

Low-dimensional solenoidal manifolds

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Abstract. In this paper we survey n -dimensional solenoidal manifolds for $n = 1, 2$ and 3 , and present new results about them. Solenoidal manifolds of dimension n are metric spaces locally modeled on the product of a Cantor set and an open n -dimensional disk. Therefore, they can be “laminated” (or “foliated”) by n -dimensional leaves. By a theorem of A. Clark and S. Hurder, topologically homogeneous, compact solenoidal manifolds are McCord solenoids, i.e., are obtained as the inverse limit of an increasing tower of finite, regular covering spaces of a compact manifold with an infinite and residually finite fundamental group. In this case, their structure is very rich since they are principal Cantor-group bundles over a compact manifold and behave like “laminated” versions of compact manifolds, thus they share many of their properties.

*To Dennis Sullivan on the occasion of his 80th birthday,
for all the inspiration and mathematical ideas he has shared with me*

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1. Introduction

Given an integer $n \geq 0$, a topological n -dimensional *solenoidal manifold* or *solenoidal lamination* or simply *n -dimensional solenoid*, is a metric space which is locally the product of a euclidean open n -disk and an infinite, compact, perfect and totally disconnected set (i.e., a Cantor set). If the space is locally the product of an open subset of the closed n -disk and the Cantor set, the space is a *solenoidal manifold with boundary*. The spaces are laminated with n -dimensional leaves and with a Cantor set transverse structure. When

$n = 1$ sometimes we will simply call them *solenoids* and when $n = 2$ we will refer to them as *Riemann surface solenoidal laminations*. These solenoidal manifolds appear naturally in many branches of mathematics. For example, in topology, the Vietoris–van Dantzig p -adic solenoids [31, 115] are some of the fundamental examples which motivated the development of new homology and cohomology theories that could apply to these spaces, for instance in the papers by Steenrod [105, 106].

Solenoids appear naturally also as Pontryagin duals of discrete locally compact Hausdorff abelian groups [89, 90]. For instance, if \mathbb{Q} denotes additive rationals with the discrete topology then its Pontryagin dual \mathbb{Q}^* is the abelian universal 1-dimensional solenoid which is a compact abelian group that fibers over the circle \mathbb{S}^1 via an epimorphism $p: \mathbb{Q}^* \rightarrow \mathbb{S}^1$ where the fiber is the Cantor group $\hat{\mathbb{Z}}$ which is the profinite completion of the integers \mathbb{Z} . This fact has a close relationship with the *adèles* and *idèles* and its properties are the first steps in Tate’s thesis [111].

Cantor groups, i.e., topological groups homeomorphic to the Cantor set, play an important role in the present paper. In particular, the profinite groups which are the profinite completions of residually finite groups.

A compact 0-dimensional, solenoidal manifold is just a Cantor set.

Solenoids have a relevant role in dynamical systems, as basic sets of axiom A diffeomorphisms in the sense of Smale [103]. In particular, one-dimensional expanding attractors are solenoidal manifolds and were studied extensively by Robert Williams [116]. In [10] it is shown that a similar construction as that in [116] holds in higher dimensions: compact solenoidal manifolds without holonomy are obtained as inverse limits of coverings of branched manifolds.

Some solenoidal manifolds encapsulate tiling spaces [14, 19, 44, 95, 96].

Laminations play an important role in holomorphic dynamics [34, 44, 66, 67, 108]. Some results on dynamics on laminations (geodesic flows, horocycle flows, harmonic measures) can be found, for instance, in [8, 9, 18, 32, 33, 72].

The papers [28–30, 52] by A. Clark, S. Hurder, and O. Lukina deal with solenoidal manifolds which they call *matchbox manifolds* [5].

A beautiful theory, covering many topics of solenoids, in particular regarding their ergodic properties, has been developed by Vicente Muñoz and Ricardo Pérez-Marco [80–86].

Higher-dimensional laminations with hyperbolic leaves have been considered by M. Kapovich in [57].

Lastly, one of the most fascinating subjects and achievements in mathematics of the first quarter of this century is the work of Peter Scholze and his collaborators on *Perfectoid spaces* [101, 102]. These spaces are closely related to solenoids. In fact, the prototype of a perfectoid space is the p -adic solenoid.

The paper is organized as follows. In Section 2 we introduce the definition of *solenoidal manifolds* and establish some of their properties. In Section 3 we study topologically homogeneous solenoidal manifolds. Some of the most important examples of compact n -dimensional solenoidal manifolds are obtained as inverse limits of an infinite

tower of finite regular coverings of compact n -dimensional manifolds, with an infinite and residually finite fundamental group.

A theorem of M. C. McCord implies that a solenoid obtained by such a tower is topologically homogeneous, in fact, it is a principal locally trivial fiber bundle over a compact n -manifold with fiber a Cantor group [74]. These solenoidal manifolds are called McCord solenoids.

The reciprocal is a theorem of A. Clark and S. Hurder [28] (see Theorem 2, Section 3). It states that compact solenoidal manifolds which are also topologically homogeneous are McCord solenoids. If we take the complete tower of regular coverings the solenoid is the *algebraic universal covering* of M . The Cantor group fiber is the *profinite completion* of the fundamental group. All non-simply-connected compact, oriented surfaces have infinite, residually finite, fundamental groups, therefore we have a solenoidal 2-manifold obtained by an infinite tower of finite coverings of order ≥ 2 . Since the subgroups of finite index in the tower are normal, the inverse limit is a principal fiber bundle over the surface with fiber the Cantor group which is the profinite completion of the sequence of finite groups of deck transformations. In particular, this Cantor group acts transitively on the Cantor fiber and one can show that the Riemann surface lamination is topologically homogeneous. Haar measure is a holonomy-invariant transverse measure. In particular, these laminations are minimal (Definition 6). In Section 3.1 we define *differentiable McCord solenoids* and in Proposition 1 (and Corollary 1) we describe some of their properties. Differentiable McCord solenoidal manifolds endowed with a laminated Riemannian metric have a very natural “volume form” which is a measure that disintegrates into the volume form (with respect to the Riemannian metric) in the leaves and disintegrates into the Haar measure in the Cantor group transversals. In Section 3.2 we give examples of smooth 8-dimensional solenoidal manifolds that are homeomorphic but not diffeomorphic, therefore they have several inequivalent differentiable structures. We also give examples of solenoidal manifolds whose leaves are not homeomorphic to a simplicial complex and examples with leaves that can be triangulated as simplicial complexes but do not have a PL structure, therefore the leaves are non-smoothable, i.e., solenoidal manifolds without a smooth structure. In Section 3.4 we define the solenoidal manifolds which are obtained as suspensions of representations of fundamental groups of compact manifolds. In Section 3.5 we deal with the problem of lifting maps between solenoidal coverings (with the caveat that, because of lack of local connectedness, liftings of homeomorphisms to non-compact solenoids might not exist or might not be homeomorphisms).

In Section 4 we introduce the notion of harmonic maps between Riemannian solenoidal manifolds, and we establish the solenoidal version of the theorem of Eells–Sampson: a differentiable map between Riemannian solenoidal manifolds with target a solenoidal manifold with a laminated Riemannian metric having all leaves with negative sectional curvature is homotopic to a harmonic map.

Compact, *oriented*, 1-dimensional solenoids are in one-to-one correspondence with conjugacy classes of homeomorphism of the Cantor set. This is because Dennis Sullivan

proved that these solenoids are mapping tori of homeomorphisms of the Cantor set [110, 114]. In the first part of Section 5 we use this fact to give some properties of these solenoids. We show, following D. Sullivan [110], that these 1-dimensional laminations are null-cobordant, i.e., there is a compact, oriented, 2-dimensional Riemann surface lamination with boundary equal to the given 1-dimensional solenoid.

In Section 5.3 we describe these compact, oriented 1-dimensional solenoids in the case when they are topologically transitive. By the results in Section 3, compact solenoidal manifolds which are also topologically homogeneous have the property of being obtained as the inverse limit of an infinite tower of regular coverings of increasing order of a compact manifold. Therefore, a compact connected, topologically homogeneous 1-dimensional solenoidal manifold is a 1-dimensional compact abelian group, since it is obtained as the inverse limit of a tower of coverings of the circle \mathbb{S}^1 and every covering of the circle is equivalent to a homomorphism of the form $z \mapsto z^n$, $n \in \mathbb{N}$. These groups are Pontryagin duals of dense subgroups of the rationals.

Section 6 deals with solenoidal surfaces, their uniformization, Theorem 8, the Ricci flow and the uniformization of hyperbolic solenoidal surfaces by the Ricci flow, Theorem 9 in Section 6.2.

In Section 6.3 we describe Dennis Sullivan’s 2-dimensional *universal hyperbolic solenoid*. We use our version of the Eells–Sampson theorem to prove a version of the Earle–Eells theorem in Section 6.4.

In Section 6.5 we study the n -dimensional solenoidal compact abelian groups and the *universal euclidean solenoid* when $n = 2$. We indicate the complexity of their classification when $n > 1$.

2. Preliminaries

Definition 1. For the general definition of a *lamination* or *foliated space*, we refer the reader to [27, Chapter 11] or [76]. A topological *solenoidal n -dimensional manifold* M is a metrizable space M endowed with an atlas $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in \mathfrak{S}}$, where $\{U_\alpha\}_{\alpha \in \mathfrak{S}}$ is an open cover of M and \mathfrak{S} a set of indices, such that: for each $\alpha \in \mathfrak{S}$, $\varphi_\alpha: U_\alpha \rightarrow D \times T_\alpha$ is a homeomorphism from U_α to a product $D \times T_\alpha$, where D is the open unit disk in \mathbb{R}^n and T_α is homeomorphic to the Cantor set. The inverse of $\varphi_\alpha^{-1} := \bar{\varphi}_\alpha: D \times T_\alpha \rightarrow U_\alpha$ is called a *local parametrization*.

It follows from the definition that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the *change of coordinates*

$$\Psi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta) \tag{1}$$

is of the form $(x, t) \mapsto (\lambda_{\alpha\beta}(x, t), \tau_{\alpha\beta}(t))$.

Definition 2 (Smooth solenoidal manifolds). The solenoidal n -dimensional manifold M is called a *solenoidal n -dimensional manifold of class C^r* , $0 \leq r \leq \infty$, ω , if $\forall \alpha, \beta$ the map $\lambda_{\alpha\beta}(\cdot, t)$ is of class C^r and depends continuously on t in the C^r -topology.

For each $\alpha \in \mathfrak{S}$ we can orient D with the standard orientation of \mathbb{R}^n . If for each fixed t the map $(\cdot, t) \mapsto (\lambda_{\alpha\beta}(\cdot, t), \tau_{\alpha\beta}(t))$ preserves the orientation we say that the lamination is *oriented*.

Definition 3. If for each $\alpha \in \mathfrak{S}$, $\varphi_\alpha: U_\alpha \rightarrow V_\alpha \subset \bar{D} \times T_\alpha$ is a homeomorphism from U_α to an open set V_α in the product $\bar{D} \times T_\alpha$, where \bar{D} is the closed unit n -disk in \mathbb{R}^n with boundary \mathbb{S}^{n-1} , we call M a *solenoidal n -dimensional manifold with boundary*. The boundary is defined in the same fashion as the usual definition for manifolds with boundary and it is an $(n - 1)$ -solenoidal manifold.

Remark 1. Sometimes solenoidal n -dimensional manifolds are called n -dimensional Cantor laminations or *matchbox manifolds* [5], since the transversals are zero-dimensional. In fact, (smooth) solenoidal manifolds are a generalization of (smooth) manifolds and also a particular type of (leafwise smooth) laminations, transversely modeled by the Cantor set. It should be noted that both points of view will alternate throughout the text.

Of course, smooth solenoidal manifolds are generalizations of smooth foliated manifolds, but the local holonomy is represented by local homeomorphisms of Cantor sets. We could define foliations of Hölder class C^r for any $r > 0$ by imposing the changes of charts to be of class C^r .

Definition 4. The atlas $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in \mathfrak{S}}$ is called a *lamination atlas* and the maps φ_α are called *flow boxes* or *foliation charts*. The sets $\varphi_\alpha^{-1}(D \times \{t\})$ are called *plaques*. Condition (2) says that the plaques glue together to form maximal connected n -dimensional smooth manifolds, called *leaves*, which are immersed in M .

A set of the form $\varphi_\alpha^{-1}(\{y\} \times T_\alpha)$ is called *transversal*. The associated *lamination* \mathcal{L} is the partition of M into leaves. More precisely, the leaves are smooth n -dimensional manifolds whose local charts are the restrictions of the maps $p_\alpha \circ \varphi_\alpha$ to plaques, where $p_\alpha: D_\alpha \times T_\alpha \rightarrow D_\alpha$ is the projection onto the first factor.

Definition 5. If M is an n -dimensional solenoidal manifold, a *holonomy invariant measure* is a family of Radon measures in all transversal Cantor sets (see Definition 4) which are invariant under the maps $\tau_{\alpha\beta}$ in formula (1), Definition 1, when restricted to their domains and ranges. Analogously, a *holonomy invariant metric* is a metric in all transversals which is invariant under the maps $\tau_{\alpha\beta}$ when restricted to their domains and ranges. This means that these measures, or metrics, are invariant under the holonomy pseudogroup.

An important class of laminations consists of minimal laminations:

Definition 6. A lamination is said to be *minimal* if all the leaves are dense.

Remark 2. We will be dealing mostly with solenoidal manifolds which are compact, oriented, and connected. However, sometimes we consider “covering” solenoidal manifolds that are neither connected (or even locally connected) nor compact.

If the laminated atlas \mathcal{A} is understood we omit it and simply refer to M as a *solenoidal manifold*.

On a smooth lamination one can define the notions of differential topology and geometry along the leaves. For instance, one has concepts such as *laminated smooth functions*, *laminated Riemannian metrics*, *laminated curvature*, *de Rham theory*, etc. [27, 76]. In particular, we have the following:

Definition 7. $C^{\infty,0}(M)$ consists of those continuous functions on M which are smooth when restricted to the leaves with the standard definition with respect to foliated charts. Analogously we can define laminated smooth vector-valued functions $C^{\infty,0}(M, \mathbb{R}^n)$.

One can define in a natural way the “laminated” tangent bundle $T(M)$ of the lamination by considering all the tangent spaces of the leaves at all the points of such leaves. All the notions about differential topology and differential geometry of manifolds can be adapted to smooth laminations. In particular, one has the notion of smooth vector fields, differential forms, tensors, Riemannian metrics, laminated connections, curvature along the leaves, etc. We require these objects to be continuous and differentiable when restricted to each leaf. *In this paper we will mostly deal with differentiable solenoidal laminations.*

Definition 8. If M_1 and M_2 are solenoidal manifolds, a *smooth foliated map* (or *smooth laminated map*) is a continuous map $f: M_1 \rightarrow M_2$ such that f sends leaves of M_1 into leaves of M_2 and, in terms of local charts, the restrictions of f to plaques of M_1 are differentiable functions into plaques of M_2 . Such a map we also call *leafwise differentiable map*. In particular, a diffeomorphism between two solenoidal manifolds M_1 and M_2 is a homeomorphism $f: M_1 \rightarrow M_2$ such that f and f^{-1} are leafwise differentiable.

3. Topologically homogeneous solenoidal manifolds

Since all non-empty, compact, metric spaces which are perfect and totally disconnected are homeomorphic (to the standard ternary Cantor set) we simply speak about “the” Cantor set K . In this paper we will use very frequently the following:

Definition 9. $\mathcal{H}(K)$ will denote the group of homeomorphisms of the Cantor set.

The most canonical examples of compact n -dimensional solenoidal manifolds are locally-trivial fibrations $p: X \rightarrow M$ over a compact n -dimensional connected manifold M with fiber the Cantor set K . We assume that X is connected. Fibrations like p have the *unique path-lifting property* and therefore they are a natural generalization of the notion of covering spaces (Steenrod [104, Section 13]). This property allows us to define holonomy along a path $f: [0, 1] \rightarrow M$: a homeomorphism $h_f: p^{-1}(f(0)) \rightarrow p^{-1}(f(1))$. The holonomy h_f only depends on the homotopy class of f with endpoints $f(0)$

and $f(1)$ fixed. If $x \in M$ then holonomy of loops based at x gives a representation $\rho: \pi_1(M, x) \rightarrow \mathcal{H}(K)$ called the holonomy of the fibration. The leaves of the lamination are the path components of X .

Let us recall some facts about compact solenoidal manifolds \mathcal{L} which are topologically homogeneous, i.e., the group of homeomorphisms $\mathcal{H}(\mathcal{L})$ of \mathcal{L} acts transitively on \mathcal{L} .

In [74] M. C. McCord proves the following:

Theorem 1 (M. C. McCord). *If \mathfrak{S} is a compact solenoidal manifold which is obtained as the inverse limit of an infinite tower of regular finite coverings of degree $d \geq 2$ of a compact, connected, manifold M then \mathfrak{S} is topologically homogeneous [74]. The solenoid \mathfrak{S} is a connected principal fiber bundle with fiber a Cantor group.*

Definition 10 (McCord solenoids). We will call solenoidal manifolds \mathfrak{S} obtained as the previous theorem *McCord solenoidal manifold based on the manifold M* .

The reciprocal of McCord theorem for *topologically homogeneous* solenoidal manifolds \mathfrak{S} (for any two points $x, y \in \mathfrak{S}$ there exists a homeomorphism f of \mathfrak{S} such that $f(x) = y$) was obtained by A. Clark and S. Hurder in [28]. (Related results are found in [2, 10, 28, 30, 39, 59, 63].)

