Polynomially-bounded Dehn functions of groups

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Abstract. It is well known that every subqadratic Dehn function is linear. A question by Bridson asked to describe the isoperimetric spectrum of groups, that is the set of all numbers α such that n^α is equivalent to the Dehn function of a finitely presented group. The goal of this paper is to give a description of the isoperimetric spectrum. Earlier a similar description was given by Sapir, Birget and Rips for the intersection of the isoperimetric spectrum with $[4, \infty]$. Lowering the bound from 4 to 2 required significant new ideas and tools.

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1. Introduction

1.1. Formulation of the theorem and corollaries.   The minimal non-decreasing function $d\colon \mathbb{N} \to \mathbb{N}$ such that every word w vanishing in a group $G = \langle A \mid R \rangle$ and having length $\|w\| \le n$ is freely equal to a product of at most $d(n)$ conjugates of relators from $R^{\pm 1}$, is called the *Dehn function* of the presentation $G = \langle A \mid R \rangle$ [8]. In other words, the the value $d(n)$ is the smallest integer that bounds from above the areas of loops of length $\le n$ in the Cayley complex $\mathrm{Cay}(G)$, and so by van Kampen's Lemma, $d(n)$ is equal to the maximal area of minimal filling diagrams Δ with perimeter $\le n$. (See Subsection 5.2 for the definitions.)

The values $d(n)$ are defined if the set of generators A is finite. For a *finitely presented* group (i.e. both sets A and R are finite), the Dehn function exists and it is usually taken up to equivalence to get rid of the dependence on a finite presentation of G (see [12]). To introduce this *equivalence* $\sim$, let f and g be non-decreasing functions $\mathbb{N} \to \mathbb{R}_+$. We write $f \preceq g$ if there is a positive integer c such that $f(n) \le cg(cn) + cn$ for every $n \in \mathbb{N}$. Two non-decreasing functions f and g on $\mathbb{N}$ are called equivalent if $f \preceq g$ and $g \preceq f$.

Note that for many functions (for example, for n^α, $n^\alpha (\log n)^\beta (\log \log n)^\gamma$, and so on), their $\sim$-equivalence classes coincide with their Θ-*equivalence* classes, where the symbol Θ is borrowed from the theory of computational complexity: one says that $f(n) = \Theta(g(n))$ if both properties $f(n) = O(g(n))$ and $g(n) = O(f(n))$ hold.

The Dehn function $d = d_G$ of a finitely presented group G is also called an isoperimetric function of G since it is equivalent to the usual isoperimetric function of a simply connected Riemannian manifold M, provided G acts properly and co-compactly on M by isometries. So the concept of Dehn function is derived from geometry, and one can find much more regarding this connection in [9].

Another connection is to Computational Complexity. The algorithmic word problem in a finitely presented group is decidable if and only if the Dehn function is recursive, and the Dehn function of a group bounds the computational complexity of the word problem. It was shown in [2] that conversely, every recursively presented group G with decidable word problem embeds into a finitely presented group whose Dehn function is only polynomially larger than the computational complexity (the time function) of the word problem in G. In particular, groups with word problem in **NP** are precisely the subgroups of finitely presented groups G with at most polynomial Dehn functions d_G.

For every positive integer α, there are (nilpotent) groups with Dehn function n^α [1]. The first examples of Dehn functions n^α with non-integer α can be found in [6], where the description of possible exponents α, forming the *isoperimetric spectrum*, was called the most fundamental question concerning isoperimetric functions. (Obviously some conditions on the real exponent α are inevitable since the set of real numbers is uncountable, while the set of non-isomorphic finitely presented groups is countable.)

Almost all possible Dehn functions $F(n) \geq n^4$ of finitely presented groups were described in [22] in terms of time functions of non-deterministic Turing machines. By Theorem 1.2 of [22], to obtain a group with Dehn function F, it suffices to assume that the function F is super-additive (i.e. $F(m + n) \geq F(m) + F(n)$ for $m, n \in \mathbb{N}$) and the integral part of $\sqrt[4]{F(n)}$ is a time function of a non-deterministic Turing machine (see the definition in [7]). As a corollary, it was proved in [22] that if $\alpha \geq 4$ and the real number α is computable in time $\leq 2^{2^m}$, then there is a finitely presented group with Dehn function equivalent to n^α (One should use the integral part sign for functions on $\mathbb{N}$, but we omit this sign speaking on asymptotic behavior.). The computability can be defined as follows.

Definition 1.1. Let $T \colon \mathbb{N} \to \mathbb{N}$. A real number α *is computable in time* $\leq T(m)$ if there exists a Turing machine which, given a natural number m, computes a binary rational approximation of α with an error $O(2^{-m})$, and the time of this computation $\leq T(m)$.

The algebraic and many transcendental numbers are computable much faster, and so there are examples of groups with Dehn functions equivalent to $n^{\pi+e}$, and so on. If, conversely, for a real α, the function n^α is equivalent to the Dehn function of a finitely presented group, then for some c, the exponent α is computable in time $2^{2^{2^{cm}}}$ [22].

Still the class of Dehn functions $< n^4$ was unclear even though it has drawn attention (see, for example, [4–6]). We note the paper [4] (also the references there), where the Dehn functions equivalent to n^α were constructed for special non-integer exponents of the form $\alpha = 2\log_2(2p/q)$, where p and q are integers, $p > q > 0$.

Since all finitely presented groups with subquadratic Dehn functions are hyperbolic [3, 8, 14], i.e. their Dehn functions are in fact linear, the only interval $2 < \alpha < 4$ remained misty. To formulate a theorem that fills this gap, we need:

Definition 1.2. We say that a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ is *suitable* if the following properties hold.

- $f(n)^3 = O(n)$.

- For every integer $c > 0$, there is $C > 0$ such that $f(cn) \leq Cf(n)$, i.e.

$$f(O(n)) = O(f(n)).$$

- There is a (non-deterministic) Turing machine $\mathbf{M_0}$ recognizing the values of the function $f(n)$ with computation time $O(n^{1/3})$.

  It works as follows. An integer $k \geq 0$ is an input of $\mathbf{M_0}$ in the form a^k for a fixed letter a. The Turing machine $\mathbf{M_0}$ produces a value $f(n)$ for some $n \geq 1$, i.e. it obtains the word $c^{f(n)}$ on a special tape. Then $\mathbf{M_0}$ compares $f(n)$ and k, accepting k if $k = f(n)$. It can accept the input word a^k if and only if $k = f(n)$ for some natural number n.

Theorem 1.3. *For every suitable function $f(n)$ and every integer $s \geq 2$, the function $F(n) = n^s f(n)^3$ is equivalent to the Dehn function of a finitely presented group.*

Given a suitable function $f(n)$, we denote $g(n) = f(n)^3$.

Corollary 1.4. *If $\alpha \geq 2$ and the real number α is computable in time $O(2^{2^m})$, then there is a finitely presented group with Dehn function equivalent to n^α.*

Remark 1.5. It is easy to see that we have an equivalent statement when replacing 2 with any integer $d > 1$ both in the statement of Corollary 1.4 and in Definition 1.1 (resp., "binary" with "d-ary"). The formulation and the proof of Corollary 1.4 are close to those for Corollary 1.4 of [22] (up to a minor inaccuracy in the formulation of Corollary 1.4 [22]). However below we give a proof of our Corollary 1.4 since one should draw it from different assumptions of Theorem 1.3.

Proof of Corollary 1.4. Assume that α is computable in time $O(2^{2^m})$. Clearly, the same property holds for the number $\beta = \frac{1}{3}(\alpha - s)$, where $s = [\alpha]$.

Let M_0 be a (non-deterministic) Turing machine producing $r = [n^{1/3}] \geq 1$ (in unary) with time $O(r)$. Then it computes $[\log_2 r]$ and $m = [\log_2 \log_2 r]$ (in binary) with time $O(r)$ using divisions by 2.

It follows from the assumption of the corollary that one can recursively compute binary rational numbers β_m such that

$$|\beta - \beta_m| = O(2^{-m}) = O\big((\log_2 r)^{-1}\big) \tag{1.1}$$

and the time of the computation of β_m is $O(r)$. Let $\mathbf{M}_0$ accomplish this computation. In addition, one may assume that the number of digits in the binary expansion of β_m is $O(m)$. Therefore the computation of the product $[\beta_m[\log_2 r]]$ needs time at most $O((\log_2 n)^2)$. Since $r = [n^{1/3}]$, the Turing machine can now obtain the binary presentation γ_m of $[\beta_m[\log_2 n]]$ with time $O((\log_2 n))$ and error $O(1)$.

Next, let $\mathbf{M}_0$ rewrite the binary presentation of γ_m in unary (as a sequence of 1-s). This well-known rewriting (e.g. see p. 352 in [22]) has time complexity of the form $O(\gamma_m) = O(\beta_m \log_2 n)$. One more $\mathbf{M}_0$-rewriting of this type applied to the unary presentation of γ_m (considered now as binary one), will have time complexity of the form

$$\Theta(2^{\gamma_m}) = \Theta\big(2^{[\beta_m[\log_2 n]]+O(1)}\big) = \Theta\big(2^{[\beta_m[\log_2 n]]}\big).$$

One can rewrite as

$$\Theta\big(2^{\beta \log_2 n}\big) = \Theta(n^\beta) = O(n^{1/3})$$

by inequalities (1.1) and $\beta < 1/3$, because $\log_2 n = O(2^m)$.

During the last (deterministic) rewriting, one can count the number $f(n)$ of commands and obtain a word $b^{f(n)}$ on a special tape, where $f(n) = \Theta(n^\beta)$. It is easy to see that the rewriting can be defined so that the function $f(n)$ is non-decreasing.

One more tape of the Turing machine $\mathbf{M}_0$ under construction has the input word a^k. It remains to check whether the lengths of the words $b^{f(n)}$ and a^k are equal or not. This takes the time $O(f(n)) = O(n^{1/3})$. Since the time of the entire procedure is $O(n^{1/3})$, the function $f(n)$ is suitable. Now by Theorem 1.3, the function $n^s f(n)^3 = \Theta(n^s n^{3\beta}) = \Theta(n^\alpha)$ is equivalent to the Dehn function of a finitely presented group. $\square$

In particular, Corollary 1.4 implies:

Corollary 1.6. *The functions n^α for every real algebraic $\alpha \geq 2$, the functions $n^{\pi-1}$, $n^{\sqrt{e}+1}, \ldots$ are equivalent to Dehn functions of finitely presented groups.* $\square$

As we mentioned above, the analog of Corollary 1.4 for $\alpha \geq 4$ was proved in [22]. But weakening the restriction to $\alpha \geq 2$ now, although uses S-machines, as in [22], it requires substantially new ideas.

Theorem 1.3 gives a tremendous class of new Dehn functions of the form $O(n^4)$. The following examples can be validated in absolutely similar way as Corollary 1.4.

Corollary 1.7. *The functions $n^\alpha(\log n)^\beta$, $n^\alpha(\log n)^\beta(\log\log n)^\gamma, \ldots$ are equivalent to the Dehn functions of finitely presented groups, provided the real $\alpha, \beta, \gamma, \ldots$ are computable in time $O(2^{2^m})$ and $\alpha > 2$ or $\alpha = 2$ and $\beta > 0$, or $\alpha = 2$, $\beta = 0$, $\gamma > 0, \ldots$.* $\square$

Since every finitely presented group is a fundamental group of a connected closed Riemannian manifold X and therefore acts properly and co-compactly by isometries on its universal cover $\widetilde{X}$, one can use Theorem 1.3 and Corollary 1.4 to formulate one more

Corollary 1.8. *For every function $F(n)$ satisfying the assumption of Theorem 1.3, there exists a closed connected Riemannian manifold X such that the isoperimetric function of the universal cover $\widetilde{X}$ is equivalent to $F(n)$.*

In particular, if a real number $\alpha \geq 2$ is computable in time $O(2^{2^m})$, then there exists such a universal cover $\widetilde{X}$ with isoperimetric function equivalent to n^α. □

Since the condition $\alpha \geq 2$ is the best possible and the obtained upper bound for the Dehn function must be equivalent to the lower one, all inequalities throughout this paper should be uniformly sharp, up to multiplicative constants.

We collect all the definitions and terms at the end of the paper (see Subject index). The next subsection presents a short outline of the plan.

1.2. Brief description of the proof of Theorem 1.3. The idea of simulation of the commands of a Turing machine by group relations goes back to the works of P. Novikov, W. Boone and many other authors (see [20, 21]). However one has to properly code the work of a Turing machine in terms of group relations, and the interpretation problem for groups remains much harder than for semigroups, because the group theoretic simulation can execute unforeseen computations with non-positive words. Boone and Novikov secured the positiveness of admissible configurations with the help of an additional "quadratic letter" (see [20, Ch. 12]). However this old trick implies that the constructed group G contains Baumslag–Solitar subgroups $B_{1,2}$ and has at least exponential Dehn function. Since we want to obtain at most polynomial Dehn functions, we use the S-machines introduced in [22]. Those S-machines invented by M. Sapir can work with non-positive words on the tapes and they are polynomially equivalent to classical Turing machines.

According to the original version, S-machines are special rewriting systems. All necessary definitions are given in Subsection 2.1. On the other hand, the state, tape and command letters of an S-machine can be regarded as group generators, and the commands can be interpreted as defining relations (see Subsection 5.1). The obtained group M is a multiple HNN-extension of a free group. Every computation of the S-machine is simulated by van Kampen diagram over this group called trapezia (Subsection 5.2).

To construct a finitely presented groups G with desired Dehn functions, one needs to add to M a special relation called the *hub*. It consists of state letters. There are very particular van Kampen diagrams, called disks, built of the hub and many trapezia attached around.

It is proved in [22] that for every Turing machine $\mathbf{M_0}$ with time function $T(n)$, there is an equivalent S-machine $\mathbf{M_1}$ with time function (and the generalized time

function) $\sim T(n)^3$. It follows that if an accepting computation starts with an input of length $\sim n$, it has length $\sim T(n)^3$, and the computational disk has perimeter $\sim n$, and so its area $\sim nT(n)^3$. Since time functions are at least linear, this approach gave the lower bound $\geq n^4$ for the Dehn function of G.

To get the linear time of accepting computation, in comparison with the length of the initial configuration for an S-machine $\mathbf{M}_2$, one can add an additional tape, where the whole history of the forthcoming computation is written. Then every command will erase one letter on this tape. This trick gives disks with quadratic area with respect to their perimeters.

However we want to construct disks with prescribed area $F(n)$, as in Theorem 1.3. In this paper, we first prove this theorem for $s = 2$, i.e. for $F(n) = O(n^3)$, and in the final Subsection 10.2, we show that the value of s can be increased, since a non-difficult modification of the main S-machine $\mathbf{M}$ constructed in Subsection 4.1 linearly slows down the work of $\mathbf{M}$.

The main S-machine is composed of the S-machine $\mathbf{M}_2$ repeating the same computation many times and another S-machine that can stop the computations of $\mathbf{M}_2$ after $\sim g(n)$ such cycles with subsequent erasure of all the tapes and acceptance. This gives the lower bound $\sim ng(n)$ for the (general) time function and the lower bound $\sim F(n)$ for the areas of computational disks.

The obtainment of the upper bounds is the major job in this paper. First of all, to obtain quadratic upper bound for the areas of trapezia, one needs a linear bound of the space of every computation (i.e. the maximal lengths of all admissible words of it) in terms of the lengths of the first and the last word. This task is aggravated by inaccurate simulation of the work of Turing machines by S-machines and so by group relations. Standard trapezia correspond to the prescribed work of S-machines, but there are non-standard ones simulating undesired computations when the same tape is simultaneously changed at both ends. The features of standard (accurate) computations of the main S-machine and of non-standard ones are considered in Subsections 4.2 and 4.3, respectively. To reduce the effect of non-standard computations, the basic steps of the work alternate with control steps in the definition of the main S-machine given in Subsection 4.1.

Whereas a non-standard computation has linearly bounded space in terms of the lengths of the first and the last words (and so the width of corresponding trapezia is linearly bounded too), there exist much wider trapezia in the standard case. Hence standard trapezia can have too large areas in the group M. The new idea is decrease the area of their boundary labels in the quotient group G. We do this in Subsection 10.1 applying the properties of long computations obtained in Subsection 4.5. It turned out that wide trapezia with super-quadratic areas in M can be replaced, preserving the boundary label, by diagrams of quadratic areas over G, i.e. by diagrams containing hubs.

However before that, we prescribe artificial (but quadratic!) G-areas to special "big" subtrapezia (Subsection 6.3), which leads to the definition of G-area for an

arbitrary diagram over M or over G. A quadratic upper bound for the G-areas of diagrams over M is given in Subsection 6.3. It turns out later, that the are diagrams of quadratic area over G with the same boundary labels. The induction on the perimeter is based on a non-trivial surgery. (See the proof of Lemma 6.16 in Subsection 6.3, where we cut and paste diagrams.) In fact, we give an upper bound G-for area in terms of the perimeter and the mixture of q- and θ-letters in the boundary label, because different types of surgeries decrease either perimeter or the mixture. The mixture is defined for arbitrary necklace with beads of two colors (see Subsection 6.2), and it is bounded by the square of the number of beads, and so we finally obtain a quadratic upper bound in terms of perimeter only.

Note that instead of the combinatorial length of words (length of paths, perimeter) we consider a modified length, where different letters and syllabi have different lengths (Subsection 6.1). With respect to this modified length $|\cdot|$, the length of the top/bottom of every q-band Q is just the number of 2-cells in Q, and the rim θ-bands with bounded number of (θ, q)-cells can be removed from a diagram with decrease of perimeter. Such properties are exploited in the paper many times. It is easy to reformulate the final results in terms of the combinatorial length $\|\cdot\|$ since it is Θ-equivalent to $|\cdot|$.

Our presentation of the group G (Subsection 5.1) is highly non-aspherical, and so van Kampen diagrams with the same boundary label can differ widely. We choose minimal diagrams in Section 7, i.e. reduced diagrams with minimal number of disks and, for given number of disks, with minimal number of (θ, q)-cells. We do not claim that a minimal diagram has minimal area or minimal G-area for fixed boundary label, but to obtain the upper bound for the Dehn function, it suffices to bound from above the G-areas of minimal diagrams.

However, even one has quadratic estimates for G-areas of disk-free diagrams (i.e. diagrams over M) and the required upper bound $\sim F(n)$ for the areas of disks (see the definition in Section 7), the "snowman" decomposition of diagrams in the union of subdiagrams with single disk defined in [22], would give at least cubic upper bound for the G-area of the entire minimal diagram. Thus, without new tools one could only hope to weaken the restriction from [22] to $\alpha \geq 3$.

The helpful property is that a minimal diagram cannot contain subdiagram formed by a disk and a very special trapezium connected by a "shaft" (Section 7). At first sight, it seems that such subdiagrams are extremely rare. But they become ordinary if the work of an S-machine has sufficiently many control steps (Steps 1^-–5^- in Subsection 4.1). Hence the absence of these exotic subdiagrams can help. And it helps indeed provided the sum σ of the lengths of all "shafts" (see Definition 7.7) linearly bounded in terms of the perimeter.

To obtain such a linear estimate, in Section 8, we introduce designs formed by two finite sets of segments and prove a pure combinatorial proposition. So there are neither machines nor groups, nor van Kampen diagrams in Section 8, and the reader can start with that short section. (Is there even shorter proof or a reference

to a known property? Although the author did believe that the linear estimate took place, he wasted time devising a proof.)

Since $\sigma \leq cn$ for some constant $c > 0$, where n is the perimeter, one can estimate the G-area of a diagram in terms of the sum $n + \sigma$ instead of n (Section 9), and this is another new tool for obtaining the required upper bound for the area.

2. General properties of S-machines

2.1. S-machines as rewriting systems.   There are several interpretations of S-machines in groups. In particular, one can define an S-machine as a group that is a multiple HNN extension of a free group. Here we slightly modify the original definition [22] using [19] and define S-machines as rewriting systems working with words in group alphabets. The precise definition of an S-machine $\mathscr{S}$ is as follows.

The hardware of an S-machine $\mathscr{S}$ is a pair (Y, Q), where $Q = \sqcup_{i=0}^{N} Q_i$ and $Y = \sqcup_{i=1}^{N} Y_i$ (for convenience we always set $Y_0 = Y_{N+1} = \emptyset$). The elements from Q are called *state letters*, the elements from Y are *tape letters*. The sets Q_i (resp., Y_i) are called *parts* of Q (resp., Y).

