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On joins and intersections of subgroups in free groups

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Abstract. We study graphs of (generalized) joins and intersections of finitely generated subgroups of a free group. We show how to disprove a lemma of Imrich and Müller on these graphs, how to repair the lemma and how to utilize it.

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1. Introduction

Suppose that F is a free group of finite rank, $\mathrm{r}(F)$ denotes the rank of F, and $\bar{\mathrm{r}}(F) := \max(\mathrm{r}(F) - 1, 0)$ is the *reduced* rank of F. Let H, K be finitely generated subgroups of F and let $\langle H, K\rangle$ denote the subgroup generated by H, K, called the *join* of H, K. Hanna Neumann [21] proved that $\bar{\mathrm{r}}(H \cap K) \le 2\bar{\mathrm{r}}(H)\bar{\mathrm{r}}(K)$ and conjectured that $\bar{\mathrm{r}}(H \cap K) \le \bar{\mathrm{r}}(H)\bar{\mathrm{r}}(K)$. This problem, known as the Hanna Neumann conjecture on subgroups of free groups, was solved in the affirmative by Friedman [7] and Mineyev [19], see also Dicks's proof [3]. Relevant results and generalizations of this conjecture can be found in [2, 5, 6, 9–11, 13, 22].

Imrich and Müller [8] introduced the reduced rank $\bar{\mathrm{r}}(\langle H, K\rangle)$ of the join $\langle H, K\rangle$ in this context and attempted to prove the following inequality

$$\bar{\mathrm{r}}(H \cap K) \le 2\bar{\mathrm{r}}(H)\bar{\mathrm{r}}(K) - \bar{\mathrm{r}}(\langle H, K\rangle)\min(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K)) \tag{1.1}$$

under the assumption that if H^*, K^* are free factors of H, K, resp., then the equality $H^* \cap K^* = H \cap K$ implies that $H = H^*$ and $K = K^*$. Note that the inequality (1.1) provides a stronger bound than the Hanna Neumann conjecture's for $\bar{\mathrm{r}}(H \cap K)$ whenever $\bar{\mathrm{r}}(\langle H, K\rangle) > \max(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K))$ and this looks quite remarkable. We also note that the inequality (1.1) was introduced in [8] as an improvement of an earlier result of Burns [1], see also [20, 23], stating that

$$\bar{\mathrm{r}}(H \cap K) \le 2\bar{\mathrm{r}}(H)\bar{\mathrm{r}}(K) - \min(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K)).$$

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Later Kent [16] discovered a serious gap in the proof of a key lemma of Imrich–Müller [8, p. 195] and gave a different proof to the inequality (1.1) under the weakened assumption that $H \cap K \neq \{1\}$. Kent [16, p. 312] remarks that the key lemma in [8] "would be quite useful, and though its proof is incorrect, we do not know if the lemma actually fails." In this note we give an example that shows that the lemma of [8] is indeed false. On positive side, we suggest a repair for this lemma so that the other arguments of Imrich–Müller [8] could be saved.

Our approach seems to be of independent interest and can be outlined as follows. Suppose H and K are finitely generated subgroups of a free group F such that $H \cap K \neq \{1\}$. We deform the Stallings graphs of the free group F and its subgroups H, K, $\langle H, K\rangle$, $H \cap K$ so that the subgroups $\widetilde{H}, \widetilde{K}, \langle \widetilde{H}, \widetilde{K}\rangle, \widetilde{H} \cap \widetilde{K}$ corresponding to the modified Stallings graphs would satisfy the inequalities

$$\bar{\mathrm{r}}(\widetilde{H}) = \bar{\mathrm{r}}(H), \quad \bar{\mathrm{r}}(\widetilde{K}) = \bar{\mathrm{r}}(K), \quad \bar{\mathrm{r}}(\widetilde{H} \cap \widetilde{K}) = \bar{\mathrm{r}}(H \cap K)$$

and $\bar{\mathrm{r}}(\langle \widetilde{H}, \widetilde{K}\rangle) \geq \bar{\mathrm{r}}(\langle H, K\rangle)$. Furthermore, our modification is done in such a manner that the Stallings graphs of subgroups $\langle \widetilde{H}, \widetilde{K}\rangle$, $\widetilde{H}$, $\widetilde{K}$ would have vertices of degree 2 or 3 only and every vertex of the graph of $\langle \widetilde{H}, \widetilde{K}\rangle$ of degree 3 would have a preimage of degree 3 in one of the graphs of $\widetilde{H}$ or $\widetilde{K}$. These properties mean that the key lemma of Imrich–Müller [8, p. 195] holds in the modified setting and the arguments of Imrich–Müller could be retained with this modification.

As an application of this strategy developed in Section 3, we will prove Theorem 4.1 in Section 4 that contains specific inequalities similar to (1.1). Theorem 5.1 in Section 5 is concerned with a natural question related to our construction used in the proofs of Lemmas 3.4–3.5.

It is worthwhile to mention that the idea of deformations of Stallings graphs and changing the rank of the join $\langle H, K\rangle$, while keeping the ranks $\mathrm{r}(H), \mathrm{r}(K), \mathrm{r}(H \cap K)$ fixed, also works in the opposite direction, i.e. instead of increasing the rank $\mathrm{r}(\langle H, K\rangle)$ as in this article, one could decrease the rank $\mathrm{r}(\langle H, K\rangle)$ down to 2 under some natural restrictions, see [14] for details.

We remark that a different approach to the Hanna Neumann conjecture type of problems was presented by Louder and McReynolds [17], see also [18], who used their approach to give an alternative proof to a theorem of Dicks [2] and to answer a question of Culler and Shalen.

2. A counterexample to the lemma of Imrich and Müller

Similarly to Stallings [24], see also [2, 15, 16], we consider finite graphs associated with finitely generated subgroups of a free group F. We consider the ambient free group F as the fundamental group of a finite connected graph U, $F = \pi_1(U)$, without vertices of degree 1. Let H, K be finitely generated subgroups of F and let X, Y denote finite graphs associated with H, K, resp. Recall that there are locally injective graph maps $\varphi_X : X \to U$, $\varphi_Y : Y \to U$. Conjugating H, K if necessary, we may assume that the graphs X, Y have no vertices of degree 1, i.e. $\operatorname{core}(X) = X$ and $\operatorname{core}(Y) = Y$, where $\operatorname{core}(\Gamma)$ is the subgraph of a graph Γ consisting of all edges that can be included into circuits of Γ.

