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Rational homotopy via Sullivan models and enriched Lie algebras

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Abstract. Rational homotopy theory originated in the late 1960s and the early 1970s with the simultaneous but distinct approaches of Quillen (1969), Sullivan (1977) and Bousfield–Kan (1972). Each approach associated to a path connected space X an "algebraic object" A which is then used to construct a *rational completion* of X, $X \to X_{\mathbb{Q}}$. These constructions are homotopy equivalent for simply connected CW complexes of finite type, in which case $H_*(X_{\mathbb{Q}}) \cong H_*(X) \otimes \mathbb{Q}$ and $\pi_*(X_{\mathbb{Q}}) \cong \pi_*(X) \otimes \mathbb{Q}$. Otherwise, they may be different; in fact, Quillen's construction is only available for simply connected spaces.

In this review, discussion is limited to Sullivan's completions, and the notation $X \to X_{\mathbb{Q}}$ is reserved for these. We briefly review the construction, and follow that with a review of developments and examples over the subsequent decades, but often without the proofs. Since the explicit form of Sullivan's completion has lent itself to a wide variety of applications in a range of fields, this survey will necessarily be modest in scope.

To Dennis Sullivan, in celebration of the 80th birthday of a great mathematician, and of his seminal contributions to mathematics

1. Sullivan models

Sullivan's construction begins with a commutative differential graded algebra (cdga) functor associating to each path connected space X a rational cdga $A_{\mathrm{PL}}(X)$ which is entirely analogous to the differential forms on a manifold. In particular, for any point x, $A_{\mathrm{PL}}(x)$ is equipped with a natural augmentation $A_{\mathrm{PL}}(x) \xrightarrow{\cong} \mathbb{Q}$, and so based spaces (X, x) yield augmented cdga maps $A_{\mathrm{PL}}(X) \xrightarrow{\varepsilon_X} \mathbb{Q}$. A *Sullivan model* of X is then a quasi-isomorphism

$$\varphi_X : \left(\bigwedge V, d\right) \to A_{\mathrm{PL}}(X)$$

from a cdga satisfying two conditions:

- $\bigwedge V$ is the *free* graded commutative algebra on $V = V^{\geq 1}$;
- $\bigwedge V$ satisfies the following nilpotence condition: V is the increasing union of subspaces V_n with $V_0 := V \cap \ker d$ and $d : V_{n+1} \to \bigwedge V_n$.

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These are now called *Sullivan algebras*. For simplicity we will frequently write $\bigwedge V$ for $(\bigwedge V, d)$ when the differential is clear from the context.

Sullivan algebras have a second (*wedge*) gradation: $\bigwedge V \cong \bigoplus_{q\geq 0} \bigwedge^q V$, where $\bigwedge^q V := V \wedge \cdots \wedge V$ (q factors), and a Sullivan algebra is *minimal* if $d\colon V \to \bigwedge^{\geq 2} V$. Each cdga A with $A^0 \cong \mathbb{Q}$ has a unique (up to isomorphism) minimal Sullivan model, and the minimal model of a path connected space X is by definition the minimal model of $A_{\mathrm{PL}}(X)$.

Secondly, to each Sullivan algebra $(\bigwedge V, d)$ is associated a space $\langle \bigwedge V\rangle$, its geometric realization. The geometric realization $\langle\ \rangle$ is a functor related to A_{PL} by an adjointness property [16, § 1.6]: for any path connected space Z and any Sullivan algebra we have a natural bijection

$$\mathrm{Hom}\big(\bigwedge V, A_{\mathrm{PL}}(Z)\big) \cong \mathrm{Hom}\big(Z, \langle \bigwedge V\rangle\big).$$

Here Hom on the left denotes the set of morphisms of differential graded algebras, and Hom on the right denotes the set of simplicial maps. If $\bigwedge V$ is a Sullivan model of X, then these constructions yield a unique continuous map

$$X \to \langle \bigwedge V\rangle.$$

It follows from the nilpotence condition for any Sullivan algebra that $(\bigwedge V, d) \cong \varinjlim(\bigwedge V_\alpha, d)$ where the $V_\alpha \subset V$ are the finite-dimensional subspaces for which $d\colon V_\alpha \to \bigwedge V_\alpha$. This then implies that

$$\langle \bigwedge V\rangle \cong \varprojlim_{\alpha} \langle \bigwedge V_\alpha\rangle.$$

When $\bigwedge V$ is the minimal Sullivan model of X then, by definition, $\langle \bigwedge V\rangle \cong X_{\mathbb{Q}}$ and the map $X \to X_{\mathbb{Q}}$ is Sullivan's completion. These constructions induce isomorphisms of graded vector spaces

$$H^*(X;\mathbb{Q}) \xleftarrow{\cong} H^*\big(\bigwedge V\big) \quad \text{and} \quad \pi_*(X_{\mathbb{Q}}) \cong \mathrm{Hom}(V, \mathbb{Q}).$$

Moreover, a map $f\colon X \to Y$ induces a map $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$.

Example 1. A minimal Sullivan model $\varphi\colon \bigwedge V_X \to A_{\mathrm{PL}}(X)$ induces by adjunction a map

$$\varphi_X\colon X \to X_{\mathbb{Q}}$$

such that $H_*(\varphi_X;\mathbb{Q})\colon H_*(X;\mathbb{Q}) \to H_*(X_{\mathbb{Q}};\mathbb{Q})$ is injective.

In fact, [12, Appendix] φ_X factors as

$$\bigwedge V_X \xrightarrow{\sigma} A_{\mathrm{PL}}\langle \bigwedge V_X\rangle \xrightarrow{A_{\mathrm{PL}}(\varphi_X)} A_{\mathrm{PL}}(X),$$

where σ is the unit of the adjunction. It follows that $H^*(\varphi_X;\mathbb{Q}) \cong H^*(A_{\mathrm{PL}}(\varphi_X))$ is surjective, and therefore that $H_*(\varphi_X;\mathbb{Q})$ is injective.

1.1. LS category

The *Lusternik–Schnirelmann (LS) category*, cat X, of a space X is the least m such that X can be covered by $(m+1)$ open sets, each contractible in X. The LS category, cat$(\bigwedge V, d)$, of a Sullivan algebra $(\bigwedge V, d)$ is the least m such that $(\bigwedge V, d)$ is a retract of a minimal Sullivan model of the cdga $\bigwedge V / \bigwedge^{>m} V$. These are related [16, Theorem 9.2]: if $(\bigwedge V, d)$ is the minimal Sullivan model of a space X then

$$\mathrm{cat}\big(\bigwedge V, d\big) \leq \mathrm{cat}\, X.$$

This illustrates the importance of the interaction of the multiplicative structure of $\bigwedge V$ with the differential. In fact, there are spaces whose models satisfy $H^{\geq 1}(\bigwedge V) \cdot H^{\geq 1}(\bigwedge V) = 0$ but for which $\mathrm{cat}(\bigwedge V, d) = \infty$.

Example 2. Let $(\bigwedge W, d) = (\bigwedge(x, y, t), d)$, $dx = dy = 0$, $dt = xy$, $\deg x = \deg y = \deg t = 1$, be the minimal model of the Heisenberg manifold. In this model the cohomology in degree 3 is generated by the cycle xyt. We construct a new Sullivan minimal algebra $\bigwedge V$ by adding successively new generators to W to kill the cycles in $\bigwedge^{\geq 3} V$. In particular, dim V is infinite because V contains the sequence of elements $u_n, n \geq 0$, defined by $du_0 = txy$ and, for $n \geq 1$, $du_n = u_{n-1}xy$. The category of $\bigwedge V$ is infinite because the associated homotopy Lie algebra (see below for the definition) $L_{\geq 1}$ is abelian and infinite-dimensional [16, Chapters 9, 10]. On the other hand, by construction $H^{\geq 1}(\bigwedge V) \cdot H^{\geq 1}(\bigwedge V) = 0$.

