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# Formality and finiteness in rational homotopy theory

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**Abstract.** We explore various formality and finiteness properties in the differential graded algebra models for the Sullivan algebra of piecewise polynomial rational forms on a space. The 1-formality property of the space may be reinterpreted in terms of the filtered and graded formality properties of the Malcev Lie algebra of its fundamental group, while some of the finiteness properties of the space are mirrored in the finiteness properties of algebraic models associated with it. In turn, the formality and finiteness properties of algebraic models have strong implications on the geometry of the cohomology jump loci of the space. We illustrate the theory with examples drawn from complex algebraic geometry, actions of compact Lie groups, and 3-dimensional manifolds.

*Dedicated to Dennis Sullivan on the occasion of his 80th birthday*

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# 1. Introduction

## 1.1. Rational homotopy type

Homotopy theory is the study of topological spaces up to homotopy equivalences. Typical examples of homotopy type invariants of a space $X$ are the homology groups $H_n(X,\mathbb{Z})$ and the homotopy groups $\pi_n(X)$. The question whether one can reconstruct the homotopy type of a space from homological data goes back to the beginnings of Algebraic Topology. Poincaré realized that homology is not enough: for a path-connected space $X$, the first homology group, $H_1(X,\mathbb{Z})$, only records the abelianization of the fundamental group, $\pi_1(X)$. Even for simply-connected spaces, homology by itself fails to detect maps such as the Hopf map, $S^3 \to S^2$. On the other hand, if one considers the de Rham algebra of differential forms, one can reconstitute in a purely algebraic fashion all the higher homotopy groups of $S^n$, modulo torsion.

As founded by Quillen [118] and Sullivan [137], rational homotopy theory is the study of rational homotopy types of spaces. Instead of considering the groups $H_n(X,\mathbb{Z})$ and $\pi_n(X)$, one considers the rational homology groups $H_n(X,\mathbb{Q})$ and the rational homotopy groups $\pi_n(X)\otimes\mathbb{Q}$. These objects are $\mathbb{Q}$-vector spaces, and hence the torsion information is lost, yet this is compensated by the fact that computations are easier in this setting.

## 1.2. Models for spaces and groups

In his seminal paper, [137], Sullivan attached in a functorial way to every space $X$ a commutative differential graded algebra over $\mathbb{Q}$, denoted $A_{\text{PL}}(X)$. This CDGA is constructed from piecewise polynomial rational forms and is weakly equivalent (through DGAs) with the cochain algebra $(C^*(X,\mathbb{Q}),d)$ so that, under the resulting identification of graded algebras, $H^*(A_{\text{PL}}(X))\cong H^*(X,\mathbb{Q})$, the induced homomorphisms in cohomology correspond.

We say that two CDGAs $A$ and $B$ are weakly equivalent (written $A\simeq B$) if there is a zigzag of CDGA maps inducing isomorphisms in cohomology and connecting $A$ to $B$. If those maps only induce isomorphisms in degree at most $q$ (for some $q\ge 0$) and monomorphisms in degree $q+1$, we say $A$ and $B$ are $q$-equivalent (written $A\simeq_q B$).

Let $\Bbbk$ be a coefficient field of characteristic 0. We say that a $\Bbbk$-CDGA $(A,d)$ is a *model* for a space $X$ if $A\simeq A_{\text{PL}}(X)\otimes_{\mathbb{Q}}\Bbbk$, and likewise for a $q$-*model*. For instance, if $X$ is a smooth manifold, de Rham's algebra $\Omega^*_{\text{dR}}(X)$ is a model for $X$ over $\mathbb{R}$, leading to the following basic principle in rational homotopy theory: "The manner in which a closed form which is zero in cohomology actually becomes exact contains geometric information," cf. [37].

Given a connected CDGA $A$, Sullivan constructed a *minimal model* for it, $\rho\colon \mathcal{M}(A)\to \mathcal{A}$, where $\rho$ is a quasi-isomorphism and $\mathcal{M}(A)$ is a CDGA obtained by iterated Hirsch extensions, starting from $\Bbbk$, so that its differential is decomposable. These properties uniquely characterize the minimal model of $A$ (up to isomorphism). The $q$-minimal models $\mathcal{M}_q(A)$ are similarly defined for all $q\ge 0$; they are generated by elements of degrees at most $q$, and the structural morphisms $\rho_q \colon \mathcal{M}_q(A) \to A$ are only $q$-quasi-isomorphisms.

A minimal model for a connected space $X$, denoted $\mathcal{M}(X)$, is a minimal model for the Sullivan algebra $A_{\mathrm{PL}}(X)$. The isomorphism type of $\mathcal{M}(X)$ is uniquely defined by the rational homotopy type of $X$. The $q$-minimal models $\mathcal{M}_q(X)$ are defined analogously; moreover, if $G$ is a finitely generated group, we set $\mathcal{M}_1(G) = \mathcal{M}_1(K(G,1))$. When $X$ is a nilpotent CW-complex with finite Betti numbers, Sullivan [137] showed that $\pi_n(X) \otimes \mathbb{Q} \cong (V^n)^\vee$ for all $n \geq 2$, where $V = \bigoplus_n V^n$ and $\mathcal{M}(X) = (\bigwedge V, d)$ is a minimal model for $X$ over $\mathbb{Q}$.

### 1.3. Formality

As formulated in [37, 137], the notion of formality plays a central role in rational homotopy theory. We say that a path-connected space $X$ is *formal* if its Sullivan algebra, $A_{\mathrm{PL}}(X)$, is weakly equivalent to its cohomology algebra, $H^*(X, \mathbb{Q})$, endowed with the zero differential. The notion of *$q$-formality* (for some $q \geq 0$) is defined accordingly. In general, partial formality is a much weaker property than (full) formality; nevertheless, if $H^{\geq q+2}(X; \mathbb{Q}) = 0$, then $X$ is $q$-formal if and only if $X$ is formal, see [91]. One may also talk about ($q$-) formality over a field $\Bbbk$, but it turns out that all these formality notions are independent of the choice of the coefficient field, as long as $\operatorname{char}(\Bbbk) = 0$.

Various conditions on the connectivity of the space or the structure of its cohomology algebra guarantee formality. For instance, if $X$ is a $k$-connected CW-complex ($k \geq 1$) of dimension $n$ and $n \leq 3k + 1$, then $X$ is formal, cf. [124]; moreover, if $X$ is a closed manifold of dimension $n$, the conclusion remains valid for $n \leq 4k + 2$, cf. [99]. Also, if $H^*(X, \mathbb{Q})$ is the quotient of a free CGA by an ideal generated by a regular sequence, then $X$ is formal, cf. [137].

A classical obstruction to formality is provided by the Massey products (of order 3 and higher): If a space $X$ is formal, then all Massey products in the cohomology algebra $H^*(X, \mathbb{Q})$ vanish – in fact, vanish uniformly. Furthermore, if $X$ is $q$-formal, for some $q \geq 1$, then all Massey products in $H^{\leq q+1}(X, \mathbb{Q})$ vanish.

A simply-connected space (or, more generally, a nilpotent space) $X$ is formal if its rational homotopy type is determined by $H^*(X, \mathbb{Q})$. In the general case, the weaker 1-formality property allows one to reconstruct the rational pro-unipotent completion of the fundamental group, solely from the cup products of degree 1 cohomology classes. Formal spaces lend themselves to various algebraic computations that provide valuable homotopy information. For instance, a result of Papadima–Yuzvinsky [115], which is valid for all formal spaces $X$, states: The Bousfield–Kan completion $\mathbb{Q}_\infty X$ is aspherical if and only if $H^*(X, \mathbb{Q})$ is a Koszul algebra.

### 1.4. Finiteness in CDGA models

A recurring theme in topology is to determine the geometric and homological finiteness properties of spaces and groups. A prototypical such problem is to determine whether a path-connected space $X$ is homotopy equivalent to a CW-complex with finite $q$-skeleton, for some $1 \le q \le \infty$, in which case we say $X$ is *$q$-finite*. Another question is to decide whether a group $G$ is finitely generated, and if so, whether it admits a finite presentation; more generally, whether it has a classifying space $K(G,1)$ with finite $q$-skeleton.

A fruitful approach to this type of question is to compare the finiteness properties of the spaces or groups under consideration to the corresponding finiteness properties of algebraic models for such spaces and groups. By analogy with the aforementioned topological notion, we say that a $\Bbbk$-CDGA $A$ is *$q$-finite* if it is connected (i.e., $A^0 = \Bbbk \cdot 1$) and $\dim A^i < \infty$ for $i \le q$.

A natural question then is: When does a $q$-finite space $X$ admit a $q$-finite $q$-model $(A,d)$? A necessary criterion is given in [113]: If a space $X$ does admit such a model, then $\dim H^i(\mathcal{M}_q(X)) < \infty$, for all $i \le q+1$. For instance, if $G = F_n/F_n''$ is the free metabelian group of rank $n \ge 2$ then $b_2(\mathcal{M}_1(G)) = \infty$, and so $G$ admits no 1-finite 1-model. Other finiteness criteria, based on the nature of the cohomology jump loci (see [113, 129]), are discussed below.

### 1.5. Malcev and holonomy Lie algebras

In his landmark paper on rational homotopy theory, Quillen [118] defined the Malcev Lie algebra, $\mathfrak{m}(G)$, of a finitely generated group $G$ as the (complete, filtered) Lie algebra of primitive elements in the $I$-adic completion of the group algebra $\mathbb{Q}[G]$, where $I$ is the augmentation ideal. The associated graded Lie algebra with respect to this filtration, $\operatorname{gr}(\mathfrak{m}(G))$, is isomorphic to $\operatorname{gr}(G,\mathbb{Q})$ the rational graded Lie algebra associated to the lower central series filtration of $G$, cf. [119].

As shown by Sullivan [137] (see also [30, 66]), the Lie algebra dual to $\mathcal{M}_1(G)$ is isomorphic to the Malcev Lie algebra $\mathfrak{m}(G)$. It follows that $G$ is 1-formal if and only if $\mathfrak{m}(G)$ is the LCS completion of a finitely generated, quadratic Lie algebra. A weaker condition was given in [132]: we say that $G$ is *filtered formal* if $\mathfrak{m}(G)$ is the completion of $\operatorname{gr}(G,\mathbb{Q})$ with respect to its LCS filtration. As shown in [133], this condition is equivalent to the existence of a Taylor expansion, $G \to \widehat{\operatorname{gr}}(\mathbb{Q}[G])$.

Now suppose $G$ has a 1-finite 1-model $(A,d)$ over $\mathbb{Q}$. Building on a classical construction of K.-T. Chen [31], the *holonomy Lie algebra* $\mathfrak{h}(A)$ was defined in [92] as the quotient of the free Lie algebra on the dual vector space $A_1 = (A^1)^\vee$ by the ideal generated by the image of the map $(d+\mu)^\vee$, where $d\colon A^1 \to A^2$ is the differential and $\mu\colon A^1 \wedge A^1 \to A^2$ is the multiplication map. Then, as shown in [13, 113] (generalizing a result from [16]), the Malcev Lie algebra $\mathfrak{m}(G)$ is isomorphic to the LCS completion of $\mathfrak{h}(A)$. Moreover, the following complete finiteness criterion in degree 1 was given in [113]: A finitely generated group $G$ admits a 1-finite 1-model if and only if $\mathfrak{m}(G)$ is the LCS completion of a finitely presented Lie algebra.

### 1.6. Cohomology jump loci

The cohomology jump loci of a space $X$ are of two basic types: the characteristic varieties, $\mathcal{V}^i_k(X)$, defined in terms of homology with coefficients in rank one local systems, and the resonance varieties, $\mathcal{R}^i_k(X)$ or $\mathcal{R}^i_k(A)$, constructed from information encoded in either the cohomology ring $H^*(X,\mathbb{C})$, or in a CDGA model $A$ for $X$.

The characteristic varieties and the related Alexander invariants of spaces and groups have their origin in the study of the Alexander polynomials of knots and links. The basic topological idea in defining these invariants is to take the homology of the maximal abelian cover of a connected CW-complex $X$ and view it as a module over the group ring of $H_1(X,\mathbb{Z})$. One then studies the support loci of these modules, or, alternatively, the jump loci $\mathcal{V}^i_k(X)$, viewed as subsets of the character group $\mathrm{Char}(X) = \mathrm{Hom}(\pi_1(X),\mathbb{C}^*)$.

The formality and finiteness properties of a space and its algebraic models put strong constraints on the geometric structure of the cohomology jump loci. To start with, let $X$ be a $q$-finite space, for some $q \geq 1$. Then the tangent cone at the trivial character $\mathbf{1}$ to the variety $\mathcal{V}^i_k(X)$ is included in $\mathcal{R}^i_k(X)$, for all $i \leq q$ and $k \geq 0$, see [86].

Now suppose $X$ admits a $q$-finite $q$-model $A$; then $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^i_k(X)) = \mathcal{R}^i_k(A)$, for all $i \leq q$, see [40]. Moreover, as a consequence of [40,42], all irreducible components of these resonance varieties are rationally defined linear subspaces of $H^1(A) = H^1(X,\mathbb{C})$, and, by [25], all the components of $\mathcal{V}^i_k(X)$ passing through $\mathbf{1}$ are algebraic subtori of $\mathrm{Char}(X)$. Finally, suppose $X$ is $q$-formal. Then, for $i \leq q$, all the components of $\mathcal{R}^i_k(X)$ are rationally defined linear subspaces of $H^1(X,\mathbb{C})$.

### 1.7. Models for Kähler manifolds and smooth algebraic varieties

One of the foundational papers of rational homotopy theory is the one by Deligne, Griffiths, Morgan, and Sullivan [37], where the authors use Hodge theory and the $dd^c$-lemma to establish the formality of all compact Kähler manifolds, and thus, of all smooth, complex algebraic projective varieties.

In [100], Morgan constructed a finite-dimensional model for any smooth, complex, quasi-projective variety $X$ by using a normal crossings divisors compactification $\overline{X}$. This model was refined by Dupont in [45], by allowing those divisors to intersect like hyperplanes in a hyperplane arrangement. These models are not always formal, but still retain good partial formality properties; for instance, if $X$ is the complement of a hypersurface in $\mathbb{CP}^n$, then $X$ is 1-formal, but not formal, in general.

The structure of the characteristic varieties of compact Kähler manifolds and smooth, quasi-projective varieties was determined in [3, 12, 24, 28, 65, 123]: If $X$ is such a space, then each variety $\mathcal{V}^i_k(X)$ is a finite union of torsion-translated subtori of $\mathrm{Char}(X)$. The key to understanding the degree-1 cohomology jump loci is the (finite) set $\mathcal{E}(X)$ of holomorphic, surjective maps $f\colon X \to \Sigma$ for which the generic fiber is connected, and the target is a smooth curve $\Sigma$ with $\chi(\Sigma) < 0$, up to reparametrization at the target. As an application of these techniques, we obtained in [113] the following result.

Let $X$ be a smooth quasi-projective variety with $b_1(X) > 0$, and let $A$ be a Dupont model for $X$; then $\pi_1(X)$ surjects onto a free, non-cyclic free group if and only if $\mathcal{R}^1_1(A) \neq \{\mathbf{0}\}$.

### 1.8. Models for compact Lie group actions

Let $M$ be a compact, connected, smooth manifold that supports an almost free action by a compact, connected Lie group $K$. Under a partial formality assumption on the orbit space $M/K$ and a regularity assumption on the characteristic classes of the action, we constructed in [113] an algebraic model for $M$ with commensurate finiteness and partial formality properties. The existence of such a model has various implications on the structure of the cohomology jump loci of $M$ and of the representation varieties of $\pi_1(M)$.

In many ways, Sasakian geometry is an odd-dimensional analog of Kähler geometry. More explicitly, every compact Sasakian manifold $M$ admits an almost-free circle action with orbit space a Kähler orbifold. Furthermore, the Euler class of the action coincides with the Kähler class of the base, $h \in H^2(M/S^1, \mathbb{Q})$, and this class satisfies the Hard Lefschetz property. As shown by Tievsky in [141], every Sasakian manifold $M$ as above has a rationally defined, finite-dimensional model over $\mathbb{R}$ of the form $(H^*(N, \mathbb{R}) \otimes \bigwedge(t), d)$, where the differential $d$ vanishes on $H^*(N, \mathbb{R})$ and sends $t$ to $h$. Using this model, it is shown in [113] that compact Sasakian manifolds of dimension $2n + 1$ are $(n - 1)$-formal, and that their fundamental groups are filtered-formal.

### 1.9. Models for closed 3-manifolds

With a few exceptions (such as rational homology spheres, knot complements, or Seifert manifolds), the rational homotopy theory of 3-dimensional manifolds is very difficult to handle. Part of the reason is that not only 3-manifolds may fail to be formal, and even fail to have a 1-finite 1-model. Nevertheless, much is known about the Alexander polynomial, $\Delta_M$, of a closed, orientable 3-manifold $M$. and the way this polynomial relates to the cohomology jump loci of $M$, see [43, 61, 127, 129]. In turn, these invariants inform on the formality and finiteness properties of $M$.

For instance, we showed in [129] the following: If $b_1(M)$ is even and positive, and if $\Delta_M \neq 0$, then $M$ is not 1-formal. On the other hand, if $\Delta_M \neq 0$, yet $\Delta_M(\mathbf{1}) = 0$ and the tangent cone to the zero set of $\Delta_M$ is not a finite union of rationally defined linear subspaces, then $M$ admits no 1-finite 1-model.

When the 3-manifold $M$ fibers over $S^1$, more can be said. For instance, if $b_1(M)$ is even, then, as shown in [110], $M$ is not 1-formal. On the other hand, if $M$ is 1-formal and the algebraic monodromy has 1 as an eigenvalue, then, as shown in [109], there are an even number of $1 \times 1$ Jordan blocks for this eigenvalue, and no higher size Jordan blocks.

### 1.10. Organization

The paper in divided in roughly five parts.

Part I (Sections 2, 3, 4) treats the general theory of (commutative) differential graded algebras, formality and its variants, Massey products, descent properties, Hirsch extensions, and Sullivan minimal models.

Part II (Sections 5, 6, 7) deals with several of the Lie algebras that appear in this theory (graded and filtered Lie algebras, Malcev Lie algebras, and holonomy Lie algebras) and discusses some of their properties and interconnections.

Part III (Sections 8, 9, 10) contains the basics of rational homotopy theory, such as completions, rationalizations, and algebraic models for spaces and groups, focusing mostly on the formality and finiteness properties of such models.

Part IV (Sections 11, 12) brings into play the Alexander invariants and the cohomology jump loci of spaces and suitable algebraic models, and connects the characteristic and resonance varieties to various formality and finiteness properties.

Part V (Sections 13, 14, 15) applies these general theories in three particular contexts: that of Kähler manifolds and smooth, quasi-projective varieties; compact Lie group actions on manifolds; and closed, orientable 3-manifolds.

## 2. Differential graded algebras

### 2.1. Graded algebras

Throughout this work, $\Bbbk$ will denote a ground field of characteristic 0. Unless otherwise specified, all tensor products will be over $\Bbbk$.

A *graded* $\Bbbk$*-vector space* is a vector space $A$ over $\Bbbk$, together with a direct sum decomposition, $A = \bigoplus_{n\geq 0} A^n$, into vector subspaces. An element $a \in A^n$ is said to be homogeneous; we write $|a| = n$ for its degree, and put $\bar{a} = (-1)^{|a|}a$.

A *graded algebra* over $\Bbbk$ is a graded $\Bbbk$-vector space, $A^* = \bigoplus_{n\geq 0} A^n$, equipped with an associative multiplication map, $\cdot\colon A \times A \to A$, making $A$ into a $\Bbbk$-algebra with unit $1 \in A^0$ such that $|a \cdot b| = |a| + |b|$ for all homogeneous elements $a, b \in A$. A graded algebra $A$ is said to be *graded-commutative* (for short, a CGA), if $a \cdot b = (-1)^{|a||b|} b \cdot a$ for all homogeneous $a, b \in A$. A morphism between two graded algebras is a $\Bbbk$-linear map $\varphi\colon A \to B$ that preserves gradings and satisfies $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ for all $a, b \in A$.

A graded $\Bbbk$-algebra $A$ is of *finite-type* (or, locally finite) if all the graded pieces $A^n$ are finite-dimensional. We say that $A$ is $q$*-finite* (for some integer $q \geq 0$) if $\dim_{\Bbbk} A^n < \infty$ for $n \leq q$, and we say that $A$ is *finite-dimensional* (as a $\Bbbk$-vector space) if $\dim_{\Bbbk} A < \infty$. Finally, we say that $A$ is *connected* if $A^0$ is the $\Bbbk$-span of the unit 1 (and thus $A^0 = \Bbbk$).

The most basic example of a $\Bbbk$-CGA is the *free* commutative graded algebra on a graded $\Bbbk$-vector space $V^*$; denoted by $\bigwedge V$, this (connected) algebra is the tensor product of the symmetric algebra on $V^{\text{even}}$ with the exterior algebra on $V^{\text{odd}}$.

### 2.2. Differential graded algebras

The next notion, which plays a key role in the theory described here, unifies the concept of a graded algebra with that of a cochain complex.

**Definition 2.1.** A *differential graded algebra* (for short, a DGA) over a field $\Bbbk$ is a graded $\Bbbk$-algebra, $A^*$, equipped with a differential $d\colon A \to A$ of degree 1 satisfying the graded Leibniz rule: $d(ab) = d(a) \cdot b + \overline{a} \cdot d(b)$ for all homogeneous $a, b \in A$.

Viewing $(A, d)$ as a cochain complex, we write $Z^n(A) = \ker(d\colon A^n \to A^{n+1})$ for the space of $n$-cocycles and $B^n(A) = \operatorname{im}(d\colon A^{n-1} \to A^n)$ for the space of $n$-coboundaries, and we let $H^n(A) = Z^n(A)/B^n(A)$ be the $n$-th cohomology group of $(A, d)$. The direct sum of those vector spaces, $H^*(A) = \bigoplus_{i\geq 0} H^i(A)$, inherits the structure of a graded algebra from $A$. When $H^*(A)$ is of finite-type, we denote by $b_n(A) = \dim_{\Bbbk} H^n(A)$ the Betti numbers of $A$. Given an $n$-cocycle $a$, we denote by $[a] \in H^n(A)$ its cohomology class.

A *commutative differential graded algebra* (for short, a CDGA) is a DGA $A = (A^*, d)$ for which the underlying graded algebra is graded-commutative. In this case, the cohomology algebra $H^*(A)$ inherits the structure of a CGA.

If $A$ is a connected DGA, then the differential $d\colon A^0 \to A^1$ vanishes; indeed, $d(1) = d(1 \cdot 1) = d(1) \cdot 1 + 1 \cdot d(1)$, and so $d(1) = 0$, since $\operatorname{char}(\Bbbk) = 0$. Therefore, $H^0(A) = \Bbbk$ and the cohomology algebra $H^*(A)$ is also connected.

### 2.3. Weak equivalences

A morphism between two DGAs is a $\Bbbk$-linear map, $\varphi\colon A \to B$, which preserves gradings, multiplicative structures, and differentials; in other words, $\varphi$ is a map of graded $\Bbbk$-algebras which is also a map of cochain complexes. Such a map induces a morphism, $\varphi^*\colon H^*(A) \to H^*(B)$, between the respective cohomology algebras. We say that $\varphi$ is a *quasi-isomorphism* if $\varphi^*$ is an isomorphism.

A *weak equivalence* between two DGAs, $A$ and $B$, is a finite sequence of quasi-isomorphisms (going either way) connecting $A$ to $B$; for instance,

$$A \xleftarrow{\varphi_1} A_1 \xrightarrow{\varphi_2} \cdots \longleftarrow A_{\ell-1} \xrightarrow{\varphi_\ell} B. \tag{2.1}$$

Note that a weak equivalence induces a well-defined isomorphism $H^*(A) \cong H^*(B)$. If a weak equivalence between $A$ and $B$ exists, the two DGAs are said to be *weakly equivalent*, written $A \simeq B$. Evidently, $\simeq$ is an equivalence relation among DGAs.

An analogous notion holds in the category of commutative DGAs. Namely, if $A$ and $B$ are two CDGAs, we say that $A \simeq B$ is there is a zigzag of quasi-isomorphisms as in (2.1) that go through CDGAs $A_i$. The following theorem resolves a long-standing question, by showing that weak equivalence among CDGAs holds even if one allows the zigzags from (2.1) to go through DGAs.

**Theorem 2.2** ([26]). *Let $A$ and $B$ be two $\Bbbk$-CDGAs. Then $A \simeq B$ as DGAs if and only if $A \simeq B$ as CDGAs.*

All these concepts have partial analogs. Fix an integer $q \geq 0$. We say that a DGA (or CDGA) morphism $\varphi\colon A \to B$ is a *$q$-quasi-isomorphism* if $\varphi^*$ is an isomorphism in degrees up to $q$ and a monomorphism in degree $q+1$. A *$q$-equivalence* between two DGAs (or CDGAs), $A$ and $B$, is a zigzag of $q$-quasi-isomorphisms of DGAs (or CDGAs) connecting $A$ to $B$. If such a zigzag exists, we say that $A$ and $B$ are *$q$-equivalent* and write this as $A \simeq_q B$. Again, $\simeq_q$ is an equivalence relation among either DGAs or CDGAs. We do not know whether Theorem 2.2 holds with $\simeq$ replaced by $\simeq_q$, for arbitrary $q$, but we expect it does.

Clearly, if $A \simeq B$, then $A \simeq_q B$ for all $q \geq 0$, and if $A \simeq_q B$, then $A \simeq_n B$ for all $n \leq q$. Moreover, if $A$ is of finite-type and $A \simeq B$, then $B$ is also of finite-type and $b_n(A) = b_n(B)$ for all $n \geq 0$. Likewise, if $A$ is $q$-finite and $A \simeq_q B$, then $B$ is also $q$-finite and $b_n(A) = b_n(B)$ for all $n \leq q$. The next lemma shows that every $q$-finite CDGA may be replaced (up to $q$-equivalence) by a finite-dimensional one, whose graded pieces vanish above degree $q+1$.

**Lemma 2.3** ([113]). *Let $A$ be a $q$-finite CDGA. There is then a natural $q$-equivalence, $A \simeq_q A[q]$, where $A[q]$ is a finite-dimensional CDGA with $A[q]^i = 0$ for all $i > q+1$.*

The construction of $A[q]$ is done in two steps: first one replaces $A$ by its truncation, $\overline{A} = A/A^{>q+1} = \bigoplus_{i \leq q+1} A^i$, and then one defines a sub-CDGA $A[q] \subset \overline{A}$ by setting

$$A[q] = \bigoplus_{i \leq q} A^i \oplus \Big( dA^q + \sum_{\substack{i,j \leq q \\ i+j=q+1}} A^i \cdot A^j \Big).$$

An analogous result holds for DGAs.

## 2.4. Homotopies and equivalences

Let $A$ be a $\Bbbk$-DGA, and let $\bigwedge(t, dt)$ be the free $\Bbbk$-DGA generated by elements $t$ in degree 0 and $u$ in degree 1, and differential $d$ given by $d(t) = u$ and $d(u) = 0$. For each $s \in \Bbbk$, let $\mathrm{ev}_s\colon \bigwedge(t, dt) \to \Bbbk$ be the DGA map sending $t$ to $s$ and $dt$ to 0. This map induces another DGA map,

$$\mathrm{Ev}_s := \mathrm{id} \otimes \mathrm{ev}_s\colon A \otimes \bigwedge(t, dt) \longrightarrow A \otimes \Bbbk = A.$$

Two DGA maps, $\varphi_0, \varphi_1\colon A \to B$, are said to be *homotopic* if there is a DGA map, $\Phi\colon A \to B \otimes \bigwedge(t, dt)$, such that $\mathrm{Ev}_s \circ \Phi = \varphi_s$ for $s = 0, 1$. It is readily seen that homotopic DGA maps induce the same map in cohomology.

We say that two DGA morphisms, $\varphi\colon A \to B$ and $\varphi'\colon A' \to B'$, are *weakly equivalent* (written $\varphi \simeq \varphi'$) if there are two zigzags of weak equivalences of DGAs, and DGA maps

$\varphi_1, \ldots, \varphi_{\ell-1}$ such that the following diagram commutes, up to homotopy:

$$\begin{array}{ccccccccc} A & \xleftarrow{\psi_1} & A_1 & \xrightarrow{\psi_2} & \cdots & \longleftarrow & A_{\ell-1} & \xrightarrow{\psi_\ell} & A' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_{\ell-1}} & & \downarrow{\scriptstyle \varphi'} \\ B & \xleftarrow{\psi'_1} & B'_1 & \xrightarrow{\psi'_2} & \cdots & \longleftarrow & B'_{\ell-1} & \xrightarrow{\psi'_\ell} & B'. \end{array} \tag{2.2}$$

The notion of *q-equivalence* (written $\varphi \simeq_q \varphi'$) is defined similarly, and so are the analogous notions in the CDGA category.

### 2.5. Poincaré duality

Let $A$ be a finite-dimensional, commutative graded algebra over a field $\Bbbk$ of characteristic 0. We say that $A$ is a *Poincaré duality algebra* of dimension $n$ (for short, an $n$-PDA) if $A^i = 0$ for $i > n$ and $A^n = \Bbbk$, while the bilinear forms $A^i \otimes A^{n-i} \to A^n = \Bbbk$ given by the product are non-degenerate, for all $0 \le i \le n$ (in particular, $A$ is connected). If $M$ is a closed, connected, orientable, $n$-dimensional manifold, then, by Poincaré duality, the cohomology algebra $A = H^*(M, \Bbbk)$ is an $n$-PDA.

Now let $A = (A^*, d)$ be a CDGA. We say that $A$ is a *Poincaré duality differential graded algebra* of dimension $n$ (for short, an $n$-PD-CDGA) if the underlying algebra $A$ is an $n$-PDA, and, moreover, $H^n(A) = \Bbbk$, or, equivalently, $dA^{n-1} = 0$.

Clearly, if $A$ is an $n$-PDA, then $(A, 0)$ is an $n$-PD-CDGA. Hasegawa showed in [69] that the minimal model for the classifying space of a finitely-generated nilpotent group is a PD-CDGA. Noteworthy is the following result of Lambrechts and Stanley [82]

**Theorem 2.4** ([82]). *Let $(A, d)$ be a* CDGA *such that $H^1(A) = 0$ and $H^*(A)$ is an $n$*-PDA. *Then $A$ is weakly equivalent to an $n$*-PD-CDGA.

## 3. Formality

### 3.1. Formal DGAs

In this section, we cover the notion of *formality*. Introduced in [37, 137] and further developed in [66, 68, 91, 100, 105, 122, 132, 139], and many other works, this notion plays a central role in rational homotopy theory.

**Definition 3.1** ([37, 137]). A DGA $(A, d_A)$ is said to be *formal* if it is weakly equivalent to $(H^*(A), 0)$, its cohomology algebra endowed with the zero differential.

Note that $A$ is formal if and only if it is weakly equivalent to some DGA $B$ with zero differential, since, in that case, we necessarily have $(B, 0) \simeq (H^*(A), 0)$. In like manner, we say that a CDGA $(A, d_A)$ is formal if $(A, d_A) \simeq (H^*(A), 0)$ via a weak equivalence through CDGAs.

**Example 3.2.** Let $A = \bigwedge(a_1, a_2, b)$ be the free CGA on generators $a_1, a_2$ in degree $n$ and $b$ in degree $2n-1$, equipped with the differential $d$ given by $da_i = 0$ and $db = a_1a_2$. If $n \geq 2$, the CDGA $(A, d)$ is not formal, since $H^*(A)$ is generated by the elements $u_i = [a_i]$, and so any weak equivalence from $(A, d)$ to $(H^*(A), 0)$ would need to take the non-zero element $b$ to 0, by degree reasons.

Formality is preserved under weak equivalences; that is, if $A \simeq B$, then $A$ is formal if and only if $B$ is formal. Moreover, as shown by Halperin and Stasheff in [68], a CDGA $(A, d)$ with $H^*(A)$ of finite-type is formal if and only if the identity map of $H^*(A)$ can be realized by a weak equivalence between $(A, d)$ and $(H^*(A), 0)$.

The next result, originally proved directly by Saleh in [122], now follows at once from Theorem 2.2.

**Corollary 3.3** ([122]). *Let $A$ be a $\Bbbk$-CDGA. Then $A$ is formal as a DGA if and only if $A$ is formal as a CDGA.*

### 3.2. Intrinsic formality

We now present two variants of formality, the first being a rigid type of formality and the second formality up to a degree.

A strong form of formality was introduced by Sullivan in [137], and developed in [50, 68, 88]. We say that a $\Bbbk$-cga $H$ is *intrinsically formal* if any $\Bbbk$-DGA $(A, d)$ whose cohomology is isomorphic to $H$ must be formal, that is, $(A, d) \simeq (H, 0)$. A similar notion holds for CDGAs; by Theorem 2.2, if $H$ is intrinsically formal in the category of DGAs, it is also intrinsically formal in the category of CDGAs. Plainly, if $A$ is a DGA or a CDGA such that $H^*(A)$ is intrinsically formal, then $A$ is formal. The following results of Sullivan and Halperin–Stasheff provide large classes of intrinsically formal CGAs.

**Theorem 3.4** ([137]). *Let $H$ be the quotient of a finitely generated, free CGA by an ideal generated by a regular sequence, that is, a sequence $r_1, \dots, r_n$ of homogeneous elements in $H$ such that $r_i$ is not a zero-divisor in $H/(r_1, \dots, r_{i-1})$, for each $i \leq n$. Then $H$ is intrinsically formal.*

Algebras of the form $H = \bigwedge V/(r_1, \dots, r_n)$ as above are sometimes called *hyperformal*, see [50, 88]. In particular, exterior algebras and polynomial algebras are hyperformal, and thus intrinsically formal.