Theorem 2 (A. Clark, S. Hurder). *If \mathfrak{S} is a compact, connected, topologically homogeneous solenoidal manifold then \mathfrak{S} is a McCord solenoidal manifold. In particular, it is minimal (Definition 6).*

Remark 3. Any countable directed set (A, \preceq) contains a cofinal subset C which is totally ordered and well-ordered, under \preceq (i.e., it is a chain). Therefore, in Theorems 1 and 2 the tower of coverings can be taken to be a linear sequence of finite-sheeted coverings:

$$\dots \xrightarrow{p_4} M_3 \xrightarrow{p_3} M_2 \xrightarrow{p_2} M_1 \xrightarrow{p_1} M_0 = M,$$

corresponding to the sequence of normal subgroups of finite index of $\pi_1(M)$:

$$\dots \subset G_4 \subset G_3 \subset G_2 \subset G_1 \subset G_0 = \pi_1(M).$$

Therefore, in this paper, we can suppose that we are dealing with inverse limits of chains of finite-sheeted regular coverings. Such inverse limits are given by special spaces of sequences:

$$\mathcal{L} = \lim_{\leftarrow p_i} \{p_i: M_i \rightarrow M_{i-1}, i \in \mathbb{N}\} = \{\{x_{i-1}\}_{i \in \mathbb{N}} : \forall i \geq 1, x_i \in M_i, p_i(x_i) = x_{i-1}\},$$

where p_i ($i \in \mathbb{N}$) is a finite-sheeted regular covering map of degree $d \geq 2$. Therefore, \mathcal{L} has a natural topology as a compact subset of $\prod_{i=0}^{\infty} M_i$.

For $j \geq 1$ let

$$\begin{aligned} \mathcal{L}_j &= \varprojlim_{p_i} \{p_i: M_i \rightarrow M_{i-1}, i \in \mathbb{N}, i \geq j + 1\} \\ &= \{\{x_{i-1}\}_{i \in \mathbb{N}} : \forall i \geq j + 1, x_i \in M_i, p_i(x_i) = x_{i-1}\} \subset \prod_{i=j}^{\infty} M_i \end{aligned}$$

The sequences representing elements of \mathcal{L}_j are obtained from sequences of elements of \mathcal{L} by cropping or “erasing” the first j elements in the sequence; thus, the elements of \mathcal{L}_1 are the sequences obtained from the sequences of \mathcal{L} by omitting the first element. We have the natural map:

Definition 11. The *crop map* $\sigma: \mathcal{L} \rightarrow \mathcal{L}_1$ is the map $\{x_{i-1}\}_{i \in \mathbb{N}} \xrightarrow{\sigma} \{x_i\}_{i \in \mathbb{N}}$. The map σ is a homeomorphism, its inverse is $\sigma^{-1}(\{x_i\}_{i \in \mathbb{N}}) = \{\dots, x_i, \dots, x_1, p_1(x_1)\}$. The composition map $\sigma_j = \sigma \circ \dots \circ \sigma := \sigma_1^j$ establishes a homeomorphism:

$$\sigma_j: \mathcal{L} \rightarrow \mathcal{L}_j.$$

This means that the inverse limit space is not affected if one disregards a finite number of initial indices.

One has a *canonical projection*

$$\Pi: \mathcal{L} \rightarrow M, \quad \Pi(\{x_{i-1}\}_{i \in \mathbb{N}}) = x_0. \tag{2}$$

3.1. Differentiable McCord solenoids

Let \mathcal{L} be an n -dimensional McCord solenoid. Then, as pointed in Remark 3, \mathcal{L} is obtained as the inverse limit of an infinite chain of finite-sheeted regular coverings,

$$\dots \xrightarrow{f_3} M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = M,$$

where M is a connected topological n -manifold. By Theorem 1, the canonical projection $\Pi: \mathcal{L} \rightarrow M$ (equation (2)) is a principal Γ -bundle where Γ is a Cantor group. This group is obtained as a quotient of the profinite completion of the fundamental group of M as explained in Remark 9 in Section 3.3 below. The Cantor group Γ acts properly and freely on the right on \mathcal{L} and preserves the fibers. For $g \in \Gamma$ and x in \mathcal{L} let $x \cdot g$ denote the action of g on x .

If M admits a differentiable structure \mathcal{D} , given by the smooth subatlas $\mathcal{A} = \{\varphi_\alpha: \mathcal{U}_\alpha \rightarrow \mathbb{R}^n : \alpha \in I\}$, where $\mathcal{U}_\alpha \subset M$ is open and contractible, then \mathcal{L} can be endowed with the subatlas $\tilde{\mathcal{A}} = \{\tilde{\varphi}_\alpha: \Pi^{-1}(\mathcal{U}_\alpha) \rightarrow \mathbb{R}^n \times \Gamma : \alpha \in I\}$, where $\tilde{\varphi}_\alpha$ is a local trivialization determined by a local section $\sigma_\alpha: \mathcal{U}_\alpha \rightarrow \Pi^{-1}(\mathcal{U}_\alpha)$ (which exists since \mathcal{U}_α is contractible). If $x \in \mathcal{U}_\alpha$, then any element of the fiber $\Pi^{-1}(\{x\})$ is of the form $\sigma_\alpha(x) \cdot g$ for a unique $g \in \Gamma$ (here we use the group structure in the fiber). The local trivialization $\tilde{\varphi}_\alpha$ is obtained as follows: for each $\alpha \in I$, there exists a continuous projection $p_\alpha: \Pi^{-1}(\mathcal{U}_\alpha) \rightarrow \Gamma$ defined

by $p_\alpha(z) = g$ if $z = \sigma_\alpha(\Pi(z)) \cdot g$, then $\tilde{\varphi}_\alpha(z) = (\varphi_\alpha(\Pi(z)), p_\alpha(z))$ (see [104, Section 13, pp. 59–67]). Therefore, $\tilde{\varphi}_\alpha^{-1}(x, g) = \sigma_\alpha(\varphi_\alpha^{-1}(x)) \cdot g$, and we have:

$$\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1}(x, g) = \tilde{\varphi}_\beta(\sigma_\alpha(\varphi_\alpha^{-1}(x)) \cdot g) = (\varphi_\beta(\varphi_\alpha^{-1}(x)), p_\beta(\sigma_\alpha(\varphi_\alpha^{-1}(x))) \cdot g),$$

where $x \in \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, $g \in \Gamma$. We note that $g_{\alpha\beta} := p_\beta(\sigma_\alpha(\varphi_\alpha^{-1}(x))) \in \Gamma$ is independent of $x \in \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Hence, the change of coordinates $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1}$ is of the form:

$$\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1}(x, g) = (\varphi_\beta(\varphi_\alpha^{-1}(x)), g_{\alpha\beta} \cdot g). \quad (3)$$

We endow \mathcal{L} with the differentiable structure given by the atlas $\tilde{\mathcal{A}}$. It follows directly from equation (3) the following:

Proposition 1. *Let \mathcal{L} be an n -dimensional McCord solenoid, obtained as the inverse limit of an infinite chain of finite-sheeted regular coverings*

$$\dots \xrightarrow{f_3} M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = M,$$

where M is a differentiable n -manifold. Then \mathcal{L} has a natural differentiable structure (induced by the differentiable structure on M) so that \mathcal{L} is a smooth n -dimensional solenoidal manifold as in Definition 2. Furthermore, the canonical projection $\Pi: \mathcal{L} \rightarrow M$ is a differentiable principal Γ -bundle. More precisely:

- (1) Π is a differentiable map and the restriction of Π to a leaf L is a smooth covering map from L to M .
- (2) The crop map $\sigma: \mathcal{L} \rightarrow \mathcal{L}$ (Definition 11) is a diffeomorphism. Therefore, for all integers $k \geq 0$, the solenoids corresponding to the chains

$$\dots \xrightarrow{p_4} M_3 \xrightarrow{p_3} M_2 \xrightarrow{p_2} M_1 \xrightarrow{p_1} M_0 = M$$

and

$$\dots \xrightarrow{p_{k+4}} M_{k+3} \xrightarrow{p_{k+3}} M_{k+2} \xrightarrow{p_{k+2}} M_{k+1} \xrightarrow{p_{k+1}} M_k$$

are diffeomorphic solenoids.

- (3) The right action of Γ on \mathcal{L} , given by the formula $F_g(z) = z \cdot g$, is by solenoidal diffeomorphisms, i.e., the map $z \mapsto z \cdot g$ ($z \in \mathcal{L}$, $g \in \Gamma$), is differentiable. In particular, the restriction of R_g to a leaf L_z ($z \in L_z$) is a diffeomorphism from L_z to $L_{z \cdot g}$.

Corollary 1. *Let \mathcal{L} and M , Π and Γ be as in Proposition 1. Let us assume that M has a differentiable structure \mathcal{D} . Let $p: \tilde{M} \rightarrow M$ be the universal covering projection. We endow the universal covering manifold \tilde{M} with the pullback of the differentiable structure \mathcal{D} (i.e., the unique differentiable structure $\tilde{\mathcal{D}}$ that makes p a local diffeomorphism). Let $\tilde{\Pi}: \tilde{\mathcal{L}} \rightarrow \tilde{M}$ be the pullback under p of the principal Γ -bundle $\Pi: \mathcal{L} \rightarrow M$. Since $\tilde{\Pi}$ admits a section, $\tilde{\mathcal{L}}$ is the trivial bundle $\tilde{M} \times \Gamma$. On the other hand, since the bundle*

$\Pi: \mathcal{L} \rightarrow M$ has the unique path lifting property it follows that if we lift loops based at a point $x \in M$ they determine homeomorphisms of the fiber $F_x := \Pi^{-1}(\{x\}) \simeq \Gamma$. These homeomorphisms depend only on the homotopy classes of the loops based at x , and are right-translations on the fiber F_x . These liftings determine a homomorphism $\chi_x: \pi_1(M, x) \rightarrow \Gamma$ called the monodromy of the fibration (called characteristic class in the book by Steenrod [104, Section 13]). If $x_1, x_2 \in M$ then χ_{x_1}, χ_{x_2} differ only by inner conjugation by an element in Γ . The conjugation classes determine the principal Γ -bundles over M [104, Section 13]. The fundamental group $\pi_1(M, x)$ acts freely, differentiable, and properly discontinuously on $\tilde{\mathcal{L}} = \tilde{M} \times \Gamma$ by the formula $\Psi_g(x, h) = (\gamma_g(x), h \cdot \chi_x(g))$ (on the first factor the action is by deck transformations and on the second by right-translations). Furthermore,

$$\mathcal{L} = (\tilde{M} \times \Gamma) / \pi_1(M, x).$$

The proposition and its corollary lead to the following definition:

Definition 12. A solenoid obtained as the inverse limit of a sequence of finite smooth coverings of a smooth manifold with the differentiable structure described in Proposition 1 is called *differentiable (or smooth) McCord solenoid*.

Remark 4. The differentiable structure on \mathcal{L} in Proposition 1 depends on the differentiable structure of the base manifold M . Furthermore, there exist topological McCord solenoidal manifolds that admit several inequivalent differentiable structures and solenoidal manifolds whose leaves cannot be homeomorphic to simplicial complexes. The following examples show the subtleties of these concepts.

3.2. Exotic examples

The following examples follow from deep theorems which have already been established by many authors. In essence, we use different versions of the product structure theorem ([94, Section 1.7.1] or [61, Section 5]) which establish that a structure of a manifold M in the categories Top, PL and Diff persists for $M \times \mathbb{R}$ if $\dim(M) \geq 5$. This permits us to take our examples as a product $M \times \mathcal{S}_p$ where \mathcal{S}_p is the Vietoris–van Dantzig p -adic solenoid (see Remark 17 below). Here we use the standard definition of *simplicial triangulation* of a manifold to mean a homeomorphism from a simplicial complex to the manifold and a PL or combinatorial triangulation (or Brouwer triangulation) a simplicial triangulation such that the link of every simplex (or, equivalently, of every vertex) is piecewise-linearly homeomorphic to a sphere [68–70].

Example 1 (Differentiably exotic solenoidal manifolds). For $n \geq 7$, let Θ^n denote the Kervaire–Milnor group of homotopy n -spheres [60]. For instance, $\Theta^7 \simeq \mathbb{Z}/28\mathbb{Z}$. One has the following property: If M_1^n and M_2^n are two homotopy n -spheres which represent different elements of Θ^n then $M_1^n \times \mathbb{S}^1$ is not diffeomorphic to $M_2^n \times \mathbb{S}^1$. To show this is enough to show that $M_1^n \times \mathbb{R}$ is not diffeomorphic to $M_2^n \times \mathbb{R}$. We proceed by contradiction. If there existed a diffeomorphism $\Phi: M_2^n \times \mathbb{R} \rightarrow M_1^n \times \mathbb{R}$ then there would

exist $m \in \mathbb{N}$ such that $M_1^n \times \{0\} \cap \Phi(M_2 \times \{m\}) = \emptyset$. Then, $(M_1^n \times \{0\}) \sqcup \Phi(M_2 \times \{m\})$ is the boundary of a compact submanifold W of $M_1^n \times \mathbb{R}$. Hence, $M_1^n \times \mathbb{R} = A \sqcup W \sqcup B$ where A is homeomorphic to $M_1^n \times (-\infty, 0)$ and B is homeomorphic to $M_2^n \times (0, \infty)$. Therefore, $(M_1^n \times \mathbb{R})/W$ is a cone and therefore it is contractible and the exact sequence in homology of the pair $(M_1^n \times \mathbb{R}, W)$ implies that the inclusion $i: W \rightarrow M_1^n \times \mathbb{R}$ induces an isomorphism in homology and, since $M_1^n \times \mathbb{R}$ and W are simply connected it follows from the Hurewicz theorem that the inclusion is a homotopy equivalence. Therefore, W is an h -cobordism. By the h -cobordism theorem, M_1^n would be diffeomorphic to M_2^n , and this would be a contradiction. Now consider the $(n + 1)$ -dimensional solenoidal manifolds $\mathcal{L}_1 = M_1^n \times \mathcal{S}_p$ and $\mathcal{L}_2 = M_2^n \times \mathcal{S}_p$, where \mathcal{S}_p is the classic Vietoris–van Dantzig p -adic solenoid [31, 115] (see Remark 17). Both \mathcal{L}_1 and \mathcal{L}_2 are homeomorphic to $\mathbb{S}^n \times \mathcal{S}_p$, but not diffeomorphic, because all the corresponding leaves are $M_1^n \times \mathbb{R}$ and $M_2^n \times \mathbb{R}$, respectively, and they are not diffeomorphic. More generally we can consider $\mathcal{S}_1 = M_1^n \times \widehat{\mathbb{S}}^1$ and $\mathcal{S}_2 = M_2^n \times \widehat{\mathbb{S}}^1$ where $\widehat{\mathbb{S}}^1$ is the adèle class group or universal 1-dimensional arithmetic solenoid (the algebraic universal covering of the circle \mathbb{S}^1 , see Definition 25 below). For $n \geq 5$ the group Θ^n is a finite group, and it is non-trivial for arbitrarily large n [60]. It is an open problem to determine if Θ^4 is trivial or not. This construction provides examples in arbitrarily high dimensions of differentiable solenoidal manifolds that have many inequivalent differentiable structures, for instance at least 28 in dimension 7.

Example 2 (Solenoidal manifolds without PL or differentiable structures). Let V be Freedman’s E_8 4-manifold, i.e., a simply connected 4-manifold with second Stiefel–Whitney class $w_2(V) = 0$ and such that E_8 is the matrix of the intersection form $H^2(V, \mathbb{Z}) \otimes H^2(V, \mathbb{Z}) \rightarrow \mathbb{Z}$. In particular, the signature of V is equal to 8. Furthermore, such a manifold V is unique up to homeomorphism. M. Freedman and A. Casson show that V does not admit a simplicial triangulation as a consequence of Rokhlin’s signature theorem. However, $M = V \times \mathbb{S}^1$ is an orientable 5-manifold and therefore M has a simplicial triangulation, since, by the work of Siebenmann combined with the double suspension theorem, it follows that all 5-dimensional non-simplicially triangulable manifolds have to be compact and non-orientable (see [94, Theorem 3.5.6] and [68, Section 4.3]). However, M does not have a PL-triangulation (see [94, Corollary 1.8.4]). It follows from the product structure theorem that $V \times \mathbb{R}$ has a simplicial triangulation but it does not admit a PL triangulation and, as a consequence, it does not admit a differentiable structure. Hence, for each prime p , $M \times \mathcal{S}_p$ (where \mathcal{S}_p is the Vietoris–van Dantzig p -adic solenoid Remark 17) is a solenoidal manifold with leaves admitting a simplicial triangulation but not a PL triangulation and therefore also not a differentiable structure. Hence, $M \times \mathcal{S}_p$ does not have a differentiable structure as a solenoidal manifold. Also, the counter-examples of Kirby–Siebenmann [61] to the Hauptvermutung, for instance homotopy tori $\widehat{\mathbb{T}}^n$ of dimension $n \geq 5$ with non-equivalent PL structures, and the product structure theorem provide examples of solenoidal manifolds $\widehat{\mathbb{T}}^n \times \mathcal{S}_p$ with inequivalent PL-structures (in a laminated sense). Of course, it is an interesting fact to have counter-examples to a version of the

Hauptvermutung (a subject very dear to Dennis Sullivan [107, 109]) for solenoidal manifolds.

A more drastic example is the following.

