EMS Surv. Math. Sci. 10 (2023), 307–319
DOI 10.4171/EMSS/73

On the Whitehead nightmare and some related topics

Valentin Poénaru

Abstract. In this very fast survey I will start by reviewing my work on the QSF property in geometric group theory. While working on this topic, I met some infinitistic complications which I called the *Whitehead nightmare*. This has analogues in other areas of low-dimensional topology. Afterwards I will present some joint work in progress with Louis Funar and Daniele Otera, where we plan to show that all groups, and only finitely presented groups are concerned by this paper, can avoid the nightmare in question. And also that all groups have the GSC (geometric simple connectivity) property, much stronger than QSF. Both QSF and GSC are explained in the main text.

To Dennis Sullivan, for his 80th birthday

1. Memories

I could tell you a lot about my friend Dennis, whom I first met around 1964. But instead of telling you about the magnificent contributions which he brought to so many and diverse fields of mathematics, I will only present you some personal memories concerning our friendship.

I really came close to him during the conference "Manifolds Amsterdam 1970," organized by our common friend the late Nico Kuiper, where we were both speakers. And I spent a lot of time together with Dennis and his then big love Lucy, feeling like in a magic fairyland, going together with Tristan and Isolde.

Few years later Dennis moved to the Paris region, he became a permanent professor at the IHES, and then we really saw each other a lot. I must say that a large part of the maths I know, I have learned it from him, during innumerable informal conversations. And his very charismatic presence to our various parties, in our home was a big festive thing. I may add that Dennis and Barry Mazur who, often, together, brought big joy to our gatherings, are the most charismatic people I know. *Happy birthday, Dennis!*

2020 Mathematics Subject Classification. Primary 57M07; Secondary 57M20, 57M35, 20F65.
Keywords. Discrete groups, geometric simple connectivity, REPRESENTATIONS.

2. Introduction

Few years ago I managed to complete the proof that all groups possess the property called QSF. This property was discovered by Steve Brick and further studied by Brick, Mihalik and Stallings (see [2, 23]). I will soon redefine QSF, but it suffices to say right now that for fundamental groups of 3-manifolds, QSF is equivalent to simple connectivity at infinity.

In the quest for proving QSF, I identified a certain kind of infinitistic complication, a big difficulty for the proofs, which I called the *Whitehead nightmare*. But before I can tell you what that is all about, I have to introduce one of my tools in geometric topology (and low-dimensional topology), the REPRESENTATIONS, not to be mixed up with the well-known group representations.

3. REPRESENTATIONS

And now I will tell you very briefly what REPRESENTATIONS are, and for more details see [8, 10, 11, 13, 19]. The REPRESENTATION map is a non-degenerate simplicial map

$$f: Y \to X^3 \tag{1}$$

where Y is a GSC complex, in the sense explained in Section 5, of dimension 2 or 3 and X^3 is the space to be REPRESENTED.

The REPRESENTATION SPACE Y may not be locally finite, and to get it to be so, when one needs it, requires hard work. In a nutshell, here are the features of (1):

(1-1) The map f is immersive, except at some singularities called mortal singularities. We call *mortal* the singularities of a map, killed by the map in question, and *immortal* the singularities of spaces to be represented, like a singular manifold $M(G)$ or its universal cover $\widetilde{M}(G)$, see below.

(1-2) In $Y \times Y$ every double point of f,(x, y), can be joined to the singularities by a *zipping path* of double points.

(1-3) In a sense which is defined in [8, 19], f is "essentially surjective."

With a lot more details, here is what the (1-2) above means. Let us consider a general, simplicial map

$$f: Y \to X$$

between cell complexes. There is then, to begin with, an obvious equivalence relation $\Phi(f) \subset Y \times Y$, where

$$(x, y) \in \Phi(f) \iff f(x) = f(y).$$

But then, there is also a more subtle equivalence relation $\Psi(f) \subset Y \times Y$. Schematically speaking $\Psi(f)$ is the smallest equivalence relation, compatible with f (i.e. such that

$\Psi(f) \subset \Phi(f)$) which kills all the mortal singularities of f, meaning the non-immersive points.

