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The isoperimetric spectrum of finitely presented groups

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Abstract. The isoperimeric spectrum consists of all real positive numbers α such that n^α is equivalent to the Dehn function of a finitely presented group. In this note we show how a recent result of Olshanskii completes the description of the isoperimetric spectrum modulo the celebrated Computer Science conjecture (and one of the seven Millennium Problems) $\mathbf{P} = \mathbf{NP}$ and even a formally weaker conjecture.

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The goal of this note is to show that the recent paper by Olshanskii [11] completes a description of the isoperimetric spectrum of finitely presented groups modulo the $\mathbf{P} = \mathbf{NP}$ conjecture.

Since in this note we consider only polynomially bounded functions $\mathbb{N} \to \mathbb{R}$, we call two functions f, g *equivalent* if

$$af(n) \le g(n) \le bf(n)$$

for some positive constants a, b.

Brady and Bridson [6] called the set of all real numbers $\alpha \ge 1$ such that n^α is equivalent to the Dehn function of a finitely presented group the *isoperimetric spectrum*. When it was introduced, it was known only that all natural numbers belong to the isoperimetric spectrum (the free nilpotent group of class c with at least 2 generators has Dehn function n^{c+1} [2]), and that by Gromov's theorem the intersection of the isoperimetric spectrum with the open interval $(1, 2)$ is empty. It is obvious also that the isoperimetric spectrum is a countable set since the set of all finite group presentation is countable. Bridson [6] found the first examples of non-integral numbers in the spectrum.

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Soon after, we proved in [14] that for $\alpha \geq 4$ to be in the isoperimetric spectrum, it is enough that α is computed in time $\leq 2^{2^m}$. Recall [14] that a real number α is called computable in time $T(m)$ where $T(m)$ is a function $\mathbb{N} \to \mathbb{N}$, if there exists a deterministic Turing machine which, given a natural number n, computes a binary rational approximation of α with an error at most $1/2^{n+1}$ in at most $T(n)$ steps. Thus all algebraic numbers ≥ 4 and many transcendental numbers such as $\pi + 1$ are in the isoperimetric spectrum. On the other hand, we proved in [14] that every number in the isoperimetric spectrum can be computed in time $\leq 2^{2^{2^{cm}}}$ for some constant c. It can be seen from the proof of this result that the number of 2's in this estimate can be reduced to two as in the lower bound if we had $\mathbf{P} = \mathbf{NP}$:

Provided $\mathbf{P} = \mathbf{NP}$ every number in the isoperimetric spectrum can be computed in time $\leq 2^{2^{cn}}$ for some constant c. (+)

I mentioned (+) (without a proof) in my ICM talk [12] and my Bulletin of Mathematical Sciences survey [13].

Note that Statement (+) is weaker than Theorem 2 below where the upper and lower bound for complexity of numbers in the isoperimetric spectrum coincide (see the discussion [9]).

For $\alpha \leq 4$, the situation was more complicated. On the one hand the tools used in [14] were too weak to handle $\alpha \leq 4$. On the other hand, Brady, Bridson, Forester, and Shankar found more numbers from the interval $(2, 4)$ in the isoperimetric spectrum, showing that the set of these numbers is dense in the interval $(2, 4)$ [3] and even contains all rational numbers [4]. Their numbers from the isioperimetric spectrum were constructed using algebraic rather than computational properties. (Note also that the groups constructed in [3–6], are given by very small presentations comparing to the groups in [14] and are subgroups of CAT(0) groups which is quite remarkable.) But the paper by Olshanskii [11] showed that the intersection of the isoperimetric spectrum with $(2, 4)$ can be described in the same terms as in [14].

Let $d \geq 2$ be a natural number. We say that the first m d-ary digits of a real number $\alpha > 0$ can be computed in time $\leq T(m)$ if there is deterministic Turing machine computing for every $m \geq 1$ a (finite) d-ary number β_m such that

$$|\alpha - \beta_m| \leq \frac{1}{d^{m+1}}$$

in time $\leq T(m)$. If $d = 2$, then we simply say that α can be computed in time $\leq T(m)$.

Combining results of [14] and [11] we get the following theorem. Notice first that there is a misprint in the formulation of [14, Corollary 1.4]: the first inequality sign there should be the ordinary $\leq$, not the coarse $\preceq$ (A. Yu. Olshanskii pointed it out to me). The proof works for the $\leq$ sign (see below).