**Theorem 3.5** ([68]). *Let $H$ be a connected CGA such that $H^i = 0$ for $1 \leq i \leq k$ and for $i > 3k+1$. Then $H$ is intrinsically formal.*

### 3.3. Partial formality

The notion of formality can also be relaxed, as follows. Fix an integer $q \geq 0$. We say that a DGA (or a CDGA) $A$ is *$q$-formal* if $(A, d_A) \simeq_q (H^*(A), 0)$ as DGAs (or CDGAs).

We do not know whether the analog of Corollary 3.3 holds in this context, but we expect it does.

Clearly, if $A$ is formal, then $A$ is $q$-formal, for all $q \geq 0$, and if $A$ is $q$-formal, then it is $n$-formal for every $n \leq q$. It is readily seen that connected DGAs are 0-formal. Moreover, if $A \simeq_q B$, then $A$ is $q$-formal if and only if $B$ is $q$-formal.

**Example 3.6.** Let $A = \bigwedge(a_1, \dots, a_{2q}, b)$ be the exterior algebra on specified generators in degree 1, equipped with the differential $d$ given by $da_i = 0$ and $db = a_1a_2 + \cdots + a_{2q-1}a_{2q}$. It follows from [91, Remark 5.4] that the CDGA $(A, d)$ is $(q-1)$-formal but not $q$-formal.

We refer to Măcinic [91] for a more thorough discussion of partial formality and related notions (see also [105, 132]).

### 3.4. Field extensions and formality

As is well known, a finite-type, rationally defined CDGA is formal if and only it is formal over any field of characteristic 0. This foundational result was proved independently and in various degrees of generality by Sullivan [137], Neisendorfer and Miller [101], and Halperin and Stasheff [68]. We conclude this section with a discussion of this topic and some recent generalizations from [132] to partially formal CDGAs.

Given a DGA $(A, d)$ over a field $\Bbbk$ of characteristic 0 and a field extension $\Bbbk \subset \mathbb{K}$, we let $(A \otimes_{\Bbbk} \mathbb{K}, d \otimes \mathrm{id}_{\mathbb{K}})$ be the CDGA over $\mathbb{K}$ obtained by extending scalars.

**Theorem 3.7** ([68]). *Let $(A, d_A)$ and $(B, d_B)$ be two CDGAs over $\Bbbk$ whose cohomology algebras are connected and of finite type. Suppose there is an isomorphism of graded algebras, $f\colon H^*(A) \to H^*(B)$, and suppose $f \otimes \mathrm{id}_{\mathbb{K}}\colon H^*(A) \otimes \mathbb{K} \to H^*(B) \otimes \mathbb{K}$ can be realized by a weak equivalence between $(A \otimes \mathbb{K}, d_A \otimes \mathrm{id}_{\mathbb{K}})$ and $(B \otimes \mathbb{K}, d_B \otimes \mathrm{id}_{\mathbb{K}})$. Then $f$ can be realized by a weak equivalence between $(A, d_A)$ and $(B, d_B)$.*

This theorem has the following corollary. The result is stated without proof in [68, Corollary 6.9]; a complete proof is provided in [132, Corollary 4.17]. Self-contained proofs under some additional hypotheses were previously given in [137, Theorem 12.1] and [101, Corollary 5.2].

**Corollary 3.8** ([68]). *Let $A = (A, d_A)$ be a connected $\Bbbk$-CDGA with $H^*(A)$ of finite-type. Then $A$ is formal if and only if the $\mathbb{K}$-CDGA $(A \otimes \mathbb{K}, d_A \otimes \mathrm{id}_{\mathbb{K}})$ is formal.*

These classical formality results were generalized in [132, Theorem 4.19], which extends the descent-of-formality results of Sullivan, Neisendorfer–Miller, and Halperin–Stasheff to the partially formal setting.

**Theorem 3.9** ([132]). *Let $(A, d_A)$ be a CDGA over $\Bbbk$, and let $\Bbbk \subset \mathbb{K}$ be a field extension. Suppose $H^{\leq q+1}(A)$ is finite-dimensional and $H^0(A) = \Bbbk$. Then $(A, d_A)$ is $q$-formal if and only if $(A \otimes \mathbb{K}, d_A \otimes \mathrm{id}_{\mathbb{K}})$ is $q$-formal.*

### 3.5. Formality of DGA maps

The notion of formality may be extended from the objects to the morphisms of the DGA category, as follows.

**Definition 3.10.** A DGA morphism $\varphi\colon A \to B$ is said to be *formal* if there is a diagram of the form (2.2) connecting $\varphi$ to the induced homomorphism $\varphi^*\colon H^*(A) \to H^*(B)$ between cohomology algebras (viewed as DGAs with zero differentials). Likewise, $\varphi$ is said to be *q-formal*, for some $q \geq 0$, if $\varphi \simeq_q \varphi^*$.

Note that in the first case both $A$ and $B$ need to be formal DGAs, while in the second case both $A$ and $B$ need to be $q$-formal. Also note that if $\varphi$ is formal and $\varphi \simeq \varphi'$, then $\varphi'$ is also formal, and similarly for $q$-formality.

Completely analogous notions may be defined for CDGA morphisms, although we do not know whether a statement analogous to Theorem 2.2 holds in this context. Nevertheless, a descent of formality result analogous to Corollary 3.8 holds.

**Theorem 3.11** ([137, 146]). *Let $\varphi\colon A \to B$ be a morphism between connected $\Bbbk$-CDGAs with finite Betti numbers, and let $\Bbbk \subset \mathbb{K}$ be a field extension. Then $\varphi$ is formal if and only if $\varphi \otimes \mathrm{id}_{\mathbb{K}}\colon A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ is formal.*

We do not know whether a statement in the spirit of Theorems 3.9 and 3.11 holds for $q$-formal maps.

**Example 3.12.** Fix an even integer $n \geq 2$ and consider the CDGA morphism $\varphi\colon (A, d) \to (B, 0)$, where $A = \bigwedge(a, b)$, with $|a| = n$, $|b| = 2n - 1$, and differential given by $d(a) = 0$ and $d(b) = a^2$; $B = \bigwedge(c)$ with $|c| = 2n - 1$; and $\varphi$ is given by $\varphi(a) = 0$ and $\varphi(b) = c$. Then $H^*(A) = \bigwedge(u)$ with $|a| = n$, and the CDGA map $\psi\colon A \to H^*(A)$ given by $\psi(a) = u$ and $\psi(b) = 0$ induces the identity in cohomology. Nevertheless, the map $\varphi^*\colon \widetilde{H}^*(A) \to \widetilde{H}^*(B)$ is trivial, and so the morphism $\varphi$, which is non-trivial, cannot be formal.

### 3.6. Massey products

A well-known obstruction to formality is provided by the higher-order Massey products, introduced by W. S. Massey in [95], and studied for instance in [5, 78, 97, 116, 117, 121, 143].

Let $(A, d)$ be a $\Bbbk$-DGA and let $u_1, \dots, u_n$ be elements in $H^*(A)$; without loss of essential generality, we may assume that $n \geq 3$ and each $u_i$ is homogeneous and of positive degree. A *defining system* for $u_1, \dots, u_n$ is a collection of elements $a_{i,j} \in A$ such that $da_{i-1,i} = 0$ and $[a_{i-1,i}] = u_i$ for $1 \leq i \leq n$ and $da_{i,j} = \sum_{i<r<j} \overline{a}_{i,r} a_{r,j}$ for $0 \leq i < j \leq n$ and $(i, j) \neq (0, n)$. It is readily verified that the element

$$\alpha := \sum_{0<r<n} \overline{a}_{0,r} a_{r,n} \tag{3.1}$$

is a cocycle, of degree $|\alpha| = 2 - n + \sum_{i=1}^{n} |u_i|$. The $n$-fold Massey product $\langle u_1, \dots, u_n \rangle$, then, is the subset of $H^*(A)$ consisting of the cohomology classes $[\alpha]$ corresponding to all possible defining systems for $u_1, \dots, u_n$. We say that the Massey product is *defined* if there is at least one such defining system, or, equivalently, $\langle u_1, \dots, u_n \rangle \neq \emptyset$, in which case the indeterminacy of the Massey product is the subset $\{u - v \mid u, v \in \langle u_1, \dots, u_n \rangle\} \subseteq H^*(A)$. When a Massey product is defined, we say it *vanishes* if it contains the element 0; otherwise, it is non-vanishing.

The simplest Massey triple products are as follows. Let $u_1, u_2, u_3$ be elements in $H^1(A)$ such that $u_1 u_2 = u_2 u_3 = 0$. We may then choose 1-cocycles $a_{0,1}, a_{1,2}, a_{2,3}$ representing $u_1, u_2, u_3$ and 1-cochains $a_{0,2}$ and $a_{1,3}$ such that $da_{0,2} = -a_{0,1}a_{1,2}$ and $da_{1,3} = -a_{1,2}a_{2,3}$. The triple product $\langle u_1, u_2, u_3 \rangle$ is then the subset of $H^2(A)$ consisting of the cohomology classes $-[a_{0,1}a_{1,3} + a_{0,2}a_{2,3}]$, for all such choices of defining systems. Due to the ambiguity in the choices made, $\langle u_1, u_2, u_3 \rangle$ may be viewed as a coset of $u_1 \cdot H^1(A) + H^1(A) \cdot u_3$ in $H^2(A)$.

**Example 3.13.** Let $(A, d)$ be the CDGA from Example 3.2 with $n = 1$; namely, $A$ is the exterior algebra on generators $a_1, a_2, b$ in degree 1 and differential given by $da_i = 0$ and $db = a_1 a_2$. Letting $u_i = [a_i] \in H^1(A)$, we have that the triple Massey products $\langle u_1, u_1, u_2 \rangle = \{[a_1 b]\}$ and $\langle u_1, u_2, u_2 \rangle = \{[b a_2]\}$ are defined, have 0 indeterminacy, and are non-vanishing; in fact, the two cohomology classes generate $H^2(A)$. Therefore, $A$ is not formal.

Massey products enjoy the following (partial) functoriality properties.

**Proposition 3.14** ([78, 97]). *Let $\varphi\colon A \to B$ be a* DGA *morphism, and let $\varphi^*\colon H^*(A) \to H^*(B)$ be the induced morphism in cohomology; then*

$$\varphi^*(\langle u_1, \dots, u_n \rangle) \subseteq \langle \varphi^*(u_1), \dots, \varphi^*(u_n) \rangle. \tag{3.2}$$

*Moreover, if $\varphi$ is a quasi-isomorphism, then* (3.2) *holds as equality.*

In particular, if $\langle u_1, \dots, u_n \rangle$ is defined, then $\langle \varphi^*(u_1), \dots, \varphi^*(u_n) \rangle$ is also defined; and if, in addition, $\langle \varphi^*(u_1), \dots, \varphi^*(u_n) \rangle$ is non-vanishing, then $\langle u_1, \dots, u_n \rangle$ is also non-vanishing. As another consequence, the following holds: if $A \simeq B$, then all Massey products in $H^*(A)$ vanish if and only if all Massey products in $H^*(B)$ vanish.

Finally, if the map $\varphi\colon A \to B$ is a $q$-quasi-isomorphism, for some $q \geq 0$, then (3.2) holds as equality in degrees up to $q + 1$. Thus, if $A \simeq_q B$, then all Massey products in $H^{\leq q+1}(A)$ vanish if and only if all Massey products in $H^{\leq q+1}(B)$ vanish.

### 3.7. Massey products and formality

The vanishing of Massey products provides a well-known obstruction to formality. An analogous statement holds for partial formality. For completeness, we make a formal statement and sketch the proof.

**Proposition 3.15.** *Let $(A, d)$ be a $\Bbbk$-DGA. If $A$ formal, then all Massey products in $H^*(A)$ vanish. Furthermore, if $A$ is $q$-formal, for some $q \geq 1$, then all Massey products in $H^{\leq q+1}(A)$ vanish.*

*Proof.* First suppose $d = 0$, so that $H^*(A) = A$. Let $\langle u_1, \dots, u_n \rangle$ be a Massey product. We may then choose a defining system with $a_{i-1,i} = u_i$ and all other $a_{i,j} = 0$, which implies that the cocycle $\alpha$ from (3.1) is equal to 0; thus, $\langle u_1, \dots, u_n \rangle$ vanishes.

Now suppose $(A, d)$ is formal, that is, $(A, d) \simeq (B, 0)$. As we just saw, all Massey products vanish in $H^*(B)$; hence, they must also vanish in $H^*(A)$. The case when $(A, d)$ is $q$-formal is treated completely analogously. ∎

In general, formality is stronger than the mere vanishing of all Massey products; in fact, it is equivalent to the *uniform* vanishing of all such products. This phenomenon will be illustrated in Theorem 10.9, where we shall see examples of non-formal CDGAs for which all Massey products vanish.

## 4. Minimal models

### 4.1. Hirsch extensions

Given a graded $\Bbbk$-vector space $V^*$, recall that $\bigwedge V$ denotes the free graded, graded-commutative algebra generated by $V$. Choosing a homogeneous basis $\mathcal{X} = \{x_\alpha\}_{\alpha \in J}$ for $V$, this algebra may be identified with $\bigwedge \mathcal{X} := \bigotimes_\alpha \bigwedge(x_\alpha)$, where $\bigwedge(x)$ is the exterior (respectively, polynomial) algebra on a single generator $x$ of odd (respectively, even) degree.

Now let $A = (A^*, d_A)$ be an arbitrary CDGA. A *Hirsch extension* $A$ (of degree $n$) is an inclusion, $(A, d_A) \hookrightarrow (A \otimes \bigwedge V, d)$, where $V$ is a $\Bbbk$-vector space concentrated in degree $n$ and the differential $d$ sends $V$ into $A^{n+1}$. We say this is a *finite* Hirsch extension if $V$ is finite-dimensional. The next lemma follows straight from the definitions.

**Lemma 4.1.** *Let $\alpha \colon (A, d_A) \hookrightarrow (A \otimes \bigwedge V, d)$ be the inclusion map of a Hirsch extension of degree $n + 1$. Then $\alpha$ is an $n$-quasi-isomorphism.*

Now suppose $V$ is an oddly-graded, finite-dimensional vector space, with homogeneous basis $\{t_i \in V^{m_i}\}$. Given a degree 1 linear map, $\tau \colon V^* \to Z^{*+1}(A)$, we define the corresponding Hirsch extension as the CDGA $(A \otimes_\tau \bigwedge V, d)$ where the differential $d$ extends the differential on $A$, while $dt_i = \tau(t_i)$.

**Proposition 4.2** ([85]). *The isomorphism type of the* CDGA $(A \otimes_\tau \bigwedge V, d)$ *depends only on $A$ and the homomorphism induced in cohomology, $[\tau] \colon V^* \to H^{*+1}(A)$. Moreover, $[\tau]$ and $[\tau] \circ g$ give isomorphic extensions, for any automorphism $g$ of the graded vector space $V$.*

The above result is proved in [85, Lemmas II.2 and II.3] in the case when all the degrees $m_i$ are equal; the same argument works in the general case.

**Proposition 4.3** ([114]). *Let $A = B \otimes_\tau \bigwedge(t_i)$ be a Hirsch extension with variables $t_i$ of odd degree $m_i$. If $B$ is an $n$-PD-CDGA, then $A$ is an $m$-PD-CDGA, where $m = n + \sum m_i$.*

### 4.2. Minimal CDGAs

The following key definition is due to Sullivan [137].

**Definition 4.4** ([137]). A CDGA $A = (A^*, d)$ is said to be *minimal* if the following conditions are satisfied.

(1) $A = \bigwedge \mathcal{X}$ is the free CGA on positive-degree generators $\mathcal{X} = \{x_\alpha\}_{\alpha \in J}$ indexed by a well-ordered set $J$.

(2) $dx_\alpha \in \bigwedge(\{x_\beta \mid \beta < \alpha\})$ for all $\alpha$.

(3) $dx_\alpha \in \bigwedge^+ \mathcal{X} \cdot \bigwedge^+ \mathcal{X}$ for all $\alpha$, where $\bigwedge^+ \mathcal{X}$ is the ideal generated by $\mathcal{X}$.

Letting $V^*$ be the graded vector space generated by the set $\mathcal{X}$, we may also write $A = (\bigwedge V, d)$. We say that $A$ is $q$-*minimal* (for some $q \geq 1$) if $A$ is minimal and $V^i = 0$ for all $i > q$, or, equivalently, $\deg(x_\alpha) \leq q$ for all $\alpha$.

Here is an alternative interpretation of this notion. The CDGA $(A, d)$ is minimal if $A = \bigcup_{j \geq 0} A_j$, where $A_0 = \Bbbk$, each $A_j$ is a Hirsch extension of $A_{j-1}$, and the differential $d$ is decomposable, i.e., $dA \subset A^+ \wedge A^+$, where $A^+ = \bigoplus_{n \geq 1} A^n$. This yields an increasing, exhausting filtration of $A$ by the sub-DGAs $A_j$. The decomposability of the differential is automatically satisfied if $A$ is generated in degree 1.

The next lemma illustrates some of the usefulness of the notion of 1-minimality.

**Lemma 4.5** ([39]). *Let $A$ be a 1-minimal CDGA, and let $\varphi, \psi\colon A \to B$ be two homotopic CDGA morphisms. Suppose $d_B = 0$ and $\varphi^1\colon A^1 \to B^1$ is surjective. Then $\psi^1\colon A^1 \to B^1$ is also surjective.*

### 4.3. Minimal models

Let $A$ be a CDGA. We say that a CDGA $\mathcal{M}$ is a *minimal model* for $A$ if $\mathcal{M}$ is a minimal CDGA and there exists a quasi-isomorphism $\rho\colon \mathcal{M} \to A$. Likewise, we say that a minimal CDGA $\mathcal{M}$ is a $q$-*minimal model* for $A$ if $\mathcal{M}$ is generated by elements of degree at most $q$, and there exists a $q$-quasi-isomorphism $\rho\colon \mathcal{M} \to A$. A basic result in rational homotopy theory is the following existence and uniqueness theorem, first proved for minimal models by Sullivan [137], and for partial minimal models by Morgan [100].

**Theorem 4.6** ([100,137]). *Let $A$ be a $\Bbbk$-CDGA with $H^0(A) = \Bbbk$. Then $A$ admits a minimal model, $\mathcal{M}(A)$, unique up to isomorphism. Likewise, for each $q \geq 0$, there is a $q$-minimal model, $\mathcal{M}_q(A)$, unique up to isomorphism.*

By construction, $\mathcal{M}(A) = (\bigwedge V, d)$ and $\mathcal{M}_q(A) = (\bigwedge V^{\leq q}, d)$, for some graded vector space $V$. It follows that the minimal model $\mathcal{M}(A)$ is isomorphic to a minimal model built from the $q$-minimal model $\mathcal{M}_q(A)$ by means of Hirsch extensions in degrees $q+1$ and higher. Thus, in view of Lemma 4.1, $\mathcal{M}_q(A) \simeq_q \mathcal{M}(A)$.

Applying Lemma 2.3, we obtain the following finiteness criterion for CDGAs.

**Proposition 4.7.** *Let $A$ be a $q$-finite* CDGA*. Then $b_i(\mathcal{M}_q(A)) < \infty$ for all $i \leq q+1$.*

The minimal model comes with a structural quasi-isomorphism, $\rho\colon \mathcal{M}(A) \to A$. If $\rho'\colon \mathcal{M}'(A) \to A$ is another minimal model, there is an isomorphism $\psi\colon \mathcal{M}(A) \xrightarrow{\cong} \mathcal{M}'(A)$ such that $\rho' \circ \psi \simeq \rho$. Furthermore, the minimal model is functorial: if $\varphi\colon A \to B$ is a morphism between two CDGAs with connected homology, there is an induced morphism of CDGAs, $\mathcal{M}(\varphi)\colon \mathcal{M}(A) \to \mathcal{M}(B)$, such that $\rho_B \circ \mathcal{M}(\varphi) \simeq \varphi \circ \rho_A$. Similar results hold for the partial minimal models.

The above considerations imply the following: two CDGAs with connected homology are weakly isomorphic if and only if their minimal models are isomorphic. Alternatively, if $A$ and $A'$ are two CDGAs with connected homology, then $A \simeq A'$ if and only if there is a minimal CDGA $\mathcal{M}$ and a short zigzag of quasi-isomorphisms,

$$A \xleftarrow{\rho} \mathcal{M} \xrightarrow{\rho'} A'.$$

Analogous results hold for $q$-minimal models.

### 4.4. Minimality and formality

In [37], Deligne, Griffiths, Morgan, and Sullivan gave a very practical interpretation of formality in the context of minimal CDGAs.

**Theorem 4.8** ([37]). *Let $A = (\bigwedge V, d)$ be a minimal* CDGA*. Then $A$ is formal if and only if each subspace $V^i = A^i \cap V$ decomposes as a direct sum, $V^i = N^i \oplus C^i$, where $C^i = Z^i(A) \cap V$ and any cocycle in the ideal of $A$ generated by $\bigoplus N^i$ is a coboundary.*

As noted in [37], choosing complements $N^i$ to $C^i$ with the specified property is equivalent to choosing a CDGA-morphism $(A, d) \to (H^*(A), 0)$ inducing the identity in cohomology. Furthermore, the existence of splittings $V^i = N^i \oplus C^i$ such that any cocycle in the ideal generated by $\bigoplus_i N^i$ is a coboundary is one way of saying that one may make uniform choices of subspaces spanned by defining systems so that all the cocycles representing Massey products are coboundaries – a stronger condition than saying that each individual Massey product vanishes.

Work of Sullivan [137] and Morgan [100] shows that a CDGA $(A, d)$ is formal if and only of there exists a quasi-isomorphism $\psi\colon \mathcal{M}(A) \to (H^*(A), 0)$. Likewise, Măcinic showed in [91] that $A$ is $q$-formal if and only if there exists a $q$-quasi-isomorphism $\psi_q\colon \mathcal{M}_q(A) \to (H^*(A), 0)$. The following lemma provides a convenient criterion for partial formality.

**Lemma 4.9** ([132]). *Let $A$ be a $\Bbbk$-CDGA, and suppose that* $\dim_{\Bbbk} H^{q+1}(\mathcal{M}_q(A)) < \infty$. *Then $A$ is $q$-formal if and only if $\mathcal{M}_q(A)$ is $q$-formal.*

Minimal models are also relevant when considering the formality of morphisms of CDGAs. Indeed, let $\varphi\colon A \to B$ be a CDGA map; then $\varphi$ is formal (in the sense of Definition 3.10) if and only if there is a diagram of the form

$$\begin{array}{ccccc} A & \xleftarrow{\rho_A} & \mathcal{M}(A) & \xrightarrow{\psi_A} & (H^*(A),0) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mathcal{M}(\varphi)} & & \downarrow{\scriptstyle\varphi^*} \\ B & \xleftarrow{\rho_B} & \mathcal{M}(B) & \xrightarrow{\psi_B} & (H^*(B),0) \end{array}$$

which commutes up to homotopy.

Analogous statements hold for $q$-formal maps, with the middle arrow replaced by the morphism $\mathcal{M}_q(\varphi)\colon \mathcal{M}_q(A) \to \mathcal{M}_q(B)$.

### 4.5. The dual of a 1-minimal CDGA

Let $A = (A^*, d)$ be a minimal CDGA over $\Bbbk$, generated in degree 1. Following [53,77,100], let us consider the filtration

$$\Bbbk = A(0) \subset A(1) \subset A(2) \subset \cdots \subset A = \bigcup_{i\geq 0} A(i), \tag{4.1}$$

where $A(1)$ is the subalgebra of $A$ generated by the cocycles in $A^1$, and $A(i)$ for $i > 1$ is the subalgebra of $A$ generated by all elements $x \in A^1$ such that $dx \in A(i-1)$. Each inclusion $A(i-1) \subset A(i)$ is a Hirsch extension of the form $A(i) = A(i-1) \otimes \bigwedge V_i$, where

$$V_i := \ker\bigl(H^2(A(i-1)) \to H^2(A)\bigr).$$

Taking degree 1 pieces in the filtration (4.1), we obtain the filtration $\Bbbk = A(0)^1 \subset A(1)^1 \subset \cdots \subset A^1$. Clearly, $A^1$ is a 1-minimal CDGA.

Let us assume now that each of the aforementioned Hirsch extensions is finite, that is, $\dim V_i < \infty$ for all $i$. Using the fact that $d(V_i) \subset A(i-1)$, we infer that each dual vector space $\mathfrak{L}_i = (A(i)^1)^\vee$ acquires the structure of a $\Bbbk$-Lie algebra by setting

$$\langle [u^\vee, v^\vee], w\rangle := \langle u^\vee \wedge v^\vee, dw\rangle \tag{4.2}$$

for $u, v, w \in A(i)^1$. Clearly, $d(V_1) = 0$, and thus $\mathfrak{L}_1 = (V_1)^\vee$ is an abelian Lie algebra. Using the vector space decompositions

$$A(i)^1 = A(i-1)^1 \oplus V_i \quad \text{and} \quad A(i)^2 = A(i-1)^2 \oplus (A(i-1)^1 \otimes V_i) \oplus \bigwedge^2 V_i,$$

one easily sees that the canonical projection $\mathfrak{L}_i \twoheadrightarrow \mathfrak{L}_{i-1}$, defined as the dual of the inclusion map $A(i-1) \hookrightarrow A(i)$, has kernel $V_i^\vee$, and this kernel is central inside $\mathfrak{L}_i$.

Therefore, we obtain a tower of finite-dimensional, nilpotent $\Bbbk$-Lie algebras,

$$0 \longleftarrow \mathfrak{L}_1 \twoheadleftarrow \mathfrak{L}_2 \twoheadleftarrow \cdots \twoheadleftarrow \mathfrak{L}_i \twoheadleftarrow \cdots . \tag{4.3}$$

Let $\mathfrak{L} = \mathfrak{L}(A)$ be the inverse limit of this tower, equipped with the inverse limit filtration. Then $\mathfrak{L}$ is a complete, filtered Lie algebra with the property that $\mathfrak{L}/\widehat{\gamma}_{i+1}(\mathfrak{L}) = \mathfrak{L}_i$ for each $i \geq 1$. Conversely, from a tower as in (4.3), one can construct a sequence of finite Hirsch extensions as in (4.1). Let $A(i) = A(i-1) \otimes \bigwedge V_i$ be one of the CDGAs in this sequence, with differential given by (4.2). Then $A(i)$ coincides with the Chevalley–Eilenberg complex $\mathcal{C}(\mathfrak{L}_i) = (\bigwedge \mathfrak{L}_i^\vee, d)$ associated to the finite-dimensional Lie algebra $\mathfrak{L}_i = \mathfrak{L}(A(i))$; that is, the CDGA whose underlying graded algebra is the exterior algebra on the dual vector space $\mathfrak{L}_i^\vee$, and whose differential is the extension by the graded Leibniz rule of the dual of the signed Lie bracket on the algebra generators. By the definition of Lie algebra cohomology, then,

$$H^*(A(i)) \cong H^*(\mathfrak{L}_i, \Bbbk).$$

The direct limit of the above sequence of Hirsch extensions, $A = \bigcup_{i\geq 0} A(i)$, is a minimal $\Bbbk$-CDGA generated in degree 1. We obtain in this fashion an adjoint correspondence that sends $A$ to the pronilpotent Lie algebra $\mathfrak{L} = \mathfrak{L}(A)$ and conversely, sends a pronilpotent Lie algebra $\mathfrak{L}$ to the minimal algebra $A = A(\mathfrak{L})$. Under this correspondence, filtration-preserving CDGA morphisms $A \to B$ get sent to filtration-preserving Lie morphisms $\mathfrak{L}(B) \to \mathfrak{L}(A)$, and the other way around.

### 4.6. Positive weights

Following Body, Mimura, Shiga, and Sullivan [20], Morgan [100], and Sullivan [137], we say that a commutative graded algebra $A^*$ has *positive weights* if each graded piece admits a vector space decomposition

$$A^i = \bigoplus_{\alpha\in\mathbb{Z}} A^{i,\alpha}$$

with $A^{i,\alpha} = 0$ for $\alpha \leq 0$, such that $xy \in A^{i+j,\alpha+\beta}$ for $x \in A^{i,\alpha}$ and $y \in A^{j,\beta}$. Furthermore, we say that a CDGA $(A, d)$ has *positive weights* if the underlying CGA $A^*$ has positive weights, and the differential is homogeneous with respect to those weights, that is, $dx \in A^{i+1,\alpha}$ for $x \in A^{i,\alpha}$.

Now let $(A, d)$ be a minimal CDGA generated in degree one, endowed with the canonical filtration $\{A_i\}_{i\geq 0}$ constructed in (4.1), where each sub-CDGA $A_i$ is given by a Hirsch extension of the form $A_{i-1} \otimes \bigwedge V_i$. The underlying CGA $A$ possesses a natural set of positive weights, which we will refer to as the *Hirsch weights*: simply declare $V_i$ to have weight $i$, and extend those weights to $A$ multiplicatively. We say that the CDGA $(A, d)$ has *positive Hirsch weights* if the differential $d$ is homogeneous with respect to those weights. If this is the case, each sub-CDGA $A_i$ also has positive Hirsch weights.

**Lemma 4.10** ([132]). *Let $A$ be a minimal* CDGA *generated in degree one, with dual Lie algebra $\mathfrak{L} = \mathfrak{L}(A)$. Then $A$ has positive Hirsch weights if and only if $\mathfrak{L} = \widehat{\operatorname{gr}}(\mathfrak{L})$.*

The next example (extracted from [132]) shows that the hypothesis of Lemma 4.10 is more restrictive than the Lie algebra $\mathfrak{L} = \mathfrak{L}(A)$ being filtered-formal.

**Example 4.11.** Let $\mathfrak{g}$ be the 5-dimensional Lie algebra with basis $e_1, \dots, e_5$ and with Lie brackets given by $[e_1, e_2] = e_3 - e_4/2 - e_5$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$, and $[e_i, e_j] = 0$, otherwise. It is readily verified that $\mathfrak{g}$ is filtered-formal, although the differential of the 1-minimal CDGA $A = \bigwedge \mathfrak{g}^\vee$ is not homogeneous on the Hirsch weights.

If a minimal CDGA is generated in degree 1 and has positive weights, but these weights do not coincide with the Hirsch weights, then the dual Lie algebra need not be filtered-formal. This phenomenon is illustrated in the next example, adapted from [34, 132].

**Example 4.12.** Let $\mathfrak{g}$ be the nilpotent, 5-dimensional Lie algebra with non-zero Lie brackets given by $[e_1, e_3] = e_4$ and $[e_1, e_4] = [e_2, e_3] = e_5$. The center of $\mathfrak{g}$ is 1-dimensional, generated by $e_5$, while the center of $\operatorname{gr}(\mathfrak{g})$ is 2-dimensional, generated by $e_2$ and $e_5$. Therefore, $\mathfrak{g} \not\cong \operatorname{gr}(\mathfrak{g})$, and so $\mathfrak{g}$ is not filtered-formal. The 1-minimal CDGA $A = \bigwedge \mathfrak{g}^\vee$ does have positive weights, given by the index of the chosen basis, but $A$ does not admit positive Hirsch weights.

## 5. Lie algebras and filtered formality

### 5.1. Graded Lie algebras

Once again, let us fix a ground field $\Bbbk$ of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$; that is, a $\Bbbk$-vector space $\mathfrak{g}$ endowed with an alternating bilinear operation, $[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, that satisfies the Jacobi identity. We say that $\mathfrak{g}$ is a *graded Lie algebra* if $\mathfrak{g}$ decomposes as $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$, the Lie bracket is compatible with the grading, and the Lie identities are satisfied with the appropriate signs. A morphism of graded Lie algebras is a $\Bbbk$-linear map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ which preserves the Lie brackets and the degrees; in particular, $\varphi$ induces $\Bbbk$-linear maps $\varphi_i\colon \mathfrak{g}_i \to \mathfrak{h}_i$ for all $i \geq 1$.

The most basic example of a graded Lie algebra is constructed as follows. Let $V$ a $\Bbbk$-vector space. The tensor algebra $T(V)$ has a natural Hopf algebra structure, with comultiplication $\Delta$ and counit $\varepsilon$ the algebra maps given by $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\varepsilon(v) = 0$, for $v \in V$. The *free Lie algebra* on $V$ is the set of primitive elements in the tensor algebra; that is, $\operatorname{Lie}(V) = \{x \in T(V) \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$, with Lie bracket $[x, y] = x \otimes y - y \otimes x$ and grading induced from $T(V)$.

A Lie algebra $\mathfrak{g}$ is said to be *finitely generated* if there is an epimorphism $\varphi\colon \operatorname{Lie}(V) \to \mathfrak{g}$ for some finite-dimensional $\Bbbk$-vector space $V$. If, moreover, the Lie ideal $\mathfrak{r} = \ker(\varphi)$ is finitely generated as a Lie algebra, then $\mathfrak{g}$ is called *finitely presented.* Now suppose all elements of $V$ are assigned degree 1 in $T(V)$. Then the inclusion

$\iota\colon \mathrm{Lie}(V) \to T(V)$ identifies $\mathrm{Lie}_1(V)$ with $T_1(V) = V$. Furthermore, $\iota$ maps $\mathrm{Lie}_2(V)$ to $T_2(V) = V \otimes V$ by sending $[v, w]$ to $v \otimes w - w \otimes v$ for each $v, w \in V$; we thus may identify $\mathrm{Lie}_2(V) \cong V \wedge V$ by sending $[v, w]$ to $v \wedge w$.

