Cartwright–Sturmfels ideals
associated to graphs and linear spaces

Aldo Conca,* Emanuela De Negri* and Elisa Gorla**

Abstract. Inspired by work of Cartwright and Sturmfels, in [15] we introduced two classes of multigraded ideals named after them. These ideals are defined in terms of properties of their multigraded generic initial ideals. The goal of this paper is showing that three families of ideals that have recently attracted the attention of researchers are Cartwright–Sturmfels ideals. More specifically, we prove that binomial edge ideals, multigraded homogenizations of linear spaces, and multiview ideals are Cartwright–Sturmfels ideals, hence recovering and extending recent results of Herzog, Hibi, Hreinsdottir, Kahle, and Rauh [20], Ohtani [28], Ardila and Boocher [3], Aholt, Sturmfels, and Thomas [2], and Binglin Li [6]. We also propose a conjecture on the rigidity of local cohomology modules of Cartwright–Sturmfels ideals, that was inspired by a theorem of Brion. We provide some evidence for the conjecture by proving it in the monomial case.

Mathematics Subject Classification (2010). 13C40, 13P10, 05E40; 14M99, 68T45.

Keywords. Gröbner basis, generic Initial Ideal, multiview ideals, binomial edge ideals.

1. Introduction

Inspired by the work of Cartwright and Sturmfels [11], in [15] we introduced two families of multigraded ideals, namely Cartwright–Sturmfels ideals and Cartwright–Sturmfels* ideals. Both families are characterized by means of multigraded generic initial ideals. Cartwright–Sturmfels (CS for short) ideals are the $\mathbb{Z}^n$-graded ideals whose multigraded generic initial ideal is radical, while Cartwright–Sturmfels* (CS*) ideals are the $\mathbb{Z}^n$-graded ideals whose multigraded generic initial ideal has a system of generators which involves at most one variable of degree $e_i \in \mathbb{Z}^n$ for every $i = 1, \ldots, n$.

Ideals with a minimal system of generators that is also a universal Gröbner basis are called robust. Being robust is a very strong property and robust ideals have attracted a lot of interest, especially in recent years, see e.g. [5,8,14–16,21,31,33,34].

Cartwright–Sturmfels* ideals are "very" robust in the sense that every multigraded minimal system of generators of a CS* ideal is a universal Gröbner basis. Furthermore a Cartwright–Sturmfels* ideal has the same graded Betti numbers as its initial ideals. It turns out that CS ideals and their initial ideals are radical and very often universal Gröbner bases of CS ideals can be kept under control as well.

In [14] and [15] we presented large families of determinantal ideals that are CS or CS*, and discussed how these results generalize classical results of Bernstein, Sturmfels, and Zelevinsky [5, 33] on universal Gröbner bases for generic maximal minors. In [16] we gave a combinatorial description of their multigraded generic initial ideals.

The goal of this paper is showing that the following three families of ideals, that have recently attracted the attention of several researchers, are CS and, in some cases, also CS*.

1. Binomial edge ideals. Binomial edge ideals have been introduced and studied by Herzog, Hibi, Hreinsdottir, Kahle, and Rauh [20] and independently by Ohtani [28], who proved that they are always radical. Matsuda and Murai [25] proved that the regularity of the binomial edge ideal associated to a graph G is bounded by the number of vertices of the graph and conjectured that equality holds only when the graph is a line. Other interesting results concerning binomial edge ideals can be found, for example, in [17, 18, 22]. We prove that binomial edge ideals are CS and describe the associated generic initial ideal. As an immediate corollary we obtain the aforementioned results of [20, 25, 28].

2. Closure of linear spaces in products of projective spaces. Let V be a vector space of linear forms in $K[x_1, \ldots, x_n]$ and let $L \subset \mathbb{A}_K^n$ be the zero locus of V, i.e. a linear space containing the origin. In [3] Ardila and Boocher studied the ideal $I(\widetilde{L})$ defining the closure $\widetilde{L}$ of L in $(\mathbb{P}^1)^n$. They established several interesting structural properties of $I(\widetilde{L})$. We prove that $I(\widetilde{L})$ is both a CS and a CS* ideal. As a consequence we recover some of the results of [3]. More generally we prove that the ideal $I_a(\widetilde{L})$ defining the closure of L in the product $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_u}$ where $a = (a_1, \ldots, a_u)$ and $\sum a_i = n$ is CS (but not CS* in general). We describe $I_a(\widetilde{L})$ first as the saturation with respect to the set of homogenizing variables of a single determinantal ideal and then as a sum of several determinantal ideals. Furthermore we give a combinatorial characterization of its multidegree, or equivalently, a description of the multigraded generic initial ideal of $I_a(\widetilde{L})$. Finally, we observe that $I_a(\widetilde{L})$ defines a Cohen–Macaulay normal domain. This follows from Brion's theorem, as we discuss in Section 2.

3. Multiview ideals. A collection of matrices $A = \{A_i\}_{i=1,\ldots,m}$ with scalar entries, with A_i of size $d_i \times n$ and rank $A_i = d_i$, induces a rational map

$$\phi_A \colon \mathbb{P}^{n-1} \dashrightarrow \prod \mathbb{P}^{d_i-1}$$

sending $x \in \mathbb{P}^{n-1}$ to $(A_i x)_{i=1,\dots,m}$. The ideal J_A of the closure of the image of ϕ_A is called multiview ideal. As explained in [2], the ideal J_A plays an important role in various aspects of geometrical computer vision. In [2] it is proved that J_A is a CS ideal when $n = 4$, $d_i = 3$ for all i, and assuming that the A_i's are generic. In [6] Binglin Li proved results that, suitably interpreted, imply that J_A is a CS ideal in all cases. We show that the same conclusion can be obtained as a simple corollary of our results on CS ideals in two ways: via elimination from the fact that the ideal of 2-minors of a multigraded matrix is CS and, again, via elimination from the multigraded closure of a linear space as discussed in (2).

Notice that the results in (2) and (3) also answer some of the questions posed by Ardila and Boocher [3, p. 234].

A result of Brion [9] asserts that a prime ideal with multiplicity-free multidegree defines a Cohen–Macaulay normal domain and is a CS ideal. As a corollary of Brion's theorem, in Section 2 we show that the minimal primes of a CS ideal are CS as well. As a further consequence we get that, if P is a prime CS ideal, then every ideal with the same multigraded Hilbert function of P is Cohen Macaulay. We propose two conjectures (Conjecture 2.13 and Conjecture 2.14) concerning extremal Betti numbers and the Hilbert function of local cohomology modules of CS ideals, that are inspired by Brion's theorem and confirmed by extensive computations. We prove that both conjectures hold for CS monomial ideals.

Our results have been suggested and confirmed by extensive computations performed using CoCoA [1] and Macaulay2 [19]. We thank Michel Brion, Marc Chardin, Kohji Yanagawa, and Matteo Varbaro for useful discussions and suggestions.

2. Multidegrees and CS ideals

Let K be a field and $S = K[x_{ij} : i = 1, \dots, n, 1 \le j \le d_i]$ with the standard $\mathbb{Z}^n$-graded structure induced by $\deg(x_{ij}) = e_i \in \mathbb{Z}^n$. We assume that $d_1, \dots, d_n > 0$ and set $x_i = x_{i,1}$. Let $T = K[x_1, x_2, \dots, x_n] \subset S$ with the induced standard $\mathbb{Z}^n$-graded structure. For $m \in \mathbb{N}$ we set $[m] = \{1, \dots, m\}$ and $[m]_0 = \{0, \dots, m\}$. A prime ideal P of S is called relevant if $P \not\supset S_{(1,\dots,1)}$ and irrelevant otherwise.

For a $\mathbb{Z}^n$-graded S-module M, denote by M_a the homogeneous component of M of degree $a \in \mathbb{Z}^n$.

Definition 2.1. Let M be a finitely generated, $\mathbb{Z}^n$-graded S-module and set $c = \dim S - \dim M$. The $\mathbb{Z}^n$-*graded Hilbert series* of M is

$$\mathrm{HS}(M, z) = \sum_{a \in \mathbb{Z}^n} \big(\dim_K M_a\big) z^a \in \mathbb{Q}[\![z_1, \dots, z_n]\!][z_1^{-1}, \dots, z_n^{-1}].$$

Set

$$K_M(z) = \mathrm{HS}(M, z) \prod_{i=1}^n (1 - z_i)^{d_i}.$$

It turns out that $K_M(z) \in \mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. Then set

$$C_M(z) = K_M(1 - z_1, \ldots, 1 - z_n).$$

The multidegree $\text{Deg}_M(z)$ of M, as defined in [27, Chapter 8], is the homogeneous component of smallest total degree, i.e. of degree c, of $C_M(z)$. It turns out that $\text{Deg}_M(z) \in \mathbb{N}[z_1, \ldots, z_n]$. Note that by [27, Claim 8.54] the multidegree $\text{Deg}_M(z)$ does not change if one replaces M by a shifted copy of it. Hence it is not restrictive to assume that $M_a = 0$ unless $a \in \mathbb{N}^n$ and, under this assumption, $K_M(z)$ and $C_M(z)$ are actually polynomials. In the geometric setting multidegrees are related to Chow classes, see [27, Notes, p. 172] for details and references.

For a Laurent polynomial $G(z) \in \mathbb{R}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ let us denote by $[G(z)]_{\min}$ the sum of the terms, including their coefficients, that are minimal with respect to division in the support of $G(z)$. For example, if $G(z) = z_1^2 + 2z_1z_2^2 + 3z_1^3$, then $[G(z)]_{\min} = z_1^2 + 2z_1z_2^2$. One easily checks that:

Lemma 2.2. *Let $G_1(z), \ldots, G_v(z)$ be Laurent polynomials such that $[G_i(z)]_{\min}$ has positive coefficients for every i. Then*

$$\left[\sum_{i=1}^{v} G_i(z) \right]_{\min} = \left[\sum_{i=1}^{v} [G_i(z)]_{\min} \right]_{\min}.$$

In [14] we defined the G-multidegree of M as follows:

$$\text{GDeg}_M(z) = [C_M(z)]_{\min}.$$

Clearly the homogeneous component of degree c of $\text{GDeg}_M(z)$ is $\text{Deg}_M(z)$. In Proposition 2.8 we will show that $\text{GDeg}_M(z) = \text{Deg}_M(z)$ if all the minimal primes of M have codimension c. On the other hand, if M has minimal primes of codimension greater than c, then $\text{GDeg}_M(z)$ might contain terms of degree higher than c.