Theorem 1 (The first part of Corollary 1.4 from [14] (with the correct inequality sign), and Corollary 1.4 from [11]). *If a number $\alpha \geq 2$ can be computed in time $\leq 2^{c2^n}$ for some c, then α belongs to the isoperimetric spectrum.*

One needs to modify a little the proof of the first part of Corollary 1.4 from [14] to obtain the estimate $\leq 2^{c2^n}$ instead of $\leq 2^{2^n}$.

For this, one should take natural number $d > \log_2 c$ which is a power 2^k for some k and consider d-ary representations of numbers instead of binary representations as in [14]. Each d-ary digit of α is a binary number with k binary digits. So if the first m binary digits of α are computed in time $\leq 2^{c2^m}$, then the first m d-ary digits of α (or, equivalently, first km binary digits of α) are computed in time $\leq d^{d^m}$ and the rest of the proof of [14, Corollary 1.4] carries by replacing 2 by d everywhere.

Let us explain the modifications in more details. After replacing 2 by d, Problem A from [14] becomes

Problem $\mathbf{A}_d$. *Given a natural number n written in d-ary, compute $f(n)$ in d-ary. The size of n is the number of d-ary digits of n, that is* $[\log_d n] + 1$.

The first part of [14, Corollary 1.3] becomes the following statement:

Corollary 1.3$_d$. *Let $f(n) \geq n^4$ be a superadditive function such that the d-ary representation of $f(n)$ is computable in time*

$$O\big(\sqrt[4]{f(n)}\big)$$

by a Turing machine (*i.e. Problem A_d is solvable in time $O(\sqrt[4]{f(n)})$*). *Then $f(n)$ is equivalent to the Dehn function of a finitely presented group.*

The proof of Corollary 1.3$_d$ repeats the proof of [14, Corollary 1.3] relacing binary by d-ary everywhere.

The first part of [14, Corollary 1.4] becomes:

Corollary 1.4$_d$. *For every real number $\alpha \geq 4$ whose d-ary representation is computable in time $\leq d^{d^m}$ the function $[n^\alpha]$ is equivalent to the Dehn function of a finitely presented group.*

Proof. Notice that the function $[n^\alpha]$ is equivalent to the function $d^{[\alpha[\log_d n]]}$. The function $[\log_d n] + 1$ (i.e. the length of the d-ary expression of the number of d-digits in n) is computable in time

$$\leq O\big((\log_d n)^2\big)$$

by an obvious algorithm: scan the number n from left to right on one tape and after each step add 1 to the number on the other tape.

Since the first $[\log_d(\log_d n)] + 1$ d-ary digits of α are computable in time $O(n)$, the function $[\alpha[\log_d n]]$ is computable in time

$$O(n) \leq O\big(n^{\alpha/4}\big).$$

Notice also that Problem A_d for a function equivalent to d^m is solvable in time $O(m)$. Indeed, we can consider the unary expression of m as the d-ary expression of

$$\frac{d^{m+1} - 1}{d - 1}$$

and use the second algorithm in the proof of [14, Corollary 1.3]. It remains to apply Corollary 1.3$_d$. □

Now we will prove the main result of the note.

Theorem 2. *Provided* $\mathbf{P} = \mathbf{NP}$, *a number* α *is in the isoperimetric spectrum if and only if it can be computed in time* $\leq 2^{c2^m}$ *for some* $c \geq 1$.

Proof. Theorem 1 gives one part of Theorem 2.

To prove the other part, suppose that n^α is the Dehn function of a finitely presented group. Then by [14, Theorem 1.1], n^α is equivalent to the time function $T(n)$ of some (non-deterministic) Turing machine M which recognizes the word problem in a finitely presented group.

Let us recall the definition of the time function of a non-deterministic Turing machine.

Definition 3. Let M be a nondeterministic Turing machine. The function $t(\cdot)$ is called the time function of M iff for all x (bitstrings that represents non-negative integers in the non-signed binary notation) we have:

(1) [Bound] for every input w of M of length $\leq x$ that is accepted by M, there exists an accepting computation of M on w, with time $\leq t(x)$; and

(2) [Tightness] there exists an input w of length $\leq x$, such that:

 (2.1) every accepting computation of M on w has time $\geq t(x)$; and

 (2.2) there exists an accepting computation of M on w with time exactly $t(x)$.

As explained by Emil Jeřábek [10] (see his proof below), the following property follows from $\mathbf{P} = \mathbf{NP}$:

> There is a deterministic Turing machine M' computing a polynomially bounded function $T'(n)$ which is equivalent to $T(n)$ and having time function at most $T(n)^d$ for some constant d. $(*)$

Here we reproduce Emil Jeřábek proof from [10] with his permission.

