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A partial order on bipartitions from the generalized Springer correspondence

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Abstract. In [1], Lusztig gives an explicit formula for the bijection between the set of bipartitions and the set $\mathcal{N}$ of unipotent classes in a spin group which carry irreducible local systems equivariant for the spin group but not equivariant for the special orthogonal group. The set $\mathcal{N}$ has a natural partial order and therefore induces a partial order on bipartitions. We use the explicit formula given in [1] to prove that this partial order on bipartitions is the same as the dominance order appeared in Dipper–James–Murphy's work [2].

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1. Preliminaries

For group $G = \mathrm{Spin}_n(k)$, where k is a field of characteristic not equal to 2, let $\mathcal{N}$ be the set of unipotent classes in G which carry irreducible local systems, equivariant for the conjugation action of G, but not equivariant for the conjugation action of the special orthogonal group. Then $\mathcal{N}$ has a one-to-one correspondence with a certain set of partitions X_n (see [1, Section 14]). X_n consists of partitions

$$\lambda = \big(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\big)$$

of n, such that each $\lambda_i \in \mathbb{N}_+$, and

(1) for each integer $n \in 2\mathbb{Z}+1$, the set $\{i\,;\lambda_i = n\}$ has at most one element;

(2) for each integer $n \in 2\mathbb{Z}$, the set $\{i\,;\lambda_i = n\}$ has an even number of elements.

Let $\mathrm{Irr}\, W_s$ be the set of all bipartitions of s. Then the generalized Springer correspondence for the spin group gives a bijection

$$X_n \longleftrightarrow \bigsqcup_{t\in 4\mathbb{Z}+n} \mathrm{Irr}\, W_{\frac{1}{4}(n-2t^2+t)}. \tag{1}$$

In [1], Lusztig gives an explicit formula for this bijection. Specifically, let

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_m) \in X_n.$$

Define

$$t_i = \sum_{j \geq i+1} d(\lambda_j) \tag{2}$$

and

$$t = \sum_{j \geq 1} d(\lambda_j). \tag{3}$$

Here

$$d(\lambda_j) = \begin{cases} 0, & \text{if } \lambda_j \text{ is even,} \\ (-1)^{(\lambda_j(\lambda_j - 1))/2}, & \text{if } \lambda_j \text{ is odd.} \end{cases} \tag{4}$$

Then the image of λ under the bijection can be constructed in the following way:

(1) If $\lambda_i \in 4\mathbb{Z}+1$, then label this entry by a, and replace this entry by $\frac{1}{4}(\lambda_i - 1) - t_i$.

(2) If $\lambda_i \in 4\mathbb{Z}+3$, then label this entry by b, and replace this entry by $\frac{1}{4}(\lambda_i - 3) + t_i$.

(3) If $\lambda_i = e \in 4\mathbb{Z}+2$, then by definition it appears $2p$ times. Replace these entries by

$$\tfrac{1}{4}(e-2) + t_i, \tfrac{1}{4}(e+2) - t_i, \ldots, \tfrac{1}{4}(e+2) - t_i, \tag{5}$$

respectively, and label them as $b, a, b, \ldots, a, b, a$.

(4) If $\lambda_i = e \in 4\mathbb{Z}$, then by definition it appears $2p$ times. Replace these entries by

$$\tfrac{1}{4}e + t_i, \tfrac{1}{4}e - t_i, \ldots, \tfrac{1}{4}e - t_i, \tag{6}$$

respectively. Label them as $b, a, b, \ldots, a, b, a$.

The modified entries with label a form a decreasing sequence α. The entries with label b form a decreasing sequence β. If $t > 0$, then λ corresponds to (α, β) in the bijection. If $t \leq 0$, then λ corresponds to (β, α). Moreover, the bipartition (α, β) (when $t \geq 1$) or (β, α) (when $t \leq 0$) is an element in Irr $W_{\frac{1}{4}(n-2t^2+t)}$.