Hence, in the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad f \quad} & X \\ & \searrow \quad \nearrow_{f_1} & \\ & Y/\Psi(f) & \end{array}$$

the map f_1, naturally induced by f, is an immersion. So the f_1 is singularity-free *and*, moreover, no equivalence relation smaller that $\Psi(f)$ can do the job.

In the papers [8, 10, 13], one can find the proofs for the existence and unicity of our $\Psi(f)$.

With all these things, the existence of that zipping path in (1-2) above means:

$$\Psi(f) = \Phi(f).$$

In plain English, this also means that the cheapest way to kill all the mortal singularities is to kill all the double points. Recall that our terminology here is “mortal singularities” for the singular points of f at the source Y and “immortal singularities” for the non-manifold points of the target X.

It can be shown that if Y is simply connected, in particular if it is GSC (*geometrically simply connected*, see Section 5), then

$$\pi_1(X) = 0.$$

So, coming back to REPRESENTATIONS, the space to be REPRESENTED has to be simply connected. Several simply-connected 3-dimensional spaces can be REPRESENTED (and the very good serious reasons why I will work with the 3-dimensional group presentations $M(G)$ are presented in [19]). I actually started using REPRESENTATIONS of homotopy 3-balls in my program for proving the 3-dimensional Poincaré conjecture.

And the Whitehead manifold Wh^3 can be REPRESENTED too. Strange things happen then, to which I will come back.

When it comes to geometric group theory, then I am interested in REPRESENTATIONS

$$X \longrightarrow \widetilde{M}(G) \tag{2}$$

where X is 2-dimensional or 3-dimensional and where $M(G)$ is a 3-dimensional presentation of G, and in all this story, and the related ones [8, 13–15, 18, 19], the standard well-known 2-dimensional group presentations are totally inadequate (see [18]).

It so happens that REPRESENTATIONS like (2) are very easy to construct, see [19], but the easy approach only yields REPRESENTATION spaces which are not locally finite, hence totally useless.

Notice, finally, that while the standard, well-known, group representations go like $G \to \{\text{some other group}\}$, our REPRESENTATIONS go the other way around

$$Y \to \widetilde{M}(G) \approx G \text{ (quasi-isometry)}.$$

Then, about ten years ago I proved the following:

Theorem 1 (Po, [13]). *For any finitely presented group G we can find a 3-dimensional* REPRESENTATION

$$f\colon X^3 \longrightarrow \widetilde{M}(G), \tag{3}$$

with the following features:

(1) *The complex X^3 is locally finite.*

(2) *The group G acts freely on X^3, $G \times X^3 \to X^3$, and f is equivariant, meaning that for any $x \in X^3$ and $g \in G$, we have $f(gx) = g(f(x))$.*

(3) *The* zipping length *is uniformly bounded. This means that there is a constant $C > 0$ such that for any $(x, y) \in M^2(f)$ we have*

$$\inf_{\lambda}\{\textit{length of the zipping path } \lambda(x, y)\} < C.$$

We use here the following notation. If $\phi\colon A \to B$ is a map, then $M_2(\phi) \subset A$ is the set of double points, meaning that if $x \in M_2(\phi)$, then $\mathrm{card}\phi^{-1}(x) > 1$. And $M^2(\phi) \subset A \times A$ means pairs of points (x, y) such that $x \neq y$ and $\phi(x) = \phi(y)$.

Theorem 1 is proved in [13], which I call *Trilogy I* since it is the first of the trilogy of papers [13–15] which prove that all groups are QSF.

Now, if in the context of Theorem 1, X^2 is an appropriate, very dense 2-skeleton of X^3, then the 2-dimensional REPRESENTATION

$$f\colon X^2 \to \widetilde{M}(G) \tag{4}$$

also has the three features from the theorem.

