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A DG-extension of symmetric functions arising from higher representation theory

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Abstract. We investigate analogs of symmetric functions arising from an extension of the nilHecke algebra defined by Naisse and Vaz. These extended symmetric functions form a subalgebra of the polynomial ring tensored with an exterior algebra. We define families of bases for this algebra and show that it admits a family of differentials making it a sub-DG-algebra of the extended nilHecke algebra. The ring of extended symmetric functions equipped with this differential is quasi-isomorphic to the cohomology of a Grassmannian. We also introduce new deformed differentials on the extended nilHecke algebra that when restricted makes extended symmetric functions quasi-isomorphic to $GL(N)$-equivariant cohomology of Grassmannians.

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1. Introduction

One of the most fundamental objects in higher representation theory is the nilHecke algebra [9, 18, 27]. This object is the most basic ingredient in categorified quantum groups and is intimately related to the geometry of flag varieties and Grassmannians [15, 19]. The nilHecke algebra admits a faithful action on the polynomial ring, further relating it to the combinatorics of symmetric functions and Schubert polynomials.

The categorification, or higher representation theory, perspective has demonstrated that extensions or alternative categorifications of quantum groups often have parallel implications in geometry and combinatorics. As an example, one motivation for studying the odd (spin/super) nilHecke algebra [2, 7, 8, 29] was an attempt to supply a representation theoretic explanation for the appearance of “odd Khovanov homology” — a distinct link homology theory with similar properties to Khovanov homology. The odd nilHecke algebra shared many of the relationships of the usual nilHecke algebra, including connections to a new noncommutative Hopf algebra of symmetric functions with strikingly similar combinatorics [3]. The odd nilHecke algebra gave “odd” noncommutative analog of the cohomology of Grassmannians and Springer varieties [2, 17]. All of these developments grew out of the discovery of an odd analog of the nilHecke algebra.

Recently, Naisse and Vaz [22] have introduced an extension of the nilHecke algebra $\mathsf{NH}_n^{\mathrm{ext}}$ that we refer to as the *extended nilHecke algebra*. This algebra arose in the study of a fundamental issue in higher representation theory. The problem was the fact biadjointness for $\mathcal{E}$ and $\mathcal{F}$ in the definition of categorified quantum groups [10, 27] implied that it was only possible to categorify finite dimensional modules; in particular, categorical analogs of Verma modules were inaccessible within the existing theory. Naisse and Vaz overcame this issue in the case of $\mathfrak{sl}_2$, by omitting the biadjointness condition, enhancing the nilHecke algebra to the extended nilHecke algebra, and altering the main $\mathfrak{sl}_2$-relation to a short exact sequence, rather than a direct sum decomposition. This work allowed for the first categorification of Verma modules and may be an indication of the way forward in higher representation theory.

Given the importance of the extended nilHecke algebra in categorifying Verma modules, this article investigates the combinatorial implications of this algebra. We define analogs of symmetric functions Λ_n^{ext} arising from the extended nilHecke algebra that we call *extended symmetric functions*. We construct families of bases for these algebras and investigate their combinatorial properties. Extending the work of Naisse and Vaz, we show that the ring Λ_n^{ext} admits a family of differentials d_N such that $(\Lambda_n^{\mathrm{ext}}, d_N)$ is a sub-DG-algebra of the extended nilHecke algebra. Additionally, we show that the extended nilHecke algebra with its differential d_N is isomorphic to the Koszul complex associated to a regular sequence of central elements in NH_n. Restricting to $(\Lambda_n^{\mathrm{ext}}, d_N)$ gives a DG-algebra which is quasi-isomorphic to the cohomology ring of a Grassmannian $\mathrm{Gr}(n, N)$. The algebra Λ_n^{ext} has been independently discovered by Naisse and Vaz using different techniques [24].

Our work facilitates an explicit realization of the extended nilHecke algebra $\mathsf{NH}_n^{\mathrm{ext}}$ as a matrix ring of size $n!$ over its center, the ring of extended symmetric functions. This identifies the ring Λ_n^{ext} with the center of the DG-algebra $\mathsf{NH}_n^{\mathrm{ext}}$. The importance of the explicit isomorphism as a matrix ring over a positively graded algebra is that it allows us to define primitive idempotents decomposing the identity $1 \in \mathsf{NH}_n^{\mathrm{ext}}$. This implies $\mathsf{NH}_n^{\mathrm{ext}}$ has a unique bigraded indecomposable module up to isomorphism and grading shift. Using this fact, we prove that the family of extended nilHecke algebras $\mathsf{NH}_n^{\mathrm{ext}}$, taken for all $n \geq 0$, categorifies the bialgebra corresponding to the positive part $\mathbf{U}^+(\mathfrak{sl}_2)$ of the quantized universal enveloping algebra of $\mathfrak{sl}_2$, suggesting that the extended nilHecke algebra likely fits into a similar extension of KLR-algebras categorifying $\mathbf{U}^+(\mathfrak{g})$ for symmetrizable Kac-Moody algebras.

We also define new deformed differentials d_N^{Σ} on $\mathsf{NH}_n^{\mathrm{ext}}$ in Section 6.3. The deformed differentials also restrict to Λ_n^{ext} and the resulting cohomology of $(\Lambda_n^{\mathrm{ext}}, d_N^{\Sigma})$ is generically isomorphic to the $GL(N)$-equivariant cohomology of a Grassmannian.

Let us point out more clearly the relation between our work and [22]. In *loc. cit.*, Vaz–Naisse define bigraded algebras Ω_k ($k \in \mathbb{Z}_{\geq 0}$) and bigraded bimodules ${}_{\Omega_{k+1}}\mathcal{F}_{\Omega_k}$, ${}_{\Omega_k}\mathcal{E}_{\Omega_{k+1}}$. These bimodules generate a 2-category which categorifies the Verma module for quantum $\mathfrak{sl}_2$ with generic highest weight. In this context, the extended nilHecke $\mathsf{NH}_n^{\mathsf{ext}}$ algebra arises as the ring of bimodule endomorphisms of $\mathcal{F}^{\otimes n}$, or equivalently $\mathcal{E}^{\otimes n}$. Our work provides an idempotent decomposition of $\mathcal{E}^{\otimes n}$ (respectively, $\mathcal{F}^{\otimes n}$) as a direct sum of $n!$ copies with shifts of a bimodule $\mathcal{E}^{(n)}$ (respectively, $\mathcal{F}^{(n)}$), thereby paving the way for a "thick calculus" version of the Vaz–Naisse 2-category, similar to what was accomplished in [12]. In this context, the ring of extended symmetric functions appears as the ring of endomorphisms of $\mathcal{E}^{(n)}$ and $\mathcal{F}^{(n)}$. It occurs that the resulting endomorphism ring is isomorphic to Ω_n, so that ${}_{\Omega_{n+k}}\mathcal{F}^{(n)}_{\Omega_k}$ and ${}_{\Omega_{k-n}}\mathcal{E}^{(n)}_{\Omega_k}$ may be more appropriately referred to as *trimodules* over $(\Omega_{k\pm n}, \Omega_n, \Omega_k)$. We remark that all of the above is compatible with the differentials d_N in the appropriate sense. See 6.4 for more.

Finally, we mention an interpretation of the algebraic structures appearing in this subject in terms of Khovanov–Rozansky homology, both the doubly graded $\mathfrak{sl}_N$ version [13] and the triply graded HOMFLY-PT version [14]. The cohomology rings of Grassmannian $\mathrm{Gr}(k, N)$ can be thought of as the $\mathfrak{sl}_N$ homology of the k-colored unknot [34, 35], while the ring of extended symmetric functions Λ_k^{ext} can be thought of as the HOMFLY-PT homology of the k-colored unknot [32]. The Koszul differential d_N considered here and in [22] is then a special case of Rasmussen's $\mathfrak{sl}_N$ differential [25]. We expect that the trimodules ${}_{\Omega_{n+k}}\mathcal{F}^{(n)}_{\Omega_k}$ and ${}_{\Omega_{k-n}}\mathcal{E}^{(n)}_{\Omega_k}$ appear in this setting as the homologies of certain MOY diagrams, namely the colored theta graphs. This is likely to be related to the point of view adopted by Vaz and Naisse in [23].

2. The nilHecke algebra

Many of our constructions for the extended nilHecke algebra build off of results for the usual nilHecke algebra and its action on polynomials. Here we recall the relevant results.

2.1. The definition. Recall the nilHecke algebra NH_n defined by generators x_i for $1 \leq i \leq n$ and ∂_j for $1 \leq j \leq n-1$ and relations

$$\begin{aligned}
&x_i x_j = x_j x_i, && \\
&\partial_i x_j = x_j \partial_i \quad \text{if } |i-j| > 1, && \partial_i \partial_j = \partial_j \partial_i \quad \text{if } |i-j| > 1, \\
&\partial_i^2 = 0, && \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \\
&x_i \partial_i - \partial_i x_{i+1} = 1, && \partial_i x_i - x_{i+1} \partial_i = 1.
\end{aligned} \tag{2.1}$$

It is not hard to prove that these relations imply

$$\partial_i x_i^{a+1} - x_{i+1}^{a+1} \partial_i = \mathsf{h}_a(x_i, x_{i+1}) = x_i^{a+1} \partial_i - \partial_i x_{i+1}^{a+1} \tag{2.2}$$

for all $a \geqslant 0$.

Given any element $w \in \mathsf{S}_n$ and a reduced decomposition $w = s_{i_1} \dots s_{i_m}$ into simple transpositions we write $\partial_w := \partial_{i_1} \dots \partial_{i_m}$. The axioms ensure this definition does not depend on the choice of reduced expression. We write w_0 for the longest word in the symmetric group S_n and ∂_{w_0} for the corresponding product of divided difference operators.

The algebra NH_n acts on the polynomial ring $\mathsf{P}_n := \mathbb{Q}[x_1, \dots, x_n]$ with x_i acting by multiplication by x_i and $\partial_i \colon \mathsf{P}_n \to \mathsf{P}_n$ given by divided difference operators

$$\partial_i := \frac{1 - s_i}{x_i - x_{i+1}}. \tag{2.3}$$

We recall several important facts relating to the nilHecke algebra and its action on polynomials.

- The ring of symmetric functions can be realized strictly in terms of the divided difference operators

$$\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{\mathsf{S}_n} = \bigcap_{j=1}^{n-1} \ker \partial_i = \bigcap_{j=1}^{n-1} \operatorname{im} \partial_i .$$

- The additive basis of Λ_n given by Schur functions $\mathfrak{s}_\lambda$ can be defined using the nilHecke algebra action on polynomials via

$$\mathfrak{s}_\lambda := \partial_{w_0}(\underline{x}^{\delta+\lambda}) := \partial_{w_0}(x_1^{n-1+\lambda_1} x_2^{n-2+\lambda_2} \dots x_n^{0+\lambda_n}),$$

  for $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition with n parts.

- For $w \in \mathsf{S}_n$ define the *Schubert polynomials* of Lascoux and Schützenberger [16] as

$$\mathfrak{S}_w(x) = \partial_{w^{-1}w_0} x^\delta, \tag{2.4}$$

  where w_0 is the permutation of maximal length and $x^\delta = x_1^{a-1} x_2^{a-2} \dots x_{a-1}$. In case $w = 1 \in \mathsf{S}_n$, we have $\mathfrak{S}_{\mathrm{id}} = \partial_{w_0}(x^\delta) = 1$.

- We have

$$\operatorname{im} \partial_{w_0} = \Lambda_n \subset \mathsf{P}_n. \tag{2.5}$$

  Indeed, if $f \in \Lambda_n$, then $f = f \partial_{w_0}(x^\delta) = \partial_{w_0}(f x^\delta)$ since divided difference operators are Λ_n-linear. Conversely, if $f \in \operatorname{im} \partial_{w_0}$, then $\partial_i(f) = 0$ for $i = 1, \dots, n-1$, hence $f \in \mathsf{P}_n^{\mathsf{S}_n}$.

- The polynomial ring P_n is a free module over Λ_n of rank $n!$ [21, Proposition 2.5.5 and 2.5.5]. In particular, multiplication in P_n induces a ring isomorphism $\mathsf{P}_n \simeq \mathcal{H}_n \otimes \Lambda_n$ where $\mathcal{H}_n$ is equivalently the abelian subgroup spanned by either of the sets $\{\mathfrak{S}_w \mid w \in \mathsf{S}_n\}$ or $\{x_1^{i_1} \dots x_n^{i_n} \mid 0 \le i_k \le n-k\}$.

The last statement allows us to identify $\mathrm{End}_{\Lambda_n}(\mathsf{P}_n)$ as the matrix ring of size $n!$ with coefficients in the ring Λ_n. The ring P_n is graded with $\deg(x_i) = 2$. Taking grading into account, it follows that there is an isomorphism of graded rings $\mathrm{End}_{\Lambda_n}(\mathsf{P}_n) \cong \mathrm{Mat}((n)^!_{q^2}; \Lambda_n)$, where $(n)^!_{q^2} = q^{n(n-1)/2}[n]!$ are the symmetric quantum factorials [18, Proposition 3.5].

The action of NH_n on P_n defines a graded ring homomorphism

$$\gamma\colon \mathsf{NH}_n \to \mathrm{Mat}((n)^!_{q^2}; \Lambda_n).$$

It was shown in [18, Proposition 3.5] that γ is an isomorphism of graded rings. We recall an alternative proof from [12, Section 2.5] that we translate into algebraic language from the so-called *thick calculus*.

For any composition $\mu = (\mu_1, \ldots, \mu_n)$ write $\underline{x}^\mu := x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$. We write $\underline{x}^\delta := x_1^{n-1} x_2^{n-2} \ldots x_n^0$. The set of sequences

$$\mathrm{Sq}(n) := \{\underline{\ell} = \ell_1 \ldots \ell_{n-1} \mid 0 \le \ell_\nu \le \nu,\ \nu = 1, 2, \ldots, n-1\} \tag{2.6}$$

has size $|\mathrm{Sq}(n)| = n!$. Let $|\underline{\ell}| = \sum_\nu \ell_\nu$, and set $\widehat{\ell_j} = j - \ell_j$. Define a composition with n-parts by

$$\widehat{\underline{\ell}} = (0, \widehat{\ell}_1, \ldots, \widehat{\ell}_{n-1}) = (0, 1-\ell_1, 2-\ell_2, \ldots, n-1-\ell_{n-1})\,. \tag{2.7}$$

Let $\mathsf{e}_r^{(a)}$ denote the rth elementary symmetric polynomial in a variables. The *standard elementary monomials* are given by

$$\mathsf{e}_{\underline{\ell}} := \mathsf{e}^{(1)}_{\ell_1} \mathsf{e}^{(2)}_{\ell_2} \ldots \mathsf{e}^{(a-1)}_{\ell_{a-1}}. \tag{2.8}$$

Define elements in NH_n by

$$\sigma_{\underline{\ell}} := \mathsf{e}_{\underline{\ell}} \partial_{w_0}, \quad \lambda_{\underline{\ell}} := (-1)^{\widehat{\underline{\ell}}}\, \underline{x}^\delta \partial_{w_0} \underline{x}^{\widehat{\underline{\ell}}}. \tag{2.9}$$

Theorem 2.1 ([12]).

(1) *For all ℓ, ℓ' in $Sq(n)$, $\lambda_{\underline{\ell'}} \cdot \sigma_{\underline{\ell}} = \delta_{\underline{\ell},\underline{\ell'}}\, x^\delta \partial_{w_0}$.*

(2) *The set $\{\lambda_{\underline{\ell}} \sigma_{\underline{\ell}} \in \mathrm{Sq}(n)\}$ form a complete set of mutually orthogonal primitive idempotents in NH_n.*

(3) *The identity element $1 \in \mathsf{NH}_n$ decomposes as*

$$1 = \sum_{\underline{\ell}} (-1)^{\widehat{\underline{\ell}}} \mathsf{e}_{\underline{\ell}} \partial_{w_0} \underline{x}^{\widehat{\underline{\ell}}}. \tag{2.10}$$

(4) *Enumerate the rows and columns of $n! \times n!$-matrices by the elements $\ell \in \mathrm{Sq}(n)$. There is an isomorphism of graded algebras*

$$\mathrm{Mat}\big((n)^!_{q^2}, \Lambda_n\big) \longrightarrow \mathsf{NH}_n \tag{2.11}$$

sending an element $x \in \Lambda_n^{\mathrm{ext}}$ in the $(\underline{\ell}, \underline{\ell'})$ entry to the element $\sigma_{\underline{\ell}} x \lambda_{\underline{\ell'}}$.

The nilHecke algebra is the simplest example of a KLR-algebra, corresponding to the Lie algebra $\mathfrak{sl}_2$. The results above are critical in the categorification of positive parts of quantized universal enveloping algebras via KLR-algebras [9,11,27]. Another important construction from categorified representation theory is the so-called *cyclotomic quotients* of KLR-algebras. These are used to categorify irreducible representations of $\mathbf{U}_q(\mathfrak{g})$.

For each $N > 1$ define the cyclotomic ideal of NH_n as the two sided ideal generated by x_1^N,

$$I_N := \langle x_1^N \rangle. \tag{2.12}$$

We define the cyclotomic quotient by $\mathsf{NH}_n^N := \mathsf{NH}_n / I_N$. We have the following results.

- The isomorphism γ from (2.11) induces an isomorphism [19, Proposition 5.3]

$$\mathrm{Mat}\big((n)^!_{q^2}, H^*(\mathrm{Gr}(n,N))\big) \longrightarrow \mathsf{NH}_n^N \tag{2.13}$$

  where $H^*(\mathrm{Gr}(n,N))$ is the cohomology ring of the Grassmannian of complex n-planes in $\mathbb{C}^N$.

- The categories of graded projective modules over $\bigoplus_n \mathsf{NH}_n^N$ categorify [6,20,31] the irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$ representation V_N of highest weight N.