Remark. In Lusztig's paper [1], he gives the formula for partitions in increasing order. Here I simply translated everything in decreasing order, for convenience of the following proof. Moreover a partition in decreasing order can be extended by adding 0's.

There is a natural partial order on $\mathcal{N}$: $c \leq c'$ if c is contained in the closure of c'. This partial order is given below, in terms of elements in X_n:

Definition 1.1. For $\lambda, \mu \in X_n$ such that each is in decreasing order. We say $\lambda \leq \mu$ if and only if for all $i \in \mathbb{N}$

$$\sum_{j \leq i} \lambda_j \leq \sum_{j \leq i} \mu_j. \tag{7}$$

From the bijection (1), we have an induced partial order on the set of bipartitions $\operatorname{Irr} W_m$, for each t. This partial order is closely related to that found in Dipper–James–Murphy's paper [2], and also appears in Geck and Iancu's paper [3] as the asymptotic case for their pre-order relation on $\operatorname{Irr} W$, indexed by two parameters a, b. In the asymptotic case $b > (n-1)a$, their pre-order is a partial order, and is defined by

Definition 1.2 (Dipper–James–Murphy). The dominance order between

$$(\lambda, \mu), (\lambda', \mu') \in \operatorname{Irr} W,$$

each in decreasing order, is

$$(\lambda, \mu) \leq (\lambda', \mu') \Leftrightarrow \begin{cases} \sum_{j \leq k} \lambda_j \leq \sum_{j \leq k} \lambda'_j, & \text{for all } k, \\ |\lambda| + \sum_{j \leq k} \mu_j \leq |\lambda'| + \sum_{j \leq k} \mu'_j, & \text{for all } k. \end{cases} \tag{8}$$

The main result of this paper is the following:

Theorem 1. *For $t \geq m$, the induced partial order on* $\operatorname{Irr} W_m$ *from the inclusion* $\operatorname{Irr} W_m \hookrightarrow X_{2t^2-t+4m}$*, is the dominance order.*

2. Proof of the main result

Let $f_{m,t} \colon \operatorname{Irr} W_m \hookrightarrow X_{2t^2-t+4m}$ be the inclusion from the generalized Springer correspondence. We first make the following observation:

Lemma 1. *If $t \geq m$ and $\lambda \in f_{m,t}(\operatorname{Irr} W_m)$, then*

$$\lambda_i \in 2\mathbb{Z} \cup (4\mathbb{Z} + 1).$$

Proof. Suppose on the contrary there is an i such that $\lambda_i \in 4\mathbb{Z} + 3$. By definition,

$$t = \sum_i d(\lambda_i).$$

Each $\lambda_i \in 4\mathbb{Z} + 1$ contributes $+1$, and each $\lambda_i \in 4\mathbb{Z} + 3$ contributes -1. By definition of X_n, each odd integer appears at most once. So

$$t = |\{i; \lambda_i \in 4\mathbb{Z} + 1\}| - |\{i; \lambda_i \in 4\mathbb{Z} + 3\}|. \tag{9}$$

And then,

$$|\{i; \lambda_i \in 4\mathbb{Z} + 1\}| \geq t + 1.$$

So

$$
\begin{aligned}
2t^2 - t + 4m = |\lambda| &= \sum_i \lambda_i \\
&\geq \sum_{i,\lambda_i \in 4\mathbb{Z}+1} \lambda_i \\
&\geq \sum_{j=0}^{t} (4j+1) \\
&= 2t^2 + 3t + 1 \geq 2t^2 - t + 4m + 1.
\end{aligned}
\tag{10}
$$

This is a contradiction! The lemma also proves that there are exactly t odd integers in λ, each in $4\mathbb{Z}+1$. □

Now the picture is clear for $t \geq m$. In fact, if (α, β) corresponds to λ, then α represents the deviation of odd integers of λ from $(4t-3, 4t-7, \ldots, 1)$, and β is the even integers of λ, up to scalar. We have the following lemma:

Lemma 2. *Suppose $t \geq m$, and $(\alpha, \beta) \in \operatorname{Irr} W_m$ corresponds to λ under $f_{m,t}$. Then, λ is the re-ordering of numbers*

$$4\alpha_i + 4(t-i) + 1,\ 1 \leq i \leq t, \quad \textit{and} \quad 2\beta_1, 2\beta_1, 2\beta_2, 2\beta_2 \ldots$$

(α is extended by "0's" if necessary). For convenience, let

$$f(\alpha_i) = 4\alpha_i + 4(t-i) + 1,$$

if the underlying t causes no ambiguity.

