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Lie algebras and torsion groups with identity

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To my teacher Leonid A. Bokut on his 80th birthday.

Abstract. We prove that a finitely generated Lie algebra L such that (i) every commutator in generators is ad-nilpotent, and (ii) L satisfies a polynomial identity, is nilpotent. As a corollary we get that a finitely generated residually-p torsion group whose pro-p completion satisfies a pro-p identity is finite.

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1. Introduction

In 1941 A. G. Kurosh formulated a Burnside-type problem for algebras [19]. Let A be an associative algebra over a field F. An element $a \in A$ is said to be nilpotent if $a^{n(a)} = 0$ for some $n(a) \geq 1$. An algebra A is said to be nil if every element of A is nilpotent.

The Kurosh Problem. *Is it true that a finitely generated nil algebra is nilpotent?*

Examples by E. S. Golod [5] (see also the far reaching examples from [22]) showed that this is not always the case. However the Kurosh Problem has positive solution in the class of algebras satisfying a polynomial identity (PI-algebras).

Let $f(x_1, x_2, \dots, x_m)$ be a nonzero element of the free associative F-algebra. We say that an algebra A satisfies the polynomial identity $f = 0$ if $f(a_1, a_2, \dots, a_m) = 0$ for arbitrary elements $a_1, a_2, \dots, a_m \in A$.

One of the high points of the theory of PI-algebras was the solution of the Kurosh Problem (I. Kaplansky [14], J. Levitzki [21], A. I. Shirshov [30]) in the following form: Let A be an associative algebra generated by elements $a_1, \dots, a_m$. Let S be the multiplicative semigroup generated by the elements $a_1, \dots, a_m$. Suppose that an arbitrary element of S is nilpotent. Then the algebra A is nilpotent.

Now let L be a Lie algebra over a field F. As above, for a nonzero element $f(x_1, x_2, \dots, x_m)$ of the free Lie algebra we say that L satisfies the identity $f = 0$ if $f(a_1, a_2, \dots, a_m) = 0$ for arbitrary elements $a_1, a_2, \dots, a_m \in A$, see [2].

An element $a \in L$ is said to be *ad-nilpotent* if the linear operator

$$\mathrm{ad}(a) : L \to L, x \to [x, a]$$

is nilpotent.

A subset $S \subset L$ is called a *Lie set* if, for arbitrary elements $a, b \in S$, we have $[a, b] \in S$. For a subset $X \subset L$, the Lie set generated by X is the smallest Lie set $S\langle X\rangle$ containing X. It consists of X and of all iterated commutators in elements from X.

Theorem 1.1. *Let L be a Lie algebra satisfying a polynomial identity and generated by elements $a_1, \ldots, a_m$. If an arbitrary element $s \in S\langle a_1, \ldots, a_m\rangle$ is ad-nilpotent then the Lie algebra L is nilpotent.*

This theorem has implications in group theory: Let p be a prime number. A group G is said to be *residually-p* if there exists a family of homomorphisms $\phi_i : G \to G_i$ into finite p-groups G_i such that $\bigcap_i \mathrm{Ker}(\phi_i) = (1)$.

Let $\mathbb{Z}_p$ be the field of order p. Consider the group algebra $(\mathbb{Z}_p)[G]$ and its fundamental ideal w spanned by all elements $1 - g, g \in G$. It is easy to see that the group G is residually-p if and only if $\bigcap_{i \geq 1} w^i = (0)$. The *Zassenhaus filtration* is defined as

$$G = G_1 > G_2 > \cdots$$

where $G_i = \{g \in G \mid 1 - g \in w^i\}$. Then $[G_i, G_j] \subseteq G_{i+j}$ and each factor G_i/G_{i+1} is an elementary abelian p-group. Hence

$$L_p(G) = \bigoplus_{i \geq 1} G_i/G_{i+1}$$

is a Lie algebra over $\mathbb{Z}_p$.

Theorem 1.2. *Let G be a residually-p finitely generated torsion group such that the Lie algebra $L_p(G)$ satisfies a polynomial identity. Then G is a finite group.*

Let $g(x_1, x_2, \ldots, x_m)$ be a nonidentical element of the free pro-p group (see [3,29]) on the set of free generators $x_1, x_2, \ldots, x_m$. We say that a pro-p group G satisfies the identity $g = 1$ if $g(a_1, a_2, \ldots, a_m) = 1$ for arbitrary elements $a_1, a_2, \ldots, a_m \in G$.

Theorem 1.3. *Let G be a residually-p finitely generated torsion group such that its pro-p completion $G_{\hat{p}}$ satisfies a nontrivial identity. Then G is a finite group.*

Remark. The examples of infinite residually-p groups due to E. S. Golod [5], R. I. Grigorchuk [6], and N. Gupta–S. Sidki [8] are finitely generated and torsion.

The results above significantly extend the positive solution of the Restricted Burnside Problem [38,39] and the work of [40] on compact torsion groups. They were announced in [41,42] but no detailed proof followed. Meanwhile they were used in numerous papers. Therefore I feel compelled to present a detailed proof.

The proof essentially uses the ideas and techniques from [38,39].

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2. The case of zero characteristic

In this section we assume that char $F = 0$. This assumption allows us to avoid major difficulties but also miss major applications.

Kostrikin Lemma (A. I. Kostrikin [16], [17, Lemma 2.1.1]). *Let L be a Lie algebra, $a \in L$, $\mathrm{ad}(a)^n = 0$. If $4 \le n < \mathrm{char}\, F$ (here zero characteristic is viewed as ∞), then*

$$\mathrm{ad}\left(b\, \mathrm{ad}(a)^{n-1}\right)^{n-1} = 0$$

for an arbitrary element $b \in L$.

Choose a nonzero element $s \in S = S\langle a_1, \dots, a_m \rangle$. The element s is ad-nilpotent. Repeatedly using the Kostrikin lemma we can assume that $\mathrm{ad}(s)^3 = 0$.

Recall that a linear algebra over a field F of characteristic $\neq 2$ is called a *Jordan algebra* if it satisfies the identities:

(J1) $x \circ y = y \circ x$;

(J2) $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$.

If A is an associative algebra then $A^{(+)} = \{A, a \circ b = \frac{1}{2}(ab + ba)\}$ is a Jordan algebra. For more information on Jordan algebras see [9,25,43].

We will use a construction of a Jordan algebra from [4] which is a refined version of the Tits–Kantor–Koecher construction [13,15,32,33].

Let L be a Lie algebra over a field of characteristic $\neq 2, 3$. Let $s \in L$, $\mathrm{ad}(s)^3 = 0$. Define a new operation $a \circ b = [a, [s, b]]$, $a, b \in L$. Then the vector space $K = \{a \in L \mid a\, \mathrm{ad}(s)^2 = 0\}$ is an ideal of the algebra $(L, \circ)$.

Theorem 2.1 ([4]). *The factor algebra $(L, \circ)/K$ is a Jordan algebra.*

For a set $X = \{x_1, x_2, \dots\}$, let $FJ\langle X \rangle$ denote the free Jordan algebra (see [9,25, 43]). Consider also the free associative algebra $F\langle X \rangle$. Let ϕ be the homomorphism $\phi : FJ\langle X \rangle \to F\langle X \rangle^{(+)}$, $x \to x$, $x \in X$. An element lying in the kernal $\ker \phi$, is called an S-identity. A Jordan algebra J is said to be PI if there exists an element $f(x_1, \dots, x_n) \in FJ\langle X \rangle$ that is not an S-identity such that $f(a_1, \dots, a_n) = 0$ for all elements $a_1, \dots, a_n \in J$.

For elements x, y, z of a Jordan algebra J, define their triple product $\{x, y, z\} = (xy)z + x(yz) - y(xz)$. An element $a \in J$ is called an *absolute zero divisor* if $a^2 = 0$ and $\{a, J, a\} = (0)$.

A Jordan algebra that does not contain nonzero absolute zero divisors is called *nondegenerate*. The smallest ideal $\mathrm{Mc}(J)$ such that the factor algebra $J/\mathrm{Mc}(J)$ is nondegenerate is called the McCrimmon radical of J.

Lemma 2.2. *Let J be a Jordan algebra with PI such that every element of J is a sum of nilpotent elements. Then $J = \mathrm{Mc}(J)$.*

Proof. Let $J \neq \mathrm{Mc}(J)$. Then without loss of generality we will assume that the algebra J is nondegenerate. Moreover, since a nondegenerate Jordan algebra is a subdirect product of prime nondegenerate Jordan algebras (see [37]), we will assume that the algebra J is prime and nondegenerate.

In [36] it was shown that a prime nondegenerate PI-algebra J has nonzero *center*

$$Z(J) = \{z \in J \mid (za)b = z(ab) \text{ for arbitrary elements } a, b \in J\}$$

and the ring of fractions $\widetilde{J} = (Z(J) \setminus \{0\})^{-1} J$ is either a simple finite dimensional algebra over the field $\widetilde{Z} = (Z(J) \setminus \{0\})^{-1} Z(J)$ or else an algebra of a symmetric nondegenerate bilinear form. In both cases the algebra $\widetilde{J}$ has a nonzero linear trace $t : \widetilde{J} \to \widetilde{Z}$ such that the trace of a nilpotent element is zero. Since every element of $\widetilde{J}$ is a sum of nilpotent elements it follows that $t(\widetilde{J}) = (0)$, a contradiction that finishes the proof of the lemma. □

Lemma 2.3. *The Jordan algebra $J = (L, \circ)/K$ is McCrimmon radical, i.e. $J = \mathrm{Mc}(J)$.*

Proof. By our assumption, the Lie algebra L satisfies a nontrivial polynomial identity. Passing to the full linearization of this identity (see [43]) we can assume that the identity looks like

$$\sum_{\sigma \in S_n} \alpha_\sigma x_0 \,\mathrm{ad}(x_{\sigma(1)}) \cdots \mathrm{ad}(x_{\sigma(n)}) = 0,$$

where not all coefficients $\alpha_\sigma \in F$ are equal to 0. This implies that

$$\sum_{\sigma \in S_n} \alpha_\sigma a_0 R(a_{\sigma(1)}) \cdots R(a_{\sigma(n)}) = 0$$

for arbitrary elements $a_0, a_1, \ldots, a_n \in J$, where $R(a) : x \to xa$ denotes the multiplication operator in J.

It is easy to see that the element $\sum_{\sigma \in S_n} \alpha_\sigma a_0 R(a_{\sigma(1)}) \cdots R(a_{\sigma(n)})$ is not an S-identity. Hence J is a PI-algebra.

The Lie algebra L is spanned by the Lie set $S = S\langle a_1, \ldots, a_m\rangle$. For an arbitrary element $a \in S$ let $\bar{a} = a + K$ be its image in the Jordan algebra J. The tth power of $\bar{a}$ in J is $a\,\mathrm{ad}([s, a])^{t-1} + K$, which implies that the element $\bar{a}$ is nilpotent. By Lemma 2.2 $J = \mathrm{Mc}(J)$, which finishes the proof of Lemma 2.3. □

Following A. I. Kostrikin [16,18], we call an element a of a Lie algebra L a *sandwich* if (i) $\mathrm{ad}(a)^2 = 0$ and (ii) $\mathrm{ad}(a)\,\mathrm{ad}(b)\,\mathrm{ad}(a) = 0$ for an arbitrary element $b \in L$.

If $\mathrm{char}\, F \neq 2$ then (i) implies (ii).

If a is a nonzero absolute zero divisor of the Jordan algebra J and char $F \neq 2, 3$ then for the nonzero element $b = a\,\mathrm{ad}(s)^2$ we have $\mathrm{ad}(b)^2 = \mathrm{ad}(s)^2\,\mathrm{ad}(a)^2\,\mathrm{ad}(s)^2$. Hence $L\,\mathrm{ad}(b)^2 \subseteq \{a, J, a\}\,\mathrm{ad}(s)^2 = (0)$. Hence, b is a sandwich of the Lie algebra L.

To summarize, we showed that if $L = \langle a_1, \dots, a_m \rangle$ is a nonzero Lie algebra over a field F of zero characteristic, every element of the Lie set $S = S\langle a_1, \dots, a_m \rangle$ is ad-nilpotent, and if L satisfies a nontrivial polynomial identity, then L contains a nonzero sandwich.

Lemma 2.4. *Let $L = \langle a_1, \dots, a_m \rangle$ be a finitely generated Lie algebra such that an arbitrary element of the Lie set $S\langle a_1, \dots, a_m \rangle$ is ad-nilpotent. Let I be an ideal of L of finite codimension. Then I is finitely generated as a Lie algebra.*

Proof. The finite dimensional Lie algebra L/I is spanned by a Lie set for which every element in the set is ad-nilpotent. By the Engel–Jacobson theorem [10] the Lie algebra L/I is nilpotent. In other words, there exists $k \geq 1$ such that $L^k \subseteq I$.

Suppose that every commutator ρ in $a_1, \dots, a_m$ of length $< k$ is ad-nilpotent of degree at most t, i.e. $\mathrm{ad}(\rho)^t = 0$. Let $N = ktm^k$. In [41, Lemma 2.5] it is shown that every product $\mathrm{ad}(a_{i_1}) \cdots \mathrm{ad}(a_{i_N})$, $1 \leq i_1, \dots, i_N \leq m$, can be represented as

$$\mathrm{ad}(a_{i_1}) \cdots \mathrm{ad}(a_{i_N}) = \sum_j v_j\,\mathrm{ad}(\rho_j),$$

where the v_j's are (possibly empty) products of the $ad(a_i)$'s and the ρ_j's are commutators in $a_1, \dots, a_m$ of length $\geq k$. Each summand on the right hand side has the same degree in each a_i as the left hand side.

It follows now that the algebra L^k is generated by commutators ρ in $a_1, \dots, a_m$ such that $k \leq \mathrm{length}(\rho) < 2N$.

We have $\dim_F(L/L^k) < \infty$. Let $b_1, \dots, b_r \in I$ be a basis of I modulo L^k. Now the algebra I is generated by $b_1, \dots, b_r$ and by all commutators ρ in $a_1, \dots, a_m$ such that $k \leq \mathrm{length}(\rho) < 2N$, which proves the lemma. □

Recall that an algebra L is called *just infinite* if it is infinite dimensional but every nonzero ideal of L is of finite codimension.

Lemma 2.5. *Let L be an infinite dimensional Lie algebra generated by elements $a_1, \dots, a_m$ such that an arbitrary element from $S\langle a_1, \dots, a_m \rangle$ is ad-nilpotent. Then L has a just infinite homomorphic image.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals of infinite codimension. We claim that the union $I = \bigcup_i I_i$ also has infinite codimension. Indeed, if $\dim_F(L/I) < \infty$ then by Lemma 2.4 the ideal I is generated by a finite collection of elements, hence I is equal to one of the terms in the ascending chain, a contradiction.

By Zorn's lemma the algebra L has a maximal ideal J of infinite codimension. The factor algebra L/J is just infinite, which proves the lemma. □

Now we are ready to finish the proof of Theorem 1.1 in the case of char $F = 0$.

Let L be a Lie algebra satisfying the assumptions of the theorem. In view of Lemma 2.5 without loss of generality we will assume the algebra L to be just infinite.

We proved that L contains a nonzero sandwich. Recall that an algebra is called locally nilpotent if every finitely generated subalgebra is nilpotent. A. N. Grishkov [7] proved that in a Lie algebra over a field of zero characteristic, an arbitrary sandwich generates a locally nilpotent ideal. Since the Lie algebra L is just infinite it follows that L contains a locally nilpotent ideal I of finite codimension. By Lemma 2.2 the algebra I is finitely generated, hence nilpotent and finite dimensional. This contradicts the assumption that the algebra L is infinite dimensional and proves the theorem.

3. Divided polynomials

The main Theorem 1.1 is valid for Lie algebras over an arbitrary ground field F. The applications to Theorems 1.2, 1.3 use only the case when the ground field F is finite.

We will show that without loss of generality, we can assume that the field F is infinite. Indeed, let F' be an infinite field extension of F. The Lie algebra $L' = L \otimes_F F'$ is generated by the same elements $a_1, \ldots, a_m$ as L and an arbitrary element $s \in S\langle a_1, \ldots, a_m\rangle$ is ad-nilpotent in L'. Since the Lie algebra L satisfies a polynomial identity, it satisfies a nontrivial multilinear identity $f(x_1, \ldots, x_n) = 0$ (see [2]). Then the Lie algebra L' also satisfies the identity $f = 0$.

From now on we assume that char $F = p > 0$ and the field F is infinite. Let $L\langle X\rangle$ be the free Lie F-algebra on the set of free generators $X = \{x_1, \ldots, x_m\}$ in the variety of algebras satisfying the identity $f = 0$ (see [2]). Let P be the set of all commutators in X and let $n : P \to N$ be a function. Let J be the ideal of $L\langle X\rangle$ generated by $\bigcup_{\rho \in P} L\langle X\rangle\, \mathrm{ad}(\rho)^{n(\rho)}$.

Our aim is to show that the algebra $L' = L\langle X\rangle/J$ is nilpotent. Suppose that this is not true. Letting $\deg(x_i) = 1$, $1 \le i \le m$, we define a gradation of L' by positive integers.

We say that a graded infinite dimensional algebra is graded just infinite if every nonzero graded ideal of it is of finite codimension.

Lemma 3.1. *The algebra L' has a graded just infinite homomorphic image L.*

The proof follows the proof of Lemma 2.5 (verbatim).

Consider the adjoint embedding $L \to \mathrm{End}_F(L)$, $a \to \mathrm{ad}(a)$. Let A' be the associative subalgebra of $\mathrm{End}_F(L)$ generated by the image of L. The algebra A' is graded and we assume that $L \subseteq A'^{(-)}$. Let I be a maximal graded ideal of the algebra A' such that $L \cap I = (0)$, $A = A'/I$, $L \subseteq A^{(-)}$. If J is a nonzero graded ideal of the algebra A then the ideal $J \cap L$ has finite codimension in L. From the

Poincaré–Birkhoff–Witt theorem it follows that the factor algebra A/J is nilpotent and finite dimensional. We have proved that the algebra A is graded just infinite.

To summarize, we assume that:

(1) the graded Lie algebra L is generated by elements $s_1, \ldots, s_m$ of degree 1; every element from the Lie set $S = S\langle s_1, \ldots, s_m\rangle$ is ad-nilpotent;

(2) L satisfies a polynomial identity;

(3) L is graded just infinite.

We fix also a graded just infinite associative enveloping algebra A of L. The algebra A is a homomorphic image of the subalgebra $\langle \mathrm{ad}(L)\rangle \subseteq \mathrm{End}_F(L)$

For elements $a_1, \ldots, a_k \in L$ let $[a_1, a_2, \ldots, a_k]$ denote their left-normed commutator by $[\cdots[a_1, a_2], a_3], \ldots, a_k]$. We also denote $[a, \underbrace{b, b, \ldots, b}_{k}] = [ab^k]$.

Lemma 3.2. *Let I be an ideal of L, $s \in S$, $k \geq 2$, $[Is^k] = (0)$. Suppose that the Lie algebra L satisfies an identity of degree n. Then the subalgebra $[Is^{k-1}]$ satisfies an identity of degree $< n$.*

Proof. Let L satisfy an identity

$$\sum_{\sigma \in S_{n-1}} \alpha_\sigma \big[x_0, x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}\big] = 0,$$

where $\alpha_\sigma \in F$, $\alpha_1 = 1$. For the variables $x_0, x_1, \ldots, x_{n-1}$ choose values $x_0 = s$, $x_1 = a \in I$, $x_i = a_i \in [Is^{k-1}]$, $2 \leq i \leq n-1$. Then

$$\sum_{\sigma \in H} \alpha_\sigma \big[[s, a], a_{\sigma(2)}, \ldots, a_{\sigma(n-1)}\big] = 0,$$

where σ runs over the stabilizer H of 1 in S_{n-1}. It follows now that the Lie algebra $[Is^{k-1}]$ satisfies the identity

$$\sum_{\sigma \in H} \alpha_\sigma \big[x_0, x_{\sigma(2)}, \ldots, x_{\sigma(n-1)}\big] = 0$$

of degree $n-1$. This finishes the proof of the lemma. □

Lemma 3.3. *Let I be an ideal of L, $s \in L$, $[Is^k] = (0)$, $k \geq 2$. Then for any integer $t \geq 2$ and any elements $a_1, \ldots, a_N \in I$, $N = kt - 1$, we can write the operator $\mathrm{ad}([a_1 s^{k-1}]) \cdots \mathrm{ad}([a_N s^{k-1}])$ as a linear combination of operators of the type*

$$P' \,\mathrm{ad}(s)^{k-1} \prod_{j=0}^{t-2} \big(\mathrm{ad}(a_{i+j})\,\mathrm{ad}(s)^{k-1}\big) P''$$

where P', P'' are products of $\mathrm{ad}(a_1), \ldots, \mathrm{ad}(a_n), \mathrm{ad}(s)$, which may be empty.

