EMS Surv. Math. Sci. 10 (2023), 223–241
DOI 10.4171/EMSS/71

The diagonal of cellular spaces and effective algebro-homotopical constructions

Anibal M. Medina-Mardones

Abstract. In this survey article we discuss certain homotopy coherent enhancements of the coalgebra structure on cellular chains defined by an approximation to the diagonal. Over the rational numbers, C_∞-coalgebra structures control the $\mathbb{Q}$-complete homotopy theory of spaces, and over the integers, E_∞-coalgebras provide an appropriate setting to model the full homotopy category. Effective constructions of these structures, the focus of this work, carry geometric and combinatorial information which has found applications in various fields including deformation theory, higher category theory, and condensed matter physics.

On the occasion of Dennis Sullivan's 80th birthday

1. Introduction

There is a tense trade-off in algebraic topology having roots reaching back to the beginning of its modern form. This tension can be illustrated with the concept of cohomology. The first approaches, dating back to Poincaré, are based on the subdivision of a space into simple contractible pieces. These elementary shapes are made to generate a free graded module whose spatial relations define a differential used to compute cohomology. This definition makes fairly clear certain geometric properties of cohomology, for example, excision. Yet, it is not easy to show that a continuous map of spaces induces a map between their associated cohomologies. The functoriality just alluded to is trivial when defining cohomology in terms of homotopy classes of maps to Eilenberg–MacLane spaces, but the passage to the homotopy category erases geometric and combinatorial information and the resulting definition is not well suited for concretely presented spaces.

Cohomology as a graded abelian group is a fairly computable invariant, but it has noticeable limitations, for example, $\mathbb{C}\mathrm{P}^2$ and $S^2 \vee S^4$ are not distinguished by it. Cohomology can be refined to a graded ring by endowing it with the cup product, an enhancement that distinguishes these spaces. In the spectral context, the product structure

2020 Mathematics Subject Classification. Primary 55U15; Secondary 18N40, 18M70, 18M85, 55S30, 55S05.
Keywords. Cellular chains, diagonal map, effective constructions, C_∞-coalgebras, E_∞-coalgebras.

is defined through the wedge product of Eilenberg–MacLane spaces, whereas in the cellular setting it is obtained from a choice of cellular approximation to the diagonal map $X \to X \times X$. Such cellular map induces a chain map

$$\Delta\colon \mathrm{C}(X) \to \mathrm{C}(X) \otimes \mathrm{C}(X) \tag{1}$$

making the cellular chains of X into a (differential graded) coalgebra. The fact that the cup product on cohomology, induced by the linear dual of (1), is associative and (graded) commutative, hints at the presence of additional structure extending the coalgebra structure on $\mathrm{C}(X)$.

In this survey article we will present, from a viewpoint that emphasizes their constructive nature, C_∞ and E_∞ extensions of Δ over the rationals and integers, respectively. The resulting algebraic structures control much of the homotopy theory of spaces. For example, over the rationals, the quasi-isomorphism type of a C_∞-coalgebra extension of the symmetrization of Δ determines the $\mathbb{Q}$-completion of X under certain assumptions [10,41]. Whereas over the integers, and under similar assumptions, the quasi-isomorphism type of an E_∞-coalgebra extension of Δ determines the homotopy type of X [27].

Rational coefficients

In Section 2 we will study extensions of chain approximations to the diagonal with rational coefficients. Over this field, a chain approximation to the diagonal can be symmetrized, giving rise to a cocommutative coalgebra. This coalgebra cannot be made simultaneously coassociative, but this relation can be imposed in a derive sense through a family of coherent chain homotopies – which also respect certain symmetry constrains – and give rise to a so-called C_∞-coalgebra structure. One can think of C_∞-coalgebras in terms of the somewhat more familiar notion of A_∞-coalgebra where cocommutativity is satisfied strictly. As a manifestation of Koszul duality, a C_∞-coalgebra structure on cellular chains is equivalent to a differential on the completion of the free graded Lie algebra generated by the cells shifted downwards in degree by one. This relates C_∞-coalgebras to deformation theory, but we do not explore this deep connection here. For cell complexes whose closed cells have the $\mathbb{Q}$-homology of a point, Dennis provided in [50] a local inductive construction defining a C_∞-coalgebra structure on their cellular chains. We reprint a challenge he posted regarding the resulting structure.

> **Problem.** *Study this free differential Lie algebra attached to a cell complex, determine its topological and geometric meaning as an intrinsic object. Give closed form formulae for the differential and for the induced maps associated to subdivisions.* [25, p. 231]

As proven by Quillen, the quasi-isomorphism type of this C_∞-coalgebra is a complete invariant of the rational homotopy type of simply-connected spaces. For the C_∞-coalgebra structure on the interval, Dennis and Ruth Lawrence addressed the challenge reprinted above introducing a formula for it which can be interpreted in terms of parallel transport of flat connections [25], and for which the subdivision map is described by the Baker–Campbell–Hausdorff formula.

To generalize Quillen's equivalence of homotopy categories to one between (not necessarily 1-connected) simplicial sets, Buijs, Félix, Murillo, and Tanré [10] extended to n-simplices the Lawrence–Sullivan structure building, constructively for $n \in \{2, 3\}$ and inductively otherwise, C_∞-coalgebra structures on their chains. Their construction agrees after linear dualization with the one obtained by Cheng and Getzler in [13], where they showed that the Kontsevich–Soibelman sum-over-trees formula defining the transfer of A_∞-algebras through a chain contraction induces a transfer of C_∞-algebras. This allowed them to construct a C_∞-algebra structure on simplicial cochains by transferring Dennis' polynomial differential forms through Dupont's contraction. The resulting description is given in terms of rooted trees.

C_∞-coalgebras are controlled by the operad $\mathcal{C}om_\infty$ which is the Koszul resolution of the operad $\mathcal{C}om$, i.e., the cobar construction applied to the $\mathcal{L}ie$ cooperad, the Koszul dual cooperad of $\mathcal{C}om$. Another interesting resolution of $\mathcal{C}om$ is constructed concatenating the bar and cobar constructions. This resolution method is an algebraic version of the W-construction of Boardman–Vogt. As Dennis and Scott Wilson considered, the resulting operad can be described using rooted trees with vertices colored black or white. In [42], Daniel Robert-Nicoud and Bruno Vallette studied coalgebras over this resolution which they termed CC_∞-coalgebras. They constructed on the chain of standard simplices natural CC_∞-coalgebra structures and described them explicitly using bicolored trees.

Despite some progress – [9,18,19], [10, § 6.5] – the "closed form formulae" part of the problem quoted before remains open. One possible avenue to generalize to cubical chains the formula defining the Lawrence–Sullivan C_∞-coalgebra on $\mathrm{C}(\mathbb{I})$, is to define the tensor product of C_∞-coalgebras and then extend it monoidally to all cubes via the isomorphism $\mathrm{N}(\square^n) \cong \mathrm{C}(\mathbb{I})^{\otimes n}$. The monoidal structure on the category of A_∞-coalgebras is defined through a chain approximation to the diagonal of the Stasheff polytopes compatible with the operad structure. Unfortunately, the resulting A_∞-coalgebra on $\mathrm{C}(\mathbb{I})^{\otimes 2}$ is not C_∞. This could be corrected through an algebraic symmetrization of the associahedral diagonal, but we do not pursue this here.