The *language of admissible words* consists of all reduced words W of the form

$$q_1^{\pm 1} u_1 q_2^{\pm 1} \ldots u_k q_{k+1}^{\pm 1}, \tag{2.1}$$

where every subword $q_i^{\pm 1} u_i q_{i+1}^{\pm 1}$ either:

- belongs to $(Q_j F(Y_{j+1}) Q_{j+1})^{\pm 1}$ for some j and $u_i \in F(Y_{j+1})$, where $F(Y_i)$ is the set of reduced group words in the alphabet $Y_i^{\pm 1}$; or
- has the form quq^{-1} for some $q \in Q_j$ and $u \in F(Y_{j+1})$; or
- is of the form $q^{-1}uq$ for $q \in Q_j$ and $u \in F(Y_j)$.

(The second and third items extend the definition of admissible words in comparison with [22], and the language of admissible words is equal to the language from [19].)

We shall follow the tradition of calling state letters *q-letters* and tape letters *a-letters*, even though we shall sometimes use letters different from q and a as state or tape letters. The number of a-letters in a word W is the *a-length* $|W|_a$ of W. Usually parts of the set Q of state letters are denoted by capital letters. They may differ from Q_i for some S-machines. For example, a set P would consist of letters p with various indices. Then we shall say that letters in P are p-letters or P-letters.) The length of a word W, i.e. the number of all letters in W, is denoted by $\|W\|$.

If a group word W over $Q \cup Y$ has the form $u_0 q_1 u_1 q_2 u_2 \ldots q_s u_s$, and $q_i \in Q_{j(i)}^{\pm 1}$, $i = 1, \ldots, s$, u_i are group words in Y, then we shall say that the *base* of W is $\text{base}(W) \equiv Q_{j(1)}^{\pm 1} Q_{j(2)}^{\pm 1} \ldots Q_{j(s)}^{\pm 1}$ Here Q_i are just letters, denoting the parts of the set of state letters.   Note that the base is not necessarily a reduced word, and the sign $\equiv$ is used for letter-by-letter equality of words. The subword of W between

the $Q_{j(i)}^{\pm 1}$-letter and the $Q_{j(i+1)}^{\pm 1}$-letter will be called a $Q_{j(i)}^{\pm 1} Q_{j(i+1)}^{\pm 1}$-*sector* of W. A word can have many $Q_{j(i)}^{\pm 1} Q_{j(i+1)}^{\pm 1}$-sectors.

Instead of specifying the names of the parts of Q and their order as in

$$Q = Q_0 \sqcup Q_2 \sqcup \cdots \sqcup Q_N,$$

we say that the *standard* base of the S-machine is $Q_0 \ldots Q_N$. An admissible word with standard base is called a *configuration* of the S-machine.

An S-machine also has a set of rewriting *rules* Θ. To every $\theta \in \Theta$, two sequences of reduced words are assigned: $[U_1, \ldots, U_m]$, $[V_1, \ldots, V_m]$, and a subset $Y(\theta) = \cup Y_j(\theta)$ of Y, where $Y_j(\theta) \subseteq Y_j$. A rule has the form:

$$\left[U_1 \to V_1, \ldots, U_m \to V_m\right],$$

where the following conditions hold:

- Each of U_i and V_i is a subword of an admissible word, both U_i and V_i have base $Q_\ell Q_{\ell+1} \ldots Q_r$ ($\ell = \ell(i) \le r = r(i)$) and have a-letters from $Y(\theta)$.
- $\ell(i + 1) = r(i) + 1$ for $i = 1, \ldots, m - 1$.
- U_1 and V_1 must start with a Q_0-letter and U_m and V_m must end with a Q_N-letter.

The pair of words U_i, V_i is called a *part* of the rule, and is denoted $[U_i \to V_i]$.

The notation $\theta \colon [U_1 \to V_1, \ldots, U_m \to V_m]$ contains all the necessary information about the rule except for the sets $Y_j(\theta)$. In most cases it will be clear what these sets are, and very often the sets $Y_j(\theta)$ will be equal to either Y_j or $\emptyset$. By default $Y_j(\theta) = Y_j$.

Every S-rule

$$\theta = \left[U_1 \to V_1, \ldots, U_m \to V_m\right]$$

has an inverse

$$\theta^{-1} = \left[V_1 \to U_1, \ldots, V_m \to U_m\right],$$

which is also a rule of $\mathcal{S}$; we set $Y_i(\theta^{-1}) = Y_i(\theta)$. We always divide the set of rules Θ of an S-machine into two disjoint parts, Θ^+ and Θ^- such that for every $\theta \in \Theta^+$, $\theta^{-1} \in \Theta^-$ and for every $\theta \in \Theta^-$, $\theta^{-1} \in \Theta^+$ (in particular $\Theta^{-1} = \Theta$, that is any S-machine is symmetric). The rules from Θ^+ (resp., Θ^-) are called *positive* (resp., *negative*). In particular, $[U_1 \to U_1; \ldots; U_m \to U_m]$ is never an S-rule. It is always the case that $Y_i(\theta^{-1}) = Y_i(\theta)$ for every i.

For every word $U_i \equiv u_0 q_l u_1 q_{l+1} \ldots q_r u_{r-l+1}$ from the definition of the rule θ, we denote by $\bar{U}_i$ its trimmed form $q_l u_1 q_{l+1} \ldots q_r$ starting and ending with state letters. To apply an S-rule θ to an admissible word W (2.1) means:

- to check if all tape letters of W belong to the alphabet $Y(\theta)$ and every state letter of W is contained in some subword $\bar{U}_i^{\pm 1}$ of W;

- if W satisfies this condition, then to replace simultaneously every subword

$$\bar{U}_i^{\pm 1} \equiv (q_l u_1 q_{l+1} \ldots q_r)^{\pm 1}$$

by subword

$$(u_0^{-1} V_i u_{r-l+1}^{-1})^{\pm 1} \equiv (u_0^{-1} v_0 q_1' v_1 \ldots q_r' v_{r-l+1} u_{r-l+1}^{-1})^{\pm 1} \quad (i = 1, \ldots, m);$$

- to trim a few first and last a-letters (to obtain an admissible word starting and ending with q-letters) followed by the reduction of the resulted word.

The following convention is important in the definition of S-machine:

After every application of a rewriting rule, the word is automatically reduced. The reduction is not regarded as a separate step of an S-machine.

For example, applying the rule $\theta \colon [q_1 \to aq_1'b^{-1}, q_2 \to cq_2'd]$ to the admissible word $W \equiv q_1 bq_2 dq_2^{-1} q_1^{-1}$ one first obtains the word

$$aq_1'b^{-1}bcq_2'd d d^{-1}(q_2')^{-1}c^{-1}b(q_1')^{-1}a^{-1},$$

then after trimming and reduction one has $q_1'cq_2'd(q_2')^{-1}c^{-1}b(q_1')^{-1}$. But the rule θ would not be applicable to W if $Y_2(\theta) = \emptyset$ or $Y_2(\theta) = \{a'\}$, where $a' \neq b$.

If a rule θ is applicable to an admissible word W (i.e. W belongs to the domain of θ) then the word W is called θ-*admissible*, and $W \cdot \theta$ denotes the word obtained after the application of θ.

A *computation* of *length* or *time* $t \geq 0$ is a sequence of admissible words $W_0 \to \cdots \to W_t$ such that for every $i = 0, \ldots, t-1$ the S-machine passes from W_i to W_{i+1} by applying one of the rules θ_i from Θ. The word $H = \theta_1 \ldots \theta_t$ is called the *history* of the computation. Since W_t is determined by W_0 and the history H, we use the notation $W_t = W_0 \cdot H$.

A computation is called *reduced* if its history is a reduced word. Clearly, every computation can be made reduced (without changing the initial and final words of the computation) by removing consecutive mutually inverse rules.

An S-machine is called *recognizing* if it has the following attributes. There are admissible words with the standard base called *input configurations* and *accept (stop) configuration*. There are *input sectors* (at least one) and other sectors are empty for input configurations, and all sectors are empty for the accept one. (However in this paper, some S-machines have no input or accept configurations.) The state letters of the input (of accept) configuration form a special vector $\vec{s}_1$ (vector $\vec{s}_0$) whose letters are involved in one rule only and are completely changed by this rule.

A configuration W is said to be *accepted* by an S-machine M if there exists at least one computation, called *accepting computation*, which starts with W and ends with the accept configuration.

Assume that M is an S-machine, and there is (only one) accept configuration. Then the a-length $|W|_a$ of an input configuration W is the number of tape letters

in W. If the configuration W is accepted, denote by $T(W)$ the minimal time of computations accepting it. Then the time function $T(n) = T_M(n)$ is defined as $\max\{T(W)\}$ over all accepted input configurations W with $|W|_a \leq n$.

The *generalized time function* $T'(n)$ is defined for every S-machine having a unique accept configuration. The definition is similar to the above definition of time function but one should consider all accepted configurations W, not just input ones. Therefore $T(n) \leq T'(n)$. (Presumably, n, in the definition of $T'(n)$, corresponds to the a-length of the accepted configurations.)

Time functions and generalized time functions are taken up to Θ-equivalence.

2.2. Simplifying the rules of S-machines. We say that two recognizing S-machines are equivalent if they have the same language of acceptable words and Θ-equivalent time functions. Next lemma simplifies the rules of S-machine. In particular, one needs Property (1) to define trapezia (Definition 5.5).

Lemma 2.1. *Every S-machine $\mathcal{S}$ is equivalent to an S-machine $\mathcal{S}'$, where*

(1) *Every part $U_i \to V_i$ of every rule θ has 1-letter base:*

$$U_i \equiv v_i q_i u_{i+1}, \quad V_i \equiv v_i' q_i' u_{i+1}',$$

   where q_i, q_i' are state letters in Q_i.

(2) *In every part $v_i q_i u_{i+1} \to v_i' q_i' u_{i+1}'$, we have that*

$$\|v_i\| + \|v_i'\| \leq 1 \quad and \quad \|u_{i+1}\| + \|u_{i+1}'\| \leq 1.$$

(3) *Moreover, one can construct $\mathcal{S}'$ so that for every rule we have*

$$\sum_i \left(\|v_i\| + \|v_i'\| + \|u_i\| + \|u_i'\| \right) \leq 1.$$

Proof. (1) Property (1) can be obtained after adding auxiliary state letters and splittings the rules of $\mathcal{S}$. Assume, for example, that the part $U_1 \to V_1$ has 2-letter base: $q_1 a q_2 \to q_1' a' q_2'$. Then we introduce auxiliary state letters $q_j(1), q_j(2)$ $(j = 0, \dots, N)$ and replace the rule θ by the product of three rules θ_1, θ_2 and θ_3, where

(θ_1) For θ_1 and $j > 1$, we replace base letters q in V_j by their $q(1)$-copies and obtain the parts $U_j \to V_j(1)$, while the part $U_1 \to V_1$ is replaced by the two parts $q_1 a \to q_1(1), q_2 \to q_2(1)$;

(θ_2) For θ_2 and $j > 1$, we have now the parts $V_j(1) \to V_j(2)$, where $V_j(2)$ is a copy of $V_j(1)$ after replacement $q(1) \to q(2)$, while the part $U_1 \to V_1$ of θ is replaced by two parts $q_1(1) \to q_1(2)$ with $Y_2(\theta_2) = \emptyset$ and $q_2(1) \to q_2(2)$;

(θ_3) For θ_3 and $j > 1$, we have $V_j(2) \to V_j$, while the first part splits now as $q_1(2) \to q_1' a'$ and $q_2(2) \to q_2'$.

The key feature of the new S-machine $\widetilde{\mathcal{S}}$ is in the following obvious property:

There is a one-to-one correspondence between computations $w_0 \rightarrow \cdots \rightarrow w_t$ of $\widetilde{\mathcal{S}}$ (with any base) such that w_0, w_t do not have auxiliary q-letters and computations of $\mathcal{S}$ connecting the same words. For every history H of such computation of $\mathcal{S}$, the corresponding history of computation of $\widetilde{S}$ is obtained from H by replacing every occurrence of the rule $\theta^{\pm 1}$ by the 3-letter word $(\theta_1 \theta_2 \theta_3)^{\pm 1}$.

Clearly, by applying this transformation to an S-machine $\mathcal{S}$ several times, we obtain an equivalent S-machine satisfying Property (1).

(2) Suppose Property (2) is not satisfied for a part $U_i \rightarrow V_i$. For example, suppose a rule θ of an S-machine $\mathcal{S}$ has the ith part of the form $a v_i q_i u_{i+1} \rightarrow v_i' q_i' u_{i+1}'$, where $u_{i+1}, v_i, u_{i+1}', v_i'$ are words in the appropriate parts of the alphabet of a-letters, v_i is not empty, a is an a-letter, q_i, q_i' are q-letters (a very similar procedure can be done in all other cases).

We want to replace θ with two rules with smaller sums of lengths of their parts. For this aim, we create a new S-machine $\widetilde{\mathcal{S}}$ with the same standard base and the same a-letters as $\mathcal{S}$. In order to build $\widetilde{\mathcal{S}}$, we add one new (*auxiliary*) q-letter $\widetilde{q}_i$ to each part of the set of q-letters, and replace the rule θ by two rules θ' and θ''. The first rule θ' is obtained from θ by replacing the part $v_i a q_i u_{i+1} \rightarrow v_i' q_i' u_{i+1}'$ by $a q_i u_{i+1} \rightarrow \widetilde{q}_i u_{i+1}'$, and all other parts $U_j \rightarrow V_j$ by $U_j \rightarrow \widetilde{q}_j$ (here $\widetilde{q}_j$ is the auxiliary q-letter in the corresponding part of the set of q-letters). The second rule θ'' is obtained from θ by replacing the part $v_i a q_i u_{i+1} \rightarrow v_i' q_i' u_{i+1}'$ by $v_i \widetilde{q}_i \rightarrow v_i' q_i'$, and all other parts $U_j \rightarrow V_j$ by $\widetilde{q}_j \rightarrow V_j$.

Note that the sum of lengths of words in all parts of θ' (resp., θ'') in $\widetilde{\mathcal{S}}$ is smaller than the similar sum for θ. Therefore, applying this transformation to an S-machine $\mathcal{S}$ several times, we obtain an equivalent S-machine satisfying conditions (1) and (2).

(3) Similarly, one can obtain Property (3). $\qquad \square$

If $Y_{i+1}(\theta) = \emptyset$ for an S-machine with Property (1), then the corresponding component $U_i \rightarrow V_i$ will be denoted $U_i \xrightarrow{\ell} V_i$ and we shall say that the rule θ *locks* the $Q_i Q_{i+1}$-sectors. In that case we always assume that U_i, V_i do not have tape letters to the right of the state letters, i.e. it has the form $v_i q_i \xrightarrow{\ell} v_i' q_i'$. Similarly, these words have no tape letters to the left of the state letters if the $Q_{i-1} Q_i$-sector is locked by the rule.

Remark 2.2. The definitions of an admissible word and a rule application given in Subsection 2.1 coincide with the the definitions from [22] in case of standard base. However computations do not change the base. So to obtain the statement of Lemma 2.4, we may use the main property of computations with standard base obtained in [22].

S-machines resemble multi-tape Turing machines (or algorithms). (The main difference is that a Turing machine does not deal with negative letters.) We do not give an accurate definition of Turing machines here since from now on we will not use them in this paper (see, for example [7] or [22] for the definition). However, it is important that the S-machine $S(M)$ constructed in [22] simulates the work of a Turing machine M with time function $T(n)$ as follows. (See [22, Lemma 3.1 and Proposition 4.1], though we use simpler notation below.)

Let M have one input sector YZ, and the input configurations have the form $W \equiv yvz\ldots$, where v is a positive word in an alphabet A. Then there is an S-machine $S(M)$ with input configurations of the form

$$\sigma(W) = y_1\alpha^n z_1 \ldots xy_2 vz_2 \ldots y_3\delta^n z_3 \ldots y_4\omega^n z_4,$$

where $n = \|v\|$ (so $S(M)$ has four input sectors with tape words α^n, v, δ^n and ω^n, respectively); the S-machine $S(M)$ has time function $\Theta(T(n)^3)$, and it accepts the configuration $\sigma(W)$ if and only if the configuration W is accepted by M. Moreover, the S-machine $S(M)$ can be constructed so that for every configuration $W \in \mathcal{L}$ accepted by M with time $T(W)$, the S-machine $S(M)$ accepts this word with time $\Theta(T(W)^3)$.

Remark 2.3. The "moreover" part is not formulated in [22] explicitly, but it follows from Proposition 4.1.3 (b) since every Turing machine can be easily modified so that the length of every accepting computation is Θ-equivalent to the space of this computation.

In the present article, we will assume that the basic S-machine $\mathbf{M}_1$ has only one input sector. Therefore, $\mathbf{M}_1$ has to have a few more rules in comparison with $S(M)$. The input configurations of $\mathbf{M}_1$ have the form

$$\bar{\sigma}(W) = \bar{y}_1\bar{z}_1 \ldots \bar{x}\bar{v}\bar{y}_2\bar{z}_2 \ldots \bar{y}_3\bar{z}_3 \ldots \bar{y}_4\bar{z}_4,$$

where $\bar{v}$ is a word in an alphabet $\overline{A}$, which copies the alphabet A (so the only input word is $\bar{v}$). The following rules of $\mathbf{M}_1$ are added to the rules of $S(M)$.

For every (positive) letter $a \in A$, there is a rule

$$\rho_a: \left[\bar{y}_1 \rightarrow \bar{y}_1\alpha, \ \bar{y}_2 \rightarrow \bar{a}^{-1}\bar{y}_2 a, \ \bar{y}_3 \rightarrow \bar{y}_3\delta, \ \bar{y}_4 \rightarrow \bar{y}_4\omega \right],$$

where $\bar{a}$ is a copy of a in the alphabet $\bar{a}$ and all other sectors are locked by ρ_a. They also are locked by the connecting rule

$$\rho: \left[\bar{y}_1 \rightarrow y_1, \ldots \bar{x} \xrightarrow{\ell} x, \bar{y}_2 \rightarrow y_2, \ldots \bar{z}_4 \rightarrow z_4 \right],$$

which switches on the S-machine $S(M)$. If an input word $\sigma(W)$ is accepted by $S(M)$, then $\bar{\sigma}(W)$ is accepted by $\mathbf{M}_1$, where $\bar{v}$ is a copy of v. Indeed, if $\bar{a}$ is the last letter of the word $\bar{v}$, then the application of the rule ρ_a moves the state letter $\bar{y}_2$ leftward and

replaces $\bar{a}$ by a from the right of $\bar{y}_2$. After n rules of this type, one can obtain the word v between y_2 and z_2. These rules will also insert α^n, δ^n, and ω^n in the sectors $Y_1 Z_1$, $Y_3 Z_3$ and $Y_4 Z_4$, respectively. So it remains to apply the rule ρ to obtain the configuration $\sigma(W)$ accepted by $S(M)$. Also for the times of computations, we see that $T_{\mathbf{M}_1}(\bar{\sigma}(W)) = O(T_{S(M)}(\sigma(W)))$.

Conversely, assume that the configuration $\bar{\sigma}(W)$ is accepted by $\mathbf{M}_1$. Then the history of this accepting reduced computation has to be of the form

$$H \equiv H_0 \rho H_1 \rho^{-1} H_2 \rho \dots H_{2s-1},$$

where $H_0, H_2, \dots$ contain $\rho_a^{\pm 1}$-rules only and $H_1, H_3, \dots$ are histories of $S(M)$. However, H cannot have subwords of the form $\rho^{-1} H_i \rho$. Indeed, if here H_i is a reduced word $\rho_{a_1}^{\pm 1} \dots \rho_{a_t}^{\pm 1}$, then the tape word u' in the XY_2 sector at the end of the computation $\mathcal{C}_i$ with history H_i is obtained from the word u written there in the beginning of $\mathcal{C}_i$, after free multiplication from the right by the reduced word $\bar{a}_1^{\mp 1} \dots \bar{a}_t^{\mp 1}$. But both u and u' are empty since the rule ρ locks the XY_2-sector. Hence $t = 0$ and so H_i is empty, a contradiction.

Thus, $H \equiv H_0 \rho H_1$, and the XY_2-sector of the word $\bar{\sigma}(W) \cdot H_0$ is empty being locked by ρ. Hence the history H_0 has to be $\rho_{a_t} \dots \rho_{a_1}$ if $v \equiv a_1 \dots a_t$, and the tape word in the $Y_2 Z_2$ sector of $\bar{\sigma}(W) \cdot H_0$ has to be v while the tape words in the sectors $Y_1 Z_1$, $Y_3 Z_3$ and $Y_4 Z_4$ become α^n, δ^n and ω^n as it follows from the definition of ρ_a-rules. Hence $\bar{\sigma}(W) \cdot H_0 \rho \equiv \sigma(W)$, and so $\sigma(W)$ is accepted my $S(M)$ and $T_{\mathbf{M}_1}(\bar{\sigma}(W)) > T_{S(M)}(\sigma(W))$.