Proof. λ defined in the lemma has order

$$4|\alpha| + 4|\beta| + \sum_{i=1}^{t} \big(4(t-i)+1\big) = 2t^2 - t + 4m.$$

Since $f_{m,t}$ is a bijection, we only need to prove that λ, the reordering of numbers

$$f(\alpha_i),\ 1 \leq i \leq t, \quad \text{and} \quad 2\beta_1, 2\beta_1, 2\beta_2, 2\beta_2 \ldots,$$

indeed gives (α, β) by Lusztig's rule. Now we assume λ is sent to (α', β'). Notice that since even integers doesn't contribute to the t-function (see (2)), the t-function associated to $f(\alpha_i)$ is exactly $t-i$. So, $\alpha' = \alpha$. If $\beta = (0)$, then the lemma is automatically true. If $\beta \neq (0)$, suppose

$$4l + 1 > 2\beta_1 > 4l - 3, \quad l \geq 1.$$

We claim that $\alpha_i = 0$ for $i \geq t - l + 1$. So, $\lambda_i = f(\alpha_i)$ for $i \leq t - l$. Indeed, otherwise $\alpha_{t-l+1} \geq 1$. Since α is decreasing, we have

$$\begin{aligned} m &= |\alpha| + |\beta| \\ &\geq t - l + 1 + \beta_1 \\ &\geq t - l + 1 + (2l - 1) \geq t + 1. \end{aligned} \tag{11}$$

This is a contradiction!

Now suppose

$$4k + 1 > 2\beta_i > 4k - 3, \quad k \leq l.$$

Since we have shown $\alpha_i = 0$ for $i \geq t - l + 1$. The odd integers less than $2\beta_i$ are exactly $4k - 3, 4k - 7, \ldots, 1$. So the corresponding t-function is k. There are two cases:

(1) $2\beta_i = 4k - 2$. Then from Lusztig's rule, $2\beta_i, 2\beta_i$ are modified by

$$\tfrac{1}{4}(2\beta_i - 2) + k = \beta_i, \quad \tfrac{1}{4}(2\beta_i + 2) - k = 0,$$

with labels b, a, respectively.

(2) $2\beta_i = 4k$. Then from Lusztig's rule, $2\beta_i, 2\beta_i$ are modified by

$$\tfrac{1}{4}(2\beta_i) + k = \beta_i, \quad \tfrac{1}{4}(2\beta_i) - k = 0,$$

with labels b, a, respectively.

So indeed $\beta' = \beta$. □

Now we use the above observation to prove the main theorem. Let (α, β),(α', β') be bipartitions with order m. They correspond to λ, λ' from the inclusion

$$f_{m,t}\colon \mathrm{Irr}\, W_m \hookrightarrow X_{2t^2-t+4m}.$$

Here $t \geq m$ is a fixed integer.

Proof of the main theorem.

(a) If $(\alpha, \beta) \geq (\alpha', \beta')$ in the dominance order, then $\lambda \geq \lambda'$.