If $\mathfrak{g} = \mathrm{Lie}(V)/\mathfrak{r}$, with $V$ a (finite-dimensional) vector space concentrated in degree 1, then we say $\mathfrak{g}$ is *(finitely) generated in degree* 1. If, moreover, the Lie ideal $\mathfrak{r}$ is homogeneous, then $\mathfrak{g}$ is a graded Lie algebra. In particular, if $\mathfrak{g}$ is finitely generated in degree 1 and the homogeneous ideal $\mathfrak{r}$ is generated in degree 2, then we say $\mathfrak{g}$ is a *quadratic Lie algebra*.

## 5.2. Filtrations

A *filtration* $\mathcal{F}$ on a Lie algebra $\mathfrak{g}$ is a nested sequence of Lie ideals, $\mathfrak{g} = \mathcal{F}_1(\mathfrak{g}) \supset \mathcal{F}_2(g) \supset \cdots$. A well-known such filtration is the *derived series*, with terms $\mathcal{F}_i(\mathfrak{g}) = \mathfrak{g}^{(i-1)}$ inductively defined by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ for $i \geq 1$. Clearly, the derived series is preserved by Lie algebra maps, and the quotient Lie algebras $\mathfrak{g}/\mathfrak{g}^{(i)}$ are solvable. Moreover, if $\mathfrak{g}$ is a graded Lie algebra, all these solvable quotients inherit a graded Lie algebra structure.

The existence of a filtration $\mathcal{F}$ on a Lie algebra $\mathfrak{g}$ makes $\mathfrak{g}$ into a topological vector space, by defining a basis of open neighborhoods of an element $x \in \mathfrak{g}$ to be $\{x + \mathcal{F}_k(\mathfrak{g})\}_{k\in\mathbb{N}}$. The fact that each basis neighborhood $\mathcal{F}_k(\mathfrak{g})$ is a Lie subalgebra implies that the Lie bracket map $[\, ,\, ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous; thus, $\mathfrak{g}$ is, in fact, a topological Lie algebra. We say that $\mathfrak{g}$ is *complete* (respectively, *separated*) if the underlying topological vector space enjoys those properties.

Every ideal $\mathfrak{a}$ of $\mathfrak{g}$ inherits a filtration, given by $\mathcal{F}_k(\mathfrak{a}) := \mathcal{F}_k(\mathfrak{g}) \cap \mathfrak{a}$. Likewise, the quotient Lie algebra, $\mathfrak{g}/\mathfrak{a}$, has a naturally induced filtration with terms $\mathcal{F}_k(\mathfrak{g})/\mathcal{F}_k(\mathfrak{a})$. Letting $\overline{\mathfrak{a}}$ be the closure of $\mathfrak{a}$ in the filtration topology, we have that $\overline{\mathfrak{a}}$ is a closed ideal of $\mathfrak{g}$. Moreover, by the continuity of the Lie bracket, $\overline{[\overline{\mathfrak{a}}, \overline{\mathfrak{r}}]} = \overline{[\mathfrak{a}, \mathfrak{r}]}$. Finally, if $\mathfrak{g}$ is complete (or separated), then $\mathfrak{g}/\overline{\mathfrak{a}}$ is also complete (or separated).

For each $j \geq k$, there is a canonical projection, $\mathfrak{g}/\mathcal{F}_j(\mathfrak{g}) \to \mathfrak{g}/\mathcal{F}_k(\mathfrak{g})$, compatible with the projections from $\mathfrak{g}$ to its quotient Lie algebras $\mathfrak{g}/\mathcal{F}_k(\mathfrak{g})$. The *completion* of the Lie algebra $\mathfrak{g}$ with respect to the filtration $\mathcal{F}$ is defined as the limit of this inverse system, $\widehat{\mathfrak{g}} = \varprojlim_k \mathfrak{g}/\mathcal{F}_k(\mathfrak{g})$. Using the fact that $\mathcal{F}_k(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, it is readily seen that $\widehat{\mathfrak{g}}$ is a Lie algebra, with Lie bracket defined componentwise. Furthermore, $\widehat{\mathfrak{g}}$ has a natural inverse limit filtration, $\widehat{\mathcal{F}}$, whose terms $\widehat{\mathcal{F}}_k(\widehat{\mathfrak{g}})$ are equal to $\widehat{\mathcal{F}_k(\mathfrak{g})} = \varprojlim_{i\geq k} \mathcal{F}_k(\mathfrak{g})/\mathcal{F}_i(\mathfrak{g})$. Observe that $\widehat{\mathcal{F}}_k(\widehat{\mathfrak{g}}) = \overline{\mathcal{F}_k(\mathfrak{g})}$, and so each term of the filtration $\widehat{\mathcal{F}}$ is a closed Lie ideal of $\widehat{\mathfrak{g}}$. Furthermore, the Lie algebra $\widehat{\mathfrak{g}}$, endowed with this filtration, is both complete and separated.

Let $\iota\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ be the canonical map to the completion. Then $\iota$ is a morphism of Lie algebras, preserving the respective filtrations. Clearly, $\ker(\iota) = \bigcap_{k\geq 1} \mathcal{F}_k(\mathfrak{g})$; hence, $\iota$ is injective if and only if $\mathfrak{g}$ is separated. Moreover, $\iota$ is surjective if and only if $\mathfrak{g}$ is complete.

### 5.3. Filtered Lie algebras

A *filtered Lie algebra* (over the field $\Bbbk$) is a Lie algebra $\mathfrak{g}$ endowed with a decreasing filtration $\mathscr{F} = \{\mathscr{F}_k(\mathfrak{g})\}_{k\geq 1}$ by $\Bbbk$-vector subspaces, satisfying the condition

$$[\mathscr{F}_k(\mathfrak{g}), \mathscr{F}_\ell(\mathfrak{g})] \subseteq \mathscr{F}_{k+\ell}(\mathfrak{g}) \tag{5.1}$$

for all $k, \ell \geq 1$. This condition implies that each subspace $\mathscr{F}_k(\mathfrak{g})$ is a Lie ideal, and so, in particular, $\mathscr{F}$ is a Lie algebra filtration. Let

$$\operatorname{gr}^{\mathscr{F}}(\mathfrak{g}) := \bigoplus_{k\geq 1} \mathscr{F}_k(\mathfrak{g})/\mathscr{F}_{k+1}(\mathfrak{g}) \tag{5.2}$$

be the corresponding associated graded vector space. Condition (5.1) implies that the Lie bracket map on $\mathfrak{g}$ descends to a map $[\,,\,]\colon \operatorname{gr}^{\mathscr{F}}(\mathfrak{g}) \times \operatorname{gr}^{\mathscr{F}}(\mathfrak{g}) \to \operatorname{gr}^{\mathscr{F}}(\mathfrak{g})$ which makes $\operatorname{gr}^{\mathscr{F}}(\mathfrak{g})$ into a graded Lie algebra, with graded pieces given by the decomposition (5.2).

A morphism of filtered Lie algebras is a linear map $\phi\colon \mathfrak{g} \to \mathfrak{h}$ preserving Lie brackets and the given filtrations, $\mathscr{F}$ and $\mathscr{G}$. Such a map induces morphisms between nilpotent quotients, $\phi_k\colon \mathfrak{g}/\mathscr{F}_{k+1}(\mathfrak{g}) \to \mathfrak{h}/\mathscr{G}_{k+1}(\mathfrak{h})$, and a morphism of associated graded Lie algebras, $\operatorname{gr}(\phi)\colon \operatorname{gr}^{\mathscr{F}}(\mathfrak{g}) \to \operatorname{gr}^{\mathscr{G}}(\mathfrak{h})$.

If $\mathfrak{g}$ is a filtered Lie algebra with a multiplicative filtration $\mathscr{F}$ as in (5.1), then its completion, $\widehat{\mathfrak{g}}$, is again a filtered Lie algebra with the completed multiplicative filtration $\widehat{\mathscr{F}}$. By construction, the canonical map to the completion, $\iota\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$, is a morphism of filtered Lie algebras. It is readily seen that the induced morphism, $\operatorname{gr}(\iota)\colon \operatorname{gr}^{\mathscr{F}}(\mathfrak{g}) \to \operatorname{gr}^{\widehat{\mathscr{F}}}(\widehat{\mathfrak{g}})$, is an isomorphism. Moreover, if $\mathfrak{g}$ is both complete and separated, then the map $\iota\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ itself is an isomorphism of filtered Lie algebras. More generally, if $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of complete, separated, filtered Lie algebras, and $\operatorname{gr}(\phi)$ is an isomorphism, then, as noted in [132, Lemma 2.1], $\phi$ is also an isomorphism.

### 5.4. The degree completion

Every Lie algebra $\mathfrak{g}$ comes equipped with a lower central series (LCS) filtration, $\{\gamma_n(\mathfrak{g})\}_{n\geq 1}$. This filtration is defined inductively by $\gamma_1(\mathfrak{g}) = \mathfrak{g}$ and $\gamma_n(\mathfrak{g}) = [\gamma_{n-1}(\mathfrak{g}), \mathfrak{g}]$ for $n \geq 2$. This is a multiplicative filtration, and if $\{\mathscr{F}_n(\mathfrak{g})\}_{n\geq 1}$ is another such filtration, then $\gamma_n(\mathfrak{g}) \subseteq \mathscr{F}_n(\mathfrak{g})$, for all $n \geq 1$. Lie algebra morphisms preserve LCS filtrations, and the quotient Lie algebras $\mathfrak{g}/\gamma_n(\mathfrak{g})$ are nilpotent. We shall write $\operatorname{gr}(\mathfrak{g})$ for the associated graded Lie algebra and $\widehat{\mathfrak{g}}$ for the completion of $\mathfrak{g}$ with respect to the LCS filtration. Furthermore, we shall take $\widehat{\gamma}_n = \overline{\gamma}_n$ as the terms of the canonical filtration on $\widehat{\mathfrak{g}}$.

Every graded Lie algebra, $\mathfrak{g} = \bigoplus_{i\geq 1} \mathfrak{g}_i$, has a canonical decreasing filtration induced by the grading, $\mathscr{F}_n(\mathfrak{g}) := \bigoplus_{i\geq n} \mathfrak{g}_i$. Moreover, if $\mathfrak{g}$ is generated in degree 1, then this filtration coincides with the LCS filtration. In particular, the associated graded Lie algebra with respect to $\mathscr{F}$ coincides with $\mathfrak{g}$. In this case, the completion of $\mathfrak{g}$ with respect to the lower central series (or, degree) filtration is called the *degree completion* of $\mathfrak{g}$, and is simply denoted by $\widehat{\mathfrak{g}}$. It is readily seen that $\widehat{\mathfrak{g}} \cong \prod_{i\geq 1} \mathfrak{g}_i$. Therefore, the morphism $\iota\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ is injective, and induces an isomorphism between $\mathfrak{g}$ and $\operatorname{gr}(\widehat{\mathfrak{g}})$.

**Lemma 5.1** ([132]). *Suppose* $\mathbb{L}$ *is a free Lie algebra generated in degree* 1 *and* $\mathfrak{r}$ *is a homogeneous ideal. Then the projection* $\mathbb{L} \twoheadrightarrow \mathbb{L}/\mathfrak{r}$ *induces an isomorphism* $\widehat{\mathbb{L}}/\overline{\mathfrak{r}} \xrightarrow{\cong} \widehat{\mathbb{L}/\mathfrak{r}}$.

## 5.5. Filtered-formality

We now consider in more detail the relationship between a filtered Lie algebra $\mathfrak{g}$ and the completion of its associated graded Lie algebra, $\widehat{\mathrm{gr}}(\mathfrak{g})$, equipped with the inverse limit filtration. Note that both Lie algebras share the same associated graded Lie algebra, namely, $\mathrm{gr}(\mathfrak{g})$. In general, though, $\mathfrak{g}$ may fail to be isomorphic to $\widehat{\mathrm{gr}}(\mathfrak{g})$. Of course, this happens if $\mathfrak{g}$ is not complete or separated, but it may happen even in the case when $\mathfrak{g}$ is a (finite-dimensional) nilpotent Lie algebra.

**Definition 5.2** ([132]). A complete, separated, filtered Lie algebra $\mathfrak{g}$ is *filtered-formal* if there is a filtered Lie algebra isomorphism, $\mathfrak{g} \cong \widehat{\mathrm{gr}}(\mathfrak{g})$, which induces the identity on associated graded Lie algebras.

If $\mathfrak{g}$ is a filtered-formal Lie algebra, there exists a graded Lie algebra $\mathfrak{h}$ such that $\mathfrak{g}$ is isomorphic to $\widehat{\mathfrak{h}} = \prod_{i\geq 1} \mathfrak{h}_i$. Conversely, if $\mathfrak{g} = \widehat{\mathfrak{h}}$ is the completion of a graded Lie algebra $\mathfrak{h} = \bigoplus_{i\geq 1} \mathfrak{h}_i$, then $\mathfrak{g}$ is filtered-formal. Moreover, if $\mathfrak{h}$ has homogeneous presentation $\mathfrak{h} = \mathrm{Lie}(V)/\mathfrak{r}$, with $V$ finitely generated and concentrated in degree 1, then, by Lemma 5.1, the complete, filtered Lie algebra $\mathfrak{g} = \prod_{i\geq 1} \mathfrak{h}_i$ has presentation $\mathfrak{g} = \widehat{\mathrm{Lie}}(V)/\overline{\mathfrak{r}}$. Some sufficient conditions for filtered formality are given in the following proposition.

**Proposition 5.3** ([132]). *Let* $\mathfrak{g}$ *be a complete, separated, filtered Lie algebra. Suppose one of the following two conditions is satisfied.*

(1) *There is a graded Lie algebra* $\mathfrak{h}$ *and an isomorphism* $\mathfrak{g} \cong \widehat{\mathfrak{h}}$ *preserving filtrations.*
(2) *The graded Lie algebra* $\mathrm{gr}(\mathfrak{g})$ *is generated in degree* 1 *and there is a morphism of filtered Lie algebras,* $\phi\colon \mathfrak{g} \to \widehat{\mathrm{gr}}(\mathfrak{g})$*, such that* $\mathrm{gr}_1(\phi)$ *is an isomorphism.*

*Then* $\mathfrak{g}$ *is filtered-formal.*

As shown in [132], filtered-formality enjoys a descent property, provided some mild finiteness hypotheses are satisfied. As usual, all the ground fields will be of characteristic 0. First, let us record a straightforward lemma, which follows from the fact that completion commutes with tensor products.

**Lemma 5.4.** *Let* $\mathfrak{g}$ *be a filtered-formal Lie algebra over a field* $\Bbbk$. *If* $\Bbbk \subset \mathbb{K}$ *is a field extension, then the* $\mathbb{K}$*-Lie algebra* $\mathfrak{g} \otimes_{\Bbbk} \mathbb{K}$ *is also filtered-formal.*

The next theorem generalizes a result of Cornulier [34]; its proof is based in part on work of Enriquez [47] and Maassarani [90].

**Theorem 5.5** ([132]). *Let* $\mathfrak{g}$ *be a complete, separated, filtered* $\Bbbk$*-Lie algebra such that* $\mathrm{gr}(\mathfrak{g})$ *is finitely generated in degree* 1*, and let* $\Bbbk \subset \mathbb{K}$ *be a field extension. Then* $\mathfrak{g}$ *is filtered-formal if and only if the* $\mathbb{K}$*-Lie algebra* $\mathfrak{g} \otimes_{\Bbbk} \mathbb{K}$ *is filtered-formal.*

## 6. Lower central series and Malcev Lie algebras

### 6.1. Lower central series

Let $G$ be a group. Given subgroups $H_1, H_2 \leq G$, their commutator, $[H_1, H_2]$, is the subgroup of $G$ generated by all elements of the form $[x_1, x_2] := x_1 x_2 x_1^{-1} x_2^{-1}$ with $x_i \in H_i$. The *lower central series* (LCS) of the group, $\{\gamma_n(G)\}_{n\geq 1}$, is defined inductively by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$. This is an $N$-series, in the sense of Lazard [84], that is, $[\gamma_n(G), \gamma_m(G)] \subseteq [\gamma_{m+n}(G)]$ for all $m, n \geq 1$. It follows that each subgroup $\gamma_n(G)$ is normal in $G$; moreover, each LCS quotient $\gamma_n(G)/\gamma_{n+1}(G)$ lies in the center of $G/\gamma_{n+1}(G)$, and thus is an abelian group. For instance, $\gamma_2(G) = [G, G]$ is the derived (or, commutator) subgroup and $G/\gamma_2(G) = G_{\text{ab}}$ is the abelianization of $G$.

If $\gamma_n(G) \neq 1$ but $\gamma_{n+1}(G) = 1$, then $G$ is said to be an *$n$-step nilpotent group*; in general, though, the LCS filtration does not terminate. For each $n \geq 2$, the factor group $G/\gamma_n(G)$ is the maximal $(n-1)$-step nilpotent quotient of $G$.

The direct sum of the LCS quotients, $\operatorname{gr}(G) = \bigoplus_{n\geq 1} \operatorname{gr}_n(G)$, acquires the structure of a graded Lie algebra over $\mathbb{Z}$, called the *associated graded Lie algebra* of $G$. The addition in $\operatorname{gr}(G)$ is induced from the group multiplication and the Lie bracket is induced from the group commutator. For instance, if $G = F_n$ is a finitely generated free group of rank $n \geq 1$, then $\operatorname{gr}(F_n) = \operatorname{Lie}(\mathbb{Z}^n)$, the free Lie algebra on $n$ generators.

If $\Bbbk$ is a field of characteristic 0, then $\operatorname{gr}(G; \Bbbk) := \bigoplus_{n\geq 1} \operatorname{gr}_n(G) \otimes_{\mathbb{Z}} \Bbbk$ is a graded Lie algebra over $\Bbbk$. We note that both the assignments $G \rightsquigarrow \operatorname{gr}(G)$ and $G \rightsquigarrow \operatorname{gr}(G; \Bbbk)$ are functorial.

### 6.2. Malcev completion

A group $G$ is said to be *rational* (or, uniquely divisible) if the power map $G \to G$, $g \mapsto g^n$ is a bijection, for every $n \geq 1$. The rational abelian groups are precisely the $\mathbb{Q}$-vector spaces. A natural way to rationalize an abelian group $A$ is to map it to $A \otimes_{\mathbb{Z}} \mathbb{Q}$ via $a \mapsto a \otimes 1$, with this map being universal for homomorphisms of $G$ into uniquely divisible abelian groups.

In the works of Malcev [93], Lazard [84], and Hilton [71] (see also [22, 72, 74]), this construction was extended to arbitrary nilpotent groups. The *Malcev completion* functor is left adjoint to the embedding of the category of rational nilpotent groups into the category of nilpotent groups. Thus, if $N$ is a nilpotent group, its Malcev completion (or, rationalization) is a rational nilpotent group, denoted $N \otimes \mathbb{Q}$, that comes endowed with a map $\kappa\colon N \to N \otimes \mathbb{Q}$ which is universal for homomorphisms of $G$ into uniquely divisible nilpotent groups. Moreover, the kernel of $\kappa$ is equal to $\operatorname{Tors}(N)$, the torsion subgroup of $N$, and the induced map, $\kappa^*\colon \operatorname{Hom}(N \otimes \mathbb{Q}, K) \to \operatorname{Hom}(N, K)$, is an isomorphism for all rational nilpotent groups $K$. Malcev completion is an exact functor, which induces isomorphisms $H_*(N, \mathbb{Q}) \cong H_*(N \otimes \mathbb{Q}, \mathbb{Z})$. The quotient $N/\operatorname{Tors}(N)$ is a torsion-free nilpotent group that has the same rationalization as $N$. If $N$ is finitely generated, then

$N \otimes \mathbb{Q}$ is a nilpotent Lie group defined over $\mathbb{Q}$, with integral form $N/\operatorname{Tors}(N)$ and whose Lie algebra, $\mathfrak{Lie}(N \otimes \mathbb{Q})$, is nilpotent.

We now turn to an arbitrary group $G$. The successive nilpotent quotients of $G$ assemble into a tower of the form

$$\cdots \longrightarrow G/\gamma_4(G) \longrightarrow G/\gamma_3(G) \longrightarrow G/\gamma_2(G) \longrightarrow 1.$$

Replacing in this tower each nilpotent quotient by its rationalization and taking the inverse limit of this directed system, we obtain a prounipotent, filtered Lie group over $\mathbb{Q}$,

$$G_{\mathbb{Q}} := \varprojlim_n (G/\gamma_n(G) \otimes \mathbb{Q}),$$

which is called the *Malcev completion* (or, the *prounipotent completion*) of the group $G$. We denote by $\kappa: G \to G_{\mathbb{Q}}$ the canonical homomorphism from $G$ to its rational completion and note that the assignment $G \rightsquigarrow G_{\mathbb{Q}}$ is functorial.

The pronilpotent Lie algebra

$$\mathfrak{m}(G) := \varprojlim_n \mathfrak{Lie}(G/\gamma_n(G) \otimes \mathbb{Q}) \tag{6.1}$$

is called the *Malcev Lie algebra* of $G$. This Lie algebra comes endowed with the inverse limit filtration, which makes it a complete, separated, filtered Lie algebra over $\mathbb{Q}$. As before, the assignment $G \rightsquigarrow \mathfrak{m}(G)$ is functorial. Moreover, if $G$ is finitely generated, then $\mathfrak{m}(G)$ is a finitely generated Lie algebra.

### 6.3. Quillen's construction

A different approach was taken by Quillen in [118, Appendix A]; we recall now his construction of the Malcev completion and the Malcev Lie algebra, building on the treatment from [52, 96, 105, 106, 132].

A *Malcev Lie algebra* is a Lie algebra $\mathfrak{m}$ over a field of characteristic 0, endowed with a decreasing, complete vector space filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\geq 1}$ such that $\mathcal{F}_1 = \mathfrak{m}$ and $[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j}$, for all $i, j$, and with the property that the associated graded Lie algebra, $\operatorname{gr}(\mathfrak{m}) = \bigoplus_{i\geq 1} \mathcal{F}_i/\mathcal{F}_{i+1}$, is generated in degree 1. For example, the completion $\widehat{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ with respect to the lower central series filtration $\{\gamma_i(\mathfrak{g})\}_{i\geq 1}$, endowed with the canonical completion filtration, is a Malcev Lie algebra.

Given a group $G$, the group algebra $\mathbb{Q}[G]$ has a natural Hopf algebra structure, with comultiplication map $\Delta: \mathbb{Q}[G] \to \mathbb{Q}[G] \otimes \mathbb{Q}[G]$ given by $\Delta(g) = g \otimes g$, and counit the augmentation map $\varepsilon: \mathbb{Q}[G] \to \mathbb{Q}$ given by $\varepsilon(g) = 1$. An element $x \in \mathbb{Q}[G]$ is said to be *group-like* if $\Delta(x) = x \otimes x$ and *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$; under the inclusion $G \hookrightarrow \mathbb{Q}[G]$, the set of all group-like elements gets identified with $G$. Let $I = \ker(\varepsilon)$ be the augmentation ideal of, and let

$$\widehat{\mathbb{Q}[G]} = \varprojlim_r \mathbb{Q}[G]/I^r$$

be the completion of $\mathbb{Q}[G]$ with respect to the filtration by the powers of this ideal. Define the completed tensor product $\widehat{\mathbb{Q}[G]} \mathbin{\hat{\otimes}} \widehat{\mathbb{Q}[G]}$ as the completion of $\mathbb{Q}[G] \otimes \mathbb{Q}[G]$ with respect to the natural tensor product filtration. Applying the $I$-adic completion functor to the map $\Delta$ yields a comultiplication map, $\widehat{\Delta}\colon \widehat{\mathbb{Q}[G]} \to \widehat{\mathbb{Q}[G]} \mathbin{\hat{\otimes}} \widehat{\mathbb{Q}[G]}$, which makes $\widehat{\mathbb{Q}[G]}$ into a complete Hopf algebra. As shown by Quillen, there is a natural, filtration-preserving isomorphism,

$$\mathfrak{m}(G) \cong \operatorname{Prim}\big(\widehat{\mathbb{Q}[G]}\big),$$

between the Malcev Lie algebra of $G$ and the Lie algebra of primitive elements in $\widehat{\mathbb{Q}[G]}$, with Lie bracket given by $[x, y] = xy - yx$.

The set of all primitive elements in $\operatorname{gr}(\mathbb{Q}[G])$ forms a graded Lie algebra, which is isomorphic to $\operatorname{gr}(G) \otimes \mathbb{Q}$. An important connection between the Malcev Lie algebra and the associated graded Lie algebra of $G$ was discovered by Quillen, who showed in [119] that there is a natural isomorphism of graded Lie algebras,

$$\operatorname{gr}(\mathfrak{m}(G)) \cong \operatorname{gr}(G; \mathbb{Q}). \tag{6.2}$$

The Malcev completion $G_{\mathbb{Q}}$ may be identified with the set consisting of all group-like elements in the Hopf algebra $\widehat{\mathbb{Q}[G]}$. This is a group which comes endowed with a complete, separated filtration, whose $n$-th term is $G_{\mathbb{Q}} \cap (1 + \widehat{I}^n)$. As explained in [96], there is a one-to-one, filtration-preserving correspondence between primitive elements and group-like elements of $\widehat{\mathbb{Q}[G]}$ via the exponential and logarithmic maps,

$$G_{\mathbb{Q}} \subset 1 + \widehat{I} \;\underset{\log}{\overset{\exp}{\rightleftarrows}}\; \widehat{I} \supset \mathfrak{m}(G).$$

Passing to associated graded objects and using (6.2), we find that $\operatorname{gr}(G_{\mathbb{Q}}) \cong \operatorname{gr}(G; \mathbb{Q})$; in particular, $H_1(G_{\mathbb{Q}}) = H_1(G, \mathbb{Q})$.

### 6.4. Multiplicative expansions and Taylor expansions

Let $G$ be a group. Given a map $f\colon G \to R$, where $R$ is a ring, we will denote by $\bar{f}\colon \mathbb{Q}[G] \to R$ its linear extension to the group algebra. A *(multiplicative) expansion* of $G$ is a map

$$E\colon G \longrightarrow \widehat{\operatorname{gr}}(\mathbb{Q}[G]) \tag{6.3}$$

such that the linear extension $\bar{E}\colon \mathbb{Q}[G] \to \widehat{\operatorname{gr}}(\mathbb{Q}[G])$ is a filtration-preserving algebra morphism with the property that $\operatorname{gr}(\bar{E}) = \mathrm{id}$. Alternatively, a map as in (6.3) is an expansion if it is a (multiplicative) monoid map and the following property holds: If $f \in I^k \setminus I^{k+1}$, then $\bar{E}(f)$ starts with $[f] \in I^k/I^{k+1}$; that is, $\bar{E}(f) = (0, \dots, 0, [f], *, *, \dots)$.

**Definition 6.1** ([8, 133]). An expansion $E\colon G \to \widehat{\operatorname{gr}}(\mathbb{Q}[G])$ is called a *Taylor expansion* if it sends each element of $G$ to a group-like element of $\widehat{\operatorname{gr}}(\mathbb{Q}[G])$; that is, $\bar{\Delta}(E(g)) = E(g) \mathbin{\hat{\otimes}} E(g)$, for all $g \in G$.

It is shown in [133] that a Taylor expansion $E\colon G \to \widehat{\mathrm{gr}}(\mathbb{Q}[G])$ induces a filtration-preserving isomorphism of complete Hopf algebras, $\widehat{E}\colon \widehat{\mathbb{Q}[G]} \to \widehat{\mathrm{gr}}(\mathbb{Q}[G])$, such that $\mathrm{gr}(\widehat{E})$ is the identity on $\mathrm{gr}(\mathbb{Q}[G])$. Conversely, a filtration-preserving isomorphism of complete Hopf algebras, $\phi\colon \widehat{\mathbb{Q}[G]} \to \widehat{\mathrm{gr}}(\mathbb{Q}[G])$, induces a Taylor expansion $E\colon G \to \widehat{\mathrm{gr}}(\mathbb{Q}[G])$. These facts may be summarized as follows.

**Theorem 6.2** ([133]). *The assignment $E \rightsquigarrow \widehat{E}$ establishes a one-to-one correspondence between Taylor expansions $G \to \widehat{\mathrm{gr}}(\mathbb{Q}[G])$ and filtration-preserving isomorphisms of complete Hopf algebras $\widehat{\mathbb{Q}[G]} \to \widehat{\mathrm{gr}}(\mathbb{Q}[G])$ for which the associated graded morphism is the identity on* $\mathrm{gr}(\mathbb{Q}[G])$.

This theorem generalizes a result of Massuyeau, from finitely generated free groups to arbitrary finitely generated groups. As a corollary, we obtain the following criterion for the existence of a Taylor expansion.

**Corollary 6.3** ([133]). *A finitely generated group $G$ has a Taylor expansion if and only if there is an isomorphism of filtered Hopf algebras, $\widehat{\mathbb{Q}[G]} \cong \widehat{\mathrm{gr}}(\mathbb{Q}[G])$.*

Now suppose $G$ admits a finite presentation of the form $G = F/R$. Starting from a Taylor expansion for the finitely generated free group $F$, one may find a presentation for the Malcev Lie algebra $\mathfrak{m}(G;\Bbbk)$, using the approach of Papadima [103] and Massuyeau [96]. This is summarized in the following theorem.

**Theorem 6.4** ([96, 103]). *Let $G$ be a group with generators $x_1, \dots, x_n$ and relators $r_1, \dots, r_m$ and let $E$ be a Taylor expansion of the free group $F = \langle x_1, \dots, x_n\rangle$. There exists then a unique filtered Lie algebra isomorphism*

$$\mathfrak{m}(G) \cong \widehat{\mathrm{Lie}}(\mathbb{Q}^n)/\langle\langle W\rangle\rangle,$$

*where $\langle\langle W\rangle\rangle$ denotes the closed ideal of the completed free Lie algebra $\widehat{\mathrm{Lie}}(\mathbb{Q}^n)$ generated by the subset* $\{\log(E(r_1)), \dots, \log(E(r_m))\}$.

### 6.5. Filtered formality

Following [132], we say that a group $G$ is *filtered formal* if its Malcev Lie algebra is filtered formal, that is, $\mathfrak{m}(G)$ is isomorphic (as a filtered Lie algebra) to the degree completion of its associated graded Lie algebra, $\mathrm{gr}(\mathfrak{m}(G))$. In view of (6.2), this condition is equivalent to $\mathfrak{m}(G) \cong \widehat{\mathrm{gr}}(G;\mathbb{Q})$. It follows from Lemma 5.1 that $G$ is filtered formal if and only if $\mathfrak{m}(G)$ admits a homogeneous presentation.

For instance, if $G = F_n$, then $\mathfrak{m}(F_n) \cong \widehat{\mathrm{Lie}}(\mathbb{Q}^n)$, and so $F_n$ is filtered formal. Moreover, if $G$ is a torsion-free, 2-step nilpotent group for which $G_{\mathrm{ab}}$ is torsion-free (e.g., if $G = F_n/\gamma_3(F_n)$ with $n \geq 2$), then $G$ is filtered-formal. On the other hand, there are torsion-free, 3-step nilpotent groups that are not filtered formal; see [132].

As the next theorem shows, the Taylor expansions of a finitely generated group $G$ are closely related to the isomorphisms between the Malcev Lie algebra and the LCS completion of the associated graded Lie algebra of $G$.

**Theorem 6.5** ([133]). *There is a one-to-one correspondence between Taylor expansions* $G \to \widehat{\mathrm{gr}}(\mathbb{Q}[G])$ *and filtration-preserving Lie algebra isomorphisms* $\mathfrak{m}(G) \to \widehat{\mathrm{gr}}(G;\mathbb{Q})$ *inducing the identity on* $\mathrm{gr}(G;\mathbb{Q})$.

Using this theorem, we obtain an alternate interpretation of filtered-formality.

**Corollary 6.6** ([133]). *A finitely generated group* $G$ *is filtered-formal if and only if* $G$ *has a Taylor expansion.*

### 6.6. The RTFN property and Taylor expansions

A group $G$ is said to be *residually torsion-free nilpotent* (for short, RTFN) if for any $g \in G$, $g \neq 1$, there exists a torsion-free nilpotent group $Q$ and an epimorphism $\psi\colon G \to Q$ such that $\psi(g) \neq 1$. The property of being residually torsion-free nilpotent is inherited by subgroups and is preserved under direct products and free products.

The RTFN property may be expressed in terms of the *rational lower central series* of $G$, whose terms are given by

$$\gamma_n^{\mathbb{Q}}(G) = \{g \in G \mid g^m \in \gamma_n(G), \text{for some } m \in \mathbb{N}\}.$$

The group $G$ is RTFN if and only if the intersection of its rational lower central series, $\gamma_\omega^{\mathbb{Q}}(G) := \bigcap_{n\geq 1} \gamma_n^{\mathbb{Q}}(G)$, is the trivial subgroup. We refer to [128] for alternate definitions and more properties of this $N$-series.

As is well known, a group $G$ is residually torsion-free nilpotent if and only if the group-algebra $\mathbb{Q}[G]$ is residually nilpotent, that is, $\bigcap_{n\geq 1} I^n = \{0\}$, where $I$ is the augmentation ideal. Therefore, if $G$ is finitely generated, the RTFN condition is equivalent to the injectivity of the canonical map to the prounipotent completion, $\kappa\colon G \to G_{\mathbb{Q}}$, where recall $G_{\mathbb{Q}}$ is the set of group-like elements in $\widehat{\mathbb{Q}[G]}$.

If $G$ is residually nilpotent and $\mathrm{gr}_n(G)$ is torsion-free for all $n \geq 1$, then $G$ is residually torsion-free nilpotent. Residually torsion-free nilpotent implies residually nilpotent, which in turn, implies residually finite. Examples of residually torsion-free nilpotent groups include torsion-free nilpotent groups, free groups, and surface groups.