Let $S(H, K)$ denote a set of representatives of those double cosets $HtK \subseteq F$, $t \in F$, that have the property $H \cap tKt^{-1} \neq \{1\}$. Recall that the connected components of the core $W = \operatorname{core}(X \times_U Y)$ of the pullback $X \times_U Y$ of graph maps $\varphi_X : X \to U$, $\varphi_Y : Y \to U$ are in bijective correspondence with elements of the set $S(H, K)$, see [2, 15, 22]. Hence, we can write

$$W = \bigvee_{t \in S(H,K)} W_t,$$

where $\bigvee$ denotes a disjoint union. In addition, if W_t is a connected component of W that corresponds to $t \in S(H, K)$, then

$$\bar{\mathrm{r}}(H \cap tKt^{-1}) = |EW_t| - |VW_t|,$$

where $E\Gamma$ is the set of (nonoriented) edges of a graph Γ, $V\Gamma$ is the set of vertices of Γ and $|A|$ is the cardinality of a set A. Using the notation $\bar{\mathrm{r}}(\Gamma) := |E\Gamma| - |V\Gamma|$, we have

$$\bar{\mathrm{r}}(W) = \sum_{t \in S(H,K)} \bar{\mathrm{r}}(W_t) = \sum_{t \in S(H,K)} \bar{\mathrm{r}}(H \cap tKt^{-1}).$$

For $S_1 \subseteq S(H, K)$, where S_1 is not empty, denoting $W(S_1) := \bigvee_{s \in S_1} W_s$, we obtain

$$\bar{\mathrm{r}}(W(S_1)) = \sum_{s \in S_1} \bar{\mathrm{r}}(W_s) = \sum_{s \in S_1} \bar{\mathrm{r}}(H \cap sKs^{-1}) = \bar{\mathrm{r}}(H, K, S_1). \tag{2.1}$$

Let $\alpha_X : W(S_1) \to X$, $\alpha_Y : W(S_1) \to Y$ denote the restrictions on $W(S_1) \subseteq X \times_U Y$ of the pullback projection maps

$$\bar{\alpha}_X : X \times_U Y \to X, \quad \bar{\alpha}_Y : X \times_U Y \to Y,$$

resp. Consider the pushout P of these maps $\alpha_X : W(S_1) \to X$, $\alpha_Y : W(S_1) \to Y$. The corresponding pushout maps are denoted $\beta_X : X \to P$, $\beta_Y : Y \to P$. We also consider the Stallings graph Z corresponding to the subgroup $\langle H, K, S_1 \rangle$ of F. It is clear from the definitions that there are graph maps $\gamma : P \to Z$ and $\delta : Z \to U$ such that the diagram depicted in Figure 1 is commutative.

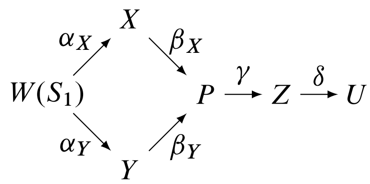

Figure 1.

As was found out by Kent [16, Figure 1], one has to distinguish between the graphs P and Z as $P \neq Z$ in general. Since the map $\delta\colon Z \to U$ is locally injective and $P \neq Z$, we see that the map $\gamma\colon P \to Z$ need not be locally injective, hence, γ factors out through a sequence of edge foldings and $\bar{\mathrm{r}}(P) \geq \bar{\mathrm{r}}(Z)$. According to Kent [16], the erroneous identification $P = Z$ is the source of a mistake in the key lemma of Imrich–Müller [8, p. 195]. Recall that this lemma claims, in our terminology, that if Z is an *almost trivalent* graph, i.e. every vertex of Z has degree 3 or 2, then every vertex in Z of degree 3 has a preimage of degree 3 in X or Y. Kent [16] explains in detail a mistake in the proof of this lemma and comments that, while the proof of lemma of [8, p. 195] can be somewhat corrected if one replaces Z by P, the subsequent arguments of Imrich–Müller rely on the property that $Z = U$ is trivalent, a property that the graph P might not possess. Furthermore, Kent [16, p. 312] points out that the lemma in [8] "would be quite useful, and though its proof is incorrect, we do not know if the lemma actually fails."

We now present an example that shows that the lemma of [8] does fail. Consider the following subgroups

$$H = \langle b_1 b_2 a_2 b_2^{-1} b_1^{-1},\ a_1^2,\ a_1 b_1 b_3 a_3 (a_1 b_1 b_3)^{-1} \rangle, \tag{2.2}$$

$$K = \langle b_1 b_2 a_2 b_2^{-1} b_1^{-1},\ a_1^3,\ a_1 b_1 b_3 a_3 (a_1 b_1 b_3)^{-1} \rangle \tag{2.3}$$

of the free group $F = \pi_1(U)$, where U is a bouquet of six circles labelled by $a_1, a_2, a_3, b_1, b_2, b_3$. The corresponding graphs X, Y, Z, resp., are depicted in Figure 2, where the base vertices of X, Y, Z are dashed circled.

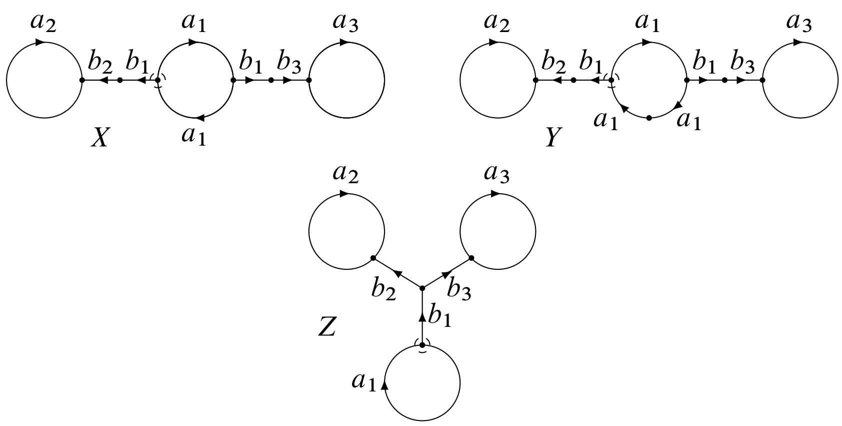

Figure 2.

It is easy to check that $S(H,K)$ has a single element, say $S(H,K) = \{1\}$, and

$$H \cap K = \langle b_1 b_2 a_2 b_2^{-1} b_1^{-1}, a_1^6, a_1 b_1 b_3 a_3 (a_1 b_1 b_3)^{-1} \rangle.$$

Furthermore, the graphs W, P look like those in Figure 3. Hence, we can see that Z is a trivalent graph that has a vertex of degree 3, which is the center of Z, without a preimage of degree 3 in $X \vee Y$, as desired.

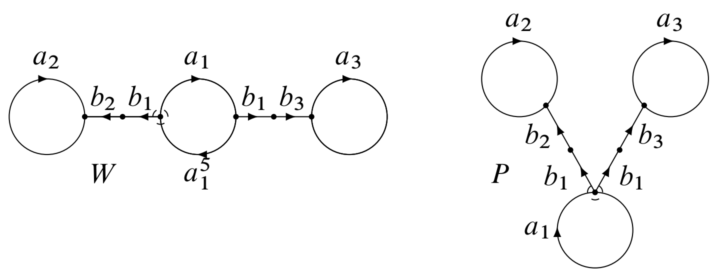

Figure 3.

3. Fixing the lemma of Imrich and Müller

We now discuss how to do certain deformations over the graphs $W(S_1)$, X, Y, P, Z, U to achieve the situation when $P = Z = U$, the graph U is almost trivalent and the lemma of [8, p. 195] would hold for Z.

Our idea could be illustrated by the remark that, when studying the graphs $W(S_1)$, X, Y, P, Z, U, or the corresponding subgroups, we can replace the ambient group $F = \pi_1(U)$ by $\widehat{F} = \pi_1(Z)$, i.e. we can replace the graph U by Z. We can go further and replace the group $F = \pi_1(U)$ with $\widetilde{F} = \pi_1(P)$ by using the pushout graph P in place of U.

Either of these replacements $U \to Z$, $U \to P$ results in some obvious cosmetic changes to subgroups H, K, to sets $S(H, K)$, $S_1 \subseteq S(H, K)$, and to subgroups $H \cap sKs^{-1}$, $s \in S_1$, $\langle H, K, S_1 \rangle$.