The Property $(*)$ is strictly weaker than $\mathbf{P} = \mathbf{NP}$, in the sense that it follows from $\mathbf{E} = \mathbf{\Sigma}_2^E$, which is not known to imply $\mathbf{P} = \mathbf{NP}$. Of course, we cannot prove this unconditionally with current technology, as it would establish $\mathbf{P} \neq \mathbf{NP}$.

Here, $\mathbf{E}$ denotes $\mathbf{DTIME}(2^{O(n)})$, $\mathbf{NE} = \mathbf{NTIME}(2^{O(n)})$, and $\mathbf{\Sigma}_2^E = \mathbf{NE}^{\mathbf{NP}}$ is the second level of the exponential hierarchy (with linear exponent), $\mathbf{EH}$ (see e.g. [7, 8]). Equivalently, we may define $\mathbf{\Sigma}_2^E$ using alternating Turing machines (see [1, Section 5.3]) as

$$\mathbf{\Sigma}_2^E = \mathbf{\Sigma_2} - \mathbf{TIME}\big(2^{O(n)}\big).$$

Note that conversely, $\mathbf{P} = \mathbf{NP}$ implies $\mathbf{P} = \mathbf{PH}$, where PH is the union of all classes in the Polynomial Hierarchy, which implies

$$\mathbf{E} = \mathbf{\Sigma}_2^E = \mathbf{EH}$$

by a padding argument similar to [1, Section 2.6.2].

To see that $\mathbf{E} = \mathbf{\Sigma}_2^{\boldsymbol{E}}$ implies ($*$), let t be the function defined exactly like T, but with the input and output integers written in binary, and let $g_t = \{(x, y) : y \leq t(x)\}$ be its hypograph. Then $g_t \in \Sigma_2^E$. Indeed, let x, y be binary strings that represents non-negative integers. It follows from Definition 3 that (x, y) satisfies $y \leq t(x)$ if and only if there exists an input w of length $\leq x$, such that:

(2.1) every accepting computation of M on w has time $\geq y$ (equivalently, no accepting computation of M on w has time $< y$); and

(2.2) there exists an accepting computation of M on w (automatically with time $\geq y$).

(Indeed: If (x,y) satisfies this, then w is accepted, but only in time $\geq y$; hence $y \leq t(x)$.) And if $y \leq t(x)$ then the Definition 3 implies that there exists w of length $\leq x$, such that (2.1) and (2.2) hold.)

Here, the non-deterministic Turing machine M, with time function $T(n) \equiv n^\alpha$. Writing n in binary as x, we have $T(n) = t(x)$.

Property (2.1) is a $\forall$ condition, which can be verified co-nondeterministically in time

$$O(x + y) = O\big(2^{|x|} + 2^{|y|}\big).$$

Property (2.2) is an $\exists$ condition, which can be verified nondeterministically in time

$$O(t(x)) \leq 2^{[\alpha]|x|}$$

(where $[\alpha]$ is the smallest integer upper-bound on α).

So, if $\mathbf{E} = \mathbf{\Sigma}_2^{\boldsymbol{E}}$, then $g_t \in \mathbf{E}$. Then we can compute t (as a function) deterministically in exponential time using binary search, and therefore we can compute $T(n)$ in time polynomial in n.

More precisely, finding $t(x)$ from x, using g_t, by binary search is done as follows.

Let $[y_0, y_1]$ be the current estimation interval for $t(x)$; initially, $[y_0, y_1] = [0, x^a]$ where $a = \lceil \alpha \rceil + 1$. Loop invariant: $(x, y_0) \in g_t$, and $(x, y_1) \notin g_t$ (the loop invariant holds during the loop, but not at the end of the loop). While $y_0 < y_1$: if

$$\big(x, \lfloor (y_1 + y_0)/2 \rfloor\big) \in g_t$$

then:

$$y_0 := \lfloor (y_1 + y_0)/2 \rfloor,$$

else

$$y_1 := \lfloor (y_1 + y_0)/2 \rfloor$$

return y_0.

The search uses $\leq \log(x^a)$ steps, each of which uses a membership test in g_t. We have $\log(x^a) = a2^{|x|}$, and membership in g_t is in $\mathbf{E}$. As a function of n (the number represented by x in binary), the search takes polynomial time.

Now let Turing machine M' with, say, k tapes and the function $T'(n)$ as in ($*$) be given. Since $\alpha \geq 2$, for any $n > 0$

$$\epsilon_1 n^\alpha \leq T'(n) \leq \epsilon_2 n^\alpha \qquad (1)$$

for some positive constants $\epsilon_1 \leq 1$ and $\epsilon_2 \geq 1$. Let number n_0 be such that

$$2^n > \log_2(\epsilon_2/\epsilon_1) \tag{2}$$

for every $n \geq n_0$.