By definition

$$\text{Deg}_M(z) = \sum e_M(a)z^a,$$

where the sum runs over all $a \in \prod_{i=1}^{n}[d_i]_0$ such that $|a| = c$. It turns out that $e_M(a) \in \mathbb{N}$. The module M has a multiplicity-free multidegree if $e_M(a) \in \{0, 1\}$ for all a. Furthermore M has a multiplicity-free G-multidegree if all the non-zero coefficients in $\text{GDeg}_M(z)$ are equal to 1. With a slight abuse of terminology we will say that a multigraded ideal I has multiplicity-free multidegree (or multiplicity-free G-multidegree) if the quotient ring S/I has that property.

Assuming that $M_a \neq 0$ for $a \gg 0$ (that is, when all the components of a are sufficiently large), there exists a non-zero polynomial $P_M(z) \in \mathbb{Q}[z_1, \ldots, z_n]$, the multigraded Hilbert polynomial of M, such that $P_M(a) = \dim_K M_a$ for $a \gg 0$. Denote by $D_M(z)$ the homogeneous component of largest degree of $P_M(z)$. If M

has irrelevant minimal primes then there is no clear relation between $D_M(z)$ and $\mathrm{Deg}_M(z)$. On the other hand if one assumes that M has at least one relevant minimal prime of minimal codimension, then the total degree of $D_M(z)$ is $\dim M - n$ and the coefficients $D_M(z)$ can be deduced from those of $\mathrm{Deg}_M(z)$. Let

$$D_M(z) = \sum \frac{f_M(a)}{a!} z^a,$$

where the sum runs over the $a \in \prod_{i=1}^{n} [d_i - 1]_0$ such that $|a| = \dim M - n$. The numbers $f_M(a)$ are actually non-negative integers and are called mixed multiplicities of M.

The polynomials $D_M(z)$ and $\mathrm{Deg}_M(z)$ are related as follows: for all the $a \in \prod_{i=1}^{n} [d_i - 1]_0$ such that $|a| = \dim M - n$ one has $f_M(a) = e_M(a')$, where $a' = (d_1 - 1 - a_1, \ldots, d_n - 1 - a_n)$. If $a \in \mathbb{N}^n$ is such that $|a| = c$ and $a_i = d_i$ for some i, then the corresponding coefficient $e_M(a)$ cannot be read off $D_M(z)$. However, if all the minimal primes of minimal codimension of M are relevant, then such coefficients are actually zero. Therefore one has:

Lemma 2.3. *Assume that all the minimal primes of minimal codimension of M are relevant. Then the polynomials $\mathrm{Deg}_M(z)$ and $D_M(z)$ are two different encodings of the same numerical data. In particular, this is the case if $M = S/P$ and P is a relevant prime.*

If K is algebraically closed and P is a relevant prime ideal then the coefficients $e_{S/P}(a)$ have a geometric interpretation. Let X denote the associated subvariety of $\mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_n-1}$. The coefficient $e_{S/P}(a)$ is the number of points of intersection of X with $L_1 \times \cdots \times L_n$ where L_i is a general linear space of $\mathbb{P}^{d_i-1}$ of dimension a_i.

Given a term order τ and a $\mathbb{Z}^n$-graded homogeneous ideal I of S, one can consider its $\mathbb{Z}^n$-*graded generic initial ideal* $\mathrm{gin}(I)$. As in the $\mathbb{Z}$-graded setting, $\mathbb{Z}^n$-graded generic initial ideals are Borel fixed. We refer to [15, Section 1] for more details on multigraded generic initial ideals. We just recall that to any $a \in \prod_{i=1}^{n} [d_i]_0$ one can associate the Borel fixed prime ideal

$$P_a = \left(x_{ij} : 1 \le i \le n \text{ and } 1 \le j \le a_i \right)$$

and that any Borel fixed prime ideal of S is of this form.

We have:

Lemma 2.4. *Let I be a $\mathbb{Z}^n$-graded ideal of S and let $\{z^{a_1}, \ldots, z^{a_s}\}$ be the support of $\mathrm{GDeg}_{S/I}(z)$. Then the minimal primes of the $\mathbb{Z}^n$-graded generic initial ideal of I are $\{P_{a_1}, \ldots, P_{a_s}\}$.*

Proof. Let $J = \mathrm{gin}(I)$. Since the G-multidegree only depends on the Hilbert series we have $\mathrm{GDeg}_{S/I}(z) = \mathrm{GDeg}_{S/J}(z)$. Then the result follows from [14, Prop. 3.12]. $\qquad\square$

Definition 2.5. Let I be a $\mathbb{Z}^n$-graded ideal of S. We say that I is a Cartwright–Sturmfels (CS) ideal if there exists a radical Borel fixed ideal J of S such that $\mathrm{HS}(I, y) = \mathrm{HS}(J, y)$.

With the notation of Definition 2.5 it turns out that $J = \mathrm{gin}(I)$, see [15, Proposition 1.6]. Notice also that by Lemma 2.4 one has the following characterization.

Proposition 2.6. *Let I be a $\mathbb{Z}^n$-graded ideal of S. One has that I is CS if and only if I has a multiplicity-free G-multidegree and $\mathrm{gin}(I)$ has no embedded primes.*

Definition 2.7. We say that I is a Cartwright–Sturmfels* (CS*) ideal if there exists a $\mathbb{Z}^n$-graded ideal J of S extended from T such that $\mathrm{HS}(I, z) = \mathrm{HS}(J, z)$.

With the notation of Definition 2.7 it turns out that $J = \mathrm{gin}(I)$ and that I and J have the same $\mathbb{Z}^n$-graded Betti numbers, see [15, Proposition 1.9, Corollary 1.10].

Notice that a $\mathbb{Z}^n$-graded homogeneous ideal of T is just a monomial ideal of T. Hence, a $\mathbb{Z}^n$-graded ideal of S which is extended from T is an ideal of S generated by monomials in $x_1, \ldots, x_n$.

We observe the following:

Proposition 2.8. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module and set*

$$F_M(z) = \sum \mathrm{length}(M_P) \mathrm{Deg}_{S/P}(z),$$

where the sum runs over the minimal primes P of M. Then:

(1) $\mathrm{GDeg}_M(z) = [F_M(z)]_{\min}$.

(2) *If all minimal primes of M have the same codimension, then*

$$F_M(z) = \mathrm{Deg}_M(z) = \mathrm{GDeg}_M(z).$$

Proof. (1) Consider a multigraded composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_v = M$$

such that for every i one has $M_i/M_{i-1} \simeq S/(P_i)(-u_i)$ where P_i is multigraded prime and $u_i \in \mathbb{Z}^n$ is a shift. Set $N_i = M_i/M_{i-1}$. Since K-polynomials are additive on short exact sequences, so are C-polynomials. Then

$$C_M(z) = \sum_{i=1}^{v} C_{N_i}(z).$$

Since $[C_{N_i}]_{\min} = \mathrm{Deg}_{S/P_i}(z)$ independently of the shift u_i, and since $\mathrm{Deg}_{S/P_i}(z)$ has positive coefficients, by Lemma 2.2 we have

$$\mathrm{GDeg}_M(z) = \left[C_M(z)\right]_{\min} = \left[\sum_{i=1}^{v} \mathrm{Deg}_{S/P_i}(z)\right]_{\min}.$$

Observe that if $P_1 \subsetneq P_2$ are multigraded prime ideals then

$$\left[\text{Deg}_{S/P_1}(z) + \text{Deg}_{S/P_2}(z)\right]_{\min} = \text{Deg}_{S/P_1}(z)$$

as can be seen by observing that $\text{gin}(P_1) \subset \text{gin}(P_2)$ and using that, by the main result of [23], all the minimal primes of $\text{gin}(P_1)$ have minimal codimension. Hence we may remove from $[\sum_{i=1}^{v} \text{Deg}_{S/P_i}(z)]_{\min}$ those summands corresponding to primes that are not minimal. Furthermore, by localization, we know that a given minimal prime of M occurs in the list $P_1, \ldots, P_v$ exactly $\text{length}(M_P)$ times. So we obtain the desired formula.

Assertion (2) follows from (1) and [27, Theorem 8.53]. $\qquad \square$

We may deduce the following important corollary:

Corollary 2.9. *Let I be a CS ideal, then*

$$\text{GDeg}_{S/I}(z) = \sum \text{Deg}_{S/P}(z),$$

where the sum runs over the minimal primes P of I.

Proof. Set $\mathbf{1} = (1, 1, \ldots, 1)$ and let $F_{S/I}(z)$ be as in Proposition 2.8. Since I is radical, we have that $F_{S/I}(\mathbf{1})$ is the geometric degree $\text{gdeg}(S/I)$ in the sense of [32]. By Proposition 2.8 we have $\text{GDeg}_{S/I}(\mathbf{1}) \leq F_{S/I}(\mathbf{1})$ and equality holds if and only if $\text{GDeg}_{S/I}(z) = F_{S/I}(z)$. Now let $J = \text{gin}(I)$. Observe that $\text{GDeg}_{S/I}(z) = \text{GDeg}_{S/J}(z)$ because GDeg just depends on the Hilbert series. Moreover $\text{GDeg}_{S/J}(\mathbf{1}) = \text{gdeg}(S/J)$ by [14, Prop. 3.12]. Both I and J are radical, hence their geometric degrees coincide with their arithmetic degrees. Therefore, combining [32, Prop. 4.1] and [32, Thm. 2.3] we have $\text{gdeg}(S/I) = \text{gdeg}(S/J)$. Summing up, we have:

$$\text{gdeg}(S/J) = \text{GDeg}_{S/J}(\mathbf{1}) = \text{GDeg}_{S/I}(\mathbf{1})$$
$$\leq F_{S/I}(\mathbf{1}) = \text{gdeg}(S/I) = \text{gdeg}(S/J).$$

Hence $\text{GDeg}_{S/I}(\mathbf{1}) = F_{S/I}(\mathbf{1})$ which implies $\text{GDeg}_{S/I}(z) = F_{S/I}(z)$. $\qquad \square$

Since every radical Borel fixed ideal has a multiplicity-free G-multidegree it follows that the same is true for every CS ideal. Furthermore every CS ideal is generated in degree $\leq \mathbf{1} \in \mathbb{Z}^n$. However, there are radical ideals generated in degree $\leq \mathbf{1}$ and with a multiplicity-free G-multidegree and that are not CS ideals, as the next example shows.