Integral coefficients

In Section 3 we will study extensions of chain approximations to the diagonal with integral coefficients. In contrast to the situation over $\mathbb{Q}$, chain approximations to the diagonal over these coefficients cannot be taken to be symmetric with respect to transposition of tensor factors. The resulting coalgebras can be made cocommutative and coassociative only up to coherent homotopies, that is to say, provided with the structure of a coalgebra over an E_∞-operad. The study of E_∞-structures has a long history, where (co)homology operations [28, 47], the recognition of infinite loop spaces [5, 29], and the complete algebraic representation of the p-adic homotopy category [26] are key milestones.

Steenrod was the first to introduce homotopy coherent corrections to the broken symmetry of a chain approximation to the diagonal [44]. He did so on simplicial chains in the form of explicit formulae defining his cup-i coproducts, with cup-0 agreeing with the Alexander–Whitney chain approximation to the diagonal. These coproducts are used to define Steenrod's mod 2 cohomology operations and to effectively compute them in specific examples.

Extending the cup-i coproducts of Steenrod, explicitly defined E_∞-coalgebra structure on simplicial chains were introduced by McClure–Smith [30] and Berger–Fresse [4]. It turns out that this structure can be described solely in terms of the Alexander–Whitney diagonal, the augmentation map and a chain version of the join of simplices [33]. This point of view can be abstracted using the language of props, which allows its application to other contexts, for example those defined by cubical chains [24] and the Adams' cobar construction [39]. We will review the resulting model of the E_∞-operad, its action on simplicial and cubical chains, and explicit generalizations of the cup-i coproducts to higher arities effectively constructing Steenrod operations at all primes [23].

We devote the final subsection to overview the use of cochain level structures in the classification of symmetry protected topological phases of matter.

2. C_∞-coalgebras

Over the rationals, the problem of extending a chain approximation to the diagonal as a C_∞-coalgebra is related to the study of Lie algebras. In this section we recall this connection, and Dennis' construction of a C_∞-coalgebra structure on the cellular chains of certain CW complexes. We also discuss the resulting structure on the cellular chains of the interval, which is presented as a formula in the work of Dennis and Ruth Lawrence. We discuss Quillen's functor from simplicial sets to complete dg Lie algebras, as extended by Buijs, Félix, Murillo, and Tanré through a cosimplicial C_∞-coalgebra, and the problem of making this construction into explicit formulae extending the Lawrence–Sullivan interval.

2.1. Quillen construction

As a motivating example illustrating the connection between cocommutative and coassociative coalgebras and dg Lie algebras, let us recall the so-called *Quillen construction*. Consider one such coalgebra C, and form the free graded Lie algebra L generated by the desuspension of C regarded as a graded vector space. Denote $s^{-1}C$ in L by L_1 and L_2 the linear span of brackets of elements in $s^{-1}C$. The boundary map and coproduct induce maps of degree -1

$$L_1 \xrightarrow{l_1} L_1, \qquad L_1 \xrightarrow{l_2} L_2,$$

respectively, and their relations ensure that $l_1 + l_2$ squares to 0. More explicitly,

$$l_1(s^{-1}c) = -s^{-1}\,\partial\, c, \qquad l_2(s^{-1}c) = \frac{1}{2}\sum_i (-1)^{|a_i|}[s^{-1}a_i, s^{-1}b_i],$$

where $\Delta(c) = \sum_i a_i \otimes b_i$.

The extension of $l_1 + l_2$ as a derivation of the Lie bracket makes L into a free dg Lie algebra naturally associated to C.

2.2. C_∞-coalgebras

The previous construction motivates the definition of C_∞-coalgebras. Before providing it, let us recall the notion of a *complete chain complex* (C, F), which is a filtered chain complex C

$$C = F_0 C \supseteq F_1 C \supseteq \cdots$$

such that

$$C = \lim_{k\to\infty} C/F_k C.$$

As expected, the *completion* of a filtered chain complex (C, F) is defined as C.

A C_∞*-coalgebra structure* on a graded vector space C is the data of a differential on the completion, with respect to the filtration by number of brackets, of the free graded Lie algebra generated by $s^{-1}C$.

2.3. C_∞-coalgebras as commutative A_∞-algebras

We can interpret a C_∞-coalgebra structure on C in terms of the somewhat more familiar notion of A_∞-coalgebra.

An A_∞*-coalgebra structure* on a graded vector space C is a family of degree $k-2$ linear maps $\Delta_k\colon C \to C^{\otimes k}$ satisfying for every $i \geq 1$ the following identity:

$$\sum_{k=1}^{i}\sum_{n=0}^{i-k}(-1)^{k+n+kn}\big(\mathrm{id}^{\otimes i-k-n} \otimes \Delta_k \otimes \mathrm{id}^{\otimes n}\big) \circ \Delta_{i-k+1} = 0. \tag{2}$$

This is equivalent to the data of a differential on $\prod_{n\geq 1}(s^{-1}C)^{\otimes n}$, the augmentation kernel of the complete tensor algebra on the desuspension of C. Indeed, such differential $d = \sum_{k\geq 1} d_k$ is determined by its restriction to $s^{-1}C$ with $d_k(s^{-1}C) \subset T^k(s^{-1}C)$, and the correspondence is explicitly given by

$$\Delta_k = -s^{\otimes k} \circ d_k \circ s^{-1}, \qquad d_k = -(-1)^{\frac{k(k+1)}{2}}(s^{-1})^{\otimes k} \circ \Delta_k \circ s.$$

Notice that (2) implies for any A_∞-coalgebra that Δ_1 squares to 0, that Δ_2 is a chain map with respect to Δ_1, and that Δ_3 is a chain homotopy between $(\Delta_2 \otimes \mathrm{id}) \circ \Delta_2$ and $(\mathrm{id} \otimes \Delta_2) \circ \Delta_2$.

A C_∞-coalgebra structure on a graded vector space C is equivalent to an A_∞-coalgebra structure on C such that the image d_k lies in the invariants of $(s^{-1}C)^{\otimes k}$ under the action of $\mathbb{S}_k$, or, expressed in terms of the coproducts Δ_k, one such that $\tau \circ \Delta_k = 0$, where

$$\tau(c_1 \otimes \cdots \otimes c_k) = \sum_{i=1}^{k} \sum_{\sigma \in \mathbb{S}(i,k-i)} \operatorname{sign}(\sigma)\,(c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(i)}) \otimes (c_{\sigma(i+1)} \otimes \cdots \otimes c_{\sigma(k)})$$

and $\mathbb{S}(i, k-i)$ denotes the set of $(i, k-i)$-shuffles.