Either of these replacements $U \to Z$, $U \to P$ preserves the graphs $W(S_1)$, X, Y, P and the maps $\alpha_X, \alpha_Y, \beta_X, \beta_Y$.

The replacement $U \to Z$ turns δ into the identity map and the replacement $U \to P$ turns both δ, γ into the identity maps. Otherwise, the structure of the diagram depicted on Figure 1 is retained. In particular, the ranks $\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K), \bar{\mathrm{r}}(W(S_1))$ do not change but the rank $\bar{\mathrm{r}}(P)$ could increase since $\bar{\mathrm{r}}(P) \geq \bar{\mathrm{r}}(Z)$ for the original graphs.

For future references, we record the idea of this replacement as follows.

Lemma 3.1. *Replacing the graph U in the diagram depicted in Figure 1 by P and changing the maps $\gamma\colon P \to Z$, $\delta\colon Z \to U$ by $\gamma = \delta = \mathrm{id}_U$, resp., and changing the maps $\varphi_X\colon X \to U$, $\varphi_Y\colon Y \to U$ by $\beta_X\colon X \to P$, $\beta_Y\colon Y \to P$, resp., preserve the properties that the graph $W(S_1) = \bigvee_{s \in S_1} W_s$ consists of connected components of* $\mathrm{core}(X \times_U Y)$ *and that $P = X \vee_{W(S_1)} Y$. In particular, this replacement does not change the ranks $\bar{\mathrm{r}}(X), \bar{\mathrm{r}}(Y), \bar{\mathrm{r}}(W(S_1))$ but could increase $\bar{\mathrm{r}}(Z)$.*

Proof. If (e_1, e_2) is an edge of $W(S_1) \subseteq \operatorname{core}(X \times_U Y)$, where $\varphi_X \alpha_X((e_1, e_2)) = e_1$ is an edge of X and $\varphi_Y \alpha_Y((e_1, e_2)) = e_2$ is an edge of Y, then the edges e_1, e_2 are identified in P, whence, (e_1, e_2) is also an edge of $X \times_P Y$. Since $\operatorname{core}(W(S_1)) = W(S_1)$, it follows that $\operatorname{core}(X \times_P Y)$ contains all edges (e_1, e_2) of $W(S_1)$, hence, $\operatorname{core}(X \times_P Y)$ will contain the graphs naturally isomorphic to connected components of the graph $W(S_1) = \bigvee_{s \in S_1} W_s$. Therefore, the original pushout $P = X \vee_{W(S_1)} Y$ will also be preserved.

Since $\bar{\mathrm{r}}(P) \geq \bar{\mathrm{r}}(Z)$ for the original graphs P, Z and since $P = Z = U$ after the replacement $U \to P$, we see that the rank $\bar{\mathrm{r}}(Z)$ could increase after the replacement. □

Suppose that v is a vertex of U. Subdividing the edges incident to v if necessary, we may assume that every oriented edge e of U whose terminal vertex, denoted e_+, is $e_+ = v$ has the initial vertex, denoted e_-, of degree 2, $\deg e_- = 2$. Hence, we may assume that U contains a subgraph $\mathsf{St}(v)$ isomorphic to a tree with $\deg v > 1$ nonoriented edges whose common vertex is v, called the *star* around v. Let T be a tree whose vertices of degree 1 are in bijective correspondence with the vertices of degree 1 of $\mathsf{St}(v)$ whose set we denote by $V_1 \mathsf{St}(v)$. Taking $\mathsf{St}(v)$ out of U and putting the tree T in place of $\mathsf{St}(v)$, using the bijective correspondence to identify the vertices of degree 1 of $(U \setminus \mathsf{St}(v)) \cup V_1 \mathsf{St}(v)$ and those of T, results in a transformation of U which we call an *elementary deformation* of U around v by means of T. It is clear that the obtained graph $U_T := (U \setminus \mathsf{St}(v)) \cup T$ is homotopically equivalent to U and $\pi_1(U_T)$ is isomorphic to $F = \pi_1(U)$.

We lift this elementary deformation of the star around v in U to all the graphs X, Y, W, Z by replacement of stars around preimages of the vertex v by trees isomorphic to suitable subtrees of T and change, accordingly, the maps α_X, α_Y, $\beta_X, \beta_Y, \varphi_X, \varphi_Y$. The new graphs and the new maps obtained this way we denote by $X_T, Y_T, W(S_1)_T, P_T, Z_T$, and by $\alpha_{X_T}, \alpha_{Y_T}, \beta_{X_T}, \beta_{Y_T}, \varphi_{X_T}, \varphi_{Y_T}, \gamma_T, \delta_T$.

Clearly, $\bar{\mathrm{r}}(Q_T) = \bar{\mathrm{r}}(Q)$, where $Q \in \{X, Y, W, Z, U\}$. As far as the new pushout graph P_T is concerned, we can only claim that $\bar{\mathrm{r}}(P_T) \geq \bar{\mathrm{r}}(P)$. The proof of this inequality would be analogous to the following.

Lemma 3.2. *Let $P = Z = U$, let γ, δ be the identity maps and let the graphs $X_T, \dots, U_T$ be obtained from $X, \dots, U$ by an elementary deformation around a vertex $v \in VU$ by means of a tree T as defined above. Then the map $\delta_T \colon Z_T \to U_T$ is an isomorphism, i.e. the graphs Z_T, U_T are naturally isomorphic. Furthermore, the restriction of the map $\delta_T \gamma_T \colon P_T \to U_T$ is bijective on the set $\gamma_T^{-1} \delta_T^{-1}(U_T \setminus T)$ and the subgraph $\widehat{T} := \gamma_T^{-1} \delta_T^{-1}(T)$ of P_T is connected. In particular, $\bar{\mathrm{r}}(P_T) \geq \bar{\mathrm{r}}(P)$ and the equality holds if and only if $\widehat{T}$ is a tree.*

Proof. It follows from the definitions that if we collapse edges of $T \subset U_T$ into a point then we obtain the graph U back. Similarly, collapsing lifts of the edges of T in Q_T, $Q \in \{X, Y, W(S_1), P\}$, into points, we obtain the original graph Q. This observation, together with the local injectivity of the map δ_T, implies that $Z_T = U_T$ and that the graph $\widehat{T} := \gamma_T^{-1}\delta_T^{-1}(T)$ is connected.

Furthermore, the graph P_T consists of a subgraph isomorphic to

$$(U_T \setminus T) \cup V_1 T,$$

where $V_1 T$ is the set of vertices of T of degree 1, along with the graph $\widehat{T} = \gamma_T^{-1}\delta_T^{-1}(T)$ which is mapped by $\delta_T \gamma_T$ to T. Surjectivity of the restriction of $\delta_T \gamma_T$ on $\widehat{T}$ follows from connectedness of $\widehat{T}$. □

The graph $\widehat{T}$ of Lemma 3.2 can be regarded as a "blow-up" of the vertex $v \in VU$. If $\widehat{T}$ turns out to be a tree, then our attempt to nontrivially "blow up" the vertex v is unsuccessful. On the other hand, if $\widehat{T}$ is not a tree, then we can increase $\bar{\mathrm{r}}(U)$ by invoking Lemma 3.1 and picking P_T in place of U.