It follows that the S-machine $\mathbf{M}_1$ with one input sector enjoys the properties of $S(M)$ from [22]:

Lemma 2.4. *Let $\mathbf{M}_0$ be a non-deterministic Turing machine accepting the language $\mathcal{L}$ with a time function $T(n)$. Then there is an S-machine $\mathbf{M}_1$ with a single input sector accepting the language $\mathcal{L}$ with time function Θ-equivalent to $T(n)^3$.*

Moreover, the S-machine $\mathbf{M}_1$ can be constructed so that for every word $W \in \mathcal{L}$ accepted by $\mathbf{M}_0$ with time $T(W)$, the S-machine $\mathbf{M}_1$ accepts this word with time $\Theta(T(W)^3)$.

Remark 2.5. Later we will assume that the Turing machine $\mathbf{M}_0$ recognizes the values of some suitable function according to Definition 1.2.

2.3. Some elementary properties of S-machines.
The base of an admissible word is not always a reduced word. However the following is an immediate corollary of the definition of admissible word.

Lemma 2.6 ([16, Lemma 3.4]). *If the ith component of the rule θ has the form*

$$v_i q_i \xrightarrow{\ell} v_i' q_i',$$

i.e. $Y_{i+1}(\theta) = \emptyset$, then the base of any θ-admissible word cannot have subwords $Q_i Q_i^{-1}$ or $Q_{i+1}^{-1} Q_{i+1}$. $\square$

In this paper we often use copies of words. If A is an alphabet and W is a word involving no letters from $A^{\pm 1}$, then to obtain a *copy* of W in the alphabet A we substitute letters from A for letters in W so that different letters from A substitute for different letters. Note that if U' and V' are copies of U and V respectively corresponding to the same substitution, and $U' \equiv V'$, then $U \equiv V$.

Lemma 2.7. *Suppose that the base of an admissible word W is $Q_i Q_{i+1}$. Suppose that each rule of a reduced computation starting with $W \equiv q_i u q_{i+1}$ and ending with $W' \equiv q_i' u' q_{i+1}'$ multiplies the $Q_i Q_{i+1}$-sector by a letter on the left (resp., right). And suppose that different rules multiply that sector by different letters. Then:*

(a) *the history of computation is a copy of the reduced form of the word $u' u^{-1}$ read from right to left (resp., of the word $u^{-1} u'$ read from left to right). In particular, if $u \equiv u'$, then the computation is empty;*

(b) *the length of the history H of the computation does not exceed $\|u\| + \|u'\|$;*

(c) *for every admissible word $q_i'' u'' q_{i+1}''$ of the computation, we have*

$$\|u''\| \le \max \big(\|u\|, \|u'\| \big).$$

Proof. (a) Part (a) is obvious.

(b) To prove part (b), we choose a word W_i of the computation with shortest tape word u_i. This factorizes the history as $H \equiv H_1 H_2$, where H_2 is the history of the subcomputation $W_i \to W_{i+1} \to \cdots \to W'$. It follows that

$$\|W_{i+1}\| = \|W_i\| + 1.$$

The next rule increases the length of admissible word again since the computation is reduced and different rules multiply the sector by different letters, i.e. $\|W_{i+2}\| = \|W_{i+1}\| + 1$. By induction, we have

$$\|u'\| \ge \|u_i\| + \|H_2\| \ge \|H_2\|.$$

To obtain the inequality $\|u\| \ge \|H_1\|$, we consider the inverse computation $W' \to \cdots \to W$. Hence

$$\|H\| = \|H_1\| + \|H_2\| \le \|u\| + \|u'\|.$$

(c) The same argument proves Statement (c) since the length of u'' is either between $\|u_i\|$ and $\|u'\|$ or between $\|u_i\|$ and $\|u\|$. $\square$

Lemma 2.8. *Suppose the base of an admissible word W is $Q_i Q_{i+1}$. Assume that each rule of a reduced computation starting with $W \equiv q_i u q_{i+1}$ and ending with $W' \equiv q_i' u' q_{i+1}'$ multiplies the $Q_i Q_{i+1}$-sector by a letter on the left and by a letter from the right. Suppose different rules multiply that sector by different letters and the left and right letters are taken from disjoint alphabets. Then:*

(a) *for every intermediate admissible word W_j of the computation, we have*

$$\|W_j\| \leq \max\big(\|W\|, \|W'\|\big);$$

(b) *the length of the history H of the computation does not exceed $\frac{1}{2}(\|u\| + \|u'\|)$.*

Proof. (a) If we choose the word W_i of minimal length, then after multiplications of the form $u_i \rightarrow u_{i+1} = au_i b$ we have no cancellation from the left or from the right. If we have the former option, then we will have no cancellation from the left after the transition $u_{i+1} \rightarrow u_{i+2}$, and therefore

$$\|u_{i+1}\| \leq \|u_{i+2}\| \leq \cdots \leq \|u'\|.$$

Hence $\|u_j\| \leq \|u'\|$ if $j \geq i$. Analogously, $\|u_j\| \leq \|u\|$ if $j \leq i$.

(b) The word u' results from u after multiplication from the left and from the right by reduced words of length $\|H\|$: i.e. u' is freely equal to AuB, where $\|A\| = \|B\| = \|H\|$. There can be cancellations in the products Au and uB but afterwards there are no cancellations since the words A and B are written in disjoint alphabets. Hence the reduced length of u' is at least

$$\|A\| + \|B\| - \|u\| = 2\|H\| - \|u\|,$$

whence $2\|H\| \leq \|u\| + \|u'\|$, as required. $\qquad\square$

The following lemma is proved in [16, Lemma 3.7].

Lemma 2.9. *Suppose the base of an admissible word W is $Q_i Q_i^{-1}$ (resp., $Q_i^{-1} Q_i$). Suppose each rule θ of a reduced computation $\mathcal{C}$ starting with $W \equiv q_i u q_i^{-1}$ (resp., $q_i^{-1} u q_i$), where $u \neq 1$, and ending with $W' \equiv q_i' u' (q_i')^{-1}$ (resp., $(q_i')^{-1} u' q_i'$) has a part $q_i \rightarrow a_\theta q_i' b_\theta$, where b_θ (resp., a_θ) is a letter, and for different θ-s the b_θ-s (resp., a_θ-s) are different. Then the history of the computation has the form $H_1 H_2^k H_3$, where $k \geq 0$,*

$$\|H_2\| \leq \min\big(\|u\|, \|u'\|\big), \quad \|H_1\| \leq \|u\|/2, \quad and \quad \|H_3\| \leq \|u'\|/2. \qquad\square$$

Lemma 2.10. *Under the assumptions of Lemma 2.9, we have*

$$|W_i|_a \leq \max\big(\|u\|, \|u'\|\big)$$

for every admissible word W_i of the computation $\mathcal{C}$.

Proof. It suffices to repeat the argument from the proof of Lemma 2.7 (c). $\qquad\square$

3. Auxiliary S-machines and constructions

3.1. Primitive S-machines. Here we define a very simple S-machine **Pr**, which has neither input nor accept configurations. As a part of other S-machines, it will be used to read the tape words and to recognize a computation by its history and also to check the order of state letters in the bases of computations.

The standard base of **Pr** has three letters $Q^1 P Q^2$, where $Q^1 = \{q^1\}$, $P = \{p^1, p^2\}$, and $Q^2 = \{q^2\}$. The alphabet Y is $Y^1 \sqcup Y^2$, where Y^2 is a copy of Y^1. The positive rules of **Pr** are defined as follows.

- $\zeta^1(a) = [q^1 \to q^1, p^1 \to a^{-1}p^1 a', q^2 \to q^2]$, where a is a positive letter from Y^1 and a' is its copy from Y^2.

  Comment. The state letter p^1 moves left replacing letters a from Y^1 by their copies a' from Y^2.

- $\zeta^{12} = [q^1 \xrightarrow{\ell} q^1, p^1 \to p^2, q^2 \to q^2]$.

  Comment. When p^1 meets q^1, it turns into p^2.

- $\zeta^2(a) = [q^1 \to q^1, p^2 \to ap^2(a')^{-1}, q^2 \to q^2]$.

  Comment. The state letter p^2 moves right towards q^2 replacing letters a' from Y^2 by their copies a from Y^1.

Lemma 3.1. *Let* $\mathcal{C}\colon W_0 \to \cdots \to W_t$ *be a reduced computation of the S-machine* **Pr** *with the standard base and with $t \geq 1$. Then:*

(1) *if $|W_i|_a > |W_{i-1}|_a$ for some $i = 1, \ldots, t-1$, then $|W_{i+1}|_a > |W_i|_a$;*

(2) *$|W_i|_a \leq \max(|W_0|_a, |W_t|_a)$ for every $i = 0, 1, \ldots, t$;*

(3) *if $W_0 \equiv q^1 u p^1 q^2$ and $W_t \equiv q^1 v p^2 q^2$ for some words u, v, then $u \equiv v$, $|W_i|_a = |W_0|_a$ for every $i = 0, \ldots, t$, $t = 2k+1$, where $k = |W_0|_a$, and p^1 (resp., p^2) meets q^1 in W_k (in W_{k+1}) and the sector $Q^1 P$ is empty in W_k and in W_{k+1}; moreover, the history H of $\mathcal{C}$ is uniquely determined by W_0 (by W_t), provided W_0 and W_t have the form $q^1 u p^1 q^2$ and $q^1 v p^2 q^2$; vice versa, the history H uniquely determines words u and v under this assumption;*

(4) *it is not possible that $W_0 \equiv q^1 u p^1 q^2$ and $W_t \equiv q^1 v p^1 q^2$ for some u, v, and it is not possible that $W_0 \equiv q^1 u p^2 q^2$ and $W_t \equiv q^1 v p^2 q^2$;*

(5) *if $W_0 \equiv q^1 u p^1 q^2$ or $W_0 \equiv q^1 p^1 u q^2$, or $W_0 \equiv q^1 u p^2 q^2$, or $W_0 \equiv q^1 p^2 u q^2$ for some word u, then $|W_i|_a \geq |W_0|_a$ for every $i = 0, \ldots, t$.*

Proof. Note that each of the rules $(\zeta^j)^{\pm 1}(a)$ ($j = 1, 2$) either moves the state letter left or moves it right, or deletes one letter from left and one letter from right, or insert letters from both sides. In the later case, the next rule of a computation must be again $\zeta(j)^{\pm 1}(b)$ for some b, and if the computation is reduced, it again must increase the length of the configuration by two. Therefore, Statement (1) is true and (2)

is also true since one can choose a shortest W_j and consider the subcomputation $W_j \to \cdots \to W_t$ and the inverse subcomputation $W_j \to \cdots \to W_0$.

Since ζ^{12} locks $Q^1 P$-sector, the p-letter must reach q_1 moving always left to change p^1 by p^2, and so $W_k \equiv q^1 p^1 \ldots$. The next rule of the form $\zeta^1(a)^{\pm 1}$ could increase the length of the configuration, which would imply that all consecutive rules have to have the same type and p^1 would never been replaced by p^2, a contradiction. Hence the next rule is ζ^{12}, and arguing in this way, one uniquely reconstructs the whole computation in case (3) for given W_0 or W_t, and vice versa, the history H determines both u and v. Property (4) holds for same reasons.

Note that no rule of **Pr** changes the projection of a word onto the free group with basis Y_1 if the state letter are mapped to 1 and the letters from Y_2 are mapped to their copies from Y_1. Since the word u is mapped to itself, we have $|q^1 u p^1 q^2|_a = \|u\| \le |W_i|_a$. The other cases of (5) are similar. $\qquad\square$

Remark 3.2. Similar tricks will later be referred to as *projection arguments*.

Lemma 3.3. *If* $W_0 \to \cdots \to W_t$ *is a reduced computation of* **Pr** *with base* $Q^1 P P^{-1}(Q^1)^{-1}$ *or* $(Q^2)^{-1} P^{-1} P Q^2$ *and* $W_0 \equiv q^i p^i u (p^i)^{-1}(q^i)^{-1}$ $(i = 1, 2)$ *or* $W_0 \equiv (q^i)^{-1}(p^i)^{-1} v p^i q^i$ $(i = 1, 2)$ *for some words* u, v, *then* $|W_j|_a \ge |W_0|_a$ *for every* $j = 0, \ldots, t$.

Proof. The statement follows from the projection argument (Remark 3.2). $\qquad\square$

Remark 3.4. Also we will use the right analog **Pr***** of **Pr**: now p^1 should move right, meet q^2 locking $P Q^2$-sector and turning into p^2, and move back towards Q^1.

Remark 3.5. Assume that a standard base has two (or more) subwords of the form $Q^1 P Q^2$, for example $Q^1 P Q^2 P' Q^3$. Then one can define *parallel* or *sequential* composition of two primitive S-machines.

For the parallel composition, the same rule changes both subwords with bases $Q^1 P Q^2$ and $Q^2 P' Q^3$. One assumes that the tape alphabet of the sector $Q^2 P'$ (of $P' Q^3$) is a copy of the tape alphabet of the sector $Q^1 P$ (of $P Q^2$, resp.) and the rules of **Pr** change simultaneously the subwords with bases $Q^1 P Q^2$ and $Q^2 P' Q^3$ (e.g. simultaneously moves left both p^1 and $(p')^1$, and so on). Every rule (e.g. $\zeta^1(a)$) is applicable to a word iff it is applicable to both these subwords.

In case of sequential work, we have a primitive S-machine working with one of these two bases, say with $Q^1 P Q^2$, while the sector $P' Q^3$ is locked with the state letter $(p')^1$. The second primitive S-machine can compute with base $Q^2 P' Q^3$ when the sector $P Q^2$ is locked (and so the first S-machine stays idle) with state p^2. For this goal, one needs a connecting rule ζ^{21}; it locks the sector $P Q^2$ and changes the state P-letters for new ones to switch off the first primitive S-machine and to switch on the second one.

It is clear that in the same way one can define a more complex compositions **P** of primitive S-machines, with several stages of parallel and sequential work. We will consider compositions **P**, where every sector can be changed at one stage only.

Lemma 3.6. *Let us have a composition* **P** *of primitive S-machines with parallel or/and sequential work, and* $\mathcal{C}\colon W_0 \to \cdots \to W_t$ *be a reduced computation of* **P** *with standard base. Then:*

(a) $|W_j|_a \leq \max(|W_0|_a, |W_t|_a)$ *for every configuration* W_j *of* $\mathcal{C}$; *moreover,*

$$|W_0|_a \leq \cdots \leq |W_t|_a$$

if every P-letter neighbors some Q-letter in the word W_0;

(b) $t \leq \|W_0\| + \|W_t\| - 4$; *moreover,*

$$t \leq 2\|W_t\| - 4$$

if every P-letter neighbors some Q-letter in the word W_0.

Proof. (a) Let W_r be a shortest word of the computation. Then either

$$|W_r|_a = |W_{r+1}|_a = \cdots = |W_t|_a \quad \text{or} \quad |W_r|_a = |W_{r+1}|_a = \cdots = |W_s| < |W_{s+1}|_a$$

for some s. It follows that the number of sectors increasing their lengths by two at the transition $W_s \to W_{s+1}$ is greater than the number of the sectors decreasing the lengths by 2. Now it follows from Lemma 3.1 (1) that the same primitive S-machine will continue increasing the lengths of the whole configurations, i.e.

$$|W_{s+1}|_a < |W_{s+2}|_a < \cdots.$$

So for every $j \geq r$, we have $|W_j|_a \leq |W_t|_a$. Similarly, we have $|W_r|_a \leq |W_0|_a$ for $j \leq r$. Under the additional assumption about P-letters, W_0 is the shortest configuration by the projection argument.

(b) If the rules of **P** do not change the lengths of configurations, then every control letter runs back and forward only one time, and the inequality follows. (One takes into account that the base has length at least 3.) If $\|W_r\| < \|W_{r+1}\|$ for some r, then every next transition keeps increasing the length by Lemma 3.1 (1), and so the inequality holds as well. $\qquad\square$

3.2. S-machine with historical sectors. To control the space of computations, we endow a given S-machine with historical sectors. Let us assume that an S-machine $\mathbf{M}_1$ satisfies the conditions of Lemma 2.1 and has hardware (Q, Y), where $Q = \sqcup_{i=0}^m Q_i$, and the set of rules Θ. The new S-machine $\mathbf{M}_2$ has hardware

$$Q_{0,r} \sqcup Q_{1,\ell} \sqcup Q_{1,r} \sqcup Q_{2,\ell} \sqcup Q_{2,r} \sqcup \cdots \sqcup Q_{m,\ell}, \quad Y_h = Y_1 \sqcup X_1 \sqcup Y_2 \sqcup \cdots \sqcup X_{m-1} \sqcup Y_m,$$

where $Q_{i,\ell}$ and $Q_{i,r}$ are (left and right) copies of Q_i, X_i is a disjoint union of two copies of Θ^+, namely $X_{i,\ell}$ and $X_{i,r}$. (There is neither $Q_{0,\ell}$, nor $Q_{m,r}$, nor X_0, nor X_m.) The positive rules of $\mathbf{M}_2$ are in one-to-one correspondence with the positive rules of $\mathbf{M}_1$.

If $\theta = [U_0 \to V_0, \ldots, U_m \to V_m]$ is a positive rule of $\mathbf{M}_1$ with parts $U_i \to V_i$ of the form $v_i q_i u_{i+1} \to v_i' q_i' u_{i+1}'$, then the corresponding two parts of the rule θ_h are $U_{i,\ell} \to V_{i,\ell}$ and $U_{i,r} \to V_{i,r}$, with

$$U_{i,\ell} \equiv v_i q_{i,\ell} a_{\theta,i}, V_{i,\ell} \equiv v_i' q_{i,\ell}' \quad \text{and} \quad U_{i,r} \equiv q_{i,r} u_{i+1}, V_{i,r} \equiv b_{\theta,i} q_{i,r}' u_{i+1}',$$

where $a_{\theta,i}$ (resp., $b_{\theta,i}$) is the copy of θ in $X_{i,\ell}$ (in $X_{i,r}$). We also claim that a sector $Q_{i,r} Q_{i+1,\ell}$ is locked by θ_h if and only if the sector $Q_i Q_{i+1}$ is locked by θ ($i = 1, \ldots, m-1$).

Comment. Every computation of the S-machine $\mathbf{M}_2$ with history H coincides with the computation of $\mathbf{M}_1$ if one observes it only in *working* sectors $Q_{i,r} Q_{i+1,l}$. In the standard base, the working sectors of $\mathbf{M}_2$ alternate with *historical* sectors $Q_{i,\ell} Q_{i,r}$. Every positive rule θ_h multiplies the content of the historical sector $Q_{i,\ell} Q_{i,r}$ by the corresponding letter $b_{\theta,i}$ from the right and by letter $a_{\theta,i}^{-1}$ from the left.

Remark 3.7. The state letters of the S-machine $\mathbf{M}_1$ split when passing to $\mathbf{M}_2$. There is a rule θ_h corresponding to the start (to the accept) rule θ of $\mathbf{M}_1$. By definition, *the set of letters $Y_h(\theta_h)$ has no letters from the right alphabets $X_{i,r}$ (from the left alphabets $X_{\ell,i}$) if θ is the start (resp., the stop) rule of $\mathbf{M}_1$.*

However we do not define input/stop configurations of $\mathbf{M}_2$ since the historical sectors are never locked. By definition, every $Q_{i-1,r} Q_{i,l}$ is the working sector of $\mathbf{M}_2$. The input sector of $\mathbf{M}_2$ is the working sector corresponding to the input sector of $\mathbf{M}_1$.

Remark 3.8. It follows from the definition of $\mathbf{M}_2$ that only Properties (1) and (2) of Lemma 2.1 hold for $\mathbf{M}_2$, but not Property (3).

The sectors of the form $Q_{i,\ell} Q_{i,\ell}^{-1}$ and $Q_{i,r}^{-1} Q_{i,r}$ (in a non-standard base) are also called *historical*. Historical sectors help to give a linear estimate of the space of every computation $W_0 \to \cdots \to W_t$ in terms of $\|W_0\|$ and $\|W_t\|$.