2.4. Sullivan's cellular C_∞-coalgebra construction

We now present Dennis' inductive construction of a local C_∞-coalgebra structure on the chains of cell complexes whose closed cells have the $\mathbb{Q}$-homology of a point [50].

Let X be one such cellular complex and $L(X) = L$ be the free Lie algebra generated by the desuspension of its rational cellular chains $s^{-1}C$.

Let us start by choosing a chain approximation $\Delta\colon C \to C \otimes C$ to the diagonal, which we assume equivariant – since we are working with rational coefficient – and local, in the sense that $\Delta(e_\alpha)$ is contained in the subcomplex generated by the tensor product of cells in the closure of e_α. We remark that (C, δ, Δ) is a cocommutative coalgebra which is in general not coassociative. Let δ_1 and δ_2 be the respective maps from L_1 to L_1 and L_2 induced from ∂ and Δ. We denote by the same symbols their extensions to L as derivations. We now quote Dennis' construction:

> *We interpret $\delta \circ \delta = 0$ as the equation $[\delta, \delta] = 0$, where $[\cdot, \cdot]$ is the graded commutator. For any δ the Jacobi identity is $[\delta, [\delta, \delta]] = 0$. Suppose $\delta^k = \delta_1 + \cdots + \delta_k$ has been defined so that $[\delta^k, \delta^k]$ has the first non-zero term in monomial degree $k+1$. Jacobi implies this error commutes with δ_1; that is, it is a closed element in the complex* $\mathrm{Der}(L)$ *of derivations of L. If we work in the closure of a cell, the homology hypothesis implies that* $\mathrm{Der}(L)$ *has homology only in degrees* 0 *and* -1*. Therefore, the error, which lives in degree* -2*, can be written as a commutator with δ_1. Using the cells to generate a linear basis of each L_k by bracketing, we choose this solution to lie in the image of the adjoint of δ_1 to make it canonical. This canonical solution is δ_{k+1} and this completes the induction, since one knows at the beginning $\delta_1 \circ \delta_1 = 0$ and δ_2 is chain mapping; that is,* $[\delta_2, \delta_1] = 0$. [50, p. 252]

Dennis' construction is such that δe_α is in the sub-Lie algebra generated by the closure of the cells in e_α, or, expressed in dual terms, the maps $\Delta_r\colon C \to C^{\otimes r}$ corresponding to the δ_r maps are local.

Recall Dennis' problem, quoted in the introduction, of determining the topological and geometric meaning of this C_∞-coalgebra as an intrinsic object, and give closed form formulae for it and the induced maps associated to subdivisions. We will next present the solution Dennis and Ruth Lawrence gave to this problem in the case of the interval.

2.5. Lawrence–Sullivan interval

Let L be a completed free graded Lie algebra with filtration given by number of brackets, and let $U(L)$ be the complete graded vector space of series on one indeterminate with values on L whose filtration is induced from that of L, i.e., the Nth-part of the filtration $U(L)$ contains series of the form

$$\sum_{n=1}^{\infty} x_n t^n$$

where x_n is in $F_N L$ for every n. Consider the linear operator given by

$$\frac{d}{dt}\Big(\sum x_n t^n\Big) = \sum n\, x_n t^{n-1}$$

and the formal differential equation

$$\frac{du}{dt} = \partial v - \mathrm{ad}_v\, u$$

where $\mathrm{ad}_v\, u = [v, u]$. By formally solving this equation one defines the *flow generated* by v for any rational time t_0.

An element $u \in L$ is said to be *flat* if it is in degree -1 and satisfies $\partial u = \frac{1}{2}[u, u]$. It is common to refer to these as *Maurer–Cartan* elements, but we do not use this terminology. We now quote the theorem of Dennis and Ruth Lawrence.

> *There is a unique completed free differential graded Lie algebra, A, with generating elements a, b and e, in degrees -1, -1 and 0, respectively, for which a and b are flat while the flow generated by e moves from a to b in unit time. The differential is specified by*
>
> $$\partial e = \mathrm{ad}_e\, b + \sum_{i=0}^{\infty} \frac{B_i}{i!} (\mathrm{ad}_e)^i (b - a),$$
>
> *where B_i denotes the ith Bernoulli number defined as coefficients in the expansion*
>
> $$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$
>
> [25, Theorem 1]

We remark that Dennis conjectured an equivalence between the description above and the one obtained by applying his inductive procedure (Section 2.4). This conjecture was verified by Parent and Tanré [40].

The Lawrence–Sullivan dg Lie algebra is described in terms of the associated C_∞-coalgebra by

$$\begin{aligned}
&\Delta_1(c) = y - z, && \Delta_1(y) = \Delta_1(z) = 0,\\
&\Delta_2(c) = -\frac{1}{2}\big(c \otimes (y+z) + (y+z) \otimes c\big), && \Delta_2(y) = -y \otimes y, \quad \Delta_2(z) = -z \otimes z,\\
&\Delta_k(c) = \sum_{p+q=k-1} \frac{B_{k-1}}{p!q!} c^{\otimes p} \otimes (y-z) \otimes c^{\otimes q}, && \Delta_k(y) = \Delta_k(z) = 0, \quad (\text{for } k \geq 3),
\end{aligned}$$

where c, y and x are generators of degree 1, 0 and 0, respectively.

2.6. Rational homotopy theory

To algebraically model the rational homotopy category of spaces, two models were introduced. On one hand, there is Dennis' commutative approach [49] based on an adjunction

$$\mathsf{sSet} \underset{|\cdot|_S}{\overset{A_{\mathrm{PL}}}{\rightleftarrows}} \mathsf{cdga}^{\mathrm{op}}.$$

On the other hand, there is Quillen's Lie approach, introduced in [41] and extended in [10, 11], which is based on an adjunction

$$\mathsf{sSet} \underset{|\cdot|_Q}{\overset{\mathscr{L}}{\rightleftarrows}} \mathsf{cdgl}$$

where cdgl denotes the category of complete dg Lie algebras. This adjunction is defined explicitly by

$$\mathscr{L}(X) = \operatorname*{colim}_{\triangle^n \to X} \mathscr{L}(\triangle^n), \qquad |L|_n = \mathsf{cdgl}(\mathscr{L}(\triangle^n), L),$$

where $\mathscr{L}(\triangle^\bullet)$ is the cosimplicial complete dg Lie algebra defined by a natural C_∞-coalgebra structures on the cellular chains of standard simplices.

Using the principles presented in the previous subsection and a careful treatment of the simplicial structure, Buijs, Félix, Murillo, and Tanré [10] introduced a construction of $\mathscr{L}(\triangle^\bullet)$, which they characterize axiomatically by requiring that the generators associated to vertices are flat, and that the linear part is induced from the boundary of chains.

We mention that this structure is isomorphic to the one obtained by dualizing the simplicial C_∞-algebra defined by the homotopy transfer theorem of C_∞-algebras applied to Dennis' polynomial differential forms and Dupont's contraction [13].