3. The extended nilHecke algebra

3.1. The definition. The extended nilHecke algebra $\mathsf{NH}_n^{\mathsf{ext}}$, first defined in [22], is a bigraded algebra with generators $x_1, \dots, x_n$, $\partial_1, \dots, \partial_{n-1}$, generators $\omega_1, \dots, \omega_n$ satisfying equations (2.1) and the following relations

$$x_i \omega_j = \omega_j x_i, \quad \omega_i \omega_j = -\omega_j \omega_i,$$
$$\partial_i \omega_j = \omega_j \partial_i - \delta_{ij} \omega_{i+1}(x_{i+1}\partial_i - \partial_i x_{i+1}).$$

For each fixed integer k the algebra $\mathsf{NH}_n^{\mathsf{ext}}$ admits a $\mathbb{Z} \times \mathbb{Z}$-grading in which the generators $x_i, \partial_i, \omega_i$ are bihomogeneous with degrees

$$\deg(x_i) = (2,0), \quad \deg(\partial_i) = (-2,0), \quad \deg(\omega_k) = (-2k, 1). \tag{3.1}$$

If $a \in \mathsf{NH}_n^{\mathsf{ext}}$ is homogeneous with $\deg(a) = (i,j)$, then $i =: \deg_q(a)$ is referred to as the *quantum degree* and $j =: \deg_h(a)$ is the *homological degree*. The *parity* of a is by definition the homological degree modulo 2.

Remark 3.1. For each $m \in \mathbb{Z}$ we may put a bigrading on $\mathsf{NH}_n^{\mathsf{ext}}$ by leaving $\deg(x_i)$ and $\deg(\partial_i)$ unchanged, while shifting the degrees of ω_i by declaring $\deg(\omega_k) = (-2(k+m), 1)$. The relations are homogeneous with respect to this bigrading, regardless of m. The resulting bigraded rings will be denoted $(\mathsf{NH}_n^{\mathsf{ext}})^{(m)}$. Note that the algebra $(\mathsf{NH}_n^{\mathrm{ext}})^{(m)}$ is naturally a graded subalgebra of $\mathsf{NH}_{m+n}^{\mathrm{ext}}$ given by restricting to the generators

$$\{x_i, \partial_j, \omega_i \mid m+1 \leq i \leq n+m,\ m+1 \leq j < n+m-1\}.$$

Remark 3.2. In [22] they consider an additional grading for their application to categorical Verma modules. Here we ignore this additional grading.

3.2. Action on polynomials. Define the *extended polynomial ring*

$$\mathsf{P}_n^{\mathrm{ext}} = \mathbb{Q}[x_1, \dots, x_n] \otimes \bigwedge[\omega_1, \dots, \omega_n], \tag{3.2}$$

bigraded via $\deg(x_i) = (2,0)$, $\deg(\omega_i) = (-2i, 1)$. Then $\mathsf{P}_n^{\mathrm{ext}}$ has the structure of a bigraded $\mathsf{NH}_n^{\mathrm{ext}}$-module, defined by letting x_i and ω_i act by left multiplication and letting ∂_i act by *extended divided difference operators*

$$\partial_i(1) = 0, \quad \partial_i(\omega_j) = -\delta_{ij}\omega_{j+1}, \quad \partial_i(x_j) = \begin{cases} 1, & \text{if } j = i, \\ -1, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

These operators are extended to arbitrary polynomials by the rule

$$\partial_i(fg) = \partial_i(f)g + f\partial_i(g) - (x_i - x_{i+1})\partial_i(f)\partial_i(g) \tag{3.3}$$

for all $f, g \in \mathbb{Q}[x_1, \dots, x_n] \otimes \bigwedge[\omega_1, \dots, \omega_n]$.

3.3. Differentials. Recall that a *differential graded algebra* (or DG-algebra) is a $\mathbb{Z}$-graded unital algebra A with $d\colon A \to A$ which is degree -1 satisfying

$$d^2 = 0, \quad d(ab) = d(a)b + (-1)^{\deg(a)\deg(b)} a d(b), \quad d(1) = 0. \tag{3.4}$$

A *left DG-module* M is a graded left A-module with differential $d_M\colon M_i \to M_{i-1}$ such that for all $a \in A$, $m \in M$,

$$d_M(am) = d(a)m + (-1)^{\deg(a)} a d_M(m). \tag{3.5}$$

Remark 3.3. In the discussion below, we will consider bigraded algebras and modules with differentials. Despite the presence of two gradings, we will continue to use the standard abbreviation and refer to them simply as DG algebras and modules.

For each $N > 0$, define a differential d_N on $\mathsf{NH}_n^{\mathrm{ext}}$ of bidegree $(2N+2, -1)$ by

$$d_N(x_i) = 0, \quad d_N(\partial_i) = 0, \quad d_N(\omega_i) = (-1)^i \mathsf{h}_{N-i+1}(\underline{x}_i), \tag{3.6}$$

where $\underline{x}_i$ denotes the set of variables $\{x_1, x_2, \dots, x_i\}$. Note the ordinary nilHecke algebra NH_n is in the kernel of this differential for all N. Furthermore, $d_N(\omega_1) = -x_1^N$. By [4, Proposition 2.8] it follows $d_N(\omega_j)$ is contained in the cyclotomic ideal $I_N := \langle x_1^N \rangle$ from (2.12).

Theorem 3.4 ([22, Proposition 8.3]). *The DG-algebra* $(\mathsf{NH}_n^{\mathrm{ext}}, d_N)$ *is quasi-isomorphic to the cyclotomic quotient of the nilHecke algebra* $\mathsf{NH}_n^N := \mathsf{NH}_n / I_N$.

4. The ring of extended symmetric polynomials

4.1. Definition.

4.1.1. Preliminary definition. The action of $\mathsf{NH}_n^{\mathsf{ext}}$ on the extended polynomial ring $\mathsf{P}_n^{\mathsf{ext}} = \mathbb{Q}[x_1, \dots, x_n] \otimes \bigwedge[\omega_1, \dots, \omega_n]$ gives rise to a homomorphism

$$\mathsf{NH}_n^{\mathsf{ext}} \to \mathrm{End}_{\mathbb{Q}} \left(\mathsf{P}_n^{\mathsf{ext}}\right).$$

By analogy with the case of symmetric polynomials, we define the ring of extended symmetric polynomials Λ_n^{ext} as

$$\Lambda_n^{\mathsf{ext}} = \bigcap_{i=1}^{n-1} \ker \partial_i = \bigcap_{i=1}^{n-1} \mathrm{im}\, \partial_i .$$

Remark 4.1. The ring $\Lambda_n^{\mathsf{ext}} \subset \mathsf{P}_n^{\mathsf{ext}}$ is bigraded and graded commutative (that is, super-commutative) with respect to the parity discussed in the comments following (3.1).

4.1.2. Action of the symmetric group on $\mathsf{P}_n^{\mathsf{ext}}$. The standard action of the symmetric group S_n on the polynomial ring $\mathsf{P}_n = \mathbb{Q}[x_1, \dots, x_n]$ lifts to an action on $\mathsf{P}_n^{\mathsf{ext}}$. Namely, one sets

$$s_i(x_j) = x_{s_i(j)} \quad \text{and} \quad s_i(\omega_j) = \omega_j + \delta_{ij}(x_j - x_{j+1})\omega_{j+1} \tag{4.1}$$

for any $1 \le i \le n-1$, $1 \ne j \le n$, and extends it to $\mathsf{P}_n^{\mathsf{ext}}$ by $s_i(fg) = s_i(f)s_i(g)$ for any $f, g \in \mathsf{P}_n^{\mathsf{ext}}$. With respect to this action, the operators ∂_i coincide with the standard divided difference operators:

$$\partial_i = \frac{\mathrm{id} - s_i}{x_i - x_{i+1}} . \tag{4.2}$$

In particular, (3.3) reduces to the standard Leibniz rule for divided difference operators

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g). \tag{4.3}$$

It follows that Λ_n^{ext} coincides with the subalgebra of S_n-invariants $\Lambda_n^{\mathsf{ext}} = \left(\mathsf{P}_n^{\mathsf{ext}}\right)^{\mathsf{S}_n}$.

We now provide an explicit description of Λ_2^{ext} and Λ_3^{ext}. The general case is discussed in 4.2 and 4.4.

Remark 4.2. The algebra $\mathsf{P}_n^{\mathsf{ext}}$ is endowed with another, more natural action of the symmetric group (which on the other hand does not respect the $\mathbb{Z}$–grading (3.1) and does not extend to an action of $\mathsf{NH}_n^{\mathsf{ext}}$). Namely, for any $w \in \mathsf{S}_n$, one can set $w(\omega_i) = \omega_{w(i)}$. The corresponding subalgebra of S_n-invariants is described by Solomon in [28], see also [5, Chapter 22]. In Section 5, we discuss the connection between these two actions and their invariants.

4.1.3. Case $n = 2$. The algebra $\mathsf{P}_2^{\text{ext}}$ is a free module of rank 4 over P_2, and it is easy to see that Λ_2^{ext} is a free module of rank 4 over Λ_2 with basis $\{1, \omega_1 + A\omega_2, \omega_2, \omega_1\omega_2\}$, where A is any solution of $\partial_1(A) = 1$. Particular choices of A are

$$\bigl\{x_1, -x_2, \tfrac{1}{2}(x_1 - x_2)\bigr\}.$$

4.1.4. Case $n = 3$. The algebra $\mathsf{P}_3^{\text{ext}}$ is a free module of rank 8 over P_3. Then

$$v = a + b\omega_1 + c\omega_2 + d\omega_3 + e\omega_1\omega_2 + f\omega_1\omega_3 + g\omega_2\omega_3 + h\omega_1\omega_2\omega_3 \in \Lambda_3^{\text{ext}}$$

if and only if $a \in \Lambda_3$, $b = \partial_1\partial_2(d)$, $c = \partial_2(d)$, $e = \partial_2\partial_1(g)$, $f = \partial_1(g)$, $h \in \Lambda_3$, and

$$\begin{aligned} \partial_1(d) &= 0, \quad \partial_1\partial_2(d) \in \Lambda_3, \\ \partial_2(g) &= 0, \quad \partial_2\partial_1(g) \in \Lambda_3. \end{aligned}$$

It is easy to show that the general solution of the system $\partial_1(d) = 0, \partial_1\partial_2(d) \in \Lambda_3$ has the form

$$d = Af_1 + Bf_2 + f_3,$$

where $f_1, f_2, f_3 \in \Lambda_3$ and $A, B \in \mathsf{P}_3$ are any solution of

$$\begin{aligned} \partial_1(A) &= 0, \quad \partial_1(B) = 0, \\ \partial_1\partial_2(A) &= 1, \quad \partial_2(B) = 1. \end{aligned}$$

Similarly for g. We conclude that Λ_3^{ext} is a free module over Λ_3 of rank 8 with basis

$$\begin{aligned} \bigl\{&1,\ \omega_1 + \partial_2(A)\omega_2 + A\omega_3,\ \omega_2 + B\omega_3,\ \omega_3, \\ &\omega_1\omega_2 + \partial_1(C)\omega_1\omega_3 + C\omega_2\omega_3,\ \omega_1\omega_3 + D\omega_2\omega_3,\ \omega_2\omega_3,\ \omega_1\omega_2\omega_3\bigr\}, \end{aligned}$$

where $A, B, C, D \in \mathsf{P}_3$ are any solution of

$$\begin{aligned} \partial_1(A) &= 0, \quad \partial_1(B) = 0, \quad \partial_2(C) = 0, \quad \partial_1(D) = 1, \\ \partial_1\partial_2(A) &= 1, \quad \partial_2(B) = 1, \quad \partial_2\partial_1(C) = 1, \quad \partial_2(D) = 0. \end{aligned}$$

Particular choices of solutions of the above system are

$$A \in \{x_1x_2, x_3^2\}, \quad B \in \{x_1 + x_2, -x_3\}, \quad C = x_1^2, \quad \text{and} \quad D = x_1.$$

4.2. The size of extended symmetric functions. We now discuss the general case for $n \geq 3$.

4.2.1. Notations. For any binary sequence $\alpha \in \mathbb{Z}_2^n$, set $\omega_\alpha = \omega_1^{\alpha_1} \dots \omega_n^{\alpha_n}$. Then

$$\mathsf{P}_n^{\mathrm{ext}} = \bigoplus_{\alpha \in \mathbb{Z}_2^n} \mathsf{P}_n \cdot \omega_\alpha.$$

The action of S_n is concisely described by the formula

$$s_i(\omega_\alpha) = \omega_\alpha + \delta_{\alpha_i,1}\delta_{\alpha_{i+1},0}(x_i - x_{i+1})\omega_{s_i(\alpha)}.$$

For $k = 1, \dots, n-1$ and $\alpha \in \mathbb{Z}_2^n$, set

$$\begin{aligned} I_k &= \{\alpha \in \mathbb{Z}_2^n \mid \alpha_k = 1, \alpha_{k+1} = 0\}, \\ J_k &= \{\alpha \in \mathbb{Z}_2^n \mid \alpha_k = 0, \alpha_{k+1} = 1\} = s_k(I_k), \\ D_\alpha &= \{k \mid \alpha \in J_k\}, \end{aligned}$$

so that, in particular,

$$s_k(\omega_\alpha) = \begin{cases} \omega_\alpha, & \text{if } \alpha \notin I_k, \\ \omega_\alpha + (x_i - x_{i+1})\omega_{s_i(\alpha)}, & \text{if } \alpha \in I_k. \end{cases}$$

For $k = 0, 1, \dots, n$, let $(\mathbb{Z}_2^n)_k$ be the subset of strings of length k

$$(\mathbb{Z}_2^n)_k = \{\alpha \in \mathbb{Z}_2^n \mid |\alpha| = \textstyle\sum_{i=1}^n \alpha_i = k\}$$

endowed with the following partial ordering $\prec$. We say that $\alpha \prec \beta$ if there exists a sequence in $(\mathbb{Z}_2^n)_k$

$$\alpha_1 = \alpha, \alpha_2, \dots, \alpha_m = \beta,$$

where $m > 1$ and for any $i = 1, \dots, m-1$, $\alpha_i \in I_r$ and $\alpha_{i+1} \in J_r$ for some r. Let $\tau^{(k)}, \lambda^{(k)}$ be, respectively, the highest and lowest element in $((\mathbb{Z}_2^n)_k, \prec)$, i.e. $\tau_i^{(k)} = 0$ if and only if $i < n-k+1$ and $\lambda_i^{(k)} = 0$ if and only if $i > k$.

4.2.2. Grassmannian permutations. A *Grassmannian permutation* w is a permutation with a unique descent. In other words there exists $k \in \{1, \dots, n-1\}$ such that $w(i) < w(i+1)$ if $i \neq k$ and $w(k) > w(k+1)$.

The Grassmannian permutations with descent $n-k$ are in canonical bijection with elements in $(\mathbb{Z}_2^n)_k$, as we now describe. Let $\alpha \in (\mathbb{Z}_2^n)_k$ be given. Let $1 \leq \mathsf{v}_1 < \dots < \mathsf{v}_{n-k} \leq n$ be the indices such that $\alpha_{\mathsf{v}_1} = \dots = \alpha_{\mathsf{v}_{n-k}} = 0$, and let $1 \leq \mathsf{u}_1 < \dots < \mathsf{u}_k \leq n$ be the indices such that $\alpha_{\mathsf{u}_1} = \dots = \alpha_{\mathsf{u}_k} = 1$. Define $\sigma_\alpha \in \mathsf{S}_n$ by

$$\sigma_\alpha(i) = \begin{cases} \mathsf{v}_i, & \text{if } 1 \leq i \leq n-k, \\ \mathsf{u}_{i-n+k}, & \text{if } n-k+1 \leq i \leq n. \end{cases}$$

More concisely, σ_α is the unique minimal length permutation which sends

$$\tau^{(k)} = (\underbrace{0,\dots,0}_{n-k},\underbrace{1,\dots,1}_{k}) \mapsto \alpha.$$

In particular, $\sigma_{\tau^{(k)}} = \mathrm{id}$. Note that σ_α is a minimal length representative of a coset in $\mathsf{S}_n/\mathsf{S}_{n-k} \times \mathsf{S}_k$.

For every $\alpha \in (\mathbb{Z}_2^n)_k$, $\alpha \neq \tau^{(k)}$, σ_α has a unique descent at $n-k$, and it is therefore Grassmannian. Conversely every Grassmannian permutation arises in this way.

4.2.3. Lehmer codes and partitions. Recall that the Lehmer code of a permutation w is the composition $L^w = (L_1^w, \dots, L_n^w)$, where

$$L_i^w = \#\{i < j : w(j) < w(i)\}.$$

We write $\lambda(w)$ for the partition obtained by sorting L^w into decreasing order. In particular, the Lehmer code of the Grassmannian permutation σ_α, $\alpha \in (\mathbb{Z}_2^n)_k$, is given by

$$L_i^\alpha = \begin{cases} m, & \text{if } \mathsf{u}_m - m < i \leq \mathsf{u}_{m+1} - (m+1), \\ 0, & \text{if } \mathsf{u}_{m+1} - (m+1) < i. \end{cases}$$

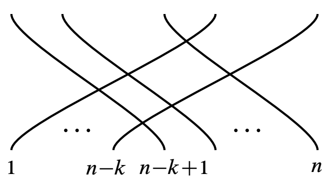
More concretely, if $1 \leq i \leq n-k$, L_i^α is the number of ones which appear to the left of the ith zero of α, and $L_i^\alpha = 0$ otherwise. In particular, $L_1^w \leq \cdots \leq L_{n-k}^w$, and $L_i^w = 0$ for $i > n-k$. The partition corresponding to σ_α is then

$$\lambda_\alpha := \lambda(\sigma_\alpha) = (m^{\mathsf{r}_m})_{m=k,\dots,1}.$$

where $\mathsf{r}_m = \mathsf{u}_{m+1} - \mathsf{u}_m - 1$ for every $m = 0,\dots,k$ (we impose $\mathsf{u}_0 = 0, \mathsf{u}_{k+1} = n+1$). Notice that λ_α has at most $n-k$ non zero terms. In fact, one sees immediately that the biggest possible size of the tableau of shape λ_α, $\alpha \in (\mathbb{Z}_2^n)_k$, is $(n-k) \times k$. The conjugate partition is $\lambda'_\alpha = (\lambda'_j)_{j=1,\dots,k}$

$$\lambda'_j = \sum_{m=j}^{k} \mathsf{r}_m = n + 1 - \mathsf{u}_j - (k - j + 1) = n - k - \mathsf{u}_j + j.$$

4.2.4. Examples. For any $1 \leq j < k \leq n$, set $c_{[j,k]} = s_j \cdots s_{k-1}$ and $c^{(k)} = c_{[k,n]} \cdots c_{[2,n-k+2]} \cdot c_{[1,n-k+1]}$. We sometimes write $c_{[j]} := c_{[j,n]}$. It may be helpful to visualize these elements

$c^{(k)} =$ 

where diagrams are read from bottom to top. Then it is easy to see that

$$c^{(k)}(\tau^{(k)}) = \lambda^{(k)}$$

and, for any $\alpha \in (\mathbb{Z}_2^n)_k$, σ_α is a subword of $c^{(k)}$.