Example 2.10. Let $S = \mathbb{Q}[x_1, x_2, x_3, y_1, y_2, y_3]$ with the $\mathbb{Z}^2$-graded structure induced by $\deg(x_i) = e_1$ and $\deg(y_i) = e_2$. The ideal

$$I = \left(x_1 y_1, x_2 y_2, x_3 y_2, x_2 y_3, x_3 y_3\right),$$

generated in degree $\mathbf{1} = (1, 1)$, is the intersection of

$$(x_1, x_2, x_3), \quad (y_1, x_2, x_3), \quad (x_1, y_2, y_3), \quad (y_1, y_2, y_3).$$

Hence by Corollary 2.9 the multidegree of S/I is

$$\mathrm{GDeg}_{S/I}(z) = \mathrm{Deg}_{S/I}(z) = z_1^3 + z_1^2 z_2 + z_1 z_2^2 + z_2^3.$$

On the other hand, its multigraded generic initial ideal is

$$\mathrm{gin}(I) = \left(x_1 y_1, x_2 y_1, x_1 y_2, x_2 y_2, x_3 y_1, x_1 x_2 y_3, x_1^2 y_3 \right)$$

so that I is not CS.

The classes of CS and CS* ideals are in a sense dual to each other. In fact, in [15, Theorem 1.14] we showed that if I is a squarefree monomial ideal, then I is CS if and only if its Alexander dual I^* is CS*. Moreover, it follows from the definitions that the families of CS and CS* ideals are closed under $\mathbb{Z}^n$-graded coordinate changes and taking initial ideals. In [14, 15] we showed that if I is CS or CS*, then its $\mathbb{Z}^n$-graded generic initial ideal does not depend on the choice of the term order but only on the total order given to the indeterminates with the same degree. We also proved that each of the two classes is closed with respect to a number of natural operations, see [15, Proposition 1.7 and Theorem 1.16]. Other interesting properties, including bounds on the projective dimension and Castelnuovo-Mumford regularity were established in [15, Proposition 1.9, Proposition 1.12, and Corollary 1.15].

A beautiful theorem of Brion [9] asserts that an irreducible subvariety X of a flag variety is normal and Cohen–Macaulay if it has multiplicity-free multidegree. Moreover Brion showed that such an X admits a flat degeneration to a reduced union of Schubert varieties that is Cohen–Macaulay as well. See also the work of Perrin [29]. Using the terminology that we have introduced and limiting ourselves to subvarieties of a product of projective spaces, Brion's result can be stated as follows:

Theorem 2.11 (Brion). *Assume K is algebraically closed. Let P be a $\mathbb{Z}^n$-graded prime ideal in the polynomial ring S. Assume that S/P has a multiplicity-free multidegree. Then:*

(1) *S/P is normal and Cohen–Macaulay,*

(2) *P is a CS ideal,*

(3) *the multigraded gin of P defines a Cohen–Macaulay ring.*

As a consequence of Brion's theorem we have:

Corollary 2.12. *Assume that K is algebraically closed.*

(1) *Let P be a prime CS ideal. Then every $\mathbb{Z}^n$-graded ideal with the same $\mathbb{Z}^n$-graded Hilbert series of P defines a Cohen–Macaulay ring.*

(2) *Let I be CS ideal and let $P_1, \ldots, P_s$ be its minimal primes. Then each P_i is CS and*

$$\mathrm{in}(I) = \cap_{i=1}^{s} \mathrm{in}(P_i)$$

for every term ordering.

Proof. Assertion (1) follows immediately from Theorem 2.11 since P, being CS, has a multiplicity-free multidegree. To prove (2) one observes that by Corollary 2.9 a minimal prime of a CS ideal has a multiplicity-free multidegree, hence it is a CS ideal by Theorem 2.11. □

Brion's theorem suggests that there might be a very strong connection between homological invariants of a CS ideal and that of its generic initial ideal. Computational experiments suggest the following conjecture:

Conjecture 2.13. *Let I be a CS ideal and J its $\mathbb{Z}^n$-graded generic initial ideal. Then the extremal (total) Betti numbers of I and J are equal. In particular, I and J have the same projective dimension and Castelnuovo-Mumford regularity.*

Even stronger, we conjecture the following.

Conjecture 2.14. *Let I be a CS ideal and J its $\mathbb{Z}^n$-graded generic initial ideal. Then one has:*

$$\dim_K H_{\mathfrak{m}}^i(S/I)_a = \dim_K H_{\mathfrak{m}}^i(S/J)_a$$

for every $i \in \mathbb{N}$ and every $a \in \mathbb{Z}^n$.

Here $H_{\mathfrak{m}}^i(S/I)$ denotes the multigraded ith local cohomology module supported on the graded maximal ideal $\mathfrak{m}$ of S. As explained by Chardin in [12] extremal Betti numbers can be characterized in terms of vanishing of graded components of the local cohomology modules. Therefore Conjecture 2.14 implies Conjecture 2.13.

We have:

Proposition 2.15. *Conjecture 2.14 holds for monomial ideals.*

Proof. Let I be a CS monomial ideal and let J be its generic initial ideal. Since J is an initial ideal of I (after a change of coordinates) we have:

$$\dim_K H_{\mathfrak{m}}^i(S/I)_a \leq \dim_K H_{\mathfrak{m}}^i(S/J)_a$$

for all i and $a \in \mathbb{Z}^n$. The Alexander duals I^* and J^* of I and J are CS* ideals with the same $\mathbb{Z}^n$-graded Hilbert function. Hence they have the same $\mathbb{Z}^n$-graded Betti numbers [15, Proposition 1.9]. In particular, I^* and J^* have the same $\mathbb{Z}$-graded Betti numbers. Then we deduce from Lemma 2.16 below that

$$\dim_K H_{\mathfrak{m}}^i(S/I)_j = \dim_K H_{\mathfrak{m}}^i(S/J)_j$$

for all $j \in \mathbb{Z}$ and all $i \geq 0$ that, in combination with the inequality above, implies the desired equality. □

Lemma 2.16. *Let I be a squarefree monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_N]$. Denote by I^* its Alexander dual. Then for every $i \geq 0$ and every $j > 0$ one has*

$$\dim_K H_m^i(R/I)_{-j} = \sum_{v=1}^{\min(i,j)} \binom{j-1}{v-1} \beta_{i-v, N-v}(I^*),$$

while $H_m^i(R/I)_j = 0$ for $j > 0$ and $\dim_K H_m^i(R/I)_0 = \beta_{i,N}(I^*)$.

Proof. Combining Hochster's formulas for Betti numbers (in the dual form) [27, Corollary 1.40] and for local cohomology [27, Theorem 13.13] one has that

$$\dim_K H_{\mathfrak{m}}^i(R/I)_{-a} = \beta_{i-|a|, 1-a}(I^*)$$

for every $a \in \{0,1\}^N$. Computing the dimension of the $\mathbb{Z}$-graded component of $H_{\mathfrak{m}}^i(R/I)$ as sum of the corresponding multigraded components and taking into account Hochster's formula for local cohomology, one obtains the desired result. $\square$

Let us conclude the section by stating a very general conjecture which is due to Jürgen Herzog. Morally speaking Herzog's conjecture asserts that a radical initial ideal behaves (homologically) as the generic initial ideal with respect to the revlex order does. The conjecture, in various forms, has been discussed in several occasions by Herzog and his collaborators but, as far as we know, never appeared in print. A special case of it appears in Varbaro's PhD thesis as [35, Question 2.1.16]. Indeed, our Conjecture 2.13 is a special case of Herzog's conjecture.

Conjecture 2.17 (Herzog). *Let I be a homogeneous ideal in a polynomial ring and J an initial ideal of I with respect to a term order. Assume J is radical. Then I and J have the same extremal Betti numbers. In particular, I and J have the same projective dimension and regularity.*

Herzog's conjecture is known to be true for toric ideals (in toric coordinates) because in that case J defines a Cohen–Macaulay ring. Furthermore it is known in few other cases as, for example, homogeneous Cohen–Macaulay ASL with discrete Buchsbaum counterpart [26, Thm. 4.4].

3. Binomial edge ideals

In this section we prove that every binomial edge ideal is a Cartwright–Sturmfels ideal.

Let K be a field, let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and let X be the $2 \times n$ matrix of variables

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

Denote by Δ_{ij} the 2-minor of X corresponding to the column indices i, j, i.e.

$$\Delta_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = x_i y_j - x_j y_i.$$

Let G be a graph on the vertex set $[n] = \{1, \ldots, n\}$ and let

$$J_G = \big(\Delta_{ij} : \{i, j\} \text{ is an edge of } G \big).$$

The ideal J_G is called the binomial edge ideal of G. Binomial edge ideals are just ideals generated by subsets of the 2-minors of X. Herzog, Hibi, Hreinsdottir, Kahle, and Rauh in [20] and, independently, Ohtani in [28] proved that J_G is radical. We will show that J_G is a Cartwright–Sturmfels ideal with respect to the natural $\mathbb{Z}^n$-graded structure induced by $\deg(x_i) = \deg(y_i) = e_i \in \mathbb{Z}^n$.

Theorem 3.1. *The $\mathbb{Z}^n$-graded generic initial ideal of J_G is generated by the monomials $y_{a_1} \ldots y_{a_v} x_i x_j$, where $i, a_1, \ldots, a_v, j$ is a path in G. In particular, J_G is a Cartwright–Sturmfels ideal and therefore all the initial ideals of J_G are radical and $\operatorname{reg}(J_G) \leq n$.*

Here by a path of G we mean a sequence of vertices without repetitions such that every pair of adjacent vertices form an edge of the graph. Note that in the description of the generators of the generic initial ideal one can assume that $i < j$ and that the path is minimal in the sense that the only edges among the vertices $i, a_1, \ldots, a_v, j$ are $(i, a_1), (a_1, a_2), \ldots, (a_v, j)$. That $\operatorname{reg}(J_G) \leq n$ has been proved originally by Matsuda and Murai in [25], where they also conjectured that equality holds if and only if G is a path of length $n - 1$. In [4] a universal Gröbner basis for J_G is described and this implies that all the initial ideals of J_G, in the given coordinates, are radical.