It is of interest to note that even when $P = Z = U$ and $\bar{\mathrm{r}}(P_T) = \bar{\mathrm{r}}(P)$, i.e. $\widehat{T}$ is a tree, the tree $\widehat{T}$ might look different from T, in this connection, see Lemma 3.4 and its proof.

The following is due to Kent [16]. We provide a proof for completeness.

Lemma 3.3. *Every vertex of P of degree at least 3 has a preimage in $X \vee Y$ of degree at least 3.*

Proof. Arguing on the contrary, assume that $u \in VP$ has degree $\deg u > 2$ and every lift $v \in VX \vee VY$ of u has degree 2. Let $v, v' \in VX \vee VY$ be two arbitrary lifts of u. It follows from the definitions of P, $W(S_1)$ that there is a sequence of vertices $v_1 = v, \dots, v_k = v'$ in $VX \vee VY$ with the following properties (A1)–(A3).

(A1) The vertices $v_1 = v, \dots, v_k = v'$ are mapped by β_X, β_Y to u and have degree 2.

(A2) For every $i = 1, \dots, k-1$, the vertices v_i, v_{i+1} belong to distinct connected components of $X \vee Y$.

(A3) For every $i = 1, \dots, k-1$, there is a vertex $w_i \in VW$ of degree 2 such that either $\alpha_X(w_i) = v_i$, $\alpha_Y(w_i) = v_{i+1}$ if $v_i \in VX$ or $\alpha_Y(w_i) = v_i$, $\alpha_X(w_i) = v_{i+1}$ if $v_i \in VY$.

It is clear from properties (A1)–(A3) and from the definitions of the graphs P, $W(S_1)$ that the vertices $v_1 = v, \dots, v_k = v' \in VX \vee VY$ will be identified in the pushout P so that the resulting vertex will have degree 2. Since $v, v' \in VX \vee VY$ were chosen arbitrarily, it follows that the degree of $u \in VP$ is also 2. This contradiction to $\deg u > 2$ proves Lemma 3.3. □

We say that T is a *trivalent tree* if T is a tree and every vertex of T has degree 1 or 3.

Lemma 3.4. *Suppose that $P = Z = U$ and v_0 is a vertex of U of degree at least 4. Then there is an elementary deformation of U around v_0 by means of a trivalent tree T such that either $\bar{\mathrm{r}}(P_T) > \bar{\mathrm{r}}(P)$ or the following are true: $\bar{\mathrm{r}}(P_T) = \bar{\mathrm{r}}(P)$, the subgraph $\widehat{T} = \gamma_T^{-1}\delta_T^{-1}(T)$ of P_T is a tree and*

$$\sum_{v \in VP_T} \max(\deg v - 3, 0) < \sum_{v \in VP} \max(\deg v - 3, 0).$$

Proof. Denote $\deg v_0 = m_0 > 3$. Let $e_1, \dots, e_{m_0}$ be all oriented edges of U that end in v_0. By Lemma 3.3, there is a vertex $v_1 \in VX \vee VY$, say $v_1 \in VX$, such that $\beta_X(v_1) = v_0$ and $\deg v_1 = m_1$, where $3 \le m_1 \le m_0$. Reindexing if necessary, we may assume that if $f_1, \dots, f_{m_1}$ are all of the edges of X that end in v_1, then $\beta_X(f_i) = e_i$, $i = 1, \dots, m_1$.

Let T be a trivalent tree with m_0 vertices of degree 1. Let U_T denote the graph obtained from U by an elementary deformation around v_0 by means of T.

Let u_i denote the vertex of T which becomes $(e_i)_-$ in U_T, $i = 1, \dots, m_0$. Here we are assuming that $\deg(e_i)_- = 2$ for every i as in the definition of an elementary deformation around v. It follows from Lemma 3.2 that if the subgraph $\widehat{T} = \gamma_T^{-1}\delta_T^{-1}(T)$ is not a tree for some T, that is, if $\bar{\mathrm{r}}(\widehat{T}) \ge 0$, then $\bar{\mathrm{r}}(\widehat{Z}_T) > \bar{\mathrm{r}}(\widehat{Z})$ and our lemma is proven. Hence, we may assume that $\widehat{T}$ is a tree for every trivalent tree T.

Suppose that $\widehat{T}$ contains no vertex of degree m_0 for some T. Then, obviously,

$$\sum_{v \in VP_T} \max(\deg v - 3, 0) < \sum_{v \in VP} \max(\deg v - 3, 0) = \sum_{v \in VU} \max(\deg v - 3, 0)$$

and our lemma is true.

Thus we may suppose that, for every trivalent tree T, $\widehat{T}$ is a tree which contains a vertex $\widehat{u}$ of degree m_0, i.e. $\widehat{T}$ is homeomorphic to a star.

It follows from the definitions of the graphs $P_T, \widehat{T}$ that the minimal subtree S of T that contains the vertices $u_1, \dots, u_{m_1}$ (recall $u_i = (e_i)_-$ in P_T) is isomorphic to a subtree $\widehat{S}$ of $\widehat{T}$. Indeed, a copy of S will show up in X_T in place of a star around the vertex v_1 of X, hence, a copy $\widehat{S}$ of S will also show up in P_T as a subgraph of $\widehat{T}$. Since T is trivalent and $\widehat{T}$ is a tree that has m_0 vertices of degree 1, one vertex $\widehat{u}$ of degree m_0 and other vertices of degree 2, it follows that $\widehat{S}$ contains a single vertex of degree ≥ 3 which implies $m_1 = 3$. Since T is an arbitrary trivalent tree, we may pick a tree $T = T_{12}$ that has adjacent vertices w_{12}, w_3 of degree 3 such that w_{12} is adjacent to both u_1, u_2 and w_3 is adjacent to u_3, see Figure 4(a). Note that, in this case, S is the subtree of $T = T_{12}$ that contains vertices $u_1, u_2, u_3, w_{12}, w_3$ and has 4 edges that connect these vertices, see Figure 4(a).

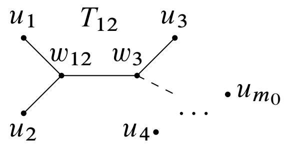

Figure 4(a).

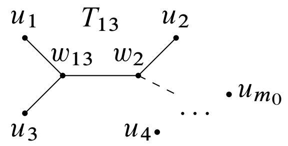

Figure 4(b).

Since the image $\widehat{w}_{12}$ of w_{12} in $\widehat{S}$ has degree 3 and $\widehat{T}_{12}$ contains a single vertex of degree at least 3, it follows that $\deg \widehat{w}_{12} = m_0$ in $\widehat{T}_{12}$. This, in particular, means that every preimage of w_3 in $\widehat{T}_{12}$ has degree 2 and adjacent to a preimage of w_{12}.

Consider a vertex $u \in VQ$, $Q \in \{X, Y\}$, such that $\beta_Q(u) = v_0$. Let $g_1, \dots, g_k$ be all oriented edges of Q that end in u. We claim that the set $\{\beta_Q(g_1), \dots, \beta_Q(g_k)\}$ may not contain both e_3 and e_j, where $j > 3$. Arguing on the contrary, assume that

$$\{e_3, e_j\} \subseteq \{\beta_Q(g_1), \dots, \beta_Q(g_k)\}. \tag{3.1}$$

Then a star neighborhood of u in Q would turn in $Q_{T_{12}}$ into a tree isomorphic to a subtree S_u of T_{12} which contains vertices w_3, u_3, u_j, see Figure 4(a). Clearly, either $\deg w_3 = 3$ in S_u or $\deg w_3 = 2$ in S_u and then S_u contains no vertex w_{12}. In either case, a copy of S_u is present in $\widehat{T}_{12}$ which is impossible because, as we saw above, every preimage of w_3 in $\widehat{T}_{12}$ has degree 2 and adjacent to a preimage of w_{12}. Thus the inclusion (3.1) is impossible.