Remark. Sullivan completions provide a computable bridge connecting $H^*(X; \mathbb{Q})$ and $\pi_*(X_{\mathbb{Q}})$. If X is a finite simply connected CW complex, then

$$H^*(X_{\mathbb{Q}}) \cong H^*(X; \mathbb{Q}) \quad \text{and} \quad \pi_*(X_{\mathbb{Q}}) \cong \pi_*(X) \otimes \mathbb{Q}.$$

However, the general situation is much more complex. For instance, the potential difference in cohomology is illustrated by an example of Ivanov and Mikhailov [22], who prove that the cohomology in degree two of the Bousfield–Kan $\mathbb{Q}$-completion of $S^1 \vee S^1$ is uncountable. Now by the Bousfield–Gugenheim theorem [2, Theorem 12.2], in this case the Bousfield–Kan completion and the Sullivan completion are equivalent, so that $H_2(S^1 \vee S^1)_{\mathbb{Q}}$ is uncountable.

Nonetheless, it is often possible to get precise information about the minimal Sullivan model of a space, and to extract interesting consequences from that. From the outset and over the subsequent decades, this has produced a wide variety of applications, including this small set of examples:

- The existence of infinitely many geometrically distinct closed geodesics on most closed manifolds [17, 29].
- Construction of Sullivan models for the space of sections of a fibration [5, 19].
- Estimates for the growth of the rational homotopy of a free loop space [15, 23, 28].

- Computing the rational homology of spaces of long knots [24].
- Rational ellipticity of almost non-negatively curved closed manifolds of cohomogeneity of most two [18].
- Properties of rational Poincaré duality for some intersection spaces [30].

2. Homotopy Lie algebras and enriched Lie algebras

A *graded Lie algebra* L is a graded vector space with a Lie bracket satisfying

$$[x, y] + (-1)^{(\deg x)(\deg y)}[y, x] = 0$$

and

$$[x, [y, z]] = [[x, y], z] + (-1)^{(\deg x)(\deg y)}[y, [x, z]].$$

All Lie algebras in this review are graded, $L = \bigoplus_{n\geq 0} L_n$ and satisfy $[L_p, L_q] \subset L_{p+q}$. For simplicity we may refer to them simply as Lie algebras. Ordinary Lie algebras will be regarded as graded Lie algebras concentrated in degree 0.

We will also need the *suspension*, sM, of a graded vector space M:

$$(sM)_{k+1} = M_k.$$

The standard convention for subscripts and superscripts defines $M^k = M_{-k}$, so that $(sM)^{k-1} = M^k$. Note as well the convention: $\mathrm{Hom}_k(M, \mathbb{Q}) = \mathrm{Hom}(M^k, \mathbb{Q})$.

Now, as already observed by Sullivan, for any Sullivan algebra $(\bigwedge V, d)$ the desuspension of the vector space $\mathrm{Hom}(V, \mathbb{Q})$ is naturally a Lie algebra $L_V = (L_V)_{\geq 0}$, which is the *homotopy Lie algebra* of $\bigwedge V$. Here we regard $\bigwedge^2 V$ as anti-commutative bilinear functions in $\mathrm{Hom}(V, \mathbb{Q}) \times \mathrm{Hom}(V, \mathbb{Q})$, and the Lie bracket is defined by the equation

$$\langle v, s[x, y]\rangle = (-1)^{\deg y+1}\langle d_1 v, sx, sy\rangle, \qquad v \in V, x, y \in L_V,$$

where $d_1 v$ is the component of dv in $\bigwedge^2 V$, and where

$$\langle v \wedge w, sx, sy\rangle = \langle v, sx\rangle \cdot \langle w, sy\rangle + (-1)^{\deg v \cdot \deg w}\langle v, sy\rangle \cdot \langle w, sx\rangle.$$

If $\bigwedge V$ is the minimal model of a space X, then the map $\pi_*(X_{\mathbb{Q}}) \xrightarrow{\cong} sL_V$ converts Whitehead products to Lie brackets.

Example 3 (Spheres). The minimal Sullivan model for S^n is given by $(\bigwedge v, 0)$ with $\deg v = n$ if n is odd, and by $(\bigwedge(v, w), dw = v^2)$ if n is even. It follows that the homotopy Lie algebra in both cases is the free graded Lie algebra $\mathbb{L}(x)$ on a generator x of degree $n - 1$.

For topological spaces X with minimal Sullivan algebra $(\bigwedge V, d)$ the interaction between $H^*(X;\mathbb{Q}) \cong H^*(\bigwedge V)$ and $\pi_*(X_{\mathbb{Q}}) \cong sL_V$ has been fundamental to applications and their proofs. For example, suppose X is a finite simply connected complex of dimension m. Then

- The center of L_V is finite-dimensional [16, Theorem 10.6].
- Either $(L_V)_k = 0$ for all $k > 2(m-1)$, or else $\sum_{i=r}^{r+m} \dim (L_V)_i$ grows exponentially in r [16, Chapters 12, 13].
- If $\dim L_V < \infty$, then $H^*(X;\mathbb{Q})$ is a Poincaré duality algebra [20].
- If X is a finite wedge of spheres S^{n_i}, with $n_i > 1$, then L_V is the free graded Lie algebra generated by $T = \bigoplus_i a_i$ with $\deg a_i = n_i - 1$ [14, Theorem 24.5].

In addition to the Lie bracket, the nilpotence condition of a minimal model $\bigwedge V$ imposes on L_V an important additional structure: L_V is a *complete enriched Lie algebra*. We define these now, noting that this is sometimes different from that given in [6].

First observe that the relation

$$\bigwedge V \cong \varinjlim_\alpha \bigwedge V_\alpha$$

in Section 1 implies that

$$L_V \cong \varprojlim_\alpha L_\alpha,$$

in which the L_α are both finite-dimensional and nilpotent.

Definition. A *complete enriched* Lie algebra is a graded Lie algebra L together with an inverse system of surjections $\rho_\alpha : L \to L_\alpha$ onto finite-dimensional and nilpotent graded Lie algebras, such that

$$L \xrightarrow{\cong} \varprojlim_\alpha L_\alpha.$$

A *complete enriched vector space* is a complete enriched Lie algebra whose Lie bracket is zero.

A *coherent morphism* of complete enriched Lie algebras $f : (L, \{\rho_\alpha\}) \to (L', \{\rho'_\beta\})$ is a morphism of graded Lie algebras such that for each β there is an α and a morphism $f_{\alpha,\beta} : L_\alpha \to L'_\beta$ such that $\rho'_\beta \circ f = f_{\alpha,\beta} \circ \rho_\alpha$.

A *quadratic Sullivan algebra* is a Sullivan algebra $(\bigwedge V, d_1)$ in which $d_1 : V \to \bigwedge^2 V$. Any minimal Sullivan algebra $(\bigwedge V, d)$ has an associated quadratic Sullivan algebra $(\bigwedge V, d_1)$ defined by $(d - d_1)v \in \bigwedge^{\geq 3} V$, $v \in V$. By definition, the homotopy Lie algebra of $(\bigwedge V, d)$ and $(\bigwedge V, d_1)$ coincide. On the other hand [12, Proposition 5] *each complete enriched Lie algebra is the homotopy Lie algebra of a unique quadratic Sullivan algebra.*

If E is a Lie subalgebra of a complete enriched Lie algebra L then its *closure*, $\overline{E} := \varprojlim_\alpha \rho_\alpha(E)$, is a complete enriched Lie subalgebra of L. In particular, let $\{L^n\}_{n\geq 1}$ be the lower central series of L (L^n is the span of the iterated Lie brackets of length n).

Then $L^{(n)} \cong \overline{L^n}$ is called the *enriched lower central series* of L. It may happen that $L \to \varprojlim_n L/L^n$ is not an isomorphism (L may not be pronilpotent) but it is always true that

$$L \xrightarrow{\cong} \varprojlim_n L/L^{(n)}.$$

However, if $\dim L/L^2 < \infty$ then $L^n \cong L^{(n)}$, $n \geq 1$, and so L is pronilpotent.