Proof. By the Jacobi identity

$$\operatorname{ad}\left([a_1 s^{k-1}]\right)\cdots\operatorname{ad}\left([a_N s^{k-1}]\right) = \sum \pm \operatorname{ad}(s)^{j_0}\operatorname{ad}(a_1)\operatorname{ad}(s)^{j_1}\cdots\operatorname{ad}(s)^{j_{N-1}}\operatorname{ad}(a_N)\operatorname{ad}(s)^{j_N},$$

and in each summand $0 \le j_0, j_1, \dots, j_N \le k-1$, $j_0 + \cdots + j_N = (k-1)N$. If, in each segment $j_\mu, j_{\mu+1}, \dots, j_{\mu+t-1}$ of length t, at least one term is $\le k-2$, then $j_\mu + \cdots + j_{\mu+t-1} \le (k-1)t - 1$. Summing all k segments we get $j_0 + \cdots + j_N \le k((k-1)t-1) < (k-1)(kt-1)$, a contradiction that proves the lemma. □

Lemma 3.4. *There exist elements $c_1, \dots, c_r \in S$ and integers $m \ge 1, N \ge 1$ such that:*

(1) $[L^i, c_1, c_2, \dots, c_r] \ne (0)$ *for arbitrary* $i \ge 1$*;*

(2) $[L^m, c_1, c_2, \dots, c_r, c_i] = (0)$ *for* $1 \le i \le r$*;*

(3) *for arbitrary elements* $a_1, \dots, a_N \in [L^m, c_1, \dots, c_r]$ *we have* $\operatorname{ad}(a_1)\cdots\operatorname{ad}(a_N) = 0$.

Proof. Choose a nonzero element $s \in S$. For an arbitrary $i \ge 1$ choose a minimal integer $k(i) \ge 1$ such that $[L^i s^{k(i)}] = (0)$. Since the Lie algebra L is graded just infinite it follows that each power of L has zero centralizer. Hence $k(i) \ge 2$. We have $k(1) \ge k(2) \ge \cdots$. There exists a sufficiently large integer m_1 such that

$$k_1 := k(m_1) = k(m_1 + 1) = \cdots .$$

In other words, $[L^{m_1} s^{k_1}] = (0)$, $[L^i s^{k_1 - 1}] \ne (0)$ for any $i \ge 1$.

Now suppose that we have found l elements $s_1 = s, s_2, \dots, s_l \in S$ and $2l$ integers $k_1, \dots, k_l \ge 2$; $1 \le m_1 \le m_2 \le \cdots \le m_l$ with the following properties:

(1) $s_i \in [L^{m_{i-1}} s_1^{k_1-1} \cdots s_{i-1}^{k_{i-1}-1}]$, $2 \le i \le l$;

(2) $[L^{m_i} s_1^{k_1-1} \cdots s_{i-1}^{k_{i-1}-1} s_i^{k_i}] = (0)$, $1 \le i \le l$;

(3) for an arbitrary $i \ge 1$ we have $[L^i s_1^{k_1-1} \cdots s_l^{k_l-1}] \ne (0)$.

Claim 3.5. *For arbitrary* $1 \le i, j \le l$ *we have* $[s_i, s_j] = 0$.

Proof. Indeed, let $i < j$. Then $s_j \in [L^{m_{j-1}} s_1^{k_1-1} \cdots s_{j-1}^{k_{j-1}-1}]$. We will show that

$$\left[L^{m_{j-1}} s_1^{k_1-1} \cdots s_{j-1}^{k_{j-1}-1} s_i\right] = (0).$$

By the inductive assumption on $i + j$ the element s_i commutes with $s_{i+1}, \dots, s_{j-1}$. Hence

$$\left[L^{m_{j-1}} s_1^{k_1-1} \cdots s_{j-1}^{k_{j-1}-1} s_i\right] = \left[L^{m_{j-1}} s_1^{k_1-1} \cdots s_i^{k_i} \cdots\right] = 0,$$

which proves the claim. □

Case 1. Suppose that there exists an element $s' \in S^{m_l} = [\underbrace{S, S, \dots, S}_{m_l}]$ such that

$$[[L^i s_1^{k_1-1} \cdots s_l^{k_l-1}], [s' s_1^{k_1-1} \cdots s_l^{k_l-1}]] \neq (0)$$

for any $i \geq 1$. Denote $s_{l+1} = [s' s_1^{k_1-1} \cdots s_l^{k_l-1}]$. As above we find integers $m_{l+1} \geq m_l$ and $k_{l+1} \geq 2$ such that

$$[[L^{m_{l+1}} s_1^{k_1-1} \cdots s_l^{k_l-1}] s_{l+1}^{k_{l+1}}] = (0),$$
$$[L^i s_1^{k_1-1} \cdots s_l^{k_l-1} s_{l+1}^{k_{l+1}-1}] \neq (0)$$

for any $i \geq 1$. The elements $s_1, \dots, s_{l+1}$ and the integers $m_1, \dots, m_{l+1}; k_1, \dots k_{l+1}$ satisfy the conditions (1), (2), and (3) above.

Case 2. Now suppose that for an arbitrary element $s' \in S^{m_l}$ there exists an integer $i(s') \geq 1$ such that

$$[[L^{i(s')} s_1^{k_1-1} \cdots s_l^{k_l-1}], [s' s_1^{k_1-1} \cdots s_l^{k_l-1}]] = (0).$$

Since S^{m_l} spans L^{m_l} it follows that for an arbitrary element $a \in [L^{m_l} s_1^{k_1-1} \cdots s_l^{k_l-1}]$ there exists $i(a) \geq 1$ such that

$$[[L^{i(a)} s_1^{k_1-1} \cdots s_l^{k_l-1}], a] = (0). \tag{1}$$

Let $t = 2k_1 \cdots k_l - 1$. Choose $2t-1$ elements $a_1, \dots, a_{2t-1} \in [L^{m_l} s_1^{k_1-1} \cdots s_l^{k_l-1}]$. Let $q = \max\{i(a_\mu), 1 \leq \mu \leq 2t-1, m_l\}$.

Our immediate aim will be to show that

$$[L^q, a_1, \dots, a_t] = (0).$$

Denote $t_j = 2k_{j+1} \cdots k_l - 1$, so $t_0 = t$. We let $t_l = 1$. From Claim 3.5 it follows that for any $1 \leq i \leq l$, $L_j := [L^q s_1^{k_1-1} \cdots s_j^{k_j-1}]$ is a subalgebra of L. Let $L_0 = L^q$.

Claim 3.6. *For t_j arbitrary elements $b_1, \dots, b_{t_j} \in \{a_1, \dots, a_t\}$ we have*

$$[L_j, b_1, \dots, b_{t_j}] = (0).$$

Proof. To prove the claim, we will use reverse induction on $j = 0, \dots, l$. For $j = l$ we have $t_l = 1$ and $[L_l, a_i] = [L^q s_1^{k_1-1} \cdots s_l^{k_l-1}, a_i] = (0)$ by the choice of q. Now suppose that the assertion is true for j, $1 \leq j \leq l$. We have $t_{j-1} = k_j(t_j + 1) - 1$. By Claim 3.5 an arbitrary element $a \in [L^q s_1^{k_1-1} \cdots s_l^{k_l-1}]$ can be represented as $a = [a' s_j^{k_j-1}]$, where $a' \in [L^q s_1^{k_1-1} \cdots s_{j-1}^{k_1-1}] = L_{j-1}$. Let $b_\mu = [b'_\mu s_j^{k_j-1}], b'_\mu \in L_{j-1}$. We apply Lemma 3.3 to the algebra $L_{j-1} + F s_j$ and its ideal L_{j-1}. By Lemma 3.3 $\operatorname{ad}(b_1)\cdots\operatorname{ad}(b_{t_{j-1}})$ is a linear combination of operators $P'\operatorname{ad}(s_j)^{k_j-1}\big(\prod_{\mu=0}^{t_j-1}\operatorname{ad}(b'_{i+\mu}\operatorname{ad}(s_j)^{k_j-1})\big)P''$. By the induction assumption

$$\begin{aligned} L_{j-1}P'\operatorname{ad}(s_j)^{k_j-1}&\left(\prod_{\mu=0}^{t_j-1}\operatorname{ad}(b'_{i+\mu})\operatorname{ad}(s_j)^{k_j-1}\right)\\ &\subseteq L_{j-1}\operatorname{ad}(s_j)^{k_j-1}\left(\prod_{\mu=0}^{t_j-1}\operatorname{ad}(b'_{i+\mu})\operatorname{ad}(s_j)^{k_j-1}\right)=L_j\prod_{\mu=0}^{t-1}\operatorname{ad}(b_{i+\mu}), \end{aligned}$$

which finishes the proof of Claim 3.6. □

In particular, for $j=0$ we have

$$\left[L^q, a_1, a_2, \dots, a_t\right] = (0).$$

Now,

$$\left[L^q, [L, a_1, a_2, \dots, a_{2t-1}]\right] \subseteq \sum \left[L^q, a_{i_1}, \dots, a_{i_\mu}, L, a_{j_1}, \dots, a_{j_\nu}\right],$$

where in each summand $\mu+\nu = 2t-1$. If $\mu \geq t$ then

$$\left[L^q, a_{i_1}, \dots, a_{i_\mu}\right] = (0)].$$

If $\nu \geq t$ then

$$\left[L^q, a_i, \dots, a_{i_\mu}, L, a_{j_1}, \dots, a_{j\nu}\right] \subseteq \left[L^q, a_{j_1}, \dots, a_{j_\nu}\right] = (0).$$

Since the power L^q has zero centralizer it follows that

$$\left[L, a_1, a_2, \dots, a_{2t-1}\right] = (0).$$

We showed that in case 2, the elements

$$c_1,\dots,c_r = \underbrace{s_1,\dots,s_1}_{k_1-1}, \underbrace{s_2,\dots,s_2}_{k_2-1}, \dots, \underbrace{s_l,\dots,s_l}_{k_l-1}$$

and the integers $m=m_l$, $N=4k_1\cdots k_l-3$ satisfy the conditions of the lemma.

Let the algebra L satisfy an identity of degree n. We will show that enlarging the system $s_1,\dots,s_l;\ k_1,\dots,k_l \geq 2;\ 1\leq m_1\leq\cdots\leq m_l$ we will encounter case 2 in $\leq n-2$ steps.

The subalgebra $I_i = [L^{m_i}s_1^{k_1-1},\dots,s_{i-1}^{k_{i-1}-1}]$ is an ideal of

$$\left[L^{m_{i-1}}s_1^{k_1-1},\dots,s_{i-1}^{k_{i-1}-1}\right]$$

and $[I_i s_i^{k_i}] = (0)$. If $[L^{m_{i-1}}s_1^{k_1-1}\cdots s_{i-1}^{k_{i-1}-1}]$ satisfies an identity of degree n_{i-1} then by Lemma 3.2 the algebra $[L^{m_i}s_1^{k_1-1},\dots,s_{i-1}^{k_{i-1}-1}]$ satisfies an identity of degree $< n_{i-1}$. This implies that $l\leq n-2$ and finishes the proof of the lemma. □

Let E be the associative commutative F-algebra presented by the countable set of generators e_i, $i \geq 1$, and relations $e_i^2 = 0$, $i \geq 1$. Ordered products $e_\pi = e_{i_1} \cdots e_{i_r}$, $\pi = \{i_1 < i_2 < \cdots < i_r\}$ form a basis of the algebra E. Notice that we don't consider empty products and therefore the algebra E does not have 1.

We will start with a short *overview* of the rest of the proof of Theorm 1.1. The crucial role is played by the "linearized" Lie algebra $\widetilde{L} = L \otimes_F E$. In Section 3, we define divided polynomials: a generalization of usual Lie polynomials that make sense in the context of the Lie algebra $\widetilde{L}$. A divided polynomial is regular if it is not identically zero on any ideal $\widetilde{L^m} = L^m \otimes_F E$, $m \geq 1$. We use Lemma 3.4 to establish existence of a regular divided polynomial whose every value is divided ad-nilpotent of degree $k \geq 3$. Then we use Kostrikin-type arguments [16,17,42] to reduce k to 3.

In Section 4, we show how such regular divided polynomials give rise to a family of quadratic Jordan algebras. This result is new only for $p = 2$ or 3. For $p \geq 5$, it follows from [4]. Using structure theory of quadratic Jordan algebras [26], we establish existence of a regular Jordan polynomial, every value of which is an absolute zero divisor. These references are an essential (hidden) part of the proof. This Jordan polynomial gives rise to a regular divided polynomial whose every value is divided ad-nilpotent of degree 2, i.e. is a sandwich (see [17,20]).

In Sections 5 and 6, we further push the envelope and construct a regular divided polynomial whose every value generates a nilpotent ideal in an associative enveloping algebra to reduce the problem to the case when the associative enveloping algebra $\widetilde{A}$ satisfies a polynomial identity and finish the proof using structure theory of PI-algebras.

For an arbitrary (not necessarily associative) F-algebra A and its Lie algebra of derivations $D = \mathrm{Der}(A)$ denote $\widetilde{A} = A \otimes_F E$, $\widetilde{D} = D \otimes_F E$. Clearly, $\widetilde{D} \subseteq \mathrm{Der}(\widetilde{A})$. Let $i \in N$ and let $\widetilde{D}_i = \sum_\pi D \otimes e_\pi$, where the sum is taken over all ordered subsets of π that contain i. Clearly, $\widetilde{D}_i \lhd \widetilde{D}$, $\widetilde{D}_i^{\,2} = (0)$, $(AD_i)(AD_i) = (0)$, and $\widetilde{D} = \sum\limits_i \widetilde{D}_i$.

Let Ω be a finite family of elements of $\widetilde{D}$ such that:

(U1) every element $d \in \Omega$ lies in some ideal $\widetilde{D}_i$;

(U2) $[d_1, d_2] = 0$ for arbitrary elements $d_1, d_2 \in \Omega$.

Consider the following linear operator on $\widetilde{A}$:

$$U_k(\Omega) = \sum d_1 \cdots d_k,$$

where the summation runs over all k-element subsets of Ω.

We further define $U_0(\Omega) = \mathrm{Id}$. Clearly, $U_1(\Omega) = \sum_{d\in\Omega} d$.

We have $k!U_k(\Omega) = (\sum_{d\in\Omega} d)^k$ and if the characteristic of the field exceeds k then

$$U_k(\Omega) = \frac{1}{k!}\left(\sum_{d\in\Omega} d\right)^k.$$

Hence the operators U_k play the role of divided powers.

The following properties of the operators $U_k(\Omega)$ are straightforward (see also [39,41]).

Lemma 3.7.

(1)
$$(ab)U_m(\Omega) = \sum_{i=0}^{m} \big(aU_i(\Omega)\big)\big(bU_{m-i}(\Omega)\big)$$

for arbitrary elements $a, b \in \widetilde{A}$, $m \geq 0$*;*

(2) *The operator*

$$\widetilde{A} \to \widetilde{A}, \quad a \to \sum_{i=0}^{\infty} aU_i(\Omega)$$

is an automorphism of the algebra $\widetilde{A}$*. We remark that the sum* $\sum_{i=0}^{\infty} aU_i(\Omega)$ *is finite;*

(3) *For an element* $a \in \widetilde{A}$ *let* $R(a)$ *denote the operator of right multiplication by* a*. Then*

$$R(aU_m(\Omega)) = \sum_{i=0}^{m} (-1)^i U_i(\Omega)R(a)U_{m-i}(\Omega);$$

(4)
$$U_i(\Omega)U_j(\Omega) = \binom{i+j}{i} U_{i+j}(\Omega).$$

Remark. This lemma will be primarily applied to Lie algebras where the operator $R(a)$ of right multiplication by an element a is the adjoint operator $\mathrm{ad}(a)$.

An arbitrary element $a \in \widetilde{A}$ can be uniquely represented as $a = \sum a_\pi$ where $a_\pi \in A \otimes e_\pi$. We call it the standard decomposition of a.

Let X be a countable set and let I be an ideal of a Lie algebra L. Consider the set $\mathrm{Map}(X, \widetilde{I})$ of mappings $X \to \widetilde{I} = I \otimes_F E$. Consider also $M(I) = \mathrm{Map}(\mathrm{Map}(X, \widetilde{I}), \widetilde{I})$. In other words, if $f \in M(I)$ and we assign values from $\widetilde{I}$ to variables from X then f takes values in $\widetilde{I}$.

Let $\mathrm{Lie}\langle X\rangle$ be the free Lie algebra on the set of free generators X. Consider the free product $L * \mathrm{Lie}\langle X\rangle$. Let (X) be the ideal of the algebra $L * \mathrm{Lie}\langle X\rangle$ generated by X. An arbitrary element from (X) gives rise to an element from $M(I)$.

We will define a subset $U(I) \subseteq M(I)$ that we will call the set of divided polynomials defined on I:

(DP1) All elements from (X) lie in $U(I)$;

(DP2) suppose that a divided polynomial w does not depend on any variables except $x_1, \dots, x_r$. We represent this fact as $w = w(x_1, \dots, x_r)$. If $v_1, \dots, v_r \in U(I)$, then $w(v_1, \dots, v_r) \in U(I)$ as well. If $w = w(x_1, \dots, x_r)$ and $v_i = v_i(y_1, \dots, y_m)$, $1 \le i \le r$, are homogeneous divided polynomials of degrees $\deg_{x_i}(w)$, $\deg_{y_k}(v_i)$ in each variable, then $w(v_1, \dots, v_r)$ is a homogeneous divided polynomial of degrees $\sum_{i=1}^r \deg_{x_i}(w) \cdot \deg_{y_k}(v_i)$ in y_k, $1 \le k \le m$;

(DP3) let $w = w(x_1, \dots, x_r) \in U(I)$. Suppose that:

(i) for arbitrary elements $a, b, a_2, \dots, a_r \in \widetilde{I}$, we have

$$\big[w(a, a_2, \dots, a_r), w(b, a_2, \dots, a_r)\big] = 0;$$

(ii) w is linear in x_1, which means that

$$w(\alpha a + \beta b, a_2, \dots, a_r) = \alpha w(a, a_2, \dots, a_r) + \beta w(b, a_2, \dots, a_r)$$

for arbitrary $\alpha, \beta \in E$; $a, b, a_2, \dots, a_r \in \widetilde{I}$ and

$$w(I \otimes e_\pi, a_2, \dots, a_r) \subseteq I \otimes e_\pi + I \otimes e_\pi E.$$

Then for an arbitary $k \ge 0$ the function $w' = x_0 \operatorname{ad}_{x_1}^{[k]}(w)$, where w' is defined as

$$w'(a_0, a_1, \dots, a_r) = a_0 U_k(\Omega), \quad \Omega = \{\operatorname{ad}(w(a_{1\pi}, a_2, \dots, a_r)\}_\pi,$$

where $a_1 = \sum_\pi a_{1\pi}$ in the standard decomposition of the element a_1, is a divided polynomial defined on I.

If w is a homogeneous divided polynomial of degrees $\deg_{x_i}(w)$ in $x_1, \dots, x_r$, then w' is a homogeneous divided polynomial of degrees $1, \kappa \cdot \deg_{x_i}(w)$, $1 \le i \le r$, in $x_0, x_1, x_2, \dots, x_r$.

An element of $M(I)$ lies in $U(I)$ if and only if starting with elements from (X) and using rules (DP2)–(DP3), it can be shown to be a divided polynomial.

A divided polynomial from $U(I)$ is a homogeneous divided polynomial if and only if starting with homogeneous elements from (X) and applying rules (DP2)–(DP3) to homogeneous polynomials, it can be shown to be a homogeneous divided polynomial.

Let's recall the definition of a polynomial map of vector spaces. Let V, W be vector spaces over an infinite field F and let

$$f : \underbrace{V \times \cdots \times V}_{m} \to W, \quad (v_1, \dots, v_m) \to f(v_1, \dots, v_m) \in W.$$

If f is multilinear then it is said to be a polynomial of degrees $(1, 1, \ldots, 1)$ in $v_1, \ldots, v_m$. Let $d_i \geq 1$, $i = 1, \ldots, m$. We say that f is a homogeneous polynomial map of degrees $(d_1, \ldots, d_m)$ in $v_1, \ldots, v_m$ if:

(1) for $d_i = 1$ f is linear in v_i;

(2) for $d_i \geq 2$ we have

$$\begin{aligned} f(v_1, \ldots, v_{i-1}, v_i' + v_i'', v_{i+1}, \ldots, v_m) - f(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_m) \\ - f(v_1, \ldots, v_{i-1}, v_i'', v_{i+1}, \ldots, v_m) \\ = \sum_{k=1}^{d_i-1} f_k(v_1, \ldots, v_{i-1}, v_i', v_i'', v_{i+1}, \ldots, v_m), \end{aligned}$$

where $f_k(v_1, \ldots, v_{i-1}, v_i', v_i'', v_{i+1}, \ldots, v_m)$ is a homogeneous polynomial map of degrees $d_1, \ldots, d_{i-1}, k, d_i - k, d_{i+1}, \ldots, d_m$ in $v_1, \ldots, v_{i-1}, v_i', v_i'', v_{i+1}, \ldots, v_m$.

Now recall the definition of a full linearization of a homogeneous polynomial map f of degrees $d_1 \geq 1, \ldots, d_m \geq 1$ in $v_1, \ldots, v_m$. For every $1 \leq i \leq m$ choose d_i elements $v_{i1}, \ldots, v_{id_i} \in V$. Let $\pi \subseteq \{v_{i1}, \ldots, v_{id_i}\}$ be a nonempty subset. Denote

$$f_\pi = f\Big(v_1, \ldots, v_{i-1}, \sum_{v\in\pi} v, v_{i+1}, \ldots, v_m\Big).$$

The mapping

$$\Delta_i(f) = \sum_{\substack{\varnothing\neq\pi \\ \subseteq\{v_{i1},\ldots,v_{id_i}\}}} (-1)^{(d_i-|\pi|)} f_\pi$$

is called the linearization of f with respect to v_i. The mapping

$$\Delta_i(f)(v_1, \ldots, v_{i-1}, v_{i1}, \ldots, v_{id_i}, v_{i+1}, \ldots, v_m)$$

is multilinear in $v_{i1}, \ldots, v_{id_i}$.

