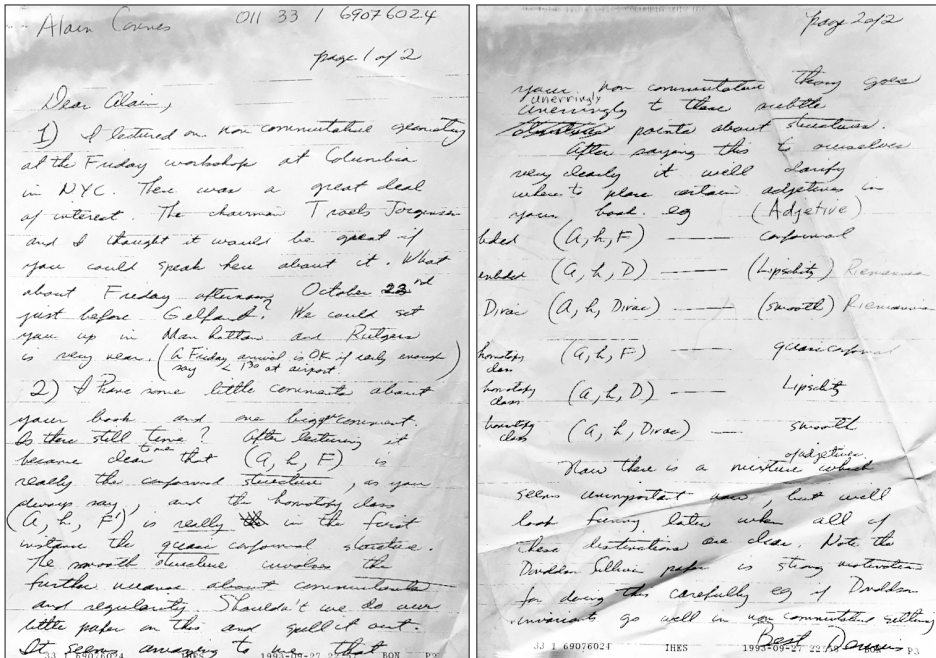
(a) Page 1

(b) Page 2

Figure 1. "… After lecturing it became clear that $(\mathcal{A}, \mathcal{H}, F)$ is really the conformal structure, as you always say, and the homotopy class $(\mathcal{A}, \mathcal{H}, F')$ is really the first instance of the quasi-conformal structure …"

Proof. One sees that by taking the log of the variation of the Schwartz kernel

$$\log(\rho(x)\rho(y)k(x, y)) = \log(k(x, y)) + \log \rho(x) + \log \rho(y)$$

and then applying $\partial_x \partial_y$, the output is independent of ρ. ∎

Let now $V = \mathbb{R}$ and $T = \mathcal{F}$ the Fourier transform, then $k_{\mathcal{F}}(x, y) = \exp(-2\pi ixy)$ and the differential form becomes $\omega(\mathcal{F}) = -2\pi i \, dx \, dy$.

Note that the Schwartz kernel of the Fourier transform already appeared in Section 3.3 (in matrix form) and there it played a crucial role in the determination of the K-theory of endomorphisms of $\bar{\mathbb{Z}}$.

Next, we compute $\omega(đf)$ for the quantized differential of a function f on $\mathbb{R}$.

Lemma 4.2. *Let f be a smooth, complex valued function on $\mathbb{R}$. Then*

$$\omega(đf) = \left(\frac{f'(x)f'(y)}{(f(x) - f(y))^2} - \frac{1}{(x - y)^2} \right) dx \, dy. \tag{10}$$

The restriction of $\omega(\bar{d} f)$ to the diagonal is $\frac{1}{6} \mathcal{S}(f) \, dx^2$, where

$$\mathcal{S}(f) = \frac{f^{(3)}(x)}{f'(x)} - \frac{3 f''(x)^2}{2 f'(x)^2}$$

is the Schwarzian derivative.

Proof. The Schwartz kernel of the quantized differential is

$$k(x, y) = \frac{i}{\pi} \frac{f(x) - f(y)}{x - y}.$$

By applying (9) one obtains the stated equality (10).   ∎

Note, in particular, that the second statement of the above lemma shows that the quantized differential determines the Schwarzian derivative.

4.2. The main lemma

The conceptual reason for the link between Weil's explicit formula and the trace of the compression of the scaling action on Sonin's space [19, 23] is rooted in the following two general facts.

Let $\mathcal{H}$ be a Hilbert space, and let $F = 2P - 1$ be the operator defining the quantized calculus. An operator T in $\mathcal{H}$ is encoded by a matrix

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

with

$$T_{11} = (1 - P)T(1 - P), \quad T_{12} = (1 - P)TP, \quad T_{21} = PT(1 - P), \quad T_{22} = PTP.$$

Proposition 4.3 ([19, Proposition 5.4]). *Let*

$$U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

be the upper-triangular matrix of an operator in $\mathcal{H}$. Then U is unitary if and only if the following conditions hold:

   (i)   *U_{11} is an isometry;*

   (ii) *U_{22} is a coisometry;*

   (iii) *U_{12} is a partial isometry from $\ker(U_{22})$ to the $\mathrm{coker}(U_{11})$.*

The next lemma (see also [22, Lemma 3.4]) relates the sign of the quantized differential of a triangular unitary operator U to the kernel of the compression PUP of U on P (corresponding to Sonin's space in the application related to Weil's explicit formula).