Moreover, if $(\bigwedge V, d_1)$ is the quadratic Sullivan algebra corresponding to L and the filtration $V_{(n)}$ is defined by

$$V_{(0)} = V \cap \ker d_1 \quad \text{and} \quad V_{(n+1)} = V \cap d_1^{-1}\big(\bigwedge^2 V_{(n)}\big),$$

then [12, Lemma 5]

$$L/L^{(n+2)} \xrightarrow{\cong} \mathrm{Hom}(V_{(n)}, \mathbb{Q}), \quad n \geq 0.$$

Furthermore, associated with any complete enriched Lie algebra L is its fundamental group G_L defined as follows: First, if $\dim L_0/L_0^2 < \infty$, then [16, Chapter 2] G_L is the group of units in the classical completion $\widehat{UL_0}$ of its universal enveloping algebra. This definition then extends to any complete enriched Lie algebra L,

$$G_L := \varprojlim_\alpha G_{L_\alpha}.$$

Thus, an inverse limit argument together with [16, Chapter 2] provides inverse bijections

$$L_0 \underset{\log}{\overset{\exp}{\rightleftarrows}} G_L.$$

In any group, denote the commutator of elements a, b by $[a, b] = aba^{-1}b^{-1}$. The subgroup generated by iterated commutators of length n, denoted by G^n, is normal. We set

$$G_L^{(n)} := \varprojlim_\alpha G_{L_\alpha}^n.$$

Then combining [16, Chapter 2] with an inverse limit argument shows that exp and log restrict to bijections

$$L_0^{(n)} \underset{\log}{\overset{\exp}{\rightleftarrows}} G_L^{(n)}, \qquad n \geq 2.$$

These in turn induce isomorphisms

$$\gamma_n \colon L_0^{(n)}/L_0^{(n+1)} \cong G_L^{(n)}/G_L^{(n+1)}$$

of abelian groups. Moreover, denoting $\exp x_i = a_i \in G_L$, we have

$$\gamma_n([x_1, [\dots [x_{n-1}, x_n] \dots]]) \equiv [a_1, [\dots [a_{n-1}, a_n] \dots]] \mod G_L^{(n+1)}. \tag{1}$$

Finally, if L is the homotopy Lie algebra of a minimal Sullivan algebra $\bigwedge V$, then an inverse limit argument combined with [16, Theorem 2.4] shows that the bijection

$$\pi_1\langle \bigwedge V\rangle \cong \mathrm{Hom}(V^1, \mathbb{Q}) \cong sL_0 \to L_0 \xrightarrow{\exp} G_L$$

is an isomorphism of groups. If $\bigwedge V$ is the minimal Sullivan model of a space X then the completion $X \to \langle \bigwedge V\rangle$ induces a homomorphism of fundamental groups. Combined with the isomorphism above this yields the homomorphism

$$\pi_1(X) \to G_L.$$

This enables Sullivan models as a method to explore the properties of $\pi_1(X)$. For a space X, we denote $L(X)$ the enriched Lie algebra associated to its minimal Sullivan model.

Example 4. If a space X satisfies $\dim H_1(X; \mathbb{Q}) < \infty$, then

$$\pi_1^n(X)/\pi_1^{n+1}(X) \otimes \mathbb{Q} \xrightarrow{\cong} G_L^n / G_L^{n+1}.$$

In fact, denoting $\pi_1(X)$ by G, we have a classifying map $X \to BG$ which induces an isomorphism in H^1 and an injection in H^2. It follows that the corresponding morphism of minimal Sullivan models is an isomorphism in degree 1. In particular,

$$L_0(X) \cong L_0(BG).$$

It is therefore sufficient to establish the isomorphism above when $X \cong BG$. This is proved in [16, Theorem 7.5].

Remark. Even finite CW complexes (eg. $S^1 \vee S^1$) can have a homotopy Lie algebra, L, in which L_0 is infinite-dimensional. Moreover, in general, inverse limits do not preserve short exact sequences. However, a theorem of Bourbaki [1] implies that the inverse limit of short exact sequences of vector spaces is exact, *if the vector spaces are all finite-dimensional*. This has enabled a full development of the properties of enriched Lie algebras [12], including the homotopy Lie algebra L_V.

3. Profree Lie algebras, wedges of spheres, and free groups

Profree Lie algebras are the analogue of free graded Lie algebras among complete enriched graded Lie algebras. In this context we consider graded vector spaces $T = \mathrm{Hom}(sS, \mathbb{Q})$ in which $S = S^{\geq 1}$. Then S is the direct limit of its finite-dimensional subspaces, S_α, and so $T \cong \varprojlim_\alpha T_\alpha$ with $T_\alpha = \mathrm{Hom}(sS_\alpha, \mathbb{Q})$.

Definition. A *profree Lie algebra* is a complete enriched graded Lie algebra $\overline{\mathbb{L}}_T$ in which $\mathbb{L}_T$ is the free graded Lie algebra generated by T (as above) and the enriched structure is provided by the surjections

$$\mathbb{L}_T \longrightarrow \mathbb{L}_{T_\alpha} \longrightarrow \mathbb{L}_{T_\alpha}/(\mathbb{L}_{T_\alpha})^n.$$

If L is a complete enriched Lie algebra and $f\colon T \to L$ is a coherent linear map, then f extends in a unique way as a coherent morphism $\overline{\mathbb{L}}_T \to L$. Moreover, any closed Lie subalgebra of a profree Lie algebra is also profree [12, Proposition 12].

Theorem 1 ([12, Theorem 1]). *The homotopy Lie algebra of a minimal Sullivan algebra* $\bigwedge V$ *is profree* ($L = \overline{\mathbb{L}}_T$) *if and only if there is a quasi-isomorphism*

$$\bigwedge V \xrightarrow{\simeq} \mathbb{Q} \oplus S$$

in which $S \cdot S = 0$.

Remark. If the homotopy Lie algebra of a finite CW complex X is a profree Lie algebra $\overline{\mathbb{L}}_T$, then $sT \cong \mathrm{Hom}(H^{\geq 1}(X), \mathbb{Q}) \cong H_{\geq 1}(X;\mathbb{Q})$.

Proposition 1. *Let* $L = L_0$ *be a complete enriched Lie algebra. Then the following conditions are equivalent:*

(i) L_0 *is profree.*

(ii) *A direct complement* T *of* $L_0^{(2)}$ *in* L_0 *freely generates a free Lie subalgebra of* L.

(iii) *If* x_i *is a basis of* T *then the elements* $\exp x_i$ *freely generate a free subgroup of* G_L.

Proof. It is straightforward to reduce to the case $\dim T < \infty$. In this case T is closed and Theorem 1 shows that (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (iii). The proof depends on the properties of a finite wedge of based circles, $X \cong \bigvee_{i=1}^n S_i^1$. Let $\bigwedge V(1)$ be the minimal Sullivan model of S^1. The base point induces an augmentation $\bigwedge V(1) \to \mathbb{Q}$, and the quasi-isomorphism

$$\bigwedge V(1) \times_{\mathbb{Q}} \cdots \times_{\mathbb{Q}} \bigwedge V(1) \to A_{\mathrm{PL}}(S^1) \times_{\mathbb{Q}} \cdots \times_{\mathbb{Q}} A_{\mathrm{PL}}(S^1) \xrightarrow{\simeq} A_{\mathrm{PL}}(S^1 \vee \cdots \vee S^1)$$

identifies a minimal Sullivan model $\bigwedge V$ of $\bigwedge V(1) \times_{\mathbb{Q}} \cdots \times_{\mathbb{Q}} \bigwedge V(1)$ as a minimal Sullivan model of $S^1 \vee \cdots \vee S^1$.

The obvious quasi-isomorphism $\bigwedge V(1) \times_{\mathbb{Q}} \cdots \times_{\mathbb{Q}} \bigwedge V(1) \xrightarrow{\cong} H^*(S^1 \vee \cdots \vee S^1;\mathbb{Q})$ provides a quasi-isomorphism

$$\bigwedge V \xrightarrow{\simeq} H^*(S^1 \vee \cdots \vee S^1;\mathbb{Q})$$

which sends $\bigwedge^{\geq 2} V \to 0$.

Thus, it follows from Theorem 1 that the homotopy Lie algebra L of $\bigwedge V$ is profree. Moreover, it follows from [16, Lemma 6.2] that $V = V^1$ and so $L = L_0$. In particular, $L = \overline{\mathbb{L}}_T$ where

$$sT = H_{\geq 1}\big(\bigvee S^1_i;\mathbb{Q}\big).$$

Thus, if $v_1,\dots,v_n \in V^1$ represent the fundamental cohomology classes of the circles then $T \cong \bigoplus_i \mathbb{Q}x_i$ in which $\langle v_i, sx_j\rangle = \delta_{ij}$.