Proof. Consider any term order such that $x_i > y_i$ for all i. To compute the generic initial ideal we first apply a multigraded upper triangular transformation ϕ to J_G, i.e. for every i we have $\phi(x_i) = x_i$ and $\phi(y_i) = \alpha_i x_i + y_i$ with $\alpha_i \in K$. We obtain a matrix

$$\phi(X) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \alpha_1 x_1 + y_1 & \alpha_2 x_2 + y_2 & \cdots & \alpha_n x_n + y_n \end{pmatrix}$$

whose 2-minors are:

$$\phi(\Delta_{ij}) = \begin{vmatrix} x_i & x_j \\ \alpha_i x_i + y_i & \alpha_j x_j + y_j \end{vmatrix} = (\alpha_j - \alpha_i) x_i x_j + \Delta_{ij}.$$

Assume that $\alpha_j \neq \alpha_i$ for $i \neq j$. We multiply $\phi(\Delta_{ij})$ by the inverse of $(\alpha_j - \alpha_i)$ and obtain:

$$F_{ij} = x_i x_j - \lambda_{ij} \Delta_{ij}$$

with

$$\lambda_{ij} = (\alpha_i - \alpha_j)^{-1}.$$

Hence

$$\phi(J_G) = \left(F_{ij} : \{i, j\} \text{ is an edge of } G\right).$$

Set

$$F = \left\{y_a F_{ij} : i, a_1, \ldots, a_v, j \text{ is a path in } G\right\},$$

where

$$y_a = y_{a_1} \cdots y_{a_v}.$$

It is enough to prove that F is a Gröbner basis for $\phi(J_G)$ for every ϕ such that $\alpha_j \neq \alpha_i$ for $i \neq j$. We first observe that $F \subset \phi(J_G)$, i.e. $y_a F_{ij} \in \phi(J_G)$ for every path $i, a_1, \ldots, a_v, j$ in G. This is proved easily by induction on v, the case $v = 0$ being trivial, applying to the matrix $\phi(X)$ the following relation

$$(z_{1i}, z_{2i})\Delta_{jk}(Z) \subseteq \left(\Delta_{ij}(Z), \Delta_{ik}(Z)\right)$$

that holds for every $2 \times n$ matrix $Z = (z_{ij})$ and every triplet of column indices i, j, k. To prove that F is a Gröbner basis we take two elements $y_a F_{ij}$ and $y_b F_{hk}$ in F and prove that the corresponding S-polynomial reduces to 0 via F. Here $a = a_1, \ldots, a_v$ and $b = b_1, \ldots, b_r$ and i, a, j and h, b, k are paths in G. We distinguish three cases:

Case 1. If $\{i, j\} = \{h, k\}$, we may assume $i = h$ and $j = k$. The corresponding S-polynomial is 0.

Case 2. If $\{i, j\} \cap \{h, k\} = \emptyset$. Let $u = \text{GCD}(y_a, y_b)$. Then $y_a F_{ij} = u(y_a/u)F_{ij}$ and $y_b F_{hk} = u(y_b/u)F_{hk}$. Note that $(y_a/u)F_{ij}$ and $(y_b/u)F_{hk}$ have coprime leading terms and hence they form a Gröbner basis. If a Gröbner basis is multiplied with a polynomial the resulting set of polynomials is still a Gröbner basis. Hence, $\{y_a F_{ij}, y_b F_{hk}\}$ is a Gröbner basis and the S-polynomial of $y_a F_{ij}, y_b F_{hk}$ reduces to 0 using only $y_a F_{ij}, y_b F_{hk}$.

Case 3. If $\#\{i, j\} \cap \{h, k\} = 1$. Renaming the column indices we may assume that $i = 1$, $h = 2$ and $j = k = n$. Hence we deal with $y_a F_{1n}$ and $y_b F_{2n}$. Let $u = \text{LCM}(y_a, y_b)$. We have:

$$S\left(y_a F_{1n}, y_b F_{2n}\right) = u\left(x_2 F_{1n} - x_1 F_{2n}\right) = u x_n F_{12}.$$

If u is divisible by a monomial $y_d = y_{d_1} \cdots y_{d_t}$ such that $1, d_1, \ldots, d_t, 2$ is a path in G then $S(y_a F_{1n}, y_b F_{2n})$ is multiple of the element $y_d F_{12}$ of F. On the other hand, if u is not divisible by a monomial $y_d = y_{d_1} \cdots y_{d_t}$ such that $1, d_1, \ldots, d_t, 2$ is a path in G then

$$\{1, a_1, \ldots, a_v\} \cap \{2, b_1, \ldots, b_r\} = \emptyset \quad \text{and} \quad u = y_a y_b.$$

In this case we proceed to reduce $S(y_a F_{1n}, y_b F_{2n}) = y_a y_b x_n F_{12}$. In doing this we observe that the reduction via an element $y_c F_{\alpha\beta}$ of F allows us to replace $x_\alpha x_\beta$

with $\lambda_{\alpha\beta}\Delta_{\alpha\beta}$ provided that the monomial $x_\alpha x_\beta$ is multiplied with a monomial multiple of y_c. We denote this operation by $\xrightarrow{\alpha\beta}$. We have:

$$S(y_a F_{1n}, y_b F_{2n}) = y_a y_b x_n F_{12} = y_a y_b(x_1 x_2 x_n - \lambda_{12} x_n \Delta_{12})$$
$$= y_a y_b(x_1 x_2 x_n - \lambda_{12} x_n x_1 y_2 + \lambda_{12} x_n x_2 y_1)$$
$$\xrightarrow{1n} y_a y_b(x_2 \lambda_{1n}\Delta_{1n} - \lambda_{12}\lambda_{1n}\Delta_{1n}y_2 + \lambda_{12}x_n x_2 y_1)$$
$$= y_a y_b(\lambda_{1n}x_1 x_2 y_n - \lambda_{1n}x_2 x_n y_1 - \lambda_{12}\lambda_{1n}\Delta_{1n}y_2 + \lambda_{12})x_n x_2 y_1$$
$$\xrightarrow{2n} y_a y_b(\lambda_{1n}x_1 x_2 y_n - \lambda_{1n}\lambda_{2n}\Delta_{2n}y_1 - \lambda_{12}\lambda_{1n}\Delta_{1n}y_2 + \lambda_{12}\lambda_{2n}\Delta_{2n}y_1)$$
$$\xrightarrow{12} y_a y_b(\lambda_{1n}\lambda_{12}\Delta_{12}y_n - \lambda_{1n}\lambda_{2n}\Delta_{2n}y_1 - \lambda_{12}\lambda_{1n}\Delta_{1n}y_2 + \lambda_{12}\lambda_{2n}\Delta_{2n}y_1)$$
$$= y_a y_b(\lambda_{1n}\lambda_{12}(\Delta_{12}y_n - \Delta_{1n}y_2) - \lambda_{1n}\lambda_{2n}\Delta_{2n}y_1 + \lambda_{12}\lambda_{2n}\Delta_{2n}y_1)$$
$$= y_a y_b(\lambda_{1n}\lambda_{12}(-\Delta_{2n}y_1) - \lambda_{1n}\lambda_{2n}\Delta_{2n}y_1 + \lambda_{12}\lambda_{2n}\Delta_{2n}y_1)$$
$$= y_a y_b y_1 \Delta_{2n}(-\lambda_{1n}\lambda_{12} - \lambda_{1n}\lambda_{2n} + \lambda_{12}\lambda_{2n}) = 0.$$

The last equality holds since $-\lambda_{1n}\lambda_{12} - \lambda_{1n}\lambda_{2n} + \lambda_{12}\lambda_{2n} = 0$, which can be checked by direct computation. This concludes the proof that the set F is a Gröbner basis. The remaining statements follow from general facts on Cartwright–Sturmfels ideals established in [15, Remark 1.5, Corollary 1.15]. $\qquad\square$

We describe now the minimal primes of the generic initial ideal of J_G. We denote by $c(G)$ the number of connected components of a graph G. For a subset T of $[n]$ let G_T be the restriction of G to T and set

$$U_T = (x_i x_j : i, j \in T \text{ and are connected by a path in } G_T) + \sum_{i \notin T}(x_i, y_i).$$

It is clear that $\mathrm{gin}(J_G) \subseteq U_T$ for every T. Furthermore let E be a subset of T such that E contains exactly one vertex for each connected component of G_T and set

$$U_{T,E} = (x_i : i \in T \setminus E) + \sum_{i \notin T}(x_i, y_i).$$

Then:

Proposition 3.2. *The minimal primes of* $\mathrm{gin}(J_G)$ *are exactly the ideals* $U_{T,E}$, *where* T *is chosen so that for every* $i \in [n] \setminus T$ *one has* $c(G_{T\cup\{i\}}) < c(G_T)$.

Proof. Let P be a minimal prime of $\mathrm{gin}(J_G)$ and let $T = \{i \in [n] : y_i \notin P\}$. Since P is Borel fixed then it follows from Theorem 3.1 that $U_T \subseteq P$ and so it follows that $\mathrm{gin}(J_G) = \bigcap_T U_T$. Now, $U_T \subseteq U_{T_1}$ if and only if $T_1 \subseteq T$ and $c(G_{T_1}) = c(G_T)$. So it follows that $\mathrm{gin}(J_G) = \bigcap_T U_T$ where the intersection is restricted to the T such that for every $i \in [n] \setminus T$ one has $c(G_{T\cup\{i\}}) < c(G_T)$. Finally, observe that $U_T = \bigcap U_{T,E}$ where the intersection ranges over $E \subset T$ such that E contains exactly one vertex for each connected component of G_T. $\qquad\square$

In [33] Schenzel and Zafar computed the structure of the local cohomology modules (indeed of the corresponding Ext-modules) of the binomial edge ideal associated to a complete bipartite graph. These results might shed some light on Conjecture 2.14 and might suggest more precise versions of it.