4. The QSF property and the Whitehead nightmare

By definition, a locally compact complex X is QSF, if for any compact $k \subset X$ there is a commutative diagram

$$\begin{array}{ccc} k & \overset{i}{\hookrightarrow} & X \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle f} & \\ & K & \end{array}$$

where K is some abstract simply-connected finite simplicial complex, j is an inclusion map, and f is a simplicial map. Moreover, the following Dehn-type condition is fulfilled

$$j(k) \cap M_2(f) = \emptyset.$$

QSF stands for quasi-simply filterable. This notion was abstracted from earlier work of Casson [24] and Po by Steve Brick [1, 2].

In particular, in my older work on $\pi_1^\infty \widetilde{M}^3$, [11, 12], I had introduced the notion of Dehn exhaustibility, which is a direct fore-runner of QSF. So, QSF has its roots in the old, classical Dehn's lemma.

And then, Brick, Michael Mihalik and the late John Stallings proved the following theorem (see [2, 23]):

Theorem 2 (Brick–Mihalik, [2]). *The following statements hold:*

- *For open 3-manifolds QSF is equivalent to* $\pi_1^\infty = 0$.
- *If* K_1 *and* K_2 *are finite simplicial complexes with the same* $\pi_1 = G$*, then*

$$K_1 \in QSF \iff K_2 \in QSF,$$

and we will say that G *is QSF.*

So, QSF is a group theoretical property, independent of the presentation of G. This is one of its important features.

And here comes now the following:

Theorem 3 (QSF theorem, Po). *All finitely presented groups are QSF.*

This is proved in the Trilogy [13–15]. But see also the rather detailed surveys [8, 18] which can be very useful.

Notice, to begin with, that the QSF is, now, at the same time universally true for all groups, and highly non-trivial, since it implies Perelman's $\pi_1^\infty \widetilde{M}^3 = 0$, consequence of the geometrization of 3-manifolds, which requires even more steam of the Ricci flow than the Poincaré conjecture (see [5, 6]). And this combination goes against standard wisdom.

And I have an "esoteric conjecture," to which I will come back, which is supposed to resolve this tension, which at least in part has motivated it.

For the proof of the QSF theorem we have to move in the very rarefied context of all groups, with no special features to hang on to, and just use the following mantra:

Discrete symmetry with compact fundamental domain.

The proof of the QSF theorem starts with the REPRESENTATION provided by Theorem 1 and then, right away, we meet the following difficulty, the *Whitehead nightmare*, in a group-theoretical context.

We want to inspect the group G, in its quasi-isometric disguise $\widetilde{M}(G)$, by a REPRESENTATION map of source the GSC space Y. And $f(Y)$ has to put its nose in every hook and nook of $\widetilde{M}(G)$. Now, when the nightmare is there, then this happens *infinitely* many times in every little spot. And then things get hard.

The "Whitehead nightmare" is a terminology which I concocted many years ago, during a walk under the starry sky, in a cold winter night. It refers to a phenomenon which occurs for the Whitehead manifold Wh^3, for the Casson Handles, for the gropes of Stanko, and which is also present in the work of Mike Freedman and Frank Quinn.

Here is a more precise formulation of what is at stake. In the context of (3), generally speaking, any compact $K \subset \widetilde{M}(G)$ is touched *infinitely* many times by f. Equivalently, in the context of (4), none of the following two subsets is *closed*:

$$f(X^2) \subset \widetilde{M}(G) \quad \text{and} \quad M_2(f) \cup \operatorname{Sing}(f) \subset X^2. \tag{5}$$

There is actually the following fact (see [14]):

Lemma 1. *The following two items are equivalent:*

(1) *For the 3-dimensional* REPRESENTATION (3) *every compact* $K \subset \widetilde{M}(G)$ *is touched only finitely many times by* f.

(2) *For the 2-dimensional* REPRESENTATION (4) *the two subsets* (5) *are closed.*

If a group G is like in the lemma above, i.e. if it manages to avoid the Whitehead nightmare, we will call it "*easy*." And do not mix this technical meaning of that word, with its standard use. It is actually easy to show that any "easy" group is QSF (see [7]). And in this sentence the first easy is meant the normal way, while the second one is with the technical meaning.