Recall that we can pick any trivalent tree T, in particular, we can pick a tree $T = T_{13}$ that has adjacent vertices w_{13}, w_2 of degree 3 such that w_{13} is adjacent to both u_1, u_3 and w_2 is adjacent to u_2, see Figure 4(b).

Repeating the above argument with indices 2 and 3 switched, we can show that

$$\{e_2, e_j\} \not\subseteq \{\beta_Q(g_1), \dots, \beta_Q(g_k)\}$$

for every $j > 3$.

Similarly, switching indices 1 and 3, we prove that

$$\{e_1, e_j\} \not\subseteq \{\beta_Q(g_1), \dots, \beta_Q(g_k)\}$$

for every $j > 3$.

Now we see that for every vertex $u \in VX \vee VY$ such that $\beta_Q(u) = v_0$, where $Q \in \{X, Y\}$, the edges $g_1, \dots, g_k$ of Q that end in u have the following property: either

$$\{\beta_Q(g_1), \dots, \beta_Q(g_k)\} \subseteq \{e_1, e_2, e_3\} \quad \text{or} \quad \{\beta_Q(g_1), \dots, \beta_Q(g_k)\} \subseteq \{e_4, \dots, e_{m_0}\}.$$

Since the edges of $X \vee Y$ are identified in the pushout $P = X \vee_{W(S_1)} Y$ if and only if their β_X-, β_Y-images in P are equal, it follows that the terminal vertex of e_1, e_2, e_3 must be different in P from the terminal vertex of an edge e_j where $j \geq 4$. This contradiction to the definition of the edges $e_1, \dots, e_{m_0}$ of U completes the proof. □

We are now ready to prove our key lemma.

Lemma 3.5. *Suppose that the graphs $X, Y, W(S_1), P, Z, U$ and the corresponding maps $\alpha_X, \dots, \delta$ are defined as in Figure 1. Then there exists a finite sequence τ of alternating replacements as in Lemma 3.1 and elementary deformations as in Lemma 3.4 that result in graphs X^τ, Y^τ, $W(S_1)^\tau$, P^τ, Z^τ, U^τ that have the following properties: $\bar{\mathrm{r}}(X^\tau) = \bar{\mathrm{r}}(X)$, $\bar{\mathrm{r}}(Y^\tau) = \bar{\mathrm{r}}(Y)$, $\bar{\mathrm{r}}(W(S_1)^\tau) = \bar{\mathrm{r}}(W(S_1))$, $P^\tau = Z^\tau = U^\tau$, $\bar{\mathrm{r}}(P^\tau) \ge \bar{\mathrm{r}}(P)$, every vertex of U^τ has degree 3 or 2 and every vertex of degree 3 of U^τ has a preimage of degree 3 in X^τ or Y^τ.*

Proof. Applying Lemma 3.1, we may assume that $P = Z = U$. If every vertex of U has degree 2 or 3, then our claim holds true as follows from Lemma 3.3. Hence, we may assume that U contains a vertex v_0 of degree at least 4. In view of Lemma 3.4, by checking all possible elementary deformations of U around v_0, we can find an elementary deformation by means of a trivalent tree T such that either $\bar{\mathrm{r}}(P_T) > \bar{\mathrm{r}}(P)$ or $\bar{\mathrm{r}}(P_T) = \bar{\mathrm{r}}(P)$ and

$$\sum_{v \in VP_T} \max(\deg v - 3, 0) < \sum_{v \in VP} \max(\deg v - 3, 0).$$

Invoking Lemma 3.1, we replace U_T with P_T and, completing one cycle of changes in graphs $X, Y, W(S_1), P, Z, U$, we rename $Q := Q_T$, where $Q \in \{X, Y, W(S_1), P, Z, U\}$, and start over.

Observe that the rank $\bar{\mathrm{r}}(Z) = \bar{\mathrm{r}}(\langle H, K, S_1 \rangle)$ is bounded above by the number of generators $\mathrm{r}(H) + \mathrm{r}(K) + |S_1|$ and this bound does not change as we perform cycles of changes of graphs $X, Y, W(S_1), P, Z, U$, because $|S_1|$ is equal to the number of connected components of $W(S_1)$ and $\bar{\mathrm{r}}(H) = \bar{\mathrm{r}}(X)$, $\bar{\mathrm{r}}(K) = \bar{\mathrm{r}}(Y)$. This implies that the number

$$\sum_{v \in VZ} \max(\deg v - 3, 0)$$

is bounded above by $2\bar{\mathrm{r}}(Z) \le 2(\mathrm{r}(H) + \mathrm{r}(K) + |S_1|)$. Thus the total number of cycles will not exceed $2(\mathrm{r}(H) + \mathrm{r}(K) + |S_1|)^2$ and our lemma is proved. □

Note that the arguments of the proof of Lemma 3.5 are constructive and provide an algorithm that deterministically constructs the desired graphs X^τ, Y^τ, $W(S_1)^\tau$, P^τ, Z^τ, U^τ in polynomial space in the size of the input which are graphs $X, Y, W(S_1)$, P, Z, U along with associated maps $\alpha_X, \dots, \delta$.

Lemma 3.6. *In the foregoing notation, we have $\bar{\mathrm{r}}(Z) \le \bar{\mathrm{r}}(X) + \bar{\mathrm{r}}(Y)$.*

Proof. In view of Lemma 3.5, we may assume that the Stallings graphs X, Y, Z of subgroups $H, K, \langle H, K, S_1 \rangle$ satisfy the conclusion of Lemma 3.5. Then all of the vertices of X, Y, Z have degree 2 or 3. Hence, $2\bar{\mathrm{r}}(U) = |V_3 U|$, where $V_3 U$ is the set of vertices of degree 3 of U. Similarly, we have $2\bar{\mathrm{r}}(X) = |V_3 X|$ and $2\bar{\mathrm{r}}(Y) = |V_3 Y|$. Now application of Lemma 3.3 yields $|V_3 Z| \le |V_3 X| + |V_3 Y|$ which implies the desired inequality $\bar{\mathrm{r}}(Z) \le \bar{\mathrm{r}}(X) + \bar{\mathrm{r}}(Y)$. □

4. Applications

As an application of Lemma 3.5, we state and prove a couple of specific inequalities for reduced ranks of the generalized intersections and joins of subgroups in free groups.