Lemma 3.9. *Let $W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{M}_2$ with base $Q_{i,\ell} Q_{i,r}$ and history H. Assume that the a-letters of W_0 belong to one of the alphabets $X_{i,\ell}$, $X_{i,r}$. Then $\|H\| \leq |W_t|_a$ and $|W_0|_a \leq |W_t|_a$.*

Proof. Let $W_0 \equiv q v_0 q'$ and assume that v_0 has no letters from $X_{i,r}$. Then $W_t \equiv q'' v_t q'''$ with $v_t = u v_0 u'$, where u is a copy of H^{-1} in the alphabet $X_{i,\ell}$ and u' is a copy of H in $X_{i,r}$. So no letter of u' is cancelled in the product $u v_0 u'$, Therefore, $|W_t|_a \geq \|u'\| = \|H\|$ and $|W_t|_a \geq |W_0|_a$ □

Lemma 3.10. *If the base of an admissible word of the S-machine $\mathbf{M}_2$ has length at least 3, then it contains a historical sector.*

Proof. The base contains a subword of the form $Q' Q'' Q'''$ with three letters from $Q^{\pm 1}$. It follows from the definition of admissible word that either $Q' Q''$-sector or $Q'' Q'''$-sector is historical since every non-historical sector of the S-machine $\mathbf{M}_2$ has to have neighbor historical sectors.   □

Lemma 3.11. *Let a reduced computation $W_0 \to W_1 \to \cdots \to W_t$ of the S-machine $\mathbf{M}_2$ have 2-letter base and the history of the form $H \equiv H_1 H_2^k H_3$ ($k \geq 0$). Then for each tape word w_i between two state letters of W_i ($i = 0, 1, \ldots, t$), we have the inequality*

$$\|w_i\| \leq \|w_0\| + \|w_t\| + 2h_1 + 3h_2 + 2h_3,$$

where $h_j = \|H_j\|$ ($j = 1, 2, 3$).

Proof. By Lemma 2.1 (2) and the definition of $\mathbf{M}_2$, we have

$$|\|w_i\| - \|w_{i-1}\|| \leq 2$$

for every $i = 1, \ldots, t$. Therefore for $i \leq h_1$, we have $\|w_i\| \leq \|w_0\| + 2h_1$. Similarly, $\|w_i\| \leq \|w_t\| + 2h_3$ for $i \geq t - h_3$. It remains to assume that $h_1 < i < t - h_3$.

Denote by W_j the words w_i with $i = h_1 + jh_2$, $j = 0, 1, \ldots, k$. If

$$W_1 = W_0 \cdot H_2 = uW_0v$$

for some words u and v depending on H_2 and on the sector, then

$$W_2 = uW_1v = u^2W_0v^2$$

in free group, since the histories of the computations $W_0 \to \cdots \to W_1$ and $W_1 \to \cdots \to W_2$ are both equal to H_2. Hence $W_j = u^j W_0 v^j$, where both u and v have length at most h_2 by Lemma 2.1 (2) and the definition of $\mathbf{M}_2$.

By Lemma 8.1 from [18], the length of an arbitrary word W_j is not greater than

$$\|u\| + \|v\| + \|W_0\| + \|W_k\|,$$

provided $0 \leq j \leq k$.

If $|i - jh_2| \leq h_2/2$ for some j, then $|\|w_i\| - \|W_j\|| \leq h_2$, and therefore

$$\|w_i\| \leq \|u\| + \|v\| + \|W_0\| + \|W_k\| + h_2.$$

Since $\|W_0\| \leq \|w_0\| + 2h_1$ and $\|W_k\| \leq \|w_t\| + 2h_3$, we obtain

$$\|w_i\| \leq \|u\| + \|v\| + \|w_0\| + \|w_t\| + 2h_1 + 2h_3 + h_2 \leq \|w_0\| + \|w_t\| + 2h_1 + 2h_3 + 3h_2$$

for every i, as required. $\qquad\qquad\square$

Lemma 3.12. *For any reduced computation $W_0 \to \cdots \to W_t$ of $\mathbf{M}_2$ with base of length at least 3, we have*

$$|W_i|_a \leq 9\big(|W_0|_a + |W_t|_a\big) \quad (0 \leq i \leq t).$$

Proof. Let $Q_{i_1}^{\pm 1} \ldots Q_{i_m}^{\pm 1}$ be the base of the computation. There are computations with the same history H and bases $Q_{i_1}^{\pm 1} \ldots Q_{i_{m_1}}^{\pm 1}, Q_{i_{m_1}}^{\pm 1} \ldots Q_{i_{m_2}}^{\pm 1}, \ldots, Q_{i_{m_{s-1}}}^{\pm 1} \ldots Q_{i_{m_s}}^{\pm 1}$, where each base has length 3 or 4. Hence it suffices to prove the lemma for any computation with base of the form $Q'Q''Q'''$ or $Q'Q''Q'''Q^{iv}$. By Lemma 3.10, every such computation contains a historical sector, say $Q''Q'''$. Consider two cases:

1. The historical sector has the form $Q_{i,\ell}Q_{i,r}$. By Lemma 2.8, we have

$$\|H\| \leq \tfrac{1}{2}\big(|W_0|_a + |W_t|_a\big).$$

It follows from Lemma 2.1 (2) that $\big||W_{i+1}|_a - |W_i|_a\big| \leq 6$ for every neighbor admissible words. Therefore,

$$\begin{aligned}|W_i|_a &\leq \max\big(|W_0|_a, |W_t|_a\big) + 3\|H\| \\ &\leq \max\big(|W_0|_a, |W_t|_a\big) + \tfrac{3}{2}\big(|W_0|_a + |W_t|_a\big) \leq \tfrac{5}{2}\big(|W_0|_a + |W_t|_a\big).\end{aligned}$$

2. The historical sector has form $Q_{i,\ell}Q_{i,\ell}^{-1}$ or $Q_{i,r}^{-1}Q_{i,r}$. Then one can apply Lemma 2.9 to the sector $Q''Q'''$ and obtain the factorization $H \equiv H_1 H_2^k H_3$, with $k \geq 0$,

$$h_2 \leq \min\big(\|u\|, \|u'\|\big), \quad h_1 \leq \|u\|/2, \quad \text{and} \quad h_3 \leq \|u'\|/2,$$

where u and u' are the a-words of W_0 and W_t, respectively in the historical sector and $h_j = \|H_j\|$ for $j = 1, 2, 3$. Since every W_i has at most three sectors, applying Lemma 3.11 to each of them, we obtain:

$$\begin{aligned}|W_i|_a &\leq |W_0|_a + |W_t|_a + 3(2h_1 + 3h_2 + 2h_3) \\ &\leq |W_0|_a + |W_t|_a + 3|W_0|_a + 9\min\big(|W_0|_a, |W_t|_a\big) + 3|W_t| \\ &\leq 9\big(|W_0|_a + |W_t|_a\big). \qquad \qquad \square\end{aligned}$$

3.3. Division S-machine. Here we start with an S-machine $\mathbf{D}_1$. This S-machine has two input sectors with words a^k and b^ℓ and checks whether $2k$ divides ℓ or not.

The standard base of $\mathbf{D}_1$ is $S(1)S(2)T(1)T(2)$. The first input sector $S(1)S(2)$ has one-letter alphabet $\{a\}$, the second input sector $T(1)T(2)$ has alphabet $\{b\}$. Also we have one-letter alphabet $\{a'\}$ for the sector $S(2)T(1)$. We omit some parts of the rules in the list below if these parts do not change configurations (e.g. $s \to s$ for $s \in S(1)$ is a part of τ_1).

- $\tau_1\colon [s_1 \to a^{-1}s_1a'], \ [t_1 \to t_1b^{-1}], \ s_1 \in S(2), t_1 \in T(1)$.

  Comment. The state letter s_1 moves left changing letter a by its copy a', while t_1 erases one letter b.

- $\tau_{12}\colon [s \xrightarrow{\ell} s, s_1 \to s_2], \ s \in S(1), s_2 \in S(2)$.

  Comment. The rule τ_{12} locks the sector $S(1)S(2)$ and replaces the state letter s_1 by s_2.

- τ_2: $[s_2 \to as_2(a')^{-1}]$, $[t_1 \to t_1b^{-1}]$.

  Comment. s_2 moves right toward $T(1)$ replacing a' by a, while t_1 erases the letter b.

- τ_{21}: $[s_2 \overset{\ell}{\to} s_1]$.

  Comment. This rule locks the sector $S(2)T(1)$ and replaces s_2 by s_1.

- τ_3: $[s_1 \overset{\ell}{\to} t_1]$, $[t_1 \overset{\ell}{\to} t_2]$.

  Comment. The state t_2 can appear if both sectors $S(2)T(1)$ and $T(1)T(2)$ are empty.

We call a transition $W \to W'$ given by the rule $\tau_1^{\pm 1}$ or $\tau_2^{\pm 1}$ *wrong* if it increases the lengths of both sectors $S(1)S(2)$ and $S(2)T(1)$.

Lemma 3.13. *Let $W_0 \to \cdots \to W_t$ be a reduced computation of the S-machine $\mathbf{D}_1$ with standard base and the first transition $W_0 \to W_1$ is wrong. Then all subsequent transitions are wrong too.*

Proof. Let the first rule θ be $\tau_1^{\pm 1}$. Then the first transition (restricted to the sectors $S(1)S(2)$ and $S(2)T(1)$) has the form $sus_1vt_1 \to sua^{\mp 1}s_1(a')^{\pm 1}vt_1$, where the words $u, v, ua^{\mp 1}, (a')^{\pm 1}v$ are reduced. It follows that the only possible rule for the next transition is θ again, W_2 contains reduced form $sua^{\mp 2}s_1(a')^{\pm 2}vt$, and so on. The case $\theta = \tau_2^{\pm 1}$ is similar. $\qquad\square$

Lemma 3.14. (1) *Suppose we have a reduced computation $W_0 \to \cdots \to W_r$ of $\mathbf{D}_1$, where $W_0 \equiv sa^ks_1t_1b^\ell t'$, where $t' \in T(2)$, and $W_r \equiv \ldots t_it'$ ($i = 1, 2$, i.e. the sector $T(1)T(2)$ is empty). Then the exponent ℓ is divisible by $2k$.*

(2) *Conversely, if $2k$ divides ℓ, then there is a computation*

$$sa^ks_1t_1b^\ell t' \to \cdots \to sa^kt_2t'$$

of length $|\ell| + |\ell/k| + 1$ for $k \neq 0$ and of length 1 for $k = 0$.

Proof. (1) If $k = 0$, then any transition given by $\tau_1^{\pm 1}$ or $\tau_2^{\pm 1}$ would be wrong, and by Lemma 3.13. one can never obtain W_r, a contradiction. So we have no such transition in the computation. But other rules do not change the exponent ℓ. Hence $\ell = 0$. Thus, we assume further that $k \neq 0$.

By Lemma 3.13, there are no wrong transitions in the computation. Therefore if the first rule is $\theta = \tau_1^{\pm 1}$, it has to move s_1 left. Moreover, we have θ^k as the prefix of the history, and

$$W_k \equiv ss_1(a')^kt_1b^{\ell-k}t'.$$

The next transition is not wrong, and the only possible next rule is τ_{12}. Now s_2 has to move right, we have $|k|$ such transitions, and obtain

$$W_{2k+1} \equiv sa^ks_2t_1b^{\ell-2k}t'.$$

Since the next rule is not wrong, it has to be τ_{21}. If $\ell - 2k \neq 0$, then the $T(1)T(2)$ sector is not locked, and the rule τ_3 does not apply. Thus the next transition $W_{2k+2} \to W_{2k+3}$ is given by θ again, and one should repeat the cycle obtaining

$$W_{4k+3} \equiv sa^k s_2 t_1 b^{\ell-4k} t',$$

and so on; the rule τ_3 will never apply if $\ell - (2k)m \neq 0$ for every $m \geq 0$.

There is another possibility for the first rule: $\theta = \tau_{21}^{-1}$. Since the second transition cannot be wrong by Lemma 3.13, it is given by $\tau_2^{\mp 1}$. Then we will obtain cycles as above, but having reverse direction. The rule θ_3 will never apply if $\ell + 2k \neq 0$, $\ell + 4k \neq 0, \ldots$ Thus, Statement (1) is proved by contradiction.

(2) If $\ell = 2km$ or $\ell = -2km$ for some $m \geq 0$, the required computations can be immediately constructed according to the samples from part (1) of the proof. $\qquad\square$

Now we want to modify the S-machine $\mathbf{D}_1$ as follows. To define the S-machine $\mathbf{D}_2$ we add one more part $T(3)$ to the standard base. The sector $T(2)T(3)$ serves to count the number of cycles of the S-machine $\mathbf{D}_1$. So it is empty for the start configuration of $\mathbf{D}_2$, and the rule τ_{21} of $\mathbf{D}_1$ extended to the rule of $\mathbf{D}_2$ has one more part: $[t' \to t'c]$, where $t' \in T(2)$. Clearly, Properties (1) and (2) of Lemma 3.14 hold for $\mathbf{D}_2$ as well. Moreover, repeating the proof of Property (2), we see that one obtains c^m in sector $T(2)T(3)$ with $m = \frac{\ell}{2k}$ when the sector $T(1)T(2)$ becomes empty.

The further modification is needed since we should check the divisibility by $(2k)^3$, which, in turn is necessitated by Lemma 2.4. The S-machine $\mathbf{D}_3$ has the same standard base as $\mathbf{D}_2$ but it checks divisibility by $2k$ three times, so its rules are subdivided in three parts.

The rules of the first part are exactly the rules of the S-machine $\mathbf{D}_2$. The rule τ_3 of $\mathbf{D}_2$ serves as a connecting rule between the rules of the first part and the rules of the second part. The difference between these two parts is that the sectors $T(1)T(2)$ and $T(2)T(3)$ interchange their roles: a state letter from $T(2)$ erases letters in the sector $T(2)T(3)$ when the analogs of τ_1 and τ_2 work, and a state letter from $T(1)$ add one letter to the sector $T(1)T(2)$ when the analog of τ_{21} is applied. (We do not introduce notation for all state letters and all rules since we do not need them.) The rules of the third part are absolutely similar to the rules of $\mathbf{D}_1$, i.e. the sector $T(2)T(3)$ is locked.

Remark 3.15. Thus, one can repeat the argument from the proof of Lemma 3.14 (1,2) three times to conclude that starting with an input configuration, the S-machine $\mathbf{D}_3$ can empty all the sectors, except for sector $S(1)S(2)$ if and only if the exponent ℓ is divisible by $(2k)^3$.

Finally, we add a rule τ erasing a-letters of the sector $S(1)S(2)$ with non-trivial part $[s \to sa^{-1}]$ for $s \in S(1)$ locking other sectors, and if all the sectors become empty, one more rule τ_0 (the stop rule) locks all the sectors and changes all the state letters for the letters of the stop configuration. Let us denote the obtained S-machine by $\mathbf{D}_4$.

Lemma 3.16. (1) *Suppose we have a reduced accepting computation* $W_0 \rightarrow \cdots \rightarrow W_r$
of $\mathbf{D}_4$, *where* $W_0 \equiv sa^k s_1 t_1 b^\ell t' t''$ *is an input configuration. Then the exponent* ℓ
is divisible by $(2k)^3$.

(2) *Conversely, if* $(2k)^3$ *divides* ℓ, *then there is an accepting computation starting
with* $W_0 \equiv sa^k s_1 t_1 b^\ell t' t''$ *of length* $\Theta(|\ell| + |k|)$.

Proof. Properties (1), (2) follows from similar properties of $\mathbf{D}_3$ mentioned in Remark 3.15. □

The next modification is obtained by adding historical sectors to the standard base
of $\mathbf{D}_4$. The approach is similar to that described in Subsection 3.2. The standard
base of $\mathbf{D}_5$ is

$$S(1)_r S(2)_\ell S(2)_r T(1)_\ell T(1)_r T(2)_\ell T(2)_r T(3)_\ell,$$

and the rules of $\mathbf{D}_5$ are the extensions of the rules of $\mathbf{D}_4$ to historical sectors as this
was defined in Subsection 3.2. The following lemma is an analog of Lemma 3.12.

Lemma 3.17. *For any reduced computation* $W_0 \rightarrow \cdots \rightarrow W_r$ *of the S-machine* $\mathbf{D}_5$
with base of length at least 3, *we have*

$$|W_i|_a \leq 9\big(|W_0|_a + |W_r|_a\big) \quad (0 \leq i \leq r).$$

3.4. Control state letters.
The work of the main S-machine will be checked by
control state letters running back and forward along the sectors from time to time.
The control letters behave as p-letters of the primitive S-machines $\mathbf{Pr}$ or $\mathbf{Pr}^*$.

Suppose M is an S-machine with a standard base $Q_0 Q_1 \ldots Q_s$. We denote
by M_c the S-machine with standard base

$$P_0 \sqcup Q_0 \sqcup R_0 \sqcup P_1 \sqcup Q_1 \sqcup R_1 \sqcup \cdots \sqcup P_s \sqcup Q_s \sqcup R_s.$$

For every rule θ of M, its ith part $[v_i q_i u_{i+1} \rightarrow v_i' q_i' u_{i+1}']$ is replaced in M_c with
three parts

$$\left[v_i p^i \xrightarrow{\ell} v_i' p^i\right], \left[q_i \xrightarrow{\ell} q_i'\right], \left[r^i u_{i+1} \rightarrow r^i u_{i+1}'\right] \tag{3.1}$$

$(i = 0, \ldots, s, \ p^i \in P_i, \ r^i \in R_i)$. Here we should use one more $\xrightarrow{\ell}$ if there is $\xrightarrow{\ell}$ in
the definition of the component of M.

Comment. Thus, the sectors $P_i Q_i$ and $Q_i R_i$ are always locked, and three state
letters p^i, q^i, r^i work together in M_c as the single q^i in M. Of course, such
a modification is useless for solo work of M. But it will be helpful when one
constructs a composition of M_c with other S-machines, because the *control letters*
from the parts P_i and R_i will work when M_c stands idle.

4. The main S-machine

4.1. Definitions of machines M_3–M_6. We use the S-machine M_2 from Subsection 3.2 and auxiliary S-machines to compose the main machine needed for this paper.

At first we add control state letters to M_2 and obtain S-machine M_3 as it was defined in Subsection 3.4. Let the standard base of M_2 be $Q_0 Q_1 \ldots Q_s$, where sectors $Q_0 Q_1, Q_2 Q_3, \ldots, Q_{s-1} Q_s$ are working sectors, $Q_{s-1} Q_s$ is the input sector, and $Q_1 Q_2, Q_3 Q_4, \ldots$ are historical sectors. Then the standard base of M_3 is

$$P_0 Q_0 R_0 P_1 Q_1 R_1 \ldots P_s Q_s R_s,$$

where P_i (resp., R_i) contains control p-letters (r-letters), $i = 0, \ldots, s$.

Since the rules of M_3 treat every syllable $P_i Q_i R_i$ as a single base letter, the working and the historical sectors for M_3 are of the form $R_{i-1} P_i$. In particular, every historical sector has the form $R_{i-1} P_i$ with even i.

The rules of the next S-machine M_4 will be partitioned in subsets corresponding to ten *steps* with auxiliary rules $\theta(ij)$ *connecting* ith and jth steps. The state letters are also disjoint for different steps. Therefore we need $\mathcal{Q}_0$, which is the disjoint union of ten subsets, $\mathcal{P}_0$, which is the disjoint union of ten subsets, and so on. Thus, the rules of different steps of a computation on M_4 must be separated by connecting rules.

We want to combine the S-machines M_3, the machine $(D_5)_c$ (i.e. the S-machine (D_5) from Subsection 3.3 endowed with control state letters), and compositions of primitive S-machines introduced in Subsection 3.1. We interbreed the input sector of M_3 and the first sectors of $(D_5)_c$. Namely, the state letters from Q_{s-1} and from $S(1)_r$ will be included in the part $\mathcal{Q}_{s-1}$ of the new S-machine M_4, Q_s and $S(2)_\ell$ will be included in $\mathcal{Q}_s$. The reader will see below that at some steps of computations, the part of base $\mathcal{Q}_{s-1} \mathcal{R}_{s-1} \mathcal{P}_s \mathcal{Q}_s \mathcal{R}_s$ works as M_3 while at other steps it works as D_5.