The problem of finding closed formulae for the C_∞-coalgebra structure on the n-simplex remains open for $n > 3$.

2.7. Operadic viewpoint

The operad C_∞ is defined as the cobar construction on the Lie cooperad, the Koszul dual cooperad of $\mathscr{C}om$. That is to say

$$C_\infty = \mathbf{\Omega}\, \mathscr{L}ie^{\mathrm{c}}.$$

The operad C_∞ is a minimal projective resolution of $\mathscr{C}om$. A larger projective resolution is defined composing the bar and cobar constructions

$$CC_\infty = \mathbf{\Omega}\, \mathbf{B}\, \mathscr{C}om.$$

Since the bar and cobar constructions are defined in terms of rooted trees the CC_∞ operad can be described using bicolored trees. In [42], Robert-Nicoud and Vallette constructed a cosimplicial CC_∞-coalgebra in terms of bicolored trees, and explored the induced adjunction between simplicial sets and L_∞-algebras.

An anecdote shared with the author by both Dennis and Bruno Vallette, is that this bicolored model brought them in contact for the first time; after a talk where Dennis used this pictorial description, Bruno, then a recent graduate, recognized it as the bar-cobar resolution of $\mathcal{C}om$.

3. E_∞-coalgebras

In this section we consider E_∞-extensions of the Alexander–Whitney and Serre coalgebras on simplicial and cubical chains, respectively. The importance of these structures is highlighted by a theorem of Mandell [27] stating that the whole homotopy type of nilpotent finite type spaces is encoded in the quasi-isomorphism type of its E_∞-algebra on cochains.

Additionally, we recall Steenrod's cup-i coproducts and their induced square operations on mod 2 cohomology, identifying generalizations of these that induce Steenrod operations on mod p cohomology for any prime p. We close this section overviewing the use of cochain level structures in the study of topological phases of matter.

3.1. Alexander–Whitney coalgebra

The first chain approximation to the diagonal was given in the simplicial context by Čech and Whitney building on independent works presented during the First International Topological Conference in 1935 in Moscow by Alexander and Kolmogorov. The original references are [2, 12, 51] and a historical account is presented by Whitney in [52, p. 110]. This chain map, referred to as the *Alexander–Whitney coproduct*, is defined on elements of the canonical basis by the formula

$$\Delta([0,\dots,n]) = \sum_{i=0}^{n} [0,\dots,i] \otimes [i,\dots,n]. \tag{3}$$

Together with the *augmentation map*

$$\varepsilon([0,\dots,n]) = \begin{cases} 1, & n = 0, \\ 0, & n > 0, \end{cases} \tag{4}$$

the Alexander–Whitney coproduct satisfies

$$(\Delta \otimes \mathsf{id}) \circ \Delta = (\mathsf{id} \otimes \Delta) \circ \Delta,$$
$$(\varepsilon \otimes \mathsf{id}) \circ \Delta = \mathsf{id} = (\mathsf{id} \otimes \varepsilon) \circ \Delta,$$

making the (normalized) chains $\mathrm{N}(X)$ of any simplicial set X into a natural coassociative counital coalgebra, referred to as the *Alexander–Whitney coalgebra* of X.

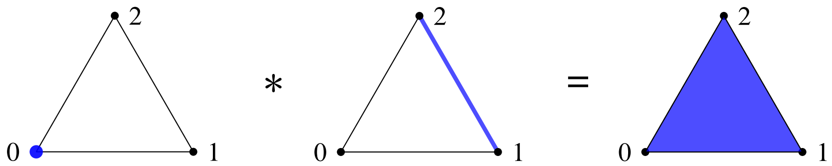

Figure 1. Geometric representation of the join product of two basis elements. It depicts the identity $*([0] \otimes [1,2]) = [0,1,2]$.

We will use the following recursively defined notation for general coalgebras:

$$\begin{aligned} \Delta^1 &= \Delta, \\ \Delta^k &= (\Delta \otimes \mathsf{id}) \circ \Delta^{k-1}. \end{aligned}$$

3.2. The join product

The *join product* $*\colon \mathrm{N}(\triangle^n)^{\otimes 2} \to \mathrm{N}(\triangle^n)$ is the natural degree 1 linear map defined by

$$* \left([v_0, \dots, v_p] \otimes [v_{p+1}, \dots, v_q]\right) = \\ \begin{cases} (-1)^p \operatorname{sign}(\pi)[v_{\pi(0)}, \dots, v_{\pi(q)}], & \forall\, i \neq j,\ v_i \neq v_j, \\ 0, & \text{otherwise,} \end{cases}$$

where π is the permutation that orders the vertices. It is an algebraic version of the usual join of faces in a simplex, please consult Figure 1 for an example.

The join product can be used in conjunction with the Alexander–Whitney coproduct to canonically construct boundaries in the chain complexes

$$\operatorname{Hom}\big(\mathrm{N}(\triangle^n)^{\otimes s}, \mathrm{N}(\triangle^n)^{\otimes r}\big).$$

For example,

$$H = (f * g) \circ \Delta$$

is a chain homotopy between any two quasi-isomorphisms $g, f\colon \mathrm{N}(\triangle^n) \to \mathrm{N}(\triangle^n)$. To see this, recall the augmentation map $\varepsilon\colon \mathrm{N}(\triangle^n) \to \Bbbk$ defined in (4) which is the counit of Δ, and notice that the join is a chain homotopy between $\varepsilon \otimes \mathsf{id}$ and $\mathsf{id} \otimes \varepsilon$, that is to say

$$\partial * = \varepsilon \otimes \mathsf{id} - \mathsf{id} \otimes \varepsilon.$$

Since f and g are quasi-isomorphisms, we have $\varepsilon \circ f = \varepsilon \circ g = \varepsilon$, so

$$\begin{aligned} \partial H &= (\varepsilon \otimes \mathsf{id} - \mathsf{id} \otimes \varepsilon) \circ (f \otimes g) \circ \Delta \\ &= (\varepsilon \otimes g - f \otimes \varepsilon) \circ \Delta \\ &= g - f. \end{aligned}$$

3.3. Steenrod cup-i coproduct structure

As it can be seen directly from (3), the Alexander–Whitney coproduct is not cocommutative. In [44], Steenrod introduced coherent higher diagonals correcting homologically this lack of cocommutativity. He used them to define the celebrated square operations, finer invariants on the mod 2 cohomology of spaces (Section 3.4). In this subsection we present an explicit recursive definition of Steenrod's higher diagonals.