**Proposition 6.7** ([133]). *A finitely generated group* $G$ *has an injective Taylor expansion if and only if* $G$ *is residually torsion-free nilpotent and filtered-formal.*

## 7. Holonomy Lie algebras

### 7.1. The holonomy Lie algebra of a CDGA

Let $A = (A^*, d)$ be a 1-finite $\Bbbk$-CDGA, that is, a CDGA over a field $\Bbbk$ of characteristic 0 with $A^0 = \Bbbk$ and $\dim_{\Bbbk} A^1 < \infty$. Writing $A_i = \mathrm{Hom}(A^i, \Bbbk)$ for the $\Bbbk$-duals of the graded pieces, we let $\mu^\vee\colon A_2 \to A_1 \wedge A_1$ be the $\Bbbk$-dual of the multiplication map $\mu\colon A^1 \wedge A^1 \to A^2$, and we let $d^\vee\colon A_2 \to A_1$ be the dual of the differential $d\colon A^1 \to A^2$.

We shall denote by $\mathrm{Lie}(A_1)$ the free Lie algebra on the $\Bbbk$-vector space $A_1$, and we will identify $\mathrm{Lie}_1(A_1) = A_1$ and $\mathrm{Lie}_2(A_1) = A_1 \wedge A_1$.

**Definition 7.1** ([92]). The *holonomy Lie algebra* of a 1-finite CDGA $A = (A^*, d)$ is the quotient of $\mathrm{Lie}(A_1)$ by the ideal generated by the image of the map $\partial_A = d^\vee + \mu^\vee$,

$$\mathfrak{h}(A) = \mathrm{Lie}(A_1)/\langle \mathrm{im}(\partial_A)\rangle.$$

Clearly, this construction is functorial. Indeed, let $\varphi\colon A \to B$ is a morphism of CDGAs as above, and write $\varphi_i = (\varphi^i)^\vee\colon B_i \to A_i$. Then the induced map, $\mathrm{Lie}(\varphi_1)\colon \mathrm{Lie}(B_1) \to \mathrm{Lie}(A_1)$, factors through a morphism of Lie algebras, $\mathfrak{h}(\varphi)\colon \mathfrak{h}(B) \to \mathfrak{h}(A)$. Observe that the Lie algebra $\mathfrak{h}(A)$ depends only on the sub-CDGA $\Bbbk \cdot 1 \oplus A^1 \oplus (d(A^1) + \mu(A^1 \wedge A^1))$ of the truncation $A^{\le 2}$. Therefore, $\mathfrak{h}(A)$ is finitely presented.

In general, though, the ideal generated by $\mathrm{im}(\partial_A)$ is not homogeneous, and so the Lie algebra $\mathfrak{h}(A)$ does not inherit a grading from $\mathrm{Lie}(A_1)$.

**Example 7.2.** Let $A = \bigwedge(a_1, a_2, a_3)$ be the exterior algebra on generators $a_i$ in degree 1, endowed with the differential $d$ given by $da_1 = da_2 = 0$ and $da_3 = a_1 \wedge a_2$. Identify $\mathrm{Lie}(A_1)$ with the free Lie algebra on dual generators $x_1, x_2, x_3$. Then the ideal $\langle \mathrm{im}(\partial_A)\rangle$ is generated by $x_3 + [x_1, x_2]$, $[x_1, x_3]$, and $[x_2, x_3]$, and thus is not homogeneous.

In the above example, $\mathfrak{h}(A)$ still admits the structure of a graded Lie algebra, with $x_1$ and $x_2$ in degree 1, and $x_3$ in degree 2. Nevertheless, using a construction from [132], we may define a minimal, finite CDGA $A$ for which $\mathfrak{h}(A)$ does not admit any grading compatible with the lower central series filtration.

**Example 7.3.** Let $A = \bigwedge(a_1, \dots, a_5)$, with $|a_i| = 1$ and differential $d$ given by $da_4 = a_1 \wedge a_3$, $da_5 = a_1 \wedge a_4 + a_2 \wedge a_3$, and $da_i = 0$, otherwise. Then, as shown in [132, Example 10.5], $\mathfrak{h}(A)$ is not isomorphic to $\mathrm{gr}(\mathfrak{h}(A))$, its associated graded Lie algebra with respect to the LCS filtration.

### 7.2. The holonomy Lie algebra of a CGA

Now suppose $d = 0$, so that $A$ is a graded, graded-commutative, 1-finite $\Bbbk$-algebra. Then $\mathfrak{h}(A) = \mathrm{Lie}(A_1)/\langle \mathrm{im}(\mu^\vee)\rangle$ is the classical holonomy Lie algebra introduced by K.-T. Chen in [31] and further studied in [77, 94, 106, 131, 132]. Clearly, $\mathfrak{h}(A)$ inherits a natural grading from the free Lie algebra $\mathrm{Lie}(A_1)$, which is compatible with the Lie bracket. Consequently, $\mathfrak{h}(A)$ is a finitely-presented, graded Lie algebra, with generators in degree 1 and relations in degree 2; in other words, $\mathfrak{h}(A)$ is a quadratic Lie algebra.

A graded, 1-finite $\Bbbk$-algebra $A$ may be realized as the quotient $T(V)/I$, where $T(V)$ is the tensor algebra on a finite-dimensional $\Bbbk$-vector space $V$ by a homogeneous, two-sided ideal $I$. The algebra $A$ is said to be *quadratic* if $A^1 = V$ and the ideal $I$ is generated in degree 2, i.e., $I = \langle I^2\rangle$, where $I^2 = I \cap (V \otimes V)$.

Given a quadratic algebra $A = T(V)/I$, we identify $V^\vee \otimes V^\vee \cong (V \otimes V)^\vee$, and define the *quadratic dual* of $A$ to be the algebra $A^! = T(V^\vee)/I^\perp$, where $I^\perp$ is the ideal of $T(V^\vee)$ generated by the vector subspace $(I^2)^\perp := \{\alpha \in V^\vee \otimes V^\vee \mid \alpha(I^2) = 0\}$. Clearly, $A^!$ is also a quadratic algebra, and $(A^!)^! = A$. For any graded algebra of the form $A = T(V)/I$, we may define its quadrature closure as $\mathrm{q}A = T(V)/\langle I^2 \rangle$.

For a Lie algebra $\mathfrak{g}$, we let $U(\mathfrak{g})$ be its universal enveloping algebra. This is the filtered, associative algebra obtained as the quotient of the tensor algebra $T(\mathfrak{g})$ by the (two-sided) ideal generated by all elements of the form $a \otimes b - b \otimes a - [a, b]$ with $a, b \in \mathfrak{g}$.

**Proposition 7.4** ([115,132]). *Let $A$ be a commutative graded $\Bbbk$-algebra such that $A^0 = \Bbbk$ and $\dim_\Bbbk A^1 < \infty$. Then $U(\mathfrak{h}(A))$ is a quadratic algebra, and $U(\mathfrak{h}(A)) = (\mathrm{q}A)^!$.*

Now suppose $\mathfrak{g}$ is a finitely generated graded Lie algebra generated in degree 1. Then, as shown in [132], there is a unique, functorially defined quadratic Lie algebra, $\mathrm{q}\mathfrak{g}$, such that $U(\mathrm{q}\mathfrak{g}) = \mathrm{q}U(\mathfrak{g})$. Therefore, by Proposition 7.4, we have that $\mathfrak{h}((\mathrm{q}U(\mathfrak{g}))^!) = \mathrm{q}\mathfrak{g}$.

Work of Löfwall [87] yields another interpretation of the universal enveloping algebra of the holonomy Lie algebra.

**Proposition 7.5** ([87]). *Let $[\mathrm{Ext}^1_A(\Bbbk, \Bbbk)] := \bigoplus_{i \geq 0} \mathrm{Ext}^i_A(\Bbbk, \Bbbk)_i$ be the linear strand in the Yoneda algebra of $A$. Then $U(\mathfrak{h}(A)) \cong [\mathrm{Ext}^1_A(\Bbbk, \Bbbk)]$.*

Applying the Poincaré–Birkhoff–Witt theorem, we infer that the graded ranks of $\mathfrak{h}(A)$ are given by

$$\prod_{n \geq 1} (1 - t^n)^{\dim_\Bbbk \mathfrak{h}_n(A)} = \sum_{i \geq 0} b_{i,i}(A) t^i,$$

where $b_{i,i}(A) = \dim_\Bbbk \mathrm{Ext}^i_A(\Bbbk, \Bbbk)_i$.

### 7.3. The completion of the holonomy Lie algebra of a CGA

Let $A$ be a connected $\Bbbk$-CGA. A 1-minimal model $\mathcal{M}_1(A)$ for $A$ may be constructed in a "formal" way, following the approach outlined by Carlson and Toledo [27] (see also [132]). For the construction of the full, bigraded minimal model of a CGA we refer to Halperin and Stasheff [68].

As in Section 4.5, start with the CDGAs $\mathcal{M}(1) = (\bigwedge(V_1), 0)$, where $V_1 = A^1$, and $\mathcal{M}(2) = (\bigwedge(V_1 \oplus V_2), d)$, where $V_2 = \ker(\mu\colon A^1 \wedge A^1 \to A^2)$ and $d\colon V_2 \hookrightarrow V_1 \wedge V_1$ is the inclusion map. Now define inductively a CDGA $\mathcal{M}(i)$ as the Hirsch extension $\mathcal{M}(i-1) \otimes \bigwedge(V_i)$, where the $\Bbbk$-vector space $V_i$ fits into the short exact sequence

$$0 \longrightarrow V_i \longrightarrow H^2(\mathcal{M}(i-1)) \longrightarrow \mathrm{im}(\mu) \longrightarrow 0,$$

while the differential $d$ includes $V_i$ into $V_1 \wedge V_{i-1} \subset \mathcal{M}(i-1)$. Setting $\mathcal{M}_1(A)$ equal to $\bigcup_{i \geq 1} \mathcal{M}(i)$, the subalgebras $\{\mathcal{M}(i)\}_{i \geq 1}$ constitute the canonical filtration (4.1) of $\mathcal{M}_1(A)$ and the differential $d$ preserves the Hirsch weights on $\mathcal{M}_1(A)$. For these reasons, we say that $\mathcal{M}_1(A)$ is the *canonical* 1-minimal model of $A$.

The next theorem relates $\mathfrak{L}(\mathcal{M}_1(A))$, the Lie algebra associated to $\mathcal{M}_1(A)$ under the adjoint correspondence from Section 4.5 to the degree completion of $\mathfrak{h}(A)$, the holonomy Lie algebra of $A$. A generalization will be given in Theorem 7.8.

**Theorem 7.6** ([94, 100, 132]). *If $A$ is a 1-finite* CGA*, then $\mathfrak{L}(\mathcal{M}_1(A))$ and $\widehat{\mathfrak{h}(A)}$ are isomorphic as complete, filtered Lie algebras.*

**Corollary 7.7.** *If $A$ is a 1-finite* CGA *and $\mathcal{M}_1(A) = \bigwedge(\bigoplus_{i\geq 1} V_i)$ is the canonical 1-minimal of $A$, then $\dim_{\Bbbk} \mathfrak{h}_i(A) = \dim V_i$ for all $i \geq 1$.*

### 7.4. Holonomy and flat connections

Given a $\Bbbk$-CDGA $(A, d)$ and a Lie algebra $\mathfrak{g}$, we let $\mathcal{F}(A, \mathfrak{g})$ be the set of $\mathfrak{g}$-valued *flat connections* on $A$, that is, the set of all elements $\omega \in A^1 \otimes \mathfrak{g}$ satisfying the Maurer–Cartan equation,

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0.$$

Suppose now that $A$ is 1-finite. As shown in [92], the natural isomorphism $A^1 \otimes \mathfrak{g} \xrightarrow{\cong} \mathrm{Hom}(A_1, \mathfrak{g})$ induces a natural identification,

$$\mathcal{F}(A, \mathfrak{g}) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{h}(A), \mathfrak{g}). \tag{7.1}$$

Assuming further that $\mathfrak{g}$ is finite-dimensional, we let $\mathcal{C}(\mathfrak{g}) = (\bigwedge \mathfrak{g}^{\vee}, d)$ be the Chevalley–Eilenberg complex of $\mathfrak{g}$. This is the CDGA whose underlying graded algebra is the exterior algebra on the dual $\Bbbk$-vector space $\mathfrak{g}^{\vee}$, and whose differential is the extension by the graded Leibniz rule of the dual of the signed Lie bracket, $d = -\beta^*$, on the algebra generators, see, e.g., [52, 68]. There is then a natural isomorphism $A^1 \otimes \mathfrak{g} \xrightarrow{\cong} \mathrm{Hom}(\mathfrak{g}^{\vee}, A^1)$, which, by [40, Lemma 3.4], induces a natural identification,

$$\mathcal{F}(A, \mathfrak{g}) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{CDGA}}(\mathcal{C}(\mathfrak{g}), A). \tag{7.2}$$

Now let $\widehat{\mathcal{C}}$ be the functor which associates to a finitely generated Lie algebra $\mathfrak{h}$ the direct limit of CDGAs

$$\widehat{\mathcal{C}}(\mathfrak{h}) = \varinjlim_n \mathcal{C}(\mathfrak{h}/\gamma_n(\mathfrak{h})).$$

This functor sends finite-dimensional central Lie extensions to Hirsch extensions of CDGAs. It follows that $\widehat{\mathcal{C}}(\mathfrak{h})$ is a 1-minimal CDGA.

Now let $(A, d)$ be a 1-finite CDGA, with holonomy Lie algebra $\mathfrak{h} = \mathfrak{h}(A)$. By (7.1), the identity map of $\mathfrak{h}$ may be identified with the "canonical" flat connection,

$$\omega = \sum_i x_i^* \otimes x_i \in \mathcal{F}(A, \mathfrak{h}(A)),$$

where $\{x_i\}$ is a basis for $A_1$ and $\{x_i^*\}$ is the dual basis for $A^1$. This gives rise to a compatible family of flat connections, $\{\omega_n \in \mathcal{F}(A, \mathfrak{h}/\gamma_n(\mathfrak{h}))\}_{n\geq 1}$. Using the correspondence (7.2), we obtain a compatible family of CDGA maps, $f_n\colon \mathcal{C}(\mathfrak{h}/\gamma_n(\mathfrak{h})) \to A$. Passing to the limit, we arrive at a natural CDGA map, $f\colon \widehat{\mathcal{C}}(\mathfrak{h}(A)) \to A$. The next theorem recovers (in a self-contained way) results from [13, 16, 18].

**Theorem 7.8** ([113]). *If $A$ is a 1-finite* CDGA*, then the classifying map $f\colon \widehat{\mathcal{C}}(\mathfrak{h}(A)) \to A$ is a 1-minimal model map for $A$.*

Consequently, we have an isomorphism $\mathcal{M}_1(A) \cong \widehat{\mathcal{C}}(\mathfrak{h}(A))$.

### 7.5. The holonomy Lie algebra of a group

A construction due to K.-T. Chen [31] and further developed in the works mentioned in Section 7.2 assigns to every finitely generated group $G$ its holonomy Lie algebra, $\mathfrak{h}(G;\Bbbk)$, which is defined as the holonomy Lie algebra of the cohomology algebra of $G$ with coefficients in a field $\Bbbk$ of characteristic 0,

$$\mathfrak{h}(G;\Bbbk) := \mathfrak{h}(H^*(G,\Bbbk)).$$

By the discussion from Section 7.2, we have that $\mathfrak{h}(G;\Bbbk) = \mathrm{Lie}(H_1(G,\Bbbk))/\langle \mu_G^\vee\rangle$, where $\mu_G\colon H^1(G,\Bbbk)\wedge H^1(G,\Bbbk)\to H^2(G,\Bbbk)$ is the cup-product map in group cohomology and $\mu_G^\vee$ is its $\Bbbk$-dual. It is readily seen that the assignment $G \rightsquigarrow \mathfrak{h}(G;\Bbbk)$ is functorial.

The Lie algebra $\mathfrak{h}(G;\Bbbk)$ is a finitely presented, quadratic Lie algebra that depends only on the cup-product map $\mu_G$. Moreover, as noted in [131], the projection map $G \twoheadrightarrow G/\gamma_n(G)$ induces isomorphisms $\mathfrak{h}(G;\Bbbk) \xrightarrow{\cong} \mathfrak{h}(G/\gamma_n(G);\Bbbk)$ for all $n\geq 3$. Consequently, the holonomy Lie algebra of $G$ depends only on its second nilpotent quotient, $G/\gamma_3(G)$.

An important feature of the holonomy Lie algebra is its relationship to the associated graded Lie algebra, as detailed in the next theorem.

**Theorem 7.9** ([94, 106, 131]). *There exists a natural epimorphism of graded $\Bbbk$-Lie algebras, $\Phi\colon \mathfrak{h}(G;\Bbbk) \twoheadrightarrow \mathrm{gr}(G;\Bbbk)$, which induces isomorphisms in degrees 1 and 2.*

Following [131, 132], we say that a finitely generated group $G$ is *graded formal* if the map $\Phi\colon \mathfrak{h}(G;\Bbbk) \twoheadrightarrow \mathrm{gr}(G;\Bbbk)$ is an isomorphism. This condition is equivalent to $\mathrm{gr}(G;\Bbbk)$ being a quadratic Lie algebra. As shown in [132], if $K\leq G$ is a retract of a graded formal group $G$, then $K$ is also graded formal.

The next result shows how to find a presentation for $\mathfrak{h}(G;\Bbbk)$, given a presentation for $\mathrm{gr}(G;\Bbbk)$.

**Proposition 7.10** ([132]). *Let $V = H_1(G;\Bbbk)$. Suppose the associated graded Lie algebra $\mathfrak{g} = \mathrm{gr}(G;\Bbbk)$ has presentation $\mathrm{Lie}(V)/\mathfrak{r}$. Then the holonomy Lie algebra $\mathfrak{h}(G;\Bbbk)$ has presentation $\mathrm{Lie}(V)/\langle \mathfrak{r}_2\rangle$, where $\mathfrak{r}_2 = \mathfrak{r}\cap \mathrm{Lie}_2(V)$. Furthermore, if $A = U(\mathfrak{g})$, then $\mathfrak{h}(G;\Bbbk) = \mathfrak{h}((\mathrm{q}A)^!)$.*

## 8. Algebraic models for spaces

### 8.1. Rational homotopy equivalences

We start with a definition that goes back to the work of Quillen [118], Bousfield–Gugenheim [21], and Halperin–Stasheff [68]. A continuous map between two topological spaces, $f\colon X\to Y$, is said to be a *rational quasi-isomorphism* if the induced map in rational cohomology, $f^*\colon H^*(Y,\mathbb{Q})\to H^*(X,\mathbb{Q})$, is an isomorphism. A *rational homotopy equivalence* between $X$ and $Y$ is a sequence of continuous maps (going either way) connecting the two spaces via rational quasi-isomorphisms. We say that $X$ and $Y$ are *rationally homotopy equivalent* (or, have the same rational homotopy type) if such a zigzag of rational quasi-isomorphisms exists, in which case we write $X \simeq_{\mathbb{Q}} Y$. The purpose of rational homotopy theory, then, is to classify topological spaces up to this equivalence relation.

One of the motivations of Sullivan's work in this field was the idea that the rational homotopy type of a simply-connected manifold, together with suitable characteristic class and integral data determines the diffeomorphism type up to finite ambiguity. For instance, he showed in [137, Theorem 13.1] that closed, simply-connected, smooth manifolds can be classified up to finite ambiguity in terms of their rational homotopy type, rational Pontrjagin classes, bounds on torsion, and certain integral lattice invariants. This important result was subsequently refined by Kreck and Triantafillou [79] (under some partial formality assumptions) and Crowley and Nordström [35] (under some higher connectivity assumptions).

### 8.2. Sullivan algebras of piecewise polynomial differential forms

Let $(C^*(X, \Bbbk), d)$ be the singular cochain complex of a space $X$, with coefficients in a field $\Bbbk$ of characteristic 0. This is, in fact, a differential graded algebra, with multiplication given by the cup-product. By definition, the cohomology of this $\Bbbk$-DGA is the cohomology algebra $H^*(X, \Bbbk)$; this is a CGA, although the cochain algebra itself is not a CDGA (except in some very special situations). More generally, we say that a $\Bbbk$-DGA $(A, d_A)$ is a DGA model for $X$ if it is weakly equivalent (through DGAs) to $(C^*(X, \Bbbk), d)$.

In his seminal paper [137], Sullivan associated in a functorial way to every space $X$ a rational, *commutative* DGA, denoted by $(A_{\mathrm{PL}}(X), d)$. When $X$ is a simplicial complex, the elements of this CDGA may be viewed as compatible collections of forms on the simplices of $X$, which are sums with rational coefficients of monomials in the barycentric coordinates. Integration defines a chain map from $A_{\mathrm{PL}}(X)$ to $C^*(X, \mathbb{Q})$ which induces an isomorphism in cohomology. In fact, the Sullivan algebra $(A_{\mathrm{PL}}(X), d)$ is weakly equivalent (through DGAs) with the cochain algebra $(C^*(X, \mathbb{Q}), d)$; moreover, under the resulting identification of graded algebras, $H^*(A_{\mathrm{PL}}(X)) \cong H^*(X, \mathbb{Q})$, the induced homomorphisms in cohomology correspond, see [52, Corollary 10.10].

We say that a $\Bbbk$-CDGA $(A, d_A)$ is a *model* over $\Bbbk$ for the space $X$ if $A$ is weakly equivalent (through CDGAs) to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$; in particular, $H^*(A) \cong H^*(X, \Bbbk)$. In view of Theorem 2.2, we may also allow the weak equivalence to go through DGAs in this definition. For instance, if $X$ is a smooth manifold, then the de Rham algebra $\Omega^*_{\mathrm{dR}}(X)$ of smooth forms on $X$ is a model of $X$ over $\mathbb{R}$, and if $X$ is a simplicial complex, then a rational model for $X$ is $A_{\mathrm{s}}(X)$, the algebra of piecewise polynomial $\mathbb{Q}$-forms on the simplices of $X$. We refer to [52–54, 139] for more details.

By the functoriality of the Sullivan algebra, a rational quasi-isomorphism $f : X \to Y$ induces a quasi-isomorphism $A_{\mathrm{PL}}(f) : A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$; therefore, if $X \simeq_{\mathbb{Q}} Y$, then $A_{\mathrm{PL}}(X) \simeq A_{\mathrm{PL}}(Y)$. Consequently, the weak isomorphism type of $A_{\mathrm{PL}}(X)$ depends only on the rational homotopy type of $X$. As another consequence, the existence of a finite model for a space $X$ is an invariant of rational homotopy type, and thus, of homotopy type.

**Remark 8.1.** In [137], Sullivan showed that there exist smooth manifolds whose rational models are not weakly isomorphic, but which become weakly isomorphic when tensored with $\mathbb{R}$. Such failure of descent from real homotopy type to rational homotopy type may even occur with models endowed with 0-differentials.

## 8.3. Sullivan minimal models

A *minimal model* for a connected space $X$, denoted $\mathcal{M}(X)$, is a minimal model for the Sullivan algebra $A_{\text{PL}}(X)$. By Theorem 4.6, this is a minimal CDGA, which always exists and is unique up to isomorphism. The Sullivan minimal model comes equipped with a CDGA map, $\rho\colon \mathcal{M}(X) \to A_{\text{PL}}(X)$, which is a quasi-isomorphism. Moreover, if $A \simeq A_{\text{PL}}(X)$ is a connected rational CDGA model for $X$, then there is a quasi-isomorphism $\mathcal{M}(X) \to A$ which corresponds to $\rho$ via the chosen weak equivalence between $A$ and $A_{\text{PL}}(X)$. By a previous remark, the isomorphism type of $\mathcal{M}(X)$ is uniquely defined by the rational homotopy type of $X$. It is an open question whether there exist spaces with weakly equivalent cochain algebras but non-isomorphic minimal models, see [51].

All these notions have partial analogs. Fix an integer $q \geq 1$. A map $f\colon X \to Y$ is said to be a *rational $q$-quasi-isomorphism* if the induced map in rational cohomology, $f^*\colon H^*(Y,\mathbb{Q}) \to H^*(X,\mathbb{Q})$, is an isomorphism in degrees up to $q$ and a monomorphism in degree $q+1$. Clearly, such a map induces a $q$-equivalence, $A_{\text{PL}}(f)\colon A_{\text{PL}}(Y) \to A_{\text{PL}}(X)$. A *rational $q$-homotopy equivalence* between $X$ and $Y$ is a zigzag of maps connecting the two spaces via rational $q$-quasi-isomorphisms; if such a zigzag exists, we say that $X$ and $Y$ are *rationally $q$-homotopy equivalent*.

A *$q$-model* for a space $X$ over a field $\Bbbk$ of characteristic 0 is a $\Bbbk$-CDGA $(A, d)$ which is $q$-equivalent to $A_{\text{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$; in particular, $H^i(A) \cong H^i(X, \Bbbk)$, for all $i \leq q$. A *$q$-minimal model* for $X$, denoted $\mathcal{M}_q(X)$, is a $q$-minimal model for $A_{\text{PL}}(X)$; this CDGA comes equipped with a $q$-quasi-isomorphism,

$$\rho_q\colon \mathcal{M}_q(X) \longrightarrow A_{\text{PL}}(X). \tag{8.1}$$

In this context, a basic question was raised in [113]: When does a $q$-finite space $X$ admit a $q$-finite $q$-model $A$? It follows from the above considerations that the existence of a $q$-finite $q$-model for a space $X$ is an invariant of rational $q$-homotopy type, and thus, of $q$-homotopy type.

## 8.4. Rational completion

In their foundational monograph [22], Bousfield and Kan associated to every space $X$ its *rational completion*, $\mathbb{Q}_\infty X$. This is a rational space (i.e., its homology groups in positive degrees are $\mathbb{Q}$-vector spaces) which comes equipped with a structure map, $k_X\colon X \to \mathbb{Q}_\infty X$, with the following property: Given a map $f\colon X \to Y$, there is an induced map, $\mathbb{Q}_\infty f\colon \mathbb{Q}_\infty X \to \mathbb{Q}_\infty Y$, such that $\mathbb{Q}_\infty f \circ k_X = k_Y \circ f$. Moreover, the map $f$ is a rational homology equivalence if and only if the map $\mathbb{Q}_\infty f$ is a weak homotopy equivalence.

A space $X$ is called $\mathbb{Q}$-*good* if the structure map $k_X\colon X \to \mathbb{Q}_\infty X$ is a rational quasi-isomorphism. It has been known for a long time that not all spaces enjoy this property. Recently, Ivanov and Mikhailov [75] gave the first examples of finite CW-complexes that are $\mathbb{Q}$-bad: if $X = \bigvee^n S^1$ is a wedge of $n \geq 2$ circles, then $H_2(\mathbb{Q}_\infty X, \mathbb{Q})$ is non-zero (in fact, it is uncountable), although of course $H_2(X, \mathbb{Q}) = 0$.

The main connection between the Sullivan minimal model $\mathcal{M}(X)$ and Bousfield and Kan's rational completion $\mathbb{Q}_\infty X$ is provided by the following theorem of Bousfield and Gugenheim [21].

**Theorem 8.2** ([21]). *Let $X$ be a connected space with finite Betti numbers, and let $\mathcal{M}(X) = (\bigwedge V, d)$ be a minimal model for $X$ over $\mathbb{Q}$. Then $\pi_n(\mathbb{Q}_\infty X) \cong (V^n)^\vee$, for all $n \geq 2$.*

A connected space $X$ is a said to be *rationally aspherical* (or, a rational $K(\pi, 1)$ space) if its rational completion is aspherical, i.e., $\pi_n(\mathbb{Q}_\infty X) = 0$ for all $n \geq 2$. As an application of the above theorem, we have the following immediate corollary.

**Corollary 8.3** ([48, 115]). *A connected space $X$ is rationally aspherical if and only if $\mathcal{M}(X) \cong \mathcal{M}_1(X)$.*

### 8.5. Nilpotent spaces

For simply-connected spaces and, more generally, for nilpotent spaces, rational homotopy theory takes a more concrete and approachable form. A path-connected space $X$ is said to be *nilpotent* if the fundamental group $G = \pi_1(X)$ is nilpotent and acts nilpotently on the homotopy groups $\pi_n(X)$ for all $n > 1$. For instance, all tori are nilpotent, but the Klein bottle is not; moreover, a real projective space $\mathbb{RP}^n$ is nilpotent if and only if $n$ is odd.

If $X$ is a nilpotent space, then, as shown in [22], $X$ is $\mathbb{Q}$-good. Moreover, $\pi_1(\mathbb{Q}_\infty X)$ is isomorphic to $\pi_1(X) \otimes \mathbb{Q}$ – the Malcev completion of the nilpotent group $\pi_1(X)$ – while $\pi_n(\mathbb{Q}_\infty X) \cong \pi_n(X) \otimes \mathbb{Q}$ for $n \geq 2$, all in a functorial way. In this context, we also have the following rational analog of Whitehead's theorem (see also [120]).

**Theorem 8.4** ([22]). *A pointed map $f\colon X \to Y$ between two nilpotent spaces is a rational homotopy equivalence if and only if it induces isomorphisms $f_*\colon \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$ for all $n \geq 1$.*

Assume now that $X$ is a nilpotent CW-complex with finite Betti numbers. Sullivan proved in [137] that the minimal model (over $\mathbb{Q}$) of such a space is of the form $\mathcal{M}(X) = (\bigwedge V, d)$, where $V$ is a graded $\mathbb{Q}$-vector space of finite type. Here are a few standard examples.

**Example 8.5.** An odd-dimensional sphere has minimal model $\mathcal{M}(S^{2n+1}) = (\bigwedge(a), 0)$, with $|a| = 2n+1$. On the other hand, an even-dimensional sphere has minimal model $\mathcal{M}(S^{2n}) = \big(\bigwedge(a,b), da = b^2\big)$, with $da = 0$, $db = a^2$, and $|a| = 2n$. Finally, an Eilenberg–MacLane space of type $\mathbb{K}(\mathbb{Z}, n)$ has minimal model $(\bigwedge(a), 0)$, with $|a| = n$.

If $H^{>n}(X) = 0$ for some $n > 0$, we can say a bit more. Pick a vector space decomposition, $\mathcal{M}^n(X) = Z^n(\mathcal{M}(X)) \oplus C^n$. Then the direct sum $J = \mathcal{M}^{\geq n+1}(X) \oplus C^n$ is an acyclic differential graded ideal of $\mathcal{M}(X)$. By construction, $A_{\text{PL}}(X)$ is weakly isomorphic to the CDGA $\mathcal{M}(X)/J$, which is finite-dimensional as a vector space. We summarize this discussion, as follows.

**Theorem 8.6** ([137]). *Let $X$ be a nilpotent CW-complex.*

(1) *If all the Betti numbers of $X$ are finite, then $X$ admits a $q$-finite $q$-model, for all $q$.*
(2) *If $\dim H_*(X, \mathbb{Q}) < \infty$, then $X$ admits a model which is finite-dimensional over $\mathbb{Q}$.*

The main application of Sullivan's theory of minimal models to the rational homotopy of nilpotent spaces is given by the following theorem.

**Theorem 8.7** ([137]). *Let $X$ be a connected, nilpotent CW-complex with finite Betti numbers, and let $\mathcal{M}(X) = (\bigwedge V, d)$ be a minimal model for $X$ over $\mathbb{Q}$. Then $\pi_n(X) \otimes \mathbb{Q} \cong (V^n)^\vee$, for all $n \geq 2$.*

An alternative proof of this foundational result was given by Lehmann in [85]. A generalization was given by Bock [19], who relaxed the hypothesis that $\pi_1(X)$ be nilpotent, thereby proving a statement first mentioned by Halperin in [67].

**Theorem 8.8** ([19]). *Let $X$ be a path-connected, triangulable space whose universal covering exists. Suppose $\pi_1(X)$ has a rationally aspherical classifying space and $\pi_n(X)$ is a finitely generated nilpotent $\pi_1(X)$-module, for each $n \geq 2$. If $\mathcal{M}(X) = (\bigwedge V, d)$ is a minimal model for $X$ over $\mathbb{Q}$, then $\pi_n(X) \otimes \mathbb{Q} \cong (V^n)^\vee$, for all $n \geq 2$.*

Consider now the rational Hurewicz homomorphisms, $\mathrm{hur}_k \colon \pi_k(X) \otimes \mathbb{Q} \to H_k(X, \mathbb{Q})$. If $X$ is $n$-connected for some $n \geq 1$, the above theorem implies that $\mathrm{hur}_k$ is an isomorphism for $k \leq 2n$, while $\ker(\mathrm{hur}_k)$ is the $\mathbb{Q}$-span of the Whitehead products for $2n+1 \leq k \leq 3n+1$, see [51]. For generalizations of Theorem 8.7 to rationally nilpotent spaces we refer to [53].

### 8.6. Models for polyhedral products

We illustrate the general theory with a class of spaces particularly amenable to study via rational homotopy methods. These spaces, variously known as polyhedral products, (generalized) moment-angle complexes, or (generalized) Davis–Januszkiewicz spaces, are constructed as follows (see, for instance, [7, 38] and references therein).

Let $K$ be a finite simplicial complex on vertex set $[n] = \{1, \dots, n\}$, and let $(\underline{X}, \underline{X}')$ be a sequence $(X_1, X_1'), \dots, (X_n, X_n')$ of pairs of spaces. The polyhedral product $\mathcal{Z}_K(\underline{X}, \underline{X}')$ is then the subspace of the Cartesian product $\prod_{i=1}^{n} X_i$ obtained as the union of all subspaces of the form $\mathcal{Z}_\sigma(\underline{X}, \underline{X}') = \prod_{i=1}^{n} Y_i$, where $\sigma$ runs through the simplices of $K$ and $Y_i = X_i$ if $i \in \sigma$ and $Y_i = X_i'$ if $i \notin \sigma$.