The new S-machine M_4 repeats the computation of M_3 many times and D_5 bounds the number of such cycles. The standard base of M_4 is

$$\mathcal{P}_0 \mathcal{Q}_0 \mathcal{R}_0 \mathcal{P}_1 \mathcal{Q}_1 \mathcal{R}_1 \ldots \mathcal{P}_{s-1} \mathcal{Q}_{s-1} \mathcal{R}_{s-1} \mathcal{P}_s \mathcal{Q}_s \mathcal{R}_s \times \mathcal{P} \mathcal{S}(2)_r \mathcal{R} \times$$
$$\mathcal{P}^{1,\ell} \mathcal{T}(1)_\ell \mathcal{R}^{1,\ell} \mathcal{P}^{1,r} \mathcal{T}(1)_r \mathcal{R}^{1,r} \mathcal{P}^{2,\ell} \mathcal{T}(2)_\ell \mathcal{R}^{2,\ell} \mathcal{P}^{2,r} \mathcal{T}(2)_r \mathcal{R}^{2,r} \mathcal{P}^{3,\ell} \mathcal{T}(3)_\ell \mathcal{R}^{3,\ell}.$$

(Starting with $\mathcal{P}_{s-1}$, this base looks looks like the base of D_5 equipped with control $\mathcal{P}$- and $\mathcal{R}$-parts.) The historical sectors of the form $\mathcal{R}_{i-1} \mathcal{P}_i$ with even i are called *big historical* sectors while $\mathcal{R}^{1,\ell} \mathcal{P}^{1,r}$ and $\mathcal{R}^{2,\ell} \mathcal{P}^{2,r}$ are *small historical* sectors. The sector $\mathcal{R}_s \mathcal{P}$ is also small historical one. It corresponds to the sector $S(2)_\ell S(2)_r$ of D_5.

The rules of M_4 will be partitioned in subsets Θ_i- and Θ_i ($i = 1, \ldots, 5$) corresponding to ten *Steps*. We will not list all state letters here since it would be

complicated and not too helpful. It suffices to define the work of $\mathbf{M}_4$ at different steps as a composition of the S-machines defined in Section 3.

The Steps $1^-, 2^-, \ldots, 5^-$ are *control* steps, where the copies of primitive S-machines work. For example, we want to put Step 2^- between Step 1 and Step 2 (see Fig. 1). So we define the composition $\mathbf{P}_{12-}$ of primitive S-machines working after the connecting rule $\theta(12^-)$ and the composition $\mathbf{P}_{22-}$ of primitive S-machines working after $\theta(2^-2)^{-1}$, provided the inverse of the canonical computation of $\mathbf{P}_{22-}$ should follow right after the canonical computation of $\mathbf{P}_{12-}$.

Thus, to define the control S-machine of Step 2^- below, one should define the order of the work of primitive components for $\mathbf{P}_{12-}$ and $\mathbf{P}_{22-}$ and choose either p-letters or r-letters to be control letters for these primitive components.

Remark 4.1. The control steps are used for double purpose. If the base of a computation is standard, then the history of a control step restores all the configurations (Lemma 4.5 (1)). If the base is not reduced, then the control steps and the right order of the work of their primitive components redound to a linear bound of the space of the computation in terms of the lengths of the first and the last words (Lemma 4.13).

By default, every *connecting* rule $\theta(ij)$ locks a sector if this sector is locked by all rules from Θ_i or if it is locked by all rules from Θ_j. It also changes all state letters used at Step i by there copies from the disjoint set of state letters used at Step j.
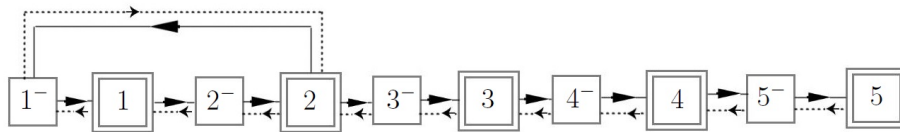

Figure 1. Graph of steps of S-machine $\mathbf{M}_4$.

Step 1^-. This is a control step between Steps 2 and 1. So we define below the canonical work of the S-machines $\mathbf{P}_{2-1}$ and $\mathbf{P}_{1-1}$, keeping in mind that the last rule (as ζ^{21} in Remark 3.5) of the canonical computation of $\mathbf{P}_{2-1}$ switches on the inverse computation for the canonical one of $\mathbf{P}_{1-1}$. This conjunction of the S-machines $\mathbf{P}_{2-1}$ and $\mathbf{P}_{1-1}$ is the S-machine of Step 1^- denoted by $\mathbf{P}_{1-}$.

Let us define $\mathbf{P}_{2-1}$. At first, we have the parallel work of primitive S-machines in all big historical sectors, and control r-letters run forward and back according the rules from Subsection 3.1. The tape alphabet for every such primitive S-machine is the left alphabet $\mathcal{X}_{i,\ell}$ of the big historical sector.

The next primitive S-machine (see Remark 3.5 for the definition of composition) starts working similarly in the input subsector of $\mathbf{M}_3$ after the above mentioned primitive S-machines stop working.

Then we have parallel work of primitive S-machines in the small historical sectors. Again, the running control letters are r-letters.

Finally, the primitive S-machine is switched on that checks the input word (of $\mathbf{D}_5$) between $\mathcal{R}^{1,r}$ and $\mathcal{P}^{2,\ell}$ (with running state letters from $\mathcal{R}_s$).

The running control letters of $\mathbf{P}_{11-}$ are r-letters again (not p-letters), and this S-machine is a copy of $\mathbf{P}_{21-}$ with another set of state letters.

The transition rule $\theta(1^-1)$ changes all state letters of Step 1^- by their copies in Step 1, which contain the letters of the start vector $\vec{s}_1$ of $\mathbf{M}_1$. It locks all sectors except for historical sectors, the sector $\mathcal{R}_{s-1}\mathcal{P}_s$ (we can call it the input since it comes from the input sectors of $\mathbf{M}_3$) and the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$.

The $\theta(1^-1)$-admissible words may involve the copies $p_i^{1,*}$ and $r_i^{1,*}$ of the letter p^1 of a primitive S-machine $\mathbf{Pr}$, but no copies of p^2. They may contains letters from alphabets $X_{i,\ell}$ but not from $X_{i,r}$.

Step 1. The rules $\theta(\mathbf{M}_4)$ from Θ_1^+ restricted to the base $\mathcal{P}_0\mathcal{Q}_0\mathcal{R}_0\ldots\mathcal{P}_s\mathcal{Q}_s\mathcal{R}_s$ are just the (copies of the) positive rules θ of $\mathbf{M}_3$. They do not change other sectors and lock the sector $\mathcal{R}^{2,r}\mathcal{P}^{3,\ell}$.

Comment. At Step 1, $\mathbf{M}_4$ works as the S-machine $\mathbf{M}_3$.

The connecting rule $\theta(12^-)$ changes the state letters by their copies in disjoint alphabet, in particular, the letters from the accept vector $\vec{s}_0$ of $\mathbf{M}_1$ are replaced by their copies. The $\theta(12^-)$-admissible words have no letters from "left" alphabets $X_{i,\ell}$. Besides, the rule $\theta(12^-)$ "removes" one letter in the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$: $[bp_{2,\ell}^1 \to (p_{2,\ell}^1)']$.

Step 2^-. This step is similar to Step 1^-, the difference in the definition of the S-machine $\mathbf{P}_{2-}$ working at Step 2^- is that alphabets $X_{i,\ell}$ should be replaced by alphabets $X_{i,r}$ ($i = 1, \ldots, s$) and the control letters are p-letters (not r-letters).

Comment. The copies of primitive S-machine check several sectors again.

The transition rule $\theta(2^-2)$ replaces all state letters of Step 2^- with their copies in Step 2, and the letters of the accept vector $\vec{s}_1$ of $\mathbf{M}_1$ are among them. It locks non-historical sectors except for the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$.

Step 2. The positive rules from Θ_2 are just copies of the negative rules from Θ_1.

Comment. $\mathbf{M}_4$ works as at Step 1, but reverses the computation procedure.

The connecting rule $\theta(21^-)$ completes the cycle.

The connecting rule $\theta(23^-)$ makes possible the final Steps 3^-–6.

Step 3^-. As at Step 1^-, the S-machine $\mathbf{P}_{3-}$ is the conjunction of two S-machines: $\mathbf{P}_{23-}$ and $\mathbf{P}_{33-}$, where the first one is just a copy of $\mathbf{P}_{21-}$ with a different set of state letters.

For the S-machine $\mathbf{P}_{33-}$, the running state letters are r-letters too. Its canonical work is as follows. At first, the primitive S-machines simultaneously check the small historical sectors. Then the next primitive S-machine checks the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$,

then the input sector $\mathcal{R}_{s-1}\mathcal{P}_s$ is checked, and finally the big historical sectors are simultaneously checked.

The connecting rule $\theta(3^-3)$ cannot be applied to an admissible word having a-letters from right alphabets of historical sectors.

Step 3. The rules from Θ_3 extend the rules of S-machine $\mathbf{D}_5$ as follows. The rules of $\mathbf{D}_5$ on the configurations with base

$$S(1)_r S(2)_\ell S(2)_r T(1)_\ell T(1)_r T(2)_\ell T(2)_r T(3)_\ell \qquad (4.1)$$

are now the rules of Step 3 on the base

$$\mathcal{P}_{s-1}\mathcal{Q}_{s-1}\mathcal{R}_{s-1}\mathcal{P}_s\mathcal{Q}_s\mathcal{R}_s\mathcal{P}\mathcal{S}(2)_r\mathcal{R} \times$$
$$\mathcal{P}^{1,\ell}\mathcal{T}(1)_\ell\mathcal{R}^{1,\ell}\mathcal{P}^{1,r}\mathcal{T}(1)_r\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}\mathcal{T}(2)_\ell\mathcal{R}^{2,\ell}\mathcal{P}^{2,r}\mathcal{T}(2)_r\mathcal{R}^{2,r}\mathcal{P}^{3,\ell}\mathcal{T}(3)_\ell\mathcal{R}^{3,\ell}$$

with control letters, according to the definition given in Subsection 3.4, but the control letters do not work at this step, and so $\mathcal{P}^{1,r}\mathcal{T}(1)_r\mathcal{R}^{1,r}$ in this base behaves as $T(1)_r$ in (4.1), and so on.

The rules of Θ_3 do not change big historical sectors and lock non-historical sectors of $\mathbf{M}_3$, except for the input sector $\mathcal{R}_{s-1}\mathcal{P}_s$.

Comment. After standard work with consecutive Steps $1^- - 2$ and control Step 3^-, the (copy of) S-machine $\mathbf{D}_5$ checks if the length of a-word in the sector $\mathcal{R}^{1,\ell}\mathcal{P}^{2,r}$ divisible by the eight cubes of the a-length of the input sector.

The rule $\theta(34^-)$ locks all sectors except for historical ones. It cannot be applied to a word having a-letters from left alphabets of small historical sectors.

Step 4^-. The tape alphabets of the control S-machine $\mathbf{P}_{4-}$ are right alphabets $\mathcal{X}_{i,r}$ for small historical sectors and left alphabets for big historical sectors. All working sectors are locked.

The first half of $\mathbf{P}_{4-}$ is the S-machine $\mathbf{P}_{34-}$. Its running control letters are p-letters, and the canonical work starts with parallel work of primitive S-machines in all small historical sectors followed by the parallel work of primitive S-machines in the big historical sectors.

The second S-machine $\mathbf{P}_{44-}$ starts with parallel work of primitive S-machines in all small historical sectors with control r-letters followed by parallel work of primitive S-machines in all big historical sectors with control p-letters.

Step 4. The rules of Θ_4 simultaneously erase the letters from small historical sectors. The corresponding parts of the positive rules are

$$r^1_{j,\ell}x \to r^1_{j,\ell} \quad (j = 1, 2, 3)$$

for every positive letter x from the right alphabet of a small historical sector.

The connecting rule $\theta(45^-)$ locks all sectors except for big historical sectors.

Step 5⁻. We define $\mathbf{P}_{5^-} = \mathbf{P}_{45^-}$. The control p-letters simultaneously check the big historical sectors, while all other sectors are locked.

The connecting rule $\theta(5^-5)$ locks all sectors except for big historical sectors.

Step 5. The rules of Θ_5 simultaneously erase the letters from big historical sectors. The corresponding parts of the positive rules are

$$xp_j^1 \to p_j^1 \quad (j = 1, \ldots, s)$$

for every positive letter x from the left alphabet of a big historical sector.

The accept command θ_0 from Θ_5 can be applied when all the sectors are empty. So it locks all the sectors, changes the state letters and terminates the work of $\mathbf{M}_4$.

So $\mathbf{M}_4$ has a unique accept configuration.

Lemma 4.2. *Let a computation $\mathcal{C}$ of an S-machine $\mathbf{P}_{i^-}$ $(i = 1, \ldots, 5)$ start with a connecting rule θ and end with a connecting rule $\theta' \neq \theta^{-1}$. Then for every sector of the standard base, there is a rule in the history of $\mathcal{C}$ locking this sector.*

Proof. We consider $\mathbf{P}_{1^-}$ only. Since the computation starts with (21^-) and ends with (1^-1), all the primitive S-machines listed in the definition of the S-machine $\mathbf{P}_{1^-}$ have to start and finish their work. So every sector of the standard base is locked either by $\theta(21^-)$ or by $\theta(1^-1)$, or by a rule of a primitive S-machine of the form ζ^{12}; see Subsection 3.1 and Lemma 3.1 there, because every sector unlocked by these connecting rules is checked by one of the primitive S-machines forming the S-machine $\mathbf{P}_{1^-}$. $\qquad\square$

Remark 4.3. Every cycle of the Steps $1^{-1}, 1, 2^-, 2$ just changes the length of the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$ by 1. (See the definition of the connecting rule $\theta(12^-)$.) If this length ℓ becomes divisible by $8k^3$, where k is the length of the input sector of $\mathbf{M}_3$, the copy of the S-machine $(\mathbf{D}_5)_c$ can accept at Step 3, and one can obtain the stop configuration of $\mathbf{M}_4$ after Steps $4^-, 4, 5^-, 5$. Hence the shortest accepting computation has at most $\Theta(k^3) = \Theta(g(n))$ cycles of the Steps $1^{-1}, 1, 2^-, 2$ if the length of the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$ is $\Theta(n)$.

This is an informal answer to the question why the division S-machine is needed. Indeed, if an auxiliary S-machine just checks if ℓ is equal to k^3, then the number of cycles as above could be $\Theta(n)$, which would lead to at least cubic Dehn function.

Another question: Since we want to repeat the cycle of the Steps $1^{-1}, 1, 2^-, 2$ $\Theta(f(n)^3)$ times, why does $\mathbf{M}_4$ recognize the values $f(n)$ instead of $g(n) = f(n)^3$? Because the Turing machine has to take time at least $\Theta(g(n))$ to recognize $g(n)$. By Lemma 2.4, the S-machine $\mathbf{M}_1$ should work as long as $\Theta(g^3(n))$ or longer for the same goal. If $g(n)$ is "almost" linear function, then this approach makes the time function of $\mathbf{M}_1$ almost cubic, and the growth of the Dehn function we are going to construct, becomes almost biquadratic.

Let B be the standard base of $\mathbf{M}_4$ and B' be its disjoint copy. By $\mathbf{M}_5$ we denote the S-machine with standard base $B(B')^{-1}$ and rules $\theta(\mathbf{M}_5) = [\theta, \theta]$, where $\theta \in \Theta$ and Θ is the set of rules of $\mathbf{M}_4$. So the rules of $\Theta(\mathbf{M}_5)$ are the same for $\mathbf{M}_4$-part of $\mathbf{M}_5$ and for the mirror copy of $\mathbf{M}_4$. Therefore we will denote $\Theta(\mathbf{M}_5)$ by Θ as well, assuming that the sector between B and $(B')^{-1}$ is locked by any rule from Θ.

Finally, the main S-machine $\mathbf{M} = \mathbf{M}_6$ is a cyclic analog of $\mathbf{M}_5$. We add one more base letter $\{t\}$. So the standard base of $\mathbf{M}_6$ it $\{t\}B(B')^{-1}\{t\}$, where the part $\{t\}$ has only one letter t and the first part $\{t\}$ is identified with the last part. (For example, $\{t\}B(B')^{-1}\{t\}B(B')^{-1}$ can be a base of an admissible word for $\mathbf{M}_6$. Furthemore, sectors involving $t^{\pm 1}$ are locked by every rule from Θ. The stop word $W_0(\mathbf{M})$ is defined accordingly: every letter in the standard base $B(B')^{-1}$ of $\mathbf{M}_5$ is replaced by the corresponding letters from the stop word of $\mathbf{M}_5$.

The "mirror" symmetry of the base of $\mathbf{M}$ will be used in Lemma 9.22. For a different purpose mirror symmetry of Turing machines was used in the papers of W. W. Boon and P. S. Novikov (see [20]).

4.2. Standard computations of M. The history H of a reduced computation of $\mathbf{M}$ can be factorized so that every factor corresponds to one of the Steps 1^-–5. If, for example, $H = H'H''H'''$, where H' is a product of rules from Step 2, H'' has only one rule $\theta(21^-)$ and H''' is a product of rules from Step 1^-, then we say that the *step history* of the computation (or its type) is $(2)(21^-)(1^-)$ or just $(2)(1^-)$ since the only rule connecting the computations of Steps 2 and 1^- is $\theta(21^-)$ and for the most asymptotic estimates of the length of steps (e.g. $\|H''\|$) or of the lengths of their admissible words, it does not matter to which of the two possible steps the connecting rule is attributed.

There are no computations of some types, say $(1)(3)$, as it immediately follows from the definition of connecting rules (and from Fig. 1). In this subsection, we eliminate some other subwords in step histories.

Lemma 4.4. (1) *There are no reduced computations of* $\mathbf{M}$ *with step histories*

$$(1^-1)(1)(1^-1),\ (21^-)(2)(21^-),\ and\ (23^-)(2)(23^-)$$
$$(with\ (12^-)(1)(12^-)\ and\ (2^-2)(2)(2^-2))$$

if the base of the computation contains at least one big historical sector $\dots \mathcal{P}_*$ *(resp., big historical sector* $\mathcal{R}_* \dots$*).*

(2) *There are no reduces computations of* $\mathbf{M}$ *with step histories*

$$(3^-3)(3)(3^-3)\quad (with\ (34^-)(3)(34^-))$$

if the base of the computation contains a small historical sectors $\dots \mathcal{P}^*$ *(resp., a small historical sectors* $\mathcal{R}^* \dots$*).*

Proof. (1) We consider only the type $(1^-1)(1)(1^-1)$ since other variants are similar. Indeed, if the history H of the part (1) is non-empty, then a state p-letter inserts a copy of H^{-1} in historical letters of the alphabet $X_{i,r}$ (see Subsection 3.2). Recall that the words with non-empty subwords over $X_{i,r}$ are not θ_{1-1}-admissible, but we should have θ_{1-1}-admissible words both in the beginning and at the end of the subcomputation with history H, a contradiction.

(2) The proof is similar. □

Lemma 4.5. (1) *There are no reduced computations of* **M** *with step histories of the form*

$$(i^- j)(i^-)(i^- j)$$

for $i = 1, \ldots, 5$ if the base is standard.

(2) *Let the base of a computation $\mathcal{C}: W_0 \to \cdots \to W_t$ be standard and $\mathcal{C}$ has one of the step histories*

$$(21^-)(1^-)(1^-1), \ (12^-)(2^-)(2^-2), \ (23^-)(3^-)(3^-3),$$
$$(34^-)(4^-)(4^-4), \ (45^-)(5^-)(5^-5).$$

Then all admissible words of $\mathcal{C}$ are uniquely defined by the history H of $\mathcal{C}$, $|W_0|_a = |W_1|_a = \cdots = |W_t|_a$, and $\|H\| \le 4\|W_0\|$.

Proof. (1) Consider only the step history $(21^-)(1^-)(21^-)$ and the work of the primitive S-machines switched on by the rule (21^-). If one of the rules $\zeta^1(a)^{\pm 1}$ of this primitive S-machine (see Subsection 3.1) does not move a control state letter right/left, but instead just insert $a'^{\pm 1}$ from the right and $a^{\mp 1}$ from the left, then the rule ζ^{12} is not applicable since the sector is not locked. So the next rule should be $\zeta^1(b)^{\pm 1}$ which is not the inverse one for $\zeta^1(a)^{\pm 1}$. Hence the control state letter has to insert letters from both sides without cancellations, and neither ζ^{12} nor $(\zeta^{12})^{-1}$ can be ever applied, a contradiction.

Therefore the primitive S-machine must work canonically, as it was described in Subsection 3.1 (also see Lemma 3.1 (2)). Hence the history of its work uniquely restores the words in the sectors controlled by this primitive S-machine.

Hence the first primitive S-machine has to complete its canonical work and switch on the next primitive S-machine according to the definition of $\mathbf{P}_{1^-}$ for Step (1^-), and so on. Thus, one never obtains a $\theta(21^-)^{-1}$-admissible word, a contradiction.

(2) By Lemma 3.1 (3), the histories of the primitive S-machines subsequently restore the tape words in all unlocked sectors. Lemma 3.6 applied to $\mathcal{C}$ and to the inverse computation, implies equalities $|W_0|_a = |W_1|_a = \cdots = |W_t|_a$, and gives $\|H\| \le 4\|W_0\|$ if one takes into account that both the S-machine $\mathbf{P}_{21^-}$ and $\mathbf{P}_{21^-}$ control each of the sectors. □

Lemma 4.6. *There are no reduced computations with standard base and step histories* $(5^-)(4)(5^-)$.