Lemma 4.4. *With the notation of Proposition* 4.3, *let* f *be a positive operator and* S *the orthogonal projection to* $\ker(U_{22})$. *Let*

$$\widetilde{S} = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},$$

and $đU$ *the quantized differential of* U, *then*

$$-\frac{1}{2}\operatorname{Tr}(f\,U^*đU) = \operatorname{Tr}(f\,\widetilde{S}) \geq 0.$$

Proof. We first show that $-\frac{1}{2}U^*đU = \widetilde{S}$. We have

$$U^*PU = \begin{bmatrix} U_{11}^* & 0 \\ U_{12}^* & U_{22}^* \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & U_{22}^* \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & U_{22}^*U_{22} \end{bmatrix}.$$

Then it follows that

$$U^*PU - P = \begin{bmatrix} 0 & 0 \\ 0 & U_{22}^*U_{22} - 1 \end{bmatrix}.$$

One also has $U_{22}^*U_{22} - 1 = -S$ since U_{22} is a coisometry. Then the claim follows from the equality $U^*đU = 2(U^*PU - P)$ together with the fact that the trace of a product of two positive operators is non-negative. ∎

4.3. The semilocal functional equation

In this part we explain the functional equation in the semilocal case, by giving a proof of the local functional equation that extends naturally to the semilocal framework. We first introduce some notations.

We denote by I the unitary inversion operator in the subspace $L^2(\mathbb{R})^{\mathrm{ev}}$ of even functions in $L^2(\mathbb{R})$ defined as

$$I(\xi)(x) := |x|^{-1}\xi(x^{-1}).$$

The scaling operator $\vartheta(\lambda)$, defined for $\lambda \in \mathbb{R}_+^*$, is the unitary operator in $L^2(\mathbb{R})^{\mathrm{ev}}$ given by

$$\vartheta(\lambda)(\xi)(x) = \lambda^{-1/2}\xi(\lambda^{-1}x).$$

One then has $I \circ \vartheta(\lambda) = \vartheta(\lambda^{-1}) \circ I$. The Fourier transform $\mathbb{F}_{e_\mathbb{R}}$ is the unitary operator in $L^2(\mathbb{R})^{\mathrm{ev}}$ defined by

$$\mathbb{F}_{e_\mathbb{R}}(\xi)(y) = \int \xi(x)\exp(-2\pi ixy)\,dx.$$

One has $\mathbb{F}_{e_\mathbb{R}} \circ \vartheta(\lambda) = \vartheta(\lambda^{-1}) \circ \mathbb{F}_{e_\mathbb{R}}$. It follows that $I \circ \mathbb{F}_{e_\mathbb{R}}$ commutes with the scaling

$$(I \circ \mathbb{F}_{e_\mathbb{R}}) \circ \vartheta(\lambda) = I \circ (\mathbb{F}_{e_\mathbb{R}} \circ \vartheta(\lambda)) = I \circ (\vartheta(\lambda^{-1}) \circ \mathbb{F}_{e_\mathbb{R}})$$

$$= (I \circ \vartheta(\lambda^{-1})) \circ \mathbb{F}_{e_\mathbb{R}} = \vartheta(\lambda) \circ (I \circ \mathbb{F}_{e_\mathbb{R}}).$$

The representation ϑ of $\mathbb{R}_+^*$ by scaling in $L^2(\mathbb{R})^{\mathrm{ev}}$ is unitarily equivalent to the multiplication action with the character λ^{-is} in $L^2(\mathbb{R})$, through the Fourier transform $\mathbb{F}_\mu \circ w$, where w is the unitary isomorphism

$$w \colon L^2(\mathbb{R})^{\mathrm{ev}} \to L^2(\mathbb{R}_+^*, dx/x), \quad w(\xi)(\lambda) = \lambda^{1/2}\xi(\lambda) \quad \forall \lambda > 0$$

and $\mathbb{F}_\mu$ denotes the multiplicative Fourier transform

$$\mathbb{F}_\mu(f)(s) := \int_0^\infty f(v)v^{-is}\, d^*v, \quad d^*v := dv/v.$$

The von Neumann algebra generated in $L^2(\mathbb{R})$ by the multiplications operators by λ^{-is} is equal to $L^\infty(\mathbb{R})$ acting by multiplication. The bicommutant theorem ensures that a unitary operator commuting with this representation is a multiplication operator by a function of modulus one on $\mathbb{R}$. Then it follows that the composite operator in $L^2(\mathbb{R})$

$$(\mathbb{F}_\mu \circ w) \circ (I \circ \mathbb{F}_{e_\mathbb{R}}) \circ (\mathbb{F}_\mu \circ w)^{-1} \tag{11}$$

is the multiplication by a function $u \in L^\infty(\mathbb{R})$ of modulus one. Next, we shall develop on the following schematic diagram (Figure 2).