Assume now that all spaces $X_i$, $X_i'$ are nilpotent CW-complexes of finite type. In [56], Félix and Tanré describe the rational homotopy type of the corresponding polyhedral product, as follows. Let $A_i$ and $A_i'$ be connected, finite-type rational models for $X_i$ and $X_i'$, so that there are quasi-isomorphisms $\mathcal{M}(X_i) \to A_i$ and $\mathcal{M}(X_i') \to A_i'$. Suppose there are surjective morphisms $\varphi_i\colon A_i \twoheadrightarrow A_i'$ modeling the inclusions $X_i' \hookrightarrow X_i$. For each simplex $\sigma$ on $[n]$, let $I_\sigma = \prod_{i=1}^{n} E_i$, with $E_i = \ker(\varphi_i)$ if $i \in \sigma$ and $E_i = A_i$ if $i \notin \sigma$.

**Theorem 8.9** ([56]). *With assumptions as above, the polyhedral product $\mathcal{Z}_K(\underline{X}, \underline{X}')$ has a connected, finite-type* CDGA *model of the form $A(K) = (\bigotimes_{i=1}^{n} A_i)/I(K)$, where $I(K)$ is the ideal $\sum_{\sigma \notin K} I_\sigma$. Moreover, if $L \subset K$ is a subcomplex, then the inclusion $\mathcal{Z}_L(\underline{X}, \underline{X}') \hookrightarrow \mathcal{Z}_K(\underline{X}, \underline{X}')$ is modeled by the projection $A(K) \twoheadrightarrow A(L)$.*

Taking homology, this theorem recovers a result from [7]: the cohomology algebra $H^*(\mathcal{Z}_K(\underline{X}, \underline{X}'), \mathbb{Q})$ is isomorphic to the quotient $(\bigotimes_{i=1}^{n} H^*(X_i, \mathbb{Q}))/J(K)$, where $J(K)$ is the Stanley–Reisner ideal generated by all the monomials $x_{j_1} \cdots x_{j_k}$ with $x_i \in H^*(X_i, \mathbb{Q})$ for which the simplex $\sigma = (j_1, \dots, j_k)$ is not in $K$.

### 8.7. Configuration spaces

A construction due to Fadell and Neuwirth associates to a space $X$ and a positive integer $n$ the space of ordered configurations of $n$ points in $X$,

$$\mathrm{Conf}(X, n) = \{(x_1, \dots, x_n) \in X^{\times n} \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The most basic example is the configuration space of $n$ ordered points in $\mathbb{C}$; this is a classifying space for $P_n$, the pure braid group on $n$ strings, whose cohomology ring was computed by Arnol'd in the late 1960s.

The $E_2$-term of the Leray spectral sequence for the inclusion $\mathrm{Conf}(X, n) \hookrightarrow X^{\times n}$ was described in concrete terms by Cohen and Taylor [33]. If $X$ is a smooth, complex projective variety of dimension $m$, then $\mathrm{Conf}(X, n)$ is a smooth, quasi-projective variety; moreover, as shown by Totaro in [142], the Cohen–Taylor spectral sequence collapses at the $E_{m+1}$-term, and the $E_m$-term is a CDGA model for the configuration space $\mathrm{Conf}(X, n)$. Other rational models for configuration spaces of smooth projective varieties were constructed by Fulton–MacPherson [62] and Kříž [80].

Now let $M$ be a closed, simply-connected smooth manifold. Under the assumption that $b_2(M) = 0$, Lambrechts and Stanley [81] showed how to construct a rational model for $\mathrm{Conf}(M, 2)$ out of a model for $M$; as a consequence, the rational homotopy type of $\mathrm{Conf}(M, 2)$ depends only on that of $M$. For configuration spaces of $n$ points, Lambrechts and Stanley [82] used Theorem 2.4 to associate to every rational model $A$ for $M$ a $\mathbb{Q}$-CDGA $G_A(n)$, which they conjectured to be a rational model for $\mathrm{Conf}(M, n)$. In [73], Idrissi proved that $G_A(n) \otimes_{\mathbb{Q}} \mathbb{R}$ is a real model for the configuration space; thus, the real homotopy type of $M$ determines the real homotopy type of $\mathrm{Conf}(M,n)$, for all $n$.

### 8.8. Rationalization

To every space $X$, Sullivan [135, 137, 138] associated in a functorial way its *rationalization*, denoted $X_{\mathbb{Q}}$; we refer to [21, 51, 53, 74, 120] for more details on this construction. The rationalization of $X$ may be viewed as a geometric realization of the Sullivan minimal model, $\mathcal{M}(X)$, for the CDGA $A_{\mathrm{PL}}(X)$. The space $X_{\mathbb{Q}}$ comes equipped with a structure map, $h\colon X \to X_{\mathbb{Q}}$, which realizes the morphism $\rho\colon \mathcal{M}(X) \to A_{\mathrm{PL}}(X)$.

Now suppose $X$ is a connected, pointed CW-complex which is a nilpotent space; then, as shown in [53], the space $X_{\mathbb{Q}}$ is again nilpotent and the map $h$ is a rational homotopy equivalence. Moreover, if $H^*(X,\mathbb{Q})$ is of finite type, then the maps $h_*\colon \pi_n(X) \otimes \mathbb{Q} \to \pi_n(X_{\mathbb{Q}})$ are isomorphisms, for all $n \geq 2$. The nilpotency condition is crucial here. Indeed, if $X = \mathbb{RP}^2$, then $\pi_1(X) = \mathbb{Z}_2$ is nilpotent but does not act nilpotently on $\pi_2(X) = \mathbb{Z}$; we also have that $X_{\mathbb{Q}} \simeq \{*\}$, and so the map $h_*\colon \pi_2(X) \otimes \mathbb{Q} \to \pi_2(X_{\mathbb{Q}})$ is the zero map.

In general, the Bousfield–Kan completion and the Sullivan rationalization do not agree, even for nilpotent spaces. Nevertheless, if $X$ is nilpotent and $H^*(X,\mathbb{Q})$ is of finite type, then $\mathbb{Q}_\infty X = X_{\mathbb{Q}}$, see [21].

When $X$ is a CW-complex, a more concrete way to construct the rationalization $X_{\mathbb{Q}}$ is via Sullivan's infinite telescopes, introduced in [135]. For instance, if $n$ is odd, then $S^n_{\mathbb{Q}} \simeq K(\mathbb{Q},n)$.

The constructions from Section 6 are related to the rationalizations of spaces, as follows. Let $X$ be a path-connected space with fundamental group $\pi_1(X) = G$. Then $\mathfrak{M}(G;\mathbb{Q}) = \pi_1(X_{\mathbb{Q}})$, the fundamental group of the rationalization of $X$.

### 8.9. Equivariant algebraic models

The study of the rational equivariant homotopy type of a space subject to the action of a finite group goes back to the work of Triantafillou [144] on equivariant minimal models. We summarize here some recent work from [113] on this subject.

Let $\Phi$ be a finite group. The category $\Phi$-CDGA (over $\Bbbk$) has objects CDGAs $A$ endowed with a compatible $\Phi$-action, while the morphisms are $\Phi$-equivariant CDGA maps $A \to B$. Given a $\Phi$-CDGA $A$, we let $A^\Phi$ be the sub-CDGA of elements fixed by $\Phi$; there is then a canonical CDGA map $A^\Phi \to A$. By definition, a $q$-equivalence $A \simeq_q B$ in $\Phi$-CDGA $(1 \leq q \leq \infty)$ is a zigzag of $\Phi$-equivariant $q$-equivalences in CDGA. It is readily seen that $A \simeq_q B$ in $\Phi$-CDGA implies that $A^\Phi \simeq_q B^\Phi$ in CDGA.

Now suppose $\Phi$ acts freely on a space $Y$, and let $X = Y/\Phi$ be the orbit space. As is well known, every CW-complex $X$ has the homotopy type of a simplicial complex $K$; moreover, if $X$ has finite $q$-skeleton, so does $K$. Fix such a triangulation of $X$, and lift it to the cover $Y$. The corresponding simplicial Sullivan algebras are then related by the equality $A_{\mathrm{s}}(X) = A_{\mathrm{s}}(Y)^\Phi$. Therefore, we have the following result.

**Proposition 8.10** ([113]). *Let $X$ be a CW-complex, and let $Y \to X$ be a finite regular cover, with group of deck transformations $\Phi$. Let $A$ be a $\Phi$-CDGA over $\Bbbk$.*

(1) *Suppose $A_{\mathrm{PL}}(Y) \otimes_{\mathbb{Q}} \Bbbk \simeq_q A$ in $\Phi$-CDGA, for some $1 \le q \le \infty$. Then $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk \simeq_q A^{\Phi}$ in CDGA.*

(2) *If, moreover, $A$ is $q$-finite, then $A^{\Phi}$ is $q$-finite.*

As a consequence, if $Y$ admits an equivariant $q$-finite $q$-model, then $X$ admits a $q$-finite $q$-model. The hypothesis from part (1) in the above proposition cannot be completely dropped. Nevertheless, we have the following conjecture regarding algebraic models for the orbit space $X = Y/\Phi$ constructed from $\Phi$-equivariant models for $Y$.

**Conjecture 8.11** ([113]). *Let $X$ be a connected CW-complex, and let $Y \to X$ be a finite, regular cover with deck group $\Phi$. Suppose that $Y$ has finite Betti numbers. Let $A$ be a $\Phi$-CDGA, and assume that there is a zigzag of quasi-isomorphisms connecting $A_{\mathrm{PL}}(Y) \otimes_{\mathbb{Q}} \Bbbk$ to $A$ in CDGA, such that the induced isomorphism between $H^*(Y, \Bbbk)$ and $H^*(A)$ is $\Phi$-equivariant. Then $A^{\Phi}$ is a model for $X$.*

### 8.10. On the Betti numbers of minimal models

We conclude this section with an obstruction to the existence of a $q$-finite CDGA model $A$ for a space $X$, an obstruction expressed in terms of Betti numbers of the $q$-minimal model $\mathcal{M}_q(X)$ associated to $X$.

**Theorem 8.12** ([113]). *Let $X$ be a connected CW-space, and assume that one of the following conditions is satisfied.*

(1) *$X$ is $(q+1)$-finite.*

(2) *$A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk \simeq_q A$, where $A$ a $q$-finite CDGA over $\Bbbk$.*

*Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \le q+1$.*

*Proof.* Recall from (8.1) that we have a $q$-quasi-isomorphism $\mathcal{M}_q(X) \to A_{\mathrm{PL}}(X)$. In case (1), the claim follows at once. In case (2), the discussion in Section 4.3 shows that $\mathcal{M}_q(X)$ is also a $q$-minimal model for $A$; thus, the claim follows from Proposition 4.7. ∎

## 9. Algebraic models for groups

### 9.1. Malcev Lie algebras and 1-minimal models

Let $G$ be a group, and let $\mathcal{M}_1(G)$ be its 1-minimal model, as described in Section 6. By definition, this is a minimal CDGA over $\mathbb{Q}$, generated in degree 1. If $G = \pi_1(X)$ is the fundamental group of a path-connected space $X$, then any classifying map $X \to K(G,1)$ induces an isomorphism between the corresponding 1-minimal models, $\mathcal{M}_1(X) \cong \mathcal{M}_1(G)$. Consequently, the existence of a 1-finite 1-model for a path-connected space $X$ is equivalent to the existence a 1-finite 1-model for its fundamental group, $G = \pi_1(X)$.

Assume now that $G$ is a finitely generated group. There is then a natural duality between the Malcev Lie algebra $\mathfrak{m}(G)$, endowed with the inverse limit filtration given by (6.1) and the 1-minimal model $\mathcal{M}_1(G)$, endowed with the increasing filtration from (4.1). Recall that the latter filtration, $\{\mathcal{M}(i)\}_{i\geq 0}$, starts with $\mathcal{M}(0) = \mathbb{Q}$. Since $G$ is finitely generated, the vector space $V_1 := H^1(G, \mathbb{Q})$ is finite-dimensional. Each sub-CDGA $\mathcal{M}(i)$ is then a Hirsch extension of the form $\mathcal{M}(i-1) \otimes \bigwedge V_i$, where $V_i = \ker(H^2(\mathcal{M}(i-1)) \to H^2(\mathcal{M}_1(G)))$ is again finite-dimensional. Let $\mathfrak{L}(G) = \varprojlim_i \mathfrak{L}_i(G)$ be the pronilpotent Lie algebra functorially associated to the 1-minimal model $\mathcal{M}_1(G)$ in the manner described in Section 4.5. We then have the following basic correspondence between the aforementioned Lie algebras.

**Theorem 9.1** ([30, 66, 137]). *There is a natural isomorphism between the towers of nilpotent Lie algebras $\{\mathfrak{m}(G/\gamma_i(G))\}_{i\geq 0}$ and $\{\mathfrak{L}_i(G)\}_{i\geq 0}$, which gives rise to a functorial isomorphism of complete, filtered Lie algebras, $\mathfrak{m}(G) \cong \mathfrak{L}(G)$.*

The functorial isomorphism $\mathfrak{m}(G) \cong \mathfrak{L}(G)$, together with the dualization correspondence $\mathcal{M}_1(G) \leftrightsquigarrow \mathfrak{L}(G)$ define adjoint functors between the category of Malcev Lie algebras of finitely generated groups and the category of 1-minimal models of finitely generated groups. Using this isomorphism and the one from (6.2), we may identify $\operatorname{gr}_n(G;\mathbb{Q})$ with $(V_n)^\vee$ for all $n \geq 1$.

### 9.2. Groups with 1-finite 1-models

The next theorem provides an effective way of computing the Malcev Lie algebra of a group $G$, under a certain finiteness assumption.

**Theorem 9.2** ([113]). *Let $G$ be a finitely generated group that admits a 1-finite 1-model $A$. Then the Malcev Lie algebra $\mathfrak{m}(G)$ is isomorphic to the LCS completion of the holonomy Lie algebra $\mathfrak{h}(A)$.*

*Proof.* By our hypothesis and by the uniqueness of 1-minimal models, we have an isomorphism $\mathcal{M}_1(G) \cong \mathcal{M}_1(A)$. By construction, the Lie algebra $\mathfrak{m}(G)$ is filtered isomorphic to the inverse limit of a tower of central extensions of finite-dimensional nilpotent Lie algebras. By Theorem 9.1, the terms $\mathfrak{m}(G/\gamma_i(G))$ of this tower are obtained by dualizing the canonical filtration of $\mathcal{M}_1(G)$.

On the other hand, by Theorem 7.8, the CDGA $\mathcal{M}_1(A)$ is isomorphic to $\widehat{\mathcal{C}}(\mathfrak{h}(A))$, the completion of the Chevalley–Eilenberg cochain functor applied to $\mathfrak{h}(A)$. Furthermore, it is shown in [113, Corollary 5.7] that the dual of the canonical filtration of $\widehat{\mathcal{C}}(\mathfrak{h}(A))$ is a tower of central extensions of finite-dimensional Lie algebras, whose terms are the nilpotent quotients $\mathfrak{h}(A)/\gamma_i(\mathfrak{h}(A))$. Putting all these facts together yields the desired isomorphism, $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}}(A)$. ∎

As an application, we have the following result, which gives a characterization of groups $G$ having a 1-finite 1-model in terms of their Malcev Lie algebras.

**Theorem 9.3** ([113]). *A finitely generated group $G$ admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(G)$ is the lower central series completion of a finitely presented Lie algebra over $\mathbb{Q}$.*

The above condition means that $\mathfrak{m}(G) = \widehat{L}$, for some finitely presented Lie algebra $L$ over $\mathbb{Q}$, where $\widehat{L} = \varprojlim_n L/\gamma_n(L)$. By Theorem 9.2, if $A$ is a 1-finite 1-model for $G$, we may take $L$ to be the holonomy Lie algebra $\mathfrak{h}(A)$.

Finally, here is a finiteness obstruction for finitely generated groups, which follows at once from Theorem 8.12.

**Corollary 9.4** ([113]). *Let $G$ be a finitely generated group. Assume that either $G$ is finitely presented or $G$ admits a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.*

### 9.3. Filtered formal groups

Recall from Section 6.5 that a finitely generated group $G$ is said to be filtered formal if its Malcev Lie algebra $\mathfrak{m}(G)$ is isomorphic to the degree completion of its associated graded Lie algebra. The next result connects certain finiteness properties of algebraic objects associated to such a group $G$.

**Proposition 9.5** ([113]). *Let $G$ be a finitely generated, filtered formal group, so that $\mathfrak{m}(G) \cong \widehat{L}$, where $L = \mathbb{L}/J$ is a graded Lie algebra over $\mathbb{Q}$ generated in degree 1 and $J$ is an ideal included in $\mathbb{L}^{\geq 2}$. If $b_2(\mathcal{M}_1(G)) < \infty$, then $\dim_{\mathbb{Q}}(J/[\mathbb{L}, J]) < \infty$.*

Here is another characterization of filtered-formality, this time in terms of minimal models.

**Theorem 9.6** ([132]). *A finitely generated group $G$ is filtered-formal over $\mathbb{Q}$ if and only if the canonical 1-minimal model $\mathcal{M}_1(G)$ is filtered-isomorphic to a 1-minimal model $\mathcal{M}$ with positive Hirsch weights.*

The notion of filtered formality over an arbitrary field $\Bbbk$ of characteristic 0 is defined analogously. It follows from Theorem 5.5 that $G$ is filtered-formal over $\Bbbk$ if and only it is filtered-formal over $\mathbb{Q}$. Another notable property of filtered formality is that it descends to maximal solvable quotients. The next theorem develops a theme started in [106].

**Theorem 9.7** ([132]). *Let $G$ be a finitely generated group. For each $i \geq 2$, the quotient map $q_i: G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\Bbbk$-Lie algebras,*

$$\Psi^{(i)}: \operatorname{gr}(G;\Bbbk)/\operatorname{gr}(G;\Bbbk)^{(i)} \longrightarrow\!\!\!\!\rightarrow \operatorname{gr}(G/G^{(i)};\Bbbk).$$

*Moreover, if $G$ is filtered-formal, then $\Psi^{(i)}$ is an isomorphism and the solvable quotient $G/G^{(i)}$ is filtered-formal.*

Taking $G = F_n$, it follows that each solvable quotient $F_n/F_n^{(i)}$ is a filtered formal group, with associated graded Lie algebra equal to $\mathbb{L}_n/\mathbb{L}_n^{(i)}$, where $\mathbb{L}_n = \mathrm{Lie}(\mathbb{Q}^n)$ denotes the free $\mathbb{Q}$-Lie algebra on $n$ generators.

### 9.4. Non-finiteness properties of certain metabelian groups

As an application of these techniques, we may construct a large class of metabelian groups that do not have good finiteness properties, either at the level of presentation complexes, or at the level of 1-models.

A finitely generated group $G$ is said to be *very large* if it has a quotient a free group $F_n$ of rank $n$ greater or equal to 2. The group $G$ is merely *large* if it has a finite-index subgroup which is very large.

**Theorem 9.8** ([113]). *Let $G$ be a metabelian group of the form $G = \pi/\pi''$, where $\pi$ is very large. Then $G$ is not finitely presentable and $G$ does not admit a 1-finite 1-model.*

*Proof.* By assumption, there is an epimorphism $\varphi\colon \pi \twoheadrightarrow F_n$, for some $n \geq 2$. Since the group $F_n$ is free, the map $\varphi$ admits a splitting, and thus, the induced homomorphism on maximal metabelian quotients, $\bar{\varphi}\colon \pi/\pi'' \twoheadrightarrow F_n/F_n''$, also has a splitting. By the homotopy functoriality of the 1-minimal model construction from Theorem 7.8, the map $\bar{\varphi}$ induces a CDGA map, $\bar{\varphi}^*\colon \mathcal{M}_1(F_n/F_n'') \to \mathcal{M}_1(\pi/\pi'')$, which is a split injection up to homotopy.

Suppose now that $\pi/\pi''$ admits a finite presentation, or a 1-finite 1-model. It then follows from Corollary 9.4 that $b_2(\mathcal{M}_1(\pi/\pi'')) < \infty$. Since the map $\bar{\varphi}^*$ is split injective (up to homotopy), and since homology is a homotopy functor, we infer that $b_2(\mathcal{M}_1(F_n/F_n'')) < \infty$. Hence, since $F_n/F_n''$ is filtered formal and $\mathbb{L}_n'' \subset \mathbb{L}^{\geq 2}$, Proposition 9.5 implies that the $\mathbb{Q}$-vector space $\mathbb{L}_n''/[\mathbb{L}_n, \mathbb{L}_n'']$ is finite-dimensional. On the other hand, a computation with Hall–Reutenauer bases done in [113, Proposition 3.2] shows that $\dim_{\mathbb{Q}}(\mathbb{L}_n''/[\mathbb{L}_n, \mathbb{L}_n'']) = \infty$. This is a contradiction, and the proof is complete. ∎

## 10. Formality of spaces, maps, and groups

### 10.1. Formal spaces

A space $X$ is said to be *formal* (over a field $\Bbbk$ of characteristic 0) if the Sullivan algebra $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$ is formal, i.e., it is weakly equivalent to the cohomology algebra $H^*(X, \Bbbk)$, equipped with the zero differential,

$$A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk \simeq (H^*(X, \Bbbk), 0).$$

If $X$ is formal (over $\mathbb{Q}$), its rationalization $X_{\mathbb{Q}}$ depends only on $H^*(X, \mathbb{Q})$.

The formality property behaves well with respect to field extensions of the form $\mathbb{Q} \subset \Bbbk$. Indeed, Halperin and Stasheff's result (Corollary 3.8) implies that a connected space $X$ with finite Betti numbers is formal over $\mathbb{Q}$ if and only if $X$ is formal over $\Bbbk$. This result was first stated and proved by Sullivan [137], using different techniques, while an independent proof was given by Neisendorfer and Miller [101] in the simply-connected case.

Formality is preserved under several standard operations on spaces. For instance, if $X$ and $Y$ are formal, then so is the product $X \times Y$ and the wedge $X \vee Y$; moreover, a retract of a formal space is formal; see [52, 54] for details. In general, a finite cover of a formal space need not be formal; nevertheless, Conjecture 8.11 holds in the formal case, and leads to the following result.

**Proposition 10.1** ([113]). *Suppose $\Phi$ is a finite group acting simplicially on a formal simplicial complex $Y$ with finite Betti numbers. Then the orbit space $X = Y/\Phi$ is again formal.*

The following result of Kreck and Triantafillou [79] fits into Sullivan's "determined up to finite ambiguity" philosophy.

**Theorem 10.2** ([79]). *Let $H$ be a finitely generated graded commutative ring over $\mathbb{Z}$. Then there are only finitely many homotopy types of simply-connected, formal, finite $CW$-complexes with integral cohomology isomorphic to $H$.*

## 10.2. Formality criteria

For nilpotent spaces, Sullivan gave a formality criterion in terms of lifting automorphisms of the cohomology algebra to the minimal model.

**Theorem 10.3** ([137]). *Let $X$ be a nilpotent CW-complex with finite Betti numbers. Then $X$ is formal if and only if every automorphism of $H^*(X, \mathbb{Q})$ can be realized by an automorphism of $\mathcal{M}(X)$.*

Roughly speaking, the more highly connected a space is, the more likely it is to be formal. This was made precise by Stasheff in [124], as follows.

**Theorem 10.4** ([124]). *Let $X$ be a $k$-connected CW-complex of dimension $n$; if $n \leq 3k+1$, then $X$ is formal.*

This is the best possible bound: attaching a cell $e^{3k+2}$ to the wedge $S^{k+1} \vee S^{k+1}$ via the iterated Whitehead product $[\iota_1, [\iota_1, \iota_2]]$ yields a non-formal CW-complex.

A powerful formality criterion was given by Sullivan in [137].

**Theorem 10.5** ([137]). *If $H^*(X, \Bbbk)$ is the quotient of a free* CGA *by an ideal generated by a regular sequence, then $X$ is a formal space. Consequently, if $H^*(M, \Bbbk)$ is a free* CGA*, then $X$ is formal.*

This result provides a large supply of formal spaces, such as: rational cohomology spheres and tori; compact connected Lie groups $G$, as well as their classifying spaces, $BG$; homogeneous spaces of the form $G/K$, with $\operatorname{rank} G = \operatorname{rank} K$; and Eilenberg–MacLane classifying spaces $K(G,n)$ for discrete groups $G$, provided $n \geq 2$. In particular, if $X$ is the complement of a knotted sphere in $S^n$, $n \geq 3$, then $X$ is a formal space.

On the other hand, not all homogeneous spaces are formal. For instance, the quotient spaces $\mathrm{SU}(pq)/(\mathrm{SU}(p) \times \mathrm{SU}(q))$ for $p,q \geq 3$; $\mathrm{SO}(n^2-1)/\mathrm{SU}(n)$ for $n \geq 3$; $\mathrm{Sp}(5)/\mathrm{SU}(5)$; and $\mathrm{SO}(78)/E_6$ are known to be non-formal. Furthermore, $K(G,1)$ spaces need not be formal. For instance, Hasegawa [69] showed that a classifying space for a torsion-free, finitely generated nilpotent group $G$ is formal if and only if $G$ is abelian. We refer to [54] for more on these topics.

A connected space $X$ is said to be *intrinsically formal* if any connected space whose rational cohomology algebra is isomorphic to $H^*(X,\mathbb{Q})$ has the same rational homotopy type as $X$; in other words, if there is a unique rational homotopy type whose rational cohomology algebra is isomorphic to that of $X$.

**Theorem 10.6** ([10, 68]). *Let $X$ be a connected space whose minimal model $\mathcal{M}(X)$ is of finite type. If $b_{2k}(X) = 0$ for all $k \geq 1$, then $X$ is intrinsically formal and has the rational homotopy type of a wedge of odd spheres.*

Although the spaces in the above theorem are intrinsically formal, they are typically not hyperformal. For instance, the space $X = S^{2k_1-1} \vee S^{2k_2-1}$ fits into this framework, but the cohomology algebra $H^*(X,\Bbbk)$ is isomorphic to $\bigwedge(x_1,x_2)/(x_1x_2)$, with $|x_i| = 2k_i - 1$, which is not hyperformal if $k_1 \neq k_2$, since in that case $\{x_1x_2\}$ is not a regular sequence.

### 10.3. Formality properties of closed manifolds

As shown by Miller [99], the dimension bound from Theorem 10.4 can be relaxed for closed manifolds, by using Poincaré duality.

**Theorem 10.7** ([99]). *Let $M$ be a closed, $k$-connected manifold ($k \geq 1$) of dimension $n \leq 4k+2$. Then $M$ is formal.*

In particular, all simply-connected closed manifolds of dimension at most 6 are formal. Again, this is the best possible: as shown by Fernández and Muñoz in [58], there exist closed, simply-connected, non-formal manifolds in each dimension $n \geq 7$. On the other hand, if $M$ is a closed, orientable, $k$-connected $n$-manifold with $b_{k+1}(M) = 1$, then the bound insuring formality can be improved to $n \leq 4k+4$, see Cavalcanti [29].

Formality also behaves well with respect to standard operations on manifolds. For instance, Stasheff [124] proved the following: If $M$ is a closed, simply-connected manifold such that the punctured manifold $M \setminus \{*\}$ is formal, then $M$ is formal. Moreover, if $M$ and $N$ are closed, orientable, formal manifolds, so is their connected sum, $M \# N$; see [52].

It has been shown by Cavalcanti [29], and, in stronger form, by Crowley and Nordström in [35], that a certain type of Hard Lefschetz property insures the intrinsic formality of highly connected manifolds.

**Theorem 10.8** ([35]). *Let $M$ be an $(n-1)$-connected manifold of dimension $4n-1$. Suppose $b_n(M) \leq 3$ and there is a cohomology class $u \in H^{2n-1}(M,\mathbb{Q})$ such that the map $H^n(M,\mathbb{Q}) \to H^{3n-1}(M,\mathbb{Q})$, $v \mapsto uv$ is an isomorphism. Then $M$ is intrinsically formal.*

In the same paper, Crowley and Nordström construct infinitely many simply-connected, non-formal manifolds all of whose Massey products vanish (the smallest dimension of such a manifold is 7). We summarize their results, as follows.

**Theorem 10.9** ([35]). *For each $k \geq 1$, there is a non-formal, $(2k-1)$-connected manifold of dimension $8k-1$ and a $(2k)$-connected manifold of dimension $8k+3$ such that all Massey products in the rational cohomology rings of these manifolds vanish.*

In [37], Deligne, Griffiths, Morgan, and Sullivan showed that every compact Kähler manifold $M$ is formal. On the other hand, symplectic manifolds need not be formal: the simplest example is the Kodaira–Thurston manifold, which is the product of the circle with the 3-dimensional Heisenberg nilmanifold (see Example 10.14 below). This led Lupton and Oprea [89] to raise the question whether compact, simply-connected symplectic manifolds are formal. The question was answered in the negative by Babenko and Taimanov [5,6], who used McDuff's symplectic blow-ups to construct non-formal, simply-connected symplectic manifolds in all even dimensions greater than 8; an 8-dimensional example was subsequently constructed by Fernández and Muñoz [60]. We refer to [54,57,83,121,143] for more on this subject.

### 10.4. Formal maps

A continuous map $f\colon X \to Y$ is said to be *formal* (over $\mathbb{Q}$) if the induced morphism between Sullivan models, $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$, is formal, in the sense of Definition 3.10. By the discussion from Section 4.4, this condition is equivalent to the existence of a diagram of the form

$$\begin{array}{ccccc} A_{\mathrm{PL}}(Y) & \xleftarrow{\rho_Y} & \mathcal{M}(Y) & \xrightarrow{\psi_Y} & (H^*(Y,\mathbb{Q}),0) \\ \downarrow{\scriptstyle A_{\mathrm{PL}}(f)} & & \downarrow{\scriptstyle \mathcal{M}(f)} & & \downarrow{\scriptstyle f^*} \\ A_{\mathrm{PL}}(X) & \xleftarrow{\rho_X} & \mathcal{M}(X) & \xrightarrow{\psi_X} & (H^*(X,\mathbb{Q}),0), \end{array}$$

which commutes up to homotopy and in which the horizontal arrows are quasi-isomorphisms. When $f$ is formal, the surjectivity of $f^*$ implies that of $\mathcal{M}(f)$.

One may define in a similar fashion formality of maps over an arbitrary field $\Bbbk$ of characteristic 0. As shown by Vigué-Poirrier in [146], a map $f\colon X \to Y$ between two nilpotent CW-complexes of finite type is formal over $\Bbbk$ if and only if it is formal over $\mathbb{Q}$. Moreover, as shown by Félix and Tanré [55], the cofiber of such a map is a formal space.

**Example 10.10.** Suppose $f\colon M \to N$ is a holomorphic map between two compact Kähler manifolds. Then, as shown in [37], $f$ is a formal map over $\mathbb{R}$.

In general, though, a map between two formal spaces need not be formal. A simple example is provided by the Hopf map $f\colon S^3 \to S^2$, for which $f^*\colon \widetilde{H}^*(S^2,\mathbb{Q}) \to \widetilde{H}^*(S^3,\mathbb{Q})$ is the zero map, yet the induced morphism $\mathcal{M}(f)\colon \mathcal{M}(S^2) \to \mathcal{M}(S^3)$ is non-trivial.

The next result, due to Arkowitz [4], delineates a class of formal spaces $X$ and $Y$ for which every map $f\colon X \to Y$ is formal.

**Theorem 10.11** ([4]). *Let $X$ and $Y$ be simply-connected, formal, rational spaces, and let $[X,Y]_{\mathrm{f}}$ be the set of homotopy classes of formal maps from $X$ to $Y$.*

(1) *The map $[X,Y]_{\mathrm{f}} \to \operatorname{Hom}(H^*(Y,\mathbb{Q}), H^*(X,\mathbb{Q}))$, $f \mapsto f^*$ is a bijection.*
(2) *Further, assume that $X$ and $Y$ are of finite type, $b_i(X) = 0$ for $i \geq 2n+1$, and $Y$ is $n$-connected. Then every map $f\colon X \to Y$ is formal.*

### 10.5. Partial formality

Let $q$ be a non-negative integer. A space $X$ is said to be *$q$-formal* (over a field $\Bbbk$ of characteristic 0) if its Sullivan algebra is $q$-formal, that is, $(A_{\mathrm{PL}}(X)\otimes_{\mathbb{Q}}\Bbbk, d) \simeq_q (H^*(X,\Bbbk), 0)$. Clearly, if $X$ is formal, then $X$ is $q$-formal for all $q \geq 0$. Under some additional hypothesis, this implication may be reversed.

**Theorem 10.12** ([91]). *Let $X$ be a space such that $H^i(X,\Bbbk) = 0$ for all $i \geq q+2$. Then $X$ is $q$-formal if and only if $X$ is formal.*

In particular, the notions of formality and $q$-formality coincide for $(q+1)$-dimensional CW-complexes.

**Example 10.13.** Let $V$ be a complex algebraic hypersurface in $\mathbb{CP}^n$, with complement $X = \mathbb{CP}^n \setminus V$. Work of Morgan [100] shows that $X$ is 1-formal, though not formal, in general. By Morse theory, $X$ has the homotopy type of a finite CW-complex of dimension $n$. Thus, if $n = 2$ (that is, $V$ is a plane curve), Theorem 10.12 implies that $X$ is formal.

**Example 10.14.** Let $M = G_{\mathbb{R}}/G_{\mathbb{Z}}$ be the 3-dimensional Heisenberg nilmanifold, where $G_{\mathbb{R}}$ is the group of real, unipotent $3\times 3$ matrices, and $G_{\mathbb{Z}} = \pi_1(M)$ is the subgroup of integral matrices in $G_{\mathbb{R}}$. This manifold has as a model the CDGA $(A,d)$, where $A = \bigwedge(a_1,a_2,b)$ with generators in degree 1, and differential given by $da_i = 0$ and $db = a_1a_2$. As noted in Example 3.6, this CDGA is not 1-formal. Alternatively, the triple Massey product $\langle [a_1],[a_1],[a_2]\rangle = \{[a_1b]\}$ is non-vanishing, with trivial indeterminacy. Therefore, $M$ is not 1-formal.