Theorem 4.1. *Let H, K be finitely generated subgroups of a free group F. Let $S(H, K) \subseteq F$ denote a set of representatives of those double cosets $HtK \subseteq F$, $t \in F$, that have the property $H \cap tKt^{-1} \neq \{1\}$, and let $S_1 \subseteq S(H, K)$ be nonempty. Then $\bar{\mathrm{r}}(\langle H, K, S_1\rangle) \le \bar{\mathrm{r}}(H) + \bar{\mathrm{r}}(K)$ and*

$$\begin{aligned} \bar{\mathrm{r}}(H, K, S_1) &:= \sum_{s \in S_1} \bar{\mathrm{r}}(H \cap sKs^{-1}) \\ &\le 2\bar{\mathrm{r}}(H)\bar{\mathrm{r}}(K) - \bar{\mathrm{r}}(\langle H, K, S_1\rangle) \min(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K)). \end{aligned} \tag{4.1}$$

Moreover,

$$\bar{\mathrm{r}}(H, K, S_1) \le \tfrac{1}{2}(\bar{\mathrm{r}}(H)+\bar{\mathrm{r}}(K)-\bar{\mathrm{r}}(\langle H, K, S_1\rangle))(\bar{\mathrm{r}}(H)+\bar{\mathrm{r}}(K)-\bar{\mathrm{r}}(\langle H, K, S_1\rangle)+1). \tag{4.2}$$

Note that the inequality (4.1) is a strengthened version of (1.1) and this strengthening is analogous to a strengthened version of the Hanna Neumann inequality introduced by Walter Neumann [22]. We also remark that the inequality (4.1) in the cases when $S_1 = \{1\}$ and $S_1 = S(H, K)$ is due to Kent [16] and the inequality (4.2) is shown by Dicks [4] who also obtained other inequalities.

It is worthwhile to mention that the natural question on the existence of a bound that would be a stronger version of (4.1), in which the first term would have the coefficient 1 in place of 2 and the second negative term would contain the factor $\bar{\mathrm{r}}(\langle H, K, S_1\rangle)$ with some coefficient, has a negative solution. Indeed, according to [12], there are finitely generated subgroups H, K of a free group F such that $\bar{\mathrm{r}}(H, K, S(H, K)) = \bar{\mathrm{r}}(H)\bar{\mathrm{r}}(K) > 0$ and $\bar{\mathrm{r}}(\langle H, K, S(H, K)\rangle) > C$ for any constant $C > 0$.

Proof. In view of Lemma 3.5, we may assume that the Stallings graphs X, Y, Z of subgroups $H, K, \langle H, K, S_1\rangle$, resp., and the graphs $W(S_1)^\tau$, P^τ satisfy the conclusion of Lemma 3.5. Recall that

$$\bar{\mathrm{r}}(H) = \bar{\mathrm{r}}(X), \quad \bar{\mathrm{r}}(K) = \bar{\mathrm{r}}(Y), \quad \text{and} \quad \bar{\mathrm{r}}(\langle H, K, S_1\rangle) \le \bar{\mathrm{r}}(Z) = \bar{\mathrm{r}}(U),$$

As before, if Q is a graph whose vertices have degree 2 or 3, then V_3Q denotes the set of vertices of Q of degree 3. Recall that $2\bar{\mathrm{r}}(Q) = |V_3Q|$.

For every $v \in VU$, denote

$$k_v := |V_3X \cap \beta_X^{-1}(v)|, \quad \ell_v := |V_3Y \cap \beta_Y^{-1}(v)|, \quad m := |V_3W(S_1)|, \quad n := |V_3U|.$$

We also let

$$V_3U := V_BU \vee V_XU \vee V_YU,$$

where V_BU is the set of vertices of U that have preimages of degree 3 in both X and Y, V_XU is the set of vertices of U that have preimages of degree 3 in X only, and V_YU is the set of vertices of U that have preimages of degree 3 in Y only. Denote

$$n_B := |V_BU|, \quad n_X := |V_XU|, \quad n_Y := |V_YU|.$$

We let

$$V_3X = V_{31}X \vee V_{32}X,$$

where $V_{31}X := \beta_X^{-1}(V_XU)$ and $V_{32}X := \beta_X^{-1}(V_BU)$. Similarly, we let

$$V_3Y = V_{31}Y \vee V_{32}Y,$$

where $V_{31}Y := \beta_Y^{-1}(V_YU)$ and $V_{32}Y := \beta_Y^{-1}(V_BU)$. Denote

$$\begin{aligned} k &:= |V_3X|, \quad k_1 := |V_{31}X|, \quad k_2 := |V_{32}X|, \\ \ell &:= |V_3Y|, \quad \ell_1 := |V_{31}Y|, \quad \ell_2 := |V_{32}Y|. \end{aligned}$$

Clearly,

$$k = k_1 + k_2, \quad \ell = \ell_1 + \ell_2, \quad n = n_B + n_X + n_Y, \quad k_1 \ge n_X, \quad \ell_1 \ge n_Y,$$

and

$$m \le \sum_{v \in VU} k_v \ell_v = \sum_{v \in V_BU} k_v \ell_v.$$

We now continue with arguments similar to those of Imrich–Müller [8] that follow the proof of their lemma. Since $\sum_{v \in V_BU} \ell_v = \ell_2$ and $\ell = \ell_1 + \ell_2$, we obtain

$$\begin{aligned} m &\le \sum_{v \in V_BU} k_v \ell_v = k\ell - \sum_{v \in V_BU} (k - k_v)\ell_v - k\ell_1 \\ &\le k\ell - \sum_{v \in V_BU} (k - k_v) - k\ell_1 = k\ell - kn_B + k_2 - k\ell_1 \\ &\le k\ell - k(n_B + n_Y - 1). \end{aligned} \tag{4.3}$$

Switching X and Y, we analogously obtain

$$m \le k\ell - \ell(n_B + n_X - 1). \tag{4.4}$$

Assume that $n_B = 0$. Then $\bar{\mathrm{r}}(W(S_1)) = 0$ and the inequality (4.1) is equivalent to

$$0 \le 2\bar{\mathrm{r}}(X)\bar{\mathrm{r}}(Y) - \min(\bar{\mathrm{r}}(X), \bar{\mathrm{r}}(Y))\bar{\mathrm{r}}(Z)$$

which is equivalent to $\bar{\mathrm{r}}(Z) \le 2\max(\bar{\mathrm{r}}(X), \bar{\mathrm{r}}(Y))$. Since $\bar{\mathrm{r}}(Z) \le \bar{\mathrm{r}}(X) + \bar{\mathrm{r}}(Y)$ by Lemma 3.6, it follows that (4.1) holds true. A reference to Lemma 3.6 also proves that

$$\bar{\mathrm{r}}(\langle H, K, S_1 \rangle) \le \bar{\mathrm{r}}(Z) \le \bar{\mathrm{r}}(X) + \bar{\mathrm{r}}(Y) = \bar{\mathrm{r}}(H) + \bar{\mathrm{r}}(K).$$

Suppose that $n_B \ge 1$. Since $n_B + n_X + n_Y = 2\bar{\mathrm{r}}(U)$, it follows that

$$n_X + n_Y \le 2\bar{\mathrm{r}}(U) - 1.$$

Hence, $\min(n_X, n_Y) \le \bar{\mathrm{r}}(U) - 1$ and $\max(n_B + n_X - 1, n_B + n_Y - 1) \ge \bar{\mathrm{r}}(U)$. Therefore, the inequalities (4.3)–(4.4) imply that

$$2\bar{\mathrm{r}}(W(S_1)) = m \le 4\bar{\mathrm{r}}(X)\bar{\mathrm{r}}(Y) - 2\min(\bar{\mathrm{r}}(X), \bar{\mathrm{r}}(Y))\bar{\mathrm{r}}(Z).$$

Dividing by 2, we obtain the required bound (4.1).