Consecutively applying linearizations with respect to all variables, we get the full linearization $\widetilde{f} : \underbrace{V \times \cdots \times V}_{\sum_{i=1}^m d_i} \to W$. Clearly $\widetilde{f}$ is a multilinear map.

Lemma 3.8. *For an arbitrary homogeneous divided polynomial $w \in U(I)$*

(1) *the full linearization $\widetilde{w}$ of w lies in $L * \mathrm{Lie}\langle X\rangle$;*

(2) *the span of all values of w on $\widetilde{I}$ is equal to the span of all values of $\widetilde{w}$ on $\widetilde{I}$.*

Proof. (1) We will use induction on the number of steps (DP2)–(DP3) needed to construct the divided polynomial w.

If $v_1, \ldots, v_r, w$ are homogeneous divided polynomials, then the full linearization of $w(v_1, \ldots, v_r)$ is a linear combination of values $\widetilde{w}(\widetilde{v_1}, \ldots, \widetilde{v_r})$ in appropriate variables.

Let $w = x_0 \operatorname{ad}_{x_1}^{[k]}(v)$, where $v = v(x_1, \ldots, x_r)$ in a homogeneous divided polynomial satisfying the conditions (DP3)(i), (ii). Since we can linearize variables in an arbitrary order, let's start with the variable x_1. Then

$$\Delta_{x_1}(w)(x_0, y_1, \ldots, y_k, x_2, \ldots, x_r) = x_0 \operatorname{ad}(v(y_1, x_2, \ldots, x_r)) \cdots \operatorname{ad}(v(y_k, x_2, \ldots, x_r)).$$

This completes the proof of assertion (1).

(2) We will prove part (2) of the lemma in a slightly more general context of polynomial maps of spaces. Consider again vector spaces V, W and a homogeneous polynomial map

$$f : \underbrace{V \times \cdots \times V}_{m} \to W, \quad (x_1, \ldots, x_m) \to f(x_1, \ldots, x_m) \in W,\ x_i \in V.$$

Let f have degrees $d_1 \geq 1, \ldots, d_m \geq 1$ with respect to $x_1, \ldots, x_m$. Choose $x_i', x_i'' \in V$, $1 \leq i \leq m$. Consider

$$\begin{aligned} f(x_1, \ldots, x_{i-1}, x_i' + x_i'', x_{i+1}, \ldots, x_m) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_m) \\ - f(x_1, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_m) \\ = \sum_{j=1}^{d_1-1} f_j(x_1, \ldots, x_{i-1}, x_i', x_i'', x_{i+1}, \ldots, x_m), \end{aligned}$$

where the summand f_j has degree j in x_i' and degree $d_i - j$ in x_i''. The homogeneous polynomial mappings f_j are called partial linearizations of f.

Consider the finite system $\mathcal{F}$ of homogeneous polynomial maps from V to W that are obtained from f by repeated partial linearizations. Let $\Omega \subset V$ be a family of elements with the following property:

if $g(x_1, \ldots, x_r) \in \mathcal{F}$ and g has degree ≥ 2 in x_i, then for an arbitrary element $v \in \Omega$ we have $g(\underbrace{V, \ldots, V}_{i-1}, v, \underbrace{V, \ldots, V}_{r-i}) = (0)$. $(*)$

We claim that for an arbitrary element $g(x_1, \ldots, x_r) \in \mathcal{F}$,

$$g(\operatorname{span}\Omega, \ldots, \operatorname{span}\Omega) \subseteq \operatorname{span} \widetilde{f}(\Omega, \ldots, \Omega).$$

Applying this inclusion to $f = w$, $\Omega = \bigcup_{i \geq 1}(I \otimes e_i + I \otimes e_i E)$, we will prove part (2) of Lemma 3.8.

If g is multilinear, then $g = \widetilde{f}$. In any case, without loss of generality, we assume that the claim is true for all partial linearizations of g. But modulo partial linearizations the mapping g is multilinear. More precisely,

$$g(\operatorname{span}\Omega, \dots, \operatorname{span}\Omega) \subseteq \operatorname{span}(g(v_1, \dots, v_r), v_i \in \Omega) + \sum g'(\operatorname{span}\Omega, \dots, \operatorname{span}\Omega),$$

where g' are partial linearizations of g. Since g has degree ≥ 2 with respect to at least one variable, we conclude that $g(v_1, \dots, v_r) = 0$. In particular,

$$\operatorname{span} f(\operatorname{span}\Omega, \dots, \operatorname{span}\Omega) = \operatorname{span}(\widetilde{f}(v_1, v_2, \dots), v_i \in \Omega).$$

If f is a homogeneous divided polynomial and $\Omega = \{a \otimes e_i, a \in L, i \geq 1\}$, then condition $(*)$ is clearly satisfied, which completes the proof of assertion (2). □

The following lemma is a linearization version of the celebrated Kostrikin Lemma ([16], [17, Lemma 2.1.1]).

Lemma 3.9. *Let L be a Lie algebra. Let $\Omega \subset \operatorname{Der}(L) \otimes E$ be a finite family of elements satisfying the conditions (U1), (U2). Suppose that $m \geq 1$ and for an arbitrary $k \geq m$, we have $U_k(\Omega) = 0$.*

(1) *Let $m \geq 2$. Then for arbitary elements $a, b \in L$, we have*

$$\left[aU_{m-1}(\Omega), bU_{m-1}(\Omega)\right] = 0.$$

(2) *Now suppose that $m \geq 4$. Let $a \in \widetilde{L}$, $a = \sum_{\pi} a_\pi$ be a standard decomposition, and $\Omega' = \{a_\pi U_{m-1}(\Omega)\}_\pi$. Then $U_k(\Omega') = 0$ for $k \geq m - 1$.*

Proof. (1) We have $2m - 2 \geq m$. Adjoint operators are right multiplications in Lie algebras. Hence Lemma 3.7(3) is applicable. By Lemma 3.7(3),

$$0 = \operatorname{ad}(bU_{2m-2}(\Omega)) = \sum_{i+j=2m-2} \pm U_i(\Omega) \operatorname{ad}(b) U_j(\Omega).$$

It implies

$$U_{m-1}(\Omega) \operatorname{ad}(b) U_{m-1}(\Omega) = \sum_{\substack{i \geq m \\ \text{or } j \geq m}} \pm U_i(\Omega) \operatorname{ad}(b) U_j(\Omega) = 0.$$

By Lemma 3.7(3), (4), we have

$$\begin{aligned}\left[aU_{m-1}(\Omega), bU_{m-1}(\Omega)\right] &= aU_{m-1}(\Omega) \operatorname{ad}\left(bU_{m-1}(\Omega)\right) \\ &= aU_{m-1}(\Omega) \operatorname{ad}(b) U_{m-1}(\Omega) \\ &= 0,\end{aligned}$$

which completes the proof of the assertion (1).

(2) We will show that

$$\mathrm{ad}(a_1 U_{m-1}(\Omega)) \cdots \mathrm{ad}(a_k U_{m-1}(\Omega)) = 0$$

for arbitrary elements $a_1, \ldots, a_k \in L$, $k \geq m-1$. Without loss of generality we will assume $k = m-1$.

By Lemma 3.7(3) and (4) the left hand side is a linear combination of operators

$$U_{i_0}(\Omega)\,\mathrm{ad}(a_1)U_{i_1}(\Omega) \cdots U_{i_{m-2}}(\Omega)\,\mathrm{ad}(a_{m-1})U_{i_{m-1}}(\Omega),$$

where $0 \leq i_0, i_1, \ldots, i_{m-1} \leq m-1$ and $i_0 + i_1 + \cdots + i_{m-1} = (m-1)^2$.

Suppose that $U_{i_0}(\Omega)\,\mathrm{ad}(a_1) \cdots \mathrm{ad}(a_{m-1})U_{i_{m-1}}(\Omega) \neq 0$ and the m-tuple $(i_0, i_1, \ldots, i_{m-1})$ is lexicographically maximal with this property.

We claim that none of the indices $i_0, i_1, \ldots, i_{m-1}$ are equal to 0. Indeed, if one of the indices is equal to 0, then all other indices have to be equal to $m-1$. Since $m \geq 4$ it follows that there exists t, $0 \leq t \leq m-1$, such that $i_t = i_{t+1} = m-1$. Now from (1) it follows that $U_{i_t}(\Omega)\,\mathrm{ad}(a_{t+1})U_{i_{t+1}}(\Omega) = U_{m-1}(\Omega)\,\mathrm{ad}(a_{t+1})U_{m-1}(\Omega) = 0$, a contradiction.

Since $(m-2)m < (m-1)^2$ it follows that at least one index i_t, $0 \leq t \leq m-1$, is equal to $m-1$. All of the indices $i_1, \ldots, i_{m-1}$ are smaller than $m-1$. Indeed, suppose that $i_t = m-1$, $1 \leq t \leq m-1$. We have $i_{t-1} \geq 1$ by the above. Now Lemma 3.7(3) implies

$$0 = \mathrm{ad}(a_t U_{i_{t-1}+i_t}(\Omega)) = \sum_{i+j=i_{t-1}+i_t} \pm U_i(\Omega)\,\mathrm{ad}(a_t)U_j(\Omega)$$

and therefore

$$U_{i_{t-1}}(\Omega)\,\mathrm{ad}(a_t)U_{i_t}(\Omega) = \sum_{\substack{i > i_t \\ j < m-1}} \pm U_i(\Omega)\,\mathrm{ad}(a_t)U_j(\Omega),$$

which contradicts lexicographical maximality of $(i_0, \ldots, i_{m-1})$.

We have proved that $i_0 = m-1$, $i_1 = i_2 = \cdots = i_{m-1} = m-2$. Now our aim will be to show that

$$U_{m-1}(\Omega)\,\mathrm{ad}(a_1)U_{m-2}(\Omega) = 0.$$

Since $(m-1) + (m-3) \geq m$, Lemma 3.7(3) implies that

$$U_{m-1}(\Omega)\,\mathrm{ad}(a_1)U_{m-3}(\Omega) - U_{m-2}(\Omega)\,\mathrm{ad}(a_1)U_{m-2}(\Omega) \\ + U_{m-3}(\Omega)\,\mathrm{ad}(a_1)U_{m-1}(\Omega) = 0.$$

Multiplying the left hand side by $U_1(\Omega)$ on the right and taking into account Lemma 3.7(4), we get

$$(m-2)U_{m-1}(\Omega)\,\mathrm{ad}(a_1)U_{m-2}(\Omega) - (m-1)U_{m-2}(\Omega)\,\mathrm{ad}(a_1)U_{m-1}(\Omega) = 0.$$

On the other hand Lemma 3.7(3) implies that

$$U_{m-1}(\Omega)\,\mathrm{ad}(a_1)U_{m-2}(\Omega) - U_{m-2}(\Omega)\,\mathrm{ad}(a_1)U_{m-1}(\Omega) = 0.$$

The system of equations implies

$$U_{m-1}(\Omega)\,\mathrm{ad}(a_1)U_{m-2}(\Omega) = 0,$$

which completes the proof of the lemma. □

Definition 3.10. We say that a divided polynomial $w(x_1, \dots, x_r)$ defined on L^s, $s \geq 1$, is *regular* if for an arbitrary $i \geq s$ we have $w(\widetilde{L}^i, \dots, \widetilde{L}^i) \neq (0)$.

By Lemma 3.8(2), a homogeneous divided polynomial w is regular if and only if its full linearization is regular.

Lemma 3.11. *There exist integers $m \geq 1, N \geq 1$ and a regular homogeneous divided polynomial $w(x_1, \dots, x_r)$ defined on L^m such that w satisfies the conditions in (DP3) and $x_0\,\mathrm{ad}_{x_1}^{[t]}(w) = 0$ holds identically on $\widetilde{L}^m$ for all $t \geq N$.*

Proof. Consider the elements $c_1, \dots, c_r \in S$ and integers $m \geq 1$, $N \geq 1$ of Lemma 3.4. By property (2), for an $i \geq m$ the subspace $L_i = [L^i, c_1, \dots, c_r]$ is a subalgebra of L. By (3) this subalgebra is nilpotent, say, of degree $d(i)$, $d(m) \geq d(m+1) \geq \cdots$. This sequence stabilizes at some step, $d = d(k) = d(k+1) = \cdots$. Thus $L_k^d = (0)$ and $L_i^{d-1} \neq (0)$ for any $i \geq m$. Let

$$w(x_1, \dots, x_{d-1}) = \big[[x_1, c_1, \dots, c_r], [x_2, c_1, \dots, c_r], \dots \\ \dots, [x_{d-1}, c_1, \dots, c_r]\big] \in L * \mathrm{Lie}\langle X \rangle.$$

The divided polynomial w is regular and linear in x_1. For arbitrary elements $a_2, \dots, a_{d-1} \in L^k$ we have

$$\big[w(L^k, a_2, \dots, a_{d-1}), w(L^k, a_2, \dots, a_{d-1})\big] = (0)$$

because the left hand side lies in L_k^d. Hence the divided polynomial w satisfies the condition (DP3). Therefore the divided polynomial $x_0\,\mathrm{ad}_{x_1}^{[t]}(w)$ is defined on L^k for any $t \geq 1$. For $t \geq N$ the polynomial $x_0\,\mathrm{ad}_{x_1}^{[t]}(w)$ is identically zero on $\widetilde{L}^k$ by Lemma 3.4(3). This finishes the proof of the lemma. □

Let $q \geq 1$ be a minimal integer with the following property: there exists an $m \geq 1$ and a regular homogeneous divided polynomial $w = w(x_1, \dots, x_r)$ defined on L^m, linear in x_1, such that:

(i) for arbitrary elements $a, b, a_2, \dots, a_r \in \widetilde{L}^m$ we have

$$\big[w(a, a_2, \dots, a_r), w(b, a_2, \dots, a_r)\big] = 0;$$

(ii) $\widetilde{L}^m\,\mathrm{ad}_{x_1}^{[t]}(w) = (0)$ holds identically on $\widetilde{L}^m$ for all $t \geq q$. Clearly, $q \leq N$.

Lemma 3.12. $q \le 3$.

Proof. Suppose that $q \ge 4$. Consider the divided polynomial $v(x_0, x_1, \dots, x_r) = x_0 \operatorname{ad}_{x_1}^{[q-1]}(w)$ defined on L^m. In view of the minimality of q, the divided polynomial v is regular.

By Lemma 3.9(1) for arbitrary elements $a, b \in \widetilde{L}^m$; $a_1, \dots, a_r \in \widetilde{L}^m$ we have

$$\big[v(a, a_1, \dots, a_r), v(b, a_1, \dots, a_r)\big] = 0.$$

We proved that the divided polynomial $y \operatorname{ad}_{x_0}^{[q-1]}(v(x_0, \dots, x_r))$ is defined on $\widetilde{L}^m$. If $q \ge 4$, then by Lemma 3.9(2), this divided polynomial is identically zero, which contradicts the minimality of q and finishes the proof of the lemma. □

Lemma 3.13. *Let L be a Lie algebra. Let $\Omega \subset \widetilde{L}$ be a finite family of elements such that* $\operatorname{ad}(\Omega)$ *satisfies the assumptions (U1), (U2). Suppose that* $U_2(\operatorname{ad}(\Omega)) = 0$. *Then* $a = \sum_{b\in\Omega} b$ *is a sandwich of the Lie algebra* $\widetilde{L}$.

Proof. We have $\operatorname{ad}(a)^2 = U_1(\operatorname{ad}(\Omega))U_1(\operatorname{ad}(\Omega)) = 2U_2(\operatorname{ad}(\Omega)) = 0$. By Lemma 3.7(3) for an arbitrary element $c \in \widetilde{L}$ we have

$$\operatorname{ad}(cU_2(\operatorname{ad}(\Omega))) = \operatorname{ad}(c)U_2(\operatorname{ad}(\Omega)) - U_1(\operatorname{ad}(\Omega)) \operatorname{ad}(c) U_1(\operatorname{ad}(\Omega)) \\ + U_2(\operatorname{ad}(\Omega)) \operatorname{ad}(c),$$

which implies $\operatorname{ad}(b) \operatorname{ad}(c) \operatorname{ad}(b) = 0$ and completes the proof of the lemma. □

In what follows, we will use the subsequent lemma.

Lemma 3.14. *Let L be a Lie algebra. Let $\Omega = \{a_1, \dots, a_n\} \subset L$ be a finite family of elements. Let $\Omega = \Omega_1 \dot\cup \cdots \dot\cup \Omega_s = \Omega'_1 \dot\cup \cdots \dot\cup \Omega'_t$ be two disjoint decompositions. Denote*

$$b_k = \sum_{a_i \in \Omega_k} a_i, \quad c_\ell = \sum_{a_j \in \Omega'_\ell} a_j.$$

Denote also

$$\operatorname{ad}[\Omega, \Omega] = \operatorname{Span}(\operatorname{ad}[a_i, a_j], 1 \le i, j \le n).$$

Suppose that if a_i, a_j lie in the same Ω_k or in the same Ω'_ℓ, then $\operatorname{ad}(a_i)\operatorname{ad}(a_j) = 0$. *Then* $\sum \operatorname{ad}(b_{k_1}) \operatorname{ad}(b_{k_2}) = \sum \operatorname{ad}(c_{\ell_1}) \operatorname{ad}(c_{\ell_2}) \mod \operatorname{ad}[\Omega, \Omega]$, *where both sums run over all 2-element subsets $\{k_1, k_2\} \subseteq \{1, \dots, s\}$ and $\{\ell_1, \ell_2\} \subseteq \{1, 2, \dots, t\}$, respectively.*

Proof. It is easy to see that

$$\sum \operatorname{ad}(b_{k_1}) \operatorname{ad}(b_{k_2}) = \sum \operatorname{ad}(c_{\ell_1}) \operatorname{ad}(c_{\ell_2}) \\ = \sum_{1\le i<j\le n} \operatorname{ad}(a_i) \operatorname{ad}(a_j) \mod \operatorname{ad}[\Omega, \Omega].$$

□

4. Jordan algebras

Let J be a vector space over a field F (of arbitrary characteristic) with two quadratic mappings $J \to J$, $x \to x^2$, and $Q : J \to \mathrm{End}_F(J)$. For elements $x, y, z \in J$ denote

$$x \circ y = (x + y)^2 - x^2 - y^2, \quad \{x, y, z\} = y(Q(x + z) - Q(x) - Q(z)).$$

Following K. McCrimmon [24] we say that $(J, x \to x^2, Q)$ is a quadratic Jordan algebra if it satisfies the identities

(M1) $\{x, x, y\} = x^2 \circ y$;

(M2) $(yQ(x)) \circ x = (y \circ x)Q(x)$;

(M3) $x^2Q(x) = (x^2)^2$;

(M4) $x^2Q(y)Q(x) = (yQ(x))^2$;

(M5) $Q(x^2) = Q(x)^2$;

(M6) $Q(yQ(x)) = Q(x)Q(y)Q(x)$;

and all their partial linearizations.

We reiterate the assumption made at the beginning of Section 3: all algebras are considered over an infinite field F of characteristic $p > 0$.

Let $w' = w'(x_1, \ldots, x_{r-1})$ be a regular homogeneous divided polynomial defined on L^m such that w' is linear in x_1 and satisfies all the assumptions of (DP3). Moreover, assume that $\widetilde{L^m}\, \mathrm{ad}_{x_1}^{[k]}(w') = 0$ holds identically for $k \geq 3$. If there exists $s \geq m$ such that $x_r\, \mathrm{ad}_{x_1}^{[2]}(w')$ holds identically on $\widetilde{L^s}$ then our goal of constructing a sandwich valued regular homogeneous divided polynomial has been achieved. We assume therefore that the divided polynomial $w(x_1, \ldots, x_r) = x_r\, \mathrm{ad}_{x_1}^{[2]}(w')$ is regular.

The divided polynomial w satisfies both assumptions of (DP3): it is clearly linear in x_r and for arbitrary elements $a, b, a_1, \ldots, a_{r-1} \in \widetilde{L^m}$, we have

$$\left[a\, \mathrm{ad}_{x_1}^{[2]}\, w'(a_1, \ldots, a_{r-1}), b\, \mathrm{ad}_{x_1}^{[2]}\, w'(a_1, \ldots, a_{r-1})\right] = 0$$

by Lemma 3.9(1).

Choose $a_1, \ldots, a_r \in \widetilde{L^m}$ and denote $a' = w'(a_1, \ldots, a_{r-1})$, $a = w(a_1, \ldots, a_r)$. Denote

$$\mathrm{ad}^{[k]}(a') = (\mathrm{ad}_{x_1}^{[k]}\, w')(a_1, \ldots, a_{r-1}).$$

If $a_r = \sum_\pi a_{r\pi}$ is the standard decomposition, then we denote

$$\mathrm{ad}^{[k]}(a) = (\mathrm{ad}_{x_r}^{[k]}\, w)(a_1, \ldots, a_r) = \sum \mathrm{ad}(a_{r\pi_1}\, \mathrm{ad}^{[2]}(a')) \cdots \mathrm{ad}(a_{r\pi_k}\, \mathrm{ad}^{[2]}(a')),$$

where the sum runs over all k-element sets $(\pi_1, \ldots, \pi_k)$.