Fact 4.5. The function u is the ratio of local archimedean factors

$$u(s) = \frac{\zeta(1/2 - is)}{\zeta(1/2 + is)} = \frac{\gamma(1/2 + is)}{\gamma(1/2 - is)}, \quad \gamma(z) := \pi^{-z/2}\Gamma(z/2). \tag{12}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$ be an even function with

$$f(0) = 0 = \int f(x)\, dx.$$

The Poisson formula (for $x > 0$)

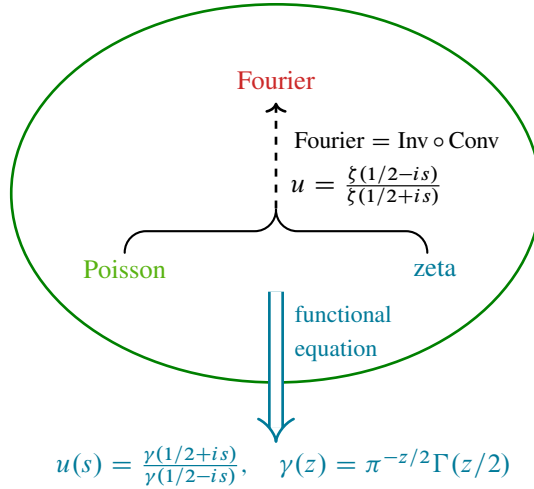$$x\sum_{n>0} f(nx) = \sum_{n>0} \hat{f}\left(\frac{n}{x}\right)$$

implies

$$x^{1/2}\sum_{n>0} f(nx) = x^{-1/2}\sum_{n>0} \hat{f}\left(\frac{n}{x}\right).$$

Define

$$\mathcal{E}(f)(x) := x^{1/2}\sum_{n>0} f(nx) \tag{13}$$

then we have: $\mathcal{E}(f)(x) = \mathcal{E}(\mathbb{F}_{e_\mathbb{R}} f)(x^{-1})$. By applying the multiplicative Fourier $\mathbb{F}_\mu$ on the right-hand side of this equality, one has (for $s \in \mathbb{R}$)

$$\mathbb{F}_\mu(\mathcal{E}(f))(s) = \zeta(1/2 - is)\mathbb{F}_\mu(wf)(s), \tag{14}$$

$$u(s) = \frac{\gamma(1/2+is)}{\gamma(1/2-is)}, \quad \gamma(z) = \pi^{-z/2}\Gamma(z/2)$$

Figure 2. The Poisson formula yields a formula for the unitary u as a ratio of the zeta function on the critical line and the functional equation for the complete zeta function expresses this ratio as the ratio of local archimedean factors.

and by the above equality: $\mathbb{F}_\mu(\mathcal{E}(f))(s) = \mathbb{F}_\mu(\mathcal{E}(\mathbb{F}_{e_\mathbb{R}} f))(-s)$. Thus, with (14), one obtains

$$\zeta(1/2 - is)\mathbb{F}_\mu(wf)(s) = \zeta(1/2 + is)\mathbb{F}_\mu(w\mathbb{F}_{e_\mathbb{R}} f)(-s)$$

Finally, in view of (11), one writes

$$(\mathbb{F}_\mu \circ w) \circ (I \circ \mathbb{F}_{e_\mathbb{R}})(f)(s) = \mathbb{F}_\mu(w\mathbb{F}_{e_\mathbb{R}} f)(-s) = \frac{\zeta(1/2 - is)}{\zeta(1/2 + is)}\mathbb{F}_\mu(wf)(s).$$

Thus, $u = \frac{\zeta(1/2-is)}{\zeta(1/2+is)}$ is the ratio of local factors. ∎

An argument similar to the one just developed in the above proof applies in the semilocal case (when finitely many places are involved, inclusive the archimedean) [22]. More precisely, one lets S be a finite set of places of $\mathbb{Q}$ containing the archimedean place and one considers the semilocal adele class space

$$X_{\mathbb{Q},S} := \mathbb{A}_{\mathbb{Q},S}/\Gamma, \tag{15}$$

where $\mathbb{A}_{\mathbb{Q},S} = \prod_{v \in S} \mathbb{Q}_v$ is the product of the local fields acted upon (by multiplication) by the subgroup

$$\Gamma = \{\pm p_1^{n_1} \cdots p_k^{n_k} : p_j \in S\setminus\{\infty\}, \, n_j \in \mathbb{Z}\} \subset \mathbb{Q}^\times. \tag{16}$$

Even though $X_{\mathbb{Q},S}$ is a noncommutative space, at the topological level and when S contains at least three places, this space is well behaved at the measure theory level

because the additive and multiplicative Haar measures are equivalent on the finite product of local fields [5, 24]. There is a natural Hilbert space $L^2(X_{\mathbb{Q},S})$ of square integrable functions on $X_{\mathbb{Q},S}$. Moreover, and very importantly, the Fourier transform $\mathbb{F}_\alpha$ on $\mathbb{A}_{\mathbb{Q},S}$ descends to a unitary operator $\underline{\mathbb{F}}_\alpha$ in $L^2(X_{\mathbb{Q},S})$. After passing to the dual of the group $C_{\mathbb{Q},S} = \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q},S})/\Gamma$ by the Fourier transform $\mathbb{F}_C$ and using the inversion I, $\mathbb{F}_\alpha$ reads as the multiplication by a function u of modulus 1 on the dual of $C_{\mathbb{Q},S}$ (see [24, Chapter 2])

$$\underline{\mathbb{F}}_\alpha = w^{-1} \circ I \circ \mathbb{F}_C^{-1} \circ u \circ \mathbb{F}_C \circ w. \tag{17}$$