Let C be a chain complex of $\mathbb{Z}$-modules and regard $\mathrm{Hom}(C, C \otimes C)$ as a chain complex of $\mathbb{Z}[\mathbb{S}_2]$-modules where $\mathbb{S}_2$ acts by permuting the factors in the target. Denote the elements $1 + (12)$ and $(12) - 1$ in $\mathbb{Z}[\mathbb{S}_2]$ by N and T, respectively. A *cup-i coproduct structure* on C is an equivariant chain map $\mathcal{W}(2) \to \mathrm{Hom}(C, C \otimes C)$ where

$$\mathcal{W}(2) = \mathbb{Z}[\mathbb{S}_2]\{e_0\} \xleftarrow{T} \mathbb{Z}[\mathbb{S}_2]\{e_1\} \xleftarrow{N} \mathbb{Z}[\mathbb{S}_2]\{e_2\} \xleftarrow{T} \cdots$$

is the minimal free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\mathbb{S}_2]$-module. The image of e_i is denoted by $\Delta_i \colon C \to C \otimes C$ and is referred to as the *cup-i coproduct* of C (with respect to the given cup-i coproduct structure).

We can use the Alexander–Whitney coproduct and the join product to give a recursive description of the natural cup-i coproduct structure on simplicial chains introduced in [44, p. 293]:

$$\begin{aligned} \Delta_0 &= \Delta, \\ \Delta_i &= (* \otimes \mathsf{id}) \circ (\mathsf{id} \otimes (12)\, \Delta_{i-1}) \circ \Delta\,. \end{aligned} \tag{5}$$

We refer to [17, 30, 37] for alternative descriptions of cup-i constructions, which have been shown to give rise to the same coproducts using an axiomatic characterization [35]. This "universal" chain level structure seems to be combinatorially fundamental. As examples illustrating this, we mention that it induces the nerve of strict infinity categories [31, 48], and that its comodules fully-faithfully model chain complex valued presheaves on simplicial complexes [36].

3.4. Steenrod square operations

Let C be equipped with a cup-i coproduct structure. The *Steenrod square operations*

$$Sq^k \colon \mathrm{H}(C^\vee) \to \mathrm{H}(C^\vee)$$

on the homology of its dual chain complex $C^\vee = \mathrm{Hom}(C, \mathbb{F}_2)$ are defined for every integer k by the formula

$$Sq^k([\alpha]) = \big[(\alpha \otimes \alpha)\, \Delta_{k-|\alpha|}(-)\big]$$

where brackets are used to denote represented elements in $\mathrm{H}(C^\vee)$.

3.5. An E_∞-coalgebra on simplicial chains

Cup-i coproducts on simplicial chains are part of an E_∞-coalgebra structure. This is a natural coalgebra structure over an operad whose arity r part is a chain complex of free

$\Bbbk[\mathbb{S}_r]$-module with the $\Bbbk$-homology of a point. Similar to Dennis' construction over $\mathbb{Q}$ of a C_∞-coalgebra structure on cellular chains (Section 2.4), the existence of an E_∞-coalgebra structure over any coefficient ring can be guaranteed using an acyclic carrier argument [14]. The goal of this subsection is to describe explicitly an E_∞-coalgebra structure on simplicial integral chains generalizing the construction of cup-i coproducts of Steenrod (Section 3.3).

The collection of all linear maps $\mathrm{N}(\triangle^n) \to \mathrm{N}(\triangle^n)^{\otimes r}$ for any r that can be expressed as an arbitrary composition of the Alexander–Whitney coproduct, the join product, and permutations of factors, define an E_∞-coalgebra structure on the chains of standard simplices. We remark that, since we are only considering maps whose domain is $\mathrm{N}(\triangle^n)$, the join is not part of this structure, although it is used in its construction.

The E_∞-operad $\mathrm{U}(\mathcal{M})$ defining this structure can be abstracted from this example. Roughly speaking, $\mathrm{U}(\mathcal{M}) = \{\mathcal{M}(1,r)\}_{r\geq 0}$ is the operad associated to the prop $\mathcal{M}$ generated by symbols $\Delta, \varepsilon, *$ in biarities $(1,2)$, $(1,0)$, and $(2,1)$ of degree $0,0,1$ with $\partial\,\Delta = 0$, $\partial\,\varepsilon = 0$, and $\partial * = \varepsilon \otimes \mathsf{id} - \mathsf{id} \otimes \varepsilon$, modulo the relations $(\varepsilon \otimes \mathsf{id}) \circ \Delta = \mathsf{id} = (\mathsf{id} \otimes \varepsilon) \circ \Delta$ and $\varepsilon \circ * = 0$. In Section 3.7 we review a family of explicit chain contractions that can be used to compute the homology of $\mathrm{U}(\mathcal{M})$. We use this family in Section 3.8 to define cup-(r,i) coproducts responsible for Steenrod operations at all primes.

Full details regarding the construction of the operad $\mathrm{U}(\mathcal{M})$ can be found in [33, 34] together with a comparison to the surjection operad [4, 30], a construction based on an earlier generalization of Steenrod's cup-i coproducts [3, § 4.5].

3.6. Monoidal extension and cubical chains

Let us consider the cellular chains on the interval $\mathrm{C}(\mathbb{I})$ as a counital coalgebra in the usual way:

$$\begin{aligned} &\Delta[01] = [0] \otimes [01] + [01] \otimes [1], && \Delta[0] = [0] \otimes [0], && \Delta[1] = [1] \otimes [1], \\ &\varepsilon[01] = 0, && \varepsilon[0] = 1, && \varepsilon[1] = 1. \end{aligned}$$

This structure can be extended to the chains of cubical sets using the isomorphism

$$\mathrm{N}(\square^n) \cong \mathrm{C}(\mathbb{I})^{\otimes n}$$

and the fact that the tensor product of counital coalgebras receives this structure canonically. Explicitly, for $i \in \{1,2\}$ let C_i be a counital coalgebra, the tensor product $C_1 \otimes C_2$ is a counital coalgebra with

$$\Delta(c_1 \otimes c_2) = (23)\big(\Delta(c_1) \otimes \Delta(c_2)\big), \tag{6}$$

$$\varepsilon(c_1 \otimes c_2) = \varepsilon(c_1)\,\varepsilon(c_2), \tag{7}$$

where the symmetric group $\mathbb{S}_4$ acts by permuting the tensor factors of $C_1 \otimes C_1 \otimes C_2 \otimes C_2$.

For any cubical set Y the induced structure on its chains agrees with that considered by Serre in [43], and we refer to it as the *Serre coalgebra* of Y.