4. Closure of linear spaces in products of projective spaces

Let $T = K[x_1, \ldots, x_n]$ be a polynomial ring with a standard $\mathbb{Z}^u$-graded structure, i.e. $\deg(x_i) \in \{e_1, \ldots, e_u\}$ for every i. Let $S = T[y_1, \ldots, y_u]$ with the $\mathbb{Z}^u$-graded structure obtained by extending that of T by letting $\deg y_i = e_i \in \mathbb{Z}^u$. Given $f = \sum_{i=1}^v \lambda_i x^{a_i} \in T \setminus \{0\}$ we consider its $\mathbb{Z}^u$-homogenization $f^{\mathrm{hom}} \in S = T[y_1, \ldots, y_u]$ defined as

$$f^{\mathrm{hom}} = \sum_{i=1}^v \lambda_i x^{a_i} y^{b-b_i},$$

where $\deg x^{a_i} = b_i \in \mathbb{Z}^u$ and $y^b = \mathrm{LCM}(y^{b_1}, \ldots, y^{b_v})$. For any $c \in \mathbb{Z}^u$ such that $\mathrm{LCM}(y^{b_1}, \ldots, y^{b_v}) | y^c$, we define

$$f^{\mathrm{hom},c} = \sum \lambda_i x^{a_i} y^{c-b_i}.$$

By construction, f^{hom} is $\mathbb{Z}^u$-homogeneous of degree b and $f^{\mathrm{hom},c} = f^{\mathrm{hom}} y^{c-b}$. Given an ideal I of T its $\mathbb{Z}^u$-homogenization is defined as

$$I^{\mathrm{hom}} = \left(f^{\mathrm{hom}} : f \in I \setminus \{0\} \right) \subset S$$

and it is clearly a $\mathbb{Z}^u$-graded ideal of S. For generalities about homogenization of ideals we refer the reader to [24]. Here we just recall that if $I = (f_1, \ldots, f_t)$ then

$$I^{\mathrm{hom}} = \left(f_1^{\mathrm{hom}}, \ldots, f_t^{\mathrm{hom}} \right) : \left(\prod_{i=1}^u y_i \right)^\infty,$$

see [24, Corollary 4.3.8]. Let $c \in \mathbb{Z}^u$ such that for every $i = 1, \ldots, t$ and for every monomial x^a in the support of f_i we have $y^v | y^c$, where $v = \deg x^a \in \mathbb{Z}^u$. Then we have

$$I^{\mathrm{hom}} = \left(f_1^{\mathrm{hom},c}, \ldots, f_t^{\mathrm{hom},c} \right) : \left(\prod_{i=1}^u y_i \right)^\infty,$$

because $f_i^{\mathrm{hom},c}$ and f_i^{hom} differ only by a monomial in the y's.

We denote by $I^\star$ the largest $\mathbb{Z}^u$-graded ideal of T contained in I, i.e. the ideal generated by the $\mathbb{Z}^u$-graded elements in I. We show that:

Theorem 4.1. *Let $T = K[x_1, \ldots, x_n]$ be a polynomial ring with a standard $\mathbb{Z}^u$-graded structure. Let V be a vector space of linear forms of T (i.e. elements of degree 1 with respect to the standard $\mathbb{Z}$-graded structure) and $J(V)$ be the ideal generated by V. Then*

(1) *$J(V)^{\mathrm{hom}}$ and $J(V)^\star$ are Cartwright–Sturmfels ideals.*

(2) *Both $J(V)^{\mathrm{hom}}$ and $J(V)^\star$ define Cohen–Macaulay normal rings.*

If K is algebraically closed, T is equipped with the natural $\mathbb{Z}^n$-graded structure, and L is the zero locus V in $\mathbb{A}_K^n$, then ideal $J(V)^{\mathrm{hom}}$ is exactly the defining ideal of the closure $\widetilde{L}$ of L in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, i.e. $J(V)^{\mathrm{hom}} = I(\widetilde{L})$ in the notation of Ardila and Boocher. If instead we equip T with a $\mathbb{Z}^u$-graded structure where a_i variables have degree e_i, then the ideal $J(V)^{\mathrm{hom}}$ is the ideal associated to the closure of L in the product $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_u}$, i.e. $J(V)^{\mathrm{hom}}$ is the ideal denoted by $I_a(\widetilde{L})$ in the introduction.

Proof. (1) The assertion on $J(V)^\star$ follows from the one on $J(V)^{\mathrm{hom}}$ and [15, Theorem 1.16] since, by [24, Tutorial 50], one has

$$J(V)^\star = J(V)^{\mathrm{hom}} \cap T.$$

To prove the assertion for $J(V)^{\mathrm{hom}}$ we argue as follows. For a matrix X and an integer t we denote by $I_t(X)$ the ideal generated by the t-minors of X. Let ℓ be a linear form of T, say $\ell = \sum_{i=1}^u \ell_i$ where ℓ_i is $\mathbb{Z}^u$-homogeneous of degree $e_i \in \mathbb{Z}^u$. Set $\mathbf{1} = \sum_1^u e_i$ and notice that $\ell^{\mathrm{hom},\mathbf{1}} = \prod_{j=1}^u y_j \sum_{i=1}^u \ell_i / y_i$ can be written as

$$H^{\mathrm{hom},\mathbf{1}} = \det \begin{pmatrix} y_1 & 0 & \cdots & \cdots & 0 & \ell_1 \\ -y_2 & y_2 & 0 & \cdots & 0 & \ell_2 \\ 0 & -y_3 & y_3 & \cdots & 0 & \ell_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & y_{u-1} & \vdots \\ 0 & 0 & \cdots & \cdots & -y_u & \ell_u \end{pmatrix}.$$

Now, if $V = \langle L_1, \ldots, L_v \rangle$, let $X_{[u]}$ be the $u \times (v + u - 1)$ matrix with block decomposition

$$X_{[u]} = \left(Y_{[u]} \mid M_{[u]} \right),$$

where

$$
Y_{[u]} = \begin{pmatrix}
y_1 & 0 & \cdots & \cdots & 0 \\
-y_2 & y_2 & 0 & \cdots & 0 \\
0 & -y_3 & y_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & y_{u-1} \\
0 & 0 & \cdots & \cdots & -y_u
\end{pmatrix}
$$

and

$$
M_{[u]} = \begin{pmatrix}
L_{11} & \cdots & L_{1v} \\
L_{21} & \cdots & L_{2v} \\
L_{31} & \cdots & L_{3v} \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
L_{u1} & \cdots & L_{uv}
\end{pmatrix}
$$

is the $u \times v$ matrix whose ith column is given by the $\mathbb{Z}^u$-homogeneous components of L_i, that is $L_i = \sum_{q=1}^{u} L_{qi}$. Let $H_{[u]}$ be the ideal generated by the u-minors $\Delta_1, \ldots, \Delta_v$ of $X_{[u]}$, where Δ_i involves the $(u-1)$ columns of $Y_{[u]}$ and the ith column of $M_{[u]}$. By construction $H_{[u]} = (L_1^{\mathrm{hom},\mathbf{1}}, \ldots, L_v^{\mathrm{hom},\mathbf{1}})$, hence

$$
J(V)^{\mathrm{hom}} = H_{[u]} : \left(\prod_{i=1}^{u} y_i \right)^{\infty}.
$$

It follows immediately from the straightening law for minors (see [10, Section 4]) that $I_{u-1}(Y_{[u]}) I_u(X_{[u]}) \subseteq H_{[u]}$ and obviously $H_{[u]} \subseteq I_u(X_{[u]})$. In this case $I_{u-1}(Y_{[u]})$ is generated by the squarefree monomials of degree $u-1$ in the variables $y_1, \ldots, y_u$. Hence,

$$
I_u(X_{[u]}) : \left(\prod_{i=1}^{u} y_i \right)^{\infty} \supseteq H_{[u]} : \left(\prod_{i=1}^{u} y_i \right)^{\infty}
$$

$$
\supseteq \left(I_{u-1}(Y_{[u]}) I_u(X_{[u]}) \right) : \left(\prod_{i=1}^{u} y_i \right)^{\infty} = I_u(X_{[u]}) : \left(\prod_{i=1}^{u} y_i \right)^{\infty}.
$$

Summing up, we have shown that

$$
J(V)^{\mathrm{hom}} = H_{[u]} : \left(\prod_{i=1}^{u} y_i \right)^{\infty} = I_u(X_{[u]}) : \left(\prod_{i=1}^{u} y_i \right)^{\infty}.
$$

The matrix $X_{[u]}$ is row-graded, i.e. the entries in its ith row are homogeneous of degree $e_i \in \mathbb{Z}^u$. Hence by [15, Corollary 1.19] its ideal of maximal minors is a

Cartwright–Sturmfels ideal. In particular $I_u(X_{[u]})$ is radical, hence

$$J(V)^{\text{hom}} = I_u(X_{[u]}) : \left(\prod_{i=1}^{u} y_i \right).$$

By [15, Theorem 1.16] it follows that $J(V)^{\text{hom}}$ is a Cartwright–Sturmfels ideal as well. This concludes the proof of (1).

(2) Since $J(V)$ is a prime ideal, then $J(V)^{\text{hom}}$ is prime (see e.g. [24, Prop. 4.3.10]). Then $J(V)^\star = J(V)^{\text{hom}} \cap T$ is prime as well. One can easily check that the ideals $J(V)^{\text{hom}}$ and $J(V)^\star$ are geometrically primes, i.e. they remain prime under field extensions. Hence we may assume without loss of generality that K is algebraically closed. So we may apply Brion's Theorem 2.11 and conclude that both $J(V)^{\text{hom}}$ and $J(V)^\star$ define Cohen–Macaulay normal rings. $\square$

In order to identify generators of $J(V)^{\text{hom}}$, we proceed as follows. For every non-empty subset A of $\{1, \ldots, u\}$ let

$$V_A = V \cap \oplus_{i \in A} T_{e_i}$$

and consider the ideal $J(V_A)$ generated by V_A. We associate to $J(V_A)$ the ideal H_A generated the homogenization $L^{\text{hom},c}$ of the generators of $J(V_A)$ with respect to the vector $c = \sum_{i \in A} e_i$ and the corresponding matrices X_A, Y_A, M_A constructed as in the proof of Theorem 4.1. In the proof of Theorem 4.1 we showed that

$$J(V_A)^{\text{hom}} = I_{|A|}(X_A) : \left(\prod_{i \in A} y_i \right) = I_{|A|}(X_A) : \left(\prod_{i=1}^{u} y_i \right)$$

and, since $J(V_A) \subseteq J(V)$, we obtain $I_{|A|}(X_A) \subseteq J(V_A)^{\text{hom}} \subseteq J(V)^{\text{hom}}$. Therefore we have

$$\sum_{A \neq \emptyset} I_{|A|}(X_A) \subseteq J(V)^{\text{hom}}.$$

We claim that equality holds. In order to prove our claim, we will need the following:

Lemma 4.2. *Let J be a $\mathbb{Z}^u$-graded Cartwright–Sturmfels ideal, let F be a product of $\mathbb{Z}^u$-graded linear forms. Let J_1 be the ideal generated by the elements of $J : (F)$ of degree smaller than $(1, \ldots, 1)$. Then $J : (F) = J + J_1$.*