4.2.5. Main result. The rest of this section is devoted to prove the following:

Theorem 4.3.

(i) *The ring of extended symmetric polynomials Λ_n^{ext} is a free module over Λ_n of rank 2^n.*

(ii) *For any collection of polynomials $\{p_\alpha\}_{\alpha\in\mathbb{Z}_2^n}$ satisfying*

$$p_\alpha \in \mathsf{P}_n^{\mathsf{S}_{n-|\alpha|}\times\mathsf{S}_{|\alpha|}} \quad \text{and} \quad \partial_{\sigma_\alpha} p_\alpha = 1 \tag{4.4}$$

there is an isomorphism of Λ_n–modules

$$\Lambda_n^{\mathsf{ext}} \simeq \bigoplus_{\alpha\in\mathbb{Z}_2^n} \Lambda_n \cdot \omega_\alpha^{\mathsf{s}}(p_\alpha) \quad \text{where } \omega_\alpha^{\mathsf{s}}(p_\alpha) := \omega_\alpha + \sum_{\beta\succ\alpha} \partial_{\sigma_\beta}(p_\alpha)\cdot\omega_\beta.$$

(iii) *Multiplication in Λ_n^{ext} induces a ring isomorphism*

$$\Lambda_n^{\mathsf{ext}} \simeq \Lambda_n \otimes \bigwedge[\omega_1^{\mathsf{s}},\dots,\omega_n^{\mathsf{s}}].$$

(iv) *Multiplication in $\mathsf{P}_n^{\mathsf{ext}}$ induces a ring isomorphism $\mathsf{P}_n^{\mathsf{ext}} \simeq \mathcal{H}_n \otimes \Lambda_n^{\mathsf{ext}}$, where $\mathcal{H}_n \subset \mathsf{P}_n$ is the subspace spanned by either of the sets $\{\mathfrak{S}_w \mid w \in \mathsf{S}_n\}$ or $\{x_1^{i_1}\dots x_n^{i_n} \mid 0 \le i_k \le n-k\}$. This gives rise to a canonical ring isomorphism*

$$\mathrm{End}_{\Lambda_n^{\mathsf{ext}}}(\mathsf{P}_n^{\mathsf{ext}}) \simeq \mathsf{Mat}(n!, \Lambda_n^{\mathsf{ext}}).$$

Remark 4.4. In 4.4 we construct examples of $p_\alpha \in \mathsf{P}_n^{\mathsf{S}_{n-|\alpha|}\times\mathsf{S}_{|\alpha|}}$ satisfying (4.4), for each α.

The proof is carried out in 4.2.6–4.2.8.

4.2.6. First characterization of Λ_n^{ext}.

Proposition 4.5. *Let $v = \sum_\alpha f_\alpha\omega_\alpha \in \mathsf{P}_n^{\mathsf{ext}}$, with $f_\alpha \in \mathsf{P}_n$. The following are equivalent.*

(i) $v \in \Lambda_n^{\mathsf{ext}}$;

(ii) *For every $i = 1,\dots,n-1$,*

$$\partial_i(f_\alpha) = \begin{cases} 0, & \text{if } \alpha \notin J_i, \\ f_{s_i(\alpha)}, & \text{if } \alpha \in J_i; \end{cases} \tag{4.5}$$

(iii) *For every* $\alpha \in \mathbb{Z}_2^n$,

$$\partial_i(f_\alpha) = \begin{cases} 0, & \textit{if } i \notin D_\alpha, \\ f_{s_i(\alpha)}, & \textit{if } i \in D_\alpha. \end{cases} \tag{4.6}$$

Proof. Clearly, (ii) and (iii) are equivalent. Now, let $v = \sum_\alpha f_\alpha \omega_\alpha$, $f_\alpha \in \mathsf{P}_n$. For every $i = 1, \ldots, n-1$,

$$\begin{aligned} s_i(v) &= \sum_{\alpha \in \mathbb{Z}_2^n} s_i(f_\alpha) + \sum_{\alpha \in I_i} (x_i - x_{i+1}) s_i(f_\alpha) \omega_{s_i(\alpha)} \\ &= \sum_{\alpha \notin J_i} s_i(f_\alpha)\omega_\alpha + \sum_{\alpha \in J_i} \big(s_i(f_\alpha) + s_i(f_{s_i(\alpha)})(x_i - x_{i+1})\big)\, \omega_\alpha. \end{aligned}$$

Therefore $v \in \Lambda_n^{\mathrm{ext}}$ if and only if, for every $i = 1, \ldots, n-1$,

$$\partial_i(f_\alpha) = \begin{cases} 0, & \text{if } \alpha \notin J_i, \\ s_i(f_{s_i(\alpha)}), & \text{if } \alpha \in J_i. \end{cases}$$

Finally, one observes that for every $\alpha \in J_i$, $s_i(\alpha) \notin J_i$. Therefore, $s_i(f_{s_i(\alpha)}) = f_{s_i(\alpha)}$ and (i) is equivalent to (ii). $\square$

4.2.7. Simplification. The system of equations (4.6) preserves $|\alpha|$, i.e. there are $n+1$ independent sets of equations, for $k = 0, 1, \ldots, n$,

$$\forall \alpha \in (\mathbb{Z}_2^n)_k \qquad \partial_i(f_\alpha) = \begin{cases} 0, & \text{if } i \notin D_\alpha, \\ f_{s_i(\alpha)}, & \text{if } i \in D_\alpha. \end{cases}$$

Let $\tau^{(k)}, \lambda^{(k)}$ be, as before, the highest and lowest element in $(\mathbb{Z}_2^n)_k$ with respect to $\prec$. Then it follows from (4.6) that $f_{\lambda^{(k)}} \in \Lambda_n$ and, for every $\alpha \in (\mathbb{Z}_2^n)_k$,

$$f_\alpha = \partial_{\sigma_\alpha}(f_{\tau^{(k)}}).$$

In particular, any solution of (4.6) is determined by the elements $f_{\tau^{(k)}} \in \mathsf{P}_n$, $k = 0, 1, \ldots, n$. More specifically, we have the following

Corollary 4.6. *Let* $v = \sum_\alpha f_\alpha \omega_\alpha \in \mathsf{P}_n^{\mathrm{ext}}$ *with* $\alpha \in \mathbb{Z}_2^n$ *and* $f_\alpha \in \mathsf{P}_n$. *Then* $v \in \Lambda_n^{\mathrm{ext}}$ *if and only if, for any* $k = 0, \ldots, n-1$, *the elements* $F_k := f_{\tau^{(k)}}$ *satisfy*

(i) $F_k \in \mathsf{P}_n^{\mathsf{S}_{n-k} \times S_k}$;

(ii) *for every* $\alpha \in (\mathbb{Z}_2^n)_k$, $f_\alpha = \partial_{\sigma_\alpha}(F_k)$.

Proof. Note that if $\alpha = \tau^{(k)}$, then $D_\alpha = \{n-k\}$. Thus, the necessity of conditions (i) and (ii) are easy consequences of condition (iii) of Proposition 4.5.

Now we show that (i) and (ii) are sufficient conditions for membership $v \in \Lambda_n^{\mathsf{ext}}$. Fix $k \in \{1, \dots, n\}$, and suppose $F_k \in \mathsf{P}_n$ is given and satisfies (i). Define $f_\alpha := \partial_{\sigma_\alpha}(F_k)$ for all $\alpha \in (\mathbb{Z}_2^n)_k$, and set $v := \sum_{\alpha \in (\mathbb{Z}_2^n)_k} f_\alpha \omega_\alpha$. We must show that $\partial_i(f_\alpha) = 0$ whenever $i \notin D_\alpha$. Let w be the longest element of $\mathsf{S}_{n-k} \times \mathsf{S}_k \subset \mathsf{S}_n$. By (i), $F_k \in \mathsf{P}_n^{\mathsf{S}_{n-k} \times \mathsf{S}_k}$ is symmetric in the first $n-k$ variables and the last k variables. It follows that $F_k = \partial_w(G_k)$ for some polynomial G_k. This is a straightforward generalization of the fact that $\Lambda_n = \operatorname{im} \partial_{w_0} \subset \mathsf{P}_n$, and follows easily from properties of the nilHecke algebra.

From the definition of D_α, it is clear that $\ell(s_i \sigma_\alpha w) = \ell(\sigma_\alpha w) - 1$ if and only if $i \notin D_\alpha$. Recall that, for any $\sigma, \sigma' \in \mathsf{S}_n$,

$$\partial_\sigma \partial_{\sigma'} = \begin{cases} \partial_{\sigma\sigma'}, & \text{if } \ell(\sigma\sigma') = \ell(\sigma) + \ell(\sigma'), \\ 0, & \text{otherwise} \end{cases}$$

(see, for example, [21, §2.3.1]). Thus, if $i \notin D_\alpha$, $\partial_i(f_\alpha) = \partial_i \partial_{\sigma_\alpha} \partial_w(G_k) = 0$. This completes the proof. □

4.2.8. Proof of Theorem 4.3. Corollary 4.6 gives us a map of Λ_n-modules

$$\Phi_k \colon \mathsf{P}_n^{\mathsf{S}_{n-k} \times \mathsf{S}_k} \to \Lambda_n^{\mathsf{ext}}$$

defined by

$$\Phi_k(F) := \sum_{\alpha \in (\mathbb{Z}_2^n)_k} \partial_{\sigma_\alpha}(F) \omega_\alpha.$$

Clearly Φ_k is injective, since F can be recovered as the coefficient of $\omega_{\tau(k)}$ in $\Phi_k(F)$. By Corollary 4.6, Φ_k surjects onto the component of Λ_n^{ext} consisting of elements which are degree k in the exterior variables ω_i. Since the dimension of $\mathsf{P}_n^{\mathsf{S}_{n-k} \times \mathsf{S}_k}$ over $\Lambda_n = \mathsf{P}_n^{\mathsf{S}_n}$ is $\binom{n}{k}$, statement (i) of Theorem 4.3 follows.

Now, let $\{p_\alpha\}_{\alpha \in \mathbb{Z}_2^n}$ be a solution of (4.4) and set

$$\omega_\alpha^{\mathsf{s}}(p_\alpha) = \omega_\alpha + \sum_{\beta \succ \alpha} \partial_{\sigma_\beta}(p_\alpha) \cdot \omega_\beta.$$

By 4.6, the elements $\omega_\alpha^{\mathsf{s}}(p_\alpha)$ belong to Λ_n^{ext} and they are linearly independent, since they are triangular with respect to $\{\omega_\alpha\}$. By a dimension argument this induces an isomorphism of Λ_n–modules

$$\Lambda_n^{\mathsf{ext}} \simeq \bigoplus_{\alpha \in \mathbb{Z}_2^n} \Lambda_n \cdot \omega_\alpha^{\mathsf{s}}(p_\alpha).$$

This proves Theorem 4.3 (ii).

For (iii), suppose we have chosen elements

$$\omega_i^{\mathsf{s}} = \omega_i + \sum_{j>i} f_j \omega_j \in \Lambda_n^{\mathsf{ext}}$$

for $i \in \{1, \dots, n\}$. Since these elements are degree 1 in the exterior variables, we have

$$\omega_i^{\mathsf{s}} \omega_j^{\mathsf{s}} = -\omega_j^{\mathsf{s}} \omega_i^{\mathsf{s}}$$

for every $1 \leq i, j \leq n$. Given the triangularity of $\{\omega_i^{\mathsf{s}}\}$ with respect to $\{\omega_i\}$, the resulting map of rings

$$\bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}] \to \Lambda_n^{\mathsf{ext}} \tag{4.7}$$

is clearly injective. Extending linearly in Λ_n gives an injective map of Λ_n-algebras

$$\Psi \colon \Lambda_n \otimes \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}] \to \Lambda_n^{\mathsf{ext}}.$$

By a dimension argument, Ψ is surjective, and we obtain (iii).

Finally, extending (4.7) by P_n-linearity gives a P_n-algebra homomorphism

$$\mathsf{P}_n \otimes \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}] \to \mathsf{P}_n \otimes \bigwedge[\omega_1, \dots, \omega_n] = \mathsf{P}_n^{\mathsf{ext}}, \tag{4.8}$$

which we claim is an isomorphism. Namely, the homomorphism is induced by the nilpotent matrix A with coefficients in P_n such that

$$\underline{\omega}^{\mathsf{s}} = (I + A)\underline{\omega} \iff \underline{\omega} = \sum_{i=0}^{n-1} (-1)^i A^i \underline{\omega}^{\mathsf{s}}$$

where $\underline{\omega}$ and $\underline{\omega}^{\mathsf{s}}$ denote, respectively, the column vectors $(\omega_1, \dots, \omega_n)^{\mathsf{T}}$ and $(\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}})^{\mathsf{T}}$. This determines the inverse to (4.8). Applying the classical identification $\mathsf{P}_n \simeq \mathcal{H}_n \otimes \Lambda_n$, we get a ring isomorphism

$$\mathsf{P}_n^{\mathsf{ext}} \simeq \mathcal{H}_n \otimes \Lambda_n \otimes \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}] \simeq \mathcal{H}_n \otimes \Lambda_n^{\mathsf{ext}}.$$

In particular, $\mathsf{P}_n^{\mathsf{ext}}$ is a free module of rank $n!$ over Λ_n^{ext} and there is a canonical isomorphism

$$\mathrm{End}_{\Lambda_n^{\mathsf{ext}}}(\mathsf{P}_n^{\mathsf{ext}}) \simeq \mathsf{Mat}(n!, \Lambda_n^{\mathsf{ext}})$$

which completes the proof of Theorem (4.3). □

4.3. Structure of the extended nilHecke algebra. The above analysis of Λ_n^{ext} also has consequences for $\mathsf{NH}_n^{\mathsf{ext}}$. Recall that $c[j] = c[j, n]$ denotes the permutation $s_j \dots s_{n-1}$. Recall also that that Λ_n^{ext} is bigraded b.

Proposition 4.7. *Let $p_j \in \mathsf{P}_n^{S_{n-1}\times S_1}$ be polynomials of degree $n-j$ such that $\partial_{c[j]}(p_j) = 1$, and set $\omega_j^{\mathsf{s}} := \sum_i \partial_{c[i]}(p_j)\omega_i$ as in Theorem 4.3. Then there is an isomorphism of algebras*

$$\mathsf{NH}_n^{\mathsf{ext}} \cong \mathsf{NH}_n \otimes_{\mathbb{Q}} \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}].$$

The induced action on $\mathsf{P}_n^{\mathsf{ext}} \cong \mathsf{P}_n \otimes_{\mathbb{Q}} \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}]$ is the standard action of NH_n on P_n, tensored with the exterior algebra. Consequently,

$$\mathsf{NH}_n^{\mathsf{ext}} \cong \mathrm{End}_{\Lambda_n^{\mathsf{ext}}}(\mathsf{P}_n^{\mathsf{ext}}) \cong \mathrm{Mat}\big((n)_{q^2}^{!}, \Lambda_n^{\mathsf{ext}}\big), \tag{4.9}$$

where Λ_n^{ext} acts on $\mathsf{P}_n^{\mathsf{ext}}$ by right multiplication.

Note that Λ_n^{ext} is graded commutative, hence in order for left multiplication by $\omega_i \in \mathsf{NH}_n$ on $\mathsf{P}_n^{\mathsf{ext}}$ to honestly commute with the action of $\omega_j^{\mathsf{s}} \in \Lambda_n^{\mathsf{ext}}$ (as opposed to commutativity up to sign), it is necessary to let Λ_n^{ext} act on $\mathsf{P}_n^{\mathsf{ext}}$ by right multiplication in (4.9).

Proof. By definition $\mathsf{NH}_n^{\mathsf{ext}}$ contains NH_n and $\mathsf{P}_n^{\mathsf{ext}}$ as subalgebras. Tensoring the inclusion maps gives us an algebra map

$$\mathsf{NH}_n \otimes_{\mathsf{P}_n} \mathsf{P}_n^{\mathsf{ext}} \to \mathsf{NH}_n^{\mathsf{ext}}.$$

By Theorem 4.3, we know that $\mathsf{P}_n^{\mathsf{ext}} \cong \mathsf{P}_n \otimes_{\mathbb{Q}} \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}]$, hence the above reduces to an algebra map

$$\mathsf{NH}_n \otimes_{\mathbb{Q}} \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}] \to \mathsf{NH}_n^{\mathsf{ext}}.$$

As a NH_n-module, the right hand side is isomorphic to $\mathsf{NH}_n \otimes_{\mathbb{Q}} \bigwedge[\omega_1, \dots, \omega_n]$. From the definitions, it is clear that the ω_i^{s} are unitriangular with respect to the ω_i, hence the above algebra map is an isomorphism. This proves the first statement. The statement regarding the action on $\mathsf{P}_n^{\mathsf{ext}}$ is easily verified. Finally (4.9) follows by combining the standard fact that $\mathsf{NH}_n \cong \mathrm{End}_{\Lambda_n}(\mathsf{P}_n)$ together with Theorem 4.3, which states that $\mathsf{P}_n^{\mathsf{ext}}$ is free of rank $[n]!$ over Λ_n^{ext}. $\square$

As an immediate corollary we have the following analogue of the usual fact that $\Lambda_n = Z(\mathsf{NH}_n)$.