Let π_v be the projection from the dual of $C_{\mathbb{Q},S}$ to the dual $\widehat{\mathbb{Q}_v^\times}$, the unitary u is of the form

$$u = \prod_S u_v \circ \pi_v, \tag{18}$$

where the $u_v \in L^\infty(\widehat{\mathbb{Q}_v^\times})$ are the functions involved in the local functional equation of Tate [35]:

$$\int_{\mathbb{Q}_v^\times} \mathbb{F}_{\alpha_v}(f)(x)\chi(x^{-1})|x|^{1/2}\,d^*x = u_v(\chi) \int_{\mathbb{Q}_v^\times} f(x)\chi(x)|x|^{1/2}\,d^*x. \tag{19}$$

Here, $\mathbb{F}_{\alpha_v}$ denotes the Fourier transform relative to the canonical additive character α_v of the local field $\mathbb{Q}_v$.

One knows that the function u_v is the ratio of the local factors of L-functions. When restricting to the "zeta sector", i.e., to the subspace of $L^2(X_{\mathbb{Q},S})$ of functions invariant under the action of the maximal compact subgroup of $C_{\mathbb{Q},S}$, the function u is the product of ratios of local factors of the Riemann zeta function:

$$u = \rho_\infty \prod \rho_p.$$

This can be proved directly using the same argument as in the above proof of Fact 4.5 (see [22]). In [19], we have developed the notion of quasi-inner function as a generalization of Beurling's notion of inner function which we first related to the main Lemma 4.4. In [19, Theorem 4.1] we showed that the product $u = \rho_\infty \prod \rho_p$ of ratios of local factors of the Riemann zeta function is a quasi-inner function

Theorem 4.6. *The product $u = \rho_\infty \prod \rho_p$ of ratios of local factors over a finite set of places of $\mathbb{Q}$ containing the archimedean place is a quasi-inner function relative to*

$$\mathbb{C}_- = \left\{ z \in \mathbb{C} \mid \Re(z) \leq \frac{1}{2} \right\}.$$

This fact shows that the quantized differential $u^* đu$ fulfills the hypothesis of the main Lemma 4.4 "modulo infinitesimals", i.e., working in the Calkin algebra (quotient of the algebra of bounded operators by the ideal of compact ones).

4.4. Quantized calculus and the semilocal trace formula

The key fact enabling the development of the semilocal framework of Section 4.3 is the equivalence of the additive and multiplicative Haar measures on a finite product of local fields. This fact fails in the global adelic case. The reason why one can go around this difficulty in order to understand the location of the zeros of the Riemann zeta function is that Weil's criterion

$$RH \iff \sum_v W_v(g * g^*) \le 0 \quad \forall g \in C_c^\infty(\mathbb{R}_+^*) \text{ with } \hat{g}\left(\pm\frac{i}{2}\right) = 0, \qquad (20)$$

only involves finitely many primes at a time. Indeed, while the sum on the right is extended to all places of $\mathbb{Q}$, for $v = p$ a non-archimedean prime, the functional

$$W_p(f) := (\log p) \sum_{m=1}^{\infty} p^{-m/2}\big(f(p^m) + f(p^{-m})\big) \qquad (21)$$

vanishes on test functions with support in the interval (p^{-1}, p). Thus, $W_v(g * g^*) \ne 0$ for only finitely many v. The functionals W_v are given by the Riemann–Weil explicit formula

$$\hat{f}\left(\frac{i}{2}\right) - \sum_{\frac{1}{2}+is \in Z} \hat{f}(s) + \hat{f}\left(-\frac{i}{2}\right) = \sum_v W_v(f),$$

$$\hat{f}(s) := \int_0^{\infty} f(x) x^{-is}\, d^*x, \quad d^*x = \frac{dx}{x}, \qquad (22)$$

where Z is the multi-set of the non-trivial zeros of the Riemann zeta function.

The archimedean distribution $W_{\mathbb{R}}$ is defined as

$$W_{\mathbb{R}}(f) = (\log 4\pi + \gamma) f(1) + \int_1^{\infty} \left(f(x) + f(x^{-1}) - 2x^{-1/2} f(1)\right) \frac{x^{1/2}}{x - x^{-1}}\, d^*x. \qquad (23)$$

The key equality now is the local trace formula of [5] (as revisited in [24])

$$\sum_{v \in S} W_v(f) = \frac{1}{2} \operatorname{Tr}(\hat{f} u^*\, đu), \qquad (24)$$

where the notations for the right-hand side are as in Section 4.3. Lemma 4.4 would imply the negativity criterion (20), if u fulfilled the required hypothesis of that lemma (in Section 4.3 we pointed out that the failure is only by an infinitesimal). When S is reduced to the single archimedean place, this difficulty can be bypassed by analyzing the effect of the infinitesimal [23], and the expected negativity of the criterion can be derived from Lemma 4.4, where the role of the orthogonal projection to $\ker(U_{22})$ is played by the orthogonal projection $\mathbf{S}$ on Sonin's space, one has

Theorem 4.7 ([23]). *Let $g \in C_c^\infty(\mathbb{R}_+^*)$ have support in the interval $[2^{-1/2}, 2^{1/2}]$ and Fourier transform vanishing at $\frac{i}{2}$ and 0. Then the following inequality holds:*

$$-W_{\mathbb{R}}(g * g^*) = W_\infty(g * g^*) \ge \operatorname{Tr}(\vartheta(g)\, \mathbf{S}\, \vartheta(g)^*). \qquad (25)$$

In the next Section 4.5 we shall see that a difficulty to extend this result in the semilocal case is that the restriction of Weil's quadratic form to test functions with support in the interval $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$ admits (for large values of λ) extremely small eigenvalues. This fact prevents the use of the approximation method developed in [23].

