J. Comb. Algebra. 1 (2017), 59–62
DOI 10.4171/JCA/1-1-3

Journal of Combinatorial Algebra

Kazhdan groups whose FC-radical is not virtually abelian

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Abstract. We construct examples of residually finite groups with Kazhdan's property (T) whose FC-radical is not virtually abelian. This answers a question of Popa and Vaes about possible fundamental groups of II_1 factors arising from Kazhdan groups.

Mathematics Subject Classification (2010). 20F05, 20F24; 20E18.

Keywords. Kazhdan's property (T), Golod–Shafarevich groups, FC-radical.

1. Introduction

If G is a group, the *FC-radical* of G, denoted by $FC(G)$, is the set of all elements of G which centralize a finite index subgroup of G. Equivalently, $FC(G)$ is the set of all elements of G with finite conjugacy class.

In [4] Popa and Vaes asked the following question:

Question 1.1. *Does there exist a residually finite (discrete) group with Kazhdan's property* (T) *whose FC-radical is not virtually abelian?*

This question was motivated by [4, Theorem 6.4(b)], which asserts that any group satisfying these conditions admits a free ergodic profinite action whose associated II_1 factor has all positive real numbers in its fundamental group. No Kazhdan group with the latter property was previously known.

In this short note we give a positive answer to Question 1.1 using Golod–Shafarevich groups.

We shall prove the following theorem:

Theorem 1.2. *Every Golod–Shafarevich group has a residually finite quotient whose FC-radical is not virtually abelian.*

In [2] it was shown that there exist Golod–Shafarevich groups with property (T). Since property (T) is preserved by quotients, applying Theorem 1.2 to any Golod–Shafarevich group with (T), we obtain a group which settles Question 1.1.

Acknowledgements. I am grateful to Mark Sapir for telling me about Question 1.1.

*The author was supported in part by the NSF grants DMS-0901703 and DMS-1201452 and the Sloan Research Fellowship grant BR 2011-105.

2. Construction

Informally speaking, a finitely generated group G is Golod–Shafarevich if it has a presentation with a "small" set of relators, where relators are counted with suitable weights. The formal definition is given below.

Definition 2.1. Let G be a finitely generated group. Given a prime p let $\mathbb{F}_p[G]$ be the $\mathbb{F}_p$-group algebra of G and I the augmentation ideal of $\mathbb{F}_p[G]$. Let $\{\omega_n G\}_{n\geq 1}$ be the Zassenhaus p-filtration of G defined by $\omega_n G = \{g \in G : g - 1 \in I^n\}$. For each $g \in G \setminus \bigcap_{n\in\mathbb{N}} \omega_n G$ we put $\deg_p(g)$ to be the largest n such that $g \in \omega_n G$.

We will need the following well-known properties of the Zassenhaus filtrations. They have been originally established by Jennings [3]; see also [1, Ch.11,12]:

(A) The subgroups $\{\omega_n G\}$ are of finite index in G. Moreover, they form a base for the pro-p topology on G, and thus $\bigcap_{n\in\mathbb{N}} \omega_n G$ is the kernel of the natural map from G to its pro-p completion $G_{\widehat{p}}$. In particular, if G is residually-p, then $\bigcap_{n\in\mathbb{N}} \omega_n G = \{1\}$, so $\deg_p(g)$ is defined for any $g \in G \setminus \{1\}$.

(B) $\omega_n G = \prod_{p^i\cdot j\geq n} (\gamma_j G)^{p^i}$ where $\gamma_j G$ is the j^{th} term of the lower central series of G and H^m is the subgroup generated by m^{th} powers of a group H. Therefore, if $\phi : G \to K$ is a group homomorphism, then $\omega_n(\phi(G)) = \phi(\omega_n G)$.

Definition 2.2. Fix a prime number p.

(a) A group presentation $\langle X|R\rangle$, with X finite, is said to satisfy the *Golod–Shafarevich* (GS) condition (with respect to p), if there is a real number $t \in (0, 1)$ such that

$$1 - H_X(t) + H_R(t) < 0 \text{ where } H_X(t) = |X|t \text{ and } H_R(t) = \sum_{r\in R} t^{\deg_p(r)}.$$

(b) A group G is called *Golod–Shafarevich* if it has a presentation satisfying the Golod–Shafarevich condition.

Any Golod–Shafarevich group G is infinite. In fact, its pro-p completion $G_{\widehat{p}}$ must be infinite and moreover satisfies a number of largeness properties (see, e.g., [2,6] and references therein for precise statements). We shall only use a very weak statement about Golod–Shafarevich groups:

Proposition 2.3. *If a group G is Golod–Shafarevich with respect to p, then its pro-p completion $G_{\widehat{p}}$ is not virtually abelian.*

Proposition 2.3 follows, for instance, from a theorem of Wilson [5] which asserts that every Golod–Shafarevich group has an infinite torsion quotient.

Proof of Theorem 1.2. Let G be a Golod–Shafarevich group, so that G has a presentation $\langle X|R\rangle$ with $1 - H_X(t) + H_R(t) < 0$ for some $t \in (0,1)$, and let $\varepsilon = -(1 - H_X(t) + H_R(t))$.