Moreover, as $X \cong S^1\vee\cdots\vee S^1$, $\pi_1(X)$ is a free group with generators g_i corresponding to the fundamental homology classes of the circles. In particular, $\pi_1(X)$ is the disjoint union of sets P_n of iterated commutators of length n in the g_i. Thus, the homomorphism

$$\varphi\colon \pi_1(X) \to G_L$$

sends g_i to the element $\exp x_i \in G_L$. It follows from Example 4 in the previous section that this homomorphism maps each P_k to a basis of G^k_L/G^{k+1}_L. Therefore, φ identifies $\pi_1(X)$ as a free subgroup of G_L, and (1) shows that this subgroup is the disjoint union of the commutators $[\exp x_{i_1}, [\dots[\exp x_{i_{s-1}}, \exp x_{i_s}]\dots]]$ corresponding to the commutators of the g'_i in $\pi_1(X)$. Therefore, (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Since the group G generated by the $\exp x'_i s$ is a free group, it is the disjoint union of sets Q_n of iterated commutators in the $\exp x'_i s$ of length n. Moreover, since $G \cong \pi_1(S^1\vee\cdots\vee S^1)$, it follows that $P_n\otimes\mathbb{Q} \cong Q_n\otimes\mathbb{Q}$. Thus, from Section 2 we obtain

$$\dim L^n/L^{n+1} = \dim P^n\otimes\mathbb{Q}.$$

On the other hand, let $\mathbb{L}$ be the free Lie algebra generated by the $x'_i s$. This gives a surjection $\mathbb{L}\to L$ for which (by the proof that (ii) $\Rightarrow$ (i))

$$\mathbb{L}^n/\mathbb{L}^{n+1} \xrightarrow{\cong} L^n/L^{n+1}.$$

Therefore, $\mathbb{L}\to L$ is an isomorphism, which proves (ii). ∎

4. Wedge and free products

Suppose $\bigwedge V'$ and $\bigwedge V''$ are the minimal Sullivan models of spaces X' and X''. Then the quasi-isomorphisms

$$\bigwedge V'\times_{\mathbb{Q}}\bigwedge V'' \xrightarrow{\simeq} A_{PL}(X')\times_{\mathbb{Q}} A_{PL}(X'') \xrightarrow{\simeq} A_{PL}(X'\vee X'')$$

identify a minimal Sullivan model $\bigwedge V$ for $\bigwedge V'\times_{\mathbb{Q}}\bigwedge V''$ as a minimal model for $X'\vee X''$.

On the other hand, denote by $L'\amalg L''$ the classical free product of graded Lie algebras L' and L''. If L, L' and L'' are respectively the homotopy Lie algebras of $X'\vee X''$, X' and X'' then the inclusions $X', X''\to X'\vee X''$ induce an inclusion $L'\amalg L''\to L$.

This [12, Proposition 23] then extends to an isomorphism of its closure $\overline{L' \amalg L''} := L' \,\widehat{\amalg}\, L''$ onto L:

$$L' \,\widehat{\amalg}\, L'' \xrightarrow{\cong} L.$$

If now L' and L'' are profree then $\bigwedge V' \xrightarrow{\simeq} \mathbb{Q} \oplus S'$ and $\bigwedge V'' \xrightarrow{\simeq} \mathbb{Q} \oplus S''$ and so

$$\bigwedge V' \times_{\mathbb{Q}} \bigwedge V'' \xrightarrow{\simeq} \mathbb{Q} \oplus S' \oplus S''.$$

Therefore, $L' \,\widehat{\amalg}\, L''$ is also profree and

$$L' \,\widehat{\amalg}\, L'' \cong \overline{\mathbb{L}}_T, \qquad sT \cong \mathrm{Hom}(S', \mathbb{Q}) \oplus \mathrm{Hom}(S'', \mathbb{Q}), \qquad H^{\geq 1}(X' \vee X''; \mathbb{Q}) \cong S' \oplus S''.$$

Example 5 ($X = S^1 \vee S^1$). In this case $L' \cong \mathbb{L}(a')$ and $L'' \cong \mathbb{L}(a'')$, so that $L_X \cong \overline{\mathbb{L}}(a', a'')$.

Example 6. Let $X = \langle \bigwedge V \rangle$ be the geometric realization of the Sullivan minimal model of $(\mathbb{Q} \oplus W, 0)$ where $W = W^2$ is a countably infinite vector space with trivial multiplication. Then the rational homotopy Lie algebra $L = L_V$ is a profree Lie algebra $L \cong \overline{\mathbb{L}}_T$ where $T = \mathrm{Hom}(sW, \mathbb{Q})$, and $L = \mathrm{Hom}(sV, \mathbb{Q})$.

Proposition 2. *With the hypotheses as above in Example 6, $L^2 \neq L^{(2)}$ and $H_3(X) \neq 0$.*

Remark. An example of a space X satisfying the hypotheses of Example 6 is given by the Sullivan completion of a wedge of infinitely many copies of the sphere S^2. In this case, surprisingly, $H_3(X) \neq 0$.

Proof of Proposition 2. Let $\{x_n\}_{n \geq 1}$ be a basis of $V^2 \cong W$. Then a basis of V^3 is given by the elements y_{ij}, $1 \leq i \leq j$, with $dy_{ij} = x_i x_j$. We associate to an element $h \in L_2 = \mathrm{Hom}(sV^3, \mathbb{Q})$ a symmetric matrix M_f defined by

$$(M_h)_{ij} = \langle y_{ij}, sh \rangle.$$

We also represent an element $f \in L_1$ by the column matrix C_f defined by

$$(C_f)_i = \langle x_i, sf \rangle.$$

If $f, g \in L_1$, then $[f, g] \in L_2$ satisfies

$$\langle y_{ij}, s[f, g] \rangle = -\langle x_i, sf \rangle \, \langle x_j, sg \rangle - \langle x_i, sg \rangle \, \langle x_j, sf \rangle.$$

It follows that

$$M_{[f,g]} = -(C_f \cdot C_g^t + C_g \cdot C_f^t).$$

Represent an element $u = \sum \alpha_i x_i$ in V^2 by the column matrix C_u with $(C_u)_i = \alpha_i$. Then, for $f \in L_1$ and $u \in V^2$, we have

$$\langle u, sf \rangle = \sum_i \alpha_i \langle x_i, sf \rangle = \sum_i \alpha_i (C_f)_i = (C_f)^t \cdot C_u.$$

If $u \in \ker sf \cap \ker sg$, then $M_{[f,g]} \cdot C_u = 0$.

More generally, for any $f_i, g_i \in L_1$, $1 \le i \le n$, there is a finite-codimensional subspace $Z \subset V$ such that for $u \in Z$,

$$M_{\sum [f_i, g_i]} \cdot C_u = 0.$$

On the other hand, let $h: V^3 \to \mathbb{Q}$ defined by $h(y_{ij}) = 0$ for $i \neq j$ and equal to 1 for $i = j$. Then M_h is the identity matrix, so that $M_h \cdot C_u = C_u$, for C_u as above. It follows that $[L_1, L_1]$ is strictly contained in L_2.

Let $\omega \in \pi_3(X)$ be a homotopy class corresponding by suspension to an element in L_2, but not in $[L_1, L_1]$. Then X contains a simply connected finite CW subcomplex Y such that ω is in the image of $\pi_3(Y)$. We denote this element by $\omega_Y \in \pi_3(Y)$. It follows that ω_Y is not decomposable, and by Lemma 1, the image a_Y of ω_Y in $H_3(Y)$ is non-zero.