4.5. The radical of the Weil quadratic form

Weil's quadratic form

$$QW(f, g) := \sum_{1/2+is \in Z} \overline{\hat{f}(\bar{s})} \hat{g}(s) \tag{26}$$

admits a non-trivial radical when working with general test functions (i.e., if one drops the compact support condition of (20)). This radical contains the range of the map $\mathcal{E}$ defined on the codimension two subspace $\mathcal{S}_0^{ev}$ of even Schwartz functions fulfilling the equalities $f(0) = 0 = \hat{f}(0)$ by the formula

$$\mathcal{E}(f)(x) = x^{1/2} \sum_{n>0} f(nx) \quad \forall f \in \mathcal{S}_0^{ev}. \tag{27}$$

Indeed, elements $h = \mathcal{E}(f)$ of this range fulfill $\hat{h}(s) = 0$ when $1/2 + is \in Z$.

The Riemann–Weil explicit formula expresses $QW(f, g)$ on test functions f, g whose support is contained in $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$ as a finite sum, thus in a form suitable for numerical testings since the sum involves primes less than λ^2. In [21] we provided numerical evidence to the fact that as λ increases the operator in $\mathcal{H}(\lambda) := L^2([\lambda^{-1}, \lambda], d^*u)$ associated to the quadratic form QW_λ (restriction of QW) admits a finite number of extremely small positive eigenvalues. For instance, we have found that when $\lambda^2 = 11$ the smallest positive eigenvalue is 2.389×10^{-48}. The presence of these minuscule positive eigenvalues is explained conceptually by the fact that the radical of Weil's quadratic form contains the range of the map $\mathcal{E}$.

In [21] we also gave an excellent approximation of the related eigenfunctions and we showed that even though RH implies $QW_\lambda > 0$, (thus that its radical is $\{0\}$), one can nevertheless construct, by making use of (27), functions g with support in $[\lambda^{-1}, \lambda]$ fulfilling:

$$QW_\lambda(g) \ll \|g\|^2.$$

Indeed, let $\mathcal{P}_\lambda$ and $\widehat{\mathcal{P}}_\lambda$ be the cutoff projections in the Hilbert space $L^2(\mathbb{R})^{ev}$, then the projection $\mathcal{P}_\lambda$ is given by the multiplication with the characteristic function of the interval $[-\lambda, \lambda] \subset \mathbb{R}$. The projection $\widehat{\mathcal{P}}_\lambda$ is the conjugate of $\mathcal{P}_\lambda$ by the Fourier transform $\mathbb{F}_{e_\mathbb{R}}$. If the even function $f \in \mathcal{S}_0^{ev}$ belongs to the range of $\mathcal{P}_\lambda$, then the support of $\mathcal{E}(f)$ is contained in $(0, \lambda] \subset \mathbb{R}_+^*$. On the other hand, when $f \in \mathcal{S}_0^{ev}$ is in the range of $\widehat{\mathcal{P}}_\lambda$ the Poisson formula

$$\mathcal{E}(f)(x) = \mathcal{E}(\mathbb{F}_{e_\mathbb{R}} f)(x^{-1})$$

shows that the support of $\mathcal{E}(f)$ is contained in $[\lambda^{-1}, \infty)$. The obstruction to obtain an element $\mathcal{E}(f)$ in the radical of QW_λ is provided by the equality $\mathcal{P}_\lambda \cap \widehat{\mathcal{P}}_\lambda = \{0\}$.

The seminal work of Slepian and Pollack [32–34] on band limited functions shows that while $\mathscr{P}_\lambda \cap \widehat{\mathscr{P}}_\lambda = \{0\}$ the prolate functions for small enough eigenvalues almost belong to $\mathscr{P}_\lambda$ and to $\widehat{\mathscr{P}}_\lambda$. Using this fact we constructed in [21] functions denoted "prolate vectors", on which QW_λ takes extremely small, non-zero values. In the same article, we verified concretely that the orthogonalization of the prolate vectors give an excellent approximation of the eigenvectors associated to the smallest eigenvalues of Weil's quadratic form. The first $k + 2$ prolate vectors, determine a k-dimensional subspace of

$$L^2([\lambda^{-1}, \lambda], d^*u) \simeq L^2(\mathbb{R}_+^*/\lambda^{2\mathbb{Z}}, d^*u)$$

on which the associated orthogonal projection $\Pi(\lambda, k)$ acts. Note that the construction of $\Pi(\lambda, k)$ only uses the prolate vectors without any reference to QW_λ.