Let $q = \lceil \alpha \rceil + 1$.

Consider the following Turing machine M'' which will calculate the first m digits of α (for every m). This Turing machine has $k+3$ tapes with tape $k+3$ being the input tape. It starts with number m in binary written on tape $k+3$ and all other tapes empty. Then it calculates the number

$$n = 2^{2^{m+n_0}} \tag{3}$$

and writes it on tape $k+1$ (using tape $k+2$ as an auxiliary tape and cleaning it after n is computed). Then M'' turns on the machine M' and produces $T'(n)$ on tape $k+2$. Then it calculates

$$p = \left[(\log_2 T'(n) + \log_2 \epsilon_1)/2^{n_0}\right]$$

and writes it on tape $k+2$. By (1),

$$\alpha \log_2 n + \log_2 \epsilon_1 \leq \log_2 T'(n) \leq \alpha \log_2 n + \log_2 \epsilon_2.$$

Therefore, by (3)

$$[\alpha 2^m] \leq p \leq \alpha 2^m + \frac{\log_2(\epsilon_2/\epsilon_1)}{2^{n_0}}. \tag{4}$$

Hence $p = [\alpha 2^m]$ because the second summand on the right hand side of inequality (4) is a positive number less than 1 by (2), so $p/2^m$ is a rational approximation of α which is within $1/2^m$ from α. From the construction of M'', it is clear that the time complexity of M'' does not exceed 2^{c2^m} for some constant c. $\square$

Emil Jeřábek's argument above shows that Property ($*$) follows also from the property $\mathbf{E} = \mathbf{\Sigma}_2^{\boldsymbol{E}}$. Thus in Theorem 2, one can replace $\mathbf{P} = \mathbf{NP}$ by a formally weaker $\mathbf{E} = \mathbf{\Sigma}_2^{\boldsymbol{E}}$.

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References

[1] S. Arora and B. Barak, *Computational complexity. A modern approach*, Cambridge University Press, Cambridge, 2009. Zbl 1193.68112 MR 2500087

[2] G. Baumslag, C. F. Miller III, and H. Short, Isoperimetric inequalities and the homology of groups, *Invent. Math.*, **113** (1993), no. 3, 531–560. Zbl 0829.20053 MR 1231836

[3] N. Brady and M. Bridson, There is only one gap in the isoperimetric spectrum, *Geom. Funct. Anal.*, **10** (2000), no. 5, 1053–1070. Zbl 0971.20019 MR 1800063

[4] N. Brady, M. R. Bridson, M. Forester, and K. Shankar, Snowflake groups, Perron–Frobenius eigenvalues and isoperimetric spectra, *Geom. Topol.*, **13** (2009), no. 1, 141–187. Zbl 1228.20031 MR 2469516

[5] N. Brady and M. Forester, Snowflake geometry in CAT(0) groups, *J. Topol.*, **10** (2017), no. 4, 883–920. Zbl 06827928 MR 3705143

[6] M. Bridson, Fractional isoperimetric inequalities and subgroup distortion, *J. Amer. Math. Soc.*, **12** (1999), no. 4, 1103–1118. Zbl 0963.20018 MR 1678924

[7] J. Hartmanis, N. Immerman, and V. Sewelson, Sparse sets in NP-P: EXPTIME versus NEXPTIME, *Inform. and Control*, **65** (1985), no. 2-3, 158–181. Zbl 0586.68042 MR 818849

[8] L. Hemaspaandra and M. Ogihara, *The complexity theory companion*, Texts in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin, 2002. Zbl 0993.68042 MR 1932447

[9] Mathoverflow, question 307526.

[10] Mathoverflow, question 307629.

[11] A. Yu. Olshanskii, Polynomially-bounded Dehn functions of groups, *J. Comb. Algebra*, **2** (2018), no. 4, 311–433.

[12] M. V. Sapir, Algorithmic and asymptotic properties of groups, in *International Congress of Mathematicians. Vol. II*, 223–244, Eur. Math. Soc., Zürich, 2006. Zbl 1110.20033 MR 2275595

[13] M. V. Sapir, Asymptotic invariants, complexity of groups and related problems, *Bull. Math. Sci.*, **1** (2011), no. 2, 277–364. Zbl 1293.20041 MR 2901003

[14] M. V. Sapir, J. C. Birget, and E. Rips, Isoperimetric and isodiametric functions of groups, *Ann. of Math. (2)*, **156** (2002), no. 2, 345–466. Zbl 1026.20018 MR 1933723

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