Proof. Assume such a computation exists. Note that the small historical sectors are empty at the both transitions (45^-). However every rule of Step 4 multiplies the word in the small historical sector by a letter x, and one obtains no cancellations of these letters since the computation is reduced and different rules multiply by different letters. Hence the part (4) is empty, and the lemma is proved by contradiction. $\square$

Below we need a rougher subdivision of the history of a reduced computation with *standard* base. We say that the Steps $1 - 4^-$ are *fundamental* steps and the Steps $4, 5^-, 5$ are *erasing* steps. So the *block history* of every computation of **M** is a subword of $(F)(E)(F)(E)\ldots$, where (F) (where (E)) are maximal parts of the history with fundamental (resp., erasing) steps only. The separating rules for neighbor blocks are $\theta(4^-4)^{\pm 1}$.

Lemma 4.7. *If the block history of a computation is* (E)*, then its step history is a subword of the word* $(4)(5^-)(5)(5^-)(4)$.

Proof. Proving by contradiction and taking into account Lemma 4.5 (1), we should get a subword $(5^-)(4)(5^-)$ in the step history, contrary to Lemma 4.6. $\square$

Lemma 4.8. *Let* $W_0 \to \cdots \to W_t$ *be a computation with block history* (E). *Then:*

(1) $|W_j|_a \leq \max\big(|W_0|_a, |W_t|_a\big)$ *for* $j = 0, 1, \ldots, t$;

(2) $t \leq 10\big(\|W_0\| + \|W_t\|\big)$.

Proof. (1) If the history has only one Step 4 or 5, then Statement (1) follows from Lemma 2.7 (c). For single Step 5^- it follows from Lemma 3.6 (a).

If there is Step 5 in the computation, then by Lemma 4.7 we have only one maximal subcomputation $W_k \to \cdots \to W_\ell$ of Step 5. Here $|W_k|_a \leq |W_0|_a$ since W_k has no non-empty sectors except for big historical sectors, which are unchanged at Steps 4, while Steps 5^- cannot decrease the sum of length of these sectors by projection argument (see Remark 3.2). Hence it suffices to prove Statement (1) for subcomputations with step histories H' and H'', where $H'(5)H''$ is the step history. Therefore we may prove Statement (1) under assumption that there are no Steps 5 in the step history.

For the step history $(4)(5^-)$, the a-length of the configuration separating two steps cannot be longer than the final configuration by projection argument, which reduce the proof to one-step histories.

(2) By Lemma 4.7, we have at most 5 steps, and by Property (1) it suffices to prove (2) for one-step histories but with coefficient 2. Indeed such estimates for the lengths of histories are obtained for Steps $4, 5^-, 5$ in Lemmas 2.7 (b) and 3.6 (b). $\square$

Lemma 4.9. *Let $W_0 \to \cdots \to W_t$ be a computation with block history (F).*

(1) *Then the step history of this computation is a subword of the word*

$$\mathcal{W}(m) \equiv (4^-)(3)(3^-)((1^-)(1)(2^-)(2))^m (3^-)(3)(4^-)$$

for some non-zero integer m.
(Here we define $((1^-)(1)(2^-)(2))^{-1} = (2)(2^-)(1)(1^-)$.)

(2) *If the step history is equal to $\mathcal{W}(m)$ for some $m \neq 0$, then the exponent m is divisible by $(2k)^3$, where k is the a-length of the input sector $\mathcal{R}_{s-1}\mathcal{P}_s$ after the application of the connecting rule $\theta(3^-3)$.*

(3) *If the step history is equal to*

$$((1^-)(1)(2^-)(2))^m (3^-)(3)(4^-)$$

and the history starts with a connecting rule, then the exponent m is congruent to l modulo $(2k)^3$, where k is as in Statement (2) and b^l is the tape word of the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$ in the beginning of the computation.

Proof. (1) Since the block history is (F), Lemmas 4.5 (1) and 4.4 forbid "reverse moves" $\theta^{\pm 1} H \theta^{\mp 1}$ in the history, where H is a one-step history and θ is a connecting rule, Statement (1) follows from the definition of connecting rules between steps (see Fig. 1).

(2) Let us restrict the subcomputation with step history $(3^-3)(3)(34^-)$ to the input sector $\mathcal{R}_{s-1}\mathcal{P}_s$ and the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$. Then we can apply Lemma 3.14 to conclude that the exponent l of the tape word b^l in the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$ at the beginning of this computation is divisible by $(2k)^3$. Similarly, for $b^{l'}$ at the end of the computation with step history $(34^-)(3)(3^-3)$, we obtain that l' is divisible by $(2k')^3$, where k' is the a-length of the sector $\mathcal{R}_{s-1}\mathcal{P}_s$.

Note that a computation with step history $(1^-1)(1)(2^-)(2)(21^-)$ (or step history $(1^-1)(1)(2^-)(2)(23^-)$) does not change the a-length of the sector $\mathcal{R}_{s-1}\mathcal{P}_s$ since it is preserved by the S-machine $\mathbf{P}_{2-}$ by Lemma 3.1 and the history of Step 2 is inverse (of the copy) of the history of Step 1 here. The same is true for the subcomputations with step history $(21^-)(1^-)(1^-1)$ and $(23^-)(3^-)(3^-3)$ by Lemma 3.1. So $k' = k$ and $l' - l$ is divisible by $(2k)^3$ by Lemma 3.14.

On the other hand, every rule $\theta(12^-)$ multiplies the tape word in the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$ by b^{-1}. Therefore $\ell' - \ell = m$, whence m is divisible by $(2k)^3$, as required.

(3) The proof is similar to the proof of Statement (2). $\qquad\square$

4.3. Computations with faulty bases.

Lemma 4.10. *If the step history of a reduced computation $\mathcal{C}$ of* **M** *is*

$$(21^-)(1^-)(1^-1),\ (12^-)(2^-)(2^-2),\ (23^-)(3^-)(3^-3),$$
$$(34^-)(4^-)(4^-4),\ or\ (45^-)(5^-)(5^-5),$$

then:

(1) *the base of $\mathcal{C}$ is a reduced word;*

(2) *the first letter of the base, the length of the base and the history H of $\mathcal{C}$ completely determine the computation $\mathcal{C}$.*

Proof. (1) This follows from Lemmas 4.2 and 2.6.

(2) Consider for example the step history $(21^-)(1^-)(1^-1)$. By Property (1) and the definition of admissible word, the base of $\mathcal{C}$ is determined by its length and the first letter. Since every sector unlocked by $\theta(21^-)$ has to be checked by a primitive S-machine, the copy of the content of this sector is contained in the history of the computation as a product of the letters of the form $\zeta^1(a)^{\pm 1}$ defined for the particular primitive S-machine. $\qquad\square$

Lemma 4.11. *Suppose that a reduced computation $W_0 \to \cdots \to W_t$ of Step 1 or 2 (of Step 3) starts with a connecting rule. Assume that the length of its base B is bounded from above by a constant N_0, and B has a big historical sector (a small historical sector, resp.) of the form $\mathcal{RP}$ (with indices). There is a constant $c = c(N_0)$ such that $|W_0|_a \le c|W_t|_a$.*

Proof. Let $V_0 \to \cdots \to V_t$ be the restriction of the computation to the sector with base $\mathcal{RP}$. By Lemma 3.9, we have $t \le |V_t|_a$ and $|V_0|_a \le |V_t|_a$.

It follows from lemma 2.1 (2) that

$$|W_0|_a \le |W_t|_a + 2N_0 t \le |W_t|_a + 2N_0|V_t|_a \le (2N_0 + 1)|W_t|_a.$$

It suffices to choose $c = 2N_0 + 1$. $\qquad\square$

Definition 4.12. We call a base of **M** *faulty* if:

- it starts and ends with the same base letter,
- it has no proper subwords with this property, and
- it is a not a reduced word.

Note that if there is a computation $\mathcal{C}$ with a faulty base $U_1 \ldots U_i \ldots U_s$ (where $U_1 = U_s$), then one can replace every admissible word of this computation by the cyclic shift of it with faulty base $U_i \ldots U_s U_1 \ldots U_{i-1} U_i$ and obtain the *cyclic shift $\mathcal{C}'$* of $\mathcal{C}$.

The following is the main lemma of this subsection.

Lemma 4.13. *There is a constant* $C = C(\mathbf{M})$, *such that for every reduced computation* $\mathcal{C}: W_0 \to \cdots \to W_t$ *of* $\mathbf{M}$ *with a faulty base and every* $j = 0, 1, \ldots, t$, *we have*

$$|W_j|_a \leq C \max\big(|W_0|_a, |W_t|_a\big).$$

Proof. **0.** If the faulty base is not of the form $(pp^{-1}p)^{\pm 1}$ for some control state letter p (or r)), then the words $W_0, \ldots, W_t$ have to contain non-control state letters. Hence we can replace the computation $\mathcal{C}$ by a cyclic shift of it and suppose further that the first and the last letters of $W_0, \ldots, W_t$ are not control letters.

1. By Lemma 4.10, the step history (and the inverse step history) has no subwords

$$(21^-)(1^-)(1^-1), \; (12^-)(2^-)(2^-2), \; (23^-)(3^-)(3^-3),$$
$$(34^-)(4^-)(4^-4), \; \text{and} \; (45^-)(5^-)(5^-5).$$

2. Assume that the history has only one Step (2^-) and the base is $(pp^{-1}p)^{-1}$ for some control state letter p (or r)). Obviously, the running letter p cannot lock any sector and so every rule has the same type (either $\zeta^1(*)$ or $\zeta^2(*)$ as in Subsection 3.1).

If $|W_{j+1}|_a > |W_j|_a$ for some j, then one of two sectors (e.g. the first one) increases its length by 2, while another one does not decrease the length under the transition $W_j \to W_{j+1}$. But then the first sector of W_{j+2} has to increase again by 2 since the computation is reduced. It follows that

$$|W_{j+1}| \leq |W_{j+2}| \leq \cdots \leq |W_t|.$$

Hence the length of every admissible word does not exceed $\max(|W_0|_a, |W_t|_a)$.

If the faulty base has no subwords of the form $(pp^{-1}p)^{\pm 1}$, the same inequality for a computation with history of Step 2^- follows from Lemmas 3.1 and 3.3. Thus, one may assume further that the step history is not (2^-). Similarly, it is neither (1^-) nor (3^-), nor (4^-), nor (5^-).

3. Assume there is (1) in the step history. Then by item 1, the set of steps is either (a) $\{(1)\}$, (b) $\{(1), (1^-)\}$, (c) $\{(1), (2^-)\}$, or (d) we have the subword $(1^-)(1)(2^-)$ in the step history (or in the inverse step history).

3a. In this case, the required inequality follows from Lemma 3.12. (Recall that by definition the connecting rule $\theta(12^-)$ inserts/deletes one letter, but this small change of length is not an obstacle.)

3c. Assume that the step history is $(2^-)(1)$ (or $(1)(2^-)$) and $W_0 \to \cdots \to W_j$ is a maximal subcomputation with step history (2^-). Then $|W_0|_a \geq |W_j|_a$ by the projection argument. Therefore it suffices to prove the statement of the lemma for a subcomputations with step histories (2^-) and (1). For the case (2^-), we refer to item 2. The case (1) is considered in item 3a. Thus, we assume further that the step history has length at least 3.

Suppose the step history has a subword $(2^-)(1)(2^-)$. Then the base (or the inverse base) has no big historical sectors $\mathcal{R} \ldots$ (with indices) by Lemma 4.4.

So the only possible bases for big historical sectors have form $\mathcal{P}^{-1}\mathcal{P}$. The state control letters of such sector should start running at Step (2^-) after the application of the connecting rule $\theta(12^-)$ but they cannot ever meet state letter from $\mathcal{R}$ and will run forever, and $|W_t|_a \geq |W_j|_a$ by Lemma 3.3, if W_j is obtained at the application of $\theta(12^-)$. Hence one can cut the computation at W_j reducing the problem to items 2 and 3a. If there are no big historical sectors, then there are no working sectors of the S-machine $\mathbf{M}_3$. The other sectors of $\mathbf{M}$ (which could come from the base of $\mathbf{D}_5$) do not work at Steps (1) and cannot decrease their length starting from a connecting rule at Steps (2^-) by Lemma 3.1. This makes the statement of the lemma obvious for them.

Hence one may assume that there are no subwords $(2^-)(1)(2^-)$, and so the step history is $(1)(2^-)(1)$.

Suppose the base has a big historical sector $\mathcal{R}\mathcal{P}$ (with indices). Then for the maximal subcomputation $W_r \to \cdots \to W_t$ of Step 1, we have $|W_r|_a \leq c|W_t|_a$ by Lemma 4.11 because the length of a faulty base is bounded by a constant N_0 depending on the S-machine $\mathbf{M}$ only. Hence one can reduce our task to the subcomputations with the step histories (1) and (2^-). (Of course, the constant in the desired inequality changes when we pass to step histories involving more types of steps.) Hence we assume further that the base has no big historical sectors $\mathcal{R}\mathcal{P}$.

Also there are no big historical sectors $\mathcal{P}^{-1}\mathcal{P}$ because state control letters of such sector should start running after the application of the connecting rule $\theta(12^-)$ but they cannot ever meet state letter from $\mathcal{R}$ and will run forever by Lemma 3.3; the last Step 1 will not be reached.

So all big historical sectors (if any) are of the form $\mathcal{R}^{-1}\mathcal{R}$. Recall that the alphabets for the $\theta(12^-)^{\pm 1}$-admissible words are $X_{i,r}$, and so the word in this alphabet will be conjugated at Step 1 by the letters from the disjoint alphabets $X_{i,\ell}$ in sectors with bases $\mathcal{R}_{i-1}^{-1}\mathcal{R}_{i-1}$. Hence after application of $\theta(12^-)^{-1}$, each rule of Step 1 will increase the length of such sector by 2. By Lemmas 3.10 and 2.1, we have $|W_r|_a \leq \cdots \leq |W_t|_a$ if the last Step 1 starts with W_r. This reduces the problem to the subcomputations with the step histories (1) and (2^-) again.

If there are no big historical sectors, then we have no working sectors except for the sectors of the S-machine $\mathbf{D}_5$ because one may assume that the left-most sector of the standard base of $\mathbf{M}_2$ is always locked and because the base is faulty. The other sectors (which could come from the base of $\mathbf{D}_5$) do not work at Steps (1) and cannot decrease their length towards W_0 or W_t at Steps (2^-) by Lemma 3.1. This makes the statement of the lemma obvious for them and completes case 3c.

3b. This case is similar to 3c up to exchange of $\mathcal{R}$ with $\mathcal{P}$ and $X_{i,\ell}$ with $X_{i,r}$.

3d. Assume that the step history has a subword $(1^-)(1)(2^-)$. Then there are no big historical sectors $\mathcal{P}^{-1}\mathcal{P}$ and $\mathcal{R}\mathcal{R}^{-1}$ since the conjugation in free group given

by Step 1 cannot transform a non-trivial word in the alphabet $X_{i,\ell}$ in a word in the alphabet $X_{i,r}$. So all big historical sectors have base of the form $\mathcal{R}\mathcal{P}$ (or inverse one).

Consider the word W_j obtained after the application of the last connecting rule $\theta(12^-)^{\pm 1}$. Only big historical sectors of W_j are not locked (except for the sectors of the S-machine $\mathbf{D}_5$, which are not touched by Step 1). If the next step is 1, then its rules cannot make the historical sector shorter by Lemma 3.9. If the next step is 2^-, then no sector becomes shorter by Lemma 3.1. The same is true for Step 1^- if it follows Step 1. Repeating this procedure, we have $|W_j|_a \leq |W_t|_a$. Therefore one can reduce Case 3d to previous cases by subdivision of the computation along the transitions between Steps 1 and 2^-. (The occurrences of $\theta(12^-)$ and $\theta(12^-)^{-1}$ in the history of the computation have to alternate in Case 3d inserting/deleting the same letter, and so this does not affect the desired inequality.)

Thus, we may assume further that there are no Steps 1 in the computation.

4. We may also assume that there are no Steps 2 in the computation. The proof copies the proof at item 3 with subcases (a) $\{(2)\}$ (b) $\{(2), (2^-)\}$, (c) $\{(2), (3^-)\}$, and (d) where one considers the subword $(2^-)(2)(3^-)$ and the word W_j provided by the last connecting rule $\theta(2^-2)^{\pm 1}$.

5. If there is Step 5 in the step history, then we cannot have steps except for 5 and 5^- by item 1. It follows from the definition of Step 5^- and Lemma 3.1 (4) that the subword $(5)(5^-)(5)$ in the step history is not possible if there is a big historical sector with base of type $\mathcal{R}\mathcal{P}$. By Lemma 3.3, historical sectors with base $\mathcal{P}^{-1}\mathcal{P}$ are not possible too. However the historical sectors with base $\mathcal{R}\mathcal{R}^{-1}$ do not change words at Step 5. It follows that the word W_j obtained after the transition from (5^-) to (5) is not longer than W_t (see similar argument at item 3d). So one reduces the task to shorter step histories.

Thus, one may assume that there are no subwords $(5)(5^-)(5)$ in the step history. Therefore assuming that there is (5) in the step history, we should consider only the history $(5^-)(5)(5^-)$ or its subwords. Again the rules $\theta(5^-5)$ defines a word W_j, which is not longer than W_t by Lemma 3.1. This reduce the task to step histories (5^-) and (5). For (5^-), the problem is solved in item 2, and it is solved by Lemma 2.7 (c) for (5).

Thus one may assume from now that there are no Steps (5) in the step history.

6. If there is (5^-) in the step history, then by items 1 and 5, there are no other steps except for (4) and (4^-). However the transition (45^-) provides us with the shortest words in the computations since neither computations of Step 5^- nor those from (4) can make big historical sectors shorter by Lemmas 3.1 and 3.3. So cutting the computation along such transitions, we can decrease the number of steps. Since a single Step 5^- is eliminated in item 2, we may further assume that there are no Steps (5^-) in the computation.

7. If there is (4) in the step history, then by items 1 and 6, there are no other steps except for (4^-). For such histories we will repeat some arguments from item 3c using now small histories instead of big ones.

The case with a single Step (4^-) is eliminated by item 2. The brief history (4) is also eliminated by Lemma 2.7 (c) (for small historical sectors $\mathcal{R}\mathcal{P}$) and Lemma 2.10 (for small historical sectors $\mathcal{R}\mathcal{R}^{-1}$). Assume that the step history is $(4^-)(4)$ and $W_0 \to \cdots \to W_j$ is a maximal subcomputation with step history (4^-). Then, $|W_0|_a \geq |W_j|_a$ by the projection argument. Therefore the problem is reduced to single step histories Thus, we assume further that the step history has length at least 3.

Assume that there is a subword $(4)(4^-)(4)$ in the step history and there is the letter $\mathcal{R}_s$ in the base. Then it cannot follow by $\mathcal{R}_s^{-1}$, because the letter from $\mathcal{R}_s$ must start running right after the connecting rule $\theta(4^-4)^{-1}$ and it cannot ever reach the part $\mathcal{P}$, so the next connecting rule $\theta(4^-4)$ cannot appear, a contradiction. Hence there is a sector $\mathcal{R}_s\mathcal{P}$. It follows that there is $\mathcal{R}^{1,\ell}$ in the base since the sectors between $\mathcal{P}$ and $\mathcal{R}^{1,\ell}$ are locked by $\theta(4^-4)$. Then we obtain $\mathcal{P}^{1,r}$ and all other parts of small historical sectors.

Also there is the part $\mathcal{P}_{s-1}$ in the base, becase all the sectors between $\mathcal{P}_{s-1}$ and $\mathcal{R}$ are locked by $\theta(4^-4)$. Hence right after the sector $\mathcal{R}_s\mathcal{P}$ is checked by the first primitive S-machine of $\mathbf{P}_{44-}$, a letter from $\mathcal{P}_{s-1}$ starts running checking the big historical sectors, and again, there should be the part $\mathcal{R}_{s-2}$ in the base, since otherwise the next occurrence of $\theta(4^-4)$ does not happen.

This implies it turn, that there is $\mathcal{P}_{s-2}$ in the base, and so on, that is we have all the sectors of the standard base of $\mathbf{M}$ and the base of our computation has no cancellation, contrary to the definition of faulty base.