4.6. Spectral triples and zeros of zeta

The notion of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ formalizes the concept of a "spectral geometry", where the underlying "space" is encoded by the algebra $\mathcal{A}$ (in general noncommutative) that acts by operators in the Hilbert space $\mathcal{H}$. The self-adjoint operator D in $\mathcal{H}$ encodes both the metric aspect of the space (by the formula $d(\mu, \nu) = \text{Sup} \, |\mu(A) - \nu(A)|$, with $\|[D, A]\| \le 1$ for the distance between states on the algebra $\mathcal{A}$) and the fundamental class in K-homology (and also in KO-homology if a real structure J is present).

In [21], we have constructed spectral triples

$$\Theta(\lambda, k) = (\mathcal{A}(\lambda), \mathcal{H}(\lambda), D(\lambda, k))$$

making use of the orthogonal projections $\Pi(\lambda, k)$ recalled at the end of Section 4.5. For this application, the algebra $\mathcal{A}$ is

$$\mathcal{A}(\lambda) := C^\infty(\mathbb{R}_+^*/\lambda^{2\mathbb{Z}})$$

acting by multiplication on $\mathcal{H}(\lambda) := L^2(\mathbb{R}_+^*/\lambda^{2\mathbb{Z}}, d^*u)$. The operator $D(\lambda, k)$ is the finite rank perturbation

$$D(\lambda, k) := (1 - \Pi(\lambda, k)) \circ D_0 \circ (1 - \Pi(\lambda, k)), \quad D_0 = -iu\partial_u \tag{28}$$

of the standard Dirac operator $D_0 = -iu\partial_u$ (with periodic boundary conditions when viewed in $L^2([\lambda^{-1}, \lambda], d^*u) \simeq L^2(\mathbb{R}_+^*/\lambda^{2\mathbb{Z}}))$. In [21, Proposition 4.2] we explain how we are able to grasp the zeros of the Riemann zeta function up to height $t = 2\pi\mu$, by computing the spectra of the operators $D(\lambda, k)$ for $\lambda^2 \le \mu$. The computation of the involved prolate vectors only requires the use of integers less than the integer part of λ, because all other terms in the sum involved in the definition of the map $\mathcal{E}$ vanish due to the support condition. This means that we only use integers n between 1 and λ. In this way, we have found a remarkable agreement with the first 31 zeros of zeta only implementing the integers 2, 3, 4!

This fact is all the more remarkable since the above restriction on the involved integers (in the sum defining the map $\mathcal{E}$) coincides exactly with the restriction in the partial sums occurring in the Riemann–Siegel formula (see [2] and [31, Section 6.1]). While these findings are largely depending on computer calculations of spectra of large matrices, we have provided their conceptual explanation by introducing the notion of zeta-cycle [21]. With Σ_μ denoting the Poincaré series operator, i.e., the linear map defined on functions $g: \mathbb{R}_+^* \to \mathbb{C}$ by the formula

$$(\Sigma_\mu g)(u) := \sum_{k \in \mathbb{Z}} g(\mu^k u). \tag{29}$$

We can state the following result.

Theorem 4.8. *For $\mu > 1$, let C be the circle $C := \mathbb{R}_+^*/\mu^{\mathbb{Z}}$.*

(i) *The spectrum of the action of the multiplicative group $\mathbb{R}_+^*$ on the orthogonal of $\Sigma_\mu \mathcal{E}(\mathcal{S}_0^{\mathrm{ev}})$ in $L^2(C)$ is formed of imaginary parts of zeros of zeta on the critical line.*

(ii) *Let $s > 0$ be such that $\zeta(1/2 + is) = 0$, then any circle of length an integral multiple of $2\pi/s$ is a zeta cycle and its spectrum contains is.*

Remark 4.9. The spectral triples $\Theta(\lambda, k) = (\mathcal{A}(\lambda), \mathcal{H}(\lambda), D(\lambda, k))$ have the same ultra-violet spectral behavior as the Dirac operator $D_0 = -iu\partial_u$ on the circle $C = \mathbb{R}_+^*/\lambda^{2\mathbb{Z}}$. In particular, the number of eigenvalues with absolute value less than E grows linearly with E. The ultraviolet behavior of the zeros of the Riemann zeta function is given by Riemann's formula for the number $N(E)$ of zeros of imaginary part between 0 and E,

$$N(E) = \frac{E}{2\pi} \log \frac{E}{2\pi} - \frac{E}{2\pi} + O(\log E). \tag{30}$$

The problem of finding a Dirac operator with the ultraviolet behavior (30) is solved by the first author and H. Moscovici in [25]. Remarkably, the solution involves the prolate spheroidal wave operator W_λ whose commutation with the projections $\mathcal{P}_\lambda$ and $\widehat{\mathcal{P}}_\lambda$ plays a key role in [32–34].

4.7. Prolate vectors and the semilocal framework

The construction of the prolate vectors (and of the projection $\Pi(\lambda, k)$) makes use of the map $\mathcal{E}$. In this part we exhibit the relation between this construction and the semilocal framework, showing that the map $\mathcal{E}$ appears naturally in the quotient $X_{\mathbb{Q},S}$ for functions with small enough support.