Actually, in the old days when Andrew Casson and I, independently of each other, were working on $\pi_1^\infty \widetilde{M}^3$, neither the QSF notion nor the other concepts like "easy" group etc. existed, but if one looks with today's eyes at what we did then (in [11, 12, 24] etc.), then one can see the following. Without any connection to 3-manifolds, or any reference to them, we had actually proved that many classes of groups, like Gromov's hyperbolic, Cannon's almost convex, Thurston's combable, automatic, and others, were "easy" and hence QSF.

Now, when one wants to prove the QSF theorem for arbitrary groups, these may *not* be "easy" and one has to face the Whitehead nightmare. This is a difficulty one has to live with.

In a later section I will give a glimpse into the proof of the QSF theorem and then, I will very briefly describe some recent developments. Together with Louis Funar and Daniele Otera we have a research program which is a direct outgrowth of the proof of the QSF theorem. Twisting around the technology of the Trilogy, we hope to be able to go far beyond QSF.

But before I can go into all that, I have to say something on geometric simple connectivity (GSC), a crucial concept for all these topics.

5. A digression on geometric simple connectivity (GSC)

Everybody knows that GSC means that "the 1-handles are cancelled by the 2-handles," like in Steve Smale's proof of the high-dimensional Poincaré conjecture. But in our context of non-compact, singular manifolds with non-empty boundary, we have to be more precise (see [13]). Our cell-complexes X^n come endowed with handlebody decompositions into handles H^λ of index λ

$$X^n = N^n(T) + \sum_{i \in I} H_i^1 + \sum_{j \in \mathcal{J}} H_j^2 + \sum H^{\lambda \geq 2}, \tag{6}$$

where $N^n(T)$ is the regular neighbourhood of an infinite tree T.

Here we have a canonical isomorphism $I \approx \mathcal{J}$ and with (6) comes a geometric intersection matrix

$$H_j^2 \cdot H_i^1 = \delta_{ji} + a_{ji}, \tag{7}$$

where $H_j^2 \cdot H_i^1$ is the number of times the attaching zone ∂H_j^2 goes through the lateral surface δH_i^1. So here $a_{ji} \in \mathbb{Z}_+$ and there are no $\pm$ signs involved. And our X^n is GSC if the matrix (7) is of the form:

easy id + nilpotent, meaning that $a_{ji} > 0$ implies $j > i$.

The Wh^3, which is *not* GSC, has a geometric intersection matrix of the form difficult id + nilpotent, see [17]. Here $a_{ji} > 0$ implies $j < i$.

Notice at this point that in the compact (i.e. finite) case difficult and easy id + nilpotent are equivalent; so we can give now a *general definition of GSC*, valid both in the compact and the non-compact case.

By definition, a handlebody or cell-complex is GSC if for a handlebody decomposition like (6), with an isomorphism $I \approx \mathcal{J}$, the geometric intersection matrix is easy id + nilpotent; in the compact case this simply means id + nilpotent.

Now, since GSC is very close to my heart, I feel like saying a few more words. To begin with, clearly GSC implies $\pi_1 = 0$. And here is the answer for the converse implication, in the case of compact smooth n-manifolds:

(i) For $n \geq 5$, answer YES (Smale).

(ii) For $n = 4$, answer NO. Andrew Casson proved that the old Mazur–Po manifolds [4,9] provide counterexamples. Casson never wrote this down, but I learned it from Mike Freedman.

(iii) For $n = 3$, answer YES (Grisha Perelman's famous work, see [5,6]).

Retain that *dimension four is very special*, and here is a (partial?) list of reasons why:

(A) The (ii) above.

(B) Then, there is a big gap between DIFF and TOP, totally beyond the control of algebraic topology, exactly in dimension four. We know this from the work of Donaldson and Freedman, combined with the Yang–Mills equations. Incidentally, my interest in the smooth 4-dimensional Schoenflies (see, for instance, [16,20]) is not unrelated with my interest in that gap.