Example 3 (Solenoidal manifolds whose leaves do not admit a simplicial triangulation). The following examples use the remarkable theorem by Ciprian Manolescu:

Theorem 3 ([68, Theorem 1.1]). *There exist non-triangulable (as simplicial complexes) n -dimensional topological manifolds for every $n \geq 5$.*

In fact, if $\dim(M) \geq 5$ and the invariant $\delta\Delta(M) \in H^5(M; \text{Ker}(\mu))$, defined in [68], does not vanish, it follows from [68, Theorem 4.3, p. 446] that M does not admit a simplicial triangulation. Here, $\Delta(M) \in H^4(M, \mathbb{Z}/2)$ is the Kirby–Siebenmann obstruction to PL structures and δ is a Bockstein homomorphism corresponding to the exact sequence

$$0 \rightarrow \text{Ker}(\mu) \rightarrow \Theta_3^H \xrightarrow{\mu} \mathbb{Z}/2 \rightarrow 0$$

where the group Θ_3^H is the three-dimensional homology cobordism group where the group operation is the connected sum; it is generated by equivalence classes of oriented integral homology 3-spheres; two homology 3-spheres are equivalent if they are H -cobordant (i.e., homology cobordant [47]). The homomorphism μ is the Rokhlin homomorphism [37, 93]:

$$\mu: \Theta_3^H \rightarrow \mathbb{Z}/2.$$

From [41] (see also [68, p. 447]) it follows that the non-vanishing of $Sq^1\Delta(M) \in H^5(M; \mathbb{Z}/2)$, where Sq^1 denotes the first Steenrod square, implies the non-vanishing of the invariant $\delta\Delta(M)$. Therefore, a topological manifold M (not necessarily compact) such that $\dim(M) \geq 5$ is not homeomorphic to a simplicial complex iff $Sq^1\Delta(M) \neq 0$. This implies, in turn, that if M is a compact manifold, $\dim(M) \geq 5$ and $Sq^1\Delta(M) \neq 0$, then $Sq^1\Delta(M \times \mathbb{R}) \neq 0$. This is proved as follows: from [77] and Künneth formula it follows that $\Delta(M \times \mathbb{R}) = \Delta(M) \otimes 1 + 1 \otimes \Delta(\mathbb{R}) = \Delta(M) \otimes 1$. Hence, $Sq^1(\Delta(M \times \mathbb{R})) \neq 0$ since, by naturality of the Steenrod operations $i^*(Sq^1(\Delta(M \times \mathbb{R}))) = Sq^1(i^*(\Delta(M \times \mathbb{R}))) = Sq^1(\Delta(M)) \neq 0$ where $i: M \rightarrow M \times \mathbb{R}$ is the inclusion map. We conclude (C. Manolescu): if M is a topological manifold which is not homeomorphic to a simplicial complex (i.e., it is not simplicially triangulable) and $\dim(M) \geq 5$, then $M \times \mathbb{R}$ is not homeomorphic to a simplicial complex. It follows that $M \times S^1$ also does not admit a simplicial triangulation and, by induction, $M \times \mathbb{T}^n$ is not simplicially triangulable.¹

Let M^n be compact, connected, topological n -manifold that is not simplicially triangulable, then $\mathcal{L}_p = M^n \times \mathcal{S}_p$ (where \mathcal{S}_p is the Vietoris–van Dantzig p -adic solenoid

¹I thank professor Ciprian Manolescu for providing me with this stability fact in a private communication, and for pointing out reference [68].

Remark 17) is an $(n + 1)$ -dimensional topological solenoidal manifold, with all leaves homeomorphic to $M \times \mathbb{R}$, such that no leaf admits a simplicial triangulation hence also no leaf admits a PL triangulation. Therefore, \mathcal{L}_p does not admit a solenoidal differentiable structure. Since there are examples of manifolds M that do not admit simplicial triangulations for $n \geq 5$, there are solenoidal manifolds \mathcal{L}^n of dimension $n \geq 6$ having all leaves without a simplicial triangulation. When $n = 5$ the manifold must be non-orientable, however for $n \geq 6$ there exist compact orientable manifolds which do not admit simplicial triangulation, hence for all $n \geq 7$ there are solenoidal manifolds \mathcal{L} such that all its leaves do not admit simplicial triangulations.

The following concrete example is due to Peter Kronheimer (see [68, Example 4.5]). Let W be the Freedman’s fake $\mathbb{C}P^2 \# (\overline{\mathbb{C}P^2})$ (where $\mathbb{C}P^2$ has the natural orientation as a complex manifold and $\overline{\mathbb{C}P^2}$ has the opposite orientation). W has the homotopy type of the blowup at one point of the complex projective plane $\mathbb{C}P^2$. By [40, Theorem 1.5], W is the unique, closed, simply connected topological 4-manifold with $H^2(W, \mathbb{Z}) \simeq \mathbb{Z}^2$, intersection form $Q = \langle 1 \rangle \oplus \langle -1 \rangle$, and non-trivial Kirby–Siebenmann invariant (it does not have a PL-structure and hence also a smooth structure).

Since the form Q is isomorphic to $-Q$, by applying [40, Theorem 1.5, Addendum], W admits an orientation-reversing homeomorphism $f: W \rightarrow W$, which induces the isomorphism between Q and $-Q$. Let M be the mapping torus of f . Then M is a topological non-orientable 5-manifold which is not homeomorphic to a simplicial complex. Explicitly, M is the orbit space of the free and properly discontinuous action of \mathbb{Z} generated by the homeomorphism:

$$F: W \times \mathbb{R} \rightarrow W \times \mathbb{R}, \text{ defined by the formula } F(x, t) = (f(x), t + 1),$$

i.e.,

$$M = \{O(x, t) : (x, t) \in W \times \mathbb{R}\},$$

where

$$O(x, t) = \{(f^n(x), t + n) : n \in \mathbb{Z}\} \text{ is the orbit of } (x, t).$$

A compact, orientable, 6-manifold N which is not homeomorphic to a simplicial complex due to Ron Stern is given in [68, Example 4.5]. The manifold N is a non-oriented circle bundle over the manifold M constructed above. Explicitly, N is constructed as follows:

Consider the action on $W \times \mathbb{R} \times \mathbb{R}$ of the group G generated by the homeomorphisms

$$F_1: W \times \mathbb{R} \times \mathbb{R} \rightarrow W \times \mathbb{R} \times \mathbb{R}, \quad F_1(x, t, s) = (f(x), t + 1, -s)$$

and

$$F_2: W \times \mathbb{R} \times \mathbb{R} \rightarrow W \times \mathbb{R} \times \mathbb{R}, \quad F_2(x, t, s) = (x, t, s + 1).$$

The group G has the presentation:

$$G = \langle F_1, F_2 \mid F_1 \cdot F_2 \cdot F_1^{-1} = F_2^{-1} \rangle,$$

so G is isomorphic to the fundamental group of the Klein bottle, so that it is a semidirect product $\mathbb{Z} \ltimes \mathbb{Z}$.

The action of G is proper, discontinuous, and free. The orbit space, $(W \times \mathbb{R} \times \mathbb{R})/G$ is homeomorphic to a compact topological 6-manifold M . The orbit of $O(x, t, s)$ of the point (x, t, s) is the set:

$$O(x, t, s) = \{(f^n(x), t + n, (-1)^n s + m) : n, m \in \mathbb{Z}\}.$$

Therefore,

$$N = \{O(x, t, s) : (x, t, s) \in W \times \mathbb{R} \times \mathbb{R}\}.$$

The map $p: N \rightarrow M$ defined by the formula:

$$p(O(x, t, s)) = p(\{(f^n(x), t + n, (-1)^n s + m) : n, m \in \mathbb{Z}\}) = \{(f^n(x), t + n) : n \in \mathbb{Z}\}$$

is an unoriented topological \mathbb{S}^1 -bundle fibration of N over M . Since M is not orientable, it follows that N is a compact *oriented* manifold which is not homeomorphic to a simplicial complex.

From the last examples, we have that $(M \times \mathbb{T}^n) \times \mathcal{S}_p$, for p prime and $n \geq 0$, are non-orientable $(5 + n + 1)$ -dimensional solenoidal manifolds with leaves homeomorphic to $(M \times \mathbb{T}^n) \times \mathbb{R}$ so the leaves do not admit simplicial triangulations. On the other hand, the solenoids $(N \times \mathbb{T}^n) \times \mathcal{S}_p$, for p prime and $n \geq 0$, are orientable $(6 + n + 1)$ -dimensional solenoidal manifolds such that their leaves do not admit simplicial triangulations.

The previous examples show that, like in the case of compact manifolds, we have to distinguish also the topological, simplicial, PL, and differentiable categories in the case of solenoidal manifolds. All the previous examples are topologically homogeneous.

Remark 5. *Henceforth, all McCord solenoidal manifolds will be assumed to be differentiable McCord solenoidal manifolds in the sense of Definition 12.*

Conjecture 1 (Differentiably homogeneous solenoidal manifolds). *The fact that the group of diffeomorphisms of a manifold M endowed with the C^∞ topology is a Polish metric space implies that Effros theorem can be applied (i.e., it acts micro-transitively on M). This fact is used fundamentally in the results in [28] and [10]. I conjecture that these methods can be adapted to show that if \mathcal{L} is a differentiable n -dimensional solenoid which is differentiably homogeneous (i.e., given $x, y \in \mathcal{L}$ there exists a diffeomorphism (Definition 8) $f: \mathcal{L} \rightarrow \mathcal{L}$ such that $f(x) = y$) then \mathcal{L} is diffeomorphic to the inverse limit of an infinite tower of smooth regular coverings of compact smooth manifolds and the differentiable structure of \mathcal{L} is equivalent to the differentiable structure given by Proposition 1.*

Remark 6. Henceforth, we will assume that all the McCord solenoids under consideration are not only differentiable in the sense of Definition 12 but also satisfy Conjecture 1, i.e., they are obtained by towers of smooth coverings over a smooth manifold and are endowed with the differentiable structure of Proposition 1.

3.3. Profinite completions and McCord solenoids

We recall, once again, that on a smooth lamination one can use all the tools of differential topology and geometry along the leaves. Concepts such as laminated smooth functions, laminated Riemannian metric, laminated curvature, de Rham theory, etc. can be defined [27, 76].

Henceforth, M will be a smooth, compact, connected, manifold whose fundamental group is infinite and residually finite. Let \tilde{M} be its universal covering. Let g be a Riemannian metric on M .

We take the tower of all of its finite pointed coverings $p_\beta: (\tilde{M}_\beta, \tilde{x}_\beta) \rightarrow (M, x_0)$ corresponding to the directed system under inclusion \mathfrak{B} of subgroups of finite index $\beta \subset \pi_1(M, x_0)$. Hence, if $\beta_1, \beta_2 \in \mathfrak{B}$ and $\beta_1 \subset \beta_2$ we simply write the order as $\beta_1 \leq \beta_2$ and one has corresponding pointed coverings $(\tilde{M}_{\beta_1}, \tilde{x}_{\beta_1})$ and $(\tilde{M}_{\beta_2}, \tilde{x}_{\beta_2})$ and a bonding pointed covering map:

$$p_{(\beta_1, \beta_2)}: (\tilde{M}_{\beta_1}, \tilde{x}_{\beta_1}) \rightarrow (\tilde{M}_{\beta_2}, \tilde{x}_{\beta_2}), \quad \beta_1 \leq \beta_2.$$

We endow each \tilde{M}_β with the pullback metric under p_β . Then the inverse limit

$$\mathbb{M} = \lim_{\substack{\leftarrow \\ p_{(\beta_1, \beta_2)}}} \{p_{(\beta_1, \beta_2)}: (\tilde{M}_{\beta_1}, \tilde{x}_{\beta_1}) \rightarrow (\tilde{M}_{\beta_2}, \tilde{x}_{\beta_2}) \mid \beta_1, \beta_2 \in \mathfrak{B}, \beta_1 \leq \beta_2\}$$

has the structure of a smooth lamination by Proposition 1. In addition, \mathbb{M} is a minimal n -dimensional solenoidal manifold (see Proposition 4 below). The pulled-back metrics induce a smooth laminated Riemannian metric \hat{g} on \mathbb{M} . The foliation is minimal: all leaves are dense and all leaves are isometric to \tilde{M} with the pullback metric \tilde{g} under the universal covering $\tilde{M} \rightarrow M$. Furthermore, by construction, each covering projection p_β is a local isometry.

Since the directed set \mathfrak{N} of all the normal subgroups of finite index of $\pi_1(M, x)$ is cofinal in \mathfrak{B} , the inverse limit corresponding to \mathfrak{N} defines the same lamination (the directed set of characteristic subgroups of finite index is cofinal so also defines the same lamination). We have:

$$\begin{aligned} \mathbb{M} &= \lim_{\substack{\leftarrow \\ p_{(\beta_1, \beta_2)}}} \{p_{(\beta_1, \beta_2)}: (\tilde{M}_{\beta_1}, \tilde{x}_{\beta_1}) \rightarrow (\tilde{M}_{\beta_2}, \tilde{x}_{\beta_2}) \mid \beta_1, \beta_2 \in \mathfrak{N}, \beta_1 \leq \beta_2\} \\ &= \left\{ \{x_\beta\}_{\beta \in \mathfrak{N}} \in \prod_{\beta \in \mathfrak{N}} \tilde{M}_\beta : x_{\beta_1} = p_{(\beta_1, \beta_2)}(x_{\beta_2}) \text{ for all } \beta_1 \leq \beta_2 \text{ in } \mathfrak{N} \right\}. \end{aligned}$$

Definition 13. \mathbb{M} is called the *algebraic universal covering* of M . For each fixed $\alpha \in \mathfrak{N}$ there is a projection

$$\Pi_\alpha: \mathbb{M} \rightarrow M_\alpha, \quad \Pi_\alpha(\{x_\beta\}_{\beta \in \mathfrak{N}}) = x_\alpha. \tag{4}$$

When $\beta_0 = \pi_1(M, x)$, we have $M_{\beta_0} = M$. Let $\Pi: \mathbb{M} \rightarrow M$ denote the corresponding projection $\Pi(\{x_\beta\}_{\beta \in \mathfrak{N}}) = x_0 \in M$. We call Π the *canonical projection*.

If $\beta \in \mathfrak{N}$ is a normal subgroup of finite index let $H_\beta = \pi_1(M)/\beta$ be the finite group which is the group of deck transformations of the finite covering $p_\beta: \tilde{M}_\beta \rightarrow M$. The finite group H_β acts by isometries on M_β .

If $\beta_1 \preceq \beta_2$, one has the epimorphism $q_{\beta_1, \beta_2}: H_{\beta_1} \rightarrow H_{\beta_2}$.

Then one has the following standard fact:

Proposition 2. *The map Π_β describes \mathbb{M} as a principal fiber bundle over M with group fiber the Cantor group $\widehat{\pi_1(M)}$ where $\widehat{\pi_1(M)}$ is the profinite group which is given as the inverse limit:*

$$\widehat{\pi_1(M)} = \varprojlim_{q_{\beta_1, \beta_2}} \{q_{\beta_1, \beta_2}: H_{\beta_1} \rightarrow H_{\beta_2} \mid \beta_1, \beta_2 \in \mathfrak{N}, \beta_1 \preceq \beta_2\}.$$

Definition 14. The group $\widehat{\pi_1(M)}$ is called the *profinite completion* of the fundamental group or *algebraic fundamental group* of M . It is a profinite Cantor group. Recall that the profinite completion of a group Γ is the inverse limit of the directed system of finite quotients of Γ . Note that $\widehat{\pi_1(M)}$ is a Cantor group and $\widehat{\pi_1(M)} \subset \prod_{\beta \in \mathfrak{N}} H_{\beta_1}$. We refer to [92] for properties of profinite groups.

Definition 15. Since $\pi_1(M)$ is residually finite there is a canonical monomorphism $j: \pi_1(M) \rightarrow \widehat{\pi_1(M)}$ given by $j(g) = ((p_\beta(g)))_{\beta \in \mathfrak{B}}$ where p_β is the canonical epimorphism $p_\beta: \pi_1(M) \rightarrow \frac{\pi_1(M)}{H_\beta}$.

Remark 7. If $\pi_1(M)$ is residually finite it follows that j is injective and $j(\pi_1(M))$ is dense in $\widehat{\pi_1(M)}$.

Let \tilde{M} be the universal covering of M corresponding to the trivial subgroup. By Corollary 1, the group $\pi_1(M)$ acts differentiably on $\tilde{\mathbb{M}} := \tilde{M} \times \widehat{\pi_1(M)}$ as follows:

$$f_\gamma(x, h) = (\gamma(x), j(\gamma)h), \tag{5}$$

where the action of $\gamma \in \pi_1(M)$ on \tilde{M} is by isometric deck transformations with respect to \tilde{g} and on $\widehat{\pi_1(M)}$ by left translation by $j(\gamma)$. The action is differentiable (Corollary 1) free and properly discontinuous. This action is a suspension (which will be defined in general later in Section 3.4).

Let $j: \pi_1(M) \rightarrow \widehat{\pi_1(M)}$ be the canonical injection. Then $\pi_1(M) \times \widehat{\pi_1(M)}$ acts on $\tilde{M} \times \widehat{\pi_1(M)}$ differentiably (by Proposition 1 and Corollary 1) as follows:

$$F_{(\gamma, g)}(\tilde{x}, h) = (\gamma(\tilde{x}), j(\gamma) \cdot h \cdot g^{-1}) \quad (\tilde{x} \in \tilde{M}, \gamma \in \Gamma, h, g \in \widehat{\pi_1(M)}). \tag{6}$$

This is indeed a left action:

$$\begin{aligned} F_{(\gamma_2 \gamma_1, g_2 g_1)}(\tilde{x}, h) &= ((\gamma_2 \gamma_1)(\tilde{x}), j(\gamma_2 \gamma_1) \cdot h \cdot (g_2 \cdot g_1)^{-1}) \\ &= (\gamma_2(\gamma_1(\tilde{x})), j(\gamma_2) \cdot j(\gamma_1) \cdot h \cdot g_1^{-1} \cdot g_2^{-1}) = F_{(\gamma_2, g_2)}(F_{(\gamma_1, g_1)}(\tilde{x}, h)). \end{aligned}$$

We have:

Proposition 3. *The solenoid \mathbb{M} is obtained from $\widetilde{\mathbb{M}} = \widetilde{M} \times \widehat{\pi_1(M)}$ as the orbit space of the left action (5) of $\pi_1(M)$, i.e.,*

$$\mathbb{M} = \pi_1(M) \backslash \widetilde{\mathbb{M}} = \pi_1(M) \backslash (\widetilde{M} \times \widehat{\pi_1(M)}).$$

\mathbb{M} is sometimes denoted as $M \times_{\pi_1(M)} \widehat{\pi_1(M)}$. Consider the canonical projection:

$$\widehat{\Pi}: \widetilde{\mathbb{M}} \rightarrow \mathbb{M}, \tag{7}$$

Then $\widehat{\Pi}$ is a solenoidal covering projection, i.e., each point in \mathbb{M} has an open neighborhood U so that $\widehat{\Pi}^{-1}(U)$ is a disjoint union of open sets and the restriction to each of these open sets is a homeomorphism onto U and $\widehat{\Pi}$ has the unique path-lifting property [104, Section 13].