Notice that $\mathrm{ad}^{[k]}(a) = 0$ for $k \geq 3$. Indeed, by Lemma 3.7(3), the equalities $\mathrm{ad}(a_{r\pi_i}\, \mathrm{ad}^{[3]}(a')) = 0$ and $\mathrm{ad}(a_{r\pi_i}\, \mathrm{ad}^{[4]}(a')) = 0$ imply

$$\mathrm{ad}^{[2]}(a')\,\mathrm{ad}(a_{r\pi_i})\,\mathrm{ad}(a') = \mathrm{ad}(a')\,\mathrm{ad}(a_{r\pi_i})\,\mathrm{ad}^{[2]}(a'), \tag{$*$}$$

$$\mathrm{ad}^{[2]}(a')\,\mathrm{ad}(a_{r\pi_i})\,\mathrm{ad}^{[2]}(a') = 0, \tag{$**$}$$

respectively.

We have

$$\mathrm{ad}^{[k]}(a) = \sum \pm\, \mathrm{ad}^{[i_1]}(a')\,\mathrm{ad}(a_{r\pi_1})\,\mathrm{ad}^{[i_2]}(a')\cdots \mathrm{ad}(a_{r\pi_k})\,\mathrm{ad}^{[i_{k+1}]}(a'),$$

where $0 \leq i_1, \dots, i_{k+1} \leq 2$, $i_1 + \cdots + i_{k+1} = 2k$. If at least one i_μ, $1 \leq \mu \leq k+1$, is equal to 0, then all other i_ν, $\nu \neq \mu$, are equal to 2. In this case, the product is equal to 0 by ($**$). Suppose that all $i_\mu \neq 0$. Then all i_μ, except two, are equal to 2. These two are equal to 1. Since $k + 1 \geq 4$, we have at least two degrees i_μ that are equal to 2. Using ($*$), we can move two operators $\mathrm{ad}^{[2]}(a')$ together and then use ($**$).

Consider the subspaces

$$K'_a = \{x \in \widetilde{L}^m \mid x\,\mathrm{ad}^{[2]}(a) = 0\} \quad \text{and} \quad K_a = \sum_i (L^m \otimes e_i + \widetilde{L}^m e_i) \cap K'_a$$

and the factor space $J_a = \widetilde{L}^m / K_a$.

Let $x = \sum_\pi x_\pi$ be the standard decomposition of an element $x \in \widetilde{L}^m$. Define

$$x^2 = a \sum \mathrm{ad}(x_{\pi_1})\,\mathrm{ad}(x_{\pi_2}) + K_a,$$

where the sum runs over all 2-element sets (π_1, π_2). The order of the factors in $\mathrm{ad}(x_{\pi_1})\,\mathrm{ad}(x_{\pi_2})$ is irrelevant since $[a, \widetilde{L}^m] \subseteq K_a$. Define further

$$yQ(x) = y\,\mathrm{ad}^{[2]}(a) \sum \mathrm{ad}(x_{\pi_1})\,\mathrm{ad}(x_{\pi_2}) + K_a$$

Again, the order of factors in $\mathrm{ad}(x_{\pi_1})\,\mathrm{ad}(x_{\pi_2})$ is irrelevant since

$$y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(\widetilde{L}^m) \subseteq K_a.$$

Linearizing the above operations, we get $x \circ y = [[a, x], y] + K_a$ for $x, y \in J_a$, and $\{x, y, z\} = [y\,\mathrm{ad}^{[2]}(a), x, z] + K_a$ for $x, y, z \in J$.

Lemma 4.1. (1) *The element* $u = y_1\,\mathrm{ad}^{[i_1]}(a)\,\mathrm{ad}(y_2)\,\mathrm{ad}^{[i_2]}(a)\cdots \mathrm{ad}(y_s)\,\mathrm{ad}^{[i_s]}(a)$, *where* $y_1, \dots, y_s \in \widetilde{L}^m$; $i_1 + \cdots + i_s \geq s + 2$, *is equal to 0;*

(2) *An operator* $\mathrm{ad}^{[i_1]}(a)\,\mathrm{ad}(y_1)\cdots \mathrm{ad}(y_s)\,\mathrm{ad}^{[i_{s+1}]}(a)$, *for* $y_1, \dots, y_s \in \widetilde{L}^m$; $i_1 + \cdots, i_{s+1} \geq s + 3$, *is equal to 0 on* $\widetilde{L}^m$*;*

(3) *Consider an operator*

$$v = \mathrm{ad}^{[i_1]}(a)\,\mathrm{ad}(y_1)\cdots\mathrm{ad}(y_s)\,\mathrm{ad}^{[i_{s+1}]}(a),$$

where $y_1,\dots,y_s \in \widetilde{L}^m$, $i_1+\cdots+i_{s+1} \geq s+2$. *Suppose that there exists* $1 \leq k \leq s-2$ *such that* $i_{k+1} = i_{k+2} = 0$*, in other words*

$$v = \sum \cdots \mathrm{ad}(y_k)\,\mathrm{ad}(y_{k+1})\,\mathrm{ad}(y_{k+2})\cdots .$$

Then v *is zero on* $\widetilde{L}^m$.

Proof. To prove (1) we will use induction on s. If $s = 1$ then $u = y_1\,\mathrm{ad}^{[i_1]}(a)$, $i_1 \geq 3$. Hence $u = 0$.

Let $s \geq 2$. If $i_1 \geq 3$ then again $u = 0$. If $i_1 \leq 1$ then choosing $y_1' = y_1\,\mathrm{ad}^{[i_1]}(a)\,\mathrm{ad}(y_2)$ we can use the induction assumption. Therefore we let $i_1 = 2$. If $i_2 = 0$ then choosing $y_1' = i_1\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(y_2)\,\mathrm{ad}(y_3)$ we again use the induction assumption. Let $i_2 = 1$. Then by Lemma 3.7(3) we have

$$\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(y_2)\,\mathrm{ad}(a) = \mathrm{ad}(a)\,\mathrm{ad}(y_2)\,\mathrm{ad}^{[2]}(a),$$

the case that has already been considered. Finally, if $i_2 = 2$ then

$$\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(y_2)\,\mathrm{ad}^{[2]}(a) = 0,$$

again by Lemma 3.7(3), which finishes the proof of part (1).

To prove (2) we consider the element

$$y_0\,\mathrm{ad}^{[i_1]}(a)\,\mathrm{ad}(y_1)\cdots\mathrm{ad}(y_s)\,\mathrm{ad}^{[i_{s+1}]}(a)$$

and use part (1).

Consider now an operator $v = \mathrm{ad}^{[i_1]}(a)\,\mathrm{ad}(y_1)\cdots\mathrm{ad}(y_s)\,\mathrm{ad}^{[i_{s+1}]}(a)$ and suppose that

$$v = v'\,\mathrm{ad}(y_k)\,\mathrm{ad}(y_{k+1})\,\mathrm{ad}(y_{k+2})v'',$$

where

$$v' = \mathrm{ad}^{[i_1]}(a)\,\mathrm{ad}(y_1)\cdots\mathrm{ad}(y_{k-1})\,\mathrm{ad}^{[i_k]}(a),$$
$$v'' = \mathrm{ad}^{[i_{k+3}]}(a)\cdots\mathrm{ad}^{[i_{s+1}]}(a).$$

By part (2) if $v \neq 0$ on $\widetilde{L}^m$ then $i_1+\cdots+i_k \leq (k-1)+2 = k+1$, $i_{k+1}+\cdots+i_{s+1} \leq (s-k-2)+2 = s-k$. However, $i_1+\cdots+i_k+i_{k+3}+\cdots+i_{s+1} \geq s+2$, a contradiction that finishes the proof of the lemma. □

Lemma 4.2. *Let* Ω *be a finite family of commuting elements from* $\widetilde{L}$ *such that every element from* Ω *lies in one of the ideals* $L \otimes e_\pi + \widetilde{L}e_\pi$. *Denote for brevity* $U_k(\mathrm{ad}(\Omega)) = \mathrm{ad}^{[k]}(\Omega)$ *and suppose that* $\mathrm{ad}^{[3]}(\Omega) = \mathrm{ad}^{[4]}(\Omega) = 0$. *Then for arbitrary elements* $y_1, y_2 \in \widetilde{L}$ *we have*

$$\mathrm{ad}(y_1\,\mathrm{ad}^{[2]}(\Omega))\,\mathrm{ad}(y_2\,\mathrm{ad}^{[2]}(\Omega)) = \mathrm{ad}^{[2]}(\Omega)\,\mathrm{ad}(y_1)\,\mathrm{ad}(y_2)\,\mathrm{ad}^{[2]}(\Omega).$$

Proof. By Lemma 3.7(3) we have

$$\operatorname{ad}(y_i \operatorname{ad}^{[2]}(\Omega)) = \operatorname{ad}(y_i)\operatorname{ad}^{[2]}(\Omega) - \operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_i)\operatorname{ad}^{[1]}(\Omega) + \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_i),$$

$i = 1, 2$. By Lemma 3.7(4) we have also

$$\operatorname{ad}^{[1]}(\Omega)\operatorname{ad}^{[1]}(\Omega) = 2\operatorname{ad}^{[2]}(\Omega),$$
$$\operatorname{ad}^{[1]}(\Omega)\operatorname{ad}^{[2]}(\Omega) = \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}^{[1]}(\Omega) = 3\operatorname{ad}^{[3]}(\Omega) = 0.$$

Again by Lemma 3.7(3) we have

$$\begin{aligned}\operatorname{ad}(y_i \operatorname{ad}^{[3]}(\Omega)) &= \operatorname{ad}(y_i)\operatorname{ad}^{[3]}(\Omega) - \operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_i)\operatorname{ad}^{[2]}(\Omega) \\ &\quad + \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_i)\operatorname{ad}^{[1]}(\Omega) - \operatorname{ad}^{[3]}(\Omega)\operatorname{ad}(y_i) = 0,\end{aligned}$$

which implies

$$\operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_i)\operatorname{ad}^{[2]}(\Omega) = \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_i)\operatorname{ad}^{[1]}(\Omega).$$

Similarly, $\operatorname{ad}(y_i \operatorname{ad}^{[4]}(\Omega)) = 0$ implies $\operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_1)\operatorname{ad}^{[2]}(\Omega) = 0$. Hence,

$$\begin{aligned}&\operatorname{ad}(y_1 \operatorname{ad}^{[2]}(\Omega))\operatorname{ad}(y_2 \operatorname{ad}^{[2]}(\Omega)) \\ &\quad = (\operatorname{ad}(y_i)\operatorname{ad}^{[2]}(\Omega) - \operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_1)\operatorname{ad}^{[1]}(\Omega) \\ &\qquad + \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_1))(\operatorname{ad}(y_2)\operatorname{ad}^{[2]}(\Omega) \\ &\qquad - \operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_2)\operatorname{ad}^{[1]}(\Omega) + \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_2)) \\ &\quad = -\operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_1)\operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_2)\operatorname{ad}^{[2]}(\Omega) \\ &\qquad + 2\operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_1)\operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_2)\operatorname{ad}^{[1]}(\Omega) \\ &\qquad + \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_1)\operatorname{ad}(y_2)\operatorname{ad}^{[2]}(\Omega) \\ &\qquad - \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_1)\operatorname{ad}^{[1]}(\Omega)\operatorname{ad}(y_2)\operatorname{ad}^{[1]}(\Omega) \\ &\quad = \operatorname{ad}^{[2]}(\Omega)\operatorname{ad}(y_1)\operatorname{ad}(y_2)\operatorname{ad}^{[2]}(\Omega),\end{aligned}$$

which proves the lemma. □

Lemma 4.3. (1) *The operations $x \to x^2$ and Q are well defined on J_a;*

(2) *Let $f : \widetilde{L}^m \times \cdots \times \widetilde{L}^m \to \widetilde{L}^m$ be a homogeneous polynomial map, and let $\widetilde{f}(x_1, \ldots, x_n)$ be its full linearization. Suppose that*

$$\widetilde{f}(L^m \otimes e_i + \widetilde{L}^m e_i, \widetilde{L}^m, \ldots, \widetilde{L}^m) \subseteq L^m \otimes e_i + \widetilde{L}^m e_i$$

for all i.

Then, if an arbitrary value of f lies in K'_a, then an arbitrary value of f lies in K_a.

Proof. (1) Choose arbitrary elements $x, y \in \widetilde{L}^m$ and $z', z \in K_a$. We need to show that $(y+z')Q(x+z) = yQ(x)$ and $(x+z)^2 = x^2$. Let $x = \sum_\pi x_\pi$ be the standard decomposition of the element x. We have $z'Q(x) = z' \operatorname{ad}^{[2]}(a) \sum \operatorname{ad}(x_{\pi_1}) \operatorname{ad}(x_{\pi_2}) + K_a = 0$ since $z' \in K_a \subseteq K'_a$. Hence $(y+z')Q(x+z) = yQ(x+z)$. Furthermore, it is easy to see that

$$yQ(x+z) = yQ(x) + y \operatorname{ad}^{[2]}(a) \operatorname{ad}(x) \operatorname{ad}(z) + yQ(z) \mod K_a.$$

By Lemma 4.2, for an arbitrary standard component x_π of the element x, we have

$$\operatorname{ad}^{[2]}(a) \operatorname{ad}(x_\pi) \operatorname{ad}(z) \operatorname{ad}^{[2]}(a) = \operatorname{ad}(x_\pi \operatorname{ad}^{[2]}(a)) \operatorname{ad}(z \operatorname{ad}^{[2]}(a)) = 0,$$

since $z \in K'_a$. Hence $y \operatorname{ad}^{[2]}(a) \operatorname{ad}(x_\pi) \operatorname{ad}(z) \in K_a$ and $y \operatorname{ad}^{[2]}(a) \operatorname{ad}(x) \operatorname{ad}(z) \in K_a$.

Let us show that $yQ(z) = 0$. We have $z = z_1 + \cdots + z_s$, where

$$z_i \in (L^m \otimes e_i + \widetilde{L}^m e_i) \cap K'_a.$$

Let $z_i = \sum_\pi z_{i\pi}$ be the standard decomposition of the element z_i. Then $z = \sum_\pi z_\pi$, $z_\pi = \sum_i z_{i\pi}$, is the standard decomposition of the element z. Consider the family of elements $\Omega = \{z_{i\pi}\}_{i,\pi}$ and two decompositions $\Omega = \bigcup \Omega_i$, $\Omega_i = \{z_{i\pi}\}_\pi$, and $\Omega = \bigcup_\pi \Omega'_\pi$, $\Omega'_\pi = \{z_{i\pi}\}_i$. By Lemma 3.14, we have

$$\begin{aligned} &y \operatorname{ad}^{[2]}(a) \sum \operatorname{ad}(z_{\pi_1}) \operatorname{ad}(z_{\pi_2}) \\ &\quad = y \operatorname{ad}^{[2]}(a) \sum_{1 \le i < j \le s} \operatorname{ad}(z_i) \operatorname{ad}(z_j) \mod y \operatorname{ad}^{[2]}(a) \operatorname{ad}(\widetilde{L}^m). \end{aligned}$$

Recall that $y \operatorname{ad}^{[2]}(a) \operatorname{ad}(\widetilde{L}^m) \subseteq K_a$. The element $y \operatorname{ad}^{[2]}(a) \operatorname{ad}(z_i) \operatorname{ad}(z_j)$ lies in $L^m \otimes e_i + \widetilde{L}^m e_i$ and

$$y \operatorname{ad}^{[2]}(a) \operatorname{ad}(z_i) \operatorname{ad}(z_j) \operatorname{ad}^{[2]}(a) = y \operatorname{ad}(z_i \operatorname{ad}^{[2]}(a)) \operatorname{ad}(z_j \operatorname{ad}^{[2]}(a)) = 0$$

by Lemma 4.2. Hence, $y \operatorname{ad}^{[2]}(a) \operatorname{ad}(z_i) \operatorname{ad}(z_j) \in K_a$. This implies $yQ(z) = 0$.

Now let us show that $(x+z)^2 = x^2$. We have $(x+z)^2 = x^2 + a \operatorname{ad}(z) \operatorname{ad}(x) + z^2 \mod K_a$. For an arbitrary standard component x_π,

$$\begin{aligned} a \operatorname{ad}(z) \operatorname{ad}(x_\pi) \operatorname{ad}^{[2]}(a) &= -z \operatorname{ad}(a) \operatorname{ad}(x_\pi) \operatorname{ad}^{[2]}(a) \\ &= -z \operatorname{ad}^{[2]}(a) \operatorname{ad}(x_\pi) \operatorname{ad}(a) \\ &= 0 \end{aligned}$$

by Lemma 3.7(3). Hence $a \operatorname{ad}(z) \operatorname{ad}(x_\pi) \in K_a$ and $a \operatorname{ad}(z) \operatorname{ad}(x) \in K_a$.

Let us show that $z^2 = 0$. We have $z^2 = a \sum \operatorname{ad}(z_{\pi_1}) \operatorname{ad}(z_{\pi_2}) + K_a$. By Lemma 3.14,

$$a \sum \operatorname{ad}(z_{\pi_1}) \operatorname{ad}(z_{\pi_2}) = a \sum_{1 \le i < j \le s} \operatorname{ad}(z_i) \operatorname{ad}(z_j) \mod a \operatorname{ad}(\widetilde{L}^m) \subseteq K_a.$$

As above,

$$\begin{aligned} a\,\mathrm{ad}(z_i)\,\mathrm{ad}(z_j)\,\mathrm{ad}^{[2]}(a) &= -z_i\,\mathrm{ad}(a)\,\mathrm{ad}(z_j)\,\mathrm{ad}^{[2]}(a)\\ &= z_i\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(z_j)\,\mathrm{ad}(a)\\ &= 0 \end{aligned}$$

by Lemma 3.7(3). Hence $a\,\mathrm{ad}(z_i)\,\mathrm{ad}(z_j) \in K_a$ and $a \sum \mathrm{ad}(z_{\pi_1})\,\mathrm{ad}(z_{\pi_2}) \in K_a$. This completes the proof of part (1).

(2) Now let $f : \widetilde{L}^m \times \cdots \times \widetilde{L}^m \to \widetilde{L}^m$ be a homogeneous polynomial map with the full linearization $\widetilde{f}$. By Lemma 3.8(2), for polynomial maps, the F-linear span of all values of f is equal to the F-linear span of all values of $\widetilde{f}$. Hence, we need to show that $\widetilde{f}(\widetilde{L}^m, \ldots, \widetilde{L}^m) \subseteq K_a$. Since $\widetilde{L}^m = \sum_i (L^m \otimes e_i + \widetilde{L}^m e_i)$, it follows that $\widetilde{f}(\widetilde{L}^m, \ldots, \widetilde{L}^m) = \sum_i \widetilde{f}(L^m \otimes e_i + \widetilde{L}^m e_i, \widetilde{L}^m, \ldots, \widetilde{L}^m)$. By our assumption, $\widetilde{f}(L^m \otimes e_i + \widetilde{L}^m e_i, \widetilde{L}^m, \ldots, \widetilde{L}^m) \subseteq K_a' \cap (L^m \otimes e_i + \widetilde{L}^m e_i) \subseteq K_a$. This completes the proof of assertion (2). □

The following proposition is a linearized an quadratic version of the construction in [4].

Proposition 4.4. *$J_a = (J_a, x \to x^2, Q)$ is a quadratic Jordan algebra.*

Since the ground field is infinite partial linearizations of the identities (M1)–(M6) follows from these identities (see [9,43]).

We will translate the identities (M1)–(M6) into the language of Lie algebras. The identities (M1)–(M6) translate as

(M1) $x\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(x)\,\mathrm{ad}(y) = a\,\mathrm{ad}(a\,\mathrm{ad}^{[2]}(x))\,\mathrm{ad}(y) \mod K_a,$

(M2) $$\begin{aligned} y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}(a)\,\mathrm{ad}(x) &= -a\,\mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\\ &= y\,\mathrm{ad}([a,x])\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x) \mod K_a, \end{aligned}$$

(M3) $a\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x) = a\,\mathrm{ad}^{[2]}(a\,\mathrm{ad}^{[2]}(x)) \mod K_a,$

(M4) $$\begin{aligned} &a\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(y)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\\ &\qquad = a\,\mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)) \mod K_a, \end{aligned}$$

(M5) $y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(a\,\mathrm{ad}^{[2]}(x)) = y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x) \mod K_a,$

(M6) $$\begin{aligned} &z\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))\\ &\qquad = z\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(y)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x) \mod K_a. \end{aligned}$$

Remark. In the formulas above, we have operators $\mathrm{ad}^{[2]}(x)$, $\mathrm{ad}^{[2]}(a\,\mathrm{ad}^{[2]}(x))$, $\mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))$ acting on elements from the space $Fa + \widetilde{L}^m\,\mathrm{ad}^{[2]}(a)$. In the definition of Jordan operations on $J_a = \widetilde{L}^m / K_a$ above, we noticed that $(Fa + \widetilde{L}^m\,\mathrm{ad}^{[2]}(a))\,\mathrm{ad}(\widetilde{L}^m) \subseteq K_a$. Hence for an arbitrary element $u \in \{x, a\,\mathrm{ad}^{[2]}(x), y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\}$, the operator $\mathrm{ad}^{[2]}(u)$ is understood as

$\sum \operatorname{ad}(u_i)\operatorname{ad}(u_j)$, where $u = \sum u_j$ is the standard decomposition, the sum runs over all 2-element sets (i, j) and the order of factors in $\operatorname{ad}(u_i)\operatorname{ad}(u_j)$ is irrelevant modulo K_a.