Corollary 4.8. *Λ_n^{ext} is isomorphic to the graded center of $\mathsf{NH}_n^{\mathsf{ext}}$ as graded algebras.* $\square$

Here, the graded center of a $\mathbb{Z}/2$ graded algebra $A = A_0 \oplus A_1$ is spanned by homogeneous elements $z \in A$ such that $za = (-1)^{\deg(a)\deg(z)}az$ for every homogeneous $a \in A$. Here the $\mathbb{Z}/2$ grading on $\mathsf{NH}_n^{\mathsf{ext}}$ is inherited from the homological grading as in the comments following (3.1).

4.4. Bases of Λ_n^{ext}. We now discuss some explicit examples of bases of Λ_n^{ext}. We adopt the following criteria. From Theorem 4.3, a basis of Λ_n^{ext} is determined by any family of elements $\{p_j\}_{1\leq j\leq n} \subset \mathsf{P}_n$ satisfying

$$\partial_{c[j]}(p_j) = 1 \quad \text{and} \quad p_j \in \mathsf{P}_n^{\mathsf{S}_{n-1}\times\mathsf{S}_1}. \tag{4.10}$$

This allows to construct a ring isomorphism

$$\Lambda_n^{\mathsf{ext}} \simeq \Lambda_n \otimes \bigwedge[\omega_1^{\mathsf{s}}, \dots, \omega_n^{\mathsf{s}}],$$

where

$$\omega_j^{\mathsf{s}} = \sum_{k\geq j} \partial_{c[k]}(p_j)\omega_k.$$

Any such collection $\{\omega_j^{\mathsf{s}}\}_{1\leq j\leq n}$ will be referred to as an exterior basis of Λ_n^{ext}.

4.4.1. Schubert polynomials. The first example we discuss involves the use of Schubert polynomials. Recall that the Schubert polynomials $\mathfrak{S}_w \in \mathsf{P}_n$, with $w \in \mathsf{S}_n$, are a collection of polynomials indexed by elements of S_n and characterized by the following conditions:

(i) $\mathfrak{S}_{\mathrm{id}} = 1$;

(ii) for every $u \in \mathsf{S}_n$

$$\partial_u \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{wu^{-1}}, & \text{if } l(wu^{-1}) = l(w) - l(u), \\ 0, & \text{otherwise.} \end{cases}$$

More explicitly, one can check that

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}\left(x_1^{n-1}x_2^{n-2}\dots x_{n-1}\right).$$

4.4.2. Schubert polynomials and Λ_n^{ext}. The above characterization implies immediately the following:

Proposition 4.9. *The elements $p_j = \mathfrak{S}_{c[j]}$, $1 \leq j \leq n$, are a solution of* (4.10)*. In particular, the elements*

$$\vartheta_j = \omega_j + \sum_{k>j} \mathfrak{S}_{c[j,k]}\omega_k$$

define an exterior basis of Λ_n^{ext}.

Proof. It is clear from the definitions that $\mathfrak{S}_{c[j]}$ satisfy (4.10). The proposition follows by an application of Theorem 4.3. $\square$

It is interesting to observe that the Schubert polynomials allow to define a solution to the full system (4.4). Specifically, for every $\alpha \in (\mathbb{Z}_2^n)_k$, we can set $p_\alpha = \mathfrak{S}_{\sigma_\alpha}$. Then, it is easy to see that $\partial_{\sigma_\beta}\mathfrak{S}_{\sigma_\alpha} = \mathfrak{S}_{\sigma_\alpha\sigma_\beta^{-1}}$,

$$\partial_{\sigma_\alpha}\mathfrak{S}_{\sigma_\alpha} = 1 \quad \text{and} \quad \partial_i\mathfrak{S}_{\sigma_\alpha} = 0$$

for every $i \neq n-k$. Therefore, we get extended symmetric polynomials

$$\vartheta_\alpha = \omega_\alpha + \sum_{\beta\succ\alpha} \mathfrak{S}_{\sigma_\alpha\sigma_\beta^{-1}}\omega_\beta \in \Lambda_n^{\mathrm{ext}}.$$

In fact, these are exactly the elements of the standard basis of $\bigwedge[\vartheta_1,\dots,\vartheta_n]$.

Proposition 4.10. *The standard basis of $\bigwedge[\vartheta_1,\dots,\vartheta_n]$ has the following description. For any $\alpha \in (\mathbb{Z}_2^n)_k$,*

$$\vartheta_1^{\alpha_1}\dots\vartheta_n^{\alpha_n} = \omega_\alpha + \sum_{\beta\succ\alpha} \mathfrak{S}_{\sigma_\alpha\sigma_\beta^{-1}}\omega_\beta.$$

The proof will be carried out in 4.4.3, 4.4.4, and 4.4.5.

4.4.3. Determinantal identities. In what follows we will make use of the following result, relating Schubert polynomials of Grassmannian permutations to Schur functions.

Proposition 4.11 ([21, Proposition 2.6.8]). *If $w \in \mathsf{S}_n$ is a Grassmannian permutation, and if r is its unique descent, then*

$$\mathfrak{S}_w = \mathfrak{s}_{\lambda(w)}(x_1, x_2, \dots, x_r),$$

where $\mathfrak{s}_{\lambda(w)}$ is the Schur function in the variables $\{x_1,\dots,x_r\}$ corresponding to the partition $\lambda(w)$.

Example 4.12.

(1) If $w = c[j] = s_js_{j+1}\dots s_{n-1} \in \mathsf{S}_n$, then w has a unique descent at position $n-1$. The Lehmer code is $(0,\dots,0,1,\dots,1,0)$ and the corresponding partition $\lambda(w) = (1^{n-j})$. Hence, $\mathfrak{S}_{c[j]} = \mathsf{e}_{n-j}(x_1,\dots,x_{n-1})$.

(2) More generally, if $j < k$ and $w = c[j,k] = s_js_{j+1}\dots s_{k-1} \in \mathsf{S}_n$, then w is a Grassmannian permutation with a unique descent in position $k-1$. The corresponding partition $\lambda(w) = (1^{k-j})$ and $\mathfrak{S}_{c[j,k]} = \mathsf{e}_{k-j}(x_1,\dots,x_{k-1})$.

(3) The permutations $c^{(k)} = c[k,n]\dots c[2,n-k+2]\cdot c[1,n-k+1]$ have a unique descent at position $n-k$. The Lehmer code for $c^{(k)}$ has $L_1^w = L_2^w = \cdots = L_{n-k}^w = k$ and $L_j^w = 0$ for $j > n-k$. It follows that $\mathfrak{S}_{c^{(k)}} = \mathfrak{s}_{(k^{n-k})}(x_1,x_2,\dots,x_{n-k})$.

(4) Generalizing all of the previous examples,

$$w = c[k,n] \cdot c[k-1,n-1] \dots c[k-j+1,n-j+1]$$

has a unique descent at $n-j$, and $\mathfrak{S}_w = \mathfrak{s}_{(j^{n-k})}(x_1,x_2,\dots,x_{n-j})$.

Recall that Schur functions satisfy the second Jacobi–Trudi identity: for every partition λ of length $l(\lambda)$

$$\mathfrak{s}_\lambda = \det(\mathsf{e}_{\lambda'_i+j-i})_{i,j=1}^{l(\lambda')} = \det(\mathsf{e}_{\lambda'_j+i-j})_{i,j=1}^{l(\lambda')} \tag{4.11}$$

where λ' is conjugate to λ. The proof of Proposition 4.10 relies on the following

Lemma 4.13. *For any $\alpha \in (\mathbb{Z}_2^n)_k$, let $\mathsf{u} = (\mathsf{u}_1,\dots,\mathsf{u}_k)$ and λ_α be, respectively, the corresponding sequence of indices and the partition defined in 4.2.2. Then*

$$\mathfrak{s}_{\lambda_\alpha} = \det(\mathsf{e}_{(n-k+i)-\mathsf{u}_j})_{i,j=1}^k.$$

Moreover,

$$\begin{aligned}\mathfrak{s}_{\lambda_\alpha}(x_1,\dots,x_{n-k}) &= \det\left(e_{n-k+i-\mathsf{u}_j}(x_1,\dots,x_{n-k})\right)_{i,j=1}^k \\ &= \det\left(e_{n-k+i-\mathsf{u}_j}(x_1,\dots,x_{n-k+i-1})\right)_{i,j=1}^k.\end{aligned}$$

Example 4.14. The result of Lemma 4.13 is addressing the following phenomenon. Set $n=2$ and consider the permutation s_2s_1. In this case we get

$$\mathfrak{S}_{s_2s_1} = x_1^2 = \det\begin{bmatrix} x_1 & 1 \\ x_1x_2 & x_1+x_2\end{bmatrix} = \det\begin{bmatrix}\mathsf{e}_1(x_1) & 1 \\ \mathsf{e}_2(x_1,x_2) & \mathsf{e}_1(x_1,x_2)\end{bmatrix}.$$

On the other hand, s_2s_1 has a unique descent at 1, its partition is [2], its conjugate is [1, 1], and, by the second Jacobi–Trudi identity,

$$\mathfrak{s}_{[2]} = \det\begin{bmatrix}\mathsf{e}_1 & 1 \\ \mathsf{e}_2 & \mathsf{e}_1\end{bmatrix} = \mathsf{e}_1^2 - \mathsf{e}_2.$$

These coincide when we input the set of variables $\{x_1\}$, namely

$$\mathfrak{S}_{s_2s_1} = x_1^2 = \mathsf{e}_1^2(x_1) - \mathsf{e}_2(x_1) = \mathfrak{s}_{[2]}(x_1).$$

4.4.4. Proof of Lemma 4.13. The first statement is immediate. Namely, the second Jacobi–Trudi identity for λ_α reads

$$\mathfrak{s}_{\lambda_\alpha} = \det(\mathsf{e}_{\lambda'_j+i-j})_{i,j=1}^k = \det(\mathsf{e}_{(n-k+i)-\mathsf{u}_j})_{i,j=1}^k$$

since

$$\lambda'_j + i - j = n-k-\mathsf{u}_j + j + i - j = (n-k+i) - \mathsf{u}_j.$$

To prove the second statement, we proceed by induction on k. For $k = 1$ there is nothing to prove. For $k > 1$, consider the expansion of

$$\mathsf{D} = \det\left(\mathsf{e}_{n-k+i-\mathsf{u}_j}(x_1, \ldots, x_{n-k+i-1})\right)$$

along the last row, i.e.

$$\mathsf{D} = \sum_{j=1}^{k} \mathsf{e}_{n-\mathsf{u}_j}(x_1, \ldots, x_{n-1}) \cdot \mathsf{M}_j,$$

where M_j is the signed minor of the matrix obtained by removing the last row and the jth column. By induction, M_j depends exclusively on the variables $x_1, \ldots, x_{n-k}$, and

$$\mathsf{M}_j = (-1)^{k+j} \det\left(\mathsf{e}_{n-k+i-\mathsf{u}_l}(x_1, \ldots, x_{n-k})\right)_{\substack{i=1,\ldots,k-1 \\ l=1,\ldots,\widehat{j},\ldots,k}}$$

Applying the usual recursive relation for elementary symmetric functions

$$\mathsf{e}_m(x_1, \ldots, x_n) = \mathsf{e}_m(x_1, \ldots, x_{n-1}) + x_n \mathsf{e}_{m-1}(x_1, \ldots, x_{n-1})$$

we get

$$\begin{aligned}
\mathsf{D} &= \sum_{j=1}^{k} \mathsf{e}_{n-\mathsf{u}_j}(x_1, \ldots, x_{n-1}) \cdot \mathsf{M}_j \\
&= \sum_{j=1}^{k} \mathsf{e}_{n-\mathsf{u}_j}(x_1, \ldots, x_{n-2}) \cdot \mathsf{M}_j + x_{n-1} \sum_{j=1}^{k} \mathsf{e}_{n-1-\mathsf{u}_j}(x_1, \ldots, x_{n-2}) \cdot \mathsf{M}_j.
\end{aligned}$$

Now we observe that

$$\sum_{j=1}^{k} \mathsf{e}_{n-1-\mathsf{u}_j}(x_1, \ldots, x_{n-2}) \cdot \mathsf{M}_j = 0$$

since it describes the determinant of a matrix with two equal rows. By iterating this process we get

$$\mathsf{D} = \sum_{j=1}^{k} \mathsf{e}_{n-\mathsf{u}_j}(x_1, \ldots, x_{n-k}) \cdot \mathsf{M}_j = \det(\mathsf{e}_{n-k+i-\mathsf{u}_j}(x_1, \ldots, x_{n-k}))_{i,j=1}^{k}.$$

4.4.5. Proof of Proposition 4.10. Let $\mathbb{S} \in \mathsf{Mat}(n \times n, \mathsf{P}_n)$ be the unipotent lower triangular matrix

$$[\mathbb{S}]_{ij} = \mathfrak{S}_{s_j \cdots s_{i-j}} = \mathfrak{S}_{c[j,i]} = \mathsf{e}_{i-j}(x_1, \ldots, x_{i-1})$$

for any $i > j$. The elements $\vartheta = (\vartheta_1, \dots, \vartheta_n)$ satisfy $\vartheta = \mathbb{S}^{\mathsf{T}}\omega$, where $\omega = (\omega_1, \dots, \omega_n)$. In particular, their wedge product can be written in terms of minors of $\mathbb{S}$. More specifically, for every $\alpha \in (\mathbb{Z}_2^n)_k$,

$$\widetilde{\vartheta}_\alpha := \vartheta_1^{\alpha_1} \cdots \vartheta_n^{\alpha_n} = \sum_{\beta \succeq \alpha} \mathsf{D}_{\beta\alpha}\omega_\beta,$$

where $\mathsf{D}_{\beta\alpha}$ is the minor of $\mathbb{S}$ corresponding to the rows identified by β and the columns identified by α.

Proposition 4.15. *For every $\alpha, \beta \in (\mathbb{Z}_2^n)_k$, $\beta \succeq \alpha$, $\mathfrak{S}_{\sigma_\alpha \sigma_\beta^{-1}} = \mathsf{D}_{\beta\alpha}$. In particular, $\vartheta_\alpha = \widetilde{\vartheta}_\alpha$.*

Proof. Since σ_α is a Grassmannian permutation with descent at $n-k$, it follows from Lemma 4.13

$$\mathfrak{S}_{\sigma_\alpha} = \mathfrak{s}_{\lambda_\alpha}(x_1, \dots, x_{n-k}) = \det(\mathsf{e}_{(n-k+i)-\mathsf{u}_j}(x_1, \dots, x_{n-k}))_{i,j=1}^k$$

and

$$\begin{aligned}\mathfrak{S}_{\sigma_\alpha} &= \det\left(\mathsf{e}_{(n-k+i)-\mathsf{u}_j}(x_1, \dots, x_{n-k})\right)_{i,j=1}^k \\ &= \det\left(\mathsf{e}_{(n-k+i)-\mathsf{u}_j}(x_1, \dots, x_{n-k+i-1})\right)_{i,j=1}^k = \mathsf{D}_{\tau^{(k)}\alpha}.\end{aligned}$$

Moreover, since the elements ϑ_j are S_n–invariant, so is $\widetilde{\vartheta}_\alpha$. Hence the coefficients $\mathsf{D}_{\beta\alpha}$ satisfy

$$\mathsf{D}_{\beta\alpha} = \partial_{\sigma_\beta}\mathsf{D}_{\tau^{(k)}\alpha}$$

and therefore

$$\mathsf{D}_{\beta\alpha} = \partial_{\sigma_\beta}\mathsf{D}_{\tau^{(k)}\alpha} = \partial_{\sigma_\beta}\mathfrak{S}_{\sigma_\alpha} = \mathfrak{S}_{\sigma_\alpha\sigma_\beta^{-1}}. \qquad \square$$

This concludes the proof of Proposition 4.10.

Remark 4.16. It follows from the discussion above that the Schubert exterior basis of Λ_n^{ext} is more concisely described in terms of elementary functions. It will be convenient to reindex these elements. Henceforth, we will adopt the following notation

$$\mathsf{e}_j^\omega = \sum_{k=0}^{j-1} \mathsf{e}_k(x_1, \dots, x_{n-j+k})\omega_{n+1-j+k} = \vartheta_{n-j+1}.$$

4.4.6. Dual Schubert polynomials. Our second example of a basis for Λ_n^{ext} relies on the notion of *dual* Schubert polynomials.

Proposition 4.17 ([21, Proposition 2.5.7]). *There is a Λ_n-bilinear form on P_n defined by $(x, y) := \partial_{w_0}(xy)$. With respect to this form the dual basis to the Schubert polynomials are given by*

$$\mathfrak{S}_w^* = (-1)^{\ell(ww_0)} w_0(\mathfrak{S}_{ww_0}), \quad w \in \mathsf{S}_n. \tag{4.12}$$

The dual Schubert polynomials are characterized by the following conditions:

(i) $\mathfrak{S}^*_{w_0} = 1$;

(ii) for every $u \in \mathsf{S}_n$

$$\partial_u \mathfrak{S}^*_w = \begin{cases} \mathfrak{S}^*_{wu^{-1}}, & \text{if } l(wu^{-1}) = l(w) + l(u), \\ 0, & \text{otherwise.} \end{cases}$$

This follows directly from the characterization of the Schubert polynomials in 4.4.1 and from the relation

$$w_o \cdot \partial_u \cdot w_0 = (-1)^{l(u)} \partial_{w_0 u w_0}.$$

In particular, we get the following result, dualizing Proposition 4.9.