Proposition 4. *\mathbb{M} is compact and the solenoidal manifold \mathbb{M} is minimal.*

Proof. It follows from the fact that $j(\pi_1(M))$ is dense in $\widehat{\pi_1(M)}$ (Remark 7). Also, $\pi_1(M)$ acts on the right without fixed points on $\widetilde{M} \times \widehat{\pi_1(M)}$ and acts simply-transitive on $\{\tilde{x}\} \times \widehat{\pi_1(M)}$ by the formula $(\tilde{x}, h) \xrightarrow{g} (\tilde{x}, hg)$ (formula (6)) when $\gamma = \text{Id}$. Thus, the leaves on \mathbb{M} are dense and isometric to the universal covering \widetilde{M} with the lifted metric \tilde{g} from the metric g on M , in particular, they are simply connected. ■

Remark 8. Of course not every laminated Riemannian metric on \mathbb{M} is obtained by pulling back a Riemannian metric g on M , using the canonical projection Π , even if the laminated metric is very homogeneous. For instance, compact 2-dimensional smooth McCord solenoids obtained from an infinite tower of smooth regular coverings of a smooth compact, orientable, surface Σ of genus h greater than one, have an infinite-dimensional Teichmüller space of metrics of constant negative curvature -1 whereas the Teichmüller space of Σ is of real dimension $6h - h$. In fact, in [25] it is shown that the Teichmüller space of Sullivan’s universal hyperbolic solenoid \mathcal{L}_h , i.e., the algebraic universal covering of Σ is homeomorphic to the space of continuous maps from the Cantor group $\widehat{\pi_1(\Sigma)}$ to the Teichmüller space $\mathcal{T}(\mathcal{L}_h)$ of Σ (which shows that $\mathcal{T}(\Sigma)$ is an infinite-dimensional Kähler manifold). See Section 6.3 below.

Let M be a compact manifold with infinite and residually finite fundamental group $\pi_1(M)$. Let $\mathcal{G} := G_1 \supset G_2 \supset \dots \supset G_i \supset \dots$ and $\mathcal{G}' := G'_1 \supset G'_2 \supset \dots \supset G'_i \supset \dots$ be two infinite chains in the directed system of normal subgroups of finite index of $\pi_1(M)$. Assume that \mathcal{G} and \mathcal{G}' correspond to the McCord solenoids \mathcal{S}_1 and \mathcal{S}_2 .

We say that \mathcal{G} is equivalent to \mathcal{G}' if there exists a chain \mathcal{G}'' of normal subgroups such that $\mathcal{G}'' \supset \mathcal{G}'$, $\mathcal{G}'' \supset \mathcal{G}$ and both \mathcal{G}' and \mathcal{G} are cofinal in \mathcal{G}'' . This is an equivalence relation. Two chains in the same equivalence class have homeomorphic corresponding inverse limits of coverings. These equivalence classes determine a quotient of the profinite completion $\widehat{\pi_1(M)}$.

By [42] there exists uncountably many non-homomorphic normal subgroups Γ_α of $\widehat{\pi_1(M)}$ if M is a hyperbolic manifold.

Furthermore, uncountably many of these subgroups are closed normal subgroups which are Cantor groups. Restricting, for each α , the action (6) to the subgroup $\pi_1(M) \times \Gamma_\alpha$, of the previous type, we obtain that the restricted action is free and proper (but not discontinuous). The orbit-space quotient

$$\mathcal{S}_\alpha := (\pi_1(M) \times \Gamma_\alpha) \backslash (\widetilde{M} \times \widehat{\pi_1(M)}) \tag{8}$$

is a McCord solenoid which is a principal bundle over M with fiber the Cantor group $\widehat{\Gamma}_\alpha = \widehat{\pi_1(M)}/\Gamma_\alpha$. So that Γ_α acts fiberwise on $\widetilde{\mathbb{M}}$ and $\mathcal{S}_\alpha = \widetilde{\mathbb{M}}/\Gamma_\alpha$. Furthermore, there is a natural solenoidal covering map $\Pi_\alpha: \mathbb{M} \rightarrow \mathcal{S}_\alpha$ (see Definition 20 below).

Summarizing:

Proposition 5 (Profinite Galois correspondence of solenoidal coverings of hyperbolic manifolds). *Let M be a compact hyperbolic manifold. Then $\widehat{\pi_1(M)}$ has uncountably many closed normal subgroups Γ_α with quotients, a Cantor group [42]. Therefore, in general, there exist uncountably many non-homeomorphic McCord solenoids \mathcal{S}_α that are principal Cantor bundles over M . This happens, for instance, if $M = S^1$. For every Γ_α there exists a solenoidal covering map $\pi_\alpha: \mathbb{M} \rightarrow \mathcal{S}_\alpha$ (see Definition 20 below).*

Remark 9 (Universal property of \mathbb{M}). Let M be a compact smooth manifold (not necessarily hyperbolic). Any McCord solenoid over M corresponds to a closed normal subgroup Γ_α of $\widehat{\pi_1(M)}$ with quotient $\widehat{\Gamma}_\alpha$ a Cantor group, and it is of the form given by the formula (8), i.e., $\mathcal{S}_\alpha = \widetilde{\mathbb{M}}/\Gamma_\alpha$, and Γ_α acts fiberwise and properly by right translations on the principal $\widehat{\pi_1(M)}$ -bundle \mathbb{M} .

Remark 10. This result is also valid for the profinite completion of the fundamental group of any 3-manifold M with infinite fundamental group. Therefore, there exist uncountably many non-homeomorphic McCord solenoids \mathcal{S}_α that are principal Cantor bundles over M , corresponding to closed normal subgroups Γ_α of $\widehat{\pi_1(M)}$ such that $\pi_1(M)/\Gamma_\alpha$ is a Cantor group.

The algebraic universal covering \mathbb{M} has an invariant normalized transverse measure obtained from the Haar measure on the Cantor group $\widehat{\pi_1(M)}$. The same holds for an invariant transverse metric; one obtains one, for instance, by the formula

$$\mathbf{d}(g, h) = \sup\{d(xgy, xhy) : x, y \in \widehat{\pi_1(M)}\},$$

where d is any distance in the Cantor group $\widehat{\pi_1(M)}$:

Definition 16 (Transverse invariant measure and distance). For each $x \in M$ let $K_x = \Pi^{-1}(\{x\})$ be the fiber of the canonical projection (4) over x . Since Π is a principal bundle, the fiber K_x is homeomorphic to the Cantor group $\widehat{\pi_1(M)}$, by choice of any point in the fiber. Thus we have the Haar measure $d\mu_x$ in K_x . This measure is called the *canonical transverse measure* at K_x . This measure is invariant under the holonomy pseudogroup.

Let \mathbf{d} be any bi-invariant distance on the Cantor group $\widehat{\pi_1(M)}$. Then \mathbf{d}_x , the corresponding distance on the fiber K_x , is a distance that is invariant under the holonomy pseudogroup.

To simplify the notation, let $\widehat{\Gamma} = \widehat{\pi_1(M)}$, $\widetilde{\mathbb{M}} = \widetilde{M} \times \widehat{\Gamma}$, and $p_2: \widetilde{M} \times \widehat{\Gamma} \rightarrow \widehat{\Gamma}$ be the projection into the second factor. Let $\mathbf{d}\mathbf{h}$ be the normalized Haar measure on $\widehat{\Gamma}$.

Let g be any laminated Riemannian metric on $\widetilde{\mathbb{M}}$. We can pull back g by the canonical projection $\widehat{\Pi}: \widetilde{M} \times \widehat{\Gamma} \rightarrow \widetilde{\mathbb{M}}$ to obtain a laminated Riemannian metric \widetilde{g} on $\widetilde{M} \times \widehat{\Gamma}$. For each \mathbf{k} in the Cantor group $\widehat{\Gamma}$ let $d\nu(\mathbf{k})$ be the measure given by the volume form (of the metric \widetilde{g}) on the leaf $p_2^{-1}(\mathbf{k}) = \widetilde{M} \times \{\mathbf{k}\}$.

Definition 17. Let $\widetilde{\mu}_g$ be the measure on $\widetilde{\mathbb{M}} = \widetilde{M} \times \widehat{\Gamma}$ such that for each continuous function with compact support $f: \widetilde{M} \times \widehat{\Gamma} \rightarrow \mathbb{R}$ one has:

$$\int_{\widetilde{\mathbb{M}}} f(\widetilde{x}, \mathbf{k}) d\widetilde{\mu}_g = \int_{\widehat{\Gamma}} \int_{p_2^{-1}(\mathbf{k})} f(\widetilde{x}, \mathbf{k}) \mathbf{d}\mathbf{h} d\nu(\mathbf{k}).$$

In particular, for each measurable bounded set $E \subset \widetilde{\mathbb{M}}$:

$$\widetilde{\mu}_g(E) = \int_{\widehat{\Gamma}} f_E(\mathbf{k}) \mathbf{d}\mathbf{h}(\mathbf{k}),$$

with $E_{\mathbf{k}} = E \cap p_2^{-1}(\mathbf{k})$ and $f_E(\mathbf{k})$ the measure of $E_{\mathbf{k}}$ with respect to $d\nu(\mathbf{k})$.

The measure $\widetilde{\mu}_g$ is invariant under the action of $\widehat{\Gamma}$ on $\widetilde{\mathbb{M}}$ and therefore descends to a measure μ_g on the algebraic covering space \mathbb{M} . In a few words: the measure μ_g is the measure that disintegrates into the Riemannian volume measure along the leaves and the Haar measure along the Cantor group transversals.

Remark 11. This type of measure can be defined for every McCord solenoid or, equivalently, for any principal bundle with fiber a Cantor group or, again equivalently, a compact topologically homogeneous solenoidal manifold.

Remark 12 (Equicontinuity of McCord solenoids). The fact that the compact topologically homogeneous solenoidal manifolds admit a transverse metric invariant under holonomy implies that these laminations are equicontinuous in the sense of [13], thus they behave very similarly to Riemannian foliations [35, 75]. In fact, it follows from S. Matsumoto in [73] that \mathbb{M} , being a minimal equicontinuous, compact lamination, is uniquely ergodic, i.e., it has a unique (up to scaling) transverse invariant Radon measure.

Definition 18 (Solenoidal volume form). If \mathcal{L} is a smooth McCord solenoid with a laminated Riemannian metric g , the measure μ_g defined above is the smooth measure given by the solenoidal volume form (with respect to the solenoidal Riemannian metric g).

The volume of \mathcal{L} with respect to the laminated volume form is:

$$\text{vol}(\mathcal{L}) = \int_{\mathcal{L}} d\mu_g.$$

3.4. The suspension of a representation

Let M be a compact n -dimensional connected smooth manifold. Let $\rho: \pi_1(M) \rightarrow \mathcal{H}(K)$ be a representation of $\pi_1(M)$ into the group of homeomorphisms of the Cantor set $\mathcal{H}(K)$. As before, let \tilde{M} be the universal covering of M . Let $\pi_1(M)$ act on $\tilde{M} \times K$ as follows:

$$\gamma(m, \mathbf{k}) = (\gamma(m), \rho(\gamma)(\mathbf{k})), \quad \gamma \in \pi_1(M), \quad \tilde{m} \in \tilde{M}, \quad \mathbf{k} \in K. \tag{9}$$

The action on \tilde{M} is by deck transformations and in the Cantor set K is via the representation ρ . The following holds:

Proposition 6. *The action of $\pi_1(M)$ given by formula (9) is free and properly discontinuous and co-compact, i.e., the orbit space is compact with respect to the quotient topology.*

Definition 19. By Proposition 6 the quotient

$$\mathbb{M}_\rho = \pi_1(M) \backslash (\tilde{M} \times \widehat{\pi_1(M)})$$

is a compact Hausdorff space called the *suspension* of the representation ρ . The suspension of ρ is of sometimes denoted $M \times_\rho K$.

There is a natural projection $\mathfrak{p}: \mathbb{M}_\rho \rightarrow M$ which is a locally trivial fibration with fiber K . The holonomy at any point $x \in M$ is conjugate to ρ .

The space $\tilde{M} \times K$ is obviously a non-compact solenoidal manifold and the canonical projection $\Pi: \tilde{M} \times K \rightarrow \mathbb{M}_\rho = M \times_\rho K$ sends the leaves $(\tilde{M}, \{\mathbf{k}\})$ into the leaves of $M \times_\rho K$. It is a solenoidal covering as in Definition 20 below.

Remark 13. As we have seen, a McCord solenoid is the suspension of a representation $\rho: \pi_1(M) \rightarrow G$, where G is a Cantor group acting on itself by left translations. The suspension of a representation $\rho: \pi_1(M) \rightarrow \mathcal{H}(K)$ is not, in general, a McCord solenoid. For this to be true it is necessary that the representation is by (left) translations on a Cantor group and every orbit in the Cantor set, under the representation, must be dense.

3.5. Lifting maps between McCord solenoids

By analogy with the usual definition of covering maps and following the definition by S.-T. Hu [51, p. 104] one has the following definition:

Definition 20 (Solenoidal covering spaces). Let \mathcal{S}_1 and \mathcal{S}_2 be two n -dimensional solenoidal manifolds (not necessarily compact or connected). A continuous map $p: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is called a *solenoidal covering map* if every point of \mathcal{S}_2 is *evenly covered* by p . This means that every point $x \in \mathcal{S}_2$ has an open neighborhood of the form: $\mathcal{U} = D \times K$, where D is an open disk in \mathbb{R}^n and K the Cantor set, such that $p^{-1}(\mathcal{U}) = \bigsqcup_{\alpha \in I} \mathcal{V}_\alpha$, where \mathcal{V}_α is open, and the restriction of p to each \mathcal{V}_α is a homeomorphism onto \mathcal{U} .

Let

$$\Pi_1: \mathbb{M}_1 \rightarrow M_1 \quad \text{and} \quad \Pi_2: \mathbb{M}_2 \rightarrow M_2$$

be the two algebraic universal coverings of the compact, connected, smooth manifolds M_1 and M_2 (both with infinite and residually finite fundamental groups), respectively.

\mathbb{M}_1 and \mathbb{M}_2 are compact solenoids which are Cantor group fiber bundles over M_1 and M_2 , respectively, and are laminated with simply connected leaves homeomorphic to the universal coverings of M_1 and M_2 , respectively.

Let $G_1 = \widehat{\pi_1(\widetilde{M}_1)}$, $\widetilde{\mathbb{M}}_1 = \widetilde{M}_1 \times G_1$, and $G_2 = \widehat{\pi_1(\widetilde{M}_2)}$, $\widetilde{\mathbb{M}}_2 = \widetilde{M}_2 \times G_2$. Let $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ be the corresponding solenoidal coverings (formula (7)):

$$\widehat{\Pi}_1: \widetilde{\mathbb{M}}_1 \rightarrow \mathbb{M}_1 \quad \text{and} \quad \widehat{\Pi}_2: \widetilde{\mathbb{M}}_2 \rightarrow \mathbb{M}_2. \tag{10}$$

Remark 14. It follows from the definitions that $\widehat{\Pi}_i$ maps leaves of $\widetilde{\mathbb{M}}_i$ homeomorphically onto the leaves of \mathbb{M}_i , $i = 1, 2$.

Definition 21. Let $f_1, f_2: \mathbb{M}_1 \rightarrow \mathbb{M}_2$ be two continuous maps. We say that f_1 is *leafwise equivalent* to f_2 if $f_1(L)$ and $f_2(L)$ are contained in the same leaf of \mathbb{M}_2 for every leaf L of \mathbb{M}_1 . Two homotopic maps are leafwise equivalent.

The mappings $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ in equation (10) are solenoidal coverings maps.

We have the following lifting theorem for maps between McCord solenoids:

Theorem 4. Let \mathbb{M}_1 and \mathbb{M}_2 be McCord solenoids and $\widehat{\Pi}_1, \widehat{\Pi}_2$ as in equation (10). Let $f: \mathbb{M}_1 \rightarrow \mathbb{M}_2$ be a continuous map. There exists a continuous map $\widetilde{f}: \widetilde{\mathbb{M}}_1 \rightarrow \widetilde{\mathbb{M}}_2$ such that $f \circ \widehat{\Pi}_1 = \widehat{\Pi}_2 \circ \widetilde{f}$. The map \widetilde{f} is of the form

$$\widetilde{f}(y, g_1) = (\widetilde{f}_{g_1}(y), \varphi(g_1)), \tag{11}$$

where $\varphi: G_1 \rightarrow G_2$ is continuous, and \widetilde{f}_{g_1} is a continuous map from $\widetilde{M}_1 \times \{g_1\}$ to $\widetilde{M}_2 \times \{\varphi(g_1)\}$.

Proof. Let $p: \widetilde{M}_1 \rightarrow M_1$ be the universal covering of M_1 . Let $x \in M_1$ and \widetilde{x} such that $p(\widetilde{x}) = x$. Let $G_1(x) = \Pi_1^{-1}(\{x\}) \sim G_1$ be the Cantor fiber over x of the fibration $\Pi_1: \widetilde{\mathbb{M}}_1 \rightarrow \mathbb{M}_1$. Let $f_x: G_1(x) \rightarrow \mathbb{M}_2$ be the restriction of f to $G_1(x)$.

We can cover the Cantor set $G_1(x)$ with a finite number of disjoint *clopen* sets (Cantor sets) of small diameter, such that their images under f have a diameter smaller than the Lebesgue number of a cover of \mathbb{M}_2 by open sets that are evenly covered. We can restrict f_x to these small Cantor sets and “lift” them using the inverses of the local homeomorphisms on the evenly covered open sets. Then, using these liftings we see that f_x admits a “lifting,” i.e., there exists a continuous function $\widetilde{f}_x: \{\widetilde{x}\} \times G_1 \subset \widetilde{\mathbb{M}}_1 \rightarrow \widetilde{M}_2 \times G_2 = \widetilde{\mathbb{M}}_2$ such that $f_x \circ i_x = \widehat{\Pi}_2 \circ \widetilde{f}_x$. Here i_x is the restriction of $\widehat{\Pi}_1$ to $\{\widetilde{x}\} \times G_1$ so that $i_x(\{\widetilde{x}\} \times G_1) = G_1(x)$.