Let $x, y \in \widetilde{L^m}$; $x = \sum_\pi x_\pi$, $y = \sum_\tau y_\tau$ the standard decompositions. At first, we will prove the identities (M1)–(M6) under the additional assumption that $[x_{\pi_i}, x_{\pi_j}] = [y_{\tau_i}, y_{\tau_j}] = 0$,

$$\operatorname{ad}(x_{\pi_i})\operatorname{ad}(x_{\pi_j})\operatorname{ad}(x_{\pi_k}) = \operatorname{ad}(y_{\tau_1})\operatorname{ad}(y_{\tau_2})\operatorname{ad}(y_{\tau_3}) = 0$$

for all i, j, k.

More precisely, let L'_0 be the Lie algebra presented by generators $a_1, \ldots, a_n$, $x_1, \ldots, x_s$, $y_1, \ldots, y_t$ and the following relations:

$$\big[\operatorname{Id}(a_i), \operatorname{Id}(a_i)\big] = \big[\operatorname{Id}(x_j), \operatorname{Id}(x_j)\big] = \big[\operatorname{Id}(y_k), \operatorname{Id}(y_k)\big] = (0),$$

where $\operatorname{Id}(a_i)$, $\operatorname{Id}(x_j)$, $\operatorname{Id}(y_k)$ denote the ideals generated by a_i, x_j, y_k respectively, $1 \le i \le n$, $1 \le j \le s$, $1 \le k \le t$; $[a_i, a_j] = 0$, $1 \le i, j \le n$; the operators $\operatorname{ad}^{[k]}(a) = \sum \operatorname{ad}(a_{i_1}) \cdots \operatorname{ad}(a_{i_k})$, where the sum is taken over all k-element subsets of $\{1, 2, \ldots, n\}$ is equal to 0 for $k \ge 3$.

Denote $a = \sum_{i=1}^n a_i$, $x = \sum_{j=1}^s x_j$.

Remark. The generators $a_1, \ldots, a_n$ should not be confused with elements $a_1, \ldots, a_r \in \widetilde{L^m}$ used to define $a' = w'(a_1, \ldots, a_{r-1})$, $a = w(a_1, \ldots, a_r)$ above.

In the algebra L'_0, define linear operators

$$\operatorname{ad}^{[2]}(a) = \sum_{1 \le i < j \le n} \operatorname{ad}(a_i)\operatorname{ad}(a_j),$$

$$\operatorname{ad}^{[2]}(x) = \sum_{1 \le i < j \le s} \operatorname{ad}(x_i)\operatorname{ad}(x_j),$$

$$\operatorname{ad}^{[2]}(y) = \sum_{1 \le i < j \le t} \operatorname{ad}(y_i)\operatorname{ad}(y_j),$$

$$\operatorname{ad}^{[2]}(a \operatorname{ad}^{[2]}(x)) = \sum_{1 \le i < j \le n} \operatorname{ad}(a_i \operatorname{ad}^{[2]}(x))\operatorname{ad}(a_j \operatorname{ad}^{[2]}(x)),$$

$$\operatorname{ad}^{[2]}(y \operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x)) = \sum_{1 \le i < j \le t} \operatorname{ad}(y_i \operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x))\operatorname{ad}(y_j \operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x))$$

and consider the elements

(M1′) $(x \operatorname{ad}^{[2]}(a)\operatorname{ad}(x)\operatorname{ad}(y) - a \operatorname{ad}(a \operatorname{ad}^{[2]}(x))\operatorname{ad}(y)\operatorname{ad}^{[2]}(a)$,

(M2′) $(y \operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x)\operatorname{ad}(a)\operatorname{ad}(x) + a \operatorname{ad}(x)\operatorname{ad}(y)\operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x))\operatorname{ad}^{[2]}(a)$,

(M3′) $(a \operatorname{ad}^{[2]}(x)\operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x) - a \operatorname{ad}^{[2]}(a \operatorname{ad}^{[2]}(x)))\operatorname{ad}^{[2]}(a)$,

(M4′) $(a \operatorname{ad}^{[2]}(x)\operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(y)\operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x)$
$\qquad - a \operatorname{ad}^{[2]}(y \operatorname{ad}^{[2]}(a)\operatorname{ad}^{[2]}(x)))\operatorname{ad}^{[2]}(a)$,

(M5′) $(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(a\,\mathrm{ad}^{[2]}(x)) - y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))\,\mathrm{ad}^{[2]}(a)$,

(M6′) $(z\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)) - z\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(y)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)$.

Now, consider the Lie algebra L_0 that is obtained from L_0' by imposing additional relations:

$$[x_i, x_j] = 0,\ 1 \le i, j \le s; \quad [y_i, y_j] = 0,\ 1 \le i, j \le t;$$
$$[L_0, x_{i_1}, x_{i_2}, x_{i_3}] = [L_0, y_{j_1}, y_{j_2}, y_{j_3}] = (0),$$

for all $1 \le i_1, i_2, i_3 \le s$, $1 \le j_1, j_2, j_3 \le t$.

We will show that the elements (M1′)–(M6′) are equal to zero in the Lie algebra L_0.

Lemma 4.5. $[a\,\mathrm{ad}^{[2]}(x), a] + [x\,\mathrm{ad}^{[2]}(a), x] \in [L_0, a, a]$.

Proof. If $p \ne 2$ then $\mathrm{ad}^{[2]}(x) = \frac{1}{2}\mathrm{ad}(x)^2$, $\mathrm{ad}^{[2]}(a) = \frac{1}{2}\mathrm{ad}^2(a)$, which makes the assertion of the lemma obvious.

Let $p = 2$. Denote $a' = a_i$, $a'' = a_j$, $x' = x_k$, $x'' = x_e$. We will show that

$$[a', x', x'', a''] + [a'', x', x'', a'] = [x', a', a'', x''] + [x'', a', a'', x'].$$

Indeed,

$$\begin{aligned}
&[a', x', x'', a''] + [a'', x', x'', a'] \\
&\quad = [[a', x'], [x'', a'']] + [a', x', a'', x''] + [[a'', x'], [x'', a']] + [a'', x', a', x''] \\
&\quad = [[a', x'], [x'', a'']] + [[a'', x'], [x'', a']],
\end{aligned}$$

since

$$[a', x', a''] + [a'', x', a'] = [[a', a''], x'] = 0.$$

Similarly,

$$[x', a', a'', x''] + [x'', a', a'', x'] = [[x', a'], [a'', x'']] + [[x'', a'], [a'', x']],$$

which finishes the proof of the lemma. □

Now (M1′) immediately follows from Lemma 4.5 since $L_0\,\mathrm{ad}(a)^2\,\mathrm{ad}(y) \subseteq K_a$. The latter inclusion follows from the following argument. The equality (see Lemma 3.7(3))

$$\begin{aligned}
0 &= \mathrm{ad}(y\,\mathrm{ad}^{[4]}(a)) \\
&= \mathrm{ad}(y)\,\mathrm{ad}^{[4]}(a) - \mathrm{ad}(a)\,\mathrm{ad}(y)\,\mathrm{ad}^{[3]}(a) + \mathrm{ad}^{[2]}(a)\,\mathrm{ad}(y)\,\mathrm{ad}^{[2]}(a) \\
&\qquad - \mathrm{ad}^{[3]}(a)\,\mathrm{ad}(y)\,\mathrm{ad}(a) + \mathrm{ad}^{[4]}(a)\,\mathrm{ad}(y)
\end{aligned}$$

implies $\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(y)\,\mathrm{ad}^{[2]}(a)=0$. Hence,

$$L_0\,\mathrm{ad}(a)^2\,\mathrm{ad}(y)\,\mathrm{ad}^{[2]}(a)\subseteq L_0\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(y)\,\mathrm{ad}^{[2]}(a)=0.$$

Let us prove (M2′). From $\mathrm{ad}^{[2]}(x)\,\mathrm{ad}(a)\,\mathrm{ad}(x)=\mathrm{ad}(x)\,\mathrm{ad}(a)\,\mathrm{ad}^{[2]}(x)$ and $\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(x)\,\mathrm{ad}(a)=\mathrm{ad}(a)\,\mathrm{ad}(x)\,\mathrm{ad}^{[2]}(a)$ (see Lemma 3.7(3)), it follows that

$$\begin{aligned} y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\,\mathrm{ad}(a)\,\mathrm{ad}(x) &= y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(x)\,\mathrm{ad}(a)\,\mathrm{ad}^{[2]}(x)\\ &= y\,\mathrm{ad}(a)\,\mathrm{ad}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\\ &= y\,\mathrm{ad}([a,x])\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x), \end{aligned}$$

since $\mathrm{ad}(a)\,\mathrm{ad}^{[2]}(a)=3\,\mathrm{ad}^{[3]}(a)=0$.

Now we will prove (M3′). We have

$$\mathrm{ad}^{[2]}(a\,\mathrm{ad}^{[2]}(x))=\sum\mathrm{ad}(a_i\,\mathrm{ad}^{[2]}(x))\,\mathrm{ad}(a_j\,\mathrm{ad}^{[2]}(x)),$$

where the sum is taken over all 2-element subsets (i,j). By applying Lemma 4.2 to $\Omega=\{x_1,\dots,x_m\}$, $y_1=a_i$, $y_2=a_j$, we get

$$\mathrm{ad}(a_i\,\mathrm{ad}^{[2]}(x))\,\mathrm{ad}(a_j\,\mathrm{ad}^{[2]}(x))=\mathrm{ad}^{[2]}(x)\,\mathrm{ad}(a_i)\,\mathrm{ad}(a_j)\,\mathrm{ad}^{[2]}(x).$$

Hence,

$$\begin{aligned}\sum\mathrm{ad}(a_i\,\mathrm{ad}^{[2]}(x))\,\mathrm{ad}(a_j\,\mathrm{ad}^{[2]}(x)) &= \mathrm{ad}^{[2]}(x)\Big(\sum\mathrm{ad}(a_i)\,\mathrm{ad}(a_j)\Big)\,\mathrm{ad}^{[2]}(x)\\ &= \mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)\end{aligned}$$

as claimed.

Let us prove (M4′). We have

$$\mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))=\sum\mathrm{ad}(y_i\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))\,\mathrm{ad}(y_j\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)).$$

By Lemma 4.2, with $\Omega=\{x_1,\dots,x_m\}$, we get

$$\begin{aligned}&\mathrm{ad}(y_i\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))\,\mathrm{ad}(y_j\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))\\ &\qquad=\mathrm{ad}^{[2]}(x)\,\mathrm{ad}(y_i\,\mathrm{ad}^{[2]}(a))\,\mathrm{ad}(y_j\,\mathrm{ad}^{[2]}(a))\,\mathrm{ad}^{[2]}(x).\end{aligned}$$

Again, by Lemma 4.2 with $\Omega=\{a_1,\dots,a_n\}$

$$\mathrm{ad}(y_i\,\mathrm{ad}^{[2]}(a))\,\mathrm{ad}(y_j\,\mathrm{ad}^{[2]}(a))=\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(y_i)\,\mathrm{ad}(y_j)\,\mathrm{ad}^{[2]}(a).$$

Finally, we get

$$\mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))=\mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(y)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x),$$

as claimed.

We will now prove (M5′). We have already shown above that by Lemma 4.2, we have

$$\mathrm{ad}^{[2]}(a\,\mathrm{ad}^{[2]}(x)) = \mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x),$$

which implies the claim.

To prove (M6′), we need only to recall the equality

$$\mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)) = \mathrm{ad}^{[2]}(x)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(y)\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x)$$

that was proved above.

Since the elements (M1′)–(M6′) are equal to zero in L_0, it follows that in the algebra L_0', the elements of (M1′)–(M6′) are linear combinations of

(1) expressions in x_i's, y_j's, $z, a_1, \dots, a_n$ involving at least one commutator $[x_i, x_j]$, $1 \le i, j \le s$ or $[y_i, y_j]$, $1 \le i, j \le t$,

(2) expressions involving $\mathrm{ad}(x_{i_1})\,\mathrm{ad}(x_{i_2})\,\mathrm{ad}(x_{i_3})$ or $\mathrm{ad}(y_{j_1})\,\mathrm{ad}(y_{j_2})\,\mathrm{ad}(y_{j_3})$, $1 \le i_1, i_2, i_3 \le s$, $1 \le j_1, j_2, j_3 \le t$.

Moreover, since the relations of the algebra L_0' are homogeneous in x_i's, y_j's, and in the total number of generators $a_1, \dots, a_n$, it follows that the presentations of (M1′)–(M6′) as linear combinations of (1), (2) preserve the degrees in x_i's, y_j's, and the total degree in $a_1, \dots, a_n$.

Now we consider arbitrary elements $x, y \in \widetilde{L^m}$ and drop the assumptions on components of standard decompositions of x, y. Let $x = x_{\pi_1} + \cdots + x_{\pi_s}$, $y = y_{\tau_1} + \cdots + y_{\tau_s}$, $a_r = a_{r1} + \cdots + a_{rn}$ be the standard decompositions of x, y, a_r respectively. Then $a = \sum_{i=1}^{n} a_i$, where $a_i = a_{ri}\,\mathrm{ad}^{[2]}(a')$.

The mapping

$$a_i \to a_{ri}\,\mathrm{ad}^{[2]}(a'),\ x_j \to x_{\pi_j},\ y_k \to y_{\tau_k},\quad 1 \le i \le n,\ 1 \le j \le s,\ 1 \le k \le t,$$

extends to a homomorphism $L_0' \to \widetilde{L^m}$. Moreover, the operators

$$\mathrm{ad}^{[2]}(a),\quad \mathrm{ad}^{[2]}(x),\quad \mathrm{ad}^{[2]}(y),\quad \mathrm{ad}^{[2]}(a\,\mathrm{ad}^{[2]}(x)),\quad \mathrm{ad}^{[2]}(y\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}^{[2]}(x))$$

project to the similar operators on $\widetilde{L^m}$, by Lemma 3.14.

Hence, the elements (M1′)–(M6′) of $\widetilde{L^m}$ are linear combinations of

(1) expressions in $x_{\pi_i}, y_{\tau_j}, a_1, \dots, a_n$ involving at least one commutator $[x_{\pi_i}, x_{\pi_j}]$ or $[y_{\tau_i}, y_{\tau_j}]$,

(2) expressions involving $\mathrm{ad}(x_{\pi_i})\,\mathrm{ad}(x_{\pi_j})\,\mathrm{ad}(x_{\pi_k})$ or $\mathrm{ad}(y_{\tau_i})\,\mathrm{ad}(y_{\tau_j})\,\mathrm{ad}(y_{\tau_k})$.

These presentations, as linear combinations of (1) and (2), preserve the degrees in x_{π_i}'s, y_{τ_j}'s and the total degree in $a_1, \dots, a_n$.

Replacing $\mathrm{ad}(a_i)$ in these expressions by

$$\mathrm{ad}(a_{ri}\,\mathrm{ad}^{[2]}(a')) = \mathrm{ad}(a_{ri})\,\mathrm{ad}^{[2]}(a') - \mathrm{ad}(a')\,\mathrm{ad}(a_{ri})\,\mathrm{ad}(a') + \mathrm{ad}^{[2]}(a')\,\mathrm{ad}(a_{ri})$$

we get expressions whose degree in a' exceeds the total degree in the other variables $x_{\pi_i}, y_{\tau_j}, z, a_{ri}$ by 1. In case (1), we merge two elements x_{π_i}, x_{π_j} or y_{τ_i}, y_{τ_j} together. Hence, the degree in a' exceeds the total degree in the other elements by 2. The only property of the element a that was used in Lemma 4.1 was $\mathrm{ad}^{[k]}(a) = 0$ for $k \geq 3$. We have $\mathrm{ad}^{[k]}(a') = 0, k \geq 3$. Hence, we can apply Lemma 4.1 to the element a'. By Lemma 4.1(1), these expressions are equal to zero. In case (2), we only need to refer to Lemma 4.1(3). We proved that the expressions (M1′)–(M6′) are equal to 0, which means that the expressions (M1)–(M6) are equal to 0 modulo K'_a. By Lemma 4.3(2), they are equal to 0 modulo K_a, which finishes the proof of Proposition 4.4.

Let us consider basic examples of quadratic Jordan algebras.

Example 1. Let A be an associative algebra. Let $yQ(x) = xyx;\ x, y \in A$. Then the vector space A with the operators $x \to x^2$ and $x \to Q(x)$ is a quadratic Jordan algebra, which is denoted as $A^{(+)}$.

Example 2. Let A be an associative algebra with an involution $* : A \to A$. Then $H(A, *) = \{a \in A \mid a^* = a\}$ is a subalgebra of the quadratic Jordan algebra $A^{(+)}$.

Example 3. Let V be a vector space and let $q : V \to F$ be a quadratic form with the associated bilinear form $q(v, w) = q(v + w) - q(v) - q(w)$. Fix an element of V that we will denote as $\mathbb{1}$ (a base point) such that $q(\mathbb{1}) = 1$. For arbitrary elements $v, w \in V$ define

$$v^2 = q(v, \mathbb{1})v - q(v)\mathbb{1},\ wQ(v) = q(v, \bar{w})v - q(v)\bar{w},$$

where $\bar{w} = q(w, \mathbb{1})\mathbb{1} - w$. These equations make V a quadratic Jordan algebra. We will denote it as $J(q, \mathbb{1})$.

Example 4. Albert algebras of a nondegenerate admissible cubic form on a 27-dimensional space (see [9,12]).

Powers of elements in a quadratic Jordan algebra J are defined inductively: we define $x^1 = x$; for an even $n = 2k$ we define $x^n = (x^k)^2$; and for an odd $n = 2k + 1$ we define $x^n = xQ(x^k)$. For arbitrary integers $i \geq 0, j \geq 0, k \geq 0$ we have $x^i Q(x^j) = x^{i+2j},\ x^i \circ x^j = 2x^{i+j},\ \{x^i, x^j, x^k\} = 2x^{i+j+k}$.

A quadratic Jordan algebra J is said to be nil of bounded degree n if $x^n = 0$ for an arbitrary element $x \in J$ and if n is a minimal integer with this property.

Just as in §2 we call an element of the free quadratic Jordan algebra $FJ\langle X\rangle$ an S-identity if it lies in the kernel of the homomorphism $FJ\langle X\rangle \to F\langle X\rangle^{(+)},\ x \to x$, where $F\langle X\rangle$ is the free associative algebra.

We say that a quadratic Jordan algebra J is PI if there exists an element $f(x_1, \ldots, x_r) \in FJ\langle X\rangle$ that is not an S-identity and that is identically zero on J.

In this paper, we call an element a of a quadratic Jordan algebra an *absolute zero divisor* if $Q(a) = 0$. This terminology is not standard. (In the standard terminology, we should have also assumed $a \neq 0$ and $a^2 = 0$.) A quadratic Jordan algebra that does not contain nonzero absolute zero divisors is called nondegenerate. The smallest ideal $M(J)$ of a Jordan algebra J such that the factor algebra $J/M(J)$ is nondegenerate is called the McCrimmon radical of the algebra J. The McCrimmon radical of an arbitrary quadratic Jordan algebra lies in the nil radical $\mathrm{Nil}(J)$ [25,35].

A nondegenerate quadratic Jordan algebra is said to be nondegenerate prime if two arbitrary nonzero ideals of J have nonzero intersection.

In [31,37] it is shown that an arbitrary nondegenerate Jordan algebra is a subdirect product of nondegenerate prime Jordan algebras.

Let $\mathrm{Sym}_n(x_1, \ldots, x_n)$ be the full linearization of x_1^n in the free Jordan algebra $FJ\langle X\rangle$.

Lemma 4.6. *There exists a function $d : \mathbb{N} \to \mathbb{N}$ such that an arbitrary nondegenerate prime quadratic Jordan algebra over a field of characteristic $p > 0$ satisfying a PI of degree n satisfies the identity* $\mathrm{Sym}_{d(n)}(x_1, \ldots, x_{d(n)}) = 0$.

Proof. Let us notice first that if J is a quadratic Jordan algebra of dimension d then J satisfies the identity $\mathrm{Sym}_{d(p-1)+1}(x_1, \ldots, x_{d(p-1)+1}) = 0$. Indeed, if $e_1, \ldots, e_d$ is a basis of J then among any $d(p-1)+1$ elements from $\{e_1, \ldots, e_d\}$ at least p elements are equal. This implies the claim.

In [26] it was shown that if J is a nondegenerate prime quadratic Jordan algebra, then one of the following possibilities holds:

(1) there exists a prime associative algebra A such that

$$A^{(+)} \subseteq J \subseteq \mathbb{Q}(A)^{(+)},$$

where $\mathbb{Q}(A)$ is the Martindale ring of the quotients of A (see [23]);

(2) there exists a prime associative algebra A with an involution $* : A \to A$, such that

$$H(A_0, *) \subseteq J \subseteq H(\mathbb{Q}(A), *)$$

where A_0 is the subalgebra of A generated by elements $a + a^*$, aa^*, $a \in A$, and $\mathbb{Q}(A)$ is the Martindale ring of quotients of the algebra A (see [23]);

(3) J is a form of an exceptional 27-dimensional Albert algebra over a field F;

(4) J is embeddable in a quadratic Jordan algebra $J(q, v_0)$ of a nondegenerate quadratic form q with a basepoint v_0 in a vector space over some extension of the base field F.

If $A^{(+)} \subseteq J \subseteq \mathbb{Q}(A)^{(+)}$, then A is a prime associative algebra satisfying an identity of degree n. Hence the center $Z(A)$ of A is nonzero and the algebra $\mathbb{Q}(A) = (Z(A) \setminus \{0\})^{-1}A$ is of dimension $\leq [\frac{n}{2}]^2$ over the field $K = (Z(A) \setminus \{0\})^{-1}Z(A)$ (see [28]). Hence the algebra $\mathbb{Q}(A)$ satisfies the identity $\mathrm{Sym}_{[\frac{n}{2}]^2(p-1)+1} = 0$.