Proposition 4.10. *Let $\mu > 1$, $\lambda = \mu^{1/2}$ and f an even function on $\mathbb{R}$ with support in $[-\lambda, \lambda]$. Let $S = \{\infty, 2, 3, \ldots, p'\}$, where p' is the largest prime less than μ. Let $\tilde{f}$ be the function on $X_{\mathbb{Q},S}$ associated to the function $\otimes_{v \in S \setminus \infty} 1_{\mathbb{Z}_v} \otimes f$ where $1_{\mathbb{Z}_v}$ is the characteristic function of the maximal compact subring $\mathbb{Z}_v \subset \mathbb{Q}_v$. Then the following*

equality holds

$$w(\widetilde{f})(u) = 2\mathcal{E}(f)(u) \quad \forall u > \lambda^{-1}. \tag{31}$$

Proof. For $u \in \mathbb{R}_+^*$, one has by construction

$$w(\widetilde{f})(u) = u^{1/2} \sum_{g \in \Gamma} (\otimes_{v \in S \backslash \infty} 1_{\mathbb{Z}_v} \otimes f)(g(1, 1, \ldots, u)).$$

The terms in the sum vanish unless g is an integer since otherwise $g(1, 1, \ldots) \notin \prod \mathbb{Z}_v$. Moreover, in that case the terms are equal to $f(gu)$. Thus, the sum becomes

$$w(\widetilde{f})(u) = u^{1/2} \sum_{g \in \mathbb{Z} \cap \Gamma} f(gu).$$

Assume $u > \lambda^{-1}$. Then for any integer n, with $|n| > \mu = \lambda^2$, one has $f(nu) = 0$ since the support of f is in $[-\lambda, \lambda]$. Moreover, the following sets are equal since all prime factors of integers less than μ are in S:

$$Y = \mathbb{Z} \cap \Gamma \cap [-\mu, \mu] = \{\pm n \mid n \in \mathbb{N}, 0 < n \le \mu\},$$

and thus for $u > \lambda^{-1}$, one obtains (since f is even)

$$w(\widetilde{f})(u) = u^{1/2} \sum_{g \in Y} f(gu) = 2\mathcal{E}(f)(u),$$

which gives the required equality. ∎

For each prime p the characteristic function $1_{\mathbb{Z}_p}$ is its own Fourier transform on $\mathbb{Q}_p$ and this implies that the semilocal Fourier transform $\underline{\mathbb{F}}_\alpha$ acts as the archimedean Fourier transform $\mathbb{F}_{e_\mathbb{R}}$ on functions $\widetilde{f}$ on the quotient $X_{\mathbb{Q},S}$, associated as above to simple tensors $\otimes 1_{\mathbb{Z}_p} \otimes f$. Thus,

$$\underline{\mathbb{F}}_\alpha(\widetilde{f}) = \widetilde{\mathbb{F}_{e_\mathbb{R}}(f)}.$$

Moreover, if the support of f is contained in the ball of radius λ the same holds for $\widetilde{f}$. Together with Proposition 4.10 this fact suggests that the minuscule eigenvalues of the Weil quadratic form of Section 4.5 can be reinterpreted intrinsically in the semilocal framework, without using the map $\mathcal{E}$, just by analyzing the relative position of the semilocal analogue of the cutoff projections $\mathcal{P}_\lambda$ and $\widehat{\mathcal{P}}_\lambda$.

References

[1] P. Baum, A. Connes, and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras. In C^*-algebras: 1943–1993 (San Antonio, TX, 1993), pp. 240–291, Contemp. Math. 167, American Mathematical Society, Providence, RI, 1994 Zbl 0830.46061 MR 1292018

[2] M. V. Berry, The Riemann–Siegel expansion for the zeta function: high orders and remainders. Proc. Roy. Soc. London Ser. A **450** (1995), no. 1939, 439–462 Zbl 0842.11030 MR 1349513

[3] J.-B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. Selecta Math. (N.S.) **1** (1995), no. 3, 411–457 Zbl 0842.46040 MR 1366621

[4] O. Bratteli and D. W. Robinson, Operator algebras and quantum-statistical mechanics. I, II. Texts Monogr. Phys., Springer, New York-Berlin, 1979–1981 Zbl 0421.46048 MR 611508

[5] A. Connes, Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. Selecta Math. (N.S.) **5** (1999), no. 1, 29–106 Zbl 0945.11015 MR 1694895

[6] A. Connes, An essay on the Riemann hypothesis. In Open problems in mathematics, pp. 225–257, Springer, Cham, 2016 Zbl 1353.11095 MR 3526936

[7] A. Connes and C. Consani, From monoids to hyperstructures: in search of an absolute arithmetic. In Casimir force, Casimir operators and the Riemann hypothesis, pp. 147–198, Walter de Gruyter, Berlin, 2010 Zbl 1234.14002 MR 2777715

[8] A. Connes and C. Consani, Schemes over $\mathbb{F}_1$ and zeta functions. Compos. Math. **146** (2010), no. 6, 1383–1415 Zbl 1201.14001 MR 2735370

[9] A. Connes and C. Consani, Characteristic 1, entropy and the absolute point. In Noncommutative geometry, arithmetic, and related topics, pp. 75–139, Johns Hopkins University Press, Baltimore, MD, 2011 Zbl 1273.11140 MR 2907005

[10] A. Connes and C. Consani, On the arithmetic of the BC-system. J. Noncommut. Geom. **8** (2014), no. 3, 873–945 Zbl 1302.11069 MR 3261604