Proof. By induction on the degree of F and by [15, Theorem 1.16], we may assume that F is a $\mathbb{Z}^u$-graded linear form, say of degree e_u. After a change of coordinates we may also assume that F is a variable, call it x. We introduce a revlex term order $<$ such that x is the smallest variable with respect to $<$. Let $G_1, \ldots, G_v$ be a Gröbner basis of J with respect to $<$. Since J is a Cartwright–Sturmfels ideal the G_i's have degrees smaller than or equal to $(1, \ldots, 1)$. Some of them, say $G_1, \ldots, G_w$,

have a leading term divisible by x and the remaining $G_{w+1}, \dots, G_v$ do not. Hence $G_j = x H_j$ for $j = 1, \dots, w$. It follows that $J : x = (H_1, \dots, H_w, G_{w+1}, \dots, G_v)$. Hence $J_1 = (H_1, \dots, H_w) + (G_j : \deg G_j < (1, \dots, 1))$ and $J : x = J + J_1$. $\qquad \square$

Lemma 4.3. *With the notation introduced above, let $F \in T$ be a $\mathbb{Z}^u$-graded polynomial of degree $a \in \mathbb{Z}^u$ with $a_u = 0$. Assume that $F y_u \in H_{[u]}$, then $F \in H_A$ with $A = [u] \setminus u$.*

Proof. Let $\pi_u : \oplus_{i=1}^u T_{e_i} \to T_{e_u}$ be the projection on the uth homogeneous component. We may choose a basis of V of the form $L_1, \dots, L_h, U_1, \dots, U_k$ so that $\pi_u(L_1), \dots, \pi_u(L_h)$ form a K-basis of $\pi_u(V)$ and $\pi_u(U_i) = 0$ for every i. By construction, $H_{[u]}$ is generated by the homogenization with respect to the vector $\mathbf{1}$ of $L_1, \dots, L_h, U_1, \dots, U_k$. Notice that for every $i = 1, \dots h$ one has that $L_i^{\mathrm{hom}, \mathbf{1}} = W_i(y_1 \cdots y_{u-1}) + y_u W_i'$, where $W_i = \pi_u(L_i)$ and W_i' is homogeneous of degree $\mathbf{1} - e_u$. Moreover, the homogenization with respect to the vector $\mathbf{1}$ of $U_1, \dots, U_k$ generates $y_u H_A$, with $A = [u] - \{u\}$. Since $F y_u$ is in $H_{[u]}$, then

$$F y_u = \sum_{i=1}^h E_i \left(W_i y_i \cdots y_{u-1} + y_u W_i' \right) + y_u C,$$

where $C \in H_A$ and the E_i's are $\mathbb{Z}^u$-homogeneous with the homogeneous component of degree e_u equal to 0. Since $W_1, \dots, W_h, y_u$ are linearly independent elements of degree e_u, it follows that $E_i = 0$ for every i and $F = C \in J_A$. $\qquad \square$

We can now give an explicit description of $J(V)^{\mathrm{hom}}$ and $J(V)^\star$ as sums of ideals of minors.

Theorem 4.4. *With the notations above one has:*

$$J(V)^{\mathrm{hom}} = \sum_{A \neq \emptyset} I_{|A|}(X_A)$$

and

$$J(V)^\star = \sum_{A \neq \emptyset} I_{|A|}(M_A).$$

Proof. In order to prove the first statement, by the proof of Theorem 4.1 it suffices to show that

$$I_u\left(X_{[u]}\right) : \left(\prod_{i=1}^u y_i \right) = \sum_{A \neq \emptyset} I_{|A|}(X_A).$$

By induction, it suffices to prove that

$$I_u\left(X_{[u]}\right) : \left(\prod_{i=1}^u y_i \right) = I_u\left(X_{[u]}\right) + W,$$

where

$$W = \sum_{j=1}^{u} \left(I_{u-1}\big(X_{[u]\setminus\{j\}}\big) : \left(\prod_{i=1}^{u} y_i \right) \right).$$

By Lemma 4.2 it suffices to show that any $\mathbb{Z}^u$-graded element $G \in I_u(X_{[u]}) :$ $(\prod_{i=1}^{u} y_i)$ of degree smaller than $(1,\ldots,1)$ is in W. We may assume that G has degree 0 in the uth coordinate. In the proof of Theorem 4.1 we observed that every squarefree monomial of degree $u-1$ in the y_i's multiplies $I_u(X_{[u]})$ in the ideal $H_{[u]} = (L_1^{\mathrm{hom},\mathbf{1}}, \ldots, L_v^{\mathrm{hom},\mathbf{1}})$. Since by assumption $G y_1 \cdots y_u \in I_u(X_{[u]})$, then

$$G\big(y_1 \cdots y_{u-1}\big)^2 y_u \in H_{[u]}.$$

Notice that the polynomial $F = G(y_1 \cdots y_{u-1})^2$ is $\mathbb{Z}^u$-graded and has degree 0 in the last coordinate and $F y_u \in H_{[u]}$. It follows from Lemma 4.3 that $F \in H_A$ with $A = [u] \setminus \{u\}$. Hence,

$$G \in H_A : \left(\prod_{i=1}^{u} y_i \right)^{\infty} = I_{|A|}(X_A) : \left(\prod_{i=1}^{u} y_i \right) \subseteq W.$$

The second statement may be deduced from the first as follows. One observes that $J(V)^{\mathrm{hom}}$ is homogeneous with respect to the $\mathbb{Z}$-graded structure induced by assigning degree 1 to the y_i's and degree 0 to the elements of T. Since $J(V)^{\star} = J(V)^{\mathrm{hom}} \cap T$, then $J(V)^{\star}$ is obtained from $J(V)^{\mathrm{hom}}$ by setting to 0 the y_i's.   □

In the next example we illustrate Theorems 4.1 and 4.4 and their proofs. We consider the linear space discussed in [3, Example 1.7] and we homogenize with respect to a different multigrading.

Example 4.5. Let $T = K[x_1, \ldots, x_6]$ with the $\mathbb{Z}^3$-graded structure induced by

$$\deg(x_i) = \begin{cases} e_1, & \text{for } i = 1, 2, \\ e_2, & \text{for } i = 3, \\ e_3, & \text{for } i = 4, 5, 6. \end{cases}$$

Let $V = \langle L_1, L_2, L_3 \rangle$ with $L_1 = x_1 + x_2 + x_6$, $L_2 = x_2 - x_3 + x_5$, $L_3 = x_3 + x_4$.
So one has $u = v = 3$ and

$$X_{\{1,2,3\}} = \begin{pmatrix} y_1 & 0 & x_1 + x_2 & x_2 & 0 \\ -y_2 & y_2 & 0 & -x_3 & x_3 \\ 0 & -y_3 & x_6 & x_5 & x_4 \end{pmatrix}.$$

In Theorem 4.1 we proved that $J(V)^{\mathrm{hom}}$ is a CS ideal and

$$J(V)^{\mathrm{hom}} = I_3\big(X_{\{1,2,3\}}\big) : (y_1 y_2 y_3).$$

In order to obtain the generators of $J(V)^{\text{hom}}$, we use Theorem 4.4. The relevant subsets $A \subseteq \{1, 2, 3\}$ are those for which $V_A \neq \{0\}$, hence in this case they correspond to

$$V_{\{1,3\}} = \langle L_1, L_2 + L_3 \rangle, \quad V_{\{2,3\}} = \langle L_3 \rangle$$

and of course $V_{\{1,2,3\}} = V$. The corresponding matrices are

$$X_{\{1,3\}} = \begin{pmatrix} y_1 & x_1 + x_2 & x_2 \\ -y_3 & x_6 & x_4 + x_5 \end{pmatrix}, \quad X_{\{2,3\}} = \begin{pmatrix} y_2 & x_3 \\ -y_3 & x_4 \end{pmatrix},$$

thus by Theorem 4.4 we have

$$J(V)^{\text{hom}} = I_2\big(X_{\{1,3\}}\big) + I_2\big(X_{\{2,3\}}\big) + I_3\big(X_{\{1,2,3\}}\big).$$

It turns out that the generators of $I_3(X_{\{1,2,3\}})$ are superfluous, so that

$$J(V)^{\text{hom}} = \big(x_4 y_2 + x_3 y_3, \; x_6 y_1 + x_1 y_3 + x_2 y_3,$$
$$x_4 y_1 + x_5 y_1 + x_2 y_3, \; x_1 x_4 + x_2 x_4 + x_1 x_5 + x_2 x_5 - x_2 x_6\big).$$

Finally by Theorem 4.4 we have $J(V)^\star = I_2(M_{\{1,3\}}) + I_2(M_{\{2,3\}}) + I_3(M_{\{1,2,3\}})$, with

$$M_{\{1,3\}} = \begin{pmatrix} x_1 + x_2 & x_2 \\ x_6 & x_4 + x_5 \end{pmatrix}, \quad M_{\{2,3\}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix},$$

$$M_{\{1,2,3\}} = \begin{pmatrix} x_1 + x_2 & x_2 & 0 \\ 0 & -x_3 & x_3 \\ x_6 & x_5 & x_4 \end{pmatrix},$$

so that one gets $J(V)^\star = (\det M_{\{1,3\}})$.

In [3], Ardila and Boocher consider $T = K[x_1, \ldots, x_n]$ with the standard $\mathbb{Z}^n$-graded structure induced by $\deg x_i = e_i \in \mathbb{Z}^n$. In this setting they prove, among other things, that all the initial ideals of $J(V)^{\text{hom}}$ (in the given coordinates) are squarefree. Moreover, the Betti numbers of $J(V)^{\text{hom}}$ and $\text{in}(J(V)^{\text{hom}})$ coincide. We can recover and generalize these results as follows.