Suppose that $H(A_0, *) \subseteq J \subseteq H(\mathbb{Q}(A), *)$. S. Amitsur [1] proved that there exists a function $h(n)$ with the following property: if an involutive associative algebra satisfies an identity of degree n with an involution then it satisfies an identity of degree $\leq h(n)$. As we have shown above, the algebra $\mathbb{Q}(A)$ in this case has dimension $\leq [\frac{h(n)}{2}]^2$ over its center and satisfies the identity $\mathrm{Sym}_{[\frac{h(n)}{2}]^2(p-1)+1} = 0$.

The same argument applies to case (3): the algebra J satisfies the identity

$$\mathrm{Sym}_{27(p-1)+1} = 0.$$

Consider now the quadratic Jordan algebra J of a quadratic form q on a vector space V where $v_0 \in V$ is a basepoint. The quadratic form q can be extended to the scalar product $V \otimes_F \widehat{E}$, $\widehat{E} = E + F \cdot 1$. For an arbitrary element $u \in V \otimes_F E$ we have $u^2 = q(u, v_0)u - q(u)v_0$. The elements $a = q(u, v_0)$, $b = q(u)$ lie in E. Hence $a^p = b^p = 0$. For an arbitrary $k \geq 1$ we have

$$u^{2k} = \sum_{i+j=k} a^i b^j u_{ij}, \quad u_{ij} \in V \otimes_F \widehat{E}.$$

Hence $u^{2(2p-1)} = 0$. This implies that the algebra J satisfies the identity $\mathrm{Sym}_{4p-2}=0$ and finishes the proof of the lemma. □

Lemma 4.7. *Let J be a quadratic Jordan F-algebra that satisfies the identity $x^n = 0$, $n \geq 2$. Then,*

(1) *for an arbitrary element $a \in J$, the elements $a^{n+1}, a^{n+2}, \ldots, a^{2n-1}$ are absolute zero divisors of J;*

(2) *if J satisfies the identities $x^n = x^{n+1} = \cdots = x^{2n-1} = 0$, then for an arbitrary $a \in J$, the element a^{n-1} is an absolute zero divisor of J.*

Proof. For $i = n+1, n+2, \ldots, 2n-1$, we have $Q(x^i) = Q(x^{i-n})Q(x^n) = 0$, which proves (1).

Suppose now that the algebra J satisfies the identities $x^{n+1} = x^{n+2} = \cdots = x^{2n-1} = 0$. Since the ground field F is infinite, the algebra J satisfies also the following partial linearization of $x^{2n-1} = 0$ (see [9,43]):

$$yQ(x^{n-1}) + x^{2(n-1)} \cdot y + \sum_{\substack{i+j=2(n-1)\\ 1\leq i<j\leq 2n-3}} \{x^i, y, x^j\} = 0.$$

Hence, $yQ(x^{n-1}) = 0$, which proves assertion of the lemma. □

Let J be a quadratic Jordan algebra, $a \in J$. Define a new structure of a quadratic Jordan algebra on J via:

$$x^{*2} = aQ(x), \quad yQ^*(x) = yQ(a)Q(x).$$

The new quadratic Jordan algebra is denoted as $J^{(a)}$ and is called a homotope of J (see [9,12,25]).

For the quadratic Jordan algebra $\widetilde{J} = J \otimes_F E$ and an element $a \in \widetilde{J}$, consider the subspaces $K_a' = \{x \in \widetilde{J} \mid xQ(a) = 0\}$ and $K_a = \sum_i (J \otimes e_i + \widetilde{J} e_i) \cap K_a'$. It is easy to see that the subspace K_a is an ideal of the algebra $\widetilde{J}^{(a)}$.

Remark. If $p \neq 2$, then K'_a is also an ideal of $\widetilde{J}^{(a)}$.

Lemma 4.8. *If $b \in J$ and $b + K_a$ is an absolute zero divisor of the algebra $J^{(a)}/K_a$, then $bQ(a)$ is an absolute zero divisor of the algebra J.*

Proof. We have $JQ(bQ(a)) = JQ^*(b)Q(a) \subseteq K_aQ(a) = (0)$, which proves the lemma. □

Lemma 4.9. *Let a be an element of a quadratic Jordan algebra J. Let $f : \widetilde{J} \times \cdots \times \widetilde{J} \to \widetilde{J}$ be a homogeneous polynomial map, let $\widetilde{f}$ be its full linearization. Suppose that $\widetilde{f}(J \otimes e_i + \widetilde{J}e_i, \widetilde{J}, \dots, \widetilde{J}) \subseteq J \otimes e_i + \widetilde{J}e_i$ for all i. If an arbitrary value of f lies in K'_a, then an arbitrary value of f lies in K_a.*

The proof is similar to the proof of Lemma 4.3(2).

Let f be an element of the free quadratic Jordan algebra $FJ\langle X\rangle$, which is not an S-identity. Let $M = M(f)$ be the variety of quadratic Jordan algebras satisfying the identity $f = 0$ (see [9,11,43]).

Definition. We say that a finite sequence of homogeneous elements $h_1, h_2, \dots, h_r \in FJ\langle X\rangle$ is an absolute zero divisor sequence for M if for an arbitrary quadratic Jordan algebra $J \in M$:

(i) every value of h_r on $\widetilde{J} = J \otimes_F E$ is an absolute zero divisor of the algebra $\widetilde{J}$;

(ii) if $h_k = h_{k+1} = \cdots = h_r = 0$ identically hold on $\widetilde{J}$, $2 \leq k \leq r$, then an arbitrary value of h_{k-1} on $\widetilde{J}$ is an absolute zero divisor of $\widetilde{J}$.

Recall that in this section we always assume that char $F = p > 0$.

Proposition 4.10. *For an arbitrary element $f \in FJ\langle X\rangle$ that is not an S-identity the variety $M(f)$ has a finite absolute zero divisor sequence $h_1, h_2, \dots, h_r$ with $h_1 = x_1 \in X$.*

Proof. Let $F_M\langle X\rangle$ be the free algebra in the variety $M = M(f)$ on the set of free generators X. Since the factor algebra of $F_M\langle X\rangle$ modulo the McCrimmon radical can be approximated by prime nondegenerate algebras [31,37], Lemma 4.6 implies that there exists $d \geq 1$ such that $y = \mathrm{Sym}_d(x_1, \dots, x_d)$ lies in the McCrimmon radical of $F_M\langle X\rangle$. Consider the homotope algebra $F_M\langle X\rangle^{(x_{d+1})}$ Since an absolute zero divisor of a Jordan algebra is an absolute zero divisor of every homotope, it follows that the McCrimmon radical of $F_M\langle X\rangle$ lies in the McCrimmon radical of $F_M\langle X\rangle^{(x_{d+1})}$. In particular, the element y lies in the McCrimmon radical of $F_M\langle X\rangle^{(x_{d+1})}$ and therefore is nilpotent.

Let $a^{(k,x_{d+1})}$ denote the kth power of an element a in the homotope algebra $F_M\langle X\rangle^{(x_{d+1})}$. There exists $m \geq 2$ such that $y^{(m-1,x_{d+1})} = 0$. Then $x_{d+1}^{(m,y)} = y^{(m-1,x_{d+1})}Q(x_{d+1}) = 0$. This implies that $x^{(m,y)} = 0$ holds identically on $F_M\langle X\rangle$.

Then by Lemmas 4.7 and 4.8, the sequence

$$y, x_{d+1}Q(y), \dots, x_{d+1}^{(m-1,y)}Q(y), x_{d+1}^{(2m-1,y)}Q(y), \dots, x_{d+1}^{(m+1,y)}Q(y)$$

is an absolute zero divisor sequence in M.

Indeed, since the Jordan algebra $F_M\langle X\rangle^{(y)}/K_y$ satisfies the identity $x^m = 0$, Lemma 4.7(1) implies that the elements

$$x_{d+1}^{(2m-1,y)} + K_y, \dots, x_{d+1}^{(m+1,y)} + K_y$$

are absolute zero divisors in $F_M\langle X\rangle^{(y)}/K_y$. By Lemma 4.8, the elements

$$x_{d+1}^{(2m-1,y)} Q(y), \dots, x_{d+1}^{(m+1,y)} Q(y)$$

are absolute zero divisors of the algebra $F_M\langle X\rangle$.

If $J \in M$ and $x_{d+1}^{(2m-1,y)} Q(y) = \cdots = x_{d+1}^{(m+1,y)} Q(y) = 0$ hold identically on $\widetilde{J}$, then for arbitrary elements $a_1, \dots, a_d \in \widetilde{J}$, $b = \mathrm{Sym}_d(a_1, \dots, a_d)$, $c \in \widetilde{J}$ the ith power $c^{(i,b)}$, $m \le i \le 2m-1$, lies in K_b'. By Lemma 4.9, we have $c^{(i,b)} \in K_b$. In other words, the Jordan algebra $\widetilde{J}^{(b)}/K_b$ satisfies the identities

$$x^m = x^{m+1} = \cdots = x^{2m-1} = 0.$$

By Lemma 4.7(2), for an arbitrary element $c \in \widetilde{J}$, the $(m-1)$th power $c^{(m-1,b)}$ is an absolute zero divisor in $\widetilde{J}^{(b)}/K_b$. By Lemma 4.8, the element $c^{(m-1,b)}Q(b)$ is an absolute zero divisor of $\widetilde{J}$.

If $J \in M$ and $x_{d+1}^{(m-1,y)} Q(y) = x_{d+1}^{(2m-1,y)} Q(y) = \cdots = x_{d+1}^{(m+1,y)} Q(y) = 0$ holds identically on $\widetilde{J}$, then using Lemma 4.9 as above, we conclude that the algebra $\widetilde{J}^{(b)}/K_b$ satisfies the identities $x^{m-1} = \cdots = x^{2m-1} = 0$.

Again, by Lemma 4.7(2) and Lemma 4.8, every value of $x_{d+1}^{(m-2,y)} Q(y)$ (and so on) is an absolute zero divisor of $\widetilde{J}$.

If an algebra J lies in M and $y = \mathrm{Sym}_d(x_1, \dots, x_d) = 0$ holds identically on J then the algebra $\widetilde{J}$ is nil of bounded index $\le d$. Again by Lemma 4.7 we conclude that

$$x_1, x_1^2, \dots, x_1^{d-1}, x_1^{2d-1}, \dots, x_1^{d+1}, y, x_{d+1}Q(y), \dots$$
$$\dots, x_{d+1}^{(m-1,y)} Q(y), x_{d+1}^{(2m-1,y)} Q(y), \dots, x_{d+1}^{(m+1,y)} Q(y)$$

is an absolute zero divisor sequence, which finishes the proof of the proposition. □

Conjecture 4.11. *If J is a quadratic Jordan PI-algebra over a field of characteristic $p > 0$ then the algebra $\widetilde{J}$ is nil of bounded index.*

Now let's come back to the Lie algebra L and the Jordan algebra $J_a = \widetilde{L}^m/K_a$.

Lemma 4.12. *Let $b + K_a$ be a nonzero absolute zero divisor of the Jordan algebra $\widetilde{L}^m/K_a$. Then the element $b\,\mathrm{ad}^{[2]}(a)$ is a nonzero sandwich of the Lie algebra $\widetilde{L}^m$.*

Proof. Let $b = \sum_\pi b_\pi$ be the standard decomposition. For an arbitrary element $c \in \widetilde{L}^m$ we have

$$(c + K_a)Q(b + K_a) = \sum c\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(b_{\pi_1})\,\mathrm{ad}(b_{\pi_2}) + K_a.$$

Hence by Lemma 4.2,

$$\sum c\,\mathrm{ad}^{[2]}(a)\,\mathrm{ad}(b_{\pi_1})\,\mathrm{ad}(b_{\pi_2})\,\mathrm{ad}^{[2]}(a) = \sum c\,\mathrm{ad}(b_{\pi_1}\,\mathrm{ad}^{[2]}(a))\,\mathrm{ad}(b_{\pi_2}\,\mathrm{ad}^{[2]}(a)) = 0.$$

Let $\Omega = \{b_{\pi_i}\,\mathrm{ad}^{[2]}(a)\}$. We showed that $\widetilde{L}^m U_2(\Omega) = (0)$. By Lemma 3.13 the element $b\,\mathrm{ad}^{[2]}(a)$ is a sandwich of the Lie algebra $\widetilde{L}^m$. □

Let $j(y_1, \dots, y_d)$ be an arbitrary Jordan polynomial, i.e. an element of the free Jordan algebra. The polynomial j defines a function $\widetilde{L}^m/K_a \times \cdots \times \widetilde{L}^m/K_a \to \widetilde{L}^m/K_a$ and, therefore, a function $\widetilde{L}^m \times \cdots \times \widetilde{L}^m \to \widetilde{L}^m/K_a$.

Lemma 4.13. *Let $j(y_1, \dots, y_q)$ be a multilinear Jordan polynomial. There exists a homogeneous divided polynomial $j'(y_1, \dots, y_q, x_1, \dots, x_r)$ defined on L^m, such that the value $j(b_1, \dots, b_q)$ in the Jordan algebra $\widetilde{L}^m/K_a$ is equal to $j'(b_1, \dots, b_q, a_1, \dots, a_r) + K_a$. In particular,*

$$j(b_1, \dots, b_q)\,\mathrm{ad}^{[2]}\,w(a_1, \dots, a_r) = j'(b_1, \dots, b_q, a_1, \dots, a_r)\,\mathrm{ad}^{[2]}\,w(a_1, \dots, a_r).$$

Proof. We will proceed by induction on the construction of the Jordan polynomial j. Let $j = \alpha j_1 + \beta j_2$, where $\alpha, \beta \in F$ and j_1, j_2 are multilinear Jordan polynomials, such that the divided polynomials j_1', j_2' exist. Then we let $j' = \alpha j_1' + \beta j_2'$. Suppose that $j = j_1 \circ j_2$, where j_1, j_2 are multilinear Jordan polynomials on disjoint variables.

We have $j(b_1, \dots, b_q) = [a, j_1(b_1, \dots, b_q), j_2(b_1, \dots, b_q)] + K_a$ and we let

$$j'(y_1, \dots, y_q, x_1, \dots, x_r) = \bigl[w(x_1, \dots, x_r), j_1'(y_1, \dots, y_q, x_1, \dots, x_r), j_2'(y_1, \dots, y_q, x_1, \dots, x_r)\bigr].$$

Finally, let $j = \{j_1, j_2, j_3\}$, where j_1, j_2, j_3 are multilinear Jordan polynomials on disjoint variables. Arguing as above, we let

$$j' = \bigl[j_2'\,\mathrm{ad}_{x_1}^{[2]}\,w(x_1, \dots, x_r), j_1', j_3'\bigr].$$

This completes the proof of the lemma. □

Proposition 4.14. *There exist integers $k \geq 1$, $t \geq 1$ and a homogeneous regular divided polynomial v defined on L^k such that every value of v on $\widetilde{L}^k$ is a sum of t sandwiches of the algebra $\widetilde{L}^k$.*

Proof. Recall that there exists a homogeneous regular divided polynomial $w = w(x_1, \dots, x_r)$ defined on L^m, $m \geq 1$, linear in x_r and such that:

(i) $[w(a_1, \dots, a_{r-1}, a), w(a_1, \dots, a_{r-1}, b)] = 0$ for arbitrary elements $a, b, a_1, \dots, a_{r-1} \in \widetilde{L}^m$;

(ii) $\widetilde{L}^m \operatorname{ad}_{x_r}^{[t]}(w) = 0$ holds identically on $\widetilde{L}^m$ for $t \geq 3$.

For an arbitrary $i \geq m$, arbitrary elements $a_1, \ldots, a_r \in \widetilde{L}^i$ consider $a = w(a_1, \ldots, a_r)$ and denote $\operatorname{ad}^{[2]}(a) = \operatorname{ad}_{x_r}^{[2]}(w(a_1, \ldots, a_r))$.

Arguing as in the proof of Lemma 2.3, we conclude that there exists an element $f \in FJ\langle X\rangle$ such that f is not an S-identity and all quadratic Jordan algebras $\widetilde{L}^i/K_a$; $i \geq m$; $a_1, \ldots, a_r \in \widetilde{L}^i$ satisfy the identity $f = 0$.

By Proposition 4.10 there exists an absolute zero divisor sequence $h_1 = x_1, h_2, \ldots, h_s$ of the variety $M(f)$.

If J is an algebra from the variety $M(f)$ such that $J = \sum_i I_i, I_i \trianglelefteq J, I_i^2 = (0)$, then J and $J \otimes_F E$ satisfy the same identities. Hence, every value of h_s on J is an absolute zero divisor of J and if $h_k = \cdots = h_s = 0$ identically hold on J, then every value of h_{k-1} on J is an absolute zero divisor.

Jordan algebras $\widetilde{L^m}/K_a$ that have been discussed above have this property. Indeed, $\widetilde{L^m}/K_a = \sum_i I_i$, where $I_i = L^m \otimes e_i + \widetilde{L^m} e_i + K_a/K_a$.

For an integer $i \geq m$ and elements $a_1, \ldots, a_r \in \widetilde{L}^i$ let $s(i, a_1, \ldots, a_r)$ be a maximal integer j, $1 \leq j \leq s$, such that h_j is not identically zero on $\widetilde{L}^i/K_a$. If $h_1 = x_1$ is identically zero on $\widetilde{L}^i/K_a$, that is, $\widetilde{L}^i = K_a$, then we let $s(i, a_1, \ldots, a_r) = 0$.

Let $s(i) = \max\{s(i, a_1, \ldots, a_r) \mid a_1, \ldots, a_r \in \widetilde{L}^i\}$. Clearly, $s(m) \geq s(m+1) \geq \cdots$. Let this decreasing sequence stabilize at $t = s(k) = s(k+1) = \cdots$.

If $t = 0$ then $\widetilde{L^k} = K_a$, which means that $\widetilde{L^k} \operatorname{ad}_{x_1}^{[2]} w(a_1, \ldots, a_r) = (0)$ for arbitrary elements $a_1, \ldots, a_r \in \widetilde{L^k}$. By Lemma 3.13 every value of w on $\widetilde{L^k}$ is a sandwich of the algebra $\widetilde{L^k}$. Therefore assume that $t \geq 1$.

Let us summarize the above. For arbitrary elements $a_1, \ldots, a_r \in \widetilde{L^k}$ let $a = w(a_1, \ldots, a_r)$, $\operatorname{ad}^{[2]}(a) = \operatorname{ad}_{x_r}^{[2]} w(a_1, \ldots, a_r)$, $K_a = \widetilde{L^k} \cap \ker \operatorname{ad}^{[2]}(a)$. Every value of the Jordan polynomial h_t on the Jordan algebra $\widetilde{L^k}/K_a$ is an absolute zero divisor. For every $k' \geq k$ there exist elements $a_1, \ldots, a_r \in \widetilde{L^{k'}}$ such that h_t is not identically zero on $\widetilde{L^{k'}}/K_a$. In particular, the Jordan polynomial h_t is regular.

Suppose that $h_t = h_t(y_1, \ldots, y_q)$. Let μ be the total degree of the homogeneous Jordan polynomial h_t. The full linearization $\widetilde{h_t}$ of the polynomial h_t depends on μ variables. An arbitrary value of the polynomial $\widetilde{h_t}$ is a linear combination of $2^\mu = \ell$ values of the polynomial h_t. Let $\widetilde{h_t}'(y_1, \ldots, y_q, x_1, \ldots, x_r)$ be the homogeneous divided polynomial of Lemma 4.13 defined on $\widetilde{L^k}$. Let $v(y_1, \ldots, y_q, x_1, \ldots, x_r) = \widetilde{h}_t'(y_1, \ldots, y_q, x_1, \ldots, x_r) \operatorname{ad}_{x_r}^{[2]} w(x_1, \ldots, x_r)$.

For arbitrary elements $b_1, \ldots, b_q, a_1, \ldots, a_r \in \widetilde{L^k}$, we have

$$\widetilde{h_t}(b_1, \ldots, b_q) \operatorname{ad}_{x_r}^{[2]} w(a_1, \ldots, a_r) = v(b_1, \ldots, b_q, a_1, \ldots, a_r).$$

We claim that the divided polynomial v is regular. Indeed, it was shown above that for arbitrary $k' \geq k$, there exist elements $a_1, \ldots, a_r \in \widetilde{L}^{k'}$ such that the Jordan polynomial h_t is not identically zero on $\widetilde{L}^{k'}/K_a$. Lemma 3.8(2) was proved for arbitrary polynomial maps that include Jordan polynomials. Hence by Lemma 3.8(2), the linear spans of the sets of values of the Jordan polynomials h_t and $\widetilde{h_t}$ on the Jordan algebra $\widetilde{L}^{k'}/K_a$ are equal. Hence $\widetilde{h_t}$ is not identically zero on $\widetilde{L}^{k'}/K_a$. By Lemma 4.13, the homogeneous divided polynomial $v = \widetilde{h_t}\, \mathrm{ad}^{[2]}_{x_r}(w)$ is not identically zero on $\widetilde{L}^{k'}$. This implies regularity of v.