We can define an E_∞-coalgebra structure extending the Serre coalgebra by describing an extension to all $\mathrm{N}(\square^n)$ of the map $*\colon \mathrm{C}(\mathbb{I})^{\otimes 2} \to \mathrm{C}(\mathbb{I})$ defined to be non-zero only for

$$*([0] \otimes [1]) = [01], \qquad *([1] \otimes [0]) = -[01].$$

For $i \in \{1, 2\}$ let A_i be a chain complex equipped with a degree 1 map $*\colon A_i^{\otimes 2} \to A_i$ and a chain map $\varepsilon\colon A_i \to \Bbbk$ such that $\varepsilon \circ * = 0$ and $\partial * = \varepsilon \otimes \mathsf{id} - \mathsf{id} \otimes \varepsilon$. The tensor product $A_1 \otimes A_2$ has the same structure, explicitly defined by (7) and

$$*\big((a_1 \otimes a_2) \otimes (a_1' \otimes a_2')\big) = (\mathsf{id} \otimes \varepsilon \otimes * + * \otimes \varepsilon \otimes \mathsf{id})(23)\big(a_1 \otimes a_2 \otimes a_1' \otimes a_2'\big) \quad (8)$$

where the right-hand side can be given more explicitly by

$$(-1)^{|a_2||a_1'|}\big(\varepsilon(a_1')\, a_1 \otimes *(a_2 \otimes a_2') + *(a_1 \otimes a_1') \otimes \varepsilon(a_2)\, a_2'\big).$$

Together, formulae (6), (7), and (8) induce on the tensor product of $\mathcal{M}$-bialgebras the same structure. In particular, $\mathrm{N}(\square^n) \cong \mathrm{C}(\mathbb{I})^{\otimes n}$ is equipped with an $\mathcal{M}$-bialgebra structure induced from that in the cellular chain on the interval. From it, a standard categorical construction – a Kan extension along the Yoneda embedding – provides the chains on any cubical set with the structure of an E_∞-coalgebra extending the Serre coalgebra structure or, more specifically, the structure of a $\mathrm{U}(\mathcal{M})$-coalgebra.

Using the monoidal structure on $\mathcal{M}$-bialgebras we also have a natural $\mathcal{M}$-bialgebra structure on any tensor product $\mathrm{N}(\triangle^{n_1}) \otimes \cdots \otimes \mathrm{N}(\triangle^{n_k})$ which defines a natural $\mathrm{U}(\mathcal{M})$-coalgebra on multisimplicial chains [20, 38].

3.7. Chain contractions for $\mathcal{M}$

We now explicitly describe chain contractions

$$\sigma \circlearrowright \mathcal{M}(s, r) \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathcal{M}(s, r - 1)$$

for every $s \geq 1$ and $r \geq 0$. In the next section we will use these to define cup-(r, i) coproducts on $\mathrm{U}(\mathcal{M})$-coalgebras.

Recall that $\mathcal{M}$ is the prop generated by Δ, ε, and $*$ modulo certain relations (Section 3.5). Let $\eta \in \mathcal{M}(s, r)$ be a composition of generators and define

$$\begin{aligned}
\iota(\eta) &= (\mathsf{id} \otimes \eta) \circ \Delta, \\
\pi(\eta) &= \big(\varepsilon \otimes \mathsf{id}^{\otimes r-1}\big) \circ \eta, \\
\sigma(\eta) &= \big(* \otimes \mathsf{id}^{\otimes r-1}\big) \circ (\mathsf{id} \otimes \eta) \circ \Delta\,.
\end{aligned}$$

These define a chain contraction as above, i.e., they satisfy

$$\pi \circ \iota - \mathsf{id} = 0, \qquad \iota \circ \pi - \mathsf{id} = \partial \sigma.$$

Using the relations defining $\mathcal{M}$ it is not hard to see that $\mathcal{M}(s,0) \cong \Bbbk$, so we have an explicit chain contraction

$$h \circlearrowright \mathcal{M}(s,r) \underset{\iota^r}{\overset{\pi^r}{\rightleftarrows}} \Bbbk$$

where

$$h = \sigma + \iota\,\sigma\,\pi + \cdots + \iota^r\,\sigma\,\pi^r\,. \tag{9}$$

In particular, given that by construction $\mathrm{U}(\mathcal{M})(r)$ is a free $\mathbb{Z}[\mathbb{S}_r]$-module, these chain contractions show that $\mathrm{U}(\mathcal{M})$ is an E_∞-operad.

3.8. Steenrod cup-(r,i) products

To generalize the notion of cup-i coproduct structure, consider the cyclic group of order r and the minimal free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\mathbb{C}_r]$-module

$$\mathcal{W}(r) = \mathbb{Z}[\mathbb{C}_r]\{e_0\} \xleftarrow{T} \mathbb{Z}[\mathbb{C}_r]\{e_1\} \xleftarrow{N} \mathbb{Z}[\mathbb{C}_r]\{e_2\} \xleftarrow{T} \cdots$$

where

$$N = 1 + \rho + \cdots + \rho^{r-1}, \qquad T = \rho - 1, \tag{10}$$

and ρ is a generator of $\mathbb{C}_r$.

Let C be a $\mathrm{U}(\mathcal{M})$-coalgebra, for example the chains on a simplicial or cubical set. For $r \geq 2$, let $\psi(r)\colon \mathcal{W}(r) \to \mathrm{End}(C, C^{\otimes r})$ be the $\mathbb{C}_r$-equivariant chain map defined recursively by

$$\begin{aligned} \psi(r)(e_0) &= \Delta^{r-1}, \\ \psi(r)(e_{2m+1}) &= h\,T\,\psi(r)(e_{2m}), \\ \psi(r)(e_{2m}) &= h\,N\,\psi(r)(e_{2m-1}), \end{aligned} \tag{11}$$

where T and N are explicitly defined in (10) and h in (9). The *Steenrod cup-(r,i) product* of C is defined for every $r, i \geq 0$ as the image in $\mathrm{End}(C, C^{\otimes r})$ of $\psi(e_i)$. We remark that for $r = 2$ the resulting cup-i coproduct structure on simplicial chains recovers Steenrod's original construction (5).

3.9. Steenrod operations

Let p be an odd prime. We now review a construction of Steenrod operations in mod p cohomology analogue to the one given in Section 3.4 for Steenrod squares. We remark that Steenrod square operations are parameterized by the mod 2 homology of $\mathbb{C}_2 = \mathbb{S}_2$. As explained for example in [1, Corollary VI.1.4], an inclusion $\mathbb{C}_p \to \mathbb{S}_p$ induces a surjection in mod p homology. Using the homological degrees where this surjection is non-zero we have the following construction explained in more detail in [28, 45–47].

Let C be a $\mathrm{U}(\mathcal{M})$-coalgebra and $C^\vee = \mathrm{Hom}(C, \mathbb{F}_p)$. We simplify notation and denote the explicit linear map $\psi(p)(e_i)\colon C \to C^{\otimes p}$ defined in (11) simply by ψ_i. For any integer s, the *Steenrod operations*

$$P_s\colon \mathrm{H}\big(C^\vee; \mathbb{F}_p\big) \to \mathrm{H}\big(C^\vee; \mathbb{F}_p\big)$$

and

$$\beta P_s \colon \mathrm{H}(C^\vee; \mathbb{F}_p) \to \mathrm{H}(C^\vee; \mathbb{F}_p)$$

are respectively defined by sending the class represented by $\alpha \in C^\vee$ of degree q to the classes represented for $\varepsilon \in \{0, 1\}$ by

$$\pm\, \alpha^{\otimes p} \circ \psi_{(2s-q)(p-1)-\varepsilon}(-),$$

where a possible sign convention is introduced and motivated in [46, (6.1)].