For any finite CW subcomplex Z of X containing Y, the element ω_Z induces also a non-zero element a_Z by the Hurewicz map, and by the naturality of the Hurewicz map, the map $H_3(Y) \to H_3(Z)$ maps a_Y to a_Z. Now X is the union of the finite CW subcomplexes Z_α containing Y. It follows that $H_3(X) \cong \varinjlim_\alpha H_3(Z_\alpha)$ and therefore the family a_{Z_α} induces a non-zero element in $H_3(X)$. ∎

Lemma 1. *Let Y be a finite type simply connected space with $L_2 \neq [L_1, L_1]$. Then the Hurewicz map* hur: $\pi_3(Y) \to H_3(Y)$ *is non-zero.*

Proof. Denote by $\bigwedge V$ the Sullivan minimal model of Y. Then V is a finite type vector space. Therefore, the bracket $[\,,\,]: L_1 \bigwedge L_1 \to L_2$ is dual to the differential $d: V^3 \to \bigwedge^2 V^2$. Since $L_2 \neq [L_1, L_1]$, d is not injective. This implies that $H^3(\bigwedge V) \to H^3(\bigwedge V / \bigwedge^{\ge 2} V)$ is non-zero. But this last map is the dual of the Hurewicz map. Thus, if $\omega \in \pi_3(Y) \otimes \mathbb{Q} \cong \mathrm{Hom}(V^3, \mathbb{Q})$ with $\omega(v) \neq 0$, then ω is not in the kernel of the Hurewicz map. ∎

Example 7 ($X = S^1 \vee S^2$). In this case $L_X \cong L' \,\hat{\amalg}\, L''$, with $L' \cong \mathbb{L}(a)$ and $L'' \cong \mathbb{L}(b)$, where $\deg a = 0$ and $\deg b = 1$. Therefore,

$$\pi_*(S^1 \vee S^2)_{\mathbb{Q}} \cong s\overline{\mathbb{L}}(a, b).$$

Now if $\widetilde{X_{\mathbb{Q}}}$ is the universal cover of $X_{\mathbb{Q}}$ then

$$\pi_*(\widetilde{X_{\mathbb{Q}}}) \cong s\big(\ker\big(\overline{\mathbb{L}}(a, b) \to \mathbb{L}(a)\big)\big).$$

The group $\pi_2(X_{\mathbb{Q}})$ is thus the suspension of the completion P of the vector space generated by the elements $\mathrm{ad}_a^n(b) = [a, [a, \dots [a, b] \dots]]$. In other words, P is the space of series $\sum_n \alpha_n \,\mathrm{ad}_a^n(b)$, with $\alpha_n \in \mathbb{Q}$.

Proposition 3. *$L_{\tilde{X}} \cong \overline{\mathbb{L}}_P$ and $H_3(\widetilde{X_{\mathbb{Q}}}) \neq 0$, whereas $H_3(\tilde{X}) = 0$.*

Proof. The Sullivan minimal model of $S^1 \vee S^2$ is the minimal model $(\bigwedge V, d)$ of the cdga $(\bigwedge(x, y)/(xy), 0)$ with $\deg x = 1$ and $\deg y = 2$. This model is quasi-isomorphic to the cdga

$$(A, d) := \left(\bigwedge x \otimes E, d\right),$$

where E is the vector space generated by the elements $y_n, n \geq 0$ with $\deg x = 1$, $\deg y_n = 2$, $E \cdot E = 0$, $d(y_0) = 0$ and for $n \geq 1$, $d(y_n) = xy_{n-1}$. This follows because

$$\left(\bigwedge V, d\right) \cong \left(\bigwedge x \otimes \bigwedge Y \otimes \bigwedge V^{\geq 3}, d\right),$$

where $Y = \bigoplus_n \mathbb{Q} y_n$. Moreover, by construction $sP \cong \mathrm{Hom}(E, \mathbb{Q})$.

Since (A, d) is a free $\bigwedge x$-module, the quotient cdga $(\bigwedge V^{\geq 2}, \bar{d}) \cong \mathbb{Q} \bigotimes_{\bigwedge x} (\bigwedge V, d)$ is quasi-isomorphic to $\mathbb{Q} \bigotimes_{\bigwedge x} (A, d) \cong (\mathbb{Q} \oplus E, 0)$. Therefore, the homotopy Lie algebra L_X is an extension

$$0 \to \overline{\mathbb{L}}_P \to L_X \to \mathbb{Q}x \to 0.$$

It follows that $\pi_* \widetilde{X_{\mathbb{Q}}} \cong s\overline{\mathbb{L}}_P$. Now it follows from Example 6 and Proposition 2 that $H_3(\widetilde{X_{\mathbb{Q}}}) \neq 0$. ■

5. Relation between $S^1_{\mathbb{Q}} \vee S^2_{\mathbb{Q}}$ and $(S^1 \vee S^2)_{\mathbb{Q}}$

First recall that $S^1_{\mathbb{Q}}$ is the infinite Sullivan telescope

$$S^1_{\mathbb{Q}} := \bigcup_{n \geq 1} (S^1 \times [0, 1])_n / \sim,$$

with $(x, 1)_n \sim (f_n(x), 0)_{n+1}$, where $f_n \colon S^1 \to S^1$ is a map of degree n. Denote $X_p = \bigcup_{n=1}^{p} (S^1 \times [0, 1])_n / \sim$, then $X_p \cong S^1$ and the injection $X_p \to X_{p+1}$ is the map f_p of degree p.

For any simply connected space Z the universal cover of $S^1 \vee Z$ is the union of the real line $\mathbb{R}$ with a space Z attached at each integer number of the line. We can thus interpret the universal cover of $X_p \vee Z$ as the union of the real line with a copy of Z at each rational point r/s where s is a divisor of $(p-1)!$. Since the universal cover Y of $S^1_{\mathbb{Q}} \vee Z$ is the union of the universal covers of the $X_p \vee Z$, the space Y is a countable wedge of copies of Z. In particular, the universal cover of $S^1_{\mathbb{Q}} \vee S^2_{\mathbb{Q}}$ is a countable wedge of copies of $S^2_{\mathbb{Q}}$.

The injections of S^1 and S^2 into $S^1 \vee S^2$ induce maps $S^1_{\mathbb{Q}} \to (S^1 \vee S^2)_{\mathbb{Q}}$ and $S^2_{\mathbb{Q}} \to (S^1 \vee S^2)_{\mathbb{Q}}$, and therefore a map

$$\varphi \colon S^1_{\mathbb{Q}} \vee S^2_{\mathbb{Q}} \to (S^1 \vee S^2)_{\mathbb{Q}}.$$

To justify the existence of φ recall that by Proposition 1, the Sullivan minimal model of $S^1 \vee S^2$ is the minimal model of the algebra $(\mathbb{Q} \oplus \mathbb{Q}a \oplus \mathbb{Q}b, 0)$, in which $ab = b^2 = 0$,

$\deg a = 1$ and $\deg b = 2$. The Sullivan minimal models of S^1 and S^2 are the minimal models of $(\mathbb{Q} \oplus \mathbb{Q}a, 0)$ and $(\mathbb{Q} \oplus \mathbb{Q}b, 0)$, the minimal models of the injections being given by the corresponding projections. The morphism φ is then obtained by taking the geometric realization of the projections.

As indicated in Example 7, $sP \cong \pi_2((S^1 \vee S^2)_{\mathbb{Q}})$ is the space of infinite series $\sum_n (\mathrm{ad}_a)^n(b)$. This implies an isomorphism

$$P \to \mathbb{Q}[[t]], \qquad \sum \alpha_i\, (\mathrm{ad}_a)^i(b) \to \sum \alpha_i t^i.$$

Recall that the action of $[a] \in \pi_1(S^1)_{\mathbb{Q}}$ on P is given by $\exp(\mathrm{ad}_a)$. It follows that the map induced by φ on π_2,

$$\pi_2(\varphi)\colon \bigoplus_{a_i \in \mathbb{Q}} \mathbb{Q} \cdot b_i \to \mathbb{Q}[[t]]$$

maps b_i to $\exp(b_i t)$. Finally, observe that $\pi_2(S^1_{\mathbb{Q}} \vee S^2_{\mathbb{Q}})$ is countably infinite but $\pi_2((S^1 \vee S^2)_{\mathbb{Q}})$ is not.