Partial formality enjoys a descent property analogous to that for full formality. Indeed, Theorem 3.9 has the following immediate corollary.

**Corollary 10.15** ([132]). *Let $X$ be a connected space such that $b_i(X) < \infty$ for $i \leq q+1$. Then $X$ is $q$-formal over $\mathbb{Q}$ if and only if $X$ is $q$-formal over $\Bbbk$.*

We may also consider a partial formality notion for maps. A continuous map $f\colon X \to Y$ is said to be *$q$-formal* if the morphism $A_{\text{PL}}(f)\colon A_{\text{PL}}(Y) \to A_{\text{PL}}(X)$ is $q$-equivalent to the induced homomorphism in cohomology, $f^*\colon H^*(Y,\mathbb{Q}) \to H^*(X,\mathbb{Q})$.

### 10.6. Koszul algebras and formality

Let $A$ be a connected, locally finite $\Bbbk$-CGA. The trivial $A$-module $\Bbbk$ has a free, graded $A$-resolution of the form

$$\cdots \xrightarrow{\varphi_3} A^{n_2} \xrightarrow{\varphi_2} A^{n_1} \xrightarrow{\varphi_1} A \longrightarrow \Bbbk \longrightarrow 0.$$

Such a resolution is *minimal* if all the non-zero entries of the matrices $\varphi_i$ have positive degrees. The algebra $A$ is said to be a *Koszul algebra* if the minimal $A$-resolution of $\Bbbk$ is linear, or, equivalently, $\operatorname{Tor}_i^A(\Bbbk,\Bbbk)_j = 0$ for all $i \neq j$. A necessary condition is that $A$ be expressed as the quotient $A = E/I$ of an exterior algebra on generators in degree 1 by an ideal $I$ generated in degree 2. A sufficient condition is that the ideal $I$ has a quadratic Gröbner basis. If $A$ is a Koszul algebra, then the quadratic dual $A^!$ is also a Koszul algebra and the following "Koszul duality" formula holds:

$$\operatorname{Hilb}(A,t)\cdot \operatorname{Hilb}(A^!,-t) = 1.$$

The following theorem of Papadima and Yuzvinsky [115] relates certain properties of the minimal model of a space $X$ to the Koszulness of its cohomology algebra.

**Theorem 10.16** ([115]). *Let $X$ be a connected space with finite Betti numbers.*

(1) *If $\mathcal{M}(X) \cong \mathcal{M}_1(X)$, then the cohomology algebra $H^*(X,\mathbb{Q})$ is a Koszul algebra.*

(2) *If $X$ is formal and $H^*(X,\mathbb{Q})$ is a Koszul algebra, then $\mathcal{M}(X) \cong \mathcal{M}_1(X)$.*

Consequently, if $X$ is formal, then $X$ is rationally aspherical if and only if $H^*(X,\mathbb{Q})$ is a Koszul algebra. When $X$ is also a nilpotent space, Berglund [15] recovers this equivalence (without assuming the cohomology algebra is generated in degree 1) and finds several alternative conditions yielding the same class of spaces, which he calls *Koszul spaces*.

As an application of Theorem 10.16, we have the following formality criterion.

**Corollary 10.17** ([107]). *Let $X$ be a connected, finite-type CW-complex, and suppose that $H^*(X,\mathbb{Q})$ is a Koszul algebra. Then $X$ is 1-formal if and only if $X$ is formal.*

**Example 10.18.** Let $\mathcal{A}$ be an arrangement of linear hyperplanes in $\mathbb{C}^n$, with complement $X = \mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H$. Work of Arnol'd and Brieskorn from the 1960s shows that $X$ is formal. Now suppose $\mathcal{A}$ is a fiber-type arrangement, or, equivalently, if its intersection lattice, $L(\mathcal{A})$, is supersolvable. Then $X$ is aspherical and $H^*(X,\mathbb{Q})$ is a Koszul algebra. Theorem 10.16 implies that $X$ is also rationally aspherical (this is a result first proved by Falk [48] by other methods). It is an open question whether the converse is true: If $X$ is rationally aspherical, is $\mathcal{A}$ necessarily of fiber-type? Put differently: If $H^*(X,\mathbb{Q})$ is a Koszul algebra, is $L(\mathcal{A})$ necessarily supersolvable?

### 10.7. The 1-formality property for groups

A finitely generated group $G$ is said to be 1-*formal* (over a field $\Bbbk$ of characteristic 0) if there is a classifying space $K(G,1)$ which is 1-formal (over $\Bbbk$). In view of the discussion from Section 8.3, we see that a connected CW-complex $X$ is 1-formal if and only if its fundamental group, $G=\pi_1(X)$, is 1-formal.

Over $\mathbb{Q}$, the 1-formality property of a group $G$ depends only on its Malcev Lie algebra, $\mathfrak{m}(G)$, or its rationalization, $G_{\mathbb{Q}}$. This is a consequence of the following well-known theorem, proved for instance in [27, 94, 132].

**Theorem 10.19.** *A finitely generated group $G$ is 1-formal if and only if $\mathfrak{m}(G)$ is isomorphic to the degree completion of a finitely generated, quadratic Lie algebra.*

Let $\mathfrak{h}(G)=\mathfrak{h}(G;\mathbb{Q})$ be the holonomy Lie algebra of $G$, as described in Section 7.5. As shown in [106], the 1-formality of $G$ is equivalent to $\mathfrak{m}(G)\cong\widehat{\mathfrak{h}}(G)$.

**Example 10.20.** Let $F_n$ be the free group of rank $n\geq 1$. We then have $H_1(F_n,\mathbb{Q})=\mathbb{Q}^n$ and $H_2(F_n,\mathbb{Q})=0$; hence, $\mu_G=0$ and so $\mathfrak{h}(F_n)=\mathrm{Lie}(\mathbb{Q}^n)$, the free Lie algebra of rank $n$. On the other hand, $\mathfrak{m}(F_n)=\widehat{\mathrm{Lie}}(\mathbb{Q}^n)$, by Theorem 6.4. Therefore, $F_n$ is 1-formal.

**Example 10.21.** Let $\Sigma_g$ be the Riemann surface of genus $g\geq 1$. The group $G=\pi_1(\Sigma_g)$ is generated by $x_1,y_1,\dots,x_g,y_g$, subject to the single relation $[x_1,y_1]\cdots[x_g,y_g]=1$. It is readily checked that $\mathfrak{h}(G,\Bbbk)$ is the quotient of the free Lie algebra on $x_1,y_1,\dots,x_g,y_g$ by the ideal generated by $[x_1,y_1]+\cdots+[x_g,y_g]$. A further computation using Theorem 6.4 shows that $\mathfrak{m}(G)\cong\widehat{\mathfrak{h}}(G)$; thus, $G$ is 1-formal.

The 1-formality property is preserved under finite free products and direct products of finitely generated groups. The following lemma (which follows at once from the discussion in Section 8.3) provides a useful 1-formality criterion.

**Lemma 10.22.** *Let $G$ a finitely generated group. Suppose there is a 1-formal group $K$ and a homomorphism $\varphi\colon G\to K$ such that $\varphi^*\colon H^1(K,\mathbb{Q})\to H^1(G,\mathbb{Q})$ is an isomorphism and $\varphi^*\colon H^1(K,\mathbb{Q})\to H^1(G,\mathbb{Q})$ is injective. Then $G$ is also 1-formal.*

**Example 10.23.** If $G$ is a finitely generated group with $b_1(G)$ equal to 0 or 1, then $G$ is 1-formal. Indeed, the claim is true for $K_0=\{1\}$ (trivially) and for $K_1=\mathbb{Z}$ (by Example 10.20). Moreover, if $b_1(G)=i\in\{0,1\}$, we may define a homomorphism $\varphi\colon G\to K_i$ satisfying the assumptions of Lemma 10.22. Therefore, the claim holds for $G$, too.

Here is another interpretation of the 1-formality notion. We say that a finitely generated group $G$ is *graded-formal* (over $\Bbbk$) if the associated graded Lie algebra $\mathrm{gr}(G;\Bbbk)$ is quadratic. It follows from Theorem 7.9 that $G$ is graded-formal precisely when the canonical surjection $\Phi\colon\mathrm{gr}(G;\Bbbk)\twoheadrightarrow\mathfrak{h}(G;\Bbbk)$ is an isomorphism. As in Section 6.5, we say that $G$ is *filtered-formal* over $\Bbbk$ if $\mathfrak{m}(G)\otimes\Bbbk\cong\widehat{\mathrm{gr}}(G;\Bbbk)$. Putting things together, we obtain the following result.

**Proposition 10.24** ([132]). *A finitely generated group $G$ is 1-formal (over $\Bbbk$) if and only if $G$ is graded-formal and filtered-formal (over $\Bbbk$).*

As a corollary, we deduce that 1-formality enjoys a descent property.

**Corollary 10.25** ([132]). *A finitely generated group $G$ is 1-formal over $\Bbbk$ if and only if $G$ is 1-formal over $\mathbb{Q}$.*

Indeed, it is easily seen that graded-formality is independent of the choice of a field $\Bbbk$ of characteristic 0. By Theorem 5.5, the same is true for filtered-formality, and the conclusion follows from Proposition 10.24. When $G$ is finitely presented, we have that $b_2(G) < \infty$, and the result also follows from Corollary 10.15.

As we saw in Example 10.20, the free group $F_n$ has vanishing cup-product map $\mu_{F_n}$ and is 1-formal. Here is a partial converse.

**Proposition 10.26** ([42]). *Let $G$ be a group admitting a finite presentation with only commutator relators. If $G$ is 1-formal and $\mu_G = 0$, then $G$ is a free group.*

**Example 10.27.** Let $G = G_{\mathbb{Z}}$ be the Heisenberg group from Example 10.14. Then $G$ is isomorphic to $F_2/\gamma_3(F_2)$, and so it has a finite presentation with only commutator relators; moreover, $\mu_G = 0$, yet $G$ is not a free group, since it is 2-step nilpotent. Therefore, we conclude once again that $G$ is not 1-formal.

### 10.8. Polyhedral products and right-angled Artin groups

We conclude this section with a discussion of the formality properties of polyhedral product spaces and some related groups. Given a finite simplicial complex $K$, it is a subtle question to decide whether the polyhedral products $\mathcal{Z}_K(\underline{X}, \underline{X}')$ from Section 8.6 are formal, even when all the spaces $X_i$ and the subspaces $X'_i$ are formal. Theorem 8.9 (together with a previous remark) yields a sufficient condition for this to happen.

**Corollary 10.28** ([56]). *Let $X_i$, $X'_i$ be nilpotent, finite-type CW-complexes. Assume that the inclusion maps $X'_i \hookrightarrow X_i$ are formal and induce epimorphisms in cohomology. Then all polyhedral products $\mathcal{Z}_K(\underline{X}, \underline{X}')$ are formal.*

We specialize now to the case when $X_i = X$ and $X'_i = X'$ for all $i$, and write $\mathcal{Z}_K(X, X')$ for the corresponding polyhedral product. If $X$ is nilpotent and formal, then the inclusion $* \to X$ satisfies the hypothesis of Corollary 10.28, and thus $\mathcal{Z}_K(X, *)$ is formal – a result first proved in [102]. In particular, the Davis–Januszkiewicz spaces $DJ_K = \mathcal{Z}_K(\mathbb{CP}^\infty, *)$ and the toric complexes $T_K = \mathcal{Z}_K(S^1, *)$ are all formal.

Letting $\Gamma$ be the 1-skeleton of $K$, it is readily seen that the fundamental group of $T_K$ is the right-angled Artin group $G_\Gamma$ associated to the graph $\Gamma$. Consequently, all right-angled Artin groups are 1-formal – a result also proved in [107], using Theorems 6.4 and 10.19. Moreover, if the flag complex of $\Gamma$ is simply connected, then, as shown in [104], the Bestvina–Brady group associated to $\Gamma$ is also 1-formal.

Finally, let us consider the moment-angle complexes $\mathcal{Z}_K = \mathcal{Z}_K(D^2, S^1)$. In this situation, Corollary 10.28 no longer applies, since the inclusion-induced homomorphism $H^1(D^2, \mathbb{Q}) \to H^1(S^1, \mathbb{Q})$ is not surjective. In fact, there are infinite families of simplicial complexes $K$ for which $H^*(\mathcal{Z}_K, \mathbb{Q})$ has non-vanishing Massey products, and thus $\mathcal{Z}_K$ is non-formal, see [9, 38, 64]. If $K$ is an $n$-vertex triangulation of $S^m$, then $\mathcal{Z}_K$ is a closed manifold of dimension $n+m+1$. Asymptotically, almost all triangulations $K$ of $S^2$ yield non-formal moment-angle manifolds $\mathcal{Z}_K$, see [38].

## 11. Alexander invariants and resonance varieties

### 11.1. A generalized Koszul complex

Given a finite-dimensional $\Bbbk$-vector space $V$, we define the corresponding *canonical element* to be tensor $\omega_V \in V^\vee \otimes V$ which corresponds to the identity automorphism of $V^\vee$ under the tensor-hom adjunction (recall that $\otimes = \otimes_\Bbbk$). In concrete terms, if we pick a basis $\{e_1, \dots, e_n\}$ for $V^\vee$ and let $\{x_1, \dots, x_n\}$ be the dual basis for $V$, then $\omega_V = \sum_{j=1}^n e_j \otimes x_j$.

Now let $A = (A^*, d)$ be a connected $\Bbbk$-CDGA, and assume that the $\Bbbk$-vector space $H^1(A)$ is finite-dimensional. Since $d(1) = 0$ and $A^0 = \Bbbk$, the differential $d\colon A^0 \to A^1$ vanishes; thus, we may identify $H^1(A)$ with $Z^1(A)$. Setting $H_1(A) = (H^1(A))^\vee$, we let $\omega_A := \omega_{H_1(A)} \in H^1(A) \otimes H_1(A)$ be the corresponding canonical element.

Let $S = \operatorname{Sym}(H_1(A))$ be the symmetric algebra on $H_1(A)$. The tensor product $A \otimes_\Bbbk S$ is both a free $S$-module and a bigraded $\Bbbk$-algebra, with product $(a \otimes s)(a' \otimes s') = aa' \otimes ss'$. It is also a $\Bbbk$-CDGA, with differential $d \otimes \mathrm{id}_S$. Left-multiplication by $\omega_A$, viewed as an element of $Z^1(A) \otimes H_1(A)$, defines an endomorphism of $A \otimes S$ of bidegree $(1,1)$. We define an $S$-linear map, $\delta_A\colon A \otimes_\Bbbk S \to A \otimes S$, by

$$\delta_A = \omega_A + d \otimes \mathrm{id}_S \,. \tag{11.1}$$

It is readily verified that $\delta_A^2 = 0$, and so the next result follows.

**Proposition 11.1** ([130]). *Let $(A^*, d)$ be a connected $\Bbbk$-CDGA with $\dim_\Bbbk H^1(A) < \infty$. There is then a cochain complex of free $S$-modules,*

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta_A^i} A^{i+1} \otimes S \xrightarrow{\delta_A^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots, \tag{11.2}$$

*with differentials given by (11.1), such that $(A^* \otimes S, \delta_A)$ is again a $\Bbbk$-CDGA.*

If we fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for $H^1(A)$ and let $\{x_1, \dots, x_n\}$ be the dual basis for $H_1(A)$, the ring $S = \operatorname{Sym}(H_1(A))$ may be identified with the polynomial ring $\Bbbk[x_1, \dots, x_n]$, viewed as the coordinate ring of the affine space $H^1(A)$. The differentials in (11.1) are then the $S$-linear maps given by

$$\delta_A^i(a \otimes s) = \sum_{j=1}^n e_j a \otimes s x_j + d(a) \otimes s$$

for all $a \in A^i$ and $s \in S$. If the CDGA $A$ has zero differential, each map $\delta^i_A$ is given by a matrix whose entries are linear forms in the variables $x_1, \dots, x_n$; in general, though, the entries of $\delta^i_A$ may also have non-zero constant terms.

## 11.2. The Alexander invariants of a CDGA

The $S$-dual of the cochain complex (11.2) is the chain complex of free $S$-modules,

$$(A_* \otimes S, \delta^A) : \cdots \longrightarrow A_2 \otimes S \xrightarrow{\delta^A_2} A_1 \otimes S \xrightarrow{\delta^A_1} A_0 \otimes S = S, \tag{11.3}$$

where the maps $\delta^A_i$ are the $S$-linear duals of the maps $\delta^i_A$. By analogy with classical the topological setting, we define the *Alexander invariants* of a CDGA $(A^*, d)$ as the homology $S$-modules of this chain complex,

$$\mathfrak{B}_i(A) := H_i(A_* \otimes S).$$

If $d = 0$, then the differentials in (11.3) are homogeneous (of degree 1), and so the $S$-modules $\mathfrak{B}_i(A)$ inherit a natural grading. For instance, if $E = \bigwedge V$ is the exterior algebra on a finite-dimensional $\Bbbk$-vector space $V$, with differential $d = 0$, then $\mathfrak{B}_i(E) = 0$ for all $i \geq 1$. In general, though, the Alexander invariants $\mathfrak{B}_i(A)$ do not have a natural grading.

An explicit finite presentation for the first Alexander invariant, $\mathfrak{B}(A) := \mathfrak{B}_1(A)$, was given in [106, Theorem 6.2] in the case when $d = 0$. This presentation is generalized in [130], as follows.

Let $(A, d)$ be a connected $\Bbbk$-CDGA with $A^1$ finite-dimensional. Set $E = \bigwedge H^1(A)$ and identify $E^1 = H^1(A)$ with $Z^1 = \ker(d \colon A^1 \to A^2)$. Let $U^1$ be its complementary $\Bbbk$-vector subspace, so that $A^1 = E^1 \oplus U^1$, and write $A_i = (A^i)^\vee$ and so forth for the $\Bbbk$-dual vector spaces. Then $U_1$ may be identified with the image of the $\Bbbk$-dual of the differential, $d^\vee \colon A_2 \to A_1$, and we have a direct sum decomposition, $A_1 = E_1 \oplus U_1$. Let $\pi_U \colon A_1 \to U_1$ be the projection onto the second summand.

**Theorem 11.2** ([130]). *The Alexander invariant of $A$, viewed as a module over the symmetric algebra $S = \operatorname{Sym}(E_1)$, has presentation*

$$(E_3 \oplus A_2) \otimes S \xrightarrow{\begin{pmatrix} \delta^E_3 & 0 \\ \mu^\vee_E \otimes \mathrm{id}_S & d^\vee_A \otimes \mathrm{id}_S + \beta^\vee_A \end{pmatrix}} (E_2 \oplus U_1) \otimes S \longrightarrow \mathfrak{B}(A) \longrightarrow 0,$$

*where* $\beta^\vee_A = (\pi_U \otimes \mathrm{id}_S) \circ (\omega_A - \mu_E \circ \omega_E)^\vee$.

Finally, let $I$ be the maximal ideal at 0 of the polynomial ring $S$. The powers of this ideal define a descending filtration, $\{I^n \mathfrak{B}(A)\}_{n \geq 0}$, on the Alexander invariant of $A$. Let $\operatorname{gr}(\mathfrak{B}(A))$ be the associated graded $S$-module with respect to this filtration.

**Proposition 11.3** ([130]). *For each $k \geq 1$, there is an isomorphism of $\Bbbk$-vector spaces,*

$$\operatorname{gr}_k(\mathfrak{B}(A))^\vee \cong \operatorname{Tor}^{\mathcal{E}}_{k-1}(A, \Bbbk)_k,$$

*where on the right side $A$ is viewed as a graded module over the exterior algebra $\mathcal{E} = \bigwedge A^1$.*

### 11.3. Resonance varieties

Let $(A, d)$ be a connected CDGA. As noted previously, $H^1(A) = Z^1(A)$. For every $\omega \in H^1(A)$, the operator $d_\omega := d + \omega\,\cdot$ is a differential on $A$. The *resonance varieties* $\mathcal{R}^i_k(A)$ are defined, for all $i, k \geq 0$, as the infinitesimal jump loci

$$\mathcal{R}^i_k(A) = \{\omega \in H^1(A) \mid \dim H^i(A, d_\omega) \geq k\}.$$

When the CDGA $A$ is $q$-finite, for some $q \geq 1$, these sets are Zariski closed subsets of the affine space $H^1(A)$, for all $i \leq q$ and $k \geq 0$.

Clearly, $H^i(A, \delta_0) = H^i(A)$; thus, the point $\mathbf{0} \in H^1(A)$ belongs to the variety $\mathcal{R}^i_1(A)$ if and only if $H^i(A) \neq 0$. Moreover, $\mathcal{R}^0_1(A) = \{\mathbf{0}\}$. When the differential of $A$ is zero, the resonance varieties $\mathcal{R}^i_k(A)$ are homogeneous subsets of $H^1(A) = A^1$. In general, though, the resonance varieties of a CDGA are not homogeneous, as we shall see in Example 11.6.

The following lemma follows quickly from the definitions.

**Lemma 11.4** ([92]). *Let $\varphi\colon A \to A'$ be a* CDGA *morphism, and assume $\varphi$ is an isomorphism up to degree $q$, and a monomorphism in degree $q + 1$, for some $q \geq 0$. Then the induced isomorphism in cohomology, $\varphi^*\colon H^1(A') \to H^1(A)$, identifies $\mathcal{R}^i_k(A)$ with $\mathcal{R}^i_k(A')$ for each $i \leq q$, and sends $\mathcal{R}^{q+1}_k(A)$ into $\mathcal{R}^{q+1}_k(A')$, for all $k \geq 0$.*

Consequently, if $A$ and $A'$ are isomorphic CDGAs, then their resonance varieties are ambiently isomorphic. As we shall see (also in Example 11.6), the conclusions of Lemma 11.4 do not always hold if we only assume that $\varphi\colon A \to A'$ is a $q$-quasi-isomorphism.

An alternative interpretation of the degree 1 resonance varieties is given by the following theorem.

**Theorem 11.5** ([130]). *Let $A$ be a connected* CDGA *with $0 < \dim A^1 < \infty$. Then, for all $k \geq 1$,*

$$\mathcal{R}^1_k(A) = \mathbf{V}\Big(\operatorname{Ann}\Big(\bigwedge^{k}(\mathfrak{B}(A))\Big)\Big),$$

*at least away from $\mathbf{0} \in H^1(A)$.*

The next example (adapted from [92] and [126]) illustrates several of the points mentioned above.

**Example 11.6.** Let $A$ be the exterior algebra over $\mathbb{C}$ on generators $a, b$ in degree 1, equipped with the differential given by $da = 0$ and $db = b \cdot a$. Then $H^1(A) = \mathbb{C}$, generated by $a$. Setting $S = \mathbb{C}[x]$, the chain complex (11.3) takes the form

$$S \xrightarrow{\delta_2 = \binom{0}{x-1}} S^2 \xrightarrow{\delta_1 = (x\ \ 0)} S.$$

It is readily seen that the Alexander invariant $\mathfrak{B}(A) = H_1(A_* \otimes S)$ is isomorphic to $S/(x-1)$. Its support is equal to $\{1\}$, yet the resonance variety $\mathcal{R}^1_1(A)$ is equal to $\{0, 1\}$; both are non-homogeneous subvarieties of $\mathbb{C}$. Finally, let $A'$ be the sub-CDGA generated by $a$. Clearly, the inclusion map, $\iota\colon A' \hookrightarrow A$, induces an isomorphism in cohomology. Nevertheless, $\mathcal{R}^1_1(A') = \{0\}$, and so the resonance varieties of $A$ and $A'$ differ, although $A$ and $A'$ are quasi-isomorphic.

### 11.4. Resonance of tensor products and Hirsch extensions

The resonance varieties behave well with respect to some natural operations on CDGAs. The next result details the behavior of the depth-1 resonance varieties with respect to tensor products.

**Proposition 11.7** ([110, 112]). *Let $(A, d)$ and $(A', d')$ be two connected, finite-type CDGAs. Then, for all $q \geq 0$,*

$$\mathcal{R}^i_1(A \otimes A') = \bigcup_{p+q=i} \mathcal{R}^p_1(A) \times \mathcal{R}^q_1(A').$$

A proof of this statement is given in [110, Proposition 13.1] under the assumption that both differentials, $d$ and $d'$, vanish (see also [112, Proposition 2]). The same approach works in this wider generality.

We conclude this section with a result that shows how the resonance varieties behave under a certain type of Hirsch extensions.

**Proposition 11.8** ([114]). *Let $B$ be a connected, finite-type CDGA. Fix an element $e \in B^2$ with $de = 0$, and let $A = (B \otimes_e \bigwedge(t), d)$ be the corresponding Hirsch extension.*

(1) *If $[e] = 0$, then $\mathcal{R}^i_1(A) = \mathcal{R}^{i-1}_1(B) \cup \mathcal{R}^i_1(B)$, for all $i$.*

(2) *If $[e] \neq 0$, then*

  (a) *$\mathcal{R}^i_k(A) \subseteq \mathcal{R}^{i-1}_1(B) \cup \mathcal{R}^i_k(B)$, for all $i$ and $k$;*

  (b) *$\mathcal{R}^1_k(A) = \mathcal{R}^1_k(B)$, for all $k$.*

## 12. Cohomology jump loci and finiteness properties

### 12.1. Characteristic varieties

Given a space $X$, the jump loci for cohomology with coefficients in rank 1 complex local systems on $X$ are powerful homotopy-type invariants, defined as follows.

We will assume that $X$ is path-connected and its fundamental group, $G = \pi_1(X)$, is finitely generated. Let $\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*)$ be the group of $\mathbb{C}$-valued multiplicative characters on $G$. This is an abelian, complex algebraic group, whose identity $\mathbf{1}$ corresponds to the trivial representation. The group $\mathbb{T}_G$ may be identified with the cohomology group

$\mathrm{Char}(X) := H^1(X, \mathbb{C}^*)$. Its identity component, $\mathbb{T}^0_G$, is isomorphic to the complex algebraic torus $(\mathbb{C}^*)^{b_1(X)}$; the other connected components of $\mathbb{T}_G$ are copies of this torus, indexed by the torsion subgroup of the finitely generated abelian group $G_{\mathrm{ab}} = H_1(X, \mathbb{Z})$.

The *characteristic varieties* of $X$ in degree $i \geq 0$ and depth $k \geq 0$ are the sets

$$\mathcal{V}^i_k(X) = \{\rho \in \mathrm{Char}(X) \mid \dim H^i(X, \mathbb{C}_\rho) \geq k\},$$

where $\mathbb{C}_\rho$ is the rank 1 local system on $X$ associated to a representation $\rho\colon G \to \mathbb{C}^*$. In other words, $\mathbb{C}_\rho$ is the vector space $\mathbb{C}$ viewed as a module over the group algebra $\mathbb{C}[G]$ via the action $g \cdot a = \rho(g)a$, for $g \in G$ and $a \in \mathbb{C}$.

When the space $X$ is $q$-finite, for some $q \geq 1$, the sets $\mathcal{V}^i_k(X)$ are Zariski closed subsets of the character group, for all $i \leq q$ and $k \geq 0$, see [111]. It is readily seen that the sets $\mathcal{V}^1_k(X)$ depend only on the group $G = \pi_1(X)$.

Now let $G$ be a finitely generated group, and set $\mathcal{V}^i_k(G) := \mathcal{V}^i_k(K(G,1))$. It is known that the sets $\mathcal{V}^1_k(G)$ with $k \geq 0$ depend only on the maximal metabelian quotient $G/G''$ (see e.g. [41]); more precisely, $\mathcal{V}^1_k(G) = \mathcal{V}^1_k(G/G'')$.

The characteristic varieties have several useful naturality properties. For instance, suppose $\varphi\colon G \twoheadrightarrow Q$ is an epimorphism. Then the induced morphism on character groups, $\varphi^*\colon \mathbb{T}_Q \to \mathbb{T}_G$, is injective and sends $\mathcal{V}^1_k(Q)$ into $\mathcal{V}^1_k(G)$ for all $k \geq 0$. Likewise, suppose that $H < G$ is a finite-index subgroup. Then the inclusion $\alpha\colon H \to G$ induces a morphism $\alpha^*\colon \mathbb{T}_G \to \mathbb{T}_H$ with finite kernel, which sends $\mathcal{V}^i_k(G)$ to $\mathcal{V}^i_k(H)$ for all $i, k \geq 0$.

For the free groups $F_n$ of rank $n \geq 2$, we have that $\mathcal{V}^1_k(F_n) = (\mathbb{C}^*)^n$ for $k \leq n-1$ and $\mathcal{V}^1_n(F_n) = \{1\}$. In general, though, the jump loci of a group can be arbitrarily complicated.

**Example 12.1.** Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be an integral Laurent polynomial with $f(1) = 0$. Then, as shown in [134], there is a finitely presented group $G$ with $G_{\mathrm{ab}} = \mathbb{Z}^n$ such that $\mathcal{V}^1_1(G)$ coincides with the variety $\mathbf{V}(f) := \{t \in (\mathbb{C}^*)^n \mid f(t) = 0\}$.

## 12.2. Algebraic models and cohomology jump loci

Work of Dimca and Papadima [40], generalizing previous work of Dimca, Papadima, and Suciu [42], establishes a tight connection between the geometry of the characteristic varieties of a space and that of resonance varieties of a model for it, around the origins of the respective ambient spaces, provided certain finiteness conditions hold.

Let $X$ be a path-connected space with $b_1(X) < \infty$, and consider the analytic map $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ induced by the coefficient homomorphism $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$. Let $(A, d)$ be a CDGA model for $X$, defined over $\mathbb{C}$. Upon identifying $H^1(A) \cong H^1(X, \mathbb{C})$, we obtain an analytic map $H^1(A) \to H^1(X, \mathbb{C}^*)$, which takes $\mathbf{0}$ to $\mathbf{1}$.

**Theorem 12.2** ([40]). *Let $X$ be a $q$-finite space, and suppose $X$ admits a $q$-finite, $q$-model $A$, for some $q \geq 1$. Then, the aforementioned map, $H^1(A) \to H^1(X, \mathbb{C}^*)$, induces a local analytic isomorphism, $H^1(A)_{(\mathbf{0})} \to H^1(X, \mathbb{C}^*)_{(\mathbf{1})}$, which identifies the germ at $\mathbf{0}$ of $\mathcal{R}^i_k(A)$ with the germ at $\mathbf{1}$ of $\mathcal{V}^i_k(X)$, for all $i \leq q$ and all $k \geq 0$.*

The work of Budur and Wang [25] builds on this theorem, providing a structural result on the geometry of the characteristic varieties of spaces satisfying the hypothesis of the above theorem. Putting together Theorem 12.2 and Corollary 12.9 yields their result, in the slightly stronger form given in [113].

**Theorem 12.3** ([25]). *Suppose $X$ is a $q$-finite space which admits a $q$-finite $q$-model. Then all the irreducible components of $\mathcal{V}^i_k(X)$ passing through* $\mathbf{1}$ *are algebraic subtori of* $\operatorname{Char}(X)$, *for all $i \leq q$ and $k \geq 0$.*

### 12.3. Finiteness obstructions

The above theorem may be used to give examples of finite CW-complexes which do not have 1-finite 1-models.

**Example 12.4.** Let $f$ be an integral Laurent polynomial in $n \geq 2$ variables, and assume its zero set in $(\mathbb{C}^*)^n$ contains the origin 1, is irreducible but is not an algebraic subtorus; for instance, take $f(t) = \sum_{i=1}^n t_i - n$. Letting $G$ be a finitely presented group with $\mathcal{V}^1_1(G) = \mathbf{V}(f)$ as in Example 12.1, we deduce from Theorem 12.3 that the finite presentation complex of $G$ admits no 1-finite 1-model.

On the other hand, as the next example shows, the existence of a 1-finite 1-model for a finitely generated group does not necessarily imply that the group is finitely presented.

**Example 12.5.** Let $Y$ be a finite, connected CW-complex which is non-simply connected yet has $b_1(Y) = 0$, and let $G$ be the Bestvina–Brady group associated to a flag triangulation of $Y$. It is proved in [108, Section 10] that $G$ is finitely generated and 1-formal, but not finitely presented.

As the next family of examples illustrates, the infinitesimal finiteness obstruction from Theorem 8.12 may be stronger than the one from Theorem 12.3, even when $q = 1$.

**Example 12.6.** Consider the free metabelian group $G = F_n/F_n''$ with $n \geq 2$. The free group $F_n = \pi_1(\bigvee^n S^1)$ admits a formal, finite CW-complex as classifying space; thus, Theorem 12.3 applies to $F_n$. It follows that the characteristic varieties $\mathcal{V}^i_k(G) \cong \mathcal{V}^i_k(F_n)$ satisfy the conditions from Theorem 12.3 for $i \leq 1$ and $k \geq 0$. On the other hand, as we saw in the proof of Theorem 9.8, we have that $b_2(\mathcal{M}_1(G)) = \infty$, and so the group $G$ admits no 1-finite 1-model.

### 12.4. Tangent cones

Before proceeding, we review two constructions that provide approximations to a subvariety $W$ of a complex algebraic torus $(\mathbb{C}^*)^n$. The first one is the classical tangent cone, while the second one is the exponential tangent cone, a construction introduced in [42] and further studied in [40, 125, 134].

Let $I$ be an ideal in the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ such that $W = V(I)$. Picking a finite generating set for $I$, and multiplying these generators with suitable monomials if necessary, we see that $W$ may also be defined by the ideal $I \cap R$ in the polynomial ring $R = \mathbb{C}[t_1, \dots, t_n]$. Let $J$ be the ideal in the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ generated by the polynomials $g(x_1, \dots, x_n) = f(x_1 + 1, \dots, x_n + 1)$, for all $f \in I \cap R$.