Proposition 4.18. *The elements* $p_j = \mathfrak{S}^*_{w_0 c[j]}$, $1 \leq j \leq n$, *are a solution of* (4.10). *In particular, the elements*

$$\vartheta^*_j = \omega_j + \sum_{k>j} \mathfrak{S}^*_{w_0 c[j,k]} \omega_k$$

define an exterior basis of Λ^{ext}_n. □

In 4.12, we showed that the Schubert polynomials involved in the exterior basis of Λ^{ext}_n are elementary symmetric functions, namely,

$$\mathfrak{S}_{c[j,k]} = \mathsf{e}_{k-j}(x_1, \ldots, x_{k-1}).$$

The dual Schubert polynomials are, instead, naturally described by complete symmetric functions. By definition, we have

$$\mathfrak{S}^*_{w_0 c[j,k]} = (-1)^{k-j} w_0\big(\mathfrak{S}_{w_0 \cdot c[j,k] \cdot w_0}\big) = (-1)^{k-j} w_0\big(\mathfrak{S}_{c[n-k+1, n-j+1]^{-1}}\big)$$

since $w_0 \cdot c[j,k] \cdot w_0 = s_{n-j} \ldots s_{n-k+1}$. The permutation $c[n-k+1, n-j+1]^{-1}$ is still a Grassmannian permutation, whose unique descent is at $n-k+1$ and whose partition is conjugate to that of $c[j,k]$. Therefore

$$\mathfrak{S}_{c[n-k+1, n-j+1]^{-1}} = \mathsf{h}_{k-j}(x_1, \ldots, x_{n-k+1})$$

and

$$\mathfrak{S}^*_{w_0 c[j,k]} = (-1)^{k-j} \mathsf{h}_{k-j}(x_k, \ldots, x_n).$$

In particular, the relation $\partial_{c[k]} \mathfrak{S}^*_{w_0 c[j]} = \mathfrak{S}^*_{w_0 c[j,k]}$ reads

$$\partial_{c[k]}\big((-1)^{n-j} \mathsf{h}_{n-j}(x_n)\big) = (-1)^{k-j} \mathsf{h}_{k-j}(x_k, \ldots, x_n)$$

providing a different proof of [1, Prop. 5.4].

As in the Schubert case, one observes that the dual Schubert polynomials give a solution of (4.4). Namely, one can set $p_\alpha = \mathfrak{S}^*_{w_0\sigma_\alpha}$. Then $\partial_{\sigma_\beta}\mathfrak{S}^*_{w_0\sigma_\alpha} = \mathfrak{S}^*_{w_0\sigma_\alpha\sigma_\beta^{-1}}$,

$$\partial_{\sigma_\alpha}\mathfrak{S}^*_{w_0\sigma_\alpha} = 1 \quad \text{and} \quad \partial_i\mathfrak{S}^*_{w_0\sigma_\alpha\sigma_\beta^{-1}} = 0$$

for every $\beta \succ \alpha$ and $i \notin D_\beta$. It follows that there are elements in Λ_n^{ext}

$$\vartheta^*_\alpha = \omega_\alpha + \sum_{\beta\succ\alpha} \mathfrak{S}^*_{w_0\sigma_\alpha\sigma_\beta^{-1}}\omega_\beta$$

which satisfy, in analogy with 4.10, $\vartheta^*_\alpha = (\vartheta^*_1)^{\alpha_1}\dots(\vartheta^*_n)^{\alpha_n}$.

Remark 4.19. It follows from the discussion above that the dual Schubert exterior basis of Λ_n^{ext} is concisely described in terms of complete functions. As before, it will be convenient to reindex these elements. Henceforth, we will adopt the following notation

$$\mathsf{h}^\omega_j = \sum_{k=0}^{j-1}(-1)^k\mathsf{h}_k(x_{n+1-j+k},\dots,x_n)\omega_{n+1-j+k} = \vartheta^*_{n-j+1}.$$

4.4.7. A family of bases for Λ_n^{ext}. We now describe a collection of bases of Λ_n^{ext} which interpolates between the Schubert basis 4.4.1 (described in terms of elementary symmetric functions) and the dual Schubert basis 4.4.6 (described in terms of complete symmetric functions). Recall that the elementary symmetric functions satisfy the relation

$$\mathsf{e}_j(x_1,\dots,x_{n-1}) = \sum_{l=0}^{j}(-1)^{j-l}x_n^{j-l}\mathsf{e}_l(x_1,\dots,x_n).$$

For every $0 \le r \le n-j$, set

$$\begin{aligned} p_j^{(r)} &= (-1)^r x_n^r \cdot \mathsf{e}_{n-j-r}(x_1,\dots,x_{n-1}) \\ &= (-1)^r x_n^r \sum_{l=0}^{n-j-r}(-1)^{n-j-r-l}x_n^{n-j-r-l}\mathsf{e}_l(x_1,\dots,x_n) \\ &= \sum_{l=0}^{n-j-r}(-1)^{n-j-l}\mathsf{h}_{n-j-l}(x_n)\mathsf{e}_l(x_1,\dots,x_n). \end{aligned}$$

Proposition 4.20. *For every choice of r, the elements $p_j^{(r)}$, $1 \le j \le n$, are a solution of* (4.10)*. In particular, the elements*

$$\vartheta_j^{(r)} = \sum_{k\ge j}\partial_{c[k]}(p_j^{(r)})\omega_k$$

define an exterior basis of Λ_n^{ext}.

Proof. Since $\partial_{c[k]}(\mathsf{h}_{n-j-l}(x_n)) = 0$ for every $k \geq j$ and $l > k - j$, we have

$$\partial_{c[k]}(p_j^{(r)}) = \sum_{l=0}^{k-j} (-1)^{k-j-l} \mathsf{e}_l(x_1, \dots, x_n) \cdot \mathsf{h}_{k-j-l}(x_k, \dots, x_n).$$

Therefore,

$$\partial_{c[j]}(p_j^{(r)}) = 1 \quad \text{and} \quad \partial_i \partial_{c[k]}(p_j^{(r)}) = 0,$$

for every $k > j$ and $i \neq k - 1$. The result follows. □

This basis interpolates between 4.4.1 and 4.4.6. Specifically, for $r = 0$,

$$p_j^{(0)} = \mathsf{e}_{n-j}(x_1, \dots, x_{n-1}) = \mathfrak{S}_{c[j]}$$

and we obtain the Schubert exterior basis 4.4.1. Instead, for $r = n - j$,

$$p_j^{(n-j)} = (-1)^{n-j} \mathsf{h}_{n-j}(x_n) = \mathfrak{S}^*_{w_0 c[j]}$$

and we obtain the dual Schubert exterior basis 4.4.6. Indeed, more precisely, we have, for any $0 \leq r \leq n - j$,

$$p_j^{(r)} = \mathfrak{S}_{c[j+r]} \cdot \mathfrak{S}^*_{w_0 c[n-r]}. \tag{4.13}$$

Example 4.21. Set $n = 3$, then we have

w	$\mathfrak{S}_w$	$\mathfrak{S}^*_{w_0 w}$
id	1	1
s_1	x_1	$-x_2 - x_3$
s_2	$x_1 + x_2$	$-x_3$
$s_1 s_2$	$x_1 x_2$	x_3^2
$s_2 s_1$	x_1^2	$x_2 x_3$
$s_1 s_2 s_1$	$x_1^2 x_2$	$x_2 x_3^2$

In particular, the Schubert exterior basis of Λ_3^{ext} is

$$\vartheta_1 = \omega_1 + x_1\omega_2 + x_1 x_2 \omega_3, \quad \vartheta_2 = \omega_2 + (x_1 + x_2)\omega_3, \quad \vartheta_3 = \omega_3.$$

Instead, the dual Schubert exterior basis is

$$\vartheta_1^* = \omega_1 - (x_2 + x_3)\omega_2 + x_3^2 \omega_3, \quad \vartheta_2^* = \omega_2 - x_3\omega_3, \quad \vartheta_3^* = \omega_3.$$

Other possible choices are obtained replacing ϑ_1 or ϑ_1^* with

$$\vartheta_1^{(1)} = \omega_1 + x_1\omega_2 - (x_1 + x_2)x_3\omega_3$$

corresponding to the choice $p_1^{(1)}$ in 4.4.7.

4.4.8. Other bases. We conclude this section with two more examples.

(i) *Power functions*. One can consider power symmetric polynomials and set

$$p_j = (-1)^{n-j}\mathsf{p}_{n-j}(x_1,\dots,x_{n-1}).$$

On the other hand,

$$\begin{aligned} p_j &= (-1)^{n-j}\mathsf{p}_{n-j}(x_1,\dots,x_{n-1}) \\ &= (-1)^{n-j}\mathsf{p}_{n-j}(x_1,\dots,x_n) + (-1)^{n-j}\mathsf{h}_{n-j}(x_n). \end{aligned}$$

Therefore it simply gives back the description in terms of complete symmetric functions.

(ii) *Symmetrizers*. The easiest example, although computationally most expensive, is obtained by full symmetrization of the exterior variables ω_j, i.e. for every $1 \le j \le n$, set

$$\omega_j^{\mathsf{s}} = \frac{1}{n!}\sum_{\sigma\in\mathsf{S}_n}\sigma(\omega_j).$$

4.5. Combinatorial identities. The following results give the relationship between $\{\mathsf{e}_i^w\}$ and $\{\mathsf{h}_i^w\}$ (see Remarks 4.16 and 4.19), where

$$\mathsf{h}_j^{\omega} = \sum_{k=0}^{j-1}(-1)^k\mathsf{h}_k(x_{n+1-j+k},\dots,x_n)\omega_{n+1-j+k} \tag{4.14}$$

$$\mathsf{e}_j^{\omega} = \sum_{k=0}^{j-1}\mathsf{e}_k(x_1,\dots,x_{n-j+k})\omega_{n+1-j+k}. \tag{4.15}$$

We use the following identity between elementary symmetric polynomials and complete homogeneous symmetric polynomials to prove the next proposition:

Lemma 4.22.

$$\begin{aligned} &\mathsf{e}_k(x_1,\dots,x_{n-j+k}) \\ &\qquad = \sum_{t=0}^{k}(-1)^{k+t}\mathsf{h}_{k-t}(x_{n-j+k+1},\dots,x_n)\mathsf{e}_t(x_1,\dots,x_n). \end{aligned} \tag{4.16}$$

Proof. Using standard facts about elementary and complete symmetric functions we have

$$\begin{aligned}
&\sum_{t=0}^{k}(-1)^t \mathsf{h}_{k-t}(x_{n-j+k+1},\dots,x_n)\mathsf{e}_t(x_1,\dots,x_n)\\
&= \sum_{t=0}^{k}(-1)^t \mathsf{h}_{k-t}(x_{n-j+k+1},\dots,x_n)\\
&\qquad\qquad \cdot \left(\sum_{a=0}^{t}\mathsf{e}_a(x_1,\dots,x_{n-j+k})\mathsf{e}_{t-a}(x_{n-j+k+1},\dots,x_n)\right)\\
&= \sum_{a=0}^{k}\sum_{t=a}^{k}(-1)^t \mathsf{h}_{k-t}(x_{n-j+k+1},\dots,x_n)\mathsf{e}_{t-a}(x_{n-j+k+1},\dots,x_n)\\
&\qquad\qquad \cdot \mathsf{e}_a(x_1,\dots,x_{n-j+k})\\
&= \sum_{a=0}^{k}\sum_{t'=0}^{k-a}(-1)^{k-t'} \mathsf{h}_{t'}(x_{n-j+k+1},\dots,x_n)\mathsf{e}_{k-a-t'}(x_{n-j+k+1},\dots,x_n)\\
&\qquad\qquad \cdot \mathsf{e}_a(x_1,\dots,x_{n-j+k})\\
&= (-1)^k\sum_{a=0}^{k}\mathsf{e}_a(x_1,\dots,x_{n-j+k})\\
&\qquad\qquad \cdot \left(\sum_{t'=0}^{k-a}(-1)^{t'} \mathsf{h}_{t'}(x_{n-j+k+1},\dots,x_n)\mathsf{e}_{k-a-t'}(x_{n-j+k+1},\dots,x_n)\right)\\
&= (-1)^k\sum_{a=0}^{k}\mathsf{e}_a(x_1,\dots,x_{n-j+k})(\delta_{a,k})\\
&= (-1)^k\mathsf{e}_k(x_1,\dots,x_{n-j+k}).
\end{aligned}$$

□

Proposition 4.23. *For any $1 \le j \le n$, we have*

$$\mathsf{e}_j^w = \sum_{k=0}^{j-1}\mathsf{e}_k\mathsf{h}_{j-k}^w.$$

Proof. Using the definition of h_j^w we have

$$\begin{aligned}
&\sum_{k=0}^{j-1}\mathsf{e}_k\mathsf{h}_{j-k}^w = \mathsf{e}_0(x_1,\dots,x_n)\mathsf{h}_j^w + \mathsf{e}_1(x_1,\dots,x_n)\mathsf{h}_{j-1}^w + \ldots + \mathsf{e}_{j-1}(x_1,\dots,x_n)\mathsf{h}_1^w\\
&= \omega_{n+1-j} + \sum_{k=1}^{j-1}\left(\sum_{l=0}^{k-1}(-1)^{k-l}\mathsf{h}_{k-l}(x_{n-j+k+1},\dots,x_n)\mathsf{e}_l(x_1,\dots,x_n)\right)\omega_{n-j+k+1}\\
&= \omega_{n+1-j} + \sum_{n+1-j<k\le n}\mathsf{e}_{k-(n+1-j)}(x_1,\dots,x_{k-1})\omega_k = \mathsf{e}_j^w,
\end{aligned}$$

where the third equality follows from Lemma 4.22 and the last step comes from a change of variables. □

5. Solomon's theorem

5.1. Superpolynomials and superinvariants. Fix an integer $n \geq 1$. Let $\mathbf{x}$ denote a set of formal even variables $x_1, \ldots, x_n$, and let $\mathbf{dx}$ denote a set of formal odd variables $dx_1, \ldots, dx_n$. Here "odd" means that these variables are assumed to anti-commute amongst themselves and square to zero. Thus, $\mathbb{Q}[\mathbf{x}, \mathbf{dx}]$ is short-hand for the *superpolynomial ring*

$$\mathbb{Q}[\mathbf{x}, \mathbf{dx}] := \mathbb{Q}[x_1, \ldots, x_n] \otimes_{\mathbb{Q}} \bigwedge[dx_1, \ldots, dx_n].$$

We make this ring bigraded by declaring that $\deg(x_i) = (1, 0)$ and $\deg(dx_i) = (0, 1)$.

The symmetric group S_n acts on $\mathbb{Q}[\mathbf{x}, \mathbf{dx}]$ by algebra automorphisms, defined by permuting indices: $w(x_i) = x_{w(i)}$ and $w(dx_i) = dx_{w(i)}$. Note that this action preserves the bidegree.

Theorem 5.1 (Solomon [28]). *For any family* $\mathbf{f} = \{f_1, \ldots, f_n\}$ *of algebraically independent generators of* $\mathbb{Q}[\mathbf{x}]^{\mathsf{S}_n}$, $\mathbb{Q}[\mathbf{x}, \mathbf{dx}]^{\mathsf{S}_n} = \mathbb{Q}[\mathbf{f}, \mathbf{df}]$.

In particular,

$$\mathbb{Q}[x_1, \ldots, x_n, dx_1, \ldots, dx_n]^{\mathsf{S}_n} = \mathbb{Q}[e_1, \ldots, e_n, de_1, \ldots, de_n],$$

where $e_i = e_i(x_1, \ldots, x_n)$ is the ith elementary symmetric polynomial, and $de_i \in \mathbb{Q}[\mathbf{x}, \mathbf{dx}]$ is to be interpreted in the usual manner for functions:

$$df := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \quad \forall f \in \mathbb{Q}[\mathbf{x}].$$

Note that $\deg(e_i) = (i, 0)$ and $\deg(de_i) = (i - 1, 1)$.

Remark 5.2. The mapping $f \mapsto df$ extends to a degree $(-1, 1)$ differential $\mathbb{Q}[\mathbf{x}, \mathbf{dx}] \to \mathbb{Q}[\mathbf{x}, \mathbf{dx}]$. This is the usual exterior derivative on polynomial differential forms.

5.2. Action of the extended nilHecke algebra. Taking a cue from higher representation theory, we would like to consider divided difference operators ∂_i acting on superpolynomials. Unlike in the case of ordinary polynomials, here it is necessary to introduce rational functions in the variables $x_1, \ldots, x_n$. So, let $\alpha_i := x_i - x_{i+1}$ for $i = 1, \ldots, n-1$, let $\boldsymbol{\alpha}^{-1} = \{\alpha_1^{-1}, \ldots, \alpha_{n-1}^{-1}\}$, and consider the algebra $\mathbb{Q}[\mathbf{x}, \mathbf{dx}, \boldsymbol{\alpha}^{-1}]$. Note that this algebra is bigraded, with $\deg((x_i - x_{i+1})^{-1}) = (-1, 0)$.

We have the divided difference operators $\partial_i \colon \mathbb{Q}[\mathbf{x}, \mathbf{dx}, \boldsymbol{\alpha}^{-1}] \to \mathbb{Q}[\mathbf{x}, \mathbf{dx}, \boldsymbol{\alpha}^{-1}]$ defined in the usual way

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}.$$

It follows from Solomon's theorem that for any tuple $\mathbf{f} = \{f_1, \dots, f_n\}$ of algebraically independent generators of $\mathbb{Q}[\mathbf{x}]^{\mathsf{S}_n}$ the subalgebra $\mathbb{Q}[\mathbf{x}, \mathbf{df}] \subset \mathbb{Q}[\mathbf{x}, \mathbf{dx}, \boldsymbol{\alpha}^{-1}]$ is closed under the action of the divided difference operators.

Consequently, $\mathbb{Q}[\mathbf{x}, \mathbf{df}]$ is a module over the extended nilHecke algebra. We wish to compare this module with the polynomial representation of the extended nilHecke algebra considered earlier. This representation can be described as follows. Let $\boldsymbol{\omega} = \{\omega_1, \dots, \omega_n\}$ be a set of formal odd variables, with bidegree

$$\deg(\omega_i) = (n - i, 1).$$

The superpolynomial ring $\mathbb{Q}[\mathbf{x}, \boldsymbol{\omega}]$ admits an S_n action via $w(x_i) = x_{w(i)}$ for all $w \in \mathsf{S}_n$, together with

$$s_j(\omega_i) = \begin{cases} \omega_i + (x_i - x_{i+1})\omega_{i+1}, & \text{if } j = i, \\ \omega_i, & \text{otherwise.} \end{cases}$$

Note that the S_n action preserves the bidegree. The actions of $\mathbb{Q}[\mathbf{x}]$ and $\mathbb{Q}[\mathsf{S}_n]$ determines uniquely that of $\mathsf{NH}_n^{\mathsf{ext}}$.