[11] A. Connes and C. Consani, The arithmetic site. C. R. Math. Acad. Sci. Paris **352** (2014), no. 12, 971–975 Zbl 1315.11054 MR 3276804

[12] A. Connes and C. Consani, Absolute algebra and Segal's Γ-rings: au dessous de $\overline{\mathrm{Spec}(\mathbb{Z})}$. J. Number Theory **162** (2016), 518–551 Zbl 1409.14046 MR 3448278

[13] A. Connes and C. Consani, Geometry of the arithmetic site. Adv. Math. **291** (2016), 274–329 Zbl 1368.14038 MR 3459019

[14] A. Connes and C. Consani, Geometry of the scaling site. Selecta Math. (N.S.) **23** (2017), no. 3, 1803–1850 Zbl 1406.11118 MR 3663595

[15] A. Connes and C. Consani, Homological algebra in characteristic one. High. Struct. **3** (2019), no. 1, 155–247 Zbl 1411.18006 MR 3939048

[16] A. Connes and C. Consani, The Riemann–Roch strategy: complex lift of the scaling site. In Advances in noncommutative geometry—on the occasion of Alain Connes' 70th birthday, pp. 53–125, Springer, Cham, 2019 Zbl 1458.11108 MR 4300552

[17] A. Connes and C. Consani, $\overline{\mathrm{Spec}\,\mathbb{Z}}$ and the Gromov norm. Theory Appl. Categ. **35** (2020), article no. 6, 155–178 Zbl 1444.16061 MR 4066805

[18] A. Connes and C. Consani, On absolute algebraic geometry the affine case. Adv. Math. **390** (2021), article no. 107909 Zbl 1478.14047 MR 4291468

[19] A. Connes and C. Consani, Quasi-inner functions and local factors. J. Number Theory **226** (2021), 139–167 Zbl 1476.46071 MR 4239718

[20] A. Connes and C. Consani, Segal's gamma rings and universal arithmetic. *Q. J. Math.* **72** (2021), no. 1–2, 1–29 Zbl 1484.14003 MR 4271378

[21] A. Connes and C. Consani, Spectral triples and ζ-cycles. *Enseign. Math.* **69** (2023), no. 1–2, pp. 93–148 Zbl 7724321 MR 4599244

[22] A. Connes and C. Consani, The scaling Hamiltonian. *J. Operator Theory* **85** (2021), no. 1, 257–276 Zbl 07606460 MR 4198972

[23] A. Connes and C. Consani, Weil positivity and trace formula the archimedean place. *Selecta Math. (N.S.)* **27** (2021), no. 4, article no. 77 Zbl 1480.46084 MR 4292787

[24] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*. Amer. Math. Soc. Colloq. Publ. 55, American Mathematical Society, Providence, RI; Hindustan Book Agency, New Delhi, 2008 Zbl 1159.58004 MR 2371808

[25] A. Connes and H. Moscovici, The UV prolate spectrum matches the zeros of zeta. *Proc. Natl. Acad. Sci. USA* **119** (2022), no. 22, article no. e2123174119 MR 4477577

[26] A. Connes, D. Sullivan, and N. Teleman, Quasiconformal mappings, operators on Hilbert space, and local formulae for characteristic classes. *Topology* **33** (1994), no. 4, 663–681 Zbl 0840.57013 MR 1293305

[27] B. I. Dundas, T. G. Goodwillie, and R. McCarthy, *The local structure of algebraic K-theory*. Algebr. Appl. 18, Springer, London, 2013 Zbl 1272.55002 MR 3013261

[28] R. Haag, N. M. Hugenholtz, and M. Winnink, On the equilibrium states in quantum statistical mechanics. *Comm. Math. Phys.* **5** (1967), 215–236 Zbl 0171.47102 MR 219283

[29] G. L. Litvinov, The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction. In *Idempotent mathematics and mathematical physics*, pp. 1–17, Contemp. Math. 377, American Mathematical Society, Providence, RI, 2005 Zbl 1096.46045 MR 2148995

[30] R. Meyer, On a representation of the idele class group related to primes and zeros of *L*-functions. *Duke Math. J.* **127** (2005), no. 3, 519–595 Zbl 1079.11044 MR 2132868

[31] S. J. Patterson, *An introduction to the theory of the Riemann zeta-function*. Cambridge Stud. Adv. Math. 14, Cambridge University Press, Cambridge, 1988 Zbl 0831.11045 MR 933558

[32] D. Slepian, Some asymptotic expansions for prolate spheroidal wave functions. *J. Math. and Phys.* **44** (1965), 99–140 Zbl 0128.29601 MR 179392

[33] D. Slepian, Some comments on Fourier analysis, uncertainty and modeling. *SIAM Rev.* **25** (1983), no. 3, 379–393 Zbl 0571.94004 MR 710468

[34] D. Slepian and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. I. *Bell System Tech. J.* **40** (1961), 43–63 Zbl 0184.08601 MR 140732

[35] J. Tate, Fourier analysis in number fields and Hecke's zeta-function. PhD thesis, Princeton University, Princeton, NJ, 1950

Alain Connes
Collège de France, 3 rue d'Ulm, 75231 Paris, France; alain@connes.org

Caterina Consani
Department of Mathematics, The Johns Hopkins University, 3400 N Charles Street, Baltimore, MD 21218, USA; cconsan1@jhu.edu