By Lemma 4.12 every value of v on $\widetilde{L}^k$ is a sum of $\ell = 2^\mu$ sandwiches of the Lie algebra $\widetilde{L}^k$. This completes the proof of the proposition. □

5. Sandwiches in L

Let $x \in \widetilde{L}$, $x = \sum_\pi x_\pi$ the standard decomposition. Suppose that

$$[x_\pi, x_\tau] = 0 \tag{5.1}$$

for arbitrary π, τ. As above denote $\mathrm{ad}^{[k]}(x) = \sum \mathrm{ad}(x_{\pi_1}) \cdots \mathrm{ad}(x_{\pi_k})$, where the sum runs over all k-element subsets $(\pi_1, \ldots, \pi_k)$. As we have already noticed in Lemma 3.7(2), $A(x) = \mathrm{Id} + \sum_{k=1}^{\infty} \mathrm{ad}^{[k]}(x)$ is an automorphism of the algebra $\widetilde{L}$.

Let elements $x_1, \ldots, x_d \in \widetilde{L}$ satisfy condition 5.1, $A = A(x_1) \cdots A(x_d) \in \mathrm{Aut}\, \widetilde{L}$. The following lemma is straightforward.

Lemma 5.1. *For arbitrary elements $a_1, \ldots, a_d \in \widetilde{L}$ we have*

$$\begin{aligned} &a_1 A \otimes \cdots \otimes a_d A - \sum_{i=1}^{d} a_1 A \otimes \cdots a_i \otimes \cdots \otimes a_d A \\ &\qquad + \sum_{1 \leq i \neq j \leq d} a_1 A \otimes \cdots \otimes a_i \cdots \otimes a_j \cdots \otimes a_d A - \cdots \pm a_1 \otimes \cdots \otimes a_d \\ &= \sum_{\sigma \in S_d} [a_1, x_{\sigma(1)}] \otimes \cdots \otimes [a_d, x_{\sigma(d)}] \\ &\qquad + \textit{terms involving at least two elements from one of the sets } \{x_{i_\pi}\}_\pi. \end{aligned}$$

A. N. Grishkov [7] showed that in a Lie algebra over a field of zero characteristic a sandwich generates the locally nilpotent ideal. We will prove an analog of this result for the algebra $\widetilde{L}$. The proof essentially depends on the following result from [20]: There exists a function $KZ : \mathbb{N} \to \mathbb{N}$ such that in an arbitrary Lie algebra if elements $a_1, \ldots, a_n$ are sandwiches then the subalgebra $\langle a_1 \cdots, a_n \rangle$ is nilpotent of degree $\leq KZ(n)$.

Let $f(m, n) = KZ((n+1)^m)$.

Lemma 5.2. *Let I be an ideal of a Lie algebra L, $a \in \widetilde{I}$ a sandwich in $\widetilde{I}$. Let $S \subset \widetilde{L}$ be a finite set of $\le n$ elements. Then the subalgebra of $\widetilde{L}$ generated by commutators $[a, b_1, \dots, b_t]$, where $b_i \in S,\ t \le m$, is nilpotent of degree $\le f(m,n)$.*

Proof. Consider a commutator

$$c = \big[[a, b_{11}, \dots, b_{1t_1}], [a, b_{21}, \dots, b_{2t_2}], \dots, [a, b_{q1}, \dots, b_{qt_q}]\big],$$

where $b_{ij} \in S$, $t_i \le m$, $1 \le i \le q$, $q = f(m,n)$. Our aim is to show that $c = 0$.

Let an element $b \in S$ occur in the commutator c $|b|$ times. Clearly $\sum_{b \in S} |b| = t_1 + \cdots + t_q$. Choose $|b|$ new elements $x_{b,1}, \dots, x_{b,|b|}$ in $\widetilde{L}$ and replace all $|b|$ occurrences of b in c by the symmetrized sum in $x_{b,1}, \dots, x_{b,|b|}$:

$$\cdots \underbrace{b \cdots b}_{|b|} \cdots \to \sum_{\sigma \in S_{|b|}} \cdots x_{b,\sigma(1)} \cdots x_{b,\sigma(|b|)} \cdots .$$

We will get a new expression c' in $a, x_{b,j}$'s, $b \in S$, $1 \le j \le |b|$. Denote it $c' = c'(a, x_{b,1}, \dots, x_{b,|b|})$. To show that $c = 0$ it is sufficient to show that $c' = 0$. Indeed, let $b = b_1 + \cdots + b_k$ be the standard decomposition of b. If $k < |b|$, then c is a sum of expressions, each containing one of the elements $b_1, \dots, b_k$ at least twice. If $k \ge |b|$, then $c = \sum c'(a, b_{i_1}, \dots, b_{i_{|b|}})$, where the sum runs over all $|b|$-element subsets of $\{b_1, \dots, b_k\}$.

Let $x_{bj} = \sum\limits_{\pi} x_{bj\pi}$, $a = \sum\limits_{\pi} a_\pi$ be the standard decompositions, $x_{bj\pi} = x'_{bj\pi} \otimes e_\pi$, $a_\pi = a'_\pi \otimes e_\pi$, $x'_{bj\pi} \in L, a'_\pi \in I$.

Let $I(b,j)$ be the ideal of the Lie algebra L generated by the subset $\{x_{bj\pi}\}_\pi$. Suppose at first that for arbitrary $b \in S$, $i \le j \le |b|$ we have

$$\big[I(b,j), I(b,j)\big] = (0). \tag{5.2}$$

Then we can consider the automorphism

$$A(x_{b,j}) = \mathrm{Id} + \sum_{k \ge 1} \mathrm{ad}^{[k]}(x_{b,j}),\ 1 \le j \le |b|$$

and the automorphism $A(b) = A(x_{b,1}) \cdots A(x_{b,|b|})$. By Lemma 5.1 for arbitrary elements $a_1, \dots, a_{|b|} \in \widetilde{L}$ we have

$$\begin{aligned} a_1 A(b) \otimes \cdots \otimes a_{|b|} A(b) - \sum a_1 A(b) \otimes \cdots \otimes a_i \otimes \cdots \otimes a_{|b|} A(b) \\ + \sum a_1 A(b) \otimes \cdots a_i \otimes \cdots \otimes a_j \otimes \cdots a_{|b|} A(b) - \cdots \\ = \sum \big[a_1, x_{b,\sigma(1)}\big] \otimes \cdots \otimes \big[a_{|b|}, x_{b,\sigma(|b|)}\big]. \end{aligned}$$

Replacing each symmetric set $x_{b,1}, \dots, x_{b,|b|}$ in the element c' by expressions of the left hand side types we can represent c' as a linear combination of commutators

$$[a\phi_{11}\cdots\phi_{1m}, \dots, a\phi_{q1}\cdots\phi_{qm}],$$

where each ϕ_{ij} is one of the automorphisms $A(b_1), \dots, A(b_n), \mathrm{Id}$. There are $\le (n+1)^m$ elements $a\phi_{i1}, \dots \phi_{im}$ and all of them are sandwiches in the algebra $\widetilde{I}$. By the choice of $q = KZ((n+1)^m)$ we conclude that

$$[a\phi_{11}, \dots, \phi_{1m}, \dots, a\phi_{q1}\cdots\phi_{qm}] = 0.$$

Now we will drop the assumption that

$$[I(b,j), I(b,j)] = (0).$$

Let $\mathrm{Lie}\langle X\rangle$ be the free Lie algebra on the set of free generators $X = \{\{x'_{b,j,\pi}\}_\pi, \{a_\pi\}_\pi\}$. Let $I(b,j)$ be the ideal of $\mathrm{Lie}\langle X\rangle$ generated by the set $\{x'_{b,j,\pi}\}_\pi$. Let $I = \sum_{b,j}[I(b,j), I(b,j)]$. Let J be the ideal of $\mathrm{Lie}\langle X\rangle$ generated by all relations needed to make $a = \sum_\pi a'_\pi e_\pi$ a sandwich in the ideal generated by a. So, J is generated by

$$[a'_\pi, \rho, a'_\tau] + [a'_\tau, \rho, a'_\pi], [a'_\pi, \rho_1, \rho_2, a'_\tau] + [a'_\tau, \rho_1, \rho_2, a'_\pi],$$

where ρ, ρ_1, ρ_2 are arbitrary commutators in X involving at least one element a'_μ.

Consider the Lie algebra $L = \mathrm{Lie}\langle X\rangle/I + J$. Since this algebra satisfies condition 5.2 the element c' computed in the algebra $\widetilde{L}$ lies in $(I+J)\otimes E$. The ideals I and J are graded with respect to each generator. The element c' has total degree one with respect to variables $\{x'_{b,j,\pi}\}_\pi$ for each b, j. Since the ideal I does not contain homogeneous elements having degree one with respect to all $\{x'_{b,j,\pi}\}_\pi$ it follows that $c' \in J \otimes E$. This finishes the proof of the lemma. □

Now, let A be an associative enveloping algebra of the algebra L, $L \subseteq A^{(-)}$, $\widetilde{L} = L \otimes_F E$, $\widetilde{A} = A \otimes_F E$.

Lemma 5.3. *Let $a \in \widetilde{L}$ be a sandwich. Then for an arbitrary element $b \in \widetilde{L}$ we have $[a,b]^p = 0$.*

Proof. N. Jacobson [10] noticed that

$$\{x_1, \dots, x_p\} = \sum_{\sigma\in S_p} x_{\sigma(1)}\cdots x_{\sigma(p)} = \sum_{\sigma\in S_{p-1}} [x_p, x_{\sigma(1)}, \dots, x_{\sigma(p-1)}].$$

Let $b = \sum_\pi b_\pi$ be the standard decomposition of the element b. Then $[a,b] = \sum_\pi [a, b_\pi]$ and $[a,b]^p = \sum\{[a, b_{\pi_1}], \dots, [a, b_{\pi_p}]\}$. Each summand on the right hand side is equal to zero since $[[a, b_{\pi_i}], [a, b_{\pi_j}]] = 0$, which proves the lemma. □

Lemma 5.4. *Let I be an ideal of the Lie algebra L. Let $a \in \widetilde{I}$ be a sandwich in $\widetilde{I}$ such that $a^p = 0$ in the algebra $\widetilde{A}$. Let $S \subset \widetilde{L}$ be a finite set of $\leq n$ elements. Then the associative subalgebra of $\widetilde{A}$ generated by commutators $[a, b_1, \dots, b_t]$, where $b_i \in S$, $t \leq m$, is nilpotent of degree $\leq p^{(n+1)^{mf(m,n)}}$.*

Proof. Denote $q = p^{(n+1)^{mf(m,n)}}$. We need to show that an arbitrary product

$$[a, b_{11}, \dots, b_{1t_1}] \cdots [a, b_{q1}, \dots, b_{qt_q}],$$

where $b_{ij} \in B$, $t_i \leq m$, $1 \leq i \leq q$, is equal to 0.

Arguing as in the proof of Lemma 5.2 we can reduce the problem to showing that an arbitrary product $(a\phi_{11} \cdots \phi_{1m}) \cdots (a\phi_{q1} \cdots \phi_{qm}) = 0$, where $\phi_{ij} \in \operatorname{Aut} \widetilde{A}$, $\phi_{ij}(\widetilde{L}) = \widetilde{L}$, $\phi_{ij}(\widetilde{I}) = \widetilde{I}$, $\#\{\phi_{ij} \mid 1 \leq i \leq q, 1 \leq j \leq m\} \leq n+1$,.

Let $Y = \{a\phi_{i1} \cdots \phi_{im}, 1 \leq i \leq q\}$, $|Y| \leq (n+1)^m$. An arbitrary element y from Y is a sandwich in $\widetilde{I}$ and $y^p = 0$ in $\widetilde{A}$.

Let L_1 be the Lie algebra generated by Y. By Lemma 5.2 $L_1^{f(m,n)} = (0)$. Let $\rho_1, \dots, \rho_r$ be left normed commutators in Y that form a basis of L_1, $r \leq |Y|^{f(m,n)} \leq (n+1)^{mf(m,n)}$.

By Lemma 5.3 for each commutator ρ_i we have $\rho_i^p = 0$. Now the Poincaré–Birkhoff–Witt theorem implies the assertion of the lemma. □

6. Proof of Theorem 1.1

Let a Lie algebra L over a field F of characteristic $p > 0$ and its associative enveloping algebra A, $L \subseteq A^{(-)}$, $A = \langle L \rangle$, satisfy the conditions outlined at the beginning of §3:

(1) L is a graded Lie algebra generated by elements $x_1, \dots, x_m$ of degree 1; every element from the Lie set $S = S\langle x_1, \dots, x_m \rangle$ is ad-nilpotent;

(2) L satisfies a polynomial identity;

(3) the grading of L extends to A; both algebras L and A are graded just infinite.

Recall that an element $g(x_1, \dots, x_r)$ of the free associative algebra is called a weak identity of the pair (L, A) if $g(a_1, \dots, a_r) = 0$ for arbitrary elements $a_1, \dots, a_r \in L$. In particular, every Lie identity of the algebra L can be viewed as a weak identity of the pair (L, A).

Let k be a minimal degree of a nonzero weak identity satisfied by (L, A). Without loss of generality we can assume that (L, A) satisfies a weak identity

$$h(x_1, \dots, x_k) = x_1 \cdots x_k + \sum_{1 \neq \sigma \in S_k} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}.$$

We remark that the pair $(\widetilde{L}, \widetilde{A})$ satisfies this weak identity as well.

Let $h(x_1,\dots,x_k) = \sum_{i=1}^{k} h_i(x_1,\dots,\hat{x_i},\dots,x_k)x_i$. The ideal M of the algebra A generated by all values of $h_k(a_1,\dots,a_{k-1})$, $a_i \in L$ is graded nonzero and therefore has finite codimension in A. Hence there exists $d \geq 1$ such that $A^d \subseteq M$. Denote $\widehat{A} = A + F1$,

Lemma 6.1. *For an arbitrary element $a \in \widetilde{L}$ we have*

$$A^d a \subseteq \sum x_{i_1}\cdots x_{i_t} a\widehat{A},\ t \leq d-1.$$

Proof. For an arbitrary product of length d we have

$$x_{i_1}\cdots x_{i_d} = \sum_j \alpha_j v_j' h_k(\rho_{j1},\dots,\rho_{j,k-1})v_j'',$$

where $\alpha_j \in F$; v_j', v_j'', and $\rho_{j1},\dots,\rho_{j,k-1}$ are monomials and commutators in generators $x_1,\dots,x_m$ of total length d.

Let $v_j'' = x_{\mu_1}\cdots x_{\mu_r}$. Then

$$v_j''a = a' + \sum_t \pm w_{jt}' a w_{jt}'',$$

where $a' = [x_{\mu_1},[x_{\mu_2},[\dots,[x_{\mu_r},a]\cdots]$; w_{jt}', w_{jt}'' are products in generators of total degree equal to the degree of v_j'' and the products w_{jt}'' are not empty. Hence,

$$x_{i_1}\cdots x_{i_d}a = \sum_j \alpha_j v_j' h_k(\rho_{j,1},\dots,\rho_{j,k-1})a' + \sum_{j,t} \alpha_j v_j' h_k(\rho_{j,1},\dots,\rho_{j,k-1})w_{jt}' a w_{jt}''.$$

Furthermore,

$$h_k(\rho_{j1},\dots,\rho_{j,k-1})a' = h(\rho_{j1},\dots,\rho_{j,k-1},a') - \sum_{i=1}^{k-1} h_i(\rho_{j1},\dots,a',\dots,\rho_{j,k-1})\rho_{ji}$$
$$= -\sum_{i=1}^{k-1} h_i(\rho_{j1},\dots,a',\dots,\rho_{j,k-1})\rho_{ji}.$$

We proved that $x_{i_1}\cdots x_{i_d}a$ is a linear combination of elements $w'aw''$, where w', w'' are products in $x_1,\dots,x_m$, with the length of w' less than d. □

Consider the function $g(m,n) = p^{(n+1)^{mf(m,n)}}$.

Lemma 6.2. *Let I be an ideal of the algebra L. Let a be a sandwich in $\widetilde{I}$ and $a^p = 0$ in $\widetilde{A}$. Let $a' = [a_1,\dots,a_d]$, where $a_i = [a,u_{i1},\dots,u_{it_i}]$, $u_{ij} \in \widetilde{L}$, $t_i \geq 0$, $i = 1,\dots,d$. Let $G = g(\max_i\{t_i + d - 1\}, t_1 + \cdots + t_d + d^{d+1}\})$. Then $(Aa')^G = (0)$. In particular, an arbitary element from $\mathrm{Id}_{\widetilde{L}}(a')^d$ generates a nilpotent ideal in $\widetilde{A}$.*

Proof. Suppose that $a'v_1a'\cdots v_{G-1}a' \neq 0$, where $v_1,\dots,v_{G-1}$ are products in generators $x_1,\dots,x_m$ of lengths $l(v_1),\dots,l(v_{G-1})$ respectively. Without loss of generality we will assume that the vector of lengths $(l(v_1),\dots,l(v_{G-1}))$ is lexicographically minimal. By Lemma 6.1 this implies that $l(v_i) \le d-1$, $1 \le i \le G-1$.

If $v = x_{j_1}\cdots x_{j_r}$ then we denote

$$[va'] = [x_{j_1},[x_{j_2},[\cdots[x_{j_r},a']\cdots] = (-1)^r[a',x_{j_r},x_{j_{r-1}},\dots,x_{j_1}].$$

Again by lexicographical minimality we have

$$a'v_1a'\cdots v_{G-1}a' = a'[v_1a']\cdots[v_{G-1}a'].$$

Lemma 5.4 is not applicable to the sandwich a' and elements $x_1,\dots,x_m$ because $x_1,\dots,x_m$ do not lie in $\widetilde{L}$. However, for an arbitrary word $v = x_{j_1}\cdots x_{j_r}$, $r<d$, we have

$$[va'] = \sum_i [[v_{i1}a_1],\dots,[v_{id}a_d]],$$

where $v_{i1},\dots,v_{id}$ are words in $x_1,\dots,x_m$ of total length $r<d$. Hence, at least one of these words is empty. Now we can apply Lemma 5.4 to the sandwich a and the set

$$S = \{[a,u_{i1},\dots,u_{it_i},x_{j1},\dots,x_{jr}] \mid 1\le i\le d,\ 1\le j_1,\dots,j_r\le m,\ 0\le r<d\} \subset \widetilde{L}.$$

By Lemma 5.4 and the choice of the number G, we have $a'[v_1a']\cdots[v_{G-1}a'] = 0$, which proves the lemma. □

Now our aim is the following:

Proposition 6.3. *There exist integers $N\ge 1$, $s\ge 1$ and a regular divided polynomial v defined on L^s such that every value of v on $\widetilde{L}^s$ generates a nilpotent ideal in $\widetilde{A}$ of degree $\le N$.*

Suppose that the algebra L satisfies an identity

$$f(x_0,x_1,\dots,x_{n-1}) = [x_0,x_1,\dots,x_{n-1}] + \sum_{1\neq\sigma\in S_{n-1}} \alpha_\sigma[x_0,x_{\sigma(1)},\dots,x_{\sigma(n-1)}],$$

where $\alpha_\sigma \in F$ and n is the minimal degree of an identity satisfied by L.

Consider the following element of degree $n-1$:

$$\begin{aligned} f_1(x_0,x_2,\dots,x_{n-1}) &= [x_0,x_2,\dots,x_{n-1}] + \sum_{\substack{1\neq\sigma\in S_{n-1}\\ \sigma(1)=1}} \alpha_\sigma[x_0,x_{\sigma(2)},\dots,x_{\sigma(n-1)}] \\ &= x_0H(\mathrm{ad}(x_2),\dots,\mathrm{ad}(x_{n-1})), \end{aligned}$$

where

$$H(y_2,\dots,y_{n-1}) = y_2\cdots y_{n-1} + \sum_{\substack{1\neq\sigma\in S_{n-1}\\ \sigma(1)=1}} \alpha_\sigma y_{\sigma(2)}\cdots y_{\sigma(n-1)}.$$

Lemma 6.4. *Let $w(x_1,\dots,x_r)$ be a regular divided polynomial defined on L^s. Then the divided polynomial*

$$w'(x_1,\dots,x_r,y_2,\dots,y_{n-1}) = f_1(w(x_1,\dots,x_r),y_2,\dots,y_{n-1})$$

is also regular.

Proof. Choose $t\geq s$ and elements $a_1,\dots,a_r\in\widetilde{L}^t$ such that $a'=w(a_1,\dots,a_r)\neq 0$. Suppose that for arbitrary elements $b_2,\dots,b_{n-1}\in\widetilde{L}^t$ we have $f_1(a',b_2,\dots,b_{n-1})=0$. Let $a'=\sum_\pi a'_\pi$ be the standard decomposition, $a'_\pi=\bar{a'_\pi}\otimes e_\pi, \bar{a'_\pi}\in L^t$. Then the assumption above means that for an arbitrary π we have $f_1(\bar{a'_\pi},L^t,\dots,L^t)=(0)$. Choose π such that $\bar{a'_\pi}\neq 0$. Let $R=\langle \mathrm{Id},\mathrm{ad}(x),x\in L\rangle\subseteq \mathrm{End}_F(L)$ be the multiplication algebra of the algebra L. Consider the ideal $\mathrm{id}_L(\bar{a'_\pi})=\bar{a'_\pi}R$ generated by the element $\bar{a'_\pi}$ in L.