Note that the graded dimensions of $\mathbb{Q}[\mathbf{x}, \boldsymbol{\omega}]$ and $\mathbb{Q}[\mathbf{x}, \mathbf{df}]$ coincide. Thus, it is natural to hope for a bidegree preserving isomorphism of $\mathsf{NH}_n^{\mathsf{ext}}$-modules $\mathbb{Q}[\mathbf{x}, \boldsymbol{\omega}] \cong \mathbb{Q}[\mathbf{x}, \mathbf{df}]$. Note that equivariance with respect to the $\mathsf{NH}_n^{\mathsf{ext}}$ action is equivalent to linearity with respect to $\mathbb{Q}[\mathbf{x}]$, together with equivariance with respect to S_n.

5.3. Preliminary computations. We say that a tuple $\mathbf{p} = \{p_1, \dots, p_n\} \subset \mathbb{Q}[\mathbf{x}]$ is *admissible* if $p_j \in \mathbb{Q}[\mathbf{x}]^{\mathsf{S}_{n-1} \times \mathsf{S}_1}$, $\deg(p_j) = n - j$, and $\partial_{c[j]} p_j \in \mathbb{Q}^\times$ for any $j = 1, \dots, n$, where $c[j] = s_j \cdot s_{j+1} \dots s_{n-1}$ and $c[n] = \mathrm{id}$. This implies, in particular, that the matrix $\mathsf{P} = [\partial_{c[j]} p_i]_{1 \le i,j \le n} \in \mathsf{Mat}(n, \mathbb{Q}[\mathbf{x}])$ is upper triangular and invertible.

We introduce the following operators. For any ring R and any $k = 1, \dots, n-1$, let γ_k be the linear operator on $\mathsf{Mat}(m \times n, R)$ defined by

$$\gamma_k(A)_{ij} = \delta_{j,k+1} A_{ik},$$

and let ρ_k be the linear operator on $\mathsf{Mat}(n \times m, R)$ defined by[1]

$$\rho_k(A)_{ij} = \delta_{i,k} A_{k+1,j}.$$

The following lemma gives a characterization of admissible tuples in terms of the corresponding matrices, obtained through divided difference operators.

[1]In other words, γ_k gives back the kth column of A in $(k+1)$th position, while ρ_k gives back the $(k+1)$th row of A in position k.

Lemma 5.3.

(i) *If* $\mathbf{p} = \{p_1, \dots, p_n\}$ *is an admissible tuple, then* P *satisfies* $\partial_k(\mathsf{P}) = \gamma_k(\mathsf{P})$ *for any* $k = 1, \dots, n-1$*, where the action of the divided difference operator is defined entrywise.*

(ii) *For any invertible* $\mathsf{Q} = [q_{ij}] \in \mathsf{Mat}(n, \mathbb{Q}[\mathbf{x}])$ *such that* $\partial_k(\mathsf{Q}) = \gamma_k(\mathsf{Q})$ *for* $k = 1, \dots, n-1$*, and* $\deg(q_{ij}) = j - i$*, the tuple* $\mathbf{q} = \{q_{1n}, \dots, q_{nn}\}$ *is admissible and* $\mathsf{Q}_{ij} = \partial_{c[j]} q_{in}$*.*

Proof. (i) follows immediately from the fact that $p_i \in \mathbb{Q}[\mathbf{x}]^{\mathsf{S}_{n-1} \times \mathsf{S}_1}$ and therefore

$$\partial_k \partial_{c[j]} p_i = \delta_{j,k+1} \partial_{c[k]} p_i.$$

Let now Q be a solution of $\partial_k(\mathsf{Q}) = \gamma_k(\mathsf{Q})$. Then, for any $k = 1, \dots, n-2$, $\partial_k q_{in} = 0$ and $q_{in} \in \mathbb{Q}[\mathbf{x}]^{\mathsf{S}_{n-1} \times \mathsf{S}_1}$, $i = 1, \dots, n$. Moreover,

$$q_{i,k} = \partial_k q_{i,k+1} = \cdots = \partial_k \partial_{k+1} \dots \partial_{n-1} q_{in} = \partial_{c[k]} q_{in}.$$

Finally, since $\deg(q_{ij}) = j - i$ and Q is invertible, it follows that $\partial_{c[j]} q_{jn} \in \mathbb{Q}^\times$. Therefore $\mathbf{q} = \{q_{1n}, \dots, q_{nn}\}$ is admissible. This proves (ii). □

We now consider the following situation. Let $\Theta = \{\theta_1, \dots, \theta_n\}$, $\Xi = \{\xi_1, \dots, \xi_n\}$ be two sets of algebraically independent elements in $\mathbb{Q}[\mathbf{x}, \mathbf{dx}]$ such that $\deg(\theta_i) = (n-i, 1) = \deg(\xi_i)$, $i = 1, \dots, n$, and let $\mathsf{P} \in \mathsf{Mat}(n, \mathbb{Q}[\mathbf{x}])$ be the invertible matrix defined by the relation

$$\Xi = \mathsf{P}\Theta. \tag{5.1}$$

Note that, necessarily, $\deg(p_{ij}) = j - i$.

Lemma 5.4. *Any two of these equations imply the third:*

(a) $\partial_k(\mathsf{P}) = \gamma_k(\mathsf{P})$*;*

(b) $\partial_k(\Xi) = 0$*;*

(c) $\partial_k(\Theta) = -\rho_k(\Theta)$.

Proof. We first show that if (a) holds, then (b) and (c) are equivalent, that is, if $\partial_k(\mathsf{P}) = \gamma_k(\mathsf{P})$, then

$$\partial_k(\Xi) = 0 \iff \partial_k \Theta = -\rho_k(\Theta). \tag{5.2}$$

Namely, since $s_k(\gamma_k(\mathsf{P})) = s_k(\partial_k(\mathsf{P})) = \partial_k(\mathsf{P}) = \gamma_k(\mathsf{P})$, one checks easily that $\gamma_k(\mathsf{P})\Theta = s_k(\mathsf{P})\rho_k(\Theta)$, where the action of s_k is defined entrywise as in the case of ∂_k. Now, the application of ∂_k to (5.1) gives

$$\begin{aligned} \partial_k(\Xi) &= \partial_k(\mathsf{P})\Theta + s_k(\mathsf{P})\partial_k(\Theta) \\ &= \gamma_k(\mathsf{P})\Theta + s_k(\mathsf{P})\partial_k(\Theta) \\ &= s_k(\mathsf{P})\big(\rho_k(\Theta) + \partial_k(\Theta)\big). \end{aligned}$$

Therefore, (5.2) follows from the invertibility of P. In particular, we proved that (a) and (b) imply (c), and (a) and (c) imply (b).

It remains to show that (b) and (c) imply (a), that is, if $\partial_k(\Xi) = 0$ and $\partial_k(\Theta) = -\rho_k(\Theta)$, then $\partial_k(\mathsf{P}) = \gamma_k(\mathsf{P})$. In this case, the application of ∂_k to (5.1) gives

$$0 = \partial_k(\mathsf{P})\Theta + s_k(\mathsf{P})\partial_k(\Theta) = \partial_k(\mathsf{P})\Theta - s_k(\mathsf{P})\rho_k(\Theta).$$

Denote by $P_1, \dots, P_n$ the column vectors of P. Since the component of $\Theta = \{\theta_1, \dots, \theta_n\}$ are algebraically independent over $\mathbb{Q}[\mathbf{x}]$, the equation $\partial_k(\mathsf{P})\Theta = s_k(\mathsf{P})\rho_k(\Theta)$ implies

$$\partial_k P_i = \delta_{i,k+1} s_k(P_k)$$

and therefore $\partial_k \mathsf{P} = \gamma_k(\mathsf{P})$. □

5.4. $\mathsf{NH}_n^{\mathsf{ext}}$–equivariant isomorphisms. Let $\mathbf{f} = \{f_1, \dots, f_n\}$ be a set of algebraically independent generators of $\mathbb{Q}[\mathbf{x}]^{\mathsf{S}_n}$, with $\deg(f_i) = n - i$, $\mathbf{p} = \{p_1, \dots, p_n\} \subset \mathbb{Q}[\mathbf{x}]$ an admissible tuple and set $\mathsf{P} = [\partial_{c[j]} p_i]_{i,j=1,\dots,n}$.

Proposition 5.5. *For any choice of* $\mathbf{f}$ *and* $\mathbf{p}$*, there is a unique* $\mathbb{Q}[\mathbf{x}]$*–linear algebra homomorphism*

$$\mathsf{J}_{\mathbf{p}}^{\mathbf{f}} \colon \mathbb{Q}[\mathbf{x}, \boldsymbol{\omega}] \to \mathbb{Q}[\mathbf{x}, \mathbf{dx}, \alpha^{-1}]$$

defined by the relation $\mathbf{df} = \mathsf{P} \cdot \mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\boldsymbol{\omega})$*. Moreover,* $\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}$ *is injective,* $\mathsf{NH}_n^{\mathsf{ext}}$*–equivariant, and degree preserving.*

Proof. Since $\mathbf{p}$ is admissible, the matrix P is invertible and the algebra homomorphism $\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}$ is uniquely determined by the condition $\mathbf{df} = \mathsf{P}\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\boldsymbol{\omega})$ and linearity in $\mathbb{Q}[\mathbf{x}]$.

The injectivity of $\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}$ follows from the invertibility of P and the algebraic independence of the elements $\mathbf{f} = \{f_1, \dots, f_n\}$ and $\mathbf{df} = \{df_1, \dots, df_n\}$.

The S_n-equivariance follows from $\mathbb{Q}[\mathbf{x}]$-linearity and Lemmas 5.3, 5.4. Namely, since $\mathbf{p}$ is admissible, it follows from Lemma 5.3 that $\partial_k(\mathsf{P}) = \gamma_k(\mathsf{P})$. Then, since $\mathbf{df} = \mathsf{P}\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\boldsymbol{\omega})$ and $\partial_k(\mathbf{df}) = 0$, it follows from Lemma 5.4 that $\partial_k(\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\boldsymbol{\omega})) = -\rho_k(\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\boldsymbol{\omega}))$, which is equivalent to

$$s_i(\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\omega_j)) = \mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\omega_j) + \delta_{ij}(x_i - x_{i+1})\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}(\omega_{i+1})$$

and implies the S_n-equivariance of $\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}$. The $\mathsf{NH}_n^{\mathsf{ext}}$-equivariance follows. Finally, the fact that $\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}$ preserves the degree is a straightforward check. □

The construction of the homomorphism $\mathsf{J}_{\mathbf{p}}^{\mathbf{f}}$ allows us to compare the description of the S_n-invariants in $\mathbb{Q}[\mathbf{x}, \boldsymbol{\omega}]$ from Theorem 4.3 and that of the S_n-invariants in $\mathbb{Q}[\mathbf{x}, \mathbf{dx}]$ from Solomon's theorem. We obtain the following corollary.

Corollary 5.6. *The homomorphisms $\mathsf{J}^{\mathbf{f}}_{\mathbf{p}}$ restricts to a canonical identification of S_n-invariants. More specifically, there is a commutative diagram*

$$\begin{tikzcd}
\mathbb{Q}[\mathbf{x},\boldsymbol{\omega}] \arrow[d, "\mathsf{J}^{\mathbf{f}}_{\mathbf{p}}"'] & \mathbb{Q}[\mathbf{x},\boldsymbol{\omega}_{\mathbf{p}}] \arrow[l, "\beta_{\mathsf{P}}"'] \arrow[d, equal] & \mathbb{Q}[\mathbf{x},\boldsymbol{\omega}_{\mathbf{p}}]^{\mathsf{S}_n} \arrow[l, hook'] \arrow[r, equal] \arrow[d, equal] & \mathbb{Q}[\mathbf{f},\boldsymbol{\omega}_{\mathbf{p}}] \arrow[d, equal] \\
\mathbb{Q}[\mathbf{x},\mathbf{dx},\alpha^{-1}] & \mathbb{Q}[\mathbf{x},\mathbf{df}] \arrow[l, hook'] & \mathbb{Q}[\mathbf{x},\mathbf{df}]^{\mathsf{S}_n} \arrow[l, hook'] \arrow[r, equal] & \mathbb{Q}[\mathbf{f},\mathbf{df}]
\end{tikzcd}$$

where β_{P} denotes the change of $\mathbb{Q}[\mathbf{x}]$–basis defined by $\boldsymbol{\omega}_{\mathbf{p}} = \mathsf{P}\boldsymbol{\omega}$ and the vertical arrows send $\boldsymbol{\omega}_{\mathbf{p}}$ to $\mathbf{df}$.

5.5. Example. Let $\mathbf{h} = \{p_1, \dots, p_n\}$ be the admissible tuple with $p_j = (-1)^{n-j}\mathsf{h}_{n-j}(x_n)$, and let H be the corresponding matrix. In particular,

$$\mathsf{H}_{ij} = \partial_{c[j]} p_i = (-1)^{j-i}\mathsf{h}_{j-i}(x_j, \dots, x_n)$$

It is easy to see that the homomorphism $\mathsf{J}^{\mathbf{f}}_{\mathbf{h}}$ is defined by

$$\mathsf{J}^{\mathbf{f}}_{\mathbf{h}}(\boldsymbol{\omega}) = \mathsf{Q}\mathbf{df} \qquad \text{where} \qquad \mathsf{Q}_{ij} = \mathsf{e}_{j-i}(x_{i+1}, \dots, x_n).$$

Similarly, let $\mathbf{e} = \{p_1, \dots, p_n\}$ be the admissible tuple with $p_j = \mathsf{e}_{n-j}(x_1, \dots, x_{n-1})$, and let E be the corresponding matrix. In particular,

$$\mathsf{E}_{ij} = \partial_{c[j]} p_i = \mathsf{e}_{j-i}(x_1, \dots, x_{j-1})$$

and the homomorphism $\mathsf{J}^{\mathbf{f}}_{\mathbf{e}}$ is defined by[2]

$$\mathsf{J}^{\mathbf{f}}_{\mathbf{e}}(\boldsymbol{\omega}) = \widetilde{\mathsf{Q}}\mathbf{de}, \quad \text{where } \widetilde{\mathsf{Q}}_{ij} = (-1)^{j-i}\mathsf{h}_{j-i}(x_1, \dots, x_i).$$

6. Differentials

In this section we show that the differential d_N on $\mathrm{NH}^{\mathrm{ext}}_n$ defined in Section 3.3 restricts to the ring of extended symmetric functions Λ^{ext}_n. We identify the resulting

[2] Both computations follow easily from the relation between the generating series of elementary and complete functions. More specifically, for $j > i$, one has

$$\Big(\sum_{k\geq 0}(-1)^k t^k \mathsf{h}_k(x_j, \dots, x_n)\Big)\Big(\sum_{k\geq 0} t^k \mathsf{e}_k(x_{i+1}, \dots, x_j, \dots, x_n)\Big) = \prod_{l=i+1}^{j-1}(1 + t x_l).$$

In particular, comparing the coefficients of t^{j-i}, we get

$$\sum_{k=i}^{j}(-1)^{j-k}\mathsf{h}_{j-k}(x_j, \dots, x_n)\mathsf{e}_{k-i}(x_{i+1}, \dots, x_n) = 0,$$

which implies that the entries of H^{-1} are the polynomials $\mathsf{e}_{j-i}(x_{i+1}, \dots, x_n)$. Similarly for E.

DG-algebra as the Koszul complex associated to a certain regular sequence of symmetric polynomials in Λ_n, whose cohomology is isomorphic to the cohomology ring of a Grassmannian. We also define new deformed differentials d_N^{Σ} on $\mathsf{NH}_n^{\mathrm{ext}}$ in Section 6.3. The deformed differentials also restrict to Λ_n^{ext} and the resulting cohomology of $(\Lambda_n^{\mathrm{ext}}, d_N^{\Sigma})$ is related to $GL(N)$-equivariant cohomology of a Grassmannian.

The reader may wish to recall the grading conventions from Section 3.3.

6.1. The standard differential. Recall that $\mathsf{NH}_n^{\mathrm{ext}}$ admits a differential d_N for each $N \geq n-1$, defined by

$$d_N(\omega_i) = (-1)^i h_{N-i+1}(x_1, \dots, x_i), \quad d_N(x_i) = 0, \quad d_N(\partial_i) = 0,$$

for all i, together with the Leibniz rule. Consequently, d_N is linear with respect to the subalgebra $\mathsf{NH}_n \subset \mathsf{NH}_n^{\mathrm{ext}}$.

Remark 6.1. With respect to the bigradings (3.1), the differential d_N is homogeneous with degree $(2-2N, -1)$.

The following states that Λ_n^{ext} is a DG-subalgebra of $\mathsf{NH}_n^{\mathrm{ext}}$ in a natural way.