3.10. Cartan and Adem relations

There is a conceptual pattern producing additional homological information from relations bounding primary structure. As an example, we have seen that Steenrod operations can be understood as structure on cohomology deduced from lifting to the cochain level the commutativity relation satisfied by the cup product. Steenrod operations, which we now regard as primary cohomological structure, satisfy relations that lead to finer structure. The first of these is the *Cartan relation* that establishes a connection between Steenrod operations and the cup product, it is given by:

$$P_s(\alpha\beta) = \sum_{i+j=s} P_i(\alpha) P_j(\beta).$$

The second is the *Adem relation*, controlling the iteration of Steenrod operations, and given by:

(1) If $p = 2$ and $a > 2b$, then

$$P_a P_b = \sum_i \binom{2i - a}{a - b - i - 1} P_{a+b-i} P_i;$$

(2) if $p > 2$ and $a > pb$, then

$$P_a P_b = \sum_i (-1)^{a+i} \binom{pi - a}{a - (p-1)b - i - 1} P_{a+b-i} P_i.$$

There are versions of these using the βP_s operations but we do not write them here, see for example [28, 47].

Steenrod operations and the above relations are homological consequences of an E_∞-structure. For cellular chains we have seen explicit cochain level constructions, the cup-(p, i) products, inducing the Steenrod operations, and it is desirable to produce cochains enforcing these relations. For the even prime case, Cartan and Adem coboundaries have been constructed effectively in [32] and [6], respectively. Cartan coboundaries for odd primes can be constructed with the tools already described, but the Adem relation requires additional techniques not yet available.

A source of motivation for these cochain level constructions comes from their use in the study of topological phases as we overview next.

3.11. Symmetry protected topological phases and cochain constructions

A central problem in physics is to define and understand the moduli "space" of quantum systems with a fixed set of invariants, for example, their dimension and symmetry type. In condensed matter physics, quantum systems are presented using *lattice models* which, intuitively, are given by a Hamiltonian presented as a sum of local terms on a Hilbert space associated to a lattice in $\mathbb{R}^n$. We think of these as defined on flat space. One such system is said to be *gapped* if the spectrum of the Hamiltonian is bounded away from 0, and two Hamiltonians represent the same *phase* if there exists a deformation between them consisting only of systems that remain bounded from below.

Given a lattice model, using cellular decompositions and state sum type constructions, one can often compute the corresponding *partition functions* on spacetime manifolds from actions expressed in terms of gauge fields represented by cochains and cochain level structures: Stiefel–Whitney cochains, cup-i products and Cartan or Adem coboundaries, for example. Subdivision invariance gives rise to a topological quantum field theory, which in the *invertible* case is expected to be controlled by a generalized cohomology theory [15, 16, 53]. The cochain level structure used in the definition of the cellular gauge theory is interpreted from this point of view as describing a cochain model of the Postnikov tower of the relevant spectrum. For example, fermionic phases protected by a G-symmetry are believed to be classified by applying to BG the Pontryagin dual of spin bordism [21, 22]. Building on these insights, A. Kapustin proposed a structural ansatz in low dimensions that Greg Brumfiel and John Morgan verified by constructing cochain models of certain connective covers of said spectrum [7,8]. The resulting models use cup-i products and Adem coboundaries to represent k-invariants and the additive structure of the spectrum.

Acknowledgments. We would like to thank Greg Brumfiel, Ruth Lawrence, John Morgan, Aniceto Murillo, Dennis Sullivan and Bruno Vallette for helpful exchanges related to their work. We are also grateful to the referees for their careful reading and many keen suggestions. We are grateful for the hospitality of both the Max Planck Institute for Mathematics and the Université Sorbonne Paris Nord.

Funding. Partial support for this project was provided by grant ANR 20 CE40 0016 01 PROJET HighAGT.

References

[1] A. Adem and R. J. Milgram, *Cohomology of finite groups*. Second edn., Grundlehren Math. Wiss. 309, Springer, Berlin, 2004 MR 2035696

[2] J. W. Alexander, On the connectivity ring of an abstract space. *Ann. of Math. (2)* **37** (1936), no. 3, 698–708 Zbl 62.0673.01 MR 1503305

[3] D. J. Benson, *Representations and cohomology. II*. Second edn., Cambridge Stud. Adv. Math. 31, Cambridge University Press, Cambridge, 1998 Zbl 0908.20002 MR 1634407

[4] C. Berger and B. Fresse, Combinatorial operad actions on cochains. *Math. Proc. Cambridge Philos. Soc.* **137** (2004), no. 1, 135–174 MR 2075046

[5] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Math. 347, Springer, Berlin–New York, 1973 Zbl 0285.55012 MR 420609

[6] G. Brumfiel, A. Medina-Mardones, and J. Morgan, A cochain level proof of Adem relations in the mod 2 Steenrod algebra. *J. Homotopy Relat. Struct.* **16** (2021), no. 4, 517–562 Zbl 1487.55027 MR 4343073

[7] G. Brumfiel and J. Morgan, The Pontrjagin dual of 3-dimensional spin bordism. 2016, arXiv:1612.02860

[8] G. Brumfiel and J. Morgan, The Pontrjagin dual of 4-dimensional spin bordism. 2018, arXiv:1803.08147

[9] U. Buijs, Y. Félix, A. Murillo, and D. Tanré, Symmetric Lie models of a triangle. *Fund. Math.* **246** (2019), no. 3, 289–300 MR 3959254

[10] U. Buijs, Y. Félix, A. Murillo, and D. Tanré, *Lie models in topology*. Progr. Math. 335, Birkhäuser/Springer, Cham, 2020 Zbl 1469.55001 MR 4212536

[11] U. Buijs and A. Murillo, Algebraic models of non-connected spaces and homotopy theory of L_∞ algebras. *Adv. Math.* **236** (2013), 60–91 Zbl 1270.55010 MR 3019716

[12] E. Čech, Multiplications on a complex. *Ann. of Math. (2)* **37** (1936), no. 3, 681–697 MR 1503304

[13] X. Z. Cheng and E. Getzler, Transferring homotopy commutative algebraic structures. *J. Pure Appl. Algebra* **212** (2008), no. 11, 2535–2542 Zbl 1158.18006 MR 2440265

[14] S. Eilenberg and S. MacLane, Acyclic models. *Amer. J. Math.* **75** (1953), no. 1, 189–199 MR 52766

[15] D. S. Freed and M. J. Hopkins, Reflection positivity and invertible topological phases. *Geom. Topol.* **25** (2021), no. 3, 1165–1330 Zbl 1508.81953 MR 4268163

[16] D. Gaiotto and T. Johnson-Freyd, Symmetry protected topological phases and generalized cohomology. *J. High Energy Phys.* (2019), no. 5, article no. 7 Zbl 1416.81168 MR 3978827

[17] R. González-Díaz and P. Real, A combinatorial method for computing Steenrod squares. *J. Pure Appl. Algebra* **139** (1999), no. 1–3, 89–108 Zbl 0931.55015 MR 1700539