The *tangent cone* of $W$ at $\mathbf{1} \in (\mathbb{C}^*)^n$ is the algebraic subset $\mathrm{TC}_{\mathbf{1}}(W) \subseteq \mathbb{C}^n$ defined by the ideal $\mathrm{in}(J) \subset S$ generated by the initial forms of all non-zero elements from $J$. The set $\mathrm{TC}_{\mathbf{1}}(W)$ is a homogeneous subvariety of $\mathbb{C}^n$, which depends only on the analytic germ of $W$ at the identity. In particular, $\mathrm{TC}_{\mathbf{1}}(W) \neq \emptyset$ if and only if $\mathbf{1} \in W$.

Let $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ be the exponential map, given in coordinates by $x_i \mapsto e^{x_i}$. The *exponential tangent cone* at $\mathbf{1}$ to a subvariety $W \subseteq (\mathbb{C}^*)^n$ is the set

$$\tau_{\mathbf{1}}(W) = \{x \in \mathbb{C}^n \mid \exp(\lambda x) \in W, \text{ for all } \lambda \in \mathbb{C}\}.$$

It is readily seen that $\tau_{\mathbf{1}}$ commutes with finite unions and arbitrary intersections. Furthermore, $\tau_{\mathbf{1}}(W)$ only depends on $W_{(\mathbf{1})}$, the analytic germ of $W$ at the identity; in particular, $\tau_{\mathbf{1}}(W) \neq \emptyset$ if and only if $\mathbf{1} \in W$. The main property of this construction is encapsulated in the following lemma.

**Lemma 12.7** ([42, 125, 134]). *The exponential tangent cone $\tau_{\mathbf{1}}(W)$ of a subvariety $W \subseteq (\mathbb{C}^*)^n$ is a finite union of rationally defined linear subspaces of the affine space $\mathbb{C}^n$.*

For instance, if $W$ is an algebraic subtorus of $(\mathbb{C}^*)^n$, then $\tau_{\mathbf{1}}(W)$ equals $\mathrm{TC}_{\mathbf{1}}(W)$, and both coincide with $T_{\mathbf{1}}(W)$, the tangent space to $W$ at the identity $\mathbf{1}$. More generally, there is always an inclusion between the two types of tangent cones associated to an algebraic subset $W \subseteq (\mathbb{C}^*)^n$, namely,

$$\tau_{\mathbf{1}}(W) \subseteq \mathrm{TC}_{\mathbf{1}}(W). \tag{12.1}$$

As we shall see, though, this inclusion is far from being an equality for arbitrary $W$. For instance, the tangent cone $\mathrm{TC}_{\mathbf{1}}(W)$ may be a non-linear, irreducible subvariety of $\mathbb{C}^n$, or $\mathrm{TC}_{\mathbf{1}}(W)$ may be a linear space containing the exponential tangent cone $\tau_{\mathbf{1}}(W)$ as a union of proper linear subspaces.

### 12.5. The Exponential Ax–Lindemann theorem

In [25], Budur and Wang establish the following version of a classical result, due to Ax and Lindemann.

**Theorem 12.8** ([25]). *Let $V \subseteq \mathbb{C}^n$ and $W \subseteq (\mathbb{C}^*)^n$ be irreducible algebraic subvarieties.*

(1) *Suppose* $\dim V = \dim W$ *and* $\exp(V) \subseteq W$. *Then $V$ is a translate of a linear subspace, and $W$ is a translate of an algebraic subtorus.*
(2) *Suppose the exponential map* $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ *induces a local analytic isomorphism $V_{(\mathbf{0})} \to W_{(\mathbf{1})}$. Then $W_{(\mathbf{1})}$ is the germ of an algebraic subtorus.*

A standard dimension argument shows the following: if $W$ and $W'$ are irreducible algebraic subvarieties of $(\mathbb{C}^*)^n$ which contain $\mathbf{1}$ and whose germs at $\mathbf{1}$ are locally analytically isomorphic, then $W \cong W'$. Using this fact, we obtain the following corollary to part (2) of the above theorem.

**Corollary 12.9.** *Let $V \subseteq \mathbb{C}^n$ and $W \subseteq (\mathbb{C}^*)^n$ be irreducible algebraic subvarieties. Suppose the exponential map* $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ *induces a local analytic isomorphism $V_{(\mathbf{0})} \cong W_{(\mathbf{1})}$. Then $W$ is an algebraic subtorus and $V$ is a rationally defined linear subspace.*

## 12.6. Tangent cones and jump loci

Let $X$ be a $q$-finite space. Its cohomology algebra, $H^*(X,\mathbb{C})$, is then $q$-finite; that is, $b_i(X) < \infty$ for $i \le q$. Thus, the resonance varieties $\mathcal{R}^i_k(X) := \mathcal{R}^i_k(H^*(X,\mathbb{C}))$ are homogeneous algebraic subsets of the affine space $H^1(X,\mathbb{C})$, for all $i \le q$ and $k \ge 0$.

The following basic relationship between the characteristic and resonance varieties was established by Libgober in [86] in the case when $X$ is a finite CW-complex and $i$ is arbitrary; a similar proof works in the generality that we work in here.

**Theorem 12.10** ([86]). *Suppose $X$ is a $q$-finite space. Then, for all $i \le q$ and $k \ge 0$,*

$$\mathrm{TC}_{\mathbf{1}}\big(\mathcal{V}^i_k(X)\big) \subseteq \mathcal{R}^i_k(X).$$

Putting together these inclusions with those from (12.1), we obtain the following corollary.

**Corollary 12.11.** *Suppose $X$ is a $q$-finite space. Then, for all $i \le q$ and $k \ge 0$,*

$$\tau_{\mathbf{1}}\big(\mathcal{V}^i_k(X)\big) \subseteq \mathrm{TC}_{\mathbf{1}}\big(\mathcal{V}^i_k(X)\big) \subseteq \mathcal{R}^i_k(X).$$

A particular case of this corollary is worth mentioning separately.

**Corollary 12.12.** *Let $G$ be a finitely generated group. Then, for all $k \ge 0$,*

$$\tau_{\mathbf{1}}\big(\mathcal{V}^1_k(G)\big) \subseteq \mathrm{TC}_{\mathbf{1}}\big(\mathcal{V}^1_k(G)\big) \subseteq \mathcal{R}^1_k(G).$$

Using now Theorems 12.2 and 12.3, we obtain the following "tangent cone formula."

**Theorem 12.13.** *Suppose $X$ is a $q$-finite space which admits a $q$-finite $q$-model $A$. Then, for all $i \le q$ and $k \ge 0$,*

$$\tau_{\mathbf{1}}\big(\mathcal{V}^i_k(X)\big) = \mathrm{TC}_{\mathbf{1}}\big(\mathcal{V}^i_k(X)\big) = \mathcal{R}^i_k(A).$$

This theorem, together with Theorem 9.3, yields the following corollary.

**Corollary 12.14.** *Suppose $G$ is a finitely generated group whose Malcev Lie algebra is the LCS completion of a finitely presented Lie algebra. Then $\tau_{\mathbf{1}}(\mathcal{V}^1_k(G)) = \mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_k(G))$, for all $k \ge 0$.*

In other words, if the first half of the tangent cone formula fails in degree 1, i.e., if $\tau_{\mathbf{1}}(\mathcal{V}^1_k(G)) \subsetneq \mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_k(G))$ for some $k > 0$, then $\mathfrak{m}(G) \not\cong \widehat{L}$, for any finitely presented Lie algebra $L$. This will happen automatically if the variety $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_k(G))$ has an irreducible component which is not a rationally defined linear subspace of $H^1(G,\mathbb{C})$.

### 12.7. Formality and cohomology jump loci

The main connection between the formality property of a space and the geometry of its cohomology jump loci is provided by the next result. This result, which was first proved in degree $i = 1$ in [42], and in arbitrary degree in [40], is now an immediate consequence of Theorem 12.13.

**Corollary 12.15.** *Let $X$ be a $q$-finite, $q$-formal space. Then, for all $i \leq q$ and $k \geq 0$,*

$$\tau_1\big(\mathcal{V}^i_k(X)\big) = \operatorname{TC}_1\big(\mathcal{V}^i_k(X)\big) = \mathcal{R}^i_k(X). \tag{12.2}$$

In particular, if $G$ is a finitely generated, 1-formal group, then, for all $k \geq 0$,

$$\tau_1\big(\mathcal{V}^1_k(G)\big) = \operatorname{TC}_1\big(\mathcal{V}^1_k(G)\big) = \mathcal{R}^1_k(G).$$

As an application of Corollary 12.15, we have the following characterization of the irreducible components of the cohomology jump loci in the formal setting.

**Corollary 12.16.** *Suppose $X$ is a $q$-finite, $q$-formal space. Then, for all $i \leq q$ and $k \geq 0$, the following hold.*

(1) *All irreducible components of the resonance varieties $\mathcal{R}^i_k(X)$ are rationally defined linear subspaces of $H^1(X, \mathbb{C})$.*
(2) *All irreducible components of the characteristic varieties $\mathcal{V}^i_k(X)$ which contain the origin are algebraic subtori of* $\operatorname{Char}(X)^0$*, of the form* $\exp(L)$*, where $L$ runs through the linear subspaces comprising $\mathcal{R}^i_k(X)$.*

## 13. Algebraic models for smooth quasi-projective varieties

### 13.1. Compactifications and formality

A complex projective variety is a subset of a complex projective space $\mathbb{CP}^n$, defined as the zero-locus of a homogeneous prime ideal in $\mathbb{C}[z_0, \ldots, z_n]$. A Zariski open subvariety of a projective variety is called a quasi-projective variety. We will only consider here projective and quasi-projective varieties which are connected and smooth.

If $M$ is a smooth, projective variety – or, more generally, a compact Kähler manifold – then the Hodge decomposition on the cohomology ring $H^*(M, \mathbb{C})$ imposes stringent constraints on the topological properties of $M$. For instance, in the famous paper of Deligne, Griffiths, Morgan, and Sullivan [37] it is shown that every such manifold is formal.

Each smooth, quasi-projective variety $X$ admits a good compactification. That is to say, there is a smooth, complex projective variety $\overline{X}$ and a normal crossings divisor $D$ such that $X = \overline{X} \setminus D$. By a well-known theorem of Deligne, each cohomology group of $X$ admits a mixed Hodge structure. This additional structure puts definite constraints on the algebraic topology of such manifolds.

For instance, if $X$ admits a smooth compactification $\overline{X}$ with $b_1(\overline{X}) = 0$, the weight 1 filtration on $H^1(X, \mathbb{C})$ vanishes; in turn, by work of Morgan [100], this implies the 1-formality of $X$. Thus, as noted by Kohno in [77], if $X$ is the complement of a hypersurface in $\mathbb{CP}^n$, then $\pi_1(X)$ is 1-formal.

In general, though, smooth quasi-projective varieties need not be 1-formal. Moreover, even when they are 1-formal, they still can be non-formal.

**Example 13.1.** Let $E^{\times n}$ be the product of $n$ copies of an elliptic curve $E$. The closed form $\frac{1}{2}\sqrt{-1}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ defines an integral cohomology class $\omega \in H^{1,1}(E^{\times n}, \mathbb{Z})$. By the Lefschetz theorem on $(1, 1)$-classes, $\omega$ can be realized as the first Chern class of an algebraic line bundle over $E^{\times n}$. Let $X_n$ be the complement of the zero-section of this bundle. Then $X_n$ is a smooth, quasi-projective variety which is not formal. In fact, $X_n$ deform-retracts onto the $(2n + 1)$-dimensional Heisenberg nilmanifold $\mathcal{H}_n$ from Example 14.19, and so $X_n$ is $(n - 1)$-formal but not $n$-formal.

## 13.2. Algebraic models

As before, let $X$ be a connected, smooth, complex quasi-projective variety, and choose a smooth compactification $\overline{X}$ such that the complement is a finite union, $D = \bigcup_{j\in J} D_j$, of smooth divisors with normal crossings. There is then a rationally defined CDGA, $A = A(\overline{X}, D)$, called the *Gysin model* (or, the *Morgan model*) of the compactification, constructed as follows. As a $\mathbb{C}$-vector space, $A^i$ is the direct sum of all subspaces

$$A^{p,q} = \bigoplus_{|S|=q} H^p\Big(\bigcap_{k\in S} D_k, \mathbb{C}\Big)(-q)$$

with $p + q = i$, where $(-q)$ denotes the Tate twist. Furthermore, the multiplication in $A$ is induced by the cup-product in $\overline{X}$, and has the property that $A^{p,q} \cdot A^{p',q'} \subseteq A^{p+p',q+q'}$, while the differential, $d\colon A^{p,q} \to A^{p+2,q-1}$, is constructed from the Gysin maps arising from intersections of divisors. The CDGA just constructed depends on the compactification $\overline{X}$; for simplicity, though, we will denote it by $A(X)$ when the compactification is understood.

An important particular case is when our variety $X$ has dimension 1. That is to say, let $\Sigma$ be a connected, possibly non-compact, smooth algebraic curve. Then $\Sigma$ admits a canonical compactification, $\overline{\Sigma}$, and thus, a canonical Gysin model, $A(\Sigma)$. We illustrate the construction of this model in a simple situation, using the very explicit description given by Bibby in [17] for complements of elliptic arrangements.

**Example 13.2.** Let $\Sigma = E^*$ be a once-punctured elliptic curve. Then $\overline{\Sigma} = E$, and the Gysin model $A(\Sigma)$ is the algebra $A = \bigwedge(a, b, e)/(ae, be)$ on generators $a, b$ in bidegree $(1, 0)$ and generator $e$ in bidegree $(0, 1)$, with differential $d\colon A \to A$ given by $da = db = 0$ and $de = ab$.

The above construction is functorial, in the following sense: If $f\colon X \to Y$ is a morphism of quasi-projective manifolds which extends to a regular map $\overline{f}\colon \overline{X} \to \overline{Y}$

between the respective good compactifications, then there is an induced CDGA morphism $f^!\colon A(Y) \to A(X)$ which respects the bigradings.

Morgan showed in [100] that the Sullivan model $A_{\text{PL}}(X)$ is connected to the Gysin model $A(X)$ by a chain of quasi-isomorphisms preserving $\mathbb{Q}$-structures. Moreover, setting the weight of $A^{p,q}$ equal to $p + 2q$ defines a positive-weight decomposition on $(A^*, d)$.

In [45], Dupont constructed a Gysin-type model for certain types of quasi-projective varieties, where the normal crossings divisors hypothesis on the compactification can be relaxed. More precisely, let $\mathcal{A}$ be an arrangement of smooth hypersurfaces in a smooth, $n$-dimensional complex projective variety $\overline{X}$, and suppose $\mathcal{A}$ locally looks like an arrangement of hyperplanes in $\mathbb{C}^n$. There is then a CDGA model for the complement, $X = \overline{X} \setminus \bigcup_{L \in \mathcal{A}} L$, which builds on the combinatorial definition of the Orlik–Solomon algebra of a hyperplane arrangement.

Finally, let $\mathcal{A}$ be an arrangement of complex linear subspaces in $\mathbb{C}^n$. Using a blow-up construction, De Concini and Procesi gave in [36] a "wonderful" CDGA model for the complement of such an arrangement. Based on a simplification of this model due to Yuzvinsky [147], Feichtner and Yuzvinsky showed in [49] the following: If the intersection poset of $\mathcal{A}$ is a geometric lattice, then the complement of $\mathcal{A}$ is a formal space. In general, though, the complement of a complex subspace arrangement need not be formal. For instance, the polyhedral product constructions of [9,38,64] mentioned in Section 10.8 yield coordinate subspace arrangements whose complements admit non-trivial Massey products over the rationals.

### 13.3. Characteristic varieties

The structure of the jump loci for cohomology in rank 1 local systems on smooth, complex projective and quasi-projective varieties (and, more generally, on Kähler and quasi-Kähler manifolds) was determined through the work of Beauville [12], Green and Lazarsfeld [65], Simpson [123], and Arapura [3]. The definitive structural result in the quasi-projective setting was obtained by Budur and Wang in [24], building on the work of Dimca and Papadima [40].

**Theorem 13.3** ([24]). *Let $X$ be a smooth quasi-projective variety. Then each characteristic variety $\mathcal{V}^i_k(X)$ is a finite union of torsion-translated subtori of* $\text{Char}(X)$.

Work of Arapura [3] explains how the non-translated subtori occurring in the above decomposition of $\mathcal{V}^1_1(X)$ arise. Let us say that a holomorphic map $f\colon X \to \Sigma$ is *admissible* if $f$ is surjective, has connected generic fiber, and the target $\Sigma$ is a connected, smooth complex curve with negative Euler characteristic. Up to reparametrization at the target, the variety $X$ admits only finitely many admissible maps; let $\mathcal{E}_X$ be the set of equivalence classes of such maps.

If $f\colon X \to \Sigma$ is an admissible map, it is readily verified that $\mathcal{V}^1_1(\Sigma) = \text{Char}(\Sigma)$. Thus, the image of the induced morphism between character groups, $f^*\colon \text{Char}(\Sigma) \to \text{Char}(X)$, is an algebraic subtorus of $\text{Char}(X)$.

**Theorem 13.4** ([3]). *The correspondence $f \mapsto f^*(\mathrm{Char}(\Sigma))$ defines a bijection between the set $\mathcal{E}_X$ of equivalence classes of admissible maps from $X$ to curves and the set of positive-dimensional, irreducible components of $\mathcal{V}^1(X)$ containing* **1**.

The positive-dimensional, irreducible components of $\mathcal{V}^1_1(X)$ which do not pass through **1** can be similarly described, by replacing the admissible maps with certain "orbifold fibrations," whereby multiple fibers are allowed.

### 13.4. Resonance varieties

We now turn to the resonance varieties associated with a quasi-projective manifold, and how they relate to the characteristic varieties. The tangent cone theorem takes a very special form in this setting.

**Theorem 13.5.** *Let $X$ be a smooth, quasi-projective variety, and let $A(X)$ be a Gysin model for $X$. Then, for each $i \geq 0$ and $k \geq 0$,*

$$\tau_{\mathbf{1}}\big(\mathcal{V}^i_k(X)\big) = \mathrm{TC}_{\mathbf{1}}\big(\mathcal{V}^i_k(X)\big) = \mathcal{R}^i_k(A(X)) \subseteq \mathcal{R}^i_k(X). \tag{13.1}$$

*Moreover, if $X$ is $q$-formal, the last inclusion is an equality, for all $i \leq q$.*

In particular, the resonance varieties $\mathcal{R}^i_k(A(X))$ are finite unions of rationally defined linear subspaces of $H^1(X,\mathbb{C})$. On the other hand, the varieties $\mathcal{R}^i_k(X)$ can be much more complicated; for instance, they may have non-linear irreducible components. If $X$ is $q$-formal, though, Theorem 13.1 guarantees this cannot happen, as long as $i \leq q$.

### 13.5. Resonance in degree 1

Once again, let $X$ be a smooth, quasi-projective variety, and let $A(X)$ be the Gysin model associated with a good compactification $\overline{X}$. The degree 1 resonance varieties $\mathcal{R}^1_1(A(X))$, and, to some extent, $\mathcal{R}^1_1(X)$, admit a much more precise description than those in higher degrees.

As in the setup from Theorem 13.4, let $\mathcal{E}_X$ be the set of equivalence classes of admissible maps from $X$ to curves, and let $f\colon X \to \Sigma$ be such map. Recall from Section 13.2 that the curve $\Sigma$ admits a canonical Gysin model, $A(\Sigma)$. As noted in [40], the induced CDGA morphism, $f^!\colon A(\Sigma) \to A(X)$, is injective. Let $f^*\colon H^1(A(\Sigma)) \to H^1(A(X))$ be the induced homomorphism in cohomology.

**Theorem 13.6** ([40, 92]). *For a smooth, quasi-projective variety $X$, the decomposition of $\mathcal{R}^1_1(A(X))$ into (linear) irreducible components is given by*

$$\mathcal{R}^1_1(A(X)) = \bigcup_{f \in \mathcal{E}_X} f^*(H^1(A(\Sigma))). \tag{13.2}$$

If $X$ admits no admissible maps, that is, if $\mathcal{E}_X = \emptyset$, formula (13.2) should be understood to mean $\mathcal{R}^1_1(A(X)) = \{\mathbf{0}\}$ if $b_1(X) > 0$ and $\mathcal{R}^1_1(A(X)) = \emptyset$ if $b_1(X) = 0$.

**Example 13.7.** Let $X = X_1$ be the complex, smooth quasi-projective surface constructed in Example 13.1. Clearly, this manifold is a $\mathbb{C}^*$-bundle over $E = S^1 \times S^1$ which deform-retracts onto the 3-dimensional Heisenberg nilmanifold $M = G_{\mathbb{R}}/G_{\mathbb{Z}}$ from Example 10.14. Hence, $\mathcal{V}^1_1(X) = \{\mathbf{1}\}$, and so $\tau_{\mathbf{1}}(\mathcal{V}^1_1(X)) = \mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(X)) = \{\mathbf{0}\}$. On the other hand, $\mathcal{R}^1_1(X) = \mathbb{C}^2$, and so $X$ is not 1-formal.

Under a 1-formality assumption, the usual resonance varieties $\mathcal{R}^1_1(X)$ admit a similar description.

**Theorem 13.8** ([42]). *Let $X$ be a smooth, quasi-projective variety, and suppose $X$ is 1-formal. Then the decomposition into irreducible components of the first resonance variety is given by*

$$\mathcal{R}^1_1(X) = \bigcup_{f \in \mathcal{E}_X} f^*(H^1(\Sigma, \mathbb{C})),$$

*with the same convention as before when $\mathcal{E}_X = \emptyset$. Moreover, all the (rationally defined) linear subspaces in this decomposition have dimension at least 2, and any two distinct ones intersect only at $\mathbf{0}$.*

If $X$ is compact, then the formality assumption in the above theorem is automatically satisfied, due to [37]. Furthermore, the conclusion of the theorem can also be sharpened in this case: each (non-trivial) irreducible component of $\mathcal{R}^1_1(X)$ is even-dimensional, of dimension at least 4. In general, though, the resonance varieties of a quasi-projective manifold can have non-linear components.

**Example 13.9** ([42]). Let $X = \mathrm{Conf}(E, n)$ be the configuration space of $n$ points on an elliptic curve $E$. Letting $\{a, b\}$ be the standard basis of $H^1(E, \mathbb{C}) = \mathbb{C}^2$, we may identify $H^*(E^{\times n}, \mathbb{C})$ with $\bigwedge(a_1, b_1, \dots, a_n, b_n)$ and find a presentation for $H^{\le 2}(X, \mathbb{C})$ from Totaro's spectral sequence [142]. A computation then gives

$$\mathcal{R}^1_1(\mathrm{Conf}(E, n)) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \le i < j < n \end{array} \right\}.$$

If $n \ge 3$, this variety is irreducible and non-linear (in fact, it is a rational normal scroll), from which we conclude that the configuration space $\mathrm{Conf}(E, n)$ is not 1-formal.

### 13.6. Large quasi-projective groups

Recall that a quasi-projective variety is a Zariski open subset of a projective variety. We will say that a space $X$ is a *quasi-projective manifold* if it is a connected, smooth, complex quasi-projective variety. Every such manifold has the homotopy type of a finite CW-complex.

A group $G$ is said to be *quasi-projective* if it can be realized as the fundamental group of a quasi-projective manifold. Clearly, every such a group admits a finite presentation. We now turn to the question of deciding whether a quasi-projective group is large. It turns out that a complete answer to this question can be given in terms of "admissible" maps to curves.

A map $f: X \to C$ from a quasi-projective manifold $X$ to a smooth complex curve $C$ is said to be *admissible* if it is regular, surjective, and has connected generic fiber. It is easy to see that the homomorphism on fundamental groups induced by such a map, $f_\sharp: \pi_1(X) \to \pi_1(C)$, is surjective. We denote by $\mathcal{E}(X)$ the family of admissible maps to curves with negative Euler characteristic, modulo automorphisms of the target.

Deep work of Arapura [3] characterizes those positive-dimensional, irreducible components of the characteristic variety $\mathcal{V}^1_1(X)$ which contain the origin of the character group $\mathrm{Char}(X)$: all such components are connected, affine subtori, which arise by pullback of the character torus $\mathrm{Char}(C)$ along the homomorphism $f_\sharp: \pi_1(X) \to \pi_1(C)$ induced by some map $f \in \mathcal{E}(X)$.

Suppose now that $C$ is a smooth complex curve with $\chi(C) < 0$. It is readily seen that the fundamental group $G = \pi_1(C)$ surjects onto a free, non-abelian group, and so $G$ is very large. More generally, we have the following characterization of large, quasi-projective groups.

**Proposition 13.10** ([113]). *Let $X$ be a smooth quasi-projective variety. Then:*

(1) *$\pi_1(X)$ is large if and only if there is a finite cover $Y \to X$ such that $\mathcal{E}(Y) \neq \emptyset$.*

(2) *$\pi_1(X)$ is very large if and only if $\mathcal{E}(X) \neq \emptyset$.*

Consequently, if $b_1(X) > 0$, then $\mathcal{E}(X) \neq \emptyset$ if and only if the analytic germ at $\mathbf{1}$ of $\mathcal{V}^1_1(X)$ is not equal to $\{\mathbf{1}\}$.

### 13.7. Resonance and largeness

To conclude this section, we rephrase the last condition in terms of resonance varieties. As shown by Morgan [100], every quasi-projective manifold $X$ admits a finite-dimensional model $A(\overline{X}, D)$; such a "Gysin" model depends on a smooth compactification $\overline{X}$ for which the complement $D = \overline{X} \setminus X$ is a normal crossings divisor. Let $A$ be a Gysin model for $X$, or any one of the more general Orlik–Solomon models constructed by Dupont in [46]. In either case, let us note that all resonance varieties of $A$ have *positive weights*, i.e., they are invariant with respect to a $\mathbb{C}^*$-action on $H^1(A)$ with positive weights.

**Proposition 13.11** ([113]). *Let $X$ be a smooth, quasi-projective variety with $b_1(X) > 0$ and let $A$ be an Orlik–Solomon model for $X$. Then $\pi_1(X)$ is very large if and only if $\mathcal{R}^1_1(A) \neq \{\mathbf{0}\}$.*

**Example 13.12.** Let $\Sigma_g$ be a compact, connected Riemann surface of genus $g$, and let $X = F_\Gamma(\Sigma_g)$ be the partial configuration space associated to a finite simple graph $\Gamma$. More concretely, if $n$ is the number of vertices of $\Gamma$, then $F_\Gamma(\Sigma_g)$ is the complement in $\Sigma_g^n$ of the union of the diagonals $z_i = z_j$, indexed by the edges of $\Gamma$. No convenient presentation is available for the fundamental group $G_{\Gamma,g} := \pi_1(F_\Gamma(\Sigma_g))$. On the other hand, the Orlik–Solomon model $A$ for $F_\Gamma(\Sigma_g)$ is much more approachable. Computing the resonance variety $\mathcal{R}^1_1(A)$ leads to a complete, explicit description of $\mathcal{E}(F_\Gamma(\Sigma_g))$. Such a description is given in [13], for all $g \geq 0$ and for all finite graphs $\Gamma$, generalizing a result from [18], valid only for chordal graphs. In particular, $\mathcal{E}(F_\Gamma(\Sigma_g)) = \emptyset$, that is, $G_{\Gamma,g}$ is not very large, if and only if either $g = 1$ and $\Gamma$ has no edges, or $g = 0$ and $\Gamma$ contains no complete subgraph on 4 vertices.

## 14. Algebraic models for Lie group actions

### 14.1. Almost free actions and Hirsch extensions

Let $K$ be a compact, connected, real Lie group. Consider the universal principal $K$-bundle, $K \to EK \to BK$, with contractible total space $EK$ and with base space the classifying space $BK = EK/K$. By a classical result of Hopf, the cohomology ring of $K$ (with coefficients in a field $\Bbbk$ of characteristic 0) is isomorphic to the cohomology ring of a finite product of odd-dimensional spheres. That is, $H^*(K, \Bbbk) \cong \bigwedge P^*$, where $P^*$ is an oddly-graded, finite-dimensional vector space, with homogeneous basis $\{t_\alpha \in P^{m_\alpha}\}$, for some odd integers $m_1, \dots, m_r$, where $r = \operatorname{rank}(K)$.

Now let $M$ be a compact, connected, differentiable manifold on which the compact, connected Lie group $K$ acts smoothly. Both $M$ and the orbit space $N = M/K$ have the homotopy type of finite $CW$-complexes. We consider the diagonal action of $K$ on the product $EK \times M$, and form the Borel construction, $M_K = (EK \times M)/K$. Let $\operatorname{pr}\colon M_K \to N$ be the map induced by the projection $\operatorname{pr}_2\colon EK \times M \to M$.

The $K$-action on $M$ is said to be *almost free* if all its isotropy groups are finite. When this assumption is met, the work of Allday and Halperin [1] provides a very useful Hirsch extension model for the manifold $M$.

**Theorem 14.1** ([1]). *Suppose $M$ admits an almost free $K$-action, with orbit space $N = M/K$. There is then a map $\sigma\colon P^* \to Z^{*+1}(A_{\mathrm{PL}}(N))$ such that $\operatorname{pr}^* \circ [\sigma]$ is the transgression in the principal bundle $K \to EK \times M \to M_K$, and*

$$A_{\mathrm{PL}}(M) \simeq A_{\mathrm{PL}}(N) \otimes_\sigma \bigwedge P.$$

This theorem may be applied, for instance, to the total space $M$ of a principal $K$-bundle over a compact manifold $N = M/K$. The next result identifies an interesting class of finite-dimensional CW-spaces that have finite CDGA models.

**Proposition 14.2** ([114]). *Let $M$ be an almost free $K$-manifold. Write $H^*(K, \Bbbk) = \bigwedge P$, for some graded $\Bbbk$-vector space $P$, and let $m$ be the maximum degree of $P^*$.*

(1) *Suppose $B$ is a $q$-finite $q$-model of the orbit space $N = M/K$, with $q \geq m + 1$. Then a suitable Hirsch extension $A = B \otimes_\tau \bigwedge P$ is a $q$-finite $q$-model for $M$.*
(2) *Suppose $N = M/K$ is $q$-formal. Then we may take $B^* = (H^*(N, \Bbbk), 0)$, and $A = B \otimes_\tau \bigwedge P$ is a $q$-finite $q$-model of $M$ with positive weights.*

Restricting to principal $K$-bundles, we can say more. As before, identify $H^*(K, \mathbb{Q})$ with $\bigwedge P = \bigwedge(t_1, \dots, t_r)$.

**Theorem 14.3** ([114]). *Let $N$ be a connected, finite CW-complex and let $K$ be a compact, connected, real Lie group. If $N$ has a finite-dimensional rational model $B$, then any Hirsch extension $A = B \otimes_\tau \bigwedge P$ can be realized as a finite-dimensional rational model of some principal $K$-bundle $M$ over $N$. When $B$ has positive weights and the image of $[\tau]$ is generated by weighted-homogeneous elements, $A$ also has positive weights.*

## 14.2. Graded regularity and partial formality

Fix an integer $q \geq 0$. Let $H^*$ be a connected commutative graded algebra over a field $\Bbbk$ of characteristic 0. Following [114], we say that a homogeneous element $e \in H^k$ is a non-zero divisor up to degree $q$ if the multiplication map $e\cdot\colon H^i \to H^{i+k}$ is injective, for all $i \leq q$. (For $q = 0$, this simply means that $e \neq 0$.)

Likewise, we say that a sequence $e_1, \dots, e_r$ of homogeneous elements in $H^+$ is *$q$-regular* if the class of each $e_\alpha$ is a non-zero divisor up to degree $q - \deg(e_\alpha) + 2$ in the quotient ring $H/\sum_{\beta<\alpha} e_\beta H$. (This implies in particular that the elements $e_1, \dots, e_r$ are linearly independent over $\Bbbk$, when $q \geq \deg(e_\alpha) - 2$ for all $\alpha$.)

**Theorem 14.4** ([114]). *Suppose $e_1, \dots, e_r$ is an even-degree, $q$-regular sequence in $H^*$. Then the Hirsch extension $A = (H \otimes_\tau \bigwedge(t_1, \dots, t_r), d)$ with $d = 0$ on $H$ and $dt_\alpha = \tau(t_\alpha) = e_\alpha$ has the same $q$-type as $(H/\sum_\alpha e_\alpha H, 0)$. In particular, $A$ is $q$-formal.*

Classical results of Borel and Chevalley provide the machinery for constructing graded algebras which satisfy the hypothesis of Theorem 14.4, in the case when $q = \infty$. Let $H^*(BK, \Bbbk)$ be the cohomology algebra of the classifying space of a compact, connected Lie group $K$. Let $T$ be a maximal torus in $K$, and let $W = NT/T$ be the Weyl group. The classifying space $BT$ is the product of $r$ copies of $\mathbb{CP}^\infty$, where $r$ is the rank of $K$. Its cohomology algebra is $H^*(BT, \Bbbk) = \Bbbk[x_1, \dots, x_r]$, with degree 2 free algebra generators, on which $W$ acts by graded algebra automorphisms.

The natural map $\kappa\colon BT \to BK$ identifies the cohomology algebra $H^*(BK, \Bbbk)$ with the invariant subalgebra of the $W$-action. More precisely, $H^*(BK, \Bbbk)$ is isomorphic to a polynomial ring of the form $\Bbbk[f_1, \dots, f_r]$, where each $f_\alpha$ is a $W$-invariant polynomial of even degree $m_\alpha + 1$, with $m_\alpha$ as in Section 14.1. Moreover, $f_1, \dots, f_r$ forms a regular sequence in $\Bbbk[x_1, \dots, x_r]$.