Proposition 6.2. *The differential d_N restricts to a differential on $\Lambda_n^{\mathrm{ext}} \subset \mathsf{NH}_n^{\mathrm{ext}}$.*

Proof. The subset $\Lambda_n^{\mathrm{ext}} = Z(\mathsf{NH}_n^{\mathrm{ext}}) \subset \mathsf{NH}_n^{\mathrm{ext}}$ can be characterized as the set consisting of those elements $z \in \mathsf{NH}_n^{\mathrm{ext}}$ such that $[\partial_i, z] = 0$ for all divided difference operators $\partial_i \in \mathsf{NH}_n^{\mathrm{ext}}$. On the other hand d_N is NH_n-linear, so

$$[\partial_i, d_N(z)] = d_N([\partial_i, z]) = 0$$

if $[\partial_i, z] = 0$. □

Example 6.3. Let us consider the differential d_N of h_j^{ω}. We will see that $d_N(\mathsf{h}_j^{\omega})$ lands in Λ_n^{ext}, by direct computation. Recall from Remark (4.19) that for $n = 3$

$$\begin{aligned}
\mathsf{h}_1^{\omega} &= \omega_3, \\
\mathsf{h}_2^{\omega} &= \omega_2 - x_3\omega_3, \\
\mathsf{h}_3^{\omega} &= \omega_1 - (x_2 + x_3)\omega_2 + {x_3}^2\omega_3.
\end{aligned}$$

Then the differentials are computed as follows.

$$\begin{aligned}
d_N(\mathsf{h}_1^{\omega}) &= d_N(\omega_3) = (-1)^3 \mathsf{h}_{N-2}(x_1, x_2, x_3), \\
d_N(\mathsf{h}_2^{\omega}) &= d_N(\omega_2 - x_3\omega_3) = \mathsf{h}_{N-1}(x_1, x_2) + x_3 \mathsf{h}_{N-2}(x_1, x_2, x_3) \\
&= \mathsf{h}_{N-1}(x_1, x_2, x_3).
\end{aligned}$$

The last equality comes from the following observation:

$$\begin{aligned}&\big\{(a,b,c)\mid a+b+c=N-1\big\}\\&\quad=\big\{(a,b,0)\mid a+b=N-1\big\}\cup\big\{(a,b,c)\mid a+b+c=N-1, c\geq 1\big\}.\end{aligned}$$

Similarly,

$$\begin{aligned}d_N(\mathsf{h}_3^{\omega})&=d_N(\omega_1-(x_2+x_3)\omega_2+x_3^2\omega_3),\\&=-x_1^N-(x_2+x_3)\mathsf{h}_{N-1}(x_1,x_2)-x_3^2\mathsf{h}_{N-2}(x_1,x_2,x_3)\\&=-\mathsf{h}_N(x_1,x_2,x_3).\end{aligned}$$

Similar to the above argument, the last equality follows from the observation:

$$\begin{aligned}&\big\{(a,b,c)\mid a+b+c=N\big\}=\big\{(a,0,0)\mid a=N\big\}\\&\quad\cup\big\{(a,b,0)\mid a+b=N,\ b\geq 1\big\}\cup\big\{(a,b,1)\mid a+b=N-1\big\}\\&\qquad\qquad\cup\big\{(a,b,c)\mid a+b+c=N \text{ and } c\geq 2\big\}.\end{aligned}$$

Before we compute $d_N(\mathsf{h}_j^{\omega})$ in general, we need the following result on symmetric functions.

Lemma 6.4. *Let* $\mathsf{h}_i(x_j,\dots,x_n)$ *denote the complete homogeneous symmetric polynomial of degree* i *in variables* $x_j,\dots,x_n$*, for* $1\leq j\leq n$*. Then for any* $1\leq i\leq n$ *and* $N\in\mathbb{N}$

$$\mathsf{h}_{N-i+1}(x_1,\dots,x_n)=\sum_{j=0}^{n-i}\mathsf{h}_{N-i-j+1}(x_1,\dots,x_{i+j})\mathsf{h}_j(x_{i+j},\dots,x_n). \tag{6.1}$$

Proof. For any $0\leq j\leq n-i$, $1\leq i\leq n$, and $N\in\mathbb{N}$

$$\begin{aligned}&\mathsf{h}_{N-i-j+1}(x_1,\dots,x_{i+j})\mathsf{h}_j(x_{i+j},\dots,x_n)\\&\quad=\Bigg(\sum_{\substack{b_1+\dots+b_{i+j}\\=N-i-j+1}}x_1^{b_1}\cdots x_{i+j}^{b_{i+j}}\Bigg)\Bigg(\sum_{a_{i+j}+\dots+a_n=j}x_{i+j}^{a_{i+j}}\cdots x_n^{a_n}\Bigg)\\&\quad=\sum_{\substack{b_1+\dots+b_{i+j}\\=N-i-j+1}}\ \sum_{a_{i+j}+\dots+a_n=j}x_1^{b_1}\cdots x_{i+j}^{b_{i+j}+a_{i+j}}x_{i+j+1}^{a_{i+j+1}}\cdots x_n^{a_n}\\&\quad=\sum_{k=0}^{j}\Bigg(\sum_{\substack{b_1+\dots+b_{i+j}\\=N-i-j+1}}\Bigg(\sum_{a_{i+j}+\dots+a_n=j-k}x_1^{b_1}\cdots x_{i+j}^{b_{i+j}+k}x_{i+j+1}^{a_{i+j+1}}\cdots x_n^{a_n}\Bigg)\Bigg).\end{aligned}$$

The exponent of each monomial in above sum is an n-tuple

$$(b_1,\dots,b_{i+j}+k,a_{i+j+1},\dots,a_n),$$

where

$$\begin{aligned} b_1 + \cdots + b_{i+j} &= N - i - j + 1, \\ a_{i+j+1} + \cdots + a_n &= j - k, \\ a_{i+j} &= k, \qquad\qquad \text{for any } 0 \le k \le j. \end{aligned}$$

As j varies in the range $0 \le j \le n - i$ these exponents exhaust uniquely all monomials appearing in $\mathsf{h}_{N-i+1}(x_1, \ldots, x_n)$. □

6.2. Koszul complex. Let R be a commutative ring, and let $a_1, \ldots, a_r \in R$ be given elements. The *Koszul complex* associated to $(a_1, \ldots, a_r)$ is the DG algebra

$$R \otimes \bigwedge[\theta_1, \ldots, \theta_r]$$

with R-linear differential uniquely characterized by $d(\theta_i) = a_i$ together with the graded Leibniz rule. For the purposes of the Leibniz rule, the grading places R in homological degree zero, and each θ_i in homological degree -1.

Proposition 6.5. *As a DG-algebra, Λ_n^{ext} is isomorphic to the Koszul complex associated to $(-1)^i h_{N-i+1} \in \Lambda_n$ ($1 \le i \le n$).*

Proof. By Theorem 4.3 we know that $\Lambda_n^{\mathsf{ext}} \simeq \Lambda_n \otimes \bigwedge[\omega_1^{\mathsf{s}}, \ldots, \omega_n^{\mathsf{s}}]$, where $\omega_j^{\mathsf{s}} = \omega_j^{\mathsf{s}}(p_j)$ are determined by any choice of $p_j \in \mathbb{Q}[x_1, \ldots, x_n]^{S_{n-1} \times S_1}$ such that $\partial_j \partial_{j+1} \cdots \partial_{n-1}(p_j) = 1$. For the purposes of computing the differential, it is especially convenient to work with the choice of p_j as constructed in 4.4.6. In this case the resulting elements ω_i^{s} are given by

$$\vartheta_i^* := \sum_{j=0}^{n-i} (-1)^j \mathsf{h}_j(x_{i+j}, \ldots, x_n)\omega_{i+j}.$$

We know that the differential d_N is linear with respect to the subalgebra Λ_n (this follows from $d_N(x_i) = 0$ and the Leibniz rule), hence to prove the Proposition we need only show that $d_N(\vartheta_i^*) = (-1)^i \mathsf{h}_{N-i+1}(x_1, \ldots, x_n)$. Compute:

$$\begin{aligned} d_N(\vartheta_i^*) &= \sum_{j=0}^{n-i} (-1)^j \mathsf{h}_j(x_{i+j}, \ldots, x_n) d_N(\omega_{i+j}) \\ &= \sum_{j=0}^{n-i} (-1)^j \mathsf{h}_j(x_{i+j}, \ldots, x_n)(-1)^{i+j} \mathsf{h}_{N-i-j+1}(x_1, \ldots, x_{i+j}) \\ &= (-1)^i \mathsf{h}_{N-i+1}(x_1, \ldots, x_n), \end{aligned}$$

where the last equality follows from Lemma 6.4. □

A sequence of elements $\mathbf{a} = (a_1, \dots, a_r) \in R$ is called a *regular sequence* if

- a_1 is not a zero divisor;
- a_i is not a zero divisor in $R/\langle a_1, \dots, a_{i-1}\rangle$ for all $2 \leq i \leq n$.

If $\mathbf{a}$ is regular, then the associated Koszul complex $K(\mathbf{a})$ has cohomology only in degree zero, where it is isomorphic to $R/\langle a_1, \dots, a_r\rangle$. Said differently, if $\mathbf{a}$ is a regular sequence then the canonical projection $K(\mathbf{a}) \to R/\langle a_1, \dots, a_r\rangle$ is a quasi-isomorphism.

Corollary 6.6. *The DG-algebra $(\Lambda_n^{\mathsf{ext}}, d_N)$ is quasi-isomorphic to the cohomology ring $H^*(\mathrm{Gr}(n, N))$.*

Proof. The sequence $h_N, h_{N-1}, \dots, h_{N-n+1} \in \Lambda_n$ is a regular sequence, see for example [34, Proposition 7.2]. Thus, the cohomology of the associated Koszul complex is isomorphic to the quotient $\Lambda_n/\langle h_N, h_{N-1}, \dots, h_{N-n+1}\rangle$, which is known to be isomorphic to $\Lambda_n/\langle \mathsf{h}_{N-n+1}\rangle \cong H^*(\mathrm{Gr}(n, N))$. □

6.3. Deformed differentials.

6.3.1. Deformed cyclotomic quotients. The cyclotomic quotients of the nilHecke algebra, and KLR algebras more generally, admit deformations called deformed cyclotomic quotients defined in [30]. For us the most relevant reference is [26, Section 3.2].

Let $\kappa_1, \dots, \kappa_N \in \mathbb{C}$ be given, and let Σ denote the root multiset consisting of pairwise distinct complex numbers $\lambda_1, \dots, \lambda_\ell$ corresponding to the roots of the polynomial

$$P(x) = x^N + \sum_{j=1}^{N} \kappa_j x^{N-j}, \tag{6.2}$$

with multiplicities $N_1, \dots, N_\ell$. For each $N > 0$ define the *deformed cyclotomic ideal I_N^Σ* associated to Σ as the ideal of NH_n defined by

$$I_N^\Sigma := \left\langle \sum_{j=0}^{N} \kappa_j x_1^{N-j} \right\rangle, \quad \kappa_i \in \mathbb{C}, \tag{6.3}$$

where we take $\kappa_0 = 1$. We define the *deformed cyclotomic quotient*

$$\mathsf{NH}_n^\Sigma := \mathsf{NH}_n/I_N^\Sigma.$$

In [26, Section 3.2] it is shown that the deformed cyclotomic quotient rings NH_n^Σ are isomorphic to matrix rings of size $n!$ with coefficients in the $GL(N)$-equivariant cohomology ring $H^*_{GL(N)}(\mathrm{Gr}(n, N))$ with equivariant parameters equal to $\underline{\kappa} = (\kappa_1, \kappa_2, \dots, \kappa_N)$. We denote this specialization by H_n^Σ. If the parameters $\underline{\kappa}$ are left generic, then the center of the deformed cyclotomic quotient is just the $GL(N)$-equivariant cohomology itself [33, Theorem 2.10].

Theorem 6.7 ([26, Theorem 13]). *There is an algebra isomorphism*

$$H_n^{\Sigma} \cong \bigoplus_{\substack{\sum n_j = n \\ 0 \le n_j \le N}} \bigotimes_{j=1}^{\ell} H^*(\mathrm{Gr}(n_j, N_j)).$$

We will realize both the deformed cyclotomic quotient NH_n^{Σ} and the rings H_n^{Σ} within the context of the extended nilHecke algebra. For these realization we make use of the following lemma.

Lemma 6.8. *The following identities hold in* NH_n^{Σ}.

(1) *For any* $1 \le i \le n$,

$$\sum_{j=0}^{N} \kappa_j x_i^{N-j} = 0;$$

(2) *For any* $m \le N$,

$$\sum_{j=0}^{N-m+1} \kappa_j \sum_{\sum a_i = (N-m+1-j)} x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} = \sum_{j=0}^{N-m+1} \kappa_j \mathsf{h}_{N-m+1-j}(x_1, \dots, x_m) = 0.$$

Proof. The first claim is proven by induction. Namely, for any $0 \le i \le n-1$, we prove the following.

(A1)$_i$ For any $y \in \mathbb{N}$,

$$\sum_{j=0}^{N} \kappa_j x_{i+1}^{y+(N-j)} \partial_i = \sum_{j=0}^{N} \kappa_j \partial_i x_{i+1}^{y+(N-j)};$$

(A2)$_i$ For any $y \in \mathbb{N}$,

$$\sum_{j=0}^{N} \kappa_j x_{i+1}^{y+(N-j)} \partial_i = 0;$$

(A3)$_i$

$$\sum_{j=0}^{N} \kappa_j x_{i+1}^{N-j} = 0.$$

Recall that by definition $\partial_0 = 1$. The case $i = 0$ holds by construction. Assume now $(\mathrm{A1})_{i-1}$, $(\mathrm{A2})_{i-1}$, $(\mathrm{A3})_{i-1}$, with $i > 0$. One has

$$\begin{aligned}
\sum_{j=0}^{N} \kappa_j x_{i+1}^{y+N-j} \partial_i &= \sum_{j=0}^{N} \kappa_j \partial_i x_i^{y+N-j} - \sum_{j=0}^{N} \kappa_j \mathsf{h}_{y+N-j-1}(x_i, x_{i+1}) \\
&= -\sum_{j=0}^{N} \kappa_j \mathsf{h}_{y+N-j-1}(x_i, x_{i+1}) \\
&= \sum_{j=0}^{N} \kappa_j x_i^{y+N-j} \partial_i - \sum_{j=0}^{N} \kappa_j \mathsf{h}_{y+N-j-1}(x_i, x_{i+1}) \\
&= \sum_{j=0}^{N} \kappa_j \partial_i x_{i+1}^{y+N-j},
\end{aligned}$$

where the first and fourth equalities follow by (2.2), the second and third ones follow by $(\mathrm{A3})_{i-1}$. This proves $(\mathrm{A1})_i$. Then, $(\mathrm{A2})_i$ holds, since

$$\begin{aligned}
\sum_{j=0}^{N} \kappa_j x_{i+1}^{y+N-j} \partial_i &= -\sum_{j=0}^{N} \kappa_j x_{i+1}^{y+N-j} \partial_i x_{i+1} \partial_i \\
&= -\sum_{j=0}^{N} \kappa_j \partial_i x_{i+1}^{y+N-j+1} \partial_i \\
&= -\sum_{j=0}^{N} \kappa_j x_{i+1}^{y+N-j+1} \partial_i^2 = 0,
\end{aligned}$$

where the first equality follows from $\partial_i = -\partial_i x_{i+1} \partial_i$, the second and third ones from $(\mathrm{A1})_i$. Finally, $(\mathrm{A3})_i$ holds, since

$$\sum_{j=0}^{N} \kappa_j x_{i+1}^{y+N-j} = \sum_{j=0}^{N} \kappa_j x_{i+1}^{y+N-j} \partial_i x_i - \sum_{j=0}^{N} \kappa_j x_{i+1}^{y+N-j+1} \partial_i = 0,$$

where the first equality follows from the nilHecke relations (2.1) and the second one from $(\mathrm{A2})_i$. This proves the first claim.

The second claim is similarly proven by induction. Using (2.1), we have

$$\sum_{j=0}^{N} \kappa_j \big(\partial_i x_i^{N-j}\big) - \sum_{j=0}^{N} \kappa_j \big(x_i^{N-j} \partial_i\big) = \sum_{j=0}^{N-1} \kappa_j \Bigg(\sum_{a+b=y+(N-j)-1} x_i^a x_{i+1}^b \Bigg).$$

The induction step is identical to Proposition 2.8 in [4]. □

6.3.2. Deformed differentials. Let Σ denote the root multiset corresponding to the roots and multiplicities of the polynomial (6.2). To each Σ define a differential d_N^{Σ} on $\mathsf{NH}_n^{\mathrm{ext}}$, which we call *deformed differential*, by

$$d_N^{\Sigma}(\partial_i) = 0, \quad d_N^{\Sigma}(x_i) = 0,$$
$$\text{and} \quad d_N^{\Sigma}(\omega_i) = \sum_{j=0}^{N-i+1} (-1)^{i+1} \kappa_j \mathsf{h}_{N-i+1-j}(x_1, \dots, x_i). \tag{6.4}$$

Note that the deformed differential d_N^{Σ} is homogeneous of degree -1 with respect to the homological grading $\deg_h$, but is in general not homogeneous with respect to $\deg_q$. Thus, we will regard $(\mathsf{NH}_n^{\mathrm{ext}}, d_N)$ as only a singly graded object (via $\deg_h$).

Proposition 6.9. *The map d_N^{Σ} satisfies the relations*

(1) $\partial_i d_N^{\Sigma}(\omega_{i+1}) = d_N^{\Sigma}(\omega_{i+1})\partial_i$;

(2) $\partial_i d_N^{\Sigma}(\omega_i) + d_N^{\Sigma}(\omega_{i+1})x_{i+1}\partial_i = d_N^{\Sigma}(\omega_i)\partial_i + \partial_i x_{i+1} d_N^{\Sigma}(\omega_{i+1})$,

for all $1 \le i \le n$.