Let $k_0 \in \mathbb{N}$ be such that $t^{k_0} < \frac{\varepsilon}{8}$. By Proposition 2.3 we can choose $x_1, y_1 \in \omega_{k_0} G$ whose images in $G_{\widehat{p}}$ do not commute. Then there exists $k_1 > k_0$ such that x_1 and y_1 do not commute modulo $\omega_{k_1} G$. By making k_1 larger we can also assume that $t^{k_1} < \frac{\varepsilon}{16}$.

Let S_1 be a finite generating set for $\omega_{k_1} G$, let $R_1 = \{[x_1, s], [y_1, s] : s \in S_1\}$ and $G_1 = G/\langle R_1\rangle^G$. Note that if $\bar{x}_1$ and $\bar{y}_1$ are the images of x_1 and y_1 in G_1, then

(i) By property (B) above we have $G/\omega_{k_1} G \cong G_1/(\omega_{k_1} G/\langle R_1\rangle^G) = G_1/\omega_{k_1} G_1$, so $\bar{x}_1$ and $\bar{y}_1$ do not commute modulo $\omega_{k_1} G_1$;

(ii) $\bar{x}_1$ and $\bar{y}_1$ lie in the FC-radical of G_1 (and the same is true for any quotient of G_1).

The group G_1 need not be Golod–Shafarevich, but it surjects onto the group $\widehat{G}_1 = G_1/\langle x_1, y_1\rangle^{G_1}$ which is Golod–Shafarevich by construction.

Thus, the pro-p completion of G_1 is not virtually abelian, so we can find elements $x_2, y_2 \in \omega_{k_1} G$ and $k_2 > k_1$ such that the images of x_2 and y_2 in G_1 do not commute modulo $\omega_{k_2} G_1$ and $t^{k_2} < \frac{\varepsilon}{32}$.

Let S_2 be a finite generating set for $\omega_{k_2} G$, let $R_2 = \{[x_2, s], [y_2, s] : s \in S_2\}$ and $G_2 = G/\langle R_1 \cup R_2\rangle^G$. By construction we have $G_1/\omega_{k_2} G_1 \cong G_2/\omega_{k_2} G_2$, the images of x_2 and y_2 in G_2 lie in the FC-radical of G_2, and G_2 surjects onto a Golod–Shafarevich group.

Continuing this process indefinitely we obtain a sequence of groups

$$G = G_0 \to G_1 \to G_2 \to \cdots,$$

elements $\{x_i, y_i\}_{i\in\mathbb{N}}$ of G and integers $k_0 < k_1 < k_2 < \cdots$ s.t.

(i) G_{i+1} is a quotient of G_i for all i

(ii) G_i surjects onto the group $G/\langle \cup_{j=1}^{i} \{x_j, y_j\}\rangle^G$

(iii) x_i and y_i lie in $\omega_{k_{i-1}} G$, and $t^{k_{i-1}} < \frac{\varepsilon}{2^{i+2}}$

(iv) The images of x_i and y_i in $G_{i-1}/\omega_{k_i} G_{i-1}$ do not commute

(v) $G_{i-1}/\omega_{k_i} G_{i-1} \cong G_j/\omega_{k_i} G_j$ for all $j \geq i$

(vi) The images of x_i and y_i in G_i lie in the FC-radical of G_i

Now let G_∞ be the inductive limit of $\{G_i\}$; in other words, if $G_i = G/N_i$, we let $N_\infty = \cup_{i\in\mathbb{N}} N_i$ and $G_\infty = G/N_\infty$. Let Q be the image of G_∞ in its pro-p completion, that is, $Q = G_\infty/\cap_{n\in\mathbb{N}} \omega_n G_\infty$.

Condition (ii) implies that G_∞ surjects onto the group $G/\langle \cup_{j=1}^{\infty} \{x_j, y_j\}\rangle^G$ which is Golod–Shafarevich by (iii). Thus, by Proposition 2.3 the group Q is infinite. Since Q is a subset of $(G_\infty)_{\widehat{p}}$, it is also residually finite.

Let $\pi : G \to Q$ be the natural projection. By condition (vi) the FC-radical of Q contains the subgroup H generated by the elements $\{\pi(x_i), \pi(y_i)\}_{i\in\mathbb{N}}$. It remains to show that H is not virtually abelian. Suppose not, so H contains a finite index abelian subgroup A. Then there exists integers $i < j$ such that $\pi(x_i x_j^{-1}) \in A$ and $\pi(y_i y_j^{-1}) \in A$. Conditions (iv) and (v) imply that $\pi(x_i)$ and $\pi(y_i)$ do not commute modulo $\omega_{k_i} Q$. Thus, if ϕ_i is the projection map $Q \to Q/\omega_{k_i} Q$, then $\phi_i\pi([x_i, y_i]) \neq 1$. On the other hand, by construction $x_j, y_j \in \omega_{k_i} G$, so $\phi_i\pi(x_j) = \phi_i\pi(y_j) = 1$. Therefore,

$$\phi_i\pi([x_i, y_i]) = \phi_i\pi([x_i x_j^{-1}, y_i y_j^{-1}]) \in \phi_i([A, A]) = \{1\}.$$

The obtained contradiction shows that the FC-radical of Q is not virtually abelian, which finishes the proof. □

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Received 31 May, 2016

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