6. $\pi_*(X_{\mathbb{Q}})$: open questions and examples

Interesting questions about $\pi_*(X_{\mathbb{Q}})$, some of which have been resolved for simply connected spaces, remain open when X is not simply connected and are particularly interesting when $X_{\mathbb{Q}}$ is a $K(\pi, 1)$. We provide here several questions, and some relevant examples. The minimal Sullivan model of a space X and its homotopy Lie algebra will be denoted throughout by $\bigwedge V_X$ and L_X.

Since part of our examples are formal spaces or spaces equipped with a weight decomposition, we recall the definitions here. A minimal model $(\bigwedge V, d)$ has a *weight decomposition* if V is equipped with a second lower gradation $V = \bigoplus_{n \geq 0} V_n$ such that $d\colon V_n^p \to (\bigwedge V)_{n-1}^{p+1}$. In that case $H^*(\bigwedge V, d)$ admits also an extra gradation, $H^*(\bigwedge V) \cong \bigoplus_{n\geq 0} H_n(\bigwedge V)$.

A space X with Sullivan minimal model $(\bigwedge V, d)$ is *formal* if there is a quasi-isomorphism $(\bigwedge V, d) \to (H^*(X;\mathbb{Q}), 0)$. Then $\bigwedge V$ has a weight decomposition such that $H_n(\bigwedge V) \cong 0$ for $n > 0$ [21]. When $\bigwedge V$ is equipped with a weight decomposition then we can form the graded Lie algebra $gL = \bigoplus_{n \geq 1} E_n$, with $E_n = \mathrm{Hom}(V_{n-1}, \mathbb{Q})$:

$$[E_p, E_q] \subset E_{p+q}, \quad \text{and} \quad L \cong \varprojlim_n gL/gL_{>n}.$$

Moreover, when $\bigwedge V$ is formal, then $E_n \cong L^n/L^{n+1}$.

In geometry, compact Kähler manifolds are formal [8], and every complex algebraic variety carries a weight decomposition [4, Theorem 4.9].

6.1. Free Lie algebras

One of the oldest unresolved (even for simply connected spaces) conjectures in rational homotopy is due to Avramov and Félix [14, Chapter 39, 4.]; in its simplest form it states the following:

Conjecture. *If X is a finite CW complex and L_X is infinite-dimensional, L_X contains a free graded Lie algebra on two generators.*

Here are three examples where the conjecture is true, and a fourth relevant example.

Example 8. If X is an *orientable Riemann surface of genus* $g \geq 2$ then L_X contains a profree ideal of codimension 1. In particular, $\pi_1(X_{\mathbb{Q}})$ contains a free group on infinitely many generators.

In this case $L_X \cong (L_X)_0$ [11] and therefore $\bigwedge V_X$ is quadratic. Moreover, $V_X \cap \ker d_1$ has a basis of the form $u_i, v_i, i \geq 1$, and these cycles, together with 1 and $u_1 v_1$ represent a basis of $H^*(\bigwedge V_X)$. Now let $I \subset L_X$ be the codimension 1 ideal defined by

$$I = \{y \in L_X \mid \langle u_1, sy\rangle = 0\}.$$

We show that I is profree.

First observe that division by the ideal generated by u_1 defines a surjection of quadratic Sullivan algebras $\bigwedge V_X \to \bigwedge V_1$ with $V_1 = V_X/u_1$. It is immediate that L_{V_1} is the homotopy Lie algebra of I, and that mapping z and u_1 to 0 yields a quasi-isomorphism

$$\rho\colon \Big(\bigwedge V_X \otimes \bigwedge z, dz = u_1\Big) \xrightarrow{\simeq} \bigwedge V_1.$$

On the other hand, let W be a direct complement of $V_X \cap \ker d_1$ in V_X. Then mapping $W, u_i u_j, v_i v_j, u_i u_j$ $(i \neq j)$ and $u_i v_i - u_j v_j$ all to zero defines a quasi-isomorphism $\bigwedge V_X \to H^*(\bigwedge V_X)$. This then extends to a quasi-isomorphism

$$\Big(\bigwedge V_X \otimes \bigwedge z, d\Big) \xrightarrow{\simeq} \Big(H^*(\bigwedge V_X) \otimes \bigwedge z, d\Big)$$

where in both cdga's $dz = u_1$.

Finally, we necessarily have $\deg z = 0$. A simple check shows that the homology of $H^{\geq 1}(H^*(\bigwedge V_X) \otimes \bigwedge z)$ is concentrated in degree 1. It follows that this is also true for $H^*(\bigwedge V_1)$ and thus that I is profree.

Example 9. If X is the *classifying space of a right-angled Artin group* then L_X is abelian or else $(L_X)_0$ contains a profree Lie subalgebra with two generators. In the second case, $\pi_1(X_{\mathbb{Q}})$ contains a free group on two generators.

A right-angled Artin group is a group A with a presentation of the form

$$A \cong \langle x_1, \dots, x_n \mid x_i x_j = x_j x_i \text{ for some possibly empty subset } S \text{ of pairs } (i,j)\rangle.$$

Thus, if S contains all the pairs (i, j) then $A \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$. In this case $X \cong K(A, 1) \cong S^1 \times \cdots \times S^1$ and $L_X \cong \mathbb{Q} \times \cdots \times \mathbb{Q}$ is abelian.

On the other hand if some pair $(i_0, j_0) \notin S$, then $A' \cong \langle x_{i_0}, x_{j_0} \rangle$ is a retract of A, and so its homotopy Lie algebra is a retract of L_X. But A' is a free group on two generators, which, by Van Kampen's theorem implies that $K(A', 1) \cong S^1 \vee S^1$. By Example 5 of Section 4, its homotopy Lie algebra is $\overline{\mathbb{L}}(a, b)$ which contains $\mathbb{L}(a, b)$.

Example 10. If X is the *complement in* $\mathbb{C}^n$ *of a central arrangement of hyperplanes* then either $(L_X)_0$ is abelian or $(L_X)_0$ contains a free Lie algebra on two generators. In the second case, $\pi_1(X_{\mathbb{Q}})$ contains a free group on two generators.

An arrangement of hyperplanes is a finite set $H_1, \ldots, H_n$ of hyperplanes in $\mathbb{C}^n$, and is central if each H_i contains the origin. In this example,

$$X \cong \mathbb{C}^n \setminus \bigcup_i H_i,$$

and there is a quasi-isomorphism $(\wedge V_X, d) \to (H^*(X; \mathbb{Q}), 0)$ [9]. The algebra $H^*(X; \mathbb{Q})$ has been computed by Orlik and Terao [25].

We fix an order on the hyperplanes H_i, and for each of them we introduce a generator e_i in degree 1. Then the cohomology $H \cong H^*(\mathbb{C}^n \setminus \bigcup_i H_i)$ is the quotient of $\wedge(e_1, \ldots, e_n)$ by an ideal I. Since the space is formal the component in degree 0 of the homotopy Lie algebra L_X depends only on the multiplication law: $\wedge^2 H^1 \to H^2$. It follows that the Lie algebra $(L_X)_0$ is the component in degree 0 of the Lie algebra of the minimal model of $H/H^{\geq 3}$.

The relations in degree 2 are obtained by family $H_{i_1}, \ldots, H_{i_3}$ such that for some j, $\bigcap_q H_{i_q} \cong \bigcap_{q \neq j} H_{i_q}$. The associated relation is $e_{i_1} e_{i_2} - e_{i_1} e_{i_3} + e_{i_2} e_{i_3}$.

Let $E = \{H_i \cap H_j, i < j\}$. If the $H_i \cap H_j$ are all different then there is no relation of degree 2, which implies that L_X is abelian. Otherwise we can suppose that $H_1 \cap H_2 = H_1 \cap H_2 \cap H_3$. Let m be the integer for which $H_1 \cap H_2 \cap H_k = H_1 \cap H_2$ for $k \leq m$ and $H_1 \cap H_2 \cap H_k \neq H_1 \cap H_2$ for $k > m$. Then for $i < j \leq m$, $H_1 \cap H_2 = H_i \cap H_j$. For each such triple H_1, H_2, H_k we have $e_1 e_2 + e_2 e_k + e_k e_1 = 0$.