Observe that Lemma 3.6 together with the foregoing classification of vertices of degree 3 in graphs $W(S), X, Y, P = Z = U$ make it possible to produce other inequalities for $\bar{\mathrm{r}}(H, K, S_1) = \bar{\mathrm{r}}(W(S_1))$ that would involve $\bar{\mathrm{r}}(\langle H, K, S_1 \rangle) = \bar{\mathrm{r}}(Z)$. For example, when maximizing the sum $\sum_{v \in VU} k_v \ell_v$ which gives an upper bound for $m = 2\bar{\mathrm{r}}(W(S_1))$, we may assume that there is a single vertex v such that both $k_v, \ell_v > 0$, whence $n_B = 1$, $k_2 = k_v$ and $\ell_2 = \ell_v$. We may further assume that $k_1 = n_X$, $\ell_1 = n_Y$. Indeed, if, say $k_1 > n_X$, then we could increase k_v by $k_1 - n_X$ and decrease k_1 by $k_1 - n_X$ making thereby $k_2\ell_2 = k_v\ell_v$ greater. Hence, to get an upper bound for m, we can maximize the product $k_2\ell_2 = (k - k_1)(\ell - \ell_1)$ subject to $k_1 + \ell_1 + 1 = n$. Since k, ℓ, n are fixed positive even integers, equal to $2\bar{\mathrm{r}}(X)$, $2\bar{\mathrm{r}}(Y)$, $2\bar{\mathrm{r}}(Z)$, resp., it follows that the product $(k - k_1)(\ell - n + 1 + k_1)$ has the maximum at $k_1 = \frac{1}{2}(k - \ell + n - 1)$. Since k_1 is an integer, it follows that $(k - k_1)(\ell - n + 1 + k_1)$ has the maximum value for an integer k_1 when $k_1 = \frac{1}{2}(k - \ell + n)$ or $k_1 = \frac{1}{2}(k - \ell + n) - 1$. This means that, unconditionally, we have

$$m \le (k - k_1)(\ell - n + 1 + k_1) \le \tfrac{1}{2}(k + \ell - n + 2) \cdot \tfrac{1}{2}(k + \ell - n).$$

Equivalently,

$$2\bar{\mathrm{r}}(W(S_1)) = m \le (\bar{\mathrm{r}}(X) + \bar{\mathrm{r}}(Y) - \bar{\mathrm{r}}(Z) + 1)(\bar{\mathrm{r}}(X) + \bar{\mathrm{r}}(Y) - \bar{\mathrm{r}}(Z))$$

which implies the bound (4.2). Theorem 4.1 is proved. □

5. One more question

One might wonder what would be the conditions that guarantee the existence of a nontrivial "blow-up" of a vertex v_0 of the graph $P = Z = U$ by an elementary deformation around v_0, as defined in Section 3. Here we present a result that provides a criterion for the existence of such a "blow-up" of a vertex v_0 of U, i.e. the existence of an elementary deformation around a vertex v_0 by means of a tree T such that $\bar{\mathrm{r}}(P_T) > \bar{\mathrm{r}}(P)$.

Assume that $P = Z = U$ and the maps γ, δ in Figure 1 are identity maps on U. Let v_0 be a fixed vertex of U and let D denote the set of all oriented edges of U that end in v_0.

A vertex v of one of the graphs $W(S_1), X, Y$ is called a v_0-*vertex* if the image of v under $\beta_X\alpha_X, \beta_X, \beta_Y$, resp., is v_0.

For every v_0-vertex $v \in VQ$, where $Q \in \{X, Y, W(S_1)\}$, define the set $D(v) \subseteq D$ so that $e \in D(v)$ if and only if there is an edge $f \in \vec{E}Q$ so that the terminal vertex f_+ of f is v and the image of the edge f in U is e.

Consider a partition $D = A \vee B$ of the set D into two nonempty subsets A, B. Recall that $\vee$ denotes the disjoint union.

A v_0-vertex $v \in VQ$, where $Q \in \{X, Y, W(S_1)\}$, is said to be of *type* $\{A, B\}$ if both intersections $D(v) \cap A$ and $D(v) \cap B$ are nonempty. The set of all vertices of type $\{A, B\}$ of a graph Q, where $Q \in \{X, Y, W(S_1)\}$, is denoted $V_{\{A,B\}}Q$.

Consider a bipartite graph $\Psi(\{A, B\})$ whose set of vertices

$$V\Psi(\{A, B\}) = V_X\Psi(\{A, B\}) \vee V_Y\Psi(\{A, B\})$$

consists of two disjoint parts

$$V_X\Psi(\{A, B\}) := V_{\{A,B\}}X \quad \text{and} \quad V_Y\Psi(\{A, B\}) := V_{\{A,B\}}Y.$$

Two vertices $v_X \in V_X\Psi(\{A, B\})$ and $v_Y \in V_Y\Psi(\{A, B\})$ are connected in $\Psi(\{A, B\})$ by an edge if and only if there is a vertex $v \in W(S_1)$ such that $\alpha_X(v) = v_X$, $\alpha_Y(v) = v_Y$ and $v \in V_{\{A,B\}}W(S_1)$.

Theorem 5.1. *Suppose $P = Z = U$, v_0 is a vertex of U. The equality*

$$\bar{\mathrm{r}}(P_T) = \bar{\mathrm{r}}(U_T) = \bar{\mathrm{r}}(U)$$

holds for every elementary deformation around v_0 if and only if the following conditions are satisfied. For every partition $D = A \vee B$ with nonempty A, B, if the graph $\Psi(\{A, B\})$ contains k connected components, then there exists a partition $D = \bigvee_{i=1}^{k+1} C_i$ such that, for every i, C_i is nonempty and either $C_i \subseteq A$ or $C_i \subseteq B$. Next, for every vertex $v \in VQ$, where $Q \in \{X, Y\}$, such that $\beta_Q(v) = v_0$, the intersection $D(v) \cap A$ is either empty or $D(v) \cap A \subseteq C_{i_{vA}}$ for some i_{vA}, $i_{vA} = 1, \dots, k+1$, and the intersection $D(v) \cap B$ is either empty or $D(v) \cap B \subseteq C_{i_{vB}}$ for some i_{vB}, $i_{vB} = 1, \dots, k + 1$.

Proof. First we introduce the notation we will need below. As above, let T be a tree whose set of vertices of degree 1 is D'. If $H' \subseteq D'$ is a nonempty subset of vertices of degree 1 in T, we let $M_T(H')$ denote the minimal subtree of T that contains H'. Clearly, the set of vertices of degree ≤ 1 of $M_T(H')$ is H'. Note that $M_T(H') = H'$ if H' consists of a single vertex. If $E \subseteq D$, then $E' \subseteq D'$ denotes the image of E under the map $d \to d'$ for every $d \in D$.

For every vertex $v \in VQ$, where $Q \in \{X, Y, W(S_1)\}$, consider a tree $M_T(D(v))$ which is isomorphic to the subtree $M_T(D(v)')$ of T and let

$$\zeta_v \colon M_T(D(v)) \to T \tag{5.1}$$

denote the natural monomorphism whose image is $M_T(D(v)')$.

If w is a vertex of $W(S_1)$, then the tree $M_T(D(\alpha_X(w)))$ contains a subtree isomorphic to $M_T(D(w))$ which we denote $M_{T,X}(D(w))$.

Similarly, the tree $M_T(D(\alpha_Y(w)))$ contains a subtree isomorphic to $M_T(D(w))$, denoted $M_{T,Y}(D(w))$.