Let $p_2: \widetilde{M}_2 \times G_2 \rightarrow G_2$ be the projection onto the second factor. The map \widetilde{f}_x induces a continuous map $\varphi: G_1 \rightarrow G_2$ as follows: $\varphi(g_1) = p_2(\widetilde{f}_x(\{\widetilde{x}\}, g_1))$.

The leaves of $\widetilde{\mathbb{M}}_1$ and $\widetilde{\mathbb{M}}_2$ are of the form $\widetilde{L}_{g_1}^1 := \widetilde{M}_1 \times \{g_1\}$ and $\widetilde{L}_{g_2}^2 := \widetilde{M}_2 \times \{g_2\}$, with $g_1 \in G_1$ and $g_2 \in G_2$, respectively.

By Remark 14 $L_{\mathfrak{g}_1}^1 := \widehat{\Pi}_1(\widetilde{L}_{\mathfrak{g}_1}^1)$ is a leaf of \mathbb{M}_1 and $L_{\varphi(\mathfrak{g}_1)}^2 := \widehat{\Pi}_2(\widetilde{L}_{\varphi(\mathfrak{g}_1)}^2)$ is a leaf of \mathbb{M}_2 , and f maps $L_{\mathfrak{g}_1}^1$ into $L_{\varphi(\mathfrak{g}_1)}^2$. The restrictions of $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ to these leaves are homeomorphisms (with respect to the manifold topologies of the leaves). We denote these restricted homeomorphisms by the same symbols.

Define $\widetilde{f}_{\mathfrak{g}_1}: \widetilde{L}_{\mathfrak{g}_1}^1 \rightarrow \widetilde{L}_{\varphi(\mathfrak{g}_1)}^2$ as the following composition:

$$\widetilde{f}_{\mathfrak{g}_1} = (\widehat{\Pi}_2)^{-1} \circ f \circ \widehat{\Pi}_1.$$

With this map $\widetilde{f}_{\mathfrak{g}_1}$, which depends continuously upon \mathfrak{g}_1 , the function given by the formula (11) is a lifting of f . ■

Remark 15. Solenoidal covering maps have the unique path-lifting property and the homotopy-lifting property.

Since solenoidal manifolds are neither connected nor locally connected, the standard lifting theorems do not apply. For instance, if $f: \mathcal{S} \rightarrow \mathcal{S}$ is a homeomorphism of a compact McCord solenoid it may happen that a lifting to $\widetilde{\mathcal{S}}$ is not bijective (see the following Remark 16).

Remark 16 (*Caveat: liftings of homeomorphisms might not be homeomorphisms*). The usual lifting properties can fail. Let M be a compact manifold with infinite residually finite fundamental group $\pi_1(M)$. Let $G = \widehat{\pi_1(M)}$. Let $\widetilde{M} = \widetilde{M} \times G$ and $\widehat{\Pi}: \widetilde{M} \rightarrow \mathbb{M}$ the canonical projection onto the algebraic universal covering $\Pi: \mathbb{M} \rightarrow M$ given in Definition 13, then $\widehat{\Pi}$ is a solenoidal covering map.

Endow $\pi_1(M)$ with the discrete topology. Let $\psi: G \rightarrow \pi_1(M)$ be a continuous map and let $F_\psi: \widetilde{M} = \widetilde{M} \times G \rightarrow \widetilde{M} = \widetilde{M} \times G$ be given by the formula:

$$F_\psi(x, \mathfrak{g}) = (\psi(\mathfrak{g})(x), j(\psi(\mathfrak{g})) \cdot \mathfrak{g}),$$

where $j: \pi_1(M) \rightarrow G$ is the canonical monomorphism. As before, the action on the left of $\pi_1(M)$ on \widetilde{M} is by deck transformations and on G by left translations.

Then $\widehat{\Pi} \circ F_\psi = \widehat{\Pi}$, therefore F_ψ is a lifting to \widetilde{M} of the identity map $I: \mathbb{M} \rightarrow \mathbb{M}$. This lifting is not invertible for non-constant ψ (compare [63, p. 255] where examples in dimension one are presented).

Henceforth, in this last part of this subsection we assume that \mathbb{M}_2 has a laminated Riemannian metric h which renders all leaves with negative sectional curvature. Then all the leaves of \mathbb{M}_2 are simply connected Hadamard manifolds and therefore two points in the same leaf are connected by a unique geodesic parametrized by arc length.

Furthermore, unit speed geodesics in Hadamard manifolds depend continuously on their endpoints. Using these unique parametrized geodesics it is very easy to obtain a homotopy between any two leafwise equivalent maps from \mathbb{M}_1 to \mathbb{M}_2 .

Since maps between solenoids necessarily send leaves into leaves, in our case, we only need to know how a map “shuffles” the leaves to determine its homotopy class.

There are uncountably many such liftings. Notice that we do not have a notion of uniqueness of these liftings, however in this paper we use only the existence of a lifting.

We have the following corollary using the construction of a lift in Theorem 4:

Corollary 2. *With the same protagonists of the previous Remark 16, suppose that \mathbb{M} has a laminated metric with leaves of strictly negative curvature. Then given any continuous map $f: \mathbb{M} \rightarrow \mathbb{M}$ such that $f(L) = L$ for every leaf of \mathbb{M} (i.e., f does not “shuffle” the leaves) there exist a lift $\tilde{f}: \tilde{\mathbb{M}} \rightarrow \tilde{\mathbb{M}}$ of the form*

$$\tilde{f}(x, \mathfrak{g}) = (\tilde{f}_{\mathfrak{g}}(x), \mathfrak{g}). \tag{12}$$

Proof. We follow the proof of Theorem 4 when $\mathbb{M}_1 = \mathbb{M}_2 = \mathbb{M}$, $G_1 = G_2 = G$. We have to prove that h can be chosen to be the identity map on G .

Let $x \in M$ and $G(x) = \Pi^{-1}(\{x\}) \subset \mathbb{M}$ the Cantor fiber over x . By choosing a point in $G(x)$ we can identify $G(x)$ with the Cantor group G .

Since the leaves of \mathbb{M} are Hadamard spaces, and f preserves each leaf, for each $\mathfrak{g} \in G(x)$ there exists a unique parametrized geodesic joining \mathfrak{g} to $f(\mathfrak{g})$. We can reparametrize the geodesic to have a map $\gamma_{\mathfrak{g}}: [0, 1] \rightarrow \mathbb{M}$, $\gamma_{\mathfrak{g}}(0) = \mathfrak{g}$, $\gamma_{\mathfrak{g}}(1) = f(\mathfrak{g})$. Since the metric is continuous in the transversal Cantor structure we can assume that the map $\mathfrak{g} \mapsto \gamma_{\mathfrak{g}}$ is a continuous map from $G(x)$ to $C^0([0, 1], \mathbb{M})$.

Let $p: \tilde{M} \rightarrow M$ be the universal covering of M and $\tilde{x} \in \tilde{M}$ be such that $p(\tilde{x}) = x$. Let i_x be the restriction of $\tilde{\Pi}$ to $\{\tilde{x}\} \times G$ so that $i_x(\{\tilde{x}\} \times \{\mathfrak{g}\}) = \mathfrak{g}$ and $i_x(\{\tilde{x}\} \times G) = G(x)$.

Each $\gamma_{\mathfrak{g}}$ can be lifted to a map $\tilde{\gamma}_{\mathfrak{g}}: [0, 1] \rightarrow \tilde{\mathbb{M}}$ so that $\tilde{\gamma}_{\mathfrak{g}}(0) = (\tilde{x}, \mathfrak{g})$. Then as in Theorem 4 we can lift the restriction f_x of f to $G(x)$ as the function $\tilde{f}_x(\tilde{x}, \mathfrak{g}) = \tilde{\gamma}_{\mathfrak{g}}(0)$. This implies that $h: G \rightarrow G$ is the identity and therefore the lifting granted by Theorem 4 is of the form given by the formula (12). ■

4. Laminated harmonic maps and the Eells–Sampson theorem

In the 1964 groundbreaking paper [38] J. Eells and J. Sampson started the theory of harmonic maps between Riemannian manifolds. They showed that for certain Riemannian manifolds, arbitrary maps could be deformed into harmonic maps. In particular, they showed that given (M, g) and (N, h) , two smooth and closed Riemannian manifolds, where the sectional curvature of (N, h) is nonpositive, then for any C^∞ map $f: M \rightarrow M$ the maximal harmonic heat flow $\{f_t : 0 < t < T\}$, with $f_0 = f$, can be prolonged to $T = \infty$. Using a result by Philip Hartman, the maps f_t converge strongly in the C^∞ topology to a harmonic map. In particular, this shows that, if (N, h) has strictly negative curvature then every continuous map is homotopic to a unique harmonic map. Their work was the inspiration for Richard Hamilton’s initial work on the Ricci flow and the crowning work of G. Perelman about the *Geometrization theorem* [62].

Let M_1 and M_2 be two compact smooth manifolds with infinite residually finite fundamental groups.

Let (\mathcal{S}_1, g) and (\mathcal{S}_2, h) be the two compact, smooth, McCord solenoids of dimensions m and n , respectively, which are algebraic coverings of the compact manifolds M_1 and M_2 , respectively.

Endow (\mathcal{S}_1, g) and (\mathcal{S}_2, h) with laminated Riemannian metrics g and h , respectively. Thus we may think of \mathcal{S}_1 and \mathcal{S}_2 as principal bundles with fibers the Cantor groups G_1 and G_2 , respectively. We recall that smooth maps between smooth solenoidal manifolds are smooth laminated maps.

Let $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ be a leafwise C^∞ map. By continuity, such a map sends leaves to leaves and it is a differentiable map from one leaf into its image. For such maps, we define the *energy functional* as follows:

Definition 22. The *Dirichlet energy* of f is defined by the equation

$$E(f) = \frac{1}{2} \int_{\mathcal{S}_1} \|df(x)\|^2 d\mu_g(x). \tag{13}$$

In this formula df is a section of the bundle $T^*(\mathcal{S}_1) \otimes f^{-1}T(\mathcal{S}_2)$ and $\mu_g(x)$ is the laminated volume form of \mathcal{S}_1 , with respect to g , as in Definition 18. The laminated metrics on \mathcal{S}_1 and \mathcal{S}_2 induce a bundle metric on this bundle and $\|\cdot\|_{(g,h)}$ is the associated norm (Hilbert–Schmidt norm). Since f sends leaves to leaves, f is expressed in terms of local coordinates as follows:

$$f(x_1, \dots, x_m; \mathfrak{g}) = (f_1(x_1, \dots, x_m; h(\mathfrak{g})), f_2(x_1, \dots, x_m; h(\mathfrak{g})), \dots, f_n(x_1, \dots, x_m; h(\mathfrak{g}))), \tag{14}$$

with $\mathfrak{g} \in G_1$ and $h: \mathcal{V} \rightarrow G_2$ a continuous map from the open set $\mathcal{V} \in G_1$ to G_2 . For each $\mathfrak{g} \in G_1$ fixed, the map in formula (14) is C^∞ -differentiable with respect to the variables x_i 's. In terms of these local laminated coordinates the Hilbert–Schmidt norm is given by the formula

$$\|df(x, \mathfrak{g})\|^2 = g^{ij}(x, \mathfrak{g}) h_{\alpha\beta}(f(x, \mathfrak{g})) \frac{\partial f_\alpha(x, \mathfrak{g})}{\partial x_i} \frac{\partial f_\beta(x, \mathfrak{g})}{\partial x_j}, \quad x = (x_1, \dots, x_m), \mathfrak{g} \in G_1. \tag{15}$$

Here, we use the usual notation in terms of coordinates: g^{ij} are the coefficients of the inverse of the matrix corresponding to the metric g and $h_{\alpha\beta}$ the coefficients of the matrix corresponding to the metric h , and we use the Einstein summation convention.

Definition 23. The *energy density* of f is the function $e(f): \mathcal{S}_1 \rightarrow \mathbb{R}^{\geq 0}$ defined, in local coordinates, by the formula

$$e(f)(x, \mathfrak{g}) = \frac{1}{2} \|df(x, \mathfrak{g})\|^2.$$

Definition 24. A leafwise C^∞ map $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is set to be *leafwise harmonic* if it is an extremal of the energy functional (13).

The Euler–Lagrange equations corresponding to this variational problem imply $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is harmonic if and only if its *tension vector field* $\tau(f)$ vanishes:

$$\tau(f) = \text{trace}_g \nabla df = 0.$$

Hence, f is harmonic if and only if $\tau(f) = 0$.

The expression of the tension field in terms of local coordinates can be found in [38].

The following is the solenoidal version of the theorem by Eells–Sampson [38]:

Theorem 5 (Solenoidal Eells–Sampson theorem). *Let (\mathcal{S}_1, g) and (\mathcal{S}_2, h) be two compact McCord solenoids of dimensions m and n , and laminated Riemannian metrics g and h , respectively. Suppose that the leaves of \mathcal{S}_2 have strictly negative sectional curvature with respect to h . Let $f_0: (\mathcal{S}_1, g) \rightarrow (\mathcal{S}_2, h)$ be a smooth map (necessarily sends leaves of (\mathcal{S}_1, g) into leaves of (\mathcal{S}_2, h)) then f_0 is leafwise homotopic to a unique harmonic map f .*

Sketch of the proof. The proof follows almost *verbatim* the steps of Eells–Sampson [38] using the heat flow. We also refer to [65, Chapter 5] or [54, Chapter 9] for other detailed presentations. Consider the evolution equation:

$$\begin{cases} \partial_t f(x, t) = \tau(f(x, t)), \\ f(x, 0) = f_0(x), \end{cases} \tag{16}$$

where $f(\cdot, t) \in C^\infty(\mathcal{S}_1, \mathcal{S}_2)$ (i.e., it is a laminated smooth map).

The proof of Theorem 5 consists of 6 steps:

- (1) The existence of $\varepsilon > 0$ such that $f(x, t)$ exists for $0 \leq t < \varepsilon$.
- (2) The existence of $f(x, t)$ for all $t > 0$.
- (3) The existence of $\lim_{t \rightarrow \infty} f(x, t) = f(x, \infty) = f(x)$.
- (4) The proof that $f(x)$ is harmonic.
- (5) The proof that $f(x)$ is homotopic to f_0 .
- (6) The uniqueness of f .

The uniqueness follows by a theorem of Philip Hartman [49]. The reason that all the steps can be achieved is that the *a priori* estimates of Eells and Sampson hold in our case because we endowed the solenoids with a very tame measure, therefore the energy $E(f(\cdot, t))$ remains bounded by a constant independent of t . The proof uses the Bochner identity which also holds in our laminated case: let $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ be a smooth map. Let df denote the derivative (pushforward) of f , ∇ the laminated gradient, i.e., the covariant derivative on $T^*(\mathcal{S}_1) \otimes f^{-1}T(\mathcal{S}_2)$.

Let Δ^g the laminated Laplace–Beltrami operator on \mathcal{S}_1 , Riem^h the laminated Riemann curvature tensor on \mathcal{S}_2 and Ric^g the Ricci curvature tensor on \mathcal{S}_1 .

Then if $\{f_t\}_t$ satisfies the evolution equation (heat flow equation) (16) then one has the *Bochner identity*:

$$\left(\frac{\partial}{\partial t} - \Delta^g\right)e(f_t) = -|\nabla(df_t)|^2 - \langle \text{Ric}^g, f_t^*h \rangle_g + \text{scal}^g(f_t^*\text{Riem}^h).$$

In terms of the local coordinates (14), $f_t = (f_{t,1}, \dots, f_{t,n})$, and using Einstein's summation, the terms are defined as follows:

$$|\nabla(df_t)(x, \mathfrak{g})|^2 = \sum_{i,j,k} \left[\frac{\partial^2 f_{t,k}}{\partial x_i \partial x_j}(x, \mathfrak{g}) \right]^2,$$

$$\langle \text{Ric}^g(x, \mathfrak{g}), (f_t^* h)(x, \mathfrak{g}) \rangle_g = \text{Ric}_{ij}^g(x, \mathfrak{g}) h_{\alpha\beta}(f_t(x, \mathfrak{g})) \frac{\partial f_{t,\alpha}(x, \mathfrak{g})}{\partial x_i} \frac{\partial f_{t,\beta}(x, \mathfrak{g})}{\partial x_j}, \quad (17)$$

$$\text{scal}^g(f_t^* \text{Riem}^h)(x, \mathfrak{g}) =$$

$$\text{Riem}_{\alpha\beta\gamma\delta}^h(f_t(x, \mathfrak{g})) \left[\frac{\partial f_{t,\alpha}(x, \mathfrak{g})}{\partial x_i} \frac{\partial f_{t,\beta}(x, \mathfrak{g})}{\partial x_j} \frac{\partial f_{t,\gamma}(x, \mathfrak{g})}{\partial x_k} \frac{\partial f_{t,\delta}(x, \mathfrak{g})}{\partial x_l} \right] g^{ik}(x, \mathfrak{g}) g^{jl}(x, \mathfrak{g}).$$

We see in equation (17) that the $m \times m$ symmetric matrix (Ric_{ij}^g) can be compared with the positive-definite symmetric $m \times m$ matrix $g_{ij}(x, \mathfrak{g})$ and, since the leafwise sectional curvature of \mathcal{S}_2 is negative comparing (15) with (17) we obtain the following:

Proposition 7. *There exists a constant $C > 0$ depending only on the leafwise Ricci curvature of \mathcal{S}_1 such that*

$$\left(\frac{\partial}{\partial t} - \Delta^g \right) e(f_t) \leq C e(f_t). \quad (18)$$

Once the Bochner identity is valid, using Proposition 7 and the theorem of Moser–Harnack lemma [65, Lemma 5.3.4, p. 115] we can prove steps (1) to (4) exactly as in references [38, 53, 54, 65]. The proofs of (5) and (6) follow Hartman's proof [49]. In fact, a direct calculation shows that if f_t is a solution of the heat equation (16) then the Dirichlet energy satisfies:

$$\frac{d}{dt} E(f_t) = - \int_{\mathcal{S}_1} \|\partial_t f_t\|^2 d\mu \leq 0,$$

$$\frac{d^2}{dt^2} E(f_t) = - \frac{\partial}{\partial t} \int_{\mathcal{S}_1} \|\partial_t f_t\|^2 d\mu = - \int_{\mathcal{S}_1} \partial_t \|\partial_t f_t\|^2 d\mu = \|\nabla \partial_t f_t\|^2 \geq 0.$$

To be sure that the leafwise evolution of the heat flow is as in the case of [38] we proceed as follows. The solenoids are of the form $\mathcal{S}_1 = \tilde{M}_1 \times G_1/G_1$ and $\mathcal{S}_2 = \tilde{M}_2 \times G_1/G_1$ where \tilde{M}_1 and \tilde{M}_2 are the universal coverings of the compact manifolds M_2 and M_2 and G_1, G_2 are the respective profinite completions of their fundamental groups.