(C) Connected to point (B), it is only in dimension four that the equations of Yang–Mills, and hence Maxwell too, make sense. We need that the Hodge dual of a 2-form should be, again, a 2-form.

We listed above our YES/NO answers for the question whether simple connectivity implies GSC, for the case of compact manifolds. The situation is very much the same in the open case, provided one adds the hypothesis "simply connected at infinity." But when we are in the case of non-compact manifolds with non-empty boundary, then there are no general YES/NO theorems, and when one still needs to prove something, this requires hard work. This is the murky context of my efforts in geometric group theory, since many years.

We end this digression on GSC with the following item, of direct interest for this survey paper. There is an implication

$$GSC \implies QSF.$$

6. A very fast glimpse into the proof of the QSF theorem

We start with the 3-dimensional REPRESENTATION provided by Theorem 1, i.e. by Trilogy I, and then, going to an appropriate 2-dimensional skeleton X^2 of X^3, we get the 2-dimensional REPRESENTATION

$$X^2 \longrightarrow \widetilde{M}(G).$$

And here we meet already the Whitehead nightmare. Here is the simplest kind of occurrence.

In our present context, the fact is that, generally speaking, our $M_2(f) \cup \mathrm{Sing}(f)$ is *not* a closed subspace of X^2. As it will be soon explained, this is one of the main difficulties for proving the QSF theorem. And since we have to live with it, we start by constructing a REPRESENTATION $f : X^2 \to \widetilde{M}(G)$, where although not closed, $M_2(f)$ should still have an accumulation pattern, as mild as still possible. To see what this means, let us start with the simplest totally non-singular situation when, locally, X^2 is a copy of $\mathbb{R}^2$ and any connected component of $M_2(f)$ is a copy of $\mathbb{R}$. Inside this $\mathbb{R}^2 \subset X^2$ the $M_2(f) \cap \mathbb{R}^2$ consists now of a collection of parallel lines, normally infinitely many of them. This should remind the reader of a one-dimensional foliation $\mathcal{F}$ or a one-dimensional lamination $\mathcal{L}$ of $\mathbb{R}^2$. These have *transversal structures*, which are $\mathbb{R}$ in the case of $\mathcal{F}$ or a Cantor set in the case of $\mathcal{L}$.

In the case of $M_2(f) \cap \mathbb{R}^2$, once this is *not* a closed subset, we can construct REPRESENTATIONS (4) which realize, as transversal structures go, *the next best thing*, once $M_2(f)$ is *not closed* (or, put otherwise, the least harmful thing). And this next best thing is a countable set which has only a *finite set of points of accumulation*. This kind of transversal structure for $M_2(f) \subset X^2$ is realized in [13] by a process which I have called the "decantorianization." But then, this structure is only the simplest of the local complications with which we have to live. More complicated ones, not to be described here, are necessary too. But the important point is that we can live with them, we can prove the QSF theorem, with them being around.

And before we continue with the proof of the QSF theorem, let us look at the simplest 2-dimensional REPRESENTATION for the Whitehead manifold Wh^3,

$$Y^2 \longrightarrow \mathrm{Wh}^3.$$

Now the nightmare in unavoidable. Even the worst things happen. The accumulation pattern transversal to $M(f)$ has a dynamics which is governed by the Julia sets [8, 22]; remember the infinite iteration of quadratic maps, the Mandelbrot set, etc.

And now back to the proof of the QSF theorem. We move to Trilogy II [14]. It starts with a ballet between 2d and 3d:

$$\begin{aligned}\text{a 3d REPRESENTATION} &\Rightarrow \text{ a 2d REPRESENTATION } X^2 \to \widetilde{M}(G)\\ &\Rightarrow \text{ another 3d REPRESENTATION } \Theta^3(X^2) \to \widetilde{M}(G).\end{aligned}$$

Here $\Theta^3(X^2)$ is some singular 3-dimensional, very thin, thickening of X^2. But $\Theta^3(f(X^2)) = f(\Theta^3(X^2))$ is not locally finite, and here the game starts.