Theorem 4.6. *Let $T = K[x_1, \ldots, x_n]$ be a polynomial ring with the standard $\mathbb{Z}^n$-graded structure. Let V be a vector space of linear forms of T (i.e. elements of degree 1 with respect to the standard $\mathbb{Z}$-graded structure) and let $J(V)$ be the ideal generated by V. Then $J(V)^{\text{hom}} \subset S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is a Cartwright–Sturmfels as well as a Cartwright–Sturmfels* ideal. Furthermore every ideal H of S with the same $\mathbb{Z}^n$-graded Hilbert function as $J(V)^{\text{hom}}$ is radical, Cohen–Macaulay and $\beta_{i,a}(H) = \beta_{i,a}(J(V)^{\text{hom}})$ for every $i \in \mathbb{N}$ and $a \in \mathbb{N}^n$.*

Proof. We proved that $J(V)^{\mathrm{hom}}$ is a Cartwright–Sturmfels ideal. Hence every ideal H of S with the same $\mathbb{Z}^n$-graded Hilbert function as $J(V)^{\mathrm{hom}}$ is radical. Let $G = \mathrm{gin}(J(V)^{\mathrm{hom}})$ be the multigraded generic initial ideal of $J(V)^{\mathrm{hom}}$. Since $J(V)^{\mathrm{hom}}$ is a $\mathbb{Z}^n$-graded prime ideal, the largest variable of each block does not appear in the generators of G. Since we have only two variables in each block it follows that the generators of G involve only one variable per block, hence $J(V)^{\mathrm{hom}}$ is a Cartwright–Sturmfels* ideal. Hence every ideal H of S with the $\mathbb{Z}^n$-graded Hilbert function of $J(V)^{\mathrm{hom}}$ satisfies $\beta_{i,a}(H) = \beta_{i,a}(J(V)^{\mathrm{hom}})$ for every $i \in \mathbb{N}$ and $a \in \mathbb{N}^n$. The Cohen–Macaulay property follows from Brion's Theorem 2.11. $\qquad\square$

Ardila and Boocher in [3] computed the multidegree of $S/J(V)^{\mathrm{hom}}$ in their setting. We are able to compute the multidegree in general.

Let $T = K[x_1, \ldots, x_n]$ with any $\mathbb{Z}^u$-graded structure. Let $V = \langle L_1, \ldots, L_v \rangle$ and consider the $v \times n$ matrix M_V, whose (i, j)-entry is the coefficient of x_j in L_i. To M_V we associate the matroid $\mathcal{M}_V$, whose elements are the subsets of $[n]$ corresponding to linearly independent columns of M_V. A basis of $\mathcal{M}_V$ is a maximal element, i.e. a set of column indices $\{b_1, \ldots, b_v\}$ corresponding to a basis of the column space of M_V. To every basis $b = \{b_1, \ldots, b_v\}$ we associate a multidegree $\deg(b) = \deg(x_{b_1} \cdots x_{b_r}) \in \mathbb{Z}^u$ and let

$$D_V = \big\{ \deg(b) : b \text{ is a basis of } \mathcal{M}_V \big\}.$$

With this notation we have:

Theorem 4.7. *Let $R = S/J(V)^{\mathrm{hom}}$. The multidegree of R is given by the formula:*

$$\mathrm{Deg}_R(z) = \sum_{w \in D_V} z^w.$$

Proof. If $W \subset \mathbb{A}^n$ is the affine $(n-v)$-dimensional linear space corresponding to $J(V)$, then $J(V)^{\mathrm{hom}}$ corresponds to the closure $\overline{W}$ of W in the product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_u}$, where $n_i = \dim_K T_{e_i}$, $1 \le i \le u$. By the results we have recalled in Section 2 we have:

$$\mathrm{Deg}_R(z) = \sum \deg(H_c \cap \overline{W}) z^c,$$

where the sum runs over the $c = (c_1, \ldots, c_u) \in \mathbb{N}^u$ with $c_i \le n_i$ and $|c| = v$. Moreover, $H_c = W_1 \times \cdots \times W_u$ and $W_i \subseteq \mathbb{P}^{n_i}$ is a generic linear subspace with $\dim W_i = c_i$. Here $\deg(H_c \cap \overline{W})$ denotes the usual intersection multiplicity of H_c and $\overline{W}$.

We claim that the intersection of H_c and $\overline{W}$ is affine (for a generic choice of H_c). In fact, since $\overline{W}$ is irreducible and not contained in the hyperplane H_i of equation $y_i = 0$ for any i, then $\dim(\overline{W} \cap H_i) = n - v - 1$. Therefore, a generic H_c has empty intersection with $\overline{W} \cap H_i$, since $\dim H_c + \dim(\overline{W} \cap H_i) = n - 1 < n$. Finally, since there are u hyperplanes H_i, the intersection of a generic H_c with $\overline{W}$ avoids them all.

Since the intersection of H_c and $\overline{W}$ is affine, then it corresponds to the intersection of W with $W_1 \times \cdots \times W_u \subset \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_u} = \mathbb{A}^n$ where W_i is a generic linear subspace of $\mathbb{A}^{n_i}$ that contains the origin.

Therefore, the defining equations of the affine part of H_c are general elements $\ell_{i,j} \in T_{e_i}$ with $1 \le i \le u$ and $1 \le j \le n_i - c_i$. In particular, $m(H_c, \overline{W}) \in \{0, 1\}$ and it is 1 if and only if the $\ell_{i,j}$'s and the L_i's are linearly independent. The associated determinant is the linear combination the maximal minors of M_V corresponding to bases of $\mathcal{M}_V$ of multidegree c whose coefficients are generic. Hence $m(H_c, \overline{W}) = 1$ if and only if $c \in D_V$. $\qquad\square$

We illustrate Theorem 4.7 by considering again Example 4.5.

Example 4.8. Let $T = K[x_1, \ldots, x_6]$ with the same $\mathbb{Z}^3$-graded structure as in Example 4.5, i.e. let

$$\deg(x_i) = \begin{cases} e_1, & \text{for } i = 1, 2, \\ e_2, & \text{for } i = 3, \\ e_3, & \text{for } i = 4, 5, 6. \end{cases}$$

Let $V = \langle L_1, L_2, L_3 \rangle$ with $L_1 = x_1 + x_2 + x_6$, $L_2 = x_2 - x_3 + x_5$, $L_3 = x_3 + x_4$, and let $R = S/J(V)^{\mathrm{hom}}$. The matrix associated to V is

$$M_V = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The set of bases of the matroid $\mathcal{M}_V$ is

$$\{123, 124, 134, 135, 145, 234, 235, 236, 245, 246, 346, 356, 456\}.$$

To every basis we associate a multidegree and a monomial in $K[z_1, z_2, z_3]$, for example 123 corresponds to the degree $\deg(x_1 x_2 x_3) = (2, 1, 0)$ and to the monomial $z_1^2 z_2$. The set of the degrees of the bases of $\mathcal{M}_V$ is

$$D_V = \{(2, 1, 0), (2, 0, 1), (1, 1, 1), (1, 0, 2), (0, 1, 2), (0, 0, 3)\},$$

so that by Theorem 4.6 one has

$$\mathrm{Deg}_R(z) = z_1^2 z_2 + z_1^2 z_3 + z_1 z_2 z_3 + z_1 z_3^2 + z_2 z_3^2 + z_3^3.$$

Each monomial in the multidegree corresponds to a minimal prime of the generic initial ideal, e.g. $(2, 1, 0)$ corresponds to the ideal generated by the first 2 variables of the first block and the first variable of the second block. Hence the multigraded generic initial ideal of $J(V)$ is the intersection of the 6 components:

$$\begin{aligned} (2, 1, 0) &\to (x_1, x_2, x_3) & (2, 0, 1) &\to (x_1, x_2, x_4) \\ (1, 1, 1) &\to (x_1, x_3, x_4) & (1, 0, 2) &\to (x_1, x_4, x_5) \\ (0, 1, 2) &\to (x_3, x_4, x_5) & (0, 0, 3) &\to (x_4, x_5, x_6) \end{aligned}$$

i.e. $\mathrm{gin}(J(V)) = (x_1 x_4, x_2 x_4, x_3 x_4, x_1 x_5, x_2 x_3 x_5, x_1 x_3 x_6)$.

5. Multiview ideals

In this section, we turn our attention to multiview ideals. Consider a collection of matrices with scalar entries $A = \{A_i\}_{i=1,\ldots,m}$ with A_i of size $d_i \times n$ and rank $A_i = d_i$. One has an induced rational map

$$\phi_A \colon \mathbb{P}^{n-1} \dashrightarrow \prod \mathbb{P}^{d_i-1}$$

sending $x \in \mathbb{P}^{n-1}$ to $(A_i x)_{i=1,\ldots,m}$. The ideal J_A associated to the closure of the image of ϕ_A is called multiview ideal. We refer to [2] for a discussion of the role played by J_A in various aspects of geometrical computer vision. Our goal is proving the following theorem.

Theorem 5.1. *For all choices of $A = \{A_i\}_{i=1,\ldots,m}$ the multiview ideal J_A is a CS ideal and it defines a Cohen–Macaulay normal domain.*

Theorem 5.1 is proved in [2] in the case $n = 4$ and $d_i = 3$ for all i, under the assumption that the A_i's are generic. Later on Binglin Li in his yet unpublished preprint [6] proved Theorem 5.1 in general. Furthermore he gave a combinatorial description of the multidegree and the generators of J_A. Our goal is giving two alternative proofs of Theorem 5.1.

First of all we introduce the algebraic objects needed to describe the problem. Notice that our point of view is somewhat dual to that of [6]. We denote by V_i the vector space of linear forms of $T = K[x_1, \ldots, x_n]$ generated by the entries of the matrix $A_i x$ where x is the column vector with entries $x_1, \ldots, x_n$. Then $V_1, \ldots, V_m$ is a collection of vectors spaces of linear forms of dimension $d_i = \dim_K V_i$. We define

$$A(V_1, \ldots, V_m) = K[V_1 y_1, \ldots, V_m y_m] \subset T[y_1, \ldots, y_m],$$

i.e. $A(V_1, \ldots, V_m)$ is the subalgebra of the polynomial ring $T[y_1, \ldots, y_m]$ generated by the elements $v y_i$ with $v \in V_i$. By construction $A(V_1, \ldots, V_m)$ is the multigraded coordinate ring of the closure of the image of ϕ_A. The $\mathbb{Z}^m$-graded structure on $A(V_1, \ldots, V_m)$ is induced by the assignment $\deg y_i = e_i \in \mathbb{Z}^m$.

We present $A(V_1, \ldots, V_m)$ as a quotient of $K[x_{ij} : i = 1, \ldots, m, \ j = 1, \ldots, d_i]$ via the K-algebra surjection

$$\phi \colon K[x_{ij} : i = 1, \ldots, m, \ j = 1, \ldots, d_i] \to A(V_1, \ldots, V_m)$$

defined by $\phi(x_{ij}) = v_{ij} y_i$, where $\{v_{ij} : j = 1, \ldots, d_i\}$ is a basis of V_i. By construction, $J_A = \ker \phi$.