By Theorem 4 applied to the solenoidal coverings $\hat{\Pi}_1: \tilde{M}_1 \times G_1 \rightarrow \mathcal{S}_1$ and $\hat{\Pi}_2: \tilde{M}_2 \times G_2 \rightarrow \mathcal{S}_2$, the map f lifts to a map $\tilde{f}: \tilde{M}_1 \times G_1 \rightarrow \tilde{M}_2 \times G_2$.

The map \tilde{f} is of the form: $\tilde{f}(x, \mathfrak{g}_1) = (\tilde{f}_{\mathfrak{g}_1}(x), h(\mathfrak{g}_1))$, where $h: G_1 \rightarrow G_2$ is continuous, $\mathfrak{g}_1 \in G_1, x \in \tilde{M}_1 \times \{\mathfrak{g}_1\}$ and

$$\tilde{f}_{\mathfrak{g}_1}: \tilde{M}_1 \times \{\mathfrak{g}_1\} \rightarrow \tilde{M}_2 \times \{h(\mathfrak{g}_1)\}$$

is a smooth map from the leaf $\tilde{M}_1 \times \{\mathfrak{g}_1\}$ into the leaf $\tilde{M}_2 \times \{h(\mathfrak{g}_1)\}$.

The Cantor group G_1 acts freely, properly discontinuously, and cocompactly on $\tilde{M}_1 \times G_1$, therefore the solenoidal covering projection $\hat{\Pi}_1: \tilde{M}_1 \times G_1 \rightarrow \mathcal{S}_1$ has a fundamental domain \mathbb{D} of the form $\mathbb{D} = \mathfrak{D} \times G_1 \subset \tilde{M}_1 \times G_1$ with $\mathfrak{D} \subset \tilde{M}_1$ compact. If we restrict \tilde{f} to \mathbb{D} we see that all the previous estimates, like Bochner's identity and Moser-Harnack inequality, hold for each \tilde{f}_{g_1} , uniformly with respect to $g_1 \in G_1$.

Hence, the heat equation evolves in each leaf as in the case of compact manifolds. The main fact is: *if the sectional curvature of the leaves of \mathcal{S}_2 is strictly negative, the Dirichlet integral (13) is a strictly decreasing, and convex function of f .*

Summarizing, we have:

Proposition 8. *Under our hypothesis, there is a unique harmonic map in each homotopy class of maps in $C^\infty(\mathcal{S}_1, \mathcal{S}_2)$. Furthermore, the harmonic map is the unique map in its homotopy class which minimizes the energy. The harmonic map is stable and depends continuously on the initial map f_0 .*

5. 1-dimensional solenoidal manifolds

5.1. 1-dimensional solenoidal manifolds as mapping tori

Part of this subsection is based on the paper by Dennis Sullivan [110] and its companion [114].

A (smooth) compact 1-dimensional, oriented, solenoidal manifold \mathcal{S} can be given a foliated Riemannian metric and thus we can define a unit vector field along the leaves and, as a consequence, one can define a nonsingular flow $\varphi_t: \mathcal{S} \rightarrow \mathcal{S}$.

Proposition 9 (Sullivan [4, 110]). *Let \mathcal{S} be a compact, oriented, 1-dimensional solenoidal manifold. Then, \mathcal{S} admits a global transversal Cantor set K . More precisely, the flow φ_t has a global Poincaré cross-section. Therefore, \mathcal{S} is the mapping torus or suspension of a homeomorphism of K .*

Proof. One can choose a finite set of transversals intersecting every leaf and such that starting at a point on one transversal and going forward (with respect to the orientation of the leaves) one first meets a different transversal.

This presents the solenoid as a *mapping torus* or *suspension* of a homeomorphism $f: K \rightarrow K$ on the Cantor set K , where f is the first-return map of the flow to the cross-section. ■

Corollary 3. *There is a surjection between conjugacy classes of the group $\mathcal{H}(K)$ of homeomorphisms of the Cantor set and homeomorphic classes of 1-dimensional compact oriented solenoids.*

More detailed results about this correspondence are given in [1–3, 39].

5.2. 1-dimensional, compact, solenoidal manifolds are null-cobordant

By Proposition 9 any oriented 1-dimensional solenoidal manifold is the suspension of a homeomorphism f of the Cantor set and this implies the following:

Proposition 10 (Sullivan [110]). *Any 1-dimensional, compact, oriented, solenoidal manifold \mathcal{S} is null-cobordant: there exists a compact 2-dimensional solenoidal manifold with boundary and this boundary is the given solenoidal 1-dimensional manifold.*

Proof. The proof is based on an idea by Thurston and it follows from the fact that any homeomorphism of the Cantor set is a finite product of commutators, i.e., $\mathcal{H}(K)$ (Definition 9) is a perfect group. The proof of the fact that the group of homeomorphisms of the Cantor set is perfect is proven in all detail in the paper [15] by R. D. Anderson.

Let \mathcal{S} be the suspension of $f \in \mathcal{H}(K)$ given by Proposition 9 and $f = [h_1, k_1] \cdots [h_g, k_g]$. Then, there exists a representation $\rho: \pi_1(\Sigma) \rightarrow \mathcal{H}(K)$, where Σ is a smooth compact surface of genus g with connected boundary a circle, such that the restriction of ρ to the element represented by the boundary circle is f .

The lamination we want is the suspension of the representation ρ given in Section 3.4. We obtain a solenoidal surface with boundary \mathcal{L}_ρ and, since the restriction of the representation to the boundary circle is the suspension of f , one has $\partial\mathcal{L}_\rho = \mathcal{S}$, and we obtain the proof of Proposition 10. ■

5.3. 1-dimensional compact topologically homogeneous solenoids

Theorem 6 (Compare R. H. Bing [20] and M. C. McCord [74]). *Every compact, connected, 1-dimensional solenoidal manifold which is topologically homogeneous is homeomorphic to a compact, connected, abelian group Γ . There exists a dense (with respect to the standard topology) subgroup $A \subset \mathbb{Q}$ of the additive rationals $(\mathbb{Q}, +)$ such the Pontryagin dual of A_δ is isomorphic to Γ , where A_δ denotes A equipped with the discrete topology.*

Proof. If \mathcal{S} is a compact, topologically homogeneous, 1-dimensional solenoid then by Theorem 2 it is a McCord solenoid and therefore it is obtained as the inverse limit of an infinite tower of regular finite coverings of the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Every finite covering map of the circle is equivalent to a map of the form $z \mapsto z^n$, $n \in \mathbb{N}$, therefore it is a group homomorphism from the circle to the circle. It follows that \mathcal{S} is the inverse limit of a sequence (see Remark 3):

$$\dots \xrightarrow{z \rightarrow z^{n_4}} \mathbb{S}^1 \xrightarrow{z \rightarrow z^{n_3}} \mathbb{S}^1 \xrightarrow{z \rightarrow z^{n_2}} \mathbb{S}^1 \xrightarrow{z \rightarrow z^{n_1}} \mathbb{S}^1, \quad n_i \geq 2. \tag{19}$$

Therefore, the inverse limit is of a sequence of homomorphisms, hence this limit is a compact abelian group. Since \mathcal{S} is of topological dimension one, it follows by Pontryagin duality that \mathcal{S} is the Pontryagin dual of a countable discrete abelian group Γ of rank one, and therefore Γ is a subgroup of the rationals \mathbb{Q} .

The non-trivial subgroups of \mathbb{Q} are either dense or infinite cyclic, the latter must be excluded since its Pontryagin dual is the circle and not a solenoid. ■

Remark 17. When in the sequence (19) we take $n_i = p^i$, where p is a prime number, the inverse limit of coverings is the classic *Vietoris–van Dantzig p -adic solenoid* which we denote \mathcal{S}_p .

By Theorem 6, compact, connected, orientable, 1-dimensional topologically homogeneous solenoidal manifolds are in one-to-one correspondence with the isomorphism classes of subgroups of the additive rationals \mathbb{Q} via the Pontryagin duality. These solenoids have a distinguished leaf, the component of the identity called the base leaf, [87]. A theorem by Reinhold Baer, [17], describes the subgroups of \mathbb{Q} up to isomorphism by equivalence classes of sequences called types (see also [112]). There are uncountably many such types. Hence, there are also uncountably many isomorphism classes of additive subgroups of \mathbb{Q} . On the other hand it is shown in [99] that two homeomorphic locally compact, connected, abelian groups are topologically isomorphic, and therefore their Pontryagin duals are isomorphic subgroups of. Hence, non-isomorphic subgroups of the rationals correspond to non-homeomorphic solenoids \mathbb{Q} (compare [45]).

Therefore, we have the following:

Corollary 4. *There exist uncountably many homeomorphism classes of compact, connected, orientable, topologically homogeneous 1-dimensional solenoidal manifolds. They are all suspensions of a minimal translation on a Cantor abelian group.*

Remark 18. Let $\mathbb{A}_{\mathbb{Z}}$ be the ring of integral *adèles* [91] then: $\mathbb{R} \times \widehat{\mathbb{Z}} = \mathbb{A}_{\mathbb{Z}} = \mathbb{R} \times \prod_p \mathbb{Z}_p$. The map $\mathbb{Z} \xrightarrow{i} \mathbb{R} \times \widehat{\mathbb{Z}}, n \mapsto (n, \mathbf{n})$ (where \mathbf{n} is the image of n by the natural inclusion of \mathbb{Z} into $\widehat{\mathbb{Z}}$) injects \mathbb{Z} into a discrete co-compact subgroup Γ . The Pontryagin dual \mathbb{Q}^* of the rationals is isomorphic, as a compact, abelian, topological group, to $\widehat{\mathbb{S}}^1 := (\mathbb{R} \times \widehat{\mathbb{Z}})/\Gamma$.

The subgroup $\{(0, \mathbf{n}) : n \in \mathbb{Z}\}$, isomorphic to \mathbb{Z} , is dense in $\{0\} \times \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of the integers \mathbb{Z} . This implies that the canonical flow whose orbits are the translations of the component of the identity (isomorphic to \mathbb{R}) is minimal.

Definition 25. The group $\widehat{\mathbb{S}}^1$ is called the *àdele class group or universal 1-dimensional arithmetic solenoid*. It is the algebraic universal covering of the circle \mathbb{S}^1 (Definition 13).

The inclusion of a subgroup $\Gamma \subset \mathbb{Q}$ of \mathbb{Q} , induces by Pontryagin duality an epimorphism of their duals $\widehat{\mathbb{S}}^1 = \mathbb{Q}^* \rightarrow \Gamma^*$. Thus the universal solenoid $\widehat{\mathbb{S}}^1$ maps epimorphically onto any 1-dimensional compact, connected, abelian group.

6. Solenoidal Riemann surfaces

6.1. Surface laminations and uniformization via the Ricci flow

Definition 26. A compact lamination by surfaces \mathcal{L} is a compact metrizable space \mathcal{L} endowed with an atlas $\mathcal{A} = (U_\alpha, \varphi_\alpha)$ such that:

- (1) Each φ_α is a homeomorphism from U_α to a product $D \times T_\alpha$, where D is the unit disk in the plane \mathbb{R}^2 and T_α is a locally compact space (not necessarily a Cantor set).
- (2) Whenever $U_\alpha \cap U_\beta \neq \emptyset$, the change of coordinates $\varphi_\beta \circ \varphi_\alpha^{-1}$ is of the form

$$(z, \zeta) \mapsto (\lambda_{\alpha\beta}(z, \zeta), \tau_{\alpha\beta}(\zeta)),$$

where $\lambda_{\alpha\beta}$ is smooth in the z variable. If the $\lambda_{\alpha\beta}$ preserve a fixed orientation of the 2-disk we say that the lamination is oriented. The solenoidal case is when T_α is a Cantor set for all α .

Surface laminations and solenoidal manifolds are particular cases of *foliated spaces*. The book [76] by C. C. Moore and C. Schochet is a good reference on the subject of analysis on foliated spaces.

In the following definition we identify \mathbb{R}^2 with \mathbb{C} via $(x, y) \mapsto z = x + \mathbf{i}y$ and thus D in Definition 1 can be considered as the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ in \mathbb{C} and one can use complex notation.

Definition 27. A 2-dimensional lamination \mathcal{L} is said to be a *lamination by Riemann surfaces* or simply a *Riemann surface lamination* if the atlas $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in \mathfrak{S}}$ satisfies that the change of coordinates $\varphi_\beta \circ \varphi_\alpha^{-1}$ is of the form $(x, t) \mapsto (\lambda_{\alpha\beta}(x, t), \tau_{\alpha\beta}(t))$, where $\lambda_{\alpha\beta}$ is a holomorphic diffeomorphism in the x variable for each t . In this case, $\lambda_{\alpha\beta}$ preserves the preferred orientation of the disk induced by the orientation of \mathbb{C} , and the lamination is necessarily orientable. An atlas satisfying this condition is called a *complex atlas*. If $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in \mathfrak{S}}$ is a complex atlas as in Definition 26 the leaves are immersed copies of *Riemann surfaces*.

Several vector bundles over \mathcal{L} can be defined in a natural way using the fact that the $\lambda_{\alpha\beta}$ are smooth in the variable x .

These include the tangent bundle $T\mathcal{L}$ to the lamination, tensor bundles, frame bundles, etc. Their fibers vary continuously in the smooth topology along the direction transverse to the laminated structure, that is, the direction given by the T_α and are smooth along the leaves.

If \mathcal{L} is a Riemann surface lamination as in Definition 27 one can define laminated complex line bundles with fiber \mathbb{C} like, for instance, the *complex tangent line bundle* $T_{\mathbb{C}}\mathcal{L}$. The complex tangent line bundle is the line bundle obtained by providing the real tangent bundle $T\mathcal{L}$ with the natural almost complex structure obtained by the complex atlas using multiplication by \mathbf{i} in each real tangent plane $T_x L_x$ of the leaf L_x through x .

One can also define the *laminated canonical line bundle* $K\mathcal{L}$ and products of tensor bundles such as $(T_{\mathbb{C}}\mathcal{L})^{\otimes m} \otimes (K\mathcal{L})^{\otimes n}$. One has $(T_{\mathbb{C}}\mathcal{L}) \otimes K\mathcal{L} \cong \mathcal{L} \times \mathbb{C}$ is the trivial line bundle.

Thus one can define the Picard group of Riemann surface lamination (i.e., the group of line bundles with multiplication given by the tensor product). In particular, the sections of $K\mathcal{L} \otimes K\mathcal{L}$ are laminated quadratic differentials which are important to study the Teichmüller laminated theory (in general the spaces are of infinite dimension).

Definition 28. A *laminated (or foliated) Riemannian metric* g is a continuous tensor such that restricted to each leaf is a Riemannian metric and it is smooth along the leaves. In terms of a foliated chart $\varphi_{\alpha}: U_{\alpha} \rightarrow D \times T_{\alpha}$ the laminated metric is determined by a continuous function $g_{\alpha}: D \times T_{\alpha} \rightarrow \text{Symm}_{+}(2)$, where $\text{Symm}_{+}(2)$ is the space of real positive definite 2×2 symmetric matrices:

$$g_{\alpha}(z, t) = \begin{pmatrix} E(z, t) & F(z, t) \\ F(z, t) & G(z, t) \end{pmatrix},$$

where $E(z, t) > 0$, $E(z, t)G(z, t) - F^2(z, t) > 0$.

The functions E , F and G are differentiable with respect to z . Thus in this chart, the metric is written in the traditional notation

$$ds^2 = E(z, t) dx^2 + 2F(z, t) dx dy + F(z, t) dy^2, \quad z = x + iy.$$

In terms of complex notation

$$ds^2 = \gamma(z, t)|dz + \mu(z, t)d\bar{z}|^2, \tag{20}$$

where $z = x + iy$, $\gamma(z, t) > 0$, and $\mu(z, t) < 1$.

Definition 29. The function μ in formula (20) is called the *Beltrami coefficient* of the metric g . In a coordinate-invariant fashion, we can regard the Beltrami coefficient as a $(-1, 1)$ -form, i.e., a section of $K^{-1}\mathcal{L} \otimes \bar{K}\mathcal{L}$.

One must impose some regularity in order to have a good description of quasi-conformal maps between Riemann surface laminations and their theory of Teichmüller spaces.

Remark 19. If g is a laminated Riemannian metric on a surface lamination \mathcal{L} then g induces an almost complex structure $J: T\mathcal{L} \rightarrow T\mathcal{L}$ as follows: if x is in the leaf L of \mathcal{L} and $v \in T_x L$, then $J(v) = w$ where w is orthogonal to v (with respect to g), $|v|_g = |w|_g$ and (v, w) is an oriented frame of $T_x L$ (with respect to the orientation induced by the almost complex structure). In dimension 2 every almost complex is integrable, therefore an almost complex structure on \mathcal{L} induces on \mathcal{L} the structure of a Riemann surface lamination and therefore the structure of a Riemann surface lamination.

This is the laminated version of the Gauss–Korn–Lichtenstein on the existence of local isothermal coordinates and it follows by the Ahlfors–Bers theory [7].