For an arbitrary element $x\in L$ we have

$$\big[\mathrm{ad}(x), H(\mathrm{ad}(L^t),\dots,\mathrm{ad}(L^t))\big]\subseteq H(\mathrm{ad}(L^t),\dots,\mathrm{ad}(L^t)).$$

Hence,

$$RH(\mathrm{ad}(L^t),\dots,\mathrm{ad}(L^t))\subseteq H(\mathrm{ad}(L^t),\dots,\mathrm{ad}(L^t))R.$$

Now for an arbitrary operator $P\in R$ we have

$$\begin{aligned} f_1(\bar{a'_\pi}P,L^t,\dots,L^t) &= \bar{a'_\pi}PH(L^t,\dots,L^t)\\ &\subseteq \bar{a'_\pi}H(L^t,\dots,L^t)R = f_1(\bar{a'_\pi},L^t,\dots,L^t)R=(0).\end{aligned}$$

Since the algebra L is graded just infinite it follows that $\mathrm{id}_L(\bar{a'_\pi})\supseteq L^k$ for sufficiently large $k\geq t$. We proved that $f_1(L^k,\dots,L^k)=(0)$.

The algebra L^k is finitely generated by Lemma 2.4. By the induction assumption on the degree of the identity f we conclude that the Lie algebra L^k is finite dimensional. Therefore the algebra L is finite dimensional as well which contradicts our assumption that the algebra L is graded just infinite, proving the lemma. □

Corollary 6.5. *Let $q\geq 1$. Choose $r+q(n-2)$ distinct variables $x_1,\dots,x_r$, y_{ij}, $1\leq i\leq q$, $2\leq j\leq n-1$. Then the divided polynomial*

$$w_q = w(x_1,\dots,x_r)H(\mathrm{ad}(y_{11}),\dots,\mathrm{ad}(y_{1,n-1}))\cdots H(\mathrm{ad}(y_{q1},\dots,\mathrm{ad}(y_{q,n-1}))$$

is regular.

Lemma 6.6. *For arbitrary elements $a, b_2, \dots, b_{n-1}, c \in \widetilde{L}$ we have*

$$\big[f_1(a, b_2, \dots, b_{n-1}), c\big] \in \sum F\big[a, b_{i_2}, \dots, \big[b_{i_k}, c\big], \dots, b_{i_{n-1}}\big],$$

where $i_2, i_3, \dots, i_{n-1}$ is a permutation of $2, \dots, n-1$, $2 \le k \le n-1$.

Proof. We have

$$\begin{aligned} aH(\mathrm{ad}(b_2), \dots, \mathrm{ad}(b_{n-1})\,\mathrm{ad}(c) &= a\,\mathrm{ad}(c)H(\mathrm{ad}(b_2), \dots, \mathrm{ad}(b_{n-1})) \\ &\quad + \sum aH(\mathrm{ad}(b_2), \dots, \mathrm{ad}([b_k, c]), \dots, \mathrm{ad}(b_{n-1})). \end{aligned}$$

Let us represent the polynomial f of minimal degree as

$$f(x_0, x_1, \dots, x_{n-1}) = \sum_{i=1}^{n-1} x_0\,\mathrm{ad}(x_i)H_i(\mathrm{ad}(x_1), \dots, \widehat{\mathrm{ad}(x_i)}, \dots, \mathrm{ad}(x_{n-1})),$$

where $H_1 = H$. Then

$$\begin{aligned} a\,\mathrm{ad}(c)H(\mathrm{ad}(b_2), \dots, \mathrm{ad}(b_{n-1})) &= -c\,\mathrm{ad}(a)H(\mathrm{ad}(b_2), \dots, \mathrm{ad}(b_{n-1})) \\ &= \sum_{i=2}^{n-1}\big[c, b_i\big]H_i(\mathrm{ad}(a), \mathrm{ad}(b_2), \dots, \widehat{\mathrm{ad}(b_i)}, \dots, \mathrm{ad}(b_{n-1})) \\ &\in \sum F\big[a, b_{i_2}, \dots, \big[b_{i_k}, c\big], \dots, b_{i_{n-1}}\big]. \quad \square \end{aligned}$$

Lemma 6.7. *Let $\nu \ge 1$. Suppose that a divided polynomial $w(x_1, \dots, x_r)$ is defined on L^s and for all $t \le q(n-2) + \nu$ the divided polynomial*

$$\big[w(x_1, \dots, x_r), y_1, \dots, y_t, w(x_1, \dots, x_r)\big]$$

is identically zero on $\widetilde{L}^s$. Then for arbitrary elements $a_k, b_{ij} \in \widetilde{L}^s$ we have

$$\begin{aligned} \big[w_q(a_k, b_{ij}, 1 \le k \le r,\ 1 \le i \le q,\ 2 \le j \le n-1), \\ \underbrace{L, L, \dots, L}_{\mu}, \underbrace{L^s, \dots, L^s}_{\nu}, w(a_1, \dots, a_r)\big] = (0) \end{aligned}$$

for $\mu \le q$.

Proof. Applying Lemma 6.6 q times we get

$$\big[w_q(a_k, b_{ij}), \underbrace{L, \dots, L}_{\mu}\big] \subseteq \big[w(a_1, \dots, a_r), \underbrace{\widetilde{L}^s, \dots, \widetilde{L}^s}_{q(n-2)}\big],$$

which implies the assertion of the lemma. $\square$

Lemma 6.8. *Let*

$$a \in \widetilde{L} \quad \text{and} \quad [a, \underbrace{\widetilde{L}, \dots, \widetilde{L}}_{\mu}, a] = (0)$$

for $\mu \le 2d$. Then for an arbitrary element $b \in \widetilde{L}$ the commutator $[a, b]$ generates a nilpotent ideal in $\widetilde{A}$ of degree $\le m^d(p-1)+1$.

Proof. Recall that the algebra L is generated by m elements $x_1, \dots, x_m$. Suppose that $[a, b]v_1[a, b]\cdots v_{N-1}[a, b] \ne 0$, where v_i are products of the generators and the vector of lengths $(l(v_1), \dots, l(v_{N-1}))$ is lexicographically minimal among all vectors with this property. Then by Lemma 6.1 $l(v_i) < d$ for $i = 1, \dots, N-1$.

As above, for a product $v = x_{i_1}, \dots, x_{i_k}$ we denote

$$[v[a, b]] = [x_{i_1}[x_{i_2}, [\cdots[x_{i_k}, [a, b]]\cdots].$$

We have

$$[a, b]v_1\cdots v_{N-1}[a, b] = [a, b][v_1, [a, b]]\cdots[v_{N-1}[a, b]].$$

By our assumption the commutators $[a, b], [v_i[a, b]], 1 \le i \le N-1$, commute. There are $< m^d$ such commutators. Hence at least one commutator $[v_i[a, b]]$ occurs $\ge p$ times. If $b = \sum_\pi b_\pi$ is the standard decomposition then $[v_i[a, b]]^p$ is a sum of expressions

$$\begin{aligned}\{[v_i[a, b_{\pi_0}]], \dots, [v_i[a, b_{\pi_{p-1}}]]\} \\ = \sum_{\sigma \in S_{p-1}} [[v_i, [a, b_{\pi_0}]], [v_i[a, b_{\pi_{\sigma(1)}}]], \dots, [v_i[a, b_{\pi_{\sigma(p-1)}}]]] = 0.\end{aligned}$$

Hence $[v_i[a, b]]^p = 0$, which finishes the proof of the lemma. □

Lemma 6.9. *For an arbitrary sandwich of the algebra $\widetilde{L}^s$ we have $a^p = 0$ in $\widetilde{A}$.*

Proof. Recall that the algebra A is a homomorphic image of the subalgebra $\langle \mathrm{ad}(x), x \in L\rangle \subseteq \mathrm{End}_F(L)$.

Hence it is sufficient to show that $\mathrm{ad}(a)^p = 0$ in $\widetilde{L}$. If $p \ge 3$ then

$$\widetilde{L}\,\mathrm{ad}(a)^p \subseteq [\widetilde{L}^s, a, a] = (0).$$

Let $p = 2$. We have

$$[\widetilde{L}^s, a, a] = (0).$$

Since the mapping $\widetilde{L} \to \widetilde{L}, x \to [x, a, a]$ is a derivation of $\widetilde{L}$ it follows that $[\widetilde{L}, a, a]$ lies in the centralizer of $\widetilde{L}^s$. In a graded just infinite algebra L the centralizer of L^s is zero. Hence $[\widetilde{L}, a, a] = (0)$ and again $\mathrm{ad}(a)^2 = 0$, which finishes the proof of the lemma. □

Proof of Proposition 6.3. We will start with the regular divided polynomial

$$v(x_1, \dots, x_r)$$

of Proposition 4.14 defined on L^s. Every value of v on $\widetilde{L}^s$ is a sum of t sandwiches of the algebra $\widetilde{L}^s$.

We will construct a sequence of finite sets M_i of divided polynomials defined on L^s, $i \geq 0$.

Let $M_0 = \{v\}$,

$$M_{i+1} = \big\{[w, y_1, \dots, y_\mu, w] \mid w \in M_i,\ \mu \leq 4d(n-2);\\ y_1, \dots, y_\mu \text{ are variables not involved in } w\big\}.$$

Let $2^i \geq (d-1)t + 1$, $w \in M_i$. Let b be a value of the divided polynomial v on $\widetilde{L}^s$. By Proposition 4.14, $b = b_1 + \cdots + b_t$, where for $1 \leq j \leq t$, each element b_j is a sandwich of the algebra $\widetilde{L}^s$. Since $2^i \geq (d-1)t + 1$, it follows that the value c of the divided polynomial w that is obtained by iterating the value b of v lies in $\sum_{j=1}^{t} \mathrm{id}_{\widetilde{L}}(b_j)^d$. By Lemma 6.2 there exists an integer $G \geq 1$ such that every value of the divided polynomial w on $\widetilde{L}^s$ generates a nilpotent ideal in $\widetilde{A}$ of degree $\leq G$.

If at least one divided polynomial in M_i is regular, then we are done.

If none of the divided polynomials in M_i is regular and i is the minimal integer with this property then there exists a regular divided polynomial $w(x_1, \dots, x_r)$ defined on $\widetilde{L}^s$ and an integer $t \geq s$ such that all the divided polynomials

$$\big[w(x_1, \dots, x_r), y_1, \dots, y_\mu, w(x_1, \dots, x_r)\big], \quad 1 \leq \mu \leq 4d(n-2),$$

are identically zero on $\widetilde{L}^t$.

Consider the regular divided polynomial

$$w_{2d}(x_1, \dots, x_r, y_{ij},\ 1 \leq i \leq 2d,\ 2 \leq j \leq n-1).$$

By Lemma 6.7 we have

$$\big[w_{2d}(a_1, \dots, a_r, b_{ij}), \underbrace{L, L, \dots, L}_{\mu}, \underbrace{L^t, \dots, L^t}_{\nu}, w(a_1, \dots, a_r)\big] = (0),$$

$\mu \leq 2d$, $\nu \leq 2d(n-2)$ for arbitrary elements $a_1, \dots, a_r, b_{ij} \in \widetilde{L}^t$. We have

$$w_{2d}(a_1, \dots, a_r, b_{ij}) \in \sum_{\nu=1}^{2d(n-2)} \big[w(a_1, \dots, a_r), \underbrace{L^t, \dots, L^t}_{\nu}\big].$$

Therefore

$$\big[w_{2d}(a_1, \dots, a_r, b_{ij}), \underbrace{L, L, \dots, L}_{\mu}, w_{2d}(a_1, \dots, a_r, b_{ij})\big] = (0)$$

for $\mu \leq 2d$.

The divided polynomial

$$w'_{2d}(x_0, x_1, \dots, x_r, y_{ij}) = \big[w_{2d}(x_1, \dots, x_r, y_{ij}), x_0\big]$$

is regular. Indeed, regularity of w_{2d} has been established in Corollary 6.5. If the divided polynomial w'_{2d} vanishes on some power of L, then some power of L has a nonzero centralizer. Since the algebra L is graded just infinite, it follows that this centralizer is of finite codimension. Hence the algebra L is solvable and therefore finite dimensional, a contradiction. By Lemma 6.8 an arbitrary value of w'_{2d} on $\widetilde{L}^t$ generates a nilpotent ideal in $\widetilde{A}$ of degree $\leq m^d(p-1)+1$. This finishes the proof of Proposition 6.3. □

By Proposition 6.3 there exist integers $s \geq 1$, $N_1 \geq 1$ and a regular divided polynomial w defined on L^s such that every value of w on $\widetilde{L}^s$ generates a nilpotent ideal in $\widetilde{A}$ of degree $\leq N_1$. Let $f(x_1, x_2, \dots, x_r) \in \widetilde{L} * \mathrm{Lie}\langle X\rangle$ be the linearization of the divided polynomial w. If l is the degree of w then every value of f in $\widetilde{L}^s$ is a linear combination of 2^l values of w. Hence for arbitrary $a_1, a_2, \dots a_r \in \widetilde{L}^s$ the element $f(a_1, \dots, a_r)$ generates a nilpotent ideal in $\widetilde{A}$ of degree $\leq N_2 = 2^l(N_1-1)+1$.

Choose variables $x_{ij} \in X$, $1 \leq i \leq r$, $1 \leq j \leq N_2$. The pair (L, A) satisfies the system of weak identities

$$F_{N_2} = \Big\{F_{N_2}(x_{ij}, y_k) = \sum_{\sigma_1,\dots,\sigma_r \in S_{N_2}} f(x_{1\sigma_1(1)}, x_{2\sigma_2(1)}, \dots, x_{r\sigma_r(1)})y_1 \\ f(x_{1\sigma_1(2)}, x_{2\sigma_2(2)}, \dots, x_{r\sigma_r(2)})y_2 \cdots y_{N_2-1} \\ f(x_{1\sigma_1(N_2)}, x_{2\sigma_2(N_2)}, \dots, x_{r\sigma_r(N_2)}) = 0\Big\},$$

where x_{ij} take values in L^s and y_k's are arbitrary products of independent variables taking values in L. Thus values of y_k's lie in A.

Indeed, these weak identities are satisfied by the pair $(\widetilde{L}, \widetilde{A})$. It remains to notice that the pair (L, A) and $(\widetilde{L}, \widetilde{A})$ satisfy the same multilinear weak identities.

Let N be the minimal integer such that (L, A) satisfies the weak identities $F_N = 0$.

Let $a = F_{N-1}(a_{ij}, b_k) \neq 0$, $a_{ij} \in L^s$, $1 \leq i \leq r$, $1 \leq j \leq N-1$; $b_k \in A$. Choose $t > \max_{i,j} \deg(a_{ij})$. By regularity of the polynomial f, there exist homogeneous elements $a_1, \dots, a_r \in L^t$ such that $f(a_1, \dots, a_r) \neq 0$. The identities $F_N = 0$ immediately imply the following lemma.

Lemma 6.10. *Let $b_\mu, c_\mu \in A + F1$. Then*

$$\Big(\sum_\mu b_\mu f(a_1, \dots, a_r)c_\mu\Big)\, a$$

is a linear combination of elements

$$\Big(\sum_\mu b_\mu f(a'_1, \dots, a'_r)c_\mu\Big)\, a',$$

where $a' \in A$, $a'_1 \in \{a_1, a_{1k},\ 1 \leq k \leq N-1\}$, $a'_2 \in \{a_2, a_{2k},\ 1 \leq k \leq N-1\}, \dots$, and at least one a'_i lies in $\{a_{ik},\ 1 \leq k \leq N-1\}$.

The ideal I generated by $f(a_1, \ldots, a_r)$ in A has finite codimension. Let $A^l \subseteq I$.

Corollary 6.11. *Let u be a homogeneous element of degree $\geq l$. Then for arbitrary $b_\mu, c_\mu \in A + F1$ the element*

$$\Big(\sum_\mu b_\mu u c_\mu\Big)\, a$$

is a linear combination of elements

$$\Big(\sum_\mu b_\mu u' c_\mu\Big)\, a',$$

where $u' \in A$ are homogeneous elements, $\deg u' < \deg u$.

Indeed, from $u \in A^\ell \subseteq I$ it follows that

$$u = \sum_\nu b'_\nu f(a_1, \ldots, a_r)\, c'_\nu,$$

where b'_ν, c'_ν are homogeneous elements,

$$\deg b'_\nu + \deg c'_\nu + \sum_{i=1}^{r} \deg a_i = \deg u.$$

Lemma 6.10 implies that

$$\Big(\sum_{\mu,\nu} b_\mu b'_\nu f(a_1, \ldots, a_r) c'_\nu c_\mu\Big)\, a$$

is a linear combination of elements

$$\Big(\sum_{\mu,\nu} b_\mu b'_\nu f(a'_1, \ldots, a'_r) c'_\nu c_\mu\Big)\, a',$$

where $\deg a'_i \leq \deg a_i$ for all i and at least for one i, we have $\deg a'_i < \deg a_i$. Hence for

$$u' = \sum_\nu b'_\nu f(a'_1, \ldots, a'_r)\, c'_\nu$$

we have $\deg u' < \deg u$.

Lemma 6.12. *Let $h(x_1, \ldots, x_q)$ be a multilinear element of the free associative algebra such that for arbitrary homogeneous elements $u_1, \ldots, u_q \in A$ of degrees $\deg u_1 < l, \ldots, \deg u_q < l$ we have $h(u_1, \ldots, u_q) = 0$. Then $h = 0$ holds identically on A.*

Proof. Let $v_1, \ldots, v_q \in A$ be homogeneous elements of A such that $h(v_1, \ldots, v_q) \neq 0$. Let the total degree $\sum_{i=1}^{q} \deg(v_i)$ be minimal among all q-tuples with this property. At least one element v_i has degree $\geq l$.

Let us show that the graded just infinite algebra A is graded prime. Indeed, if I_1, I_2 are nonzero graded ideals of A then $A^{t_1} \subseteq I_1, A^{t_2} \subseteq I_2$ for some integers $t_1, t_2 \geq 1$. If $I_1 I_2 = (0)$ then $A^{t_1+t_2} = (0)$, a contradiction.

Hence there exists an element $b \in A$ such that $h(v_1, \ldots, v_q)ba \neq 0$.

By Corollary 6.11 the element $h(v_1, \ldots, v_q)ba$ is a linear combination of elements $h(v_1, \ldots, v_{i-1}, v', v_{i+1}, \ldots, v_q)ba'$, where $\deg v' < \deg v_i$. This contradicts the minimality of $\sum_{i=1}^{q} \deg(v_i)$ and finishes the proof of the lemma. □

Remark. A nonzero element $h(x_1, \ldots, x_q)$ satisfying the hypothesis of Lemma 6.12 exists for an arbitrary finitely generated algebra. Moreover, for an arbitrary associative algebra A and a finite dimensional subspace $V \subset A$, there exists a multilinear element $h(x_1, \ldots, x_q)$ of the free associative algebra such that $h(u_1, \ldots, u_q) = 0$ for all elements $u_1, \ldots, u_q \in V$. Indeed, it is sufficient to choose an element that is skew-symmetric in $x_1, \ldots, x_q$, where $q = \dim_F V + 1$, for example the element

$$h(x_1, \ldots, x_q) = \sum_{\sigma \in S_q} (-1)^{|\sigma|} x_{\sigma(1)} \cdots x_{\sigma(q)}.$$

Proof of Theorem 1.1. It is known (see [27]) that a graded prime algebra is prime. By Lemma 6.12, A is a PI-algebra. The prime PI-algebra A has a nonzero center Z and the ring of fractions $(Z \setminus \{0\})^{-1}A$ is a finite dimensional associative algebra over the field $(Z \setminus \{0\})^{-1}Z$ (see [28]). Now the Engel–Jacobson theorem [10] implies that the algebra A is nilpotent, a contradiction. Thus Theorem 1.1 is proved. □

Proof of Theorem 1.2. Let G be a residually-p finitely generated torsion group. Let $G = G_1 \geq G_2 \geq \cdots$ be the Zassenhaus filtration. Consider the Lie algebra $L_p(G) = \bigoplus_{i \geq 1} G_i/G_{i+1}$. Because of the torsion property of elements of G, an arbitrary homogeneous element $a \in G_i/G_{i+1}$ of the Lie algebra $L_p(G)$ is ad-nilpotent (see [18,34]).

Consider the subalgebra L of $L_p(G)$ generated by G_1/G_2. If the Lie algebra $L_p(G)$ satisfies a polynomial identity then by Theorem 1.1 the finitely generated Lie algebra L is nilpotent. This implies that the pro-p completion $G_{\hat{p}}$ of the group G is p-adic analytic and therefore linear (see [3]). Now finiteness of G follows from theorems of Burnside and Schur [11]. □

Proof of Theorem 1.3. Let Fr be the free pro-p group. Let $Fr = Fr_1 > Fr_2 > \cdots$ be the Zassenhaus filtration of Fr. Suppose that the pro-p completion $G_{\hat{p}}$ satisfies the pro-p identity $w = 1$, $w \in Fr_n \setminus Fr_{n+1}$, hence $w = \rho_1^{p^{s_1}} \cdots \rho_r^{p^{s_r}} w'$, where each ρ_i is a left normed group commutator of length l_i, $p^{s_i} \cdot l_i = n$, $w' \in Fr_{n+1}$.

Considering, if necessary, $[w, x_0]$ instead of w, we can assume that n is not a multiple of p, and $w = \rho \cdots \rho_r w'$, where all commutators $\rho_1, \dots, \rho_r$ are of length n.

Let $\bar{\rho_i}$ be the commutator from the free Lie algebra that mimics the group commutator ρ_i. Then the Lie algebra $L_p(G)$ satisfies the polynomial identity $\sum_i \bar{\rho_i} = 0$. By Theorem 1.2 we conclude that $|G| < \infty$. □

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