The space of relations in degree 2 is therefore the sum of two components, the first one in $\wedge^2(e_1, \ldots, e_m)$ and the second one in the ideal generated by $e_{m+1}, \ldots, e_n$. This implies that $H/(H^{\geq 3}, e_{m+1}, \ldots, e_n)$ is a retract of $H/H^{\geq 3}$. We are therefore reduced to prove that the homotopy Lie algebra of the minimal model of $H' := (H/(H^{\geq 3}, e_{m+1}, \ldots, e_n), 0)$ contains a free Lie algebra.

In H', we have for $r < s$,

$$e_r e_s = e_1 e_r - e_1 e_s.$$

It follows that H' is the free module on $\wedge e_1$ generated by $1, e_2, \ldots, e_m$. Let $\wedge T$ be its minimal model. Then, since the homology of $\wedge(T/e_1)$ is concentrated in degree 1, L_T contains a profree Lie algebra on $m - 1$ generators. By construction this profree Lie algebra is a Lie subalgebra of L_X.

Example 11 (Configuration spaces in $\mathbb{R}^n$). The rational homotopy Lie algebra of the configuration space $F(\mathbb{R}^{q+2}; k)$ of k points in $\mathbb{R}^{q+2}$, $q \geq 1$, denoted by $\mathcal{L}_k(q)$, has been computed by Cohen and Gitler [7, Theorem 2.3].

The graded Lie algebra $\mathcal{L}_k(q)$ is the quotient of the free graded Lie algebra on the elements $B_{i,j}$, $1 \leq j < i \leq k$ of degree q by the following relations, called infinitesimal braid relations:

(i) $[B_{i,j}, B_{s,t}] = 0$ if $\{i, j\} \cap \{s, t\} = \emptyset$,

(ii) $[B_{i,j}, B_{i,t} + (-1)^q B_{t,j}] = 0$ if $1 \leq j < t < i \leq k$,

(iii) $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ if $1 \leq j < t < i \leq k$.

According to [7, Theorem 4.1], the Lie subalgebra generated by the $B_{k,i}$ is a free ideal, and we have a short exact sequence of graded Lie algebras

$$0 \to \mathbb{L}(x_{k,i}) \to \mathcal{L}_k(q) \to \mathcal{L}_{k-1}(q) \to 0.$$

Since $\mathcal{L}_k(q)$ is a graded Lie algebra of finite type with no elements of degree 0, $\mathbb{L}(x_{k,i})$ is profree.

Example 12 (An extension to universal enveloping algebras). The *classical completion* of the universal enveloping algebra of a graded Lie algebra, L, is defined by $\widehat{UL} = \varprojlim_n UL/I^n$, where I denotes the maximal ideal of UL. Then, we have the following:

Problem 1. With the same hypotheses as in the Avramov–Félix conjecture, does $\widehat{UL}$ contain a tensor algebra on 2 generators?

The following proposition is a partial answer to this question:

Proposition 4. *Let $\bigwedge V$ be a Sullivan minimal model with associated homotopy Lie algebra L. Suppose that $\dim L/L^2 < \infty$, that $V = V^1$, that $(\bigwedge V, d)$ has a weight decomposition and that $H^*(\bigwedge V, d)$ is a finite-dimensional vector space with non-zero Euler characteristic. Then $\widehat{UL}$ contains a tensor algebra on two generators.*

Proof. Denote by gL the graded Lie algebra associated to the weight decomposition and observe that we have injections

$$U(gL) \subset UL \subset \widehat{UL}.$$

It is therefore enough to construct an injection $T(x, y) \subset U(gL)$.

By hypothesis, $V = V^1$ and $V \cong \bigoplus_{n \geq 0} V_n$, with $d(V_n) \subset \left(\bigwedge^2 V\right)_{n-1}$. We equip V with an extra upper gradation

$$V^{(2n+3)} = V_n.$$

It follows that $d(V^{(r)}) \subset (\bigwedge^2 V)^{(r+1)}$. In particular, with this new upper gradation $\bigwedge V$ is a minimal Sullivan algebra, and $V = V^{(odd)}$ and $V = V^{(\geq 2)}$. It follows that the dimensions of the vector spaces $H^{even}(\bigwedge V)$ and $H^{odd}(\bigwedge V)$ have not been modified.

In particular, the Euler characteristic of the cohomology of $\bigwedge V$ with the new grading is non-zero. By [10], gL does not contain any non-zero solvable ideal and $U(gL)$ contains a tensor algebra $T(x, y)$. Forgetting this new grading, this defines an injection $T(x, y) \subset U(gL)$. ■

6.2. The growth of $\pi_1(X)_{\mathbb{Q}}$

When X is a simply connected CW complex with finite Betti numbers and finite category, the classical dichotomy theorem says that either $\pi_*(X) \otimes \mathbb{Q}$ is finite-dimensional or else the sequence $\sum_{i=1}^{s} \dim \pi_i(X) \otimes \mathbb{Q}$ has an exponential growth [14, Chapter 33].

A similar dichotomy problem concerns the fundamental group. Suppose $\operatorname{cat} X < \infty$. Then denote by G the fundamental group of $X_{\mathbb{Q}}$ and set $L \cong (L_X)_0$. Denote also as usual

$$G^1 = G, \quad \text{and} \quad G^n = [G^{n-1}, G], \quad n \geq 1,$$

and recall from Section 2,

$$L^1 = L, \quad \text{and} \quad L^n = [L^{n-1}, L], \quad n \geq 1.$$

Problem 2. Suppose $\sum \dim G^n/G^{n+1} = \infty$ and $\dim G^1/G^2 < \infty$. Do the series

$$\sum_n \dim G^n/G^{n+1} t^n \quad \text{and} \quad \sum_n \dim L^n/L^{n+1} t^n$$

have exponential growth?

This is trivially the case when L contains a free Lie algebra on two generators, but there is no global answer. The following proposition is a first step in that direction for a special family of groups.

Proposition 5. *Suppose X is formal, $V = V^1$,* $\operatorname{cat} X = m < \infty$ *and* $\dim L/L^2 < \infty$. *Then the elements* $\dim L^n/L^{n+1}$ *are unbounded, and the sequence* $\sum_{i=1}^{n} \dim L^i/L^{i+1}$ *has exponential growth.*

Proof. Since $[L/L^2, L^p/L^{p+1}] \cong L^{p+1}/L^{p+2}$, if $\dim L = \infty$, then $L^p/L^{p+1} \neq 0$ for all p.

For $p < q$, write $V_{[p,q]} = \bigoplus_{j=p}^{q} V_j$ and denote $\lambda(k) = \dim V_{[k,2k-1]}$. By the mapping theorem [16, Theorem 9.3] the category of the quotient minimal model

$$\left(\bigwedge V_{\geq k}, \bar{d}\right) := \left(\bigwedge V, d\right) \bigotimes_{(\bigwedge V_{<k}, d)} \mathbb{Q}$$

is $\leq m$, and $V_{[k,2k-1]} \subset \ker \bar{d}$. Therefore, $\bigwedge^{m+1} V_{[k,2k-1]} \subset \operatorname{Im} \bar{d}$. Since the differential d is quadratic, it follows that

$$\bigwedge^{m+1} V_{[k,2k-1]} \subset \bar{d}\left(V_{[2k,4k-1]}\right) \cdot \bigwedge^{m-1} V_{[k,2k-1]}.$$

Since $\lambda(k) \geq k$, we can choose k such that $\lambda(k) \geq C^2$ with $C = 2^{m+1}(m+1)!$. Then for $q \leq m+1$, $\lambda(k) - q \geq \frac{\lambda(k)}{2}$ and so

$$\frac{\lambda(k)^{m+1}}{(m+1)!2^{m+1}} \leq \dim \bigwedge^{m+1} V_{[k,2k-1]}.$$

Therefore,

$$\frac{\lambda(k)^{m+1}}{C} \leq \lambda(2k)\lambda(k)^{m-2}.$$

We deduce that $\lambda(2k) \geq \frac{\lambda(k)^2}{C}$. Iterating the process yields

$$\lambda(2^r k) \geq \left(\frac{\lambda(2^{r-1}k)}{C}\right)^2 \geq \left(\frac{\lambda(k)}{C}\right)^{2^r}.$$

Let now s and r in $\mathbb{N}$ be such that $2^{r-1}k \leq s < 2^r k$. Then, with $A = (\frac{\lambda(k)}{C})^{\frac{1}{4k}}$, we have

$$\sum_{i=0}^{s} \dim V_i \geq \lambda(2^{r-2}k) \geq \left(\frac{\lambda(k)}{C}\right)^{\frac{2^{r-2}k}{k}} = A^{2^r k} \geq A^s.$$

It follows directly that the elements $\dim V_n$ are unbounded. ■

7. Inert attachments

A cell attachment via $f\colon S^n \to X$ is called *inert* if the induced maps $\pi_*(X) \to \pi_*(X \bigcup_f D^{n+1})$ are surjective. We say that the attachment is *rationally inert* if the morphism $L_X \to L_{X \bigcup_f D^{n+1}}$ is surjective.