Remark 20 (Reeb stability). If a connected surface lamination \mathcal{L} contains a leaf homeomorphic to the sphere S^2 , then by Reeb’s stability theorem (which is valid for laminations by [26, Theorem 5.3]), the leaf space of \mathcal{L} is a compact Hausdorff space and \mathcal{L} homeomorphic to a fiber bundle over the compact leaf space with fiber S^2 . *To avoid this trivial case we will assume heretofore that no leaf of the surface laminations considered in this paper are spherical.*

Definition 30. Two laminated Riemannian metrics g_1 and g_2 with induced norms $|\xi|_{g_1}$, $|\xi|_{g_2}$, respectively, on a smooth surface lamination \mathcal{L} , are said to be *quasi-isometric* if there exists a constant $a \geq 1$ such that:

$$a^{-1}|\xi|_{g_1} \leq |\xi|_{g_2} \leq a|\xi|_{g_1}, \quad \forall \xi \in T\mathcal{L}.$$

As explained in Remark 19, laminated Riemannian metrics on \mathcal{L} induce complex structures on the leaves of \mathcal{L} so that they become Riemann surfaces and \mathcal{L} is a Riemann surface lamination. By the Koebe–Poincaré uniformization theorem, the universal cover \tilde{L} of a leaf L , with the lifted complex structure, is conformally equivalent to either the Riemann sphere $\bar{\mathbb{C}}$, the complex plane \mathbb{C} or the Poincaré disk Δ ; we say that L is of *elliptic, parabolic* or *hyperbolic* type, respectively. We exclude the case of spherical leaves (Remark 20). Let S be a *simply connected* non-compact surface endowed with a complete Riemannian metric g . Let $x \in S$ and $r > 0$; define $A_g(r, x)$ to be the area of the geodesic disk of radius r centered at x . One says that g has *polynomial area growth* if there exists a constant $c > 0$ and an integer n , such that $A_g(r, x) \leq cr^n$. We say that the area *grows exponentially* if there exist positive constants c, b such that $A_g(r, x) \geq ce^{br}$. These two properties are independent of the point x . By M. Kanai [56, Theorem 3.3], these growth properties depend only on the quasi-isometry class of the Riemannian metric provided that the injectivity radius is positive and the Ricci curvature is bounded below. Thus, under these conditions, the conformal type of a non-compact simply connected surface is determined by the growth rate of the area. If \mathcal{L} is a compact surface lamination with a laminated Riemannian metric g then, by compactness, there exists $\delta > 0$ such that the injectivity radius of every leaf is greater than δ . In addition, the Ricci curvature of the leaves is uniformly bounded below.

Leaves of parabolic type have polynomial growth (i.e., their universal covering is conformally equivalent to \mathbb{C}) and leaves of a hyperbolic type have exponential area growth in their universal covering surface, i.e., the universal covering surface is conformally equivalent to the unit disk Δ . The surface itself might have sub-exponential growth, for instance when the surface is an infinite cyclic covering of a compact hyperbolic surface (*Jacob’s Ladder*) the volume growth is linear. Therefore, we have the unambiguous notion of the *conformal type* of a leaf L of a smooth *compact* lamination by surfaces:

Proposition 11. *Let \mathcal{L} be a compact, smooth, surface lamination. Let L be a leaf of \mathcal{L} , then the conformal type of L , with respect to the complex structure induced by a laminated Riemannian metric g , is independent of g . This allows us to speak about hyperbolic or*

parabolic leaves, independently of the laminated metric (recall that we excluded spherical leaves).

Definition 31 (Hyperbolic lamination). If all leaves are of hyperbolic type with respect to a laminated metric g we refer to \mathcal{L} as a *hyperbolic lamination* (the property is independent of g).

We recall that by Remark 19 on an *oriented* lamination a laminated Riemannian metric g determines a conformal structure on every leaf, that is, it turns every leaf into a Riemann surface.

A complex leaf L is hyperbolic if and only if it can be uniformized by the unit disk, i.e., there is a conformal covering map $\varphi: \Delta \rightarrow L$. There are examples of surface laminations with leaves of mixed types [43, 44, 78], in fact even holomorphic foliations by Riemann surfaces on compact complex surfaces with mixed types of leaves can be constructed [46, 113].

The following theorem due to Étienne Ghys (a proof can be found in [12] and [11, Théorème 6.5]) shows that for compact laminations with a Riemannian metric g , whose leaves are of hyperbolic type there exists a laminated Riemannian metric, conformally equivalent to g , whose leaves have negative curvature.

Theorem 7 (É. Ghys). *Let \mathcal{L} be a compact lamination by surfaces of hyperbolic type. Then, in each conformal class of \mathcal{L} , there exists a Riemannian laminated metric in such a way that the leaves of \mathcal{L} have negative curvature.*

When all leaves are hyperbolic, the uniformization maps of individual leaves vary continuously from leaf to leaf. More precisely, the following *uniformization theorem* holds (see [26, 113]):

Theorem 8. *Let \mathcal{L} be a compact lamination by hyperbolic surfaces endowed with a laminated Riemannian metric g . Then there exists a laminated Riemannian metric g' which is conformally equivalent to g and for which every leaf has constant curvature -1 . The metric g' has a continuous variation in the smooth topology in the direction transverse to the leaves of \mathcal{L} .*

Definition 32 (Hyperbolic laminations). Laminated Riemannian metrics g of \mathcal{L} with all leaves of constant Gaussian curvature -1 are called *laminated hyperbolic metrics*.

6.2. Laminated Ricci flow

In this subsection, we will state a stronger version of Theorem 8 established in [79], which uses the geometric flow determined by the Ricci flow in the spirit of Hamilton [48]. Given a compact lamination by surfaces \mathcal{L} with a laminated Riemannian metric g_0 with all leaves of negative curvature, we find a laminated metric g_∞ of class $C^{\infty,0}$ (Definition 7) which renders each leaf of constant negative curvature conformally equivalent to g_0 .

Let \mathcal{L} be a compact lamination and g_0 a $C^{\infty,0}$ laminated metric on it. In two dimensions, the Ricci curvature for a given metric g is equal to $\frac{1}{2}Rg$, where R is the scalar curvature or $R = 2K$ where K is the Gauss curvature. We can consider the “normalized laminated Ricci flow” as the evolution of the metric under the equation

$$\begin{cases} \frac{\partial g(t)}{\partial t} = (c - R(t))g(t), \\ g(0) = g_0. \end{cases} \tag{21}$$

Here $R(t)$ is the scalar curvature of the leaves for the metric $g(t)$ and c is a normalizing constant, independent of t , which is chosen conveniently. Let us denote by R_0 the curvature (in the leaf direction) of the metric g_0 . Since M is compact the leaves are complete and R_0 , being a continuous function on M , is bounded. From this, it is possible to conclude that there exists $\varepsilon > 0$ such that for each time t in an interval $[0, \varepsilon)$ there is a solution $g(t)$ to the Ricci flow equation; for $g(t) := g_t$ to be a solution to (21) on \mathcal{L} it has to vary continuously in the transverse direction; this is a consequence of the continuous dependence of the solution to (21) with respect to the initial condition.

The curvature of a family of metrics $g(t)$ satisfying (21) evolves under the equation:

$$\frac{\partial}{\partial t} R(t) = \Delta_t R(t) + (R(t) - c)R(t).$$

Here Δ_t denotes the Laplacian in the leaf direction (with respect to $g(t)$), or equivalently, we consider the above equation on each leaf.

In the 2-dimensional case the metrics given by equation (21) leave invariant the conformal class of the initial metric g_0 , hence we can write the evolution as an evolution of a single function u .

More precisely, by writing $g = e^u g_0$ for a metric in the conformal class of g_0 , we have that under the normalized Ricci flow u evolves according to

$$\frac{\partial u}{\partial t} = c - R = \Delta_t u - e^{-u} R_0 + c = e^{-u} (\Delta_0 u - R_0) + c,$$

where Δ_0 denotes the Laplacian operator associated with g_0 . Since $\Delta_t u = e^{-u} \Delta_0$ the equation follows from the well-known fact that $\Delta_{e^u g_0} = e^{-u} \Delta_0$.

The following theorem in [79] shows that if we start with an initial laminated metric g_0 such that every leaf has negative curvature at every point then the Ricci flow exists for all positive time and defines a one-parameter family of metrics $\{g_t\}_{t \geq 0}$ such that $\lim_{t \rightarrow \infty} g_t = g_\infty$ exists and defines a smooth metric with all leaves of constant negative curvature (rescaling we can assume that this constant curvature is -1).

The metric g_∞ is conformally equivalent to g_0 . By Theorem 7 we can choose as the initial metric for the Ricci flow a globally hyperbolic metric.

Remark 21. Let g_0 be an initial metric as in Theorem 7. Let $R_0: M \rightarrow \mathbb{R}_{<0}$ be the Ricci curvature of g_0 . Then, by compactness there exist two constants R_{\min} and R_{\max} such that

$R_{\min} \leq R_0(x) \leq R_{\max} < 0, \forall x \in M$. We may assume that R_0 is not a (negative) constant since in such a case there is nothing to prove, hence we may suppose $R_{\min} < R_{\max}$. The normalizing constant c is chosen such that $c \in (R_{\min}, R_{\max})$.

In [79] the following is proven:

Theorem 9 (R. Muñiz, A. Verjovsky). *Let g_0 be a laminated Riemannian metric of negative curvature (not necessarily constant) on a compact, connected, surface lamination \mathcal{L} . Let $g_t = g(t) = e^{ut} g_0$ be the leafwise solution to the normalized Ricci flow equation (21) on \mathcal{L} , with a constant $c \in (R_{\min}, R_{\max})$ and initial condition g_0 .*

The function $u_t = u(\cdot, t)$ belongs to $C^{\infty,0}(\mathcal{L})$. Furthermore, $\lim_{t \rightarrow \infty} g_t = g_\infty$ exists and defines a laminated metric of class $C^{\infty,0}$ with all leaves of constant negative curvature (rescaling we can assume that this constant curvature is -1). Furthermore, the metric g_∞ is conformally equivalent to g_0 .

Theorem 8 is a corollary of Theorem 9.

Corollary 5. *The hyperbolic metric g_∞ depends continuously on the initial metric g_0 . Therefore, if $\mathfrak{M}(\mathcal{L})$ denotes the Fréchet manifold of laminated metrics of class $C^{\infty,0}$ and $\text{Hyp}(\mathcal{L})$ denotes the closed subset of hyperbolic metrics, then $\mathfrak{M}(\mathcal{L})$ retracts strongly onto $\text{Hyp}(\mathcal{L})$ via the Ricci flow. Hence, there is a retraction map $\Pi: \mathfrak{M}(\mathcal{L}) \rightarrow \text{Hyp}(\mathcal{L})$ such that the fiber $\Pi^{-1}(\{g\})$ consists of laminated Riemannian metrics conformally equivalent to the hyperbolic metric g .*

Proof. The metric obtained by É. Ghys’ theorem (Theorem 7) can be chosen to depend continuously on the conformal class, i.e., given g with leaves of hyperbolic type there exists g_0 with leaves of negative curvature and the map $g \rightarrow g_0$ depends continuously (in the smooth topology of metrics) on g . The metrics $g_t = tg + (1 - t)g_0, 0 \leq t \leq 1$ are conformally equivalent to g . Therefore, we can strongly retract all Riemannian metrics on a compact hyperbolic lamination to metrics that are conformally equivalent to metrics with all leaves of negative curvature. Then, using the Ricci flow given by Theorem 9 we obtain a strong deformation retract to the space of hyperbolic metrics. ■

Corollary 6. *Let \mathcal{L} be a compact, connected, hyperbolic surface lamination and g any $C^{\infty,0}$ laminated Riemannian metric. Then using Theorem 9 we can endow \mathcal{L} with a $C^{\infty,0}$ laminated complex structure which turns \mathcal{L} into a Riemann surface lamination.*

All metrics g' conformally equivalent to g determine biholomorphic laminations. Reciprocally, if \mathcal{L} is a hyperbolic Riemann surface lamination it determines a conformal class of laminated metrics.

As in the case of compact Riemann surfaces it is possible to define the corresponding Teichmüller space:

Definition 33. The laminated Teichmüller space $T(\mathcal{L})$ of the hyperbolic compact lamination \mathcal{L} is defined as the set of equivalence classes $\text{Hyp}(\mathcal{L})/\sim$, where $\text{Hyp}(\mathcal{L})$ is as in

Corollary 5 and \sim is the equivalence relation $g_1 \sim g_2$ if there exists a laminated smooth diffeomorphism $f: M \rightarrow M$ which is isotopic to the identity (therefore it preserves each leaf) and f is an isometry on each leaf, i.e., g_2 is the *pushforward* by f of g_1 .

6.3. Universal hyperbolic solenoid

In [108] Dennis Sullivan starts the study of the Teichmüller space of hyperbolic Riemann surface laminations \mathcal{L} . For more information on this topic, see also the papers [21–23, 25, 71, 87, 88, 97, 98]. Because of the correspondence between conformal classes of metrics, complex structures, and hyperbolic structures of compact hyperbolic laminations, one can define the Teichmüller space of \mathcal{L} as the space of all continuous conformal structures along the leaves with transversally continuous Beltrami coefficients (Definition 29), modulo the group of quasi-conformal isotopies leaving each leaf invariant. There is a “laminated” Bers embedding theorem.

The natural topology on the Teichmüller space is Hausdorff, and it is biholomorphic to a nonempty open subset of the Banach space of holomorphic quadratic differentials along the leaves, which are continuous on M . In general, the Teichmüller space of a compact lamination is infinite-dimensional. In fact, in [32] B. Deroin proves that the Teichmüller space of a compact surface lamination with one simply connected leaf is infinite-dimensional.

Solenoidal surfaces are important due to their connections with complex analysis and complex dynamics [34, 44, 66, 67, 108].

Dennis Sullivan has constructed an important lamination whose Teichmüller space is remarkable: *The universal commensurability Teichmüller space*. The construction of *Sullivan’s universal hyperbolic solenoid* is based on profinite constructions. More precisely: if Σ is a compact orientable surface of genus $g \geq 2$ and if we consider the inverse limit corresponding to the tower of *all* the finite-sheeted coverings of Σ we obtain a 2-dimensional solenoidal manifold or surface lamination \mathcal{L}_h .

Remark 22. \mathcal{L}_h is the McCord solenoidal manifold which is the algebraic universal covering of Σ discussed in Section 3.3. However, following Dennis Sullivan, we call this solenoid *universal hyperbolic solenoid* to emphasize the fact that \mathcal{L}_h , together with its laminated complex or hyperbolic structures, encapsulates many of the features of Riemann surfaces (or, equivalently, hyperbolic surfaces).

We can consider complex structures in this lamination so that each leaf has a complex structure and the complex structures vary continuously in the transversal direction. There exists a canonical projection $\pi: \mathcal{L}_h \rightarrow \Sigma$. For a dense set of laminated complex structures, the restriction of π to each leaf is a conformal map onto Σ endowed with a complex structure independent of the leaf. Moreover, the inverse limit of a point $K_z := \pi^{-1}\{z\}$, $z \in \Sigma$ is a Cantor set. In this construction, one gets the same inverse limit using any co-final set of finite coverings, for instance, normal subgroups or even characteristic subgroups. In these cases, K_z is canonically a nonabelian Cantor group [23, 76, 98, 108].

The lamination \mathcal{L}_h is the suspension of a homeomorphism $\rho: \pi_1(\Sigma) \rightarrow \mathcal{H}(K)$ (see Section 3.4). If Σ is a surface of genus two we can consider a simple closed curve γ in Σ which separates the surface into two surfaces of genus one with boundary γ .

The restriction of the representation to γ produces an oriented 1-dimensional solenoid. Thus there exists four homeomorphisms f_1, f_2, g_1, g_2 of the Cantor group K_z such that $[f_1, f_2] = [g_1, g_2] := h$ and the 1-dimensional solenoid is the suspension of h . These four homeomorphisms of the Cantor set satisfying the commutator relations above determine the universal solenoid. Sullivan constructs the universal Teichmüller space of the “pointed” lamination \mathcal{L}_h obtained by taking the inverse limit of all finite pointed unbranched coverings of a compact surface of genus greater than one and a chosen base point. The base points upstairs in the coverings determine a point and a distinguished leaf called the *base leaf*.

Definition 34. The lamination \mathcal{L}_h defined in the previous paragraph as the inverse limit of the tower of pointed coverings of an orientable compact surface of genus 2 is called the *universal hyperbolic solenoid*. It was defined by Dennis Sullivan in [108]. Note that if we take the inverse limit of the tower of all regular finite covering surfaces over any, oriented, compact, surface Σ_g of genus $g \geq 2$ we obtain a lamination that is diffeomorphic to \mathcal{L}_h . The diffeomorphism is given by a finite iteration of the *crop map* (defined in Remark 3, Definition 11). Thus for each $g \geq 2$ there is a fibration

$$\Pi_g: \mathcal{L}_h \rightarrow \Sigma_g. \tag{22}$$

Theorem 10. *The space of hyperbolic structures on a hyperbolic compact solenoidal surface (as in Definition 33) up to isometries isotopic to the identity has the structure of a separable complex Banach manifold. The metric is the natural Teichmüller metric based on the minimal conformal distortion of a map between structures. The isotopy classes of homeomorphisms preserving a chosen leaf L (the base leaf) in \mathcal{L}_h act by isometries on this Banach manifold.*

This lamination is a *universal compact surface*, i.e., it is the universal object in the category of finite coverings of a compact surface. This is the first example of a Teichmüller space that is separable but not finite-dimensional. We recall that Teichmüller spaces of compact Riemann surfaces of finite type are finite-dimensional complex manifolds and that Teichmüller spaces of geometrically infinite Riemann surfaces are non-separable infinite-dimensional complex Banach manifolds.

In this space the commensurability automorphism group of the fundamental group of any higher genus compact surface acts by isometries. This group is independent of the genus by definition [23, 24, 87, 98, 108].

Theorem 11 (Sullivan). *The space of hyperbolic structures up to isometry preserving the distinguished leaf on this solenoidal surface \mathcal{L}_h is non-Hausdorff and any Hausdorff quotient is a point.*

The proof of this result relies on deep results by Jeremy Kahn and Vladimir Marković on the validity of the Ehrenpreis conjecture [55]. Sullivan’s observation is that the action of the *universal commensurability automorphism group of the fundamental group* is by isometries (and thus biholomorphic modular transformations) and the action is minimal. The action is described in [23] (see Remark 23 below).