We construct a locally finite, singular and very high-dimensional space

$$\mathcal{S}_u \widetilde{M}(G), \tag{8}$$

which is {a singular high-dimensional thickening of $\Theta^3(X^2)$ from which various things are deleted, i.e. sent to infinity} + {some more non-compact additions}.

And the main result of Trilogy II is the following:

Theorem 4 (GSC theorem, Po, [14]). *$\mathcal{S}_u \widetilde{M}(G)$ is GSC.*

This is actually the main step in the proof of the QSF theorem. And I have to tell you some words regarding its proof, because it concerns our present topic.

To begin with, our $\mathcal{S}_u$ is a functor, the $\mathcal{S}_u M(G)$ makes sense too, and we have:

$$\mathcal{S}_u \widetilde{M}(G) = (\mathcal{S}_u M(G))^{\sim}.$$

One builds up the $\mathcal{S}_u \widetilde{M}(G)$ by starting with the zipping of $f(X_0^2)$ where X_0^2 is the REPRESENTATION space of the 2d REPRESENTATION $f : X_0^2 = X \to \widetilde{M}(G)$ above,

$$X_0^2 \to X_1^2 \to X_2^2 \to \cdots \to f(X_0^2).$$

There is no reason why this process should be GSC preserving. In the case of an "easy" G, starting from our Y satisfying GSC, one can actually produce a GSC-preserving, high-dimensional thickening of (7), see [7]. But in the general case, the Whitehead nightmare is in the way for this procedure.

So we need a second functor $\mathcal{S}_b$, which uses now an infinite sequence of quite naturally GSC-preserving inclusions, instead of our former quotient maps, which the inclusions replace:

$$GSC \ni X_0^N \to X_1^N \to X_2^N \to \cdots \to \mathcal{S}_b\widetilde{M}(G).$$

Here, again we have

$$\mathcal{S}_b\widetilde{M}(G) = (\mathcal{S}_b M(G))^{\sim}.$$

Incidentally, "u" stands for *usual* and "b" for *bizarre*. Anyway, in the context of (8) we find now that the following things happen:

(11-1) Quite naturally, now $\mathcal{S}_b\widetilde{M}(G)$ is GSC.

(11-2) It is only $\mathcal{S}_b\widetilde{M}(G)$ which sees the infinite complications, see below, none of the intermediary $X_{i<\infty}^N$ do.

What we want to do now is to relate $\mathcal{S}_u\widetilde{M}(G)$ to $\mathcal{S}_b\widetilde{M}(G)$. We start by going down, from $\widetilde{M}(G)$ to $M(G)$, and now we are in the compact surrounding of $M(G)$. But, careful, the $\mathcal{S}_u M(G)$ is not compact. Then, by various compactness arguments and also making big use of the uniform boundedness of the zipping length (Trilogy I, remember), we can prove the diffeomorphism

$$\mathcal{S}_u M(G) \overset{\text{DIFF}}{=} \mathcal{S}_b M(G),$$

and hence we also have

$$\mathcal{S}_u M(G)^{\sim} = \mathcal{S}_b M(G)^{\sim}.$$

Then, finally, by functoriality we get:

$$\mathcal{S}_u\widetilde{M}(G) = \mathcal{S}_u M(G)^{\sim} = \mathcal{S}_b M(G)^{\sim} = \mathcal{S}_b\widetilde{M}(G),$$

and we conclude that $\mathcal{S}_u\widetilde{M}(G)$ is GSC, because $\mathcal{S}_b\widetilde{M}(G)$ is so.

7. A very fast survey of recent joined work, in progress, with Louis Funar and Daniele Otera

What we hope to prove is the following result, and I put a question mark to the statement below, since we do not yet have a complete proof, but I am quite optimistic about it.