Proof. Let $A(k) = \{\lambda_1, \ldots, \lambda_k\}$ (repetitions are allowed, with multiplicity specified), and define $A'(k)$ similarly. $A(k), A'(k)$ are defined for all positive integers k, and λ, λ' are extended by 0's. Let $|A(k)|$ denote the sum of elements in $A(k)$, and similarly for $|A'(k)|$. Suppose $\lambda \geq \lambda'$ does not hold. Then since $|\lambda| = |\lambda'|$, there is a largest k such that

$$|A(k)| < |A'(k)|.$$

For convenience, let $g_1 \geq g_2 \geq \cdots$ be the decreasing sequence of even integers in λ, and similarly define g'_i for λ'. It is clear that

$$(4\alpha, g) \geq (4\alpha', g'),$$

as bipartitions of $4m$. In fact, for l even, the inequality is equivalent to

$$|\alpha| + \beta_1 + \cdots + \beta_{\frac{l}{2}} \geq |\alpha'| + \beta'_1 + \cdots + \beta'_{\frac{l}{2}}.$$

The inequalities for odd l is deduced from the average of those of $l-1$ and $l+1$.

Suppose $A'(k)$ consists of elements $f(\alpha'_i)$, $1 \leq i \leq u$, and $g'_1, \ldots, g'_l$. So, $k = u + l$. If $l = 0$, then

$$\begin{aligned} |A(k)| - |A'(k)| &\geq \sum_{i=1}^{u} f(\alpha_i) - \sum_{i=1}^{u} f(\alpha'_i) \\ &= 4\big(\alpha_1 + \alpha_2 + \alpha_u - \alpha'_1 - \cdots - \alpha'_u\big) \geq 0. \end{aligned} \tag{12}$$

This is a contradiction! Here we used that $A(k)$ consists of the largest k elements of λ. If $l \neq 0$, and $\alpha_{u+1} = 0$. Then we can choose k elements in λ:

$$f(\alpha_i),\ i \leq u, \quad \text{and} \quad g_1, \ldots, g_l.$$

These k elements have sum greater than or equal to $|A'(k)|$. So,

$$|A(k)| \geq |A'(k)|.$$

A contradiction!

If $\alpha_{u+1} \neq 0$. Suppose $A(k)$ consists of elements

$$f(\alpha_i),\ 1 \leq i \leq u + s, \quad \text{and} \quad g_1, \ldots, g_{l-s}.$$

If $s = l$, then we claim that $f(\alpha_{u+l}) < \lambda'_k$. Otherwise, assume $f(\alpha_{u+l}) \geq \lambda'_k$. From Lemma 2,

$$f(\alpha_{u+s-1}) - f(\alpha_{u+s}) \geq 4 \geq \lambda'_{k-1} - \lambda'_k$$

(since terms beyond $2\beta'_1$ contain all the positive integers in $4\mathbb{Z}+1$ smaller than $2\beta'_1$). So we get

$$f(\alpha_{u+l-1}) \geq \lambda'_{k-1}.$$

This method proceeds, so we get

$$\lambda_i = f(\alpha_i) \geq \lambda'_i, \quad i \geq x,$$

where $\lambda'_x = g'_1$. But from the case $l = 0$ above, we know that

$$|A(x-1)| \geq |A'(x-1)|.$$

So, $|A(k)| \geq |A(k)'|$, which is a contradiction! Hence,

$$\lambda_k = f(\alpha_{u+l}) < \lambda'_k.$$

If λ_{k+1} is odd, we have

$$\lambda_{k+1} \leq \lambda_k - 4 \leq \lambda'_{k+1}.$$

But then $|A(k+1)| < |A'(k+1)|$, and this contradicts with the fact that k is the largest. So λ_{k+1} is even.

From Lemma 2, we conclude that $\alpha_i = 0$ for $i \geq k+1 = u+s+1$, and

$$4(t-u-s)+1 > 2\beta_1 > 4(t-u-s)-3.$$

If $s \neq l$, Lemma 2 also implies that $\alpha_i = 0$ for $i \geq u+s+1$. In either case, we have