Hence there are no parts $\mathcal{R}_s$ in the base, provided $(4)(4^-)(4)$ is a subword of the step history. Similar argument shows that there are no $\mathcal{R}^{1,\ell}$ and $\mathcal{R}^{2,\ell}$. (For example, if there is $\mathcal{R}^{1,\ell}$, then there is a rule locking the sector $\mathcal{R}^{1,\ell}\mathcal{P}^{1,r}$ by the definition of primitive S-machines; this rule has to lock sector $\mathcal{R}_s\mathcal{P}$ too, and so the part $\mathcal{R}_s$ occurs in the base too.) Hence nothing changes at Steps 4, since only $\mathcal{R}$-letters can erase the small historical sectors. The transition (44^-) provides us with the shortest words in the computations since neither computations of Step 4^- nor those from (4) can make small historical sectors shorter by Lemmas 3.1 and 3.3. So cutting the computation along such transitions, we can decrease the number of steps. Therefore we may further assume that there are no subwords $(4)(4^-)(4)$ in the step history.

Thus, the step history is $(4^-)(4)(4^-)$, and as above it can be subdivided in one-step histories. Therefore we may assume from now that there are no Steps 4 in the step history.

8. Now we assume that there is Step 4^- in the step history.

Suppose the step history has a subword $(4^-)(3)(4^-)$. Then the base has no small historical sectors $\mathcal{R}^* \ldots$ by Lemma 4.4 (2). So the only possible bases for

small historical sectors are $(\mathcal{P}^{i,r})^{-1}\mathcal{P}^{i,r}$ or $\mathcal{P}^{-1}\mathcal{P}$. The state control letters of such sectors should start running after the application of the connecting rule $\theta(34^-)$ but they cannot ever meet state $\mathcal{R}$-letters and will run forever, whence $|W_t|_a \geq |W_j|_a$ by Lemma 3.3, if W_j is obtained at the application of $\theta(34^-)$. So the whole step history is $(4^-)(3)(4^-)$. Hence one can cut the computation at W_j reducing the problem to item 2 and Lemma 3.17. If there are no small historical sectors, then there are no sectors of the S-machine $\mathbf{D}_5$ changed at Step 3. The other sectors (which could come from the base of $\mathbf{M}_2$) do not work at Steps 3 and cannot decrease their length when starting from a connecting rule at Steps 4^- by Lemma 3.1. As usual, this allows induct on the number of steps.

Hence one may assume that there are no subwords $(4^-)(3)(4^-)$. The subwords $(3^-)(3)(3^-)$ are eliminated in the same way.

Assume that the step history has a subword $(3^-)(3)(4^-)$. Then there are no small historical sectors with non-reduced bases UU^{-1} since the conjugation in free group given by Step 3 cannot transform a non-trivial word in the alphabet $X_{i,\ell}$ in a word in the alphabet $X_{i,r}$. So all small historical sectors have base of the form $\mathcal{R}\mathcal{P}$ (or inverse ones).

Consider the word W_j obtained after the application of the last connecting rule $\theta(34^-)^{\pm 1}$. Only small historical sectors of W_j are not locked (except for the sectors of S-machine $\mathbf{M}_3$, which are not touched by Step 3). If the next step is (3), then its rules cannot make the historical sector shorter by Lemma 3.9. If the next Step (4^-), then no sector becomes shorter by Lemma 3.1. The same is true for Step 3^- if it follows Step 3. Repeating this procedure, we have $|W_j|_a \leq |W_t|_a$. Therefore one can reduce the problem to shorter step histories.

Assume that the step history is $(3)(4^-)(3)$. This resembles item 3c, but below we consider small historical sectors instead of the big ones.

Suppose the base has a small historical sector $\mathcal{R}^{i,\ell}\mathcal{P}^{i,r}$ or $\mathcal{R}_s\mathcal{P}$. Let $W_r \to \cdots \to W_t$ be the maximal subcomputation with step history (3). Then we obtain inequality $|W_r|_a \leq c|W_t|_a$ by Lemma 4.11. Hence one can reduce our task to the subcomputations with the step histories (3) and (4^-). Therefore we assume further that the base has no small historical sectors of the form $\mathcal{R}\mathcal{P}$.

Also there are no small historical sectors $(\mathcal{P}^{i,r})^{-1}\mathcal{P}^{i,r}$ (or $\mathcal{P}^{-1}\mathcal{P}$), because state control letters of such sector should start running after the application of the connecting rule $\theta(34^-)$ but they cannot ever meet state letter from $\mathcal{R}$ and will run forever by Lemma 3.3; the last Step 3 will not be reached.

So all small historical sectors are of the form $\mathcal{R}^{i,\ell}(\mathcal{R}^{i,\ell})^{-1}$ (or $\mathcal{R}_s\mathcal{R}_s^{-1}$) But then the word from such a sector in "right" alphabet will be conjugated at Step 3 by the letters from a left alphabet. Hence after application of $\theta(34)^{-1}$, each rule of Step 3 will increase the length of such sector by 2. By Lemmas 3.10 and 2.1, we have $|W_r|_a \leq \cdots \leq |W_t|_a$ if the last Step 3 starts with W_r. This reduces the problem to the subcomputations with the step histories (3) and (4^-) again.

If there are no such sectors, then there are no sectors of the S-machine $\mathbf{D}_5$ changed at Step 3 since the base is faulty. The other sectors (which could come from the base of $\mathbf{M}_2$) do not work at Steps 3 and cannot decrease the length towards W_0 or W_t at Steps 3. This makes the statement of the lemma obvious.

It remains to consider the brief history $(3)(4^-)$ and apply the projection argument to the part (4^-). (Compare with case $(2^-)(1)$ in item 3c.)

Hence one may assume from now that there are no Steps 4^- in the computation.

9. If there is Step 3 in the step history, then there are no steps except for 3 and (3^-). As in item 8, one may assume that the length of step history is at least 3. Then the subwords $(3^-)(3)(3^-)$ can be eliminated by the same argument we used in item 8 to eliminate subwords $(4^-)(3)(4^-)$. It remains to consider computations with step history $(3)(3^-)(3)$. Again, one refer to item 8 since one can eliminate such history in the way the histories $(3)(4^-)(3)$ were eliminated in item 8.

The lemma is proved. $\qquad\square$

4.4. Space and time of M-computations with standard base.

Lemma 4.14. *Let* $\mathcal{C}: W_0 \to \cdots \to W_t$ *be a computation with standard base and step history* $(21^-)(1^-)(1)(2^-)(2)(21^-)$. *Then the configuration* W_t *is a copy of* W_0 *except for the sector* $\mathcal{R}^{1,r}\mathcal{P}^{2,l}$ *and the mirror copy of it, whose lengths in* W_0 *and* W_t *differ by one.*

If $H(1^-)$ *and* $H(2^-)$ *are the histories of the subcomputations* $\mathcal{C}(1^-)$ *and* $\mathcal{C}(2^-)$ *of* $\mathcal{C}$ *with step histories* $(21^-)(1^-)(1^-1)$ *and* $(12^-)(2^-)(21^-)$, *respectively, then*

$$|\|H(1^-)\| - \|H(2^-)\|| = 2.$$

Proof. The subcomputations $\mathcal{C}(1^-)$ and $\mathcal{C}(2^-)$ do not change the a-words in the historical sectors by Lemma 3.6(a), and so the histories of Steps 1 and 2 are inverse copies of each other. Taking into account that the transition (12^-) changes the length of the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,l}$ (and the mirror it) by 1, we obtain the first statement of the lemma.

After the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,l}$ changes length by one, the primitive S-machine checking this sector changes the computation time by 2, as it follows from Lemma 3.1 (3). This proves the second statement. $\qquad\square$

Recall that the blocks (E) and (F) of a history were defined before Lemma 4.7.

Lemma 4.15. (1) *If a configuration* W_0 *is accepted and* θ-*admissible for a rule* θ *from block* (E), *then there is a reduced accepting computation* $W_0 \to \cdots \to W_t$ *with block history* (E) *and* $t \le 3\|W_0\|$;

(2) *There is a constant* c_1 *depending on* $\mathbf{M}$ *only, such that for any computation* $\mathcal{C}: W_0 \to \cdots \to W_t$ *of* $\mathbf{M}$, *which is the beginning of a reduced accepting computation with block history* (E) *or* $(F)(E)$, *we have*

$$\|W_j\| \le c_1 (\|W_0\| + s)$$

for every $j = 0, \ldots, t$, *where* s *is the length of the step history.*

Proof. (1) Note that all the sectors of W_t are empty. It follows that all the words in big historical sectors of W_0 are copies of the same word since every rule of $\mathbf{M}$ multiplies the tape words of these sectors by the copies of the same letter or does not change all these words. Similarly, the tape words in the small historical sectors (and in their mirror sectors) are copies of each other.

If the small historical sectors of W_0 are non-empty, then θ is a rule of Step 4. So there is an accepting computation erasing all the letters of this sector (and of its mirror copy). The length of the next control Step 5^- will be at most $2\|W_0\|$ by the definition of the S-machine $\mathbf{P}_{5-}$ and Lemma 3.1 (3). Then the rules of Step 5 can erase all tape letters in the big historical sectors (and their mirror copies). This gives the total upper estimate $t \leq 2\|W_0\| + \|W_0\|$, as required. If θ is a rule of Step 5^- or 5, then the estimate is even better.

(2) If the computation is accepting and has type (E), then the step history is a suffix of the word $(4)(5^-)(5)$ by Lemma 4.7. The rules of Step 5^- cannot increase the lengths of configurations by Lemma 3.1. Clearly, the rules of Steps 4 and 5 cannot insert letters too. Hence $\|W_j\| \leq |W_0\|$, and so it suffices to prove the same inequality under the assumption that $j \leq r$, where $W_0 \to \cdots \to W_r$ is the maximal subcomputation with block history (F).

By Lemma 4.9, the step history of this subcomputation is a suffix of the word

$$\mathcal{W}(m) \equiv (4^-)(3)(3^-)((1^-)(1)(2^-)(2))^m(3^-)(3)(4^-)$$

for some non-zero integer m.

At first we consider the subcomputations $\mathcal{C}_i$ with step histories

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}.$$

By Lemma 4.14, we conclude that in the beginning and at the end of $\mathcal{C}_i$, the difference of lengths of configurations is equal to ± 2. The number of such subcomputations $\mathcal{C}_i$ does not exceed $s/2$.

The number of one-step subcomputations, which are not subcomputations of any $\mathcal{C}_i$, is at most 7. The transitions of some of them $(1^-, 2^-, 3^-, 4^-)$ do not increase the lengths of configurations by Lemma 3.6 (a). All transitions of each other step $(1, 2, 3)$ can increase the a-length but Lemma 4.11 bounds possible enlargement from above. This proves Statement (2). $\qquad\square$

Lemma 4.16. *Let a history H of a computation $\mathcal{C}\colon W_0 \to \cdots \to W_t$ with standard base have type (F) and end with a connecting rule. Suppose that $\mathcal{C}$ is a beginning of a reduced accepting computation and there are at most 10 steps in $\mathcal{C}$. Then for a constant $c_2 = c_2(\mathbf{M})$, we have*

$$\|H\| \leq c_2\|W_0\|.$$

Proof. Recall that the length of the history is linearly bounded for each of the steps in terms of the lengths of their configurations (Lemma 2.7 for Steps 4 and 5, Lemma 3.9 for Steps 1, 2 and 3, Lemmas 4.5(2) and 3.1 (1) for control steps.) The lengths of these configurations are linearly bounded in Lemma 4.15 (2). Taking into account that the number of steps is at most 10 we come to the desired inequality. □

Lemma 4.17. *There is a constant c_3 such that for any accepted configuration W_0, which is $\theta(2, 1^-)^{\pm 1}$-admissible, there exists an accepting computation with block history $(F)(E)$ of length at most $c_3(k^3 + 1)(\|W_0\| + k^3)$, where k is the a-length of the input sector $\mathcal{R}_s \mathcal{P}$ of the word W_0. The number of steps in this computation is less than $32k^3 + 4$.*

Proof. By Lemma 4.15 (1), an accepted word θ-admissible for a rule θ from block (E) can be accepted by a computation having only one block. Hence it suffices to consider computations with block histories $(F)(E)$.

By Lemma 4.9, the step history of block (F) is a word

$$((1^-)(1)(2^-)(2))^m (3^-)(3)(4^-)$$

for some integer m since W_0 is $\theta(21^-)^{\pm 1}$ admissible.

Assume that $|m| > (2k)^3$. Then the rule $\theta(12^-)$ occurs at least $(2k)^3 + 1$ times in the history. The rule $\theta(12^-)$ permanently changes the length of the sector $\mathcal{R}^{1,r} \mathcal{P}^{2,\ell}$ by one multiplying it by the same letter. On the other hand, by Lemma 4.14, the subcomputations with step histories

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}$$

do not change the a-word of length k in the input sector $\mathcal{R}_s \mathcal{P}$. Therefore there is a transition (not the last one), where the a-length of the sector $\mathcal{R}^{1,r} \mathcal{P}^{2,\ell}$ is divisible by $(2k)^3$, and so the transition to Step 3 was possible earlier by Lemma 3.16 (2). Hence it suffices to prove the lemma under the assumption $|m| \le (2k)^3$, and so the number of steps does not exceed $4|m| + 3 \le 32k^3 + 3$ by Lemma 4.7.

The length of the history is linearly bounded for each of the steps in terms of the lengths of the configurations (Lemma 2.7 for Steps 4 and 5, Lemma 3.9 for Steps 1, 2 and 3, Lemmas 4.5(2) and 3.1 (1) for control steps.) The lengths of these configurations are linearly bounded in Lemma 4.15 (2). Taking into account that the number of steps is linearly bounded in terms of m and $|m| \le (2k)^3$, we come to the desired inequality. □

We will consider a suitable function function $f(n)$ and the functions $g(n)$ and $F(n)$ from Definition 1.2 under the assumption that $s = 2$ in that definition. (The inequality $s \ge 3$ will appear in the last Subsection 10.2.) So the recognizing Turing S-machine $\mathbf{M_0}$ is taken from that definition and $\mathbf{M_1}$ is given by Lemma 2.4.

Without loss of generality, one may assume that the values of $f(n)$ are greater than some constant, for instance, for every integer $n \geq 1$, we have

$$f(n) \geq 5. \qquad (4.2)$$

Besides, it is convenient to enlarge the domain of those functions assuming that they are defined for every real $x \geq 0$. One may assume that $f(x)$ is still positive for $x > 0$ and non-decreasing.

Lemma 4.18. *For every $x > 0$ and $d \in [0, x)$, we have*

$$F(x) - F(x - d) \geq \frac{d}{x} F(x) = dxg(x).$$

Proof. Note that $F(x)/x^2 \geq F(x - d)/(x - d)^2$ since the function $g(x)$ is non-decreasing. Hence

$$F(x) - F(x - d) \geq F(x)\left(1 - \frac{(x - d)^2}{x^2}\right)$$
$$\geq F(x)\frac{2dx - d^2}{x^2} \geq F(x)\frac{dx}{x^2} = \frac{d}{x}F(x). \qquad \square$$

Lemma 4.19. *Let $\mathcal{C}: W_0 \to \cdots \to W_t$ be a computation of $\mathbf{M}$ with block history (F) and the step history of this computation contains a subword*

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}.$$

Let k be the a-length of the input sector $\mathcal{R}_s \mathcal{P}$ of a $\theta(2^-1)^{\pm 1}$-admissible configuration W_j in $\mathcal{C}$. Then (a) $k = f(n)$, for some $n \geq 1$, (b) $k = O(f(\|W_0\|))$, and (c) $k^3 = O(\|W_0\|)$.

Proof. Claim (c) follows from (b) since $f(n)^3 = O(n)$.

To prove (b) statement, it suffices to prove that $k = O(f(r))$, where r is the length of a big historical sector of W_0. We have $r' \leq r$, where r' is the length of a big historical sector of W_j by Lemma 4.9; indeed, the computation $W_j \to \cdots \to W_0$ cannot decrease the length of it by Lemma 3.6 (a) applied to the control steps and Lemma 3.9 applied to Steps 1 and 2. (Step 3 does not change this sector.) For the same reason $r'' \leq r'$, and $k = k''$ where r'' is the length of a big historical sector of the first configuration D_0 of a subcomputation $\mathcal{D}$ of $\mathcal{C}^{\pm 1}$ with the step history $(1^-1)(1)(12^-)$ and k'' is the a-length of the input sector $\mathcal{R}_s \mathcal{P}$ of D_0. Hence it suffices to prove that $k'' = O(f(r''))$.

The subcomputation of Step 1 (restricted to the base of $\mathbf{M}_3$) is actually the computation of $\mathbf{M}_2$. If we ignore the historical sectors, we have the accepting computation of $\mathbf{M}_1$ with input sector $a^{k''}$. By the definition of $\mathbf{M}_0$ and $\mathbf{M}_1$, $k'' = f(n)$ for some $n \geq 1$, the accepting computation of $\mathbf{M}_0$ has length $\Theta(n^{1/3})$,

and so the number of rules at Step 1 is $\Theta(n)$ by Lemma 2.4. Hence $r'' \geq \Theta(n)$, because D_0 contains the history of Step 1 written in big historical sectors. Hence, $k'' = f(n) = O(f(r''))$. Since $k = k''$, Property (a) is obtained as well. $\square$

Lemma 4.20. *Let W_0 be an accepted word and $\mathcal{C} \colon W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{M}$ with block history (E), where the computation either length-non-increasing or length-non-decreasing, or $(F)(E)$, where the block (E) is a length-non-increasing subcomputation, or (F). Then:*

(a) $\|W_j\| \leq c_4 \max \big(\|W_0\|, \|W_t\| \big) \ (j = 0, \dots, t)$, *where $c_4 = c_4(\mathbf{M})$; or*

(b) *there are accepting computations for W_0 and W_t with block histories (E) or $(F)(E)$ and histories H_0 and H_t such that*

$$\|H_0\| + \|H_t\| < t/100,$$

the history H of $\mathcal{C}$ has a factorization $H = H(1)H(2)H(3)$, where

$$\|H(1)\|, \|H(2)\| < t/100,$$

$H(2)$ is of type

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^m$$

with $|m| \geq c_3$ and the lengths of the subhistories of $H(2)$ of type

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}$$

are less than $t/10$.

Proof. If $\mathcal{C}$ is of type (E), then Property (a) follows since $\mathcal{C}$ either non-increases or non-decreases the lengths of configurations.

Let $\mathcal{C}$ have type $(F)(E)$. Again we obtain Property (a) by Lemma 4.15 (2) if the number of steps in block (F) is less than 10.

Otherwise, by Lemmas 4.7 and 4.9, block (F) contains a subword

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}$$

in the step history, and by Lemma 4.14, the corresponding computations $\mathcal{C}_i$ just multiply the words in the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$ (and in its mirror copy) by a letter b or by b^{-1} depending on the sign of the exponent, while the length k of the word in the sector $\mathcal{R}_s\mathcal{P}$ is not changed.

The history corresponding to the block (F) is $H_1 H_2 H_3$, where the length of the step history at most 4 for H_1 and H_3, H_2 has the form

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^m,$$

and H_2 is the history of the subcomputation $W_{s_0} \to \cdots \to W_{s_1} \to \cdots \to W_{s_m}$ starting with $\theta(21^-)^{\pm 1}$, and every W_{s_i} is a $\theta(21^-)^{\pm 1}$-admissible word. If

$$w = \max\left(\|W_{s_0}\|, \|W_{s_m}\|\right),$$

then by Lemma 4.14,

$$\max\left(\sum_{i=0}^{m-1} \|W_{s_i}\|, \sum_{i=1}^{m} \|W_{s_i}\|\right) \geq w + |w-2| + \cdots + |w-2|m-1|| \geq |m|w/2. \quad (4.3)$$

Hence the sum of the lengths of the subcomputations $\mathcal{C}_0, \ldots, \mathcal{C}_{m-1}$ is at least $|m|w$ by Lemma 3.1 for the Steps 1^- and 2^- of these subcomputations. Therefore, $\|H_2\| > |m|w$.

Now if $\max(\|W_0\|, \|W_t\|) = w' \geq w$, then the length of every W_j with $j \leq s_0$ or $j \geq s_m$ is bounded by $c_1(w' + 10)$ by Lemma 4.15 (2), and the same estimate works if W_j is a configuration of some subcomputation $\mathcal{C}_i$. Hence we have the inequality of item (a) of the lemma.

So we assume now that $w' < w$. Let k be the a-length of the input sector $\mathcal{R}_s\mathcal{P}$ of the word W_{s_0}. If $m \geq c_3^2 k^3$, then $t \geq \|H_2\| > c_3^2 k^3 w$. We also have that each of the subcomputations $\mathcal{C}_i$ with step histories

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}$$

have length less than $\frac{t}{10}$. This follows from the property, that the difference of the lengths of $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ is at most 4 (the computational time for control step can change by 2 by Lemma 4.14, but the number of such subcomputations is at least $c_3^2 > 1000$).