Proof of Theorem 5.1. First proof. We take the point of view of [13]. Observe that $A(V_1, \ldots, V_m)$ is a subring of the Segre product

$$K[x_i y_j : i = 1, \ldots, n \text{ and } j = 1, \ldots, m]$$

of the polynomial rings T and $K[y_1, \ldots, y_m]$. The latter is defined as a quotient of the polynomial ring

$$K[x_{ij} : i = 1, \ldots, m, \ j = 1, \ldots, m]$$

by the ideal $I_2(X)$ of 2-minors of the matrix $X = (x_{ij})$. Hence J_A is obtained from $I_2(X)$ by performing a $\mathbb{Z}^m$-graded change of variables and then eliminating the variables x_{ij} with $d_i < j \leq m$. Since $I_2(X)$ is a CS ideal, by [15, Theorem 1.6] we may conclude that J_A is a CS ideal.

Second proof. We consider the K-algebra map

$$\phi_0 : K[x_{ij} : i = 1, \ldots, m, \ j = 1, \ldots, d_i] \to T$$

defined by $\phi_0(x_{ij}) = v_{ij}$. Clearly $\operatorname{Ker}\phi_0$ is generated by linear forms, indeed by $\sum_{i=1}^m d_i - \dim_K \sum_{i=1}^m V_i$ linear forms. By construction $\operatorname{Ker}\phi$ is the ideal generated by the $\mathbb{Z}^m$-homogeneous elements of $\operatorname{Ker}\phi_0$. With the notation introduced above:

$$\operatorname{Ker}\phi = (\operatorname{Ker}\phi_0)^{\star}$$

and by Theorem 4.6 we conclude that $\operatorname{Ker}\phi$ is CS, i.e. J_A is CS.

Finally, Cohen–Macaulayness and normality follow from Brion's Theorem 2.11. $\square$

Example 5.2. Let $m \leq d$ and $n = (m-1)d$. Let $A = \{A_i\}_{i=1,\ldots,m}$ with A_i generic of size $d \times n$. By genericity, we may choose coordinates such that

$$V_j = \langle x_{d(j-1)+1}, x_{d(j-1)+2}, \ldots, x_{jd} \rangle$$

for $j = 1, \ldots, m-1$, and

$$V_m = \langle v_1, \ldots, v_d \rangle, \quad \text{with } v_h = -\sum_{j=1}^{m-1} x_{d(j-1)+h}$$

for $h = 1, \ldots, d$. Then

$$\operatorname{Ker}\phi_0 = \left(\sum_{i=1}^m x_{ik} : k = 1, \ldots, d\right).$$

It follows that the multiview ideal J_A, i.e. the ideal of the closure of the image of the rational map

$$\phi_A : \mathbb{P}^{n-1} \dashrightarrow \prod_{i=1}^m \mathbb{P}^{d-1},$$

is defined by the m-minors of the generic $m \times d$ matrix

$$\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1d} \\ x_{21} & x_{22} & \ldots & x_{2d} \\ \ldots & \ldots & \ldots & \ldots \\ x_{m1} & x_{m2} & \ldots & x_{md} \end{pmatrix}.$$

References

[1] J. Abbott, A. M. Bigatti, and G. Lagorio, CoCoA: a system for doing Computations in Commutative Algebra. Available at: http://cocoa.dima.unige.it

[2] C. Aholt, B. Sturmfels, and R. Thomas, A Hilbert schemes in computer vision, *Canad. J. Math.*, **65** (2013), no. 5, 961–988. Zbl 1284.13035 MR 3095002

[3] F. Ardila and A. Boocher, The closure of a linear space in a product of lines, *J. Algebraic Combin.*, **43** (2016), no. 1, 199–235. Zbl 1331.05051 MR 3439307

[4] M. Badiane, I. Burke, and E. Sköldberg, The universal Gröbner basis of a binomial edge ideal, *Electron. J. Combin.*, **24** (2017), no. 4, Paper 4.11, 12pp. Zbl 06791426 MR 3711044

[5] D. Bernstein and A. Zelevinsky, Combinatorics of maximal minors, *J. Algebraic Combin.*, **2** (1993), no. 2, 111–121. Zbl 0794.13021 MR 1229427

[6] Binglin Li, Images of rational maps of projective spaces, *Int. Math. Res. Not.*, (2017). DOI: 10.1093/imrn/rnx003

[7] A. Boocher, Free resolutions and sparse determinantal ideals, *Math. Res. Lett.*, **19** (2012), no. 4, 805–821. Zbl 1273.14097 MR 3008416

[8] A. Boocher and E. Robeva, Robust toric ideals, *J. Symbolic Comput.*, **68** (2015), no. 1, 254–264. Zbl 1315.13041 MR 3283846

[9] M. Brion, Multiplicity-free subvarieties of flag varieties, in *Commutative algebra (Grenoble/Lyon, 2001)*, 13–23, Contemp. Math., 331, Amer. Math. Soc., Providence, RI, 2003. Zbl 1052.14055 MR 2011763

[10] W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics, 1327, Springer-Verlag, Berlin, 1988. Zbl 0673.13006 MR 953963

[11] D. Cartwright and B. Sturmfels, The Hilbert scheme of the diagonal in a product of projective spaces, *Int. Math. Res. Not. IMRN*, (2010), no. 9, 1741–1771. Zbl 1213.14010 MR 2643580

[12] M. Chardin, Some results and questions on Castelnuovo–Mumford regularity, in *Syzygies and Hilbert functions*, 1–40, Lect. Notes Pure Appl. Math., 254, Chapman & Hall/CRC, Boca Raton, FL, 2007. Zbl 1127.13014 MR 2309925

[13] A. Conca, Linear spaces, transversal polymatroids and ASL domains, *J. Algebraic Combin.*, **25** (2007), no. 1, 25–41. Zbl 1108.13022 MR 2284146

[14] A. Conca, E. De Negri, and E. Gorla, Universal Gröbner bases for maximal minors, *Int. Math. Res. Not. IMRN*, (2015), no. 11, 3245–3262. Zbl 1325.13028 MR 3373050

[15] A. Conca, E. De Negri, and E. Gorla, Universal Gröbner bases and Cartwright–Sturmfels ideals, to appear in *Int. Math. Res. Not. IMRN*, 2018.

[16] A. Conca, E. De Negri, and E. Gorla, Multigraded generic initial ideals of determinantal ideals, in *Homological and Computational Methods in Commutative Algebra*, 81–96, Springer INdAM Series 20, Springer, 2017. Zbl 1387.13002 MR 3751880

[17] V. Ene and A. Zarojanu, On the regularity of binomial edge ideals, *Math. Nachr.*, **288** (2015), 19–24. Zbl 1310.13021 MR 3310496

[18] V. Ene, Viviana, J. Herzog, T. Hibi, and A. A. Qureshi, The binomial edge ideal of a pair of graphs, *Nagoya Math. J.*, **213** (2014), 105–125. Zbl 1298.13020 MR 3290687

[19] D. Grayson and M. Stillman, Macaulay2: a software system for research in algebraic geometry. Available at: http://www.math.uiuc.edu/Macaulay2/

[20] J. Herzog, T. Hibi, F. Hreinsdottir, T. Kahle, and J. Rauh, Binomial edge ideals and conditional independence statements, *Adv. in Appl. Math.*, **45** (2010), 317–333. Zbl 1196.13018 MR 2669070

[21] M. Y. Kalinin, Universal and comprehensive Gröbner bases of the classical determinantal ideal, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, **373** (2009), Teoriya Predstavlenii, Dinamicheskie Sistemy, Kombinatornye Metody. XVII, 134–143, 348; translation in *J. Math. Sci. (N. Y.)*, **168** (2010), no. 3, 385–389. Zbl 1288.13023 MR 2749259

[22] D.Kiani and S. Saeedi Madani, The Castelnuovo–Mumford regularity of binomial edge ideals, *J. Combin. Theory Ser. A*, **139** (2016), 80–86. Zbl 1328.05087 MR 3436053

[23] M. Kalkbrener and B. Sturmfels, Initial complexes of prime ideals, *Adv. in Math.*, **116** (1995), 365–376. Zbl 0877.13025 MR 1363769

[24] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra. 2*, Springer Verlag, Heidelberg, 2005. Zbl 1090.13021 MR 2159476

[25] K. Matsuda and S. Murai, Regularity bounds for binomial edge ideals, *J. Commut. Algebra*, **5** (2013), no. 1, 141–149. Zbl 1272.13018 MR 3084125

[26] M. Miyazaki, On the discrete counterparts of algebras with straightening laws, *J. Commut. Algebra*, **2** (2010), 79–89. Zbl 1237.13041 MR 2607102

[27] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, 227, Springer-Verlag, 2005. Zbl 1090.13001 MR 2110098

[28] M. Ohtani, Graphs and ideals generated by some 2-minors, *Comm. Alg.*, **39** (2011), 905–917. Zbl 1225.13028 MR 2782571

[29] N. Perrin, On the geometry of spherical varieties, *Transform. Groups*, **19** (2014), no. 1, 171–223. Zbl 1309.14001 MR 3177371

[30] P. Schenzel and S. Zafar, Algebraic properties of the binomial edge ideal of a complete bipartite graph, *An. Stiint Univ. "Ovidius" Constanza Ser. Mat.*, **22** (2014), 217–237. Zbl 1313.05363 MR 3195706

[31] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Amer. Math. Soc., Providence, RI, 1995. Zbl 0856.13020 MR 1363949

[32] B. Sturmfels, N. V. Trung, and W. Vogel, Bounds on degrees of projective schemes, *Math. Ann.*, **302** (1995), 417–432. Zbl 0828.14040 MR 1339920

[33] B. Sturmfels and A. Zelevinsky, Maximal minors and their leading terms, *Adv. Math.*, **98** (1993), no. 1, 65–112. Zbl 0776.13009 MR 1212627

[34] S. Sullivant, Strongly robust toric ideals in codimension 2, 2016. arXiv:1610.07476

[35] M. Varbaro, *Cohomological and Combinatorial Methods in the Study of Symbolic Powers and Equations defining Varieties*, Ph.D. thesis, Genova, 2011. arXiv:1105.5507

A. Conca, Dipartimento di Matematica, Università di Genova,
Via Dodecaneso 35, I-16146 Genova, Italy

E-mail: conca@dima.unige.it

E. De Negri, Dipartimento di Matematica, Università di Genova,
Via Dodecaneso 35, I-16146 Genova, Italy

E-mail: denegri@dima.unige.it

E. Gorla, Institut de Mathématiques, Université de Neuchâtel,
Rue Emile-Argand 11, CH-2000 Neuchâtel, Switzerland

E-mail: elisa.gorla@unine.ch