Write $Y = X \bigcup_f D^{n+1}$. Then we have the following:

Theorem 2 ([13]). *The following three conditions are equivalent:*

(i) *The attachment f is inert.*

(ii) *The homotopy Lie algebra of the homotopy fibre of the map $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is a profree Lie algebra.*

(iii) *The element $[f]$ generates a profree ideal I in $\pi_*(\Omega X_{\mathbb{Q}})$ and there is an isomorphism*

$$I/I^{(2)} \cong \overline{U(\pi_*(\Omega X_{\mathbb{Q}})/[f])}.$$

When an attachment is not inert, the attachment often creates new profree Lie algebras. Here are two examples:

Theorem 3 ([13]). *Let Y be a space obtained by adding 2-cells to a wedge of circles X. Denote by E the image of the map $L_X \to L_Y$. Then the injection $E \to L_Y$ admits a retraction r whose kernel is a profree Lie algebra.*

Proposition 6. *Let $Y = X \bigcup_f D^{n+1}$ where f corresponds to a non-zero element β in the center of $(L_X)_{n-1}$. We suppose that $\pi_1(X_{\mathbb{Q}}) \neq 0$ and that $\dim \pi_1(X_{\mathbb{Q}}) \geq 2$. Then L_Y contains a profree Lie algebra $\overline{\mathbb{L}}(x, y)$ with $\deg x = 0$ and $\deg y = n$.*

Proof. To simplify we suppose that n is odd. The proof is similar when n is even. Let $(\bigwedge V, d)$ be the Sullivan minimal model of X and let $\rho\colon (\bigwedge V, d) \to (\mathbb{Q} \oplus \mathbb{Q} \cdot a, 0)$ be a Sullivan representative of $f\colon S^n \to X$. Then choose $v \in V^1$ and $w \in V^n$ with v and w linearly independent, $dv = 0$, $\rho(w) = a$ and $\langle w, s\beta\rangle = 1$. Denote by E a direct complement of the vector space generated by v and w in V. Since β is in the center of L_X, there is no $z \in V$ with $dz = vw + \omega$ with ω in the ideal generated by E.

Write $B = dw$ and let $\varphi\colon \bigwedge T \xrightarrow{\simeq} (\mathbb{Q} \oplus \ker \rho)$ be a Sullivan minimal model. We can suppose that $V^{\leq m} \subset T$ and $B \in \bigwedge T$. There is then an element $z \in T^{m+1}$ with $dz = Bv$ and $\varphi(z) = wv$.

Now denote by $\psi\colon (\bigwedge W, \overline{d}) \to (\mathbb{Q} \oplus (z, v), 0)$ the Sullivan minimal model of $(\mathbb{Q} \oplus (z, v), 0)$. Since this is a formal space we can suppose that W is equipped with a second lower gradation $W \cong \bigoplus_q W_q$ such that $d(W_q) \subset (\bigwedge W)_{q-1}$ and $H_q(\bigwedge W) \cong 0$ for $q \neq 0$. By construction $W_0 \cong (z, w)$ and $W_1 \cong (y_1, z_1)$ with $\overline{d}y_1 = zv$ and $\overline{d}z_1 = z^2$. For $r < s$, we write $W_{[r,s]} \cong \bigoplus_{q=r}^{s} W_q$.

We construct by induction on the lower gradation a differential d and a morphism of cdga's

$$\varphi\colon \left(\bigwedge E_{\leq n} \otimes \bigwedge W, d\right) \to \mathbb{Q} \oplus (\ker \rho, d)$$

with the following properties:

- $\varphi(W_{\geq 1}) = 0$.
- On W, the differential d has the following form. If $x \in W_r$, then

$$d = \overline{d} + \delta, \qquad \text{where} \quad \delta(x) = \sum_{q=1}^{\infty} \theta_q(x) B^q, \quad \text{and} \quad \theta_q(x) \in W_{[1,r]}.$$

On $\bigwedge x \otimes \bigwedge E_{\leq n}$, φ is the natural injection. The morphism φ and the differential on $W_{\leq 1}$ are given by $\varphi(v) = v$, $\varphi(z) = wv$, $dz = vB$, $d(y_1) = zv$, $d(z_1) = z^2 - 2y_1 B$, $\varphi(y_1) = \varphi(z_1) = 0$.

Suppose we have constructed d on $W_{<q}$ for some q, with $\varphi(W_{[1,q]}) = 0$ and $d^2 = 0$. In particular, for $m \geq 1$,

$$\theta_m \overline{d} + \overline{d}\theta_m + \sum_{i+j=m} \theta_i \theta_j = 0.$$

Now let $x \in W_q$. Since $\overline{d}\theta_1 \overline{d}x = -\theta_1 \overline{d}^2 x = 0$, and since ψ is a quasi-isomorphism, there is a $\theta_1(x) \in W_{[1,q]}$ with $\theta_1 \overline{d}x = -\overline{d}\theta_1 x$. We construct inductively the θ_p. First we verify that $\bar d(\theta_p \bar d + \sum_{i+j=p} \theta_i\theta_j)x = 0$. Indeed,

$$\begin{aligned}\bar d\Big(\theta_n \bar d + \sum_{i+j=p} \theta_i\theta_j\Big)x &= -\theta_p \bar d^2 x - \sum_{r+s=p} \theta_r\theta_s \bar d x = \bar d \sum_{i+j=p} \theta_i\theta_j(x)\\ &= -\sum_{r+s=p} \theta_r\Big(\bar d\theta_s - \sum_{k+\ell=s}\theta_k\theta_\ell\Big)(x) - \bar d\sum_{i+j=p}\theta_i\theta_j(x)\\ &= \bar d\sum_{r+s=p}\theta_r\theta_s x + \sum_{r+s=p,a+b=r}\theta_a\theta_b\theta_s x\\ &\quad - \sum_{r+s=p,k+\ell=s}\theta_r\theta_k\theta_\ell x - \bar d\sum_{i+j=p}\theta_i\theta_j(x) = 0.\end{aligned}$$

Since $\psi(\theta_p\bar d + \sum\theta_i\theta_j)x = 0$, we can define $\theta_p(x) \in W_{[1,q]}$ with

$$\Big(\theta_p\bar d + \sum_{i+j=p}\theta_i\theta_j\Big)x = -\bar d\theta_p x.$$

Finally, we define $\varphi(x) = 0$.

Denote by L the rational homotopy Lie algebra of $(\bigwedge v \otimes \bigwedge E_{\le n} \otimes \bigwedge W, d)$, and by α and β elements with $\langle v, s\alpha\rangle = 1 = \langle w, s\beta\rangle$. Since we have a surjection

$$(\bigwedge v \otimes \bigwedge E_{\le n} \otimes \bigwedge W, d) \to (\bigwedge W, \bar d),$$

L contains the profree Lie algebra on α and β.

Now we extend φ to a quasi-isomorphism

$$(\bigwedge Z, d) := (\bigwedge v \otimes \bigwedge E_{\le n-1} \otimes \bigwedge W \otimes \bigwedge Z', d) \to \mathbb{Q} \oplus \ker\rho.$$

By this construction $(\bigwedge Z, d)$ is not necessarily minimal. We denote its homotopy Lie algebra by L_Z, and by construction we have a not necessarily surjective morphism $L_Z \to L$. However, since v and z are not boundaries in Z for the linear part of the differential, the elements α and β are in the image of $L_Z \to L$. Therefore, L_Z contains the free Lie algebra on α and β. ∎

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