Let $W_{s_0} \equiv V_0 \to \cdots \to V_d$ be a shortest accepting computation for W_{s_0} with a history H_0. To estimate d from above, we may assume by Lemma 4.15 (1) that its block history is either (E) or $(F)(E)$. The step history of block (E) has length at most 3 by Lemma 4.7.

If the number of steps in H_0 is at most 10, then $\|H_0\| = O(\|W_0\|)$ by Lemma 4.16. Otherwise by Lemma 4.9, $H_0 = H'H''$, where H' has step history

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}$$

and $H'H''$ starts with a $\theta(21^-)^{\pm 1}$-admissible configuration W_{s_0}. By Lemma 4.17,

$$\begin{aligned}
\|H'H''\| &\leq c_3(k^3 + 1)\left(\|W_{s_0}\| + k^3\right) \\
&\leq c_3(k^3 + 1)(w + k^3) = c_3(k^3 + 1)O(w)
\end{aligned} \quad (4.4)$$

since $k^3 = O(w)$ by Lemma 4.19.

Thus by (4.4),

$$\|H'H''\| \le c_3^2 k^3 w/400 \le \|H_2\|/400 \le (s-r)/400$$

if c_3 is large enough. To estimate the length of the shortest accepting computation for W_0, it remains to estimate $s_0 = \|H_1\|$, but this value does not exceed $c_2 w \le c_3^2 k^3 w/800 \le t/400$ by Lemma 4.16. Therefore,

$$\|H_0\| \le t/400 + t/400 = t/200 \le \|H_2\|/200.$$

Similarly, we obtain

$$\|H_t\| \le t/200 \le \|H_2\|/200.$$

Now define $H(1) = H_1$, $H(2) = H_2$, and $H(3) = H_3\bar{H}$, where $\bar{H}$ is of type (E) and so has at most 3 steps, and $H(3)$ has less than 10 steps. To obtain Property (2) of the lemma if $m \ge c_3^2 k^3$, it remains to estimate $H(3)$. Indeed, by Lemmas 4.15 (2) and 4.16, we have

$$\|H(3)\| = \|H_3\| + \|\bar{H}\| \le c_2 w + 3w \le t/100.$$

Assume now that $|m| < c_3^2 k^3$. Then the number of steps s in the subcomputation $\mathcal{C}$ does not exceed $10c_3^2 k^3 + 20$. Therefore for every configuration W_j of $\mathcal{C}$, we obtain from Lemmas 4.15 (2) and 4.19 that

$$\|W_j\| \le c_1(w' + 10c_3^2 k^3 + 20) = c_1 c_3^2 O(w') \le c_4 w',$$

if c_4 is big enough, and we have Property (a).

The same argument works if the block history is just (F).   □

Lemma 4.21. *Let W_0 be an accepted word and $\mathcal{C}: W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{M}$ with block history of the form $(E)(F)\dots(F)(E)$, where the first (the last) block (E) is a length-non-decreasing (resp., length-non-increasing) subcomputation. Then there are accepting computations for W_0 and W_t with block histories (E) or $(F)(E)$ and histories H_0 and H_t such that $\|H_0\| + \|H_t\| < t/100$.*

Proof. Since the word W_0 is θ-admissible for a rule θ of block E, we have $\|H_0\| \le 3\|W_0\|$ by Lemma 4.15 (1), and $\|W_0\| \le \|W_s\|$, where $\mathcal{C}': W_s \to \cdots \to W_r$ is the subcomputation corresponding to the first occurrence of (F) in the block history of $\mathcal{C}$, because the rules of the first block (E) does not decrease the lengths.

Let k and m be the parameters of $\mathcal{C}'$ defined as in the proof of Lemma 4.20. Note that $m \ne 0$ by Lemmas 4.9 (1) and 4.5 (1) since $\mathcal{C}'$ starts with $\theta(4^-4)^{-1}$. Due to the maximality of $\mathcal{C}'$, one can apply Lemmas 4.9 (2) and 4.19 (a) to $\mathcal{C}'$ and obtain

$$|m| \ge (2k)^3 \ge 1000$$

since the length k of any word accepted by $\mathbf{M}_1$ and $\mathbf{M}_2$ is at least 5 by (4.2). Then, as in Lemma 4.20, we obtain that the length of $\mathcal{C}'$ is at least

$$2\|W_s\| + 2|\|W_s\| - 2| + \cdots + 2|\|W_s\| - 1998|,$$

which is at least $1000\|W_s\|$ and at least $1000\|W_0\| > 300\|H_0\|$.

Similarly we have that the length of $\mathcal{C}''$ is greater than $300\|H_t\|$, where the computation $\mathcal{C}''$ corresponds to the last occurrence of (F) in the block history of $\mathcal{C}$. It follows that $\|H_0\| + \|H_t\| < t/100$. $\qquad\square$

Lemma 4.22. *Let W_0 be an accepted word, $\mathcal{C}\colon W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{M}$ and H_0, H_t be the histories of the shortest computations accepting W_0 and W_t, respectively. Then:*

(1) $\mathcal{C}$ *is a product of at most three subcomputations*

$$\mathcal{C}_1\colon W_0 \to \cdots \to W_{n_1}, \quad \mathcal{C}_2\colon W_{n_1} \to \cdots \to W_{n_1+n_2},$$

and $\quad \mathcal{C}_3\colon W_{n_1+n_2} \to \cdots \to W_{n_1+n_2+n_3} \quad (n_1 + n_2 + n_3 = t),$

where
$$\max\left(\|W_{n_1}\|, \|W_{n_1+n_2}\|\right) \leq \max\left(\|W_0\|, \|W_t\|\right)$$

and for every $\mathcal{C}_i$ $(i = 1, 2, 3)$ either:

(a) $\|W_j\| \leq c_4 \max\left(\|W_0\|, \|W_t\|\right)$, *for every configuration W_j of $\mathcal{C}_i$, where $c_4 = c_4(\mathbf{M})$; or*

(b) *there are accepting computations for the first and the last configuration of $\mathcal{C}_i$ with block histories (E) or $((F)(E)$ and histories H_i' and H_i'' such that $\|H_i'\| + \|H_i''\| < n_i/100$.*

(2) *The sum of lengths of all maximal subcomputations of $\mathcal{C}$ with block history (E) does not exceed $3(\|W\|_0 + \|W_t\|) + t/100$.*

Proof. (1) If the block history of $\mathcal{C}$ is (F), then the entire computation $\mathcal{C}$ satisfies either (a) or (b) by Lemma 4.20. If the block history of $\mathcal{C}$ is $(E)(F)$, we consider the subcomputation

$$\mathcal{C}'\colon W_0 \to \cdots \to W_s$$

corresponding to the first block (E) of the step history. If W_{n_1} has minimal length in $\mathcal{C}'$, then the subcomputation

$$\mathcal{C}''\colon W_{n_1} \to \cdots \to W_s$$

is length-non-decreasing; this follows from Lemma 2.7 for Steps 4 and 5 and from Lemma 3.6 for Step 4^-. Similarly, the subcomputation

$$W_0 \to \cdots \to W_{n_1}$$

is length-non-increasing. Since $\|W_{n_1}\| \leq \|W_0\|$ we can again apply Lemma 4.20 to the subcomputations

$$W_0 \to \cdots \to W_{n_1} \quad \text{and} \quad W_{n_1} \to \cdots \to W_t.$$

Therefore we assume that there are at least three blocks.

Consider the subcomputation

$$\mathcal{C}': W_r \to \cdots \to W_s$$

corresponding to the first block (E) occured in the step history. It has a maximal length-non-increasing part $W_r \to \cdots \to W_{n_1}$ as in the previous paragraph.

Observe that $\|W_r\| \leq \|W_0\|$. Indeed, only historical sectors can be unlocked in W_r, but neither control S-machine can increase the lengths of these sectors in the computation $W_r \to \cdots \to W_0$ with block history (F) by Lemma 3.1, nor the computations of Steps 1, 2, 3 can do this by Lemma 3.9.

Since $\|W_{n_1}\| \leq \|W_r\| \leq \|W_0\|$, we can apply Lemma 4.20 to the subcomputation

$$\mathcal{C}_1: W_0 \to \cdots \to W_{n_1}$$

and obtain one of the properties (a) or (b) for it. Similarly, we consider the last block (E) in the block history of $\mathcal{C}$ and define the subcomputation $\mathcal{C}_3$ starting with $W_{n_1+n_2}$ whose length does not exceed the length of W_t, and so either (a) or (b) holds for $\mathcal{C}_3$. We have $n_2 = 0$ if the block history is $(F)(E)(F)$.

If there are at least two blocks (E), then the middle computation $\mathcal{C}_2$ satisfies the assumptions of Lemma 4.21, and so Property (b) holds for it.

(2) The subdivision of each subcomputation corresponding to a (E) according the sample of part (1) gives the required estimate. Namely, if a subcomputation $\mathcal{D}$ is a product $\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3$ with block history $(E)(F)(E)$ satisfying Lemma 4.21, then we obtain that the length of $\mathcal{D}_1$ plus the length of $\mathcal{D}_3$ is less than 0.01 of the length of $\mathcal{D}_2$. If $\mathcal{D}$ has brief history (E) or $(F)(E)$, or $(E)(F)$, the we refer to Lemma 4.20. □

Lemma 4.23. *For every accepted word W_0 of length at most n there is an accepting computation of length $O(nf(n)^3)$ with number of steps $O(f(n)^3)$. The generalized time function $T'(n)$ of $\mathbf{M}$ is equivalent to $\Theta(nf(n)^3)$.*

Proof. By Lemma 4.15 (1), given an accepted word W_0 of length n, there is a shortest accepting computation $W_0 \to \cdots \to W_t$ with block history either (E) or $(F)(E)$. We denote by H its history. The step history of block (E) has length at most $3\|W_0\|$ by Lemma 4.7 and contains at most three steps.

If the number of steps in H is at most 10, then

$$\|H\| = O\big(\|W_0\|\big) = O(n)$$

by Lemma 4.16. Otherwise by Lemma 4.9, $H = H_1 H_2 H_3$, where H_1 has less than 10 steps and H_2 has step history

$$((1^-)(1)(2^-)(2))^{\pm 1}$$

and $H_2 H_3$ starts with a $\theta(21^-)^{\pm 1}$-admissible configuration W_j. By Lemma 4.17,

$$\|H_2 H_3\| \leq c_3(k^3 + 1)\big(\|W_j\| + k^3\big),$$

where k is the length of the sector $\mathcal{R}_s \mathcal{P}$ in W_j, and the number of steps in $H_2 H_3$ is $O(k^3)$ (and so the number of steps in the entire H is $O(k^3)$). Here,

$$\|W_j\| \leq c_1\big(|W_0|_a + 10\big) = O(n)$$

by Lemma 4.15 (2).

Since $k = O(f(\|W_j\|)) = O(n)$ and $k^3 = O(\|W_j\|)$ by Lemma 4.19, we have

$$O(k^3) = O\big(f(n)^3\big)$$

for the number of steps and

$$\|H_2 H_3\| \leq c_3(k^3 + 1)\big(\|W_j\| + k^3\big) = c_3\big(f(n)^3\big)O(n) = O(n)g(n)$$

by the definition of the functions $f(n)$ and $g(n)$.

The length of each of each one-step subhistory of H_1 is bounded by $4c_1(|W_0|_a + 10)$ (use Lemma 3.6 (b) for Steps $1^-, 2^-, 3^-, 4^-, 5^-$, Lemma 2.8 (b) for Steps 1, 2, and 3, and Lemma 2.7 for Steps 4 and 5). Hence the length of the whole history H is also $O(n)g(n)$, as required.

To bound $T'(n)$ from below, we will construct a series of accepted words $V(n)$ of length $\Theta(n)$. The base of every $V(n)$ is standard, and $V(n)$ is $\theta(2^-1)$-acceptable. The input sector $\mathcal{R}_s \mathcal{P}$ contains a^k, where $k = f(n) > 0$, the word in the sector $\mathcal{R}^{2,\ell} \mathcal{P}^{3,\ell}$ is b^l, where $l = \Theta(n) > 8k^3$ and l congruent to $4k^3$ modulo $8k^3$. (There is such l since $k^3 = f(n)^3 = O(n)$.) Each of the big historical sectors of $V(n)$ contains the history of an accepting computation for $\mathbf{M}_2$, written in the alphabets $X_{i,\ell}$. The length of this history is $O(n)$ by the definition of the suitable function $f(n)$, the definition of the machines $\mathbf{M}_0 - \mathbf{M}_2$ and Lemma 2.4). Each of the small historical sectors contains the history of the computation of $\mathbf{D}_5$ (also in left alphabets) that checks that $l - 4k^3$ is divisible by $8k^3$. Since $l = \Theta(n)$, this history has length $O(n)$ by Lemma 3.16. Thus, we have $\|V_n\| = \Theta(n)$.

Every word $V(n)$ is accepted. Indeed, the rules of Step 1^- can check all the sectors since the base is standard. Then the rules of Step 1 can accept $f(n)$, $\theta(12^-)$ replaces b^l with b^{l-1}. The rules of Step 2^- check the sectors again, the history of Step 2 copies the inverse history of Step 1, it restores the alphabets $X_{i,\ell}$ in big historical sectors. Then we repeat the cycle decreasing the exponent at b by one

again. After $4k^3$ such cycles we obtain b^{l-4k^3} in the sector $\mathcal{R}^{2,\ell}\mathcal{P}^{3,\ell}$, where $l - 4k^3$ is divisible by $8k^3$, and therefore after Step 3^-, Step 3 can complete its work by Lemma 3.16 (b). It remains to erase all tape letters using the rules of block (E) and stop computing after the rule θ_0 is applied.

Now let us estimate from below the length of arbitrary (reduced) computation

$$V(n) \equiv W_0 \to \cdots \to W_t.$$

By Lemma 4.9, we have the block (F) of the form

$$((1^-)(1)(2^-)(2))^m (3^-)(3)(4^-)$$

in the history, where $m \equiv 4k^3 \pmod{8k^3}$ by Lemma 4.9 (2). Hence,

$$|m| \geq 4k^3 = 4f(n)^3.$$

The history of every subcomputation with step history $((1^-)(1)(2^-)(2))^{\pm 1}$ has length at least $\Theta(n)$ for the following reason. Every configuration of it has a word in the sector $\mathcal{R}^{2,\ell}\mathcal{P}^{3,\ell}$ of length $\Theta(n)$ since this length belongs to the segment $[l - 4k^3, l]$. So by Lemma 3.1, one needs $\Theta(n)$ rules to check this sector at the control steps 1^- and 2^-.

Therefore the length of the computation $W_0 \to \cdots \to W_t$ is at least $4k^3\Theta(n) = \Theta(nf(n)^3)$, as desired. Since we obtain the required lower bound for every n and $\|V_n\| = \Theta(n)$, the lemma is proved.   □

Remark 4.24. A subcomputation with step history

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}$$

does not change the length of the sector $\mathcal{R}^{2,\ell}\mathcal{P}^{3,\ell}$ by Lemma 3.6 applied to steps (1^-) and (2^-). Hence we have the same property for computations $((1^-)(1)(2^-)(2))^m$ starting and ending with connecting rules. Thus, above we obtained $\Theta(nf(n)^3)$ configurations of length at least $\Theta(n)$ for any computation accepting the word $V(n)$.

We call a base B of a reduced computation (and the computation itself) *revolving* if $B \equiv xvx$ for some letter x and a word v, and B has no proper subword of this form.

If $v \equiv v_1zv_2$ for some letter z, then the word zv_2xv_1z is also revolving. One can cyclically permute the sectors of revolving computation with base xvx and obtain a uniquely defined computation with the base zv_2xv_1z, which is called a cyclic permutation of the original computation. The history and lengths of configurations do not change when one cyclically permutes a computation.

Lemma 4.25. *There is a constant $c_4 = c_4(\mathbf{M})$ such that following holds. For any computation $\mathcal{C}: W_0 \to \cdots \to W_t$ of $\mathbf{M}$ with a revolving base xvx either:*

(1) *we have the inequality*

$$\|W_j\| \le c_4 \max \left(\|W_0\|, \|W_t\| \right),$$

for every word W_j of $\mathcal{C}$, where $c_4 = c_4(\mathbf{M})$; or

(2) *we have the following properties:*

(a) *the word xv is a cyclic permutation of the standard base $B = B(\mathbf{M})$ or of B^{-1};*

(b) *the corresponding cyclic permutations W_0' and W_t' of the words W_0 and W_t are accepted words;*

(c) *the step history of $\mathcal{C}$ (or of the inverse computation) contains subwords*

$$(21^-)(1^-)(1^-1) \text{ and } (12^-)(2^-)(2^-2);$$

and

(d) *$\mathcal{C}$ is a product of at most three subcomputations*

$$\mathcal{C}_1: W_0' \to \cdots \to W_{n_1}', \quad \mathcal{C}_2: W_{n_1}' \to \cdots \to W_{n_1+n_2}',$$

and $\quad \mathcal{C}_3: W_{n_1+n_2}' \to \cdots \to W_{n_1+n_2+n_3}' \quad (n_1 + n_2 + n_3 = t),$

where

$$\max \left(\|W_{n_1}'\|, \|W_{n_1+n_2}'\| \right) \le \max \left(\|W_0\|, \|W_t\| \right)$$

and for each $\mathcal{C}_i$, either:

(d1) *$\|W_j'\| \le c_4 \max(\|W_0'\|, \|W_t'\|)$, for every configuration W_j of $\mathcal{C}_i$; or*

(d2) *there are accepting computations for the first and the last configuration of $\mathcal{C}_i$ with histories H_i' and H_i'' such that $\|H_i'\| + \|H_i''\| < n_i$ and the corresponding block histories are either (E) or $(F)(E)$.*

Proof. If the computation is faulty, then Property (1) is given by Lemma 4.13 since $c_4 > C$. If it is non-faulty, then we have all sectors of the base in the same order as in the standard base (or its inverse), and we obtain Property (2a). Therefore we may assume now that the base xv is standard and Property (1) does not hold.

If the block history of $\mathcal{C}$ is (E), we obtain a contradiction with Lemma 4.8 since $c_4 > 1$.

If the computation has only one step of type (F), then Property (1) follows from Lemmas 3.6, 3.12 and 3.17, a contradiction again. So there is a connecting rule θ from block (F) in the history.

Assume there is a block (F) in the block history of $\mathcal{C}$, and this block has at least 8 steps. Then by Lemma 4.9, the step history of $\mathcal{C}$ has a subword

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1},$$

and Property (c) follows. Moreover the words at the big (small) history sectors are copies of the same word since the subcomputations of Step (1^-) (or 2^-) have simultaneously controlled these sectors. Therefore after a number of such cycles one can obtain the length of the sector $\mathcal{R}^{1,r}\mathcal{P}^{2,\ell}$ divisible by $8k^3$ (where k is the length of the sector $\mathcal{R}_s\mathcal{P}$), which by Lemma 4.9 (2), makes possible to accept after the Steps $3^-, 3, 4^-, 4, 5$. So one obtains Properties (a), (b), and (c). Then Property (d) follows from Lemma 4.22.

If there are no such blocks (F), then there are no subwords $(E)(F)(E)$ in the block history by Lemmas 4.9 and 4.5. Hence the block history is $(F)(E)(F)$ or a subword of this word. Let configuration W'_r and W'_s subdivide $\mathcal{C}$ in single block computations. Then $\|W'_r\| < c_3\|W'_0\|$, because there are at most 7 steps in the subcomputation $W'_r \to \cdots \to W'_0$, and each step transition from W'_j towards W'_0 can multiply the length by at most c. (See Lemma 3.6 for control steps and Lemma 4.11 for Steps 1, 2, and 3.) Analogously, we have $\|W'_s\| < c_3\|W'_0\|$. Since for every step the lengths of all configurations are linearly bounded in terms the first and the last configurations (see Lemmas 3.12, 3.17 3.6 (a)) we have

$$\|W'_j\| \le c_3 \max\left(\|W'_0\|, \|W'_t\|\right)$$

if $j \le r$ or $j \ge s$. So to obtain Property (1) (and a contradiction), it suffices to linearly bound the configurations in the subcomputation $W'_r \to \cdots \to W'_s$ in terms of $\max(\|W'_r\|, \|W'_s\|)$. This is done in Lemma 4.8 (1). Thus, the proof is complete. $\square$

4.5. Two more properties of standard computations. Here we prove two lemmas needed for the estimates in Section 9. The first one says (due to Lemma 4.5 (2)) that