Proposition 1. *For any finitely presented group G (and no other groups will concern us here) there exist two twin locally-compact cell complexes $\mathcal{X}^N$ (with high N) and $\mathcal{X}_0^3$, both geometrically simply connected (GSC), such that the following things should happen:*

(1) *There is a free and co-compact action*

$$G \times \mathcal{X}^N \to \mathcal{X}^N.$$

(2) • *There is a 3-dimensional* REPRESENTATION

$$f : \mathcal{X}_0^3 \to \widetilde{M}(G),$$

where $M(G)$ is a compact singular 3-manifold with $\pi_1 M(G) = G$, such that any compact $K \subset \widetilde{M}(G)$ should be touched only finitely *many times by $f(\mathcal{X}_0^3)$.*

• *Or, equivalently, there is a 2-dimensional* REPRESENTATION, *from a well-chosen sufficiently fine 2-skeleton $\mathcal{X}_0^2$ of $\mathcal{X}_0^3$*

$$f : \mathcal{X}_0^2 \to \widetilde{M}(G),$$

such that both

$$f(\mathcal{X}_0^2) \subset \widetilde{M}(G) \quad \textit{and} \quad M_2(f) \cup \mathrm{Sing}(f) \subset \mathcal{X}_0^2$$

(where M_2 means double points and Sing *are the non-immersive points) are* closed subsets.

Our (1) above means that all finitely presented G's are GSC and hence Tucker too[1] (see also [3]). (Just like $\pi_1^\infty = 0$ means asymptotically simply connected, *Tucker* means asymptotic with finitely presented π_1.)

Our (2) above means that all G's can avoid the *Whitehead nightmare*. Or, to put it another way:

"All groups are 'easy',"

something I had conjectured years ago [19], calling this the impertinent conjecture.

And I also have a much more esoteric conjecture, already mentioned, about the existence of a much larger category than groups, where the "difficult" objects should hide. I believe that this new category manifests itself already via things like the Penrose aperiodic tilings and the quasi-crystals of condensed matter physics. Inside it, the finitely presented groups should live like the rationals among reals, or periodic functions among the quasi-periodic ones.

And now that Proposition 1 tells us that all G's are GSC, I also suspect that those difficult guys should brutally violate GSC. And this should bring the non-commutative geometry of Alain Connes into the picture, see [19, 21].

In the proof of the QSF theorem we have learned how to live with the Whitehead nightmare, and the idea now is that this has created enough technology so that we should be able to circumvent it altogether, for all G's.

We have now a free action (actually everything above was equivariant, see Theorem 1)

$$G \times \mathcal{S}_u \widetilde{M}(G) \to \mathcal{S}_u \widetilde{M}(G), \tag{9}$$

and the $\mathcal{S}_u \widetilde{M}(G)$ is GSC. But our (9) is not co-compact. And here the work starts.

[1] L. Funar, comments on WGSC, Tucker and GSC; letter, 2015.

References

[1] S. G. Brick, Quasi-isometries and ends of groups. *J. Pure Appl. Algebra* **86** (1993), no. 1, 23–33 Zbl 0780.57001 MR 1213151

[2] S. G. Brick and M. L. Mihalik, The QSF property for groups and spaces. *Math. Z.* **220** (1995), no. 2, 207–217 Zbl 0843.57003 MR 1355026

[3] L. Funar and D. E. Otera, On the wgsc and qsf tameness conditions for finitely presented groups. *Groups Geom. Dyn.* **4** (2010), no. 3, 549–596 Zbl 1218.57012 MR 2653975

[4] B. Mazur, A note on some contractible 4-manifolds. *Ann. of Math. (2)* **73** (1961), 221–228 Zbl 0127.13604 MR 125574

[5] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*. Clay Math. Monogr. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007 Zbl 1179.57045 MR 2334563

[6] J. Morgan and G. Tian, *The geometrization conjecture*. Clay Math. Monogr. 5, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014 Zbl 1302.53001 MR 3186136

[7] D. E. Otera and V. Poénaru, "Easy" representations and the QSF property for groups. *Bull. Belg. Math. Soc. Simon Stevin* **19** (2012), no. 3, 385–398 Zbl 1277.57005 MR 3027350