Proof. The first identity holds since $d_N^{\Sigma}(\omega_{i+1})$ is symmetric in x_i and x_{i+1}. For the second identity, we show that

$$d_N^{\Sigma}(\omega_{i+1})x_{i+1}\partial_i - \partial_i x_{i+1} d_N^{\Sigma}(\omega_{i+1}) = d_N^{\Sigma}(\omega_i)\partial_i - \partial_i d_N^{\Sigma}(\omega_i).$$

One has

$$\begin{aligned}
&d_N^{\Sigma}(\omega_{i+1})x_{i+1}\partial_i \\
&\quad = \sum_{j=0}^{N-i} (-1)^{i+2}\kappa_j \big(\mathsf{h}_{N-i-j}(x_1, \dots, x_{i+1})x_{i+1}\partial_i\big) \\
&\quad = \sum_{j=0}^{N-i} (-1)^{i+2}\kappa_j \Bigg(\sum_{a+b=N-i-j} \mathsf{h}_a(x_1, \dots, x_{i-1})\mathsf{h}_b(x_i, x_{i+1})x_{i+1}\partial_i\Bigg) \\
&\quad = \sum_{j=0}^{N-i} (-1)^{i+2}\kappa_j \Bigg(\sum_{a+b=N-i-j} \mathsf{h}_a(x_1, \dots, x_{i-1}) \sum_{k+\ell=b} x_i^k x_{i+1}^{\ell+1}\partial_i\Bigg) \\
&\quad = \sum_{j=0}^{N-i} (-1)^{i+2}\kappa_j \Bigg(\sum_{a+b=N-i-j} \mathsf{h}_a(x_1, \dots, x_{i-1})x_{i+1}^{b+1}\partial_i\Bigg) \\
&\qquad + \sum_{j=0}^{N-i} (-1)^{i+2}\kappa_j \Bigg(\sum_{a+b=N-i-j} \mathsf{h}_a(x_1, \dots, x_{i-1}) \sum_{k'+\ell'=b-1} x_i^{k'+1} x_{i+1}^{\ell'+1}\partial_i\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\Bigg(\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i x_i^{b+1}\Bigg)\\
&\quad - \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\Bigg(\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\mathsf{h}_b(x_i,x_{i+1})\Bigg)\\
&\quad + \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\Bigg(\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\sum_{k'+\ell'=b-1}x_i^{k'+1}x_{i+1}^{\ell'+1}\partial_i\Bigg)\\
&= \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\Bigg(\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i x_i^{b+1}\Bigg)\\
&\quad - \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\mathsf{h}_{N-i-j}(x_1,\dots,x_{i+1})\\
&\quad + \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\Bigg(\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\sum_{k'+\ell'=b-1}x_i^{k'+1}x_{i+1}^{\ell'+1}\Bigg)\partial_i,
\end{aligned}
$$

where the fifth equality follows by applying (2.2) to the first summand. A similar computation gives

$$
\begin{aligned}
\partial_i x_{i+1} d_N^{\Sigma}(\omega_{i+1}) &= \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i x_{i+1}^{b+1}\\
&\quad + \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\Bigg(\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\sum_{k'+\ell'=b-1}x_i^{k'+1}x_{i+1}^{\ell'+1}\Bigg)\partial_i.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&d_N^{\Sigma}(\omega_{i+1})x_{i+1}\partial_i - \partial_i x_{i+1} d_N^{\Sigma}(\omega_{i+1})\\
&\quad = \sum_{j=0}^{N-i}(-1)^{i+2}\kappa_j\sum_{a+b=N-i-j}\mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i(x_i^{b+1}-x_{i+1}^{b+1})\\
&\qquad\qquad + \sum_{j=0}^{N-i}(-1)^{i+1}\kappa_j\mathsf{h}_{N-i-j}(x_1,\dots,x_{i+1}).
\end{aligned}
$$

On the other hand, one has

$$
\begin{aligned}
d_N^{\Sigma}(\omega_i)\partial_i - \partial_i d_N^{\Sigma}(\omega_i) &= \sum_{j=0}^{N-i+1}(-1)^{i+1}\kappa_j\mathsf{h}_{N-i+1-j}(x_1,\dots,x_i)\partial_i - \partial_i\\
&\qquad\cdot\sum_{j=0}^{N-i+1}(-1)^{i+1}\kappa_j\mathsf{h}_{N-i+1-j}(x_1,\dots,x_i)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\sum_{a+b=N-i+1-j} \mathsf{h}_a(x_1,\dots,x_{i-1})\mathsf{h}_b(x_i) \Bigg)\partial_i \\
&\qquad - \partial_i \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\sum_{a+b=N-i+1-j} \mathsf{h}_a(x_1,\dots,x_{i-1})\mathsf{h}_b(x_i) \Bigg) \\
&= \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\mathsf{h}_{N-i+1-j}(x_1,\dots,x_{i-1})\partial_i + \sum_{a+b'=N-i-j} \mathsf{h}_a(x_1,\dots,x_{i-1})x_i^{b'+1}\partial_i \Bigg) \\
&\qquad - \partial_i \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\mathsf{h}_{N-i+1-j}(x_1,\dots,x_{i-1}) + \sum_{a+b'=N-i-j} \mathsf{h}_a(x_1,\dots,x_{i-1})x_i^{b'+1} \Bigg) \\
&= \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\sum_{a+b'=N-i-j} \mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i x_{i+1}^{b'+1} \Bigg) \\
&\qquad + \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\sum_{a+b'=N-i-j} \mathsf{h}_a(x_1,\dots,x_{i-1})\mathsf{h}_{b'}(x_i,x_{i+1}) \Bigg) \\
&\qquad - \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\sum_{a+b'=N-i-j} \mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i x_i^{b'+1} \Bigg) \\
&= \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \mathsf{h}_{N-i-j}(x_1,\dots,x_{i+1}) \\
&\qquad + \sum_{j=0}^{N-i+1} (-1)^{i+1}\kappa_j \Bigg(\sum_{a+b'=N-i-j} \mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i ({x_{i+1}}^{b'+1} - x_i^{b'+1}) \Bigg),
\end{aligned}
$$

where the fourth equality follows again from (2.2). Finally, since there is no contribution for $j = N-i+1$, one has

$$
\begin{aligned}
&d_N^{\Sigma}(\omega_i)\partial_i - \partial_i d_N^{\Sigma}(\omega_i) \\
&\qquad = \sum_{j=0}^{N-i} (-1)^{i+2}\kappa_j \Bigg(\sum_{a+b'=N-i-j} \mathsf{h}_a(x_1,\dots,x_{i-1})\partial_i ({x_i}^{b'+1} - x_{i+1}^{b'+1}) \Bigg) \\
&\qquad\qquad\qquad + \sum_{j=0}^{N-i} (-1)^{i+1}\kappa_j \mathsf{h}_{N-i-j}(x_1,\dots,x_{i+1})
\end{aligned}
$$

and the second identity follows. □

Corollary 6.10. *The deformed differential d_N^{Σ} defines a degree -1 differential on $\mathrm{NH}_n^{\mathrm{ext}}$.*

Proof. The only nontrivial relations to verify are proven in Proposition 6.9. □

Theorem 6.11. *The DG–algebra $(\mathsf{NH}_n^{\mathrm{ext}}, d_N^{\Sigma})$ is quasi-isomorphic to deformed cyclotomic quotient of the nilHecke algebra $\mathsf{NH}_n^{\Sigma} = \mathsf{NH}_n/\langle(\sum_{j=0}^{N} \kappa_j x_1^{N-j})\rangle$.*

Proof. The statement follows immediately from the identity (2) of Lemma 6.8. That is, since

$$\sum_{j=0}^{N-m+1} \kappa_j \mathsf{h}_{N-m+1-j}(x_1,\dots,x_m) = 0$$

in NH_n^{Σ}, the same holds for the image of $d_N^{\Sigma}(\omega_i)$ in NH_n^{Σ}. □

Proposition 6.12. *For each $N > 0$, the pair $(\Lambda_n^{\mathrm{ext}}, d_N^{\Sigma})$ is a DG-subalgebra of $(\mathsf{NH}_n^{\mathrm{ext}}, d_N^{\Sigma})$.*

Proof. This is immediate since the differential d_N^{Σ} acting on h_i^{ω} can be expressed as a linear combination of undeformed differentials, each of which preserves the ring Λ_n^{ext}. □

Theorem 6.13. *The DG-algebra $(\Lambda_n^{\mathrm{ext}}, d_N^{\Sigma})$ is quasi-isomorphic to the ring H_n^{Σ} from Theorem 6.7.*

Proof. This follows from [26, Lemma 11] and Proposition 6.5. □

6.4. Categorification. Let $\mathbf{f}$ denote the positive part $\mathbf{U}^{+}(\mathfrak{sl}_2)$ of the quantized universal enveloping algebra of $\mathfrak{sl}_2$. This $\mathbb{Q}(q)$-algebra is a polynomial ring in the generator E. This algebra is $\mathbb{N}$-graded with E in degree 2. We equip the tensor product $\mathbf{f} \otimes \mathbf{f}$ with the twisted algebra structure

$$(E^a \otimes E^b)(E^c \otimes E^d) = q^{-2cd} E^a E^c \otimes E^b E^d.$$

The algebra $\mathbf{f}$ contains a subring ${}_{\mathcal{A}}\mathbf{f}$ which is the $\mathbb{Z}[q, q^{-1}]$-lattice generated by all products of quantum divided powers

$$E^{(n)} := \frac{E^n}{[n]!}. \tag{6.5}$$

Hence, a categorification of ${}_{\mathcal{A}}\mathbf{f}$ amounts to identifying objects $\mathcal{E}^{(n)}$ and $\mathcal{E}^n$ in a graded category and lifting the divided power relation (6.5) to an explicit isomorphism

$$\mathcal{E}^n \cong \bigoplus_{[n]!} \mathcal{E}^{(n)} = \mathcal{E}^{(n)}\langle n-1\rangle \oplus \mathcal{E}^{(n)}\langle n-3\rangle \oplus \cdots \oplus \mathcal{E}^{(n)}\langle 1-n\rangle. \tag{6.6}$$

The extended nilHecke algebra has been studied in connection with Verma modules by Naisse and Vaz [22, 24]. Here we show that the results from the previous section allow us to define a categorification of ${}_{\mathcal{A}}\mathbf{f}$, and in particular, define categorifications of quantum divided powers. For this it suffices only to consider only the quantum grading on the extended nilHecke algebra, as we will do throughout this section, regarding $\mathsf{NH}_n^{\mathsf{ext}}$ as a $\mathbb{Z}$-graded algebra. Consider the $\mathbb{Z}$- graded ring

$$\mathsf{NH}^{\mathsf{ext}} := \bigoplus_{n\geq 0} \mathsf{NH}_n^{\mathsf{ext}},$$

and denote by $\mathsf{NH}^{\mathsf{ext}}$-gmod the category of projective graded $\mathsf{NH}^{\mathsf{ext}}$-modules. Recall from Proposition 4.7 the isomorphism $\mathsf{NH}_n^{\mathsf{ext}} \cong \mathrm{Mat}((n)^!_{q^2}, \Lambda_n^{\mathsf{ext}})$. One can easily show that $e_n = \underline{x}^{\delta}\partial_{w_0}$ is the minimal idempotent projecting onto the lowest degree column of $\mathsf{NH}_n^{\mathsf{ext}}$. The graded module $\mathsf{NH}_n^{\mathsf{ext}} e_n$ is the unique indecomposable projective of $\mathsf{NH}_n^{\mathsf{ext}}$ up to isomorphism and grading shift. The regular representation then decomposes into $n!$ isomorphic copies of $\mathsf{NH}_n^{\mathsf{ext}} e_n$. Taking gradings into account, if we define

$$\mathcal{E}^{(n)} := \mathsf{NH}_n^{\mathsf{ext}} e_n \langle -n(n-1)/2\rangle, \quad \mathcal{E}^n := \mathsf{NH}_n^{\mathsf{ext}},$$

then we have an isomorphism of graded projective left modules

$$\mathcal{E}^n := \mathsf{NH}_n^{\mathsf{ext}} \cong \bigoplus_{[n]!} \mathsf{NH}_n^{\mathsf{ext}} e_n =: \bigoplus_{[n]!} \mathcal{E}^{(n)}.$$

Hence, we have proven the following.

Proposition 6.14. *There is an isomorphism of $\mathcal{A}$-modules*

$$\gamma\colon {}_{\mathcal{A}}\mathbf{f} \to K_0(\mathsf{NH}^{\mathsf{ext}}) \tag{6.7}$$

sending $E^{(n)}$ to the class of the indecomposable projective module $\mathcal{E}^{(n)}$.

There are inclusions of graded rings

$$\iota_{n,m}\colon \mathsf{NH}_n^{\mathsf{ext}} \otimes \mathsf{NH}_m^{\mathsf{ext}} \to \mathsf{NH}_{n+m}^{\mathsf{ext}} \tag{6.8}$$

given diagrammatically by placing diagrams side-by-side with those in $\mathsf{NH}_n^{\mathsf{ext}}$ appearing above $\mathsf{NH}_m^{\mathsf{ext}}$. In order to make this inclusion graded, it is necessary to adjust the gradings of the odd generators in $\mathsf{NH}_m^{\mathsf{ext}}$ by an appropriate amount, as in Remark 3.1. In the notation of the aforementioned remark, the above map should be written

$$\iota_{n,m}\colon (\mathsf{NH}_n^{\mathsf{ext}})^{(0)} \otimes (\mathsf{NH}_m^{\mathsf{ext}})^{(n)} \to (\mathsf{NH}_{n+m}^{\mathsf{ext}})^{(0)}.$$

These inclusions give rise to induction and restriction functors

$$\begin{aligned}
\mathrm{Ind}_{n,m}\colon & \big((\mathsf{NH}_n^{\mathsf{ext}})^{(0)} \otimes (\mathsf{NH}_m^{\mathsf{ext}})^{(n)}\big)\text{-gmod} \to (\mathsf{NH}_{n+m}^{\mathsf{ext}})^{(0)}\text{-gmod},\\
\mathrm{Res}_{n,m}\colon & (\mathsf{NH}_{n+m}^{\mathsf{ext}})^{(0)}\text{-gmod} \to \big((\mathsf{NH}_n^{\mathsf{ext}})^{(0)} \otimes (\mathsf{NH}_m^{\mathsf{ext}})^{(n)}\big)\text{-gmod}.
\end{aligned}$$

By the basis theorem 4.3 for $\mathsf{NH}_{n+m}^{\mathsf{ext}}$ it follows that the super module $\mathsf{NH}_{n+m}^{\mathsf{ext}}$ is a free graded left super $(\mathsf{NH}_n^{\mathsf{ext}})^{(0)} \otimes (\mathsf{NH}_m^{\mathsf{ext}})^{(n)}$-module. A basis is given by the crossing diagrams in $\mathsf{NH}^{\mathrm{ext}}_{n+m}$ corresponding to the minimal representative of a left $\mathsf{S}_n \times S_m$-coset in S_{n+m}, see for example [9, Proposition 2.16]. It follows that $\mathrm{Res}_{n,m}$ takes projectives to projectives, and therefore descends to a map in the Grothendieck group. Similarly, by a version of the Mackey induction-restriction theorem it follows that $\mathrm{Ind}_{n,m}$ also sends projectives to projectives.

At the level of Grothendieck groups we have

$$[\mathrm{Ind}_{n,m}]\colon K_0\big((\mathsf{NH}^{\mathrm{ext}}_n)^{(0)} \otimes (\mathsf{NH}^{\mathrm{ext}}_m)^{(n)}\big) \to K_0((\mathsf{NH}^{\mathrm{ext}}_{n+m})^{(0)}),$$
$$[\mathrm{Res}_{n,m}]\colon K_0((\mathsf{NH}^{\mathrm{ext}}_{n+m})^{(0)}) \to K_0\big((\mathsf{NH}^{\mathrm{ext}}_n)^{(0)} \otimes (\mathsf{NH}^{\mathrm{ext}}_m)^{(n)}\big).$$

Since $K_0((\mathsf{NH}^{\mathrm{ext}}_n)^{(m)})$ is canonically isomorphic to $K_0((\mathsf{NH}^{\mathrm{ext}}_n)^{(0)}) = K_0(\mathsf{NH}^{\mathrm{ext}}_n)$, these maps induce maps

$$[\mathrm{Ind}_{n,m}]\colon K_0(\mathsf{NH}^{\mathrm{ext}}_n) \otimes K_0(\mathsf{NH}^{\mathrm{ext}}_m) \to K_0(\mathsf{NH}^{\mathrm{ext}}_{n+m}),$$
$$[\mathrm{Res}_{n,m}]\colon K_0(\mathsf{NH}^{\mathrm{ext}}_{n+m} \to K_0(\mathsf{NH}^{\mathrm{ext}}_n) \otimes K_0(\mathsf{NH}^{\mathrm{ext}}_m).$$

Summing over all $n, m \in \mathbb{Z}_{\geq 0}$ these functors induce maps

$$[\mathrm{Ind}]\colon K_0(\mathsf{NH}^{\mathrm{ext}}) \otimes K_0(\mathsf{NH}^{\mathrm{ext}}) \to K_0(\mathsf{NH}^{\mathrm{ext}}),$$
$$[\mathrm{Res}]\colon K_0(\mathsf{NH}^{\mathrm{ext}}) \to K_0(\mathsf{NH}^{\mathrm{ext}}) \otimes K_0(\mathsf{NH}^{\mathrm{ext}}).$$

Just as in the case of the nilHecke algebra, see [9], induction and restriction equip ${}_{\mathcal{A}}\mathbf{f}$ with the structure of a twisted bialgebra and we have the following result.

Theorem 6.15. *The isomorphism*

$$\gamma\colon {}_{\mathcal{A}}\mathbf{f} \to K_0(\mathsf{NH}^{\mathrm{ext}}) \tag{6.9}$$

is an isomorphism of twisted bialgebras.

Remark 6.16. In [24, Section 3.6] the authors independently considered a related construction where they sum the algebras $(\mathsf{NH}^{\mathrm{ext}}_n)^{(t)}$ over both $n, t \in \mathbb{Z}$. They then take the sum over $t \in \mathbb{Z}$ of induction and restriction functors

$$\mathrm{Ind}^{(t)}_{n,m}\colon \big((\mathsf{NH}^{\mathrm{ext}}_n)^{(t)} \otimes (\mathsf{NH}^{\mathrm{ext}}_m)^{(n+t)}\big)\text{-gmod} \to (\mathsf{NH}^{\mathrm{ext}}_{n+m})^{(t)}\text{-gmod},$$
$$\mathrm{Res}^{(t)}_{n,m}\colon (\mathsf{NH}^{\mathrm{ext}}_{n+m})^{(t)}\text{-gmod} \to \big((\mathsf{NH}^{\mathrm{ext}}_n)^{(t)} \otimes (\mathsf{NH}^{\mathrm{ext}}_m)^{(n+t)}\big)\text{-gmod}.$$

At the level of Grothendieck rings, this corresponds to a direct sum over $t \in \mathbb{Z}$ many copies of ${}_{\mathcal{A}}\mathbf{f}$. They regard this as a copy of the positive part of $\mathfrak{sl}(2)$ inside the Beilinson–Lusztig–MacPherson idempotent form of the quantum group, since their construction effectively includes idempotents indexed by the weight lattice $t \in \mathbb{Z}$.

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