[18] I. Griniasty and R. Lawrence, An explicit symmetric DGLA model of a triangle. *High. Struct.* **3** (2019), no. 1, 1–16 Zbl 1473.17048 MR 3939044

[19] I. Griniasty and R. Lawrence, Explicit symmetric DGLA models of 3-cells. *Canad. Math. Bull.* **64** (2021), no. 2, 368–382 Zbl 1507.17040 MR 4273206

[20] V. K. A. M. Gugenheim, On supercomplexes. *Trans. Amer. Math. Soc.* **85** (1957), no. 1, 35–51 MR 86299

[21] A. Kapustin and R. Thorngren, Fermionic SPT phases in higher dimensions and bosonization. *J. High Energy Phys.* (2017), no. 10, article no. 80 Zbl 1383.83180 MR 3731133

[22] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, Fermionic symmetry protected topological phases and cobordisms. *J. High Energy Phys.* (2015), no. 12, article no. 52 Zbl 1388.81845 MR 3464750

[23] R. M. Kaufmann and A. M. Medina-Mardones, Cochain level May–Steenrod operations. *Forum Math.* **33** (2021), no. 6, 1507–1526 MR 4333989

[24] R. M. Kaufmann and A. M. Medina-Mardones, A combinatorial E_∞-algebra structure on cubical cochains and the Cartan–Serre map. *Cah. Topol. Géom. Différ. Catég.* **63** (2022), no. 4, 387–424 Zbl 1512.55013 MR 4556595

[25] R. Lawrence and D. Sullivan, A formula for topology/deformations and its significance. *Fund. Math.* **225** (2014), no. 1, 229–242 MR 3205571

[26] M. A. Mandell, E_∞ algebras and p-adic homotopy theory. *Topology* **40** (2001), no. 1, 43–94 MR 1791268

[27] M. A. Mandell, Cochains and homotopy type. *Publ. Math. Inst. Hautes Études Sci.* (2006), no. 103, 213–246 MR 2233853

[28] J. P. May, A general algebraic approach to Steenrod operations. In *The Steenrod Algebra and its Applications*, pp. 153–231, Lecture Notes in Math. 168, Springer, Berlin–New York, 1970 Zbl 0242.55023 MR 281196

[29] J. P. May, *The geometry of iterated loop spaces*. Lecture Notes in Math. 271, Springer, Berlin–New York, 1972 Zbl 0244.55009 MR 420610

[30] J. E. McClure and J. H. Smith, Multivariable cochain operations and little n-cubes. *J. Amer. Math. Soc.* **16** (2003), no. 3, 681–704 Zbl 1014.18005 MR 1969208

[31] A. M. Medina-Mardones, An algebraic representation of globular sets. *Homology Homotopy Appl.* **22** (2020), no. 2, 135–150 Zbl 1440.55014 MR 4093174

[32] A. M. Medina-Mardones, An effective proof of the Cartan formula: the even prime. *J. Pure Appl. Algebra* **224** (2020), no. 12, article no. 106444 Zbl 1472.55012 MR 4102178

[33] A. M. Medina-Mardones, A finitely presented E_∞-prop I: algebraic context. *High. Struct.* **4** (2020), no. 2, 1–21 Zbl 1459.55010 MR 4133162

[34] A. M. Medina-Mardones, A finitely presented E_∞-prop II: cellular context. *High. Struct.* **5** (2021), no. 1, 186–203 Zbl 1481.55010 MR 4367220

[35] A. M. Medina-Mardones, An axiomatic characterization of Steenrod's cup-i products. 2022, arXiv:1810.06505

[36] A. M. Medina-Mardones, Ranicki–Weiss assembly and the Steenrod construction. 2022, arXiv:2207.13115; to appear in *Proc. Amer. Math. Soc.*

[37] A. M. Medina-Mardones, New formulas for cup-i products and fast computation of Steenrod squares. *Comput. Geom.* **109** (2023), article no. 101921 Zbl 07618075 MR 4473678

[38] A. M. Medina-Mardones, A. Pizzi, and P. Salvatore, Multisimplicial chains and configuration spaces. 2023, arXiv:2012.02060v2; to appear in *J. Homotopy Relat. Struct.*

[39] A. M. Medina-Mardones and M. Rivera, Adams' cobar construction as a monoidal E_∞-coalgebra model of the based loop space. 2021, arXiv:2108.02790

[40] P.-E. Parent and D. Tanré, Lawrence–Sullivan models for the interval. *Topology Appl.* **159** (2012), no. 1, 371–378 MR 2852982

[41] D. Quillen, Rational homotopy theory. *Ann. of Math. (2)* **90** (1969), no. 2, 205–295 MR 258031

[42] D. Robert-Nicoud and B. Vallette, Higher Lie theory. 2020, arXiv:2010.10485

[43] J.-P. Serre, Homologie singulière des espaces fibrés. Applications. *Ann. of Math. (2)* **54** (1951), no. 3, 425–505 MR 45386

[44] N. E. Steenrod, Products of cocycles and extensions of mappings. *Ann. of Math. (2)* **48** (1947), no. 2, 290–320 MR 22071

[45] N. E. Steenrod, Reduced powers of cohomology classes. *Ann. of Math. (2)* **56** (1952), no. 1, 47–67 Zbl 0048.41301 MR 48026

[46] N. E. Steenrod, Cyclic reduced powers of cohomology classes. *Proc. Nat. Acad. Sci. U.S.A.* **39** (1953), no. 3, 217–223 Zbl 0050.39501 MR 54965

[47] N. E. Steenrod, *Cohomology operations*. Ann. of Math. Stud. 50, Princeton University Press, Princeton, NJ, 1962 MR 145525

[48] R. Street, The algebra of oriented simplexes. *J. Pure Appl. Algebra* **49** (1987), no. 3, 283–335 Zbl 0661.18005 MR 920944

[49] D. Sullivan, Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 269–331 (1978) MR 646078

[50] D. Sullivan, Appendix A in 'T. Tradler and M. Zeinalian, Infinity structure of Poincaré duality spaces.' *Algebr. Geom. Topol.* **7** (2007), no. 1, 251–253 MR 2308943

[51] H. Whitney, On products in a complex. *Ann. of Math. (2)* **39** (1938), no. 2, 397–432 MR 1503416

[52] H. Whitney, Moscow 1935: topology moving toward America. In *A century of mathematics in America, Part I*, pp. 97–117, Hist. Math. 1, American Mathematical Society, Providence, RI, 1988 Zbl 0666.01008 MR 1003166

[53] C. Z. Xiong, Minimalist approach to the classification of symmetry protected topological phases. *J. Phys. A* **51** (2018), no. 44, article no. 445001 Zbl 1412.82005 MR 3863285

Received 1 November 2021; revised 5 December 2022.

Anibal M. Medina-Mardones
Department of Mathematics, Western University, 1151 Richmond St, London, ON, N6A 3K7, Canada; anibal.medina.mardones@uwo.ca