[8] D. E. Otera and V. Poénaru, Topics in geometric group theory I. In *Handbook of group actions. V*, pp. 301–346, Adv. Lect. Math. (ALM) 48, International Press, Somerville, MA, 2020 Zbl 1464.57034 MR 4237896

[9] V. Poenaru, Les décompositions de l'hypercube en produit topologique. *Bull. Soc. Math. France* **88** (1960), 113–129 Zbl 0135.41704 MR 125572

[10] V. Poénaru, On the equivalence relation forced by the singularities of a nondegenerate simplicial map. *Duke Math. J.* **63** (1991), no. 2, 421–429 Zbl 0738.57008 MR 1115115

[11] V. Poénaru, Almost convex groups, Lipschitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds. *J. Differential Geom.* **35** (1992), no. 1, 103–130 Zbl 0777.57001 MR 1152227

[12] V. Poénaru, Geometry "à la Gromov" for the fundamental group of a closed 3-manifold M^3 and the simple connectivity at infinity of $\widetilde{M}^3$. *Topology* **33** (1994), no. 1, 181–196 Zbl 0830.57010 MR 1259521

[13] V. Poénaru, Equivariant, locally finite inverse representations with uniformly bounded zipping length, for arbitrary finitely presented groups. *Geom. Dedicata* **167** (2013), 91–121 Zbl 1354.20027 MR 3128772 (Trilogy I)

[14] V. Poénaru, Geometric simple connectivity and finitely presented groups. 2014, arXiv:1404.4283 (Trilogy II)[2]

[15] V. Poénaru, All finitely presented groups are QSF. 2014, arXiv:1409.7325 (Trilogy III)[2]

[16] V. Poénaru, All smooth four-dimensional Schoenflies balls are geometrically simply connected. 2016, arXiv:1609.05094

[17] V. Poénaru, A glimpse into the problems of the fourth dimension. In *Geometry in history*, pp. 687–703, Springer, Cham, 2019 Zbl 1451.57001 MR 3965777

[2]There are now improved, more reader-friendly versions of Trilogy II and III (2021), soon to be available online.

[18] V. Poénaru, Topics in geometric group theory II. In *Handbook of group actions. V*, pp. 347–398, Adv. Lect. Math. (ALM) 48, International Press, Somerville, MA, 2020 Zbl 07274145 MR 4237897

[19] V. Poénaru, On geometric group theory. In *Topology and geometry – A collection of essays dedicated to Vladimir G. Turaev*, pp. 399–432, IRMA Lect. Math. Theor. Phys. 33, EMS Press, Berlin, 2021 MR 4394517

[20] V. Poénaru, All 4-dimensional smooth Schoenflies balls are geometrically simply-connected. In *Surveys in Geometry I*, pp. 269–307, Springer, Cham, 2022 Zbl 1502.57013 MR 4404398

[21] V. Poénaru, Classical differential topology and non-commutative geometry. In *Surveys in geometry I*, pp. 309–341, Springer, Cham, 2022 Zbl 1502.57014 MR 4404399

[22] V. Poénaru and C. Tanasi, Representations of the Whitehead manifold Wh^3 and Julia sets. *Ann. Fac. Sci. Toulouse Math. (6)* **4** (1995), no. 3, 655–694 Zbl 0878.57015 MR 1607516

[23] J. R. Stallings, Brick's quasi-simple filtrations for groups and 3-manifolds. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, pp. 188–203, London Math. Soc. Lecture Note Ser. 181, Cambridge University Press, Cambridge, 1993 Zbl 0838.57001 MR 1238526

[24] J. R. Stallings and S. M. Gersten, Casson's idea about 3-manifolds whose universal cover is $\mathbb{R}^3$. *Internat. J. Algebra Comput.* **1** (1991), no. 4, 395–406 Zbl 0739.57010 MR 1154440

Received 14 September 2021; revised 19 November 2022.

Valentin Poénaru
Faculté des Sciences, Université Paris-Sud 11, F-91405 Orsay, France; valpoe@hotmail.com