6.4. The Earle–Eells theorem for the universal hyperbolic solenoid

The fiber bundle description of Teichmüller space of a smooth, compact orientable surface of genus $g \geq 2$ given in [36] has a version for the universal hyperbolic lamination \mathcal{L}_h in Definition 34.

Let $\mathcal{D}_0(\mathcal{L}_h)$ be the topological group of diffeomorphisms of \mathcal{L}_h which are isotopic to the identity (hence they preserve each leaf). Endow this group with the C^∞ -topology of uniform convergence of differentials of all orders. The group becomes a Fréchet manifold [64].

Let $M(\mathcal{L}_h)$ be the set of all smooth complex structures on \mathcal{L}_h compatible with the standard orientation of \mathcal{L}_h induced by an orientation on the surface of genus 2 and give $M(\mathcal{L}_h)$ the C^∞ -topology. Then (viewing the elements of $M(\mathcal{L}_h)$ as smooth tensor fields on \mathcal{L}_h) we have a natural right action: $M(\mathcal{L}_h) \times \mathcal{D}_0(\mathcal{L}_h) \rightarrow M(\mathcal{L}_h)$, via pullback. Recall that a complex structure on \mathcal{L}_h is a smooth bundle automorphism $J: T(\mathcal{L}_h) \rightarrow T(\mathcal{L}_h)$, such that for each $x \in \mathcal{L}_h$ the linear map $J_x: T_x(\mathcal{L}_h) \rightarrow T_x(\mathcal{L}_h)$ satisfies $J_x^2 = -I_x$ (I_x is the identity map on $T_x(\mathcal{L}_h)$).

The structure $J \in M(\mathcal{L}_h)$ defines a conformal class of laminated Riemannian metrics. These are those metrics such that for every point $x \in \mathcal{L}_h$ and each non-zero vector $V_x \in T_x(\mathcal{L}_h)$, the vectors V_x and $J_x(V_x)$ are orthogonal.

The uniformization theorem (Theorem 9) associates to each conformal class of metrics in \mathcal{L}_h a unique laminated hyperbolic metric with all leaves of curvature -1 .

Reciprocally, the laminated global isothermal coordinates of Corollary 6 associate a complex structure on the lamination for each hyperbolic metric in $\text{Hyp}(\mathcal{L}_h)$ (the set of hyperbolic metrics on \mathcal{L}_h). If $\text{Conf}(\mathcal{L}_h)$ denotes the set of conformal equivalence classes of laminated metrics on \mathcal{L}_h , we have canonical identifications:

$$\text{Conf}(\mathcal{L}_h) \sim M(\mathcal{L}_h) \sim \text{Hyp}(\mathcal{L}_h).$$

With these identifications each of the sets becomes a Fréchet manifold.

Theorem 12 (Fiber bundle description of the Teichmüller space of \mathcal{L}_h). *The following holds:*

- (1) $\text{Hyp}(\mathcal{L}_h)$ is a contractible complex analytic manifold modeled on a Fréchet space.
- (2) $\mathcal{D}_0(\mathcal{L}_h)$ acts (via pullback) continuously, effectively, and properly on $\text{Hyp}(\mathcal{L}_h)$.
- (3) The quotient map

$$\Phi: \text{Hyp}(\mathcal{L}_h) \rightarrow \mathcal{T}(\mathcal{L}_h) = \text{Hyp}(\mathcal{L}_h) / \mathcal{D}_0(\mathcal{L}_h) (= M(\mathcal{L}_h) / \mathcal{D}_0(\mathcal{L}_h)),$$

where $\mathcal{T}(\mathcal{L}_h)$ the space of orbits (endowed with the quotient topology) is a universal principal $\mathcal{D}_0(\mathcal{L}_h)$ bundle.

(4) Both $\mathcal{D}_0(\mathcal{L}_h)$ and the Teichmüller space $\mathcal{T}(\mathcal{L}_h)$ are contractible.

Proof. The proof uses harmonic maps and follows the steps in [36, Section 8E].

Let $g_0, g_1 \in \text{Hyp}(\mathcal{L}_h)$, be two hyperbolic metrics. Let $f \in \mathcal{D}_0(\mathcal{L}_h)$ and consider the Dirichlet energy of f with respect to the two metrics, i.e., $f: (\mathcal{L}_h, g_1) \rightarrow (\mathcal{L}_h, g_0)$:

$$E(f) = \frac{1}{2} \int_{\mathcal{L}_h} \|df(x)\|^2 d\mu_{g_0}(x),$$

where $d\mu_{g_0}$ is the laminated volume form (Definition 18) on \mathcal{L}_h with respect to g_0 , and $\|\cdot\|$ the Hilbert–Schmidt norm with respect to g_0 and g_1 (15).

By Theorem 5 there exists a unique harmonic map $f_{(g_1, g_0)}: (\mathcal{L}_h, g_1) \rightarrow (\mathcal{L}_h, g_0)$ homotopic to f and depending continuously on (g_1, g_0) . An adaptation of a theorem by R. Schoen and S. T. Yau in [100] implies that $f_{(g_1, g_0)}$ is a diffeomorphism. Their proof is for compact surfaces with negative curvature but it can be applied in our case because their proof uses the fact that the Jacobian has only isolated zeros and the maximum principle shows that the Jacobian does not vanish at any point.

Therefore, the Jacobian of $f_{(g_1, g_0)}$ does not vanish and it must be a local (laminated) diffeomorphism. The fact that \mathcal{L}_h is compact and connected and that $f_{(g_1, g_0)}$ is homotopic to the identity implies it is a global diffeomorphism.

In what follows we fix the hyperbolic metric g_1 .

Let

$$F: \text{Hyp}(\mathcal{L}_h) \rightarrow \mathcal{D}_0(\mathcal{L}_h)$$

be the map $F(g_0) = f_{(g_1, g_0)}$. If $h \in \mathcal{D}_0(\mathcal{L}_h)$ and $g_0 \cdot h$ is the pullback under h of the metric g we have that $h: (\mathcal{L}_h, g_0 \cdot h) \rightarrow (\mathcal{L}_h, g_0)$ is an isometry. The post-composition of a harmonic map with an isometry is a harmonic map, therefore by uniqueness, we have the commutative diagram:

$$\begin{array}{ccc} (\mathcal{L}_h, g_1) & \xrightarrow{f_{(g_1, g_0)}} & (\mathcal{L}_h, g_0) \\ & \searrow f_{(g_1, g_0 \cdot h)} & \uparrow h \\ & & (\mathcal{L}_h, g_0 \cdot h), \end{array}$$

so that

$$f_{(g_1, g_0)} = h \circ f_{(g_1, g_0 \cdot h)}, \quad \forall h \in \mathcal{D}_0(\mathcal{L}_h). \tag{23}$$

Define $\Psi: \text{Hyp}(\mathcal{L}_h) \rightarrow \mathcal{T}(\mathcal{L}_h) \times \mathcal{D}_0(\mathcal{L}_h)$ by the formula $\Psi(g_0) = (\Phi(g_0), f_{(g_1, g_0)}^{-1})$. The function Ψ is continuous. The group $\mathcal{D}_0(\mathcal{L}_h)$ acts on $\mathcal{T}(\mathcal{L}_h) \times \text{Hyp}(\mathcal{L}_h)$ as follows:

$$(\tau, g) \xrightarrow{h} (\tau, g \circ h) := (\tau, g) \cdot h.$$

Equation (23) implies

$$\Psi(g_0 \cdot h) = \Psi(g_0) \cdot h = (\Phi(g_0), f_{(g_1, g_0)}^{-1} \circ h), \quad \forall h \in \mathcal{D}_0(\mathcal{L}_h). \tag{24}$$

The function Ψ is injective: if $\Psi(g_1) = \Psi(g_2)$ then $\Phi(g_1) = \Phi(g_2)$. Therefore, there exists $h \in \mathcal{D}_0(\mathcal{L}_h)$ such that $g_2 = g_1 \cdot h$ so that by equation (24) $\Psi(g_1) = \Psi(g_2) = \Psi(g_1) \cdot h$ and therefore $h = \text{Id}$ and $g_1 = g_2$. On the other hand, Ψ is surjective: given $(g, h) \in \text{Hyp}(\mathcal{L}_h) \times \mathcal{D}_0(\mathcal{L}_h)$ one has

$$(\Phi(g), h) = \Psi(g) \cdot (f_{(g_1, g)} \circ h) = \Psi(g \cdot (f_{(g_1, g)} \circ h)),$$

since $\Phi(g) = \Phi(g \cdot (f_{(g_1, g)} \circ h))$. From this we conclude that Ψ is a homeomorphism, and Φ is a principal fiber bundle over $\mathcal{T}(\mathcal{L}_h)$ with fiber the Fréchet group $\mathcal{D}_0(\mathcal{L}_h)$.

Formula (24) establishes a bundle isomorphism between Φ and the trivial bundle $p_1: \mathcal{T}(\mathcal{L}_h) \times \mathcal{D}_0(\mathcal{L}_h) \rightarrow \mathcal{T}(\mathcal{L}_h)$. An explicit section $\sigma: \mathcal{T}(\mathcal{L}_h) \rightarrow \text{Hyp}(\mathcal{L}_h)$ is given by the formula: $\sigma(\Phi(g)) = \Psi^{-1}(\Phi(g), \text{Id}) = g \cdot f_{(g_1, g)}$.

The space of all smooth Riemannian metrics on \mathcal{L}_h is a convex set in the Fréchet space of sections of tensors of type $(0, 2)$, since a convex linear combination of Riemannian metrics on \mathcal{L}_h is a Riemannian metric.

In addition, each conformal class of Riemannian metrics is also convex. In each of these conformal classes, there is a unique hyperbolic metric and this metric varies continuously with the conformal class. Therefore, $\text{Hyp}(\mathcal{L}_h)$ is contractible. Hence, the fibration $\Phi: \mathcal{T}(\mathcal{L}_h) \rightarrow \mathcal{T}(\mathcal{L}_h)$ is universal. The existence of the locally trivial fiber bundle map Φ implies that both $\mathcal{T}(\mathcal{L}_h)$ and $\mathcal{D}_0(\mathcal{L}_h)$ are contractible. ■

Remark 23. The Teichmüller theory of \mathcal{L}_h has been studied from the viewpoint of quasi-conformal mappings in [71, 88, 97]. See also the results in [21–23, 25, 87] and, of course, [108].

A very important part of the Teichmüller theory of \mathcal{L}_h is played by the transversely *locally constant complex structures* (TLC). Particular examples of these are obtained by taking any compact Riemann surface of an arbitrary genus and lifting the complex structure to the leaves of \mathcal{L}_h using the projection Π_g in Remark 22, Definition 4. By [108] these TLC structures constitute a dense set of the universal Teichmüller space of \mathcal{L}_h .

The notions of *mapping class group* and *base leaf mapping class group* exist for the solenoidal surface \mathcal{L}_h and play an important role. The *abstract virtual automorphism group* (*universal commensurability automorphism group*) of the fundamental group of a compact surface of genus 2 provides a solenoidal version of Nielsen’s theorem that intertwines the fundamental group of a surface and its mapping class group. See, for instance, the paper by Chris Odden [87] as well as [21, 23, 24, 71, 108].

6.5. The universal 2-dimensional euclidean solenoid

Theorem 6 has a higher-dimensional analog for the topologically transitive McCord solenoids which are obtained by an infinite tower of coverings of tori:

Theorem 13. *Let \mathcal{S} be a compact McCord solenoid which is obtained as the inverse limit of finite coverings of an n -torus $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ (we call it a McCord solenoid over a torus). Then \mathcal{S} is a compact connected abelian group. The Pontryagin dual of \mathcal{S} is a*

dense subgroup of the additive group of the vector space \mathbb{Q}^n over \mathbb{Q} . There is a bijective correspondence between the n -dimensional McCord solenoids over \mathbb{T}^n and isomorphic classes of dense subgroups of $(\mathbb{Q}^n, +)$.

Proof. The proof is the same as the proof of Theorem 6. Every covering map $p: \mathbb{T}^n \rightarrow \mathbb{T}^n$ is regular and it is a surjective endomorphism corresponding to the image of an injective homomorphism $\varphi: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$, of the fundamental group of $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$. The injective homomorphism φ is determined by a nonsingular $n \times n$ matrix $A = (a_{ij})$, with integer coefficients. If $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, then the endomorphism induced by A is given by:

$$(z_1, \dots, z_n) \mapsto (z_1^{a_{11}} z_2^{a_{12}} \cdots z_n^{a_{1n}}, \dots, z_1^{a_{j1}} z_2^{a_{j2}} \cdots z_n^{a_{jn}}, \dots, z_1^{a_{n1}} z_2^{a_{n2}} \cdots z_n^{a_{nn}}).$$

Therefore, the inverse limit is a compact abelian group, as the bonding maps are group homomorphisms between compact abelian groups. Since the topological dimension of \mathcal{S} is n , it follows that its Pontryagin dual is a subgroup of the additive group Γ of \mathbb{Q}^n . Since the inverse limit is a solenoid it follows that Γ must be dense in \mathbb{Q}^n . ■

Again by [99], to non-isomorphic, additive, dense subgroups of \mathbb{Q}^n correspond non-homeomorphic n -dimensional solenoids. The problem of isomorphic classes of dense subgroups of \mathbb{Q}^d has been addressed in [45, Theorem 1.5]. In particular, when $\Gamma = \mathbb{Q}^n$ we have the algebraic universal covering solenoid of \mathbb{T}^n :

Definition 35. $\widehat{\mathbb{T}}^n$ will denote the algebraic universal covering of \mathbb{T}^n , i.e., the inverse limit of all its finite coverings.

Since $\widehat{\mathbb{T}}^n$ is the Pontryagin dual of $\mathbb{Q}^n = \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$ and the dual of \mathbb{Q} is the algebraic universal covering of the circle $\widehat{\mathbb{S}}^1$ we have:

Proposition 12. $\widehat{\mathbb{T}}^n$, the algebraic universal covering of \mathbb{T}^n , is isomorphic to the product $\widehat{\mathbb{S}}^1 \times \cdots \times \widehat{\mathbb{S}}^1$ (n factors). The algebraic fundamental group of $\widehat{\mathbb{T}}^n$ is $\widehat{\mathbb{Z}}^n = \widehat{\mathbb{Z}} \times \cdots \times \widehat{\mathbb{Z}}$, with $\widehat{\mathbb{Z}}$ the profinite completion of the integers.

Therefore, \mathbb{Z}^n acts freely and properly on $\mathbb{R}^n \times \widehat{\mathbb{Z}}^n$. Using the notation in Proposition 3 we have:

$$\widehat{\mathbb{T}}^n = \mathbb{Z}^n \backslash (\mathbb{R}^n \times \widehat{\mathbb{Z}}^n) = \widehat{\mathbb{S}}^1 \times \cdots \times \widehat{\mathbb{S}}^1 \quad (n \text{ factors}).$$

In particular for the special case $n = 2$ we have:

Definition 36 (Universal euclidean solenoidal surface). $\widehat{\mathbb{T}}^2$, the algebraic universal covering of the 2-torus, will be called the universal euclidean solenoidal surface. $\widehat{\mathbb{T}}^2 = \widehat{\mathbb{S}}^1 \times \widehat{\mathbb{S}}^1$ and its algebraic fundamental group is $\widehat{\pi_1(\mathbb{T}^2)} = \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$. It is the Pontryagin dual of $(\mathbb{Q}^2, +)$. The component of the identity is called the base leaf.

Remark 24. As shown in Theorem 6, the classification of compact topologically homogeneous 1-dimensional solenoids is equivalent to the classification of the dense subgroups

of the additive rationals and there is a good understanding of these subgroups due to Baer [17]. One may try to classify the n -dimensional solenoids described in Theorem 13 for $n > 1$ but this could be an impossible task. The classification is equivalent to the classification of abelian, torsion-free, groups of finite rank $n > 1$, whose Pontryagin dual is a solenoid, i.e., dense subgroups of the additive group $(\mathbb{Q}^n, +)$. For $n > 1$, there is no hope for a classification, the complexity increases with n . We refer to work due to S. Adams [6, 16], G. Hjorth [50], A. S. Kechris [58] and S. Thomas [112]. They work in the context of both abelian group theory and countable Borel equivalence relations and provide examples of countable Borel equivalence relations as well as a new way of thinking about the complexity of the classification problem for torsion-free abelian groups of finite rank.

Remark 25. By Bieberbach's theorem, given a compact flat n -manifold N with a flat metric g , there exists a flat n -torus $(\mathbb{T}^n, \tilde{g})$ and a locally isometric covering projection $\pi_1: \mathbb{T}^n \rightarrow N$. If $\mathcal{S}_{\mathbb{T}^n}^n$ is the solenoid obtained from an infinite tower of regular coverings of $(\mathbb{T}^n, \tilde{g})$ then we can endow $\mathcal{S}_{\mathbb{T}^n}^n$ with the metric \hat{g} obtained by lifting the metric on the leaves by the canonical projection $\Pi: \mathcal{S}_{\mathbb{T}^n}^n \rightarrow \mathbb{T}^n$.

The metric \hat{g} renders all leaves flat and it is homogeneous: the group of isometries of $(\mathcal{S}_{\mathbb{T}^n}^n, \hat{g})$ acts transitively. This is an example of a *compact, homogeneous, flat n -dimensional solenoid*. In fact, it is a compact n -dimensional abelian group.

Remark 24 implies that the task of classifying flat, homogeneous, n -dimensional solenoids is hopeless if $n > 1$. However, the algebraic universal covering of a flat torus cover, in the solenoidal sense, every such flat solenoid: all are of the form $(\mathbb{R}^n \times \widehat{\mathbb{Z}}^n)/(\mathbb{Z}^n \times \Gamma_\alpha)$ where Γ_α is a closed subgroup of $\widehat{\mathbb{Z}}^n$ with quotient $\widehat{\mathbb{Z}}^n/\Gamma_\alpha$ a Cantor group.

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