It follows from the definitions that the graph $\widehat{T} = \gamma_T^{-1}\delta_T^{-1}(T)$ can be described as follows. $\widehat{T}$ is the union of graphs $M_T(D(v))$ over all $v \in VQ$, $Q \in \{X, Y\}$, which are identified along their subgraphs of the form $M_{T,X}(D(w)) = M_{T,Y}(D(w))$ over all $w \in VW(S_1)$.

Now assume that the equality $\bar{\mathrm{r}}(P_T) = \bar{\mathrm{r}}(U_T)$ holds for every tree T or, equivalently, see Lemma 3.2, the graph $\widehat{T}$ is a tree for every T. Note that $\widehat{T}$ being a tree need not imply that $\widehat{T}$ is naturally isomorphic to T.

Let $D = A \vee B$ be a partition of D with nonempty A, B. Our goal is to find a partition $D = \bigvee_{i=1}^{k+1} C_i$ with the properties described in Theorem 5.1.

Consider a trivalent tree $T = T(\{A, B\})$ such that T contains a nonoriented edge e so that the graph $T \setminus \{e\}$ consists of two connected components T_{eA}, T_{eB} such that $A' \subseteq VT_{eA}$ and $B' \subseteq VT_{eB}$.

Note that the tree $M_T(D(v))$, where $v \in VQ$, $Q \in \{X, Y, W(S_1)\}$, contains an edge f with $\zeta_v(f) = e$, where ζ_v is defined by (5.1), if and only if v has type $\{A, B\}$. Hence, the set $\gamma_T^{-1}\delta_T^{-1}(e)$ consists of images of such edges f of $M_T(D(v))$ with $\zeta_v(f) = e$.

Furthermore, let $v, v' \in V_{\{A,B\}}X \vee V_{\{A,B\}}Y$ be two vertices that belong to the same connected component of $\Psi(\{A, B\})$. Then there exists a sequence $v = v_1, v_2, \dots, v_\ell = v'$ of vertices in the graph $\Psi(\{A, B\})$ so that v_i, v_{i+1} are connected by an edge. It follows from the definitions that the edges $f_1, f_2, \dots, f_\ell$ such that $\zeta_{v_i}(f_i) = e$, $i = 1, \dots, \ell$, are identified in $\widehat{T}$. Conversely, it follows from the definitions in a similar fashion that if $v, v' \in V_{\{A,B\}}X \vee V_{\{A,B\}}Y$ and the edges f, f' such that $\zeta_v(f) = e = \zeta_{v'}(f')$ are identified in $\widehat{T}$, then v, v' belong to the same connected component of $\Psi(\{A, B\})$. These remarks prove that there is a bijection between the set $\gamma_T^{-1}\delta_T^{-1}(e) = \{f_1, \dots, f_k\} \subseteq \widehat{T}$ and the set of connected components of the graph $\Psi(\{A, B\})$. Since $\widehat{T}$ is a tree with $|D|$ vertices of degree 1 whose set we denote D'', it follows that the graph $\widehat{T} - (\gamma_T^{-1}\delta_T^{-1}(e))$ splits into $k+1$ connected components which induce a partition $D'' = \bigvee_{i=1}^{k+1} C_i''$, where for every i, C_i'' is nonempty and either $C_i'' \subseteq A''$ or $C_i \subseteq B''$. Then $D = \bigvee_{i=1}^{k+1} C_i$, where C_i'' is the image of C_i under the map $d \to d''$ for every $d \in D$, provides a desired partition for D because, for every v_0-vertex $v \in VQ$, where $Q \in \{X, Y\}$, the vertices of $A'' \cap D(v)'' \subseteq D''$ and those of $B'' \cap D(v)''$ are connected in $\widehat{T} - (\gamma_T^{-1}\delta_T^{-1}(e))$ by images of edges of $M_T(D(v)) - \zeta_v^{-1}(\{e\})$ of $\widehat{T}$. Therefore, $D(v) \cap A \subseteq C_{i_{vA}}$ for some i_{vA} whenever the intersection $D(v) \cap A$ is not empty and $D(v) \cap B \subseteq C_{i_{vB}}$ for some i_{vB} whenever the intersection $D(v) \cap B$ is nonempty.

Now we will prove the converse. Assume that for every partition $D = A \vee B$ there exists a partition $D = \bigvee_{i=1}^{k+1} C_i$ with the properties of Theorem 5.1. Arguing on the contrary, suppose that there is a tree T for which the graph $\widehat{T}$ is not a tree. Pick a shortest circuit p in $\widehat{T}$ of positive length. Let e be a nonoriented edge of T such that $e \in \delta_T\gamma_T(p)$. The graph $T - \{e\}$ consists of two connected components T_{eA}, T_{eB} which define a partition $D' = A' \vee B'$, where $A' \subseteq T_{eA}$, $B' \subseteq T_{eB}$, and both A', B' are nonempty. Let k denote the number of connected components of the graph $\Psi(\{A, B\})$. It is clear from Lemma 3.2 and the definitions that $k \ge 1$. According to our assumption, there exists a partition $D = \bigvee_{i=1}^{k+1} C_i$, where, for every i, C_i is nonempty, $C_i \subseteq A$ or $C_i \subseteq B$, and, for every v_0-vertex $v \in VQ$, where $Q \in \{X, Y\}$, we have that $D(v) \cap A \subseteq C_{i_{vA}}$ for some i_{vA} whenever $D(v) \cap A$ is nonempty and we have that $D(v) \cap B \subseteq C_{i_{vB}}$ for some i_{vB} whenever $D(v) \cap B$ is nonempty.

As above, we observe that if $v, v' \in V_{\{A,B\}}X \vee V_{\{A,B\}}Y$ are two vertices, then the edges $\zeta_v^{-1}(e)$, $\zeta_{v'}^{-1}(e)$ are identified in $\widehat{T}$ if and only if v, v' belong to the same connected component of the graph $\Psi(\{A, B\})$. This implies that the edges $\gamma_T^{-1}\delta_T^{-1}(e) \subseteq \widehat{T}$ are in bijective correspondence with connected components of $\Psi(\{A, B\})$. Hence, there are k edges in $\gamma_T^{-1}\delta_T^{-1}(e)$. Denote $\gamma_T^{-1}\delta_T^{-1}(e) = \{f_1, \dots, f_k\}$. Since identification of subgraphs

$$M_{T,X}(D(w)) \subseteq M_T(D(\alpha_X(w))) \quad \text{and} \quad M_{T,Y}(D(w)) \subseteq M_T(D(\alpha_Y(w)))$$

in process of construction of $\widehat{T}$ respects the partition $D' = \bigvee_{i=1}^{k+1} C_i'$, it follows that the graph $\widehat{T}$ consists of pairwise disjoint subgraphs $S_1, \dots, S_{k+1}$, where

$$\delta_T\gamma_T(S_j) \cap D' = C_j', \quad j = 1, \dots, k+1,$$

which are joined by k nonoriented edges of $\gamma_T^{-1}\delta_T^{-1}(e)$. Collapsing subgraphs $S_1, \dots, S_{k+1}$ into points, we see that $\widehat{T}$ turns into a tree with k nonoriented edges and $k + 1$ vertices. Consequently, a shortest circuit in $\widehat{T}$ of positive length may not contain any edge of $\gamma_T^{-1}\delta_T^{-1}(e)$. This contradiction to the choice of the path p and the edge e in $\delta_T\gamma_T(p)$ completes the proof of Theorem 5.1. □

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