Let $U \subseteq K$ be a closed, connected subgroup of a compact, connected Lie group. As shown in [140], the Sullivan minimal model of the homogeneous space $K/U$ is a Hirsch extension of the form $A = H \otimes_\tau \bigwedge(t_1, \dots, t_s)$, where $H^*$ is a free graded algebra on finitely many even-degree generators, with zero differential, as in Theorem 14.4. As is well known, not all homogeneous spaces $K/U$ are formal. Nevertheless, the criterion from Theorem 14.4 may be used to gain information on their partial formality properties.

**Example 14.5.** For the homogeneous space $\mathrm{Sp}(5)/\mathrm{SU}(5)$, the aforementioned algebra $H^*$ has two free generators, $x_6$ and $x_{10}$, where subscripts denote degrees, and the sequence from Theorem 14.4 is $\{x_6^2, x_{10}^2, x_6x_{10}\}$, see [54]. It follows that $\mathrm{Sp}(5)/\mathrm{SU}(5)$ is 19-formal. On the other hand, a computation with Massey triple products shows that this estimate is sharp, that is, $\mathrm{Sp}(5)/\mathrm{SU}(5)$ is not 20-formal.

### 14.3. Partial formality of $K$-manifolds

Let $M$ be an almost free $K$-manifold. We write $H^*(K,\Bbbk) = \bigwedge(t_1,\dots,t_r)$, and denote the transgression of $t_\alpha$ by $e_\alpha \in H^{m_\alpha+1}(M/K,\Bbbk)$. As before, set $m = \max\{m_\alpha\}$.

**Theorem 14.6** ([114]). *Suppose the $K$-action on $M$ is almost free, the orbit space $N = M/K$ is $k$-formal, for some $k \geq m+1$, and $e_1,\dots,e_r$ form a $q$-regular sequence in $H^*(N,\Bbbk)$, for some $q \leq k$. Then the quotient algebra $H^*(N,\Bbbk)/\sum_{\alpha=1}^{r} e_\alpha H^*(N,\Bbbk)$, equipped with the zero differential, is a finite-dimensional $q$-model for $M$; in particular, $M$ is $q$-formal.*

As illustrated in the next two examples, the $q$-regularity assumption from Theorem 14.6 is optimal with respect to the $q$-formality conclusion for the manifold $M$, at least in the case when $K = S^1$ or $S^3$.

**Example 14.7.** Let $M = \mathscr{H}_1$ be the 3-dimensional Heisenberg nilmanifold from Example 10.14. This manifold is the total space of the principal $S^1$-bundle over the formal manifold $N = S^1 \times S^1$, with Euler class $e \in H^2(N,\mathbb{Z})$ equal to the orientation class. In this case, the sequence $\{e\}$ is 0-regular, but not 1-regular in $H^*(N,\Bbbk)$. In fact, as mentioned previously, $M$ is not 1-formal. As explained in Example 14.19, this is the first manifold in a series, $\mathscr{H}_n$, where $(n-1)$-regularity implies $(n-1)$-formality in an optimal way.

**Example 14.8.** Let $M$ to be the total space of the principal $S^3$-bundle over $N = S^2 \times S^2$ obtained by pulling back the Hopf bundle $S^7 \to S^4$ along a degree-one map $N \to S^4$. As above, $N$ is formal, and the Euler class $e \in H^4(N,\mathbb{Z})$ is the orientation class. In this case, $\{e\}$ is 3-regular, but not 4-regular in $H^*(N,\Bbbk)$, and Theorem 14.6 says that $M$ is 3-formal. Direct computation with the minimal model of $M$ shows that, in fact, $M$ is not 4-formal.

### 14.4. Malcev completion and representation varieties

Let $H$ be a 2-finite CDGA with zero differential, and let $A = H \otimes_\tau \bigwedge P$ be a Hirsch extension, where $P$ is an oddly-graded, finite-dimensional vector space.

**Theorem 14.9** ([114]). *The holonomy Lie algebra $\mathfrak{h}(A)$ admits a finite presentation with generators in degree 1 and relations in degrees 2 and 3.*

**Corollary 14.10** ([114]). *Suppose $M$ supports an almost free $K$-action with 2-formal orbit space. Then:*

(1) *The group* $\pi = \pi_1(M)$ *is filtered-formal. More precisely, the Malcev Lie algebra* $\mathfrak{m}(\pi)$ *is isomorphic to the LCS completion of* $\mathrm{Lie}(H_1(\pi, \Bbbk))/\mathfrak{r}$, *where* $\mathfrak{r}$ *is a homogeneous ideal generated in degrees* 2 *and* 3.

(2) *For every complex linear algebraic group* $G$, *the germ at the origin of the representation variety* $\mathrm{Hom}_{\mathrm{gr}}(\pi, G)$ *is defined by quadrics and cubics only.*

The second statement in the above corollary is analogous to the quadraticity obstruction for fundamental groups of compact Kähler manifolds obtained by Goldman–Millson in [63]. Note that the corollary applies to principal $K$-bundles over formal manifolds.

### 14.5. Orbifold fundamental groups

Assume now that $M$ is an almost free $K$-manifold. By [23, Theorem 4.3.18], the projection $p: M \to M/K$ induces a natural epimorphism $f: \pi_1(M) \twoheadrightarrow \pi_1^{\mathrm{orb}}(M/K)$ between orbifold fundamental groups.

**Theorem 14.11** ([114]). *Suppose that the* $K$*-action on* $M$ *is almost free and the transgression* $P^* \to H^{*+1}(M_K, \Bbbk) \cong H^{*+1}(M/K, \Bbbk)$ *is injective in degree* 1. *Then the following hold.*

(1) *If the orbit space* $N = M/K$ *has a* 2*-finite* 2*-model over* $\Bbbk \subseteq \mathbb{C}$, *then the homomorphism* $f: \pi_1(M) \twoheadrightarrow \pi_1^{\mathrm{orb}}(N)$ *induces an analytic isomorphism between the germs at* 1 *of* $\mathcal{V}_k^1(\pi_1^{\mathrm{orb}}(N))$ *and* $\mathcal{V}_k^1(\pi_1(M))$, *for all* $k$.

(2) *If* $N$ *is* 2*-formal, then* $f$ *induces an analytic isomorphism between the germs at* 1 *of* $\mathrm{Hom}(\pi_1^{\mathrm{orb}}(N), \mathrm{SL}_2(\mathbb{C}))$ *and* $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))$.

**Example 14.12.** Let $K$ be a compact, connected Lie group, and identify $H^*(K, \mathbb{Q})$ with $\bigwedge P_K^*$. Let $N$ be a compact, formal manifold, and assume $b_2(N) \geq s$, where $s = \dim P_K^1$ (for instance, take $N$ to be the product of at least $s$ compact Kähler manifolds). There is then a degree-preserving linear map, $\tau: P_K^* \to H^{*+1}(N, \mathbb{Q})$, which is injective in degree 1. By Theorem 14.3, such a map can be realized as the transgression in a principal $K$-bundle, $M_\tau \to N$, and the manifold $M_\tau$ satisfies the assumptions from Theorem 14.11.

Theorem 14.11 may also be applied to a Seifert fibered 3-manifold with non-zero Euler class, $p: M \to M/S^1 = \Sigma_g$. In this case, $\mathcal{V}_k^1(M)_{(\mathbf{1})}$ is isomorphic to $\mathcal{V}_k^1(\Sigma_g)_{(\mathbf{1})}$, for all $k$, while $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))_{(\mathbf{1})} \cong \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{C}))_{(\mathbf{1})}$.

### 14.6. Sasakian geometry

The machinery outlined above has some noteworthy consequences for the topology of compact Sasakian manifolds, which are related to formality properties, representation varieties and cohomology jump loci. A comprehensive reference for Sasakian geometry is the book of Boyer and Galicki [23].

Let $M^{2n+1}$ be a compact Sasakian manifold of dimension $2n + 1$. Without loss of essential generality, we may assume that the Sasakian structure is quasi-regular. A basic structural result in Sasakian geometry guarantees that, in this case, $M$ supports an almost free circle action. Furthermore, the quotient space, $N = M/S^1$, is a compact Kähler orbifold, with Kähler class $h \in H^2(N, \Bbbk)$ satisfying the Hard Lefschetz property, that is, multiplication by $h^k$ defines an isomorphism

$$H^{n-k}(N, \Bbbk) \xrightarrow{\cong} H^{n+k}(N, \Bbbk)$$

for each $1 \leq k \leq n$; see [23, Proposition 7.2.2 and Theorem 7.2.9]. The thesis of Tievsky [141, Section 4.3] provides a very useful model for a Sasakian manifold.

**Theorem 14.13** ([141]). *Every compact Sasakian manifold $M$ admits as a finite model over $\mathbb{R}$ the Hirsch extension $A^*(M) = (H^*(N, \mathbb{R}) \otimes_h \bigwedge(t), d)$, where $d$ is zero on $H^*(N, \mathbb{R})$ and $dt = h$, the Kähler class of $N$.*

Sasakian geometry is an odd-dimensional analog of Kähler geometry. From this point of view, the above theorem is a rough analog of the main result on the algebraic topology of compact Kähler manifolds from [37], guaranteeing that such manifolds are formal. Theorem 14.13 only says that $M$ behaves like an almost free compact $S^1$-manifold with formal orbit space. A result from [11] establishes the formality of the orbifold de Rham algebra of a compact Kähler orbifold. Unfortunately, this is not enough for applying Theorem 14.6, since the authors of [11] do not prove that the orbifold de Rham algebra is weakly equivalent to the Sullivan de Rham algebra.

By construction, the Tievsky model $A^*(M)$ is a real CDGA defined over $\mathbb{Q}$. Nevertheless, in view of Remark 8.1, it does *not* follow from [141] that $A^*(M)$ is a model for $M$ over $\mathbb{Q}$.

However, we can say something very useful regarding rational models for Sasakian manifolds. We start with a lemma and will come back to this point in Theorem 14.18.

**Lemma 14.14** ([114]). *The Tievsky model $A^*_{\mathbb{R}}(M) = (H^*(N, \mathbb{R}) \otimes_h \bigwedge(t), d)$ is a finite model with positive weights for $M$.*

**Corollary 14.15** ([114]). *Let $M$ be a compact Sasakian manifold. For each $i, k \geq 0$, all irreducible components of the characteristic variety $\mathcal{V}^i_k(M)$ passing through $1$ are algebraic subtori of the character group $H^1(M, \mathbb{C}^*)$.*

A well-known, direct relationship between Kähler and Sasakian geometry is as follows. Let $N$ be a compact Kähler manifold such that the Kähler class is integral, i.e., $h \in H^2(N, \mathbb{Z})$, and let $M$ be the total space of the principal $S^1$-bundle classified by $h$. Then $M$ is a regular Sasakian manifold. A concrete class of examples is provided by the Heisenberg manifolds $\mathcal{H}_n$ from Example 14.19 below.

### 14.7. Partial formality of Sasakian manifolds

Let $M^{2n+1}$ be a compact Sasakian manifold, with fundamental group $G = \pi_1(M)$. One may ask: Is the group $\pi$ (or, equivalently, the manifold $M$) 1-formal? When $n = 1$, the answer is clearly negative, a simple example being provided by the Heisenberg manifold $\mathcal{H}_1$. In [76, Theorem 1.1], Kasuya claims that the case $n = 1$ is exceptional, in the following sense.

**Claim 14.16.** *Every compact Sasakian manifold of dimension $2n + 1$ is 1-formal over $\mathbb{R}$, provided $n > 1$.*

As pointed out in [114], the proof from [76] has a gap, which we briefly explain. Given a CDGA $A$, the decomposable part of $H^2(A)$ is the linear subspace $DH^2(A)$ defined as the image of the product map in cohomology, $H^1(A) \wedge H^1(A) \to H^2(A)$. What Kasuya actually shows is that

$$DH^2(\mathcal{M}_1(M)) = H^2(\mathcal{M}_1(M)), \tag{14.1}$$

for a compact Sasakian manifold $M^{2n+1}$ with $n > 1$, where $\mathcal{M}_1(M)$ is the 1-minimal model of $M$ over $\mathbb{R}$. Equality (14.1) is an easy consequence of 1-formality. Kasuya deduces the 1-formality of $M$ from (14.1), by invoking as a crucial tool [2, Lemma 3.17]. Unfortunately, though, this lemma is false, as shown by Măcinic in [91]. Nevertheless, the next theorem proves Claim 14.16 in a stronger form, while also recovering equality (14.1).

**Theorem 14.17** ([114]). *Every compact Sasakian manifold $M$ of dimension $2n + 1$ is $(n - 1)$-formal, over an arbitrary field $\Bbbk$ of characteristic 0.*

The next result makes Theorem 14.17 more precise, by constructing an explicit finite, $(n - 1)$-model with zero differential for $M$ over any field of characteristic 0.

**Theorem 14.18** ([114]). *Let $M$ be a compact Sasakian manifold $M$ of dimension $2n + 1$. The Sullivan model of $M$ over a field $\Bbbk$ of characteristic 0 has the same $(n - 1)$-type over $\Bbbk$ as the CDGA $(H^*(N, \Bbbk)/h \cdot H^*(N, \Bbbk), 0)$, where $N = M/S^1$ and $h \in H^2(N, \Bbbk)$ is the Kähler class.*

As illustrated by the next example, the conclusion of Theorem 14.17 is optimal.

**Example 14.19.** Let $E = S^1 \times S^1$ be an elliptic complex curve, and let $N = E^{\times n}$ be the product of $n$ such curves, with Kähler form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. The corresponding Sasakian manifold is the $(2n + 1)$-dimensional Heisenberg nilmanifold $\mathcal{H}_n$. Theorem 14.17 guarantees that $\mathcal{H}_n$ is $(n - 1)$-formal. As noted in [91], though (see also Example 3.6), the manifold $\mathcal{H}_n$ is *not* $n$-formal.

### 14.8. Sasakian groups

A group $\pi$ is said to be a *Sasakian group* if it can be realized as the fundamental group of a compact, Sasakian manifold. A major open problem in the field (see, e.g., [23, Chapter 7] or [32]) is: "Which finitely presented groups are Sasakian?"

A first, well-known obstruction is that the first Betti number $b_1(\pi)$ must be even, see, for instance, the references listed in [32]. Much more subtle obstructions are provided by the following result. Fix a field $\Bbbk$ of characteristic 0.

**Corollary 14.20** ([114]). *Let $\pi = \pi_1(M^{2n+1})$ be a Sasakian group. Then:*

(1) *The Malcev Lie algebra $\mathfrak{m}(\pi, \Bbbk)$ is the LCS completion of the quotient of the free Lie algebra $\mathrm{Lie}(H_1(\pi, \Bbbk))$ by an ideal generated in degrees 2 and 3. Moreover, this Lie algebra presentation can be explicitly described in terms of the graded ring $H^*(M/S^1, \Bbbk)$ and the Kähler class $h \in H^2(M/S^1, \Bbbk)$.*

(2) *The group $\pi$ is filtered-formal.*

(3) *For every complex linear algebraic group $G$, the germ at the origin of the representation variety $\mathrm{Hom}(\pi, G)$ is defined by quadrics and cubics only.*

As an application of Corollary 14.15, we obtain another (independent) obstruction to Sasakianity.

**Corollary 14.21** ([114]). *Let $\pi$ be a Sasakian group. For each $k \geq 0$, all irreducible components of the characteristic variety $\mathcal{V}^1_k(\pi)$ passing through $\mathbf{1}$ are algebraic subtori of the character group $\mathrm{Hom}(\pi, \mathbb{C}^*)$.*

By Theorem 14.13, the $\mathbb{R}$-homotopy type of a compact Sasakian manifold $M$ depends only on the cohomology ring $H^*(M/S^1, \mathbb{R})$ and the Kähler class $h \in H^2(M/S^1, \mathbb{Q})$. Surprisingly enough, it turns out that the germs at $\mathbf{1}$ of certain representation varieties and jump loci of $\pi_1(M)$ depend only on the graded cohomology ring of $M/S^1$.

**Corollary 14.22** ([114]). *Let $M$ be a compact Sasakian manifold, and let $G = \mathrm{SL}_2(\mathbb{C})$. Then the germ at $\mathbf{1}$ of $\mathrm{Hom}(\pi_1(M), G)$ depends only on the graded ring $H^*(M/S^1, \mathbb{C})$ and the Lie algebra of $G$, in an explicit way. Similarly, the germs at $\mathbf{1}$ of the characteristic varieties $\mathcal{V}^1_k(\pi_1(M))$ depend (explicitly) only on $H^*(M/S^1, \mathbb{C})$.*

## 15. Algebraic models for closed 3-manifolds

In this final section we give a partial characterization of the formality and finiteness properties for rational models of closed 3-manifolds.

### 15.1. The intersection form of a 3-manifold

Let $M$ be a compact, connected 3-manifold without boundary. For short, we shall refer to $M$ as being a *closed* 3-manifold. Throughout, we will also assume that $M$ is orientable.

Fix an orientation class $[M] \in H_3(M, \mathbb{Z}) \cong \mathbb{Z}$. With this choice, the cup product on $M$ determines an alternating 3-form $\mu_M$ on $H^1(M, \mathbb{Z})$, given by

$$\mu_M(a \wedge b \wedge c) = \langle a \cup b \cup c, [M] \rangle, \tag{15.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing. In turn, the cup-product map $\bigwedge^2 H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is determined by the intersection form $\mu_M$ via $\langle a \cup b, \gamma \rangle = \mu_M(a \wedge b \wedge c)$, where $c$ is the Poincaré dual of $\gamma \in H_2(M, \mathbb{Z})$.

In [136], Sullivan proved the following result.

**Theorem 15.1** ([136]). *For every finitely generated, torsion-free abelian group $H$ and every 3-form $\mu \in \bigwedge^3 H^\vee$, there is a closed, oriented 3-manifold $M$ with $H^1(M,\mathbb{Z}) = H$ and cup-product form $\mu_M = \mu$.*

Such a 3-manifold can be constructed by a process known as "Borromean surgery." More precisely, if $n = \operatorname{rank} H$, a manifold $M$ with the claimed properties may be defined as 0-framed surgery on a link in $S^3$ obtained from the trivial $n$-component link by replacing a collection of trivial 3-string braids by the corresponding collection of 3-string braids whose closures are the Borromean rings. For instance, 0-surgery on the Borromean rings produces the 3-torus $T^3$.

### 15.2. Poincaré duality and Koszul complex

We now fix a basis $\{e_1,\dots,e_n\}$ for the free abelian group $H^1(M,\mathbb{Z})$, and we choose $\{e_1^\vee,\dots,e_n^\vee\}$ as basis for the torsion-free part of $H^2(M,\mathbb{Z})$, where $e_i^\vee$ denotes the Kronecker dual of the Poincaré dual of $e_i$. Writing

$$\mu_M = \sum_{1\le i<j<k\le n} \mu_{ijk} e_i e_j e_k,$$

where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$ and using formula (15.1), we find that $e_i e_j = \sum_{k=1}^n \mu_{ijk} e_k^\vee$.

In order to identify the resonance varieties of the cohomology algebra $A^* = H^*(M,\mathbb{C})$, we let $S = \operatorname{Sym}(A_1)$ be the symmetric algebra on $A_1 = H_1(M,\mathbb{C})$, and we identify $S$ with the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$. The Koszul complex from (11.2) then has the form

$$A^0 \otimes_{\mathbb{C}} S \xrightarrow{\delta_A^0} A^1 \otimes_{\mathbb{C}} S \xrightarrow{\delta_A^1} A^2 \otimes_{\mathbb{C}} S \xrightarrow{\delta_A^2} A^3 \otimes_{\mathbb{C}} S,$$

where the differentials are the $S$-linear maps given by $\delta_A^q(u) = \sum_{j=1}^n e_j u \otimes x_j$ for $u \in A^q$. In our chosen basis, the matrix of $\delta_A^2$ is the transpose of $\delta_A^0 = (x_1 \cdots x_n)$, while the matrix of $\delta_A^1$ is an $n \times n$ matrix of linear forms in the variables $x_i$, given by $\delta_A^1(e_i) = \sum_{j=1}^n \sum_{k=1}^n \mu_{jik} e_k^\vee \otimes x_j$.

Note that the matrix $\delta_M := \delta_A^1$ is skew-symmetric; moreover, it is singular, since the vector $(x_1,\dots,x_n)$ is in its kernel. Hence, both the determinant $\det(\delta_A^1)$ and the Pfaffian $\operatorname{pf}(\delta_M)$ vanish. Let $\delta_M(i;j)$ be the sub-matrix obtained from $\delta_M$ by deleting the $i$-th row and $j$-th column. We then have the following lemma, due to Turaev [145].

**Lemma 15.2** ([145]). *Assume $n \ge 3$. There is then a polynomial $\operatorname{Det}(\mu) \in S$ such that $\det \delta_M(i;j) = (-1)^{i+j} x_i x_j \operatorname{Det}(\mu)$. Moreover, if $n$ is even, then $\operatorname{Det}(\mu) = 0$, while if $n$ is odd, then $\operatorname{Det}(\mu) = \operatorname{Pf}(\mu)^2$, where $\operatorname{pf}(\delta_M(i;i)) = (-1)^{i+1} x_i \operatorname{Pf}(\mu)$.*

### 15.3. Resonance varieties of 3-manifolds

Let $\mathcal{R}_k^i(M)$ be the resonance varieties associated to the cohomology algebra $A = H^*(M,\mathbb{C})$ of a closed, orientable 3-manifold $M$. As shown in [127], Poincaré duality implies that $\mathcal{R}_k^2(M) = \mathcal{R}_k^1(M)$ for $1 \le k \le n$, while $\mathcal{R}_1^3(M) = \mathcal{R}_1^0(M) = \{\mathbf{0}\}$ if $n > 0$.

The basic structure of the degree 1, depth 1 resonance varieties is given by the following theorem.

**Theorem 15.3** ([127, 129]). *Let $M$ be a closed, orientable 3-manifold. Set $n = b_1(M)$ and let $\mu_M$ be the associated alternating 3-form. Then*

$$\mathcal{R}^1_1(M) = \begin{cases} \emptyset & \text{if } n = 0; \\ \{0\} & \text{if } n = 1 \text{ or } n = 3 \text{ and } \mu_M \text{ has rank } 3; \\ \mathbf{V}(\operatorname{Pf}(\mu_M)) & \text{if } n \text{ is odd, } n > 3, \text{ and } \mu_M \text{ is generic}; \\ H^1(M;\mathbb{C}) & \text{otherwise.} \end{cases}$$

In the case when $n = 2g + 1$ with $g > 1$, we say that the alternating form $\mu_M$ is *generic* (in the sense of Berceanu and Papadima [14]) if there is an element $c \in A^1$ such that the 2-form $\gamma_c \in A_1 \wedge A_1$ defined by $\gamma_c(a \wedge b) = \mu_M(a \wedge b \wedge c)$ for $a, b \in A^1$ has maximal rank, that is, $\gamma_c^g \neq 0$ in $\bigwedge^{2g} A_1$. For detailed information on the resonance varieties $\mathcal{R}^1_k(M)$ in depth $k > 1$ we refer to [127].

### 15.4. Characteristic varieties of 3-manifolds

As noted in [129], Poincaré duality with local coefficients imposes the same type of constraints on the characteristic varieties of a closed, orientable 3-manifold $M$; for instance, $\mathcal{V}^1_k(M) \cong \mathcal{V}^2_k(M)$, for all $k \geq 0$. Best understood is the variety $\mathcal{V}^1_1(M)$, due to its close connection to both the resonance variety $\mathcal{R}^1_1(M)$ and to the Alexander polynomial $\Delta_M$, which we define next.

Let $H = G_{\mathrm{ab}}/\operatorname{Tors}(G_{\mathrm{ab}})$ be the maximal torsion-free abelian quotient of the group $G = \pi_1(M, x_0)$, and let $q\colon M^H \to M$ be the regular cover corresponding to the projection $G \twoheadrightarrow H$. The Alexander module of $M$ is defined as the relative homology group $A_M = H_1(M^H, q^{-1}(x_0);\mathbb{Z})$, viewed as a module over the Noetherian ring $\mathbb{Z}H$. Finally, let $E_1(A_M) \subseteq \mathbb{Z}H$ be the ideal of codimension 1 minors in a $\mathbb{Z}H$-presentation for $A_M$. The *Alexander polynomial* of $M$ is then defined as the greatest common divisor of the elements in this determinantal ideal, $\Delta_M = \gcd(E_1(A_M))$.

As noted in [41] and [129], works of McMullen [98] and Turaev [145] yield the following relationship between the first characteristic variety and the Alexander polynomial of $M$.

**Proposition 15.4** ([41, 129]). *Let $M$ be a closed, orientable, 3-dimensional manifold. Then*

$$\mathcal{V}^1_1(M) \cap \operatorname{Char}(M)^0 = \mathbf{V}(\Delta_M) \cup \{\mathbf{1}\}.$$

*Moreover, if $b_1(M) \geq 4$, then $\mathcal{V}^1_1(M) \cap \operatorname{Char}(M)^0 = \mathbf{V}(\Delta_M)$.*

The next theorem shows that the second half of the tangent cone formula (12.2) holds for a large class of closed 3-manifolds with odd first Betti number (regardless of whether those manifolds are 1-formal), yet fails for most 3-manifolds with even first Betti number.

**Theorem 15.5** ([129]). *Let $M$ be a closed, orientable 3-manifold, and set $n = b_1(M)$.*

(1) *If $n \leq 1$, or $n$ is odd, $n \geq 3$, and $\mu_M$ is generic, then $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$.*

(2) *If $n$ is even, $n \geq 2$, then $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$ if and only if $\Delta_M = 0$.*

The information contained in the cohomology jump loci and the Alexander polynomials provides a method for determining which 3-manifold groups can also be realized as fundamental groups of Kähler manifolds, or smooth, quasi-projective varieties. We summarize the relevant results from [43, 44, 61], as follows.

**Theorem 15.6.** *Let $G = \pi_1(M)$ be the fundamental group of a closed, orientable 3-manifold $M$. Then:*

[44] *$G \cong \pi_1(X)$, for some compact Kähler manifold $X$ if and only if $G$ is a finite subgroup of $\mathrm{SO}(4)$, acting freely on $S^3$.*

[43] *$G$ is 1-formal and $G \cong \pi_1(X)$, for some smooth quasi-projective variety $X$ if and only if $\mathfrak{m}(G) \cong \mathfrak{m}(F_n)$ or $\mathfrak{m}(G) \cong \mathfrak{m}(\mathbb{Z} \times \pi_1(\Sigma_g))$.*

[61] *If $G \cong \pi_1(X)$, for some smooth quasi-projective variety $X$, then all the prime components of $M$ are graph manifolds.*

### 15.5. Finite models for 3-manifolds

The previous theorem leads to obstructions to the existence of CDGA models for closed 3-manifolds with specified finiteness properties. These obstructions are quite effective since they are expressed solely in terms of the Alexander polynomial of the manifold.

**Theorem 15.7** ([129]). *Let $M$ be a closed, orientable, 3-manifold, and set $n = b_1(M)$.*

(1) *If $n \leq 1$, then $M$ is formal, and has the rational homotopy type of $S^3$ or $S^1 \times S^2$.*

(2) *If $n$ is even, $n \geq 2$, and $\Delta_M \neq 0$, then $M$ is not 1-formal.*

(3) *If $\Delta_M \neq 0$, yet $\Delta_M(\mathbf{1}) = 0$ and $\mathrm{TC}_{\mathbf{1}}(V(\Delta_M))$ is not a finite union of rationally defined linear subspaces, then $M$ admits no 1-finite 1-model.*

*Proof.* For completeness, we give a proof of this result. As shown in [59], the 1-formality of $M$ is equivalent to formality. On the other hand, we saw in Example 10.23 that any finitely generated group $G$ with $b_1(G) \leq 1$ is 1-formal. Thus, if $b_1(M) = 0$ or 1, then $M$ is formal, and so, as noted in [105], $M$ must be rationally homotopy equivalent to either $S^3$ or $S^1 \times S^2$.

Now suppose $b_1(M)$ is even and positive, and $\Delta_M \neq 0$. Then, by Theorem 15.5, we have that $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(M)) \neq \mathcal{R}^1_1(M)$, and so, by Corollary 12.15, $M$ is not 1-formal.

Finally, if $\Delta_M \neq 0$ and $\Delta_M(\mathbf{1}) = 0$, it follows from Proposition 15.4 that $\mathcal{V}^1_1(M)$ and $\mathbf{V}(\Delta_M)$ share the same tangent cone and exponential tangent cone at $\mathbf{1}$. On the other hand, if not all the irreducible components of $\mathrm{TC}_{\mathbf{1}}(\mathbf{V}(\Delta_M))$ are rational linear subspaces, then, by Lemma 12.7, $\tau_{\mathbf{1}}(\mathbf{V}(\Delta_M)) \neq \mathrm{TC}_{\mathbf{1}}(\mathbf{V}(\Delta_M))$. Therefore, if both assumptions are satisfied, $\tau_1(\mathcal{V}_1^1(M)) \subsetneq \mathrm{TC}_1(\mathcal{V}_1^1(M))$, and so, by Theorem 12.13, $M$ cannot have a 1-finite 1-model. ∎

Consequently, if $\mathfrak{m} = \mathfrak{m}(G)$ is the Malcev Lie algebra of $G = \pi_1(M)$, then the following hold in the three cases delineated in Theorem 15.7: (1) $\mathfrak{m} = 0$ (if $n = 0$) or $\mathfrak{m} = \mathbb{Q}$ (if $n = 1$); (2) $\mathfrak{m}$ is not the LCS completion of a finitely generated, quadratic Lie algebra; and (3) $\mathfrak{m}$ is not the LCS completion of a finitely presented Lie algebra.

The next two examples illustrate how the finiteness obstructions provided by Theorem 15.7 work in cases (2) and (3).

**Example 15.8.** The Heisenberg 3-dimensional nilmanifold $M$ admits a finite model, for instance, $A = (\bigwedge(a, b, c), d)$ with $da = db = 0$ and $dc = ab$. Nevertheless, $M$ is not 1-formal, since $b_1(M) = 2$ and $\Delta_M = 1$. Furthermore, $\mu_M = 0$, and so $\mathrm{TC}_1(\mathcal{V}_1^1(M)) = \{\mathbf{0}\}$, whereas $\mathcal{R}_1^1(M) = \mathbb{C}^2$.

**Example 15.9.** Let $M$ be a closed, orientable 3-manifold with $H_1(M, \mathbb{Z}) = \mathbb{Z}^2$ and $\Delta_M = (t_1 + t_2)(t_1 t_2 + 1) - 4t_1 t_2$ (such a manifold exists by [145, Chapter VIII Section 5.3]). Then $\mathrm{TC}_1(\mathcal{V}_1^1(M)) = \{x_1^2 + x_2^2 = 0\}$ decomposes as the union of two lines defined over $\mathbb{C}$, but not over $\mathbb{Q}$; hence, $M$ admits no 1-finite 1-model. Furthermore, $\tau_1(\mathcal{V}_1^1(M)) = \{\mathbf{0}\}$ is properly contained in $\mathrm{TC}_1(\mathcal{V}_1^1(M))$.

## 15.6. 3-manifolds fibering over the circle

We conclude this section with a discussion of the 1-formality property for closed 3-manifolds that fiber over $S^1$. We start with a result which relates the notion of 1-formality of a semidirect product of the form $G = K \rtimes \mathbb{Z}$ to the algebraic monodromy of the extension.

**Theorem 15.10** ([109]). *Let $1 \to K \to G \to \mathbb{Z} \to 1$ be a short exact sequence of groups. Suppose $G$ is finitely presented and 1-formal, and $b_1(K) < \infty$. Then the eigenvalue 1 of the monodromy action on $H_1(K, \mathbb{C})$ has only $1 \times 1$ Jordan blocks.*

This theorem yields as an immediate corollary a substantial extension of a result of Fernández, Gray, and Morgan [57], where the non-formality of the total spaces of certain bundles is established by a different method, using Massey products.

**Corollary 15.11** ([109]). *Let $F \to X \to S^1$ be a smooth fibration whose fiber $F$ is connected and has the homotopy type of a CW-complex with finite 2-skeleton, and for which the monodromy on $H_1(F, \mathbb{C})$ has eigenvalue 1, with a Jordan block of size greater than 1. Then the group $G = \pi_1(X)$ is not 1-formal.*

Next, we recall a result from [110], which is based on the interplay between the Bieri–Neumann–Strebel invariant, $\Sigma^1(G)$, and the (first) resonance variety, $\mathcal{R}_1^1(G)$, of a 1-formal group $G$.

**Proposition 15.12** ([110]). *Let $M$ be a closed, orientable 3-manifold which fibers over the circle. If $b_1(M)$ is even, then $M$ is not 1-formal.*

Combining the results above yields the following corollary, which puts strong restrictions on the algebraic monodromy of a formal 3-manifold fibering over the circle.

**Corollary 15.13** ([109]). *Let $M$ be a closed, orientable, 1-formal 3-manifold. Suppose $M$ fibers over the circle, and the algebraic monodromy has 1 as an eigenvalue. Then, there are an even number of $1 \times 1$ Jordan blocks for this eigenvalue, and no higher size Jordan blocks.*

Indeed, by Corollary 15.11, the algebraic monodromy has only $1 \times 1$ Jordan blocks for the eigenvalue 1. Let $m$ be the number of such blocks. From the Wang sequence of the fibration, we deduce that $b_1(M) = m + 1$. By Proposition 15.12, $m$ must be even.

**Example 15.14.** The 3-dimensional Heisenberg manifold $M$ from Examples 10.14 and 10.27 fibers over $S^1$ with fiber $S^1 \times S^1$ and monodromy given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since this is a Jordan block of size 2 with eigenvalue 1, we see once again that $M$ is not 1-formal.

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