$$g_i < 4(t-u-s)+1 = f(\alpha'_{u+s})$$

for $i \geq l-s+1$. Then,

$$\begin{aligned}
|A(k)| - |A'(k)| &= \sum_{i=1}^{u+s} f(\alpha_i) + \sum_{i=1}^{l} g_i - \left(\sum_{i=1}^{u+s} f(\alpha'_i) + \sum_{i=1}^{l} g'_i \right) \\
&\qquad\qquad + \sum_{i=u+1}^{u+s} f(\alpha'_i) - \sum_{i=l-s+1}^{l} g_i \\
&= \left(4|\alpha| + \sum_{i=1}^{l} g_i - 4|\alpha'| - \sum_{i=1}^{l} g'_i \right) + \sum_{i=u+1}^{u+s} f(\alpha'_i) - \sum_{i=l-s+1}^{l} g_i \\
&\geq \sum_{i=u+1}^{u+s} f(\alpha'_i) - \sum_{i=l-s+1}^{l} g_i \geq s f(\alpha'_{u+s}) - \sum_{i=l-s+1}^{l} g_i \geq 0.
\end{aligned} \tag{13}$$

Contradiction! So we have shown $\lambda \geq \lambda'$.

(b) Suppose $\lambda \geq \lambda'$, then $(\alpha, \beta) \geq (\alpha', \beta')$. Clearly $\alpha \geq \alpha'$ from Lemma 2 and the discussion at the beginning of the proof above. So if $(\alpha, \beta) \geq (\alpha', \beta')$ does not hold, then there is a smallest k, such that

$$|\alpha| + \beta_1 + \cdots + \beta_k < |\alpha'| + \beta'_1 + \cdots + \beta'_k. \tag{14}$$

We still use the notation $A(x)$ for the first x terms of λ, and use $S(x)$ to represent the sum of first x terms of β and $|\alpha|$. So $S(k) < S'(k)$, for some $k \geq 1$. By assumption of k, $\beta_k < \beta'_k$. If $\beta_k = 0$, then it is automatically a contradiction, since the left side is then $m = |\alpha'| + |\beta'|$. So,

$$0 < \beta_k < \beta'_k.$$

They both come from some even integers $2\beta_k, 2\beta'_k$ in the corresponding partition. Suppose they correspond to λ_{x-1}, λ_x and $\lambda'_{x'-1}, \lambda'_{x'}$, respectively. Suppose,

$$4u + 5 > 2\beta_k > 4u + 1 \quad \text{and} \quad 4u' + 5 > 2\beta'_k > 4u' + 1.$$

Then $u' \geq u$. So,

$$x = 2k + t - u - 1 \geq x' = 2k + t - u' - 1.$$

Now,

$$\begin{aligned} |A(x)| - |A'(x')| &= |A(x)| - |A'(x)| + \sum_{i=x'+1}^{x} \lambda'_i \\ &\geq \sum_{i=x'+1}^{x} \lambda'_i. \end{aligned} \tag{15}$$

Also notice that

$$\begin{aligned} |A(x)| - |A'(x')| &= \sum_{i=u+1}^{u'} (4i + 1) + 4\big(S(k) - S'(k)\big) \\ &< \sum_{i=u+1}^{u'} (4i + 1). \end{aligned} \tag{16}$$

This means

$$\sum_{i=u+1}^{u'} (4i + 1) > \sum_{i=x'+1}^{x} \lambda'_i. \tag{17}$$

However, this is a contradiction, since

$$\{4u + 5, 4u + 9, \ldots, 4u' + 1\} \subset \{\lambda_{x'+1}, \lambda_{x'+2}, \ldots\},$$

and $x - x' = u' - u$. □

We now give an example that violates the above partial order for $t = m - 1$. The partition

$$\lambda = (4t + 1, 4t - 3, \ldots, 9, 5, 3, 1)$$

corresponds to (α, β), where $\alpha = (1, 1, \ldots, 1)$ (t "1's") and $\beta = (1)$.

The partition

$$\lambda' = (4t + 1, 4t - 3, \ldots, 9, 5, 2, 2)$$

corresponds to (α', β'), where $\alpha' = (1, 1, \ldots, 1, 1)$ ($t + 1$ "1's".) and $\beta' = (0)$.

Then $\lambda > \lambda'$, but $(\alpha, \beta) < (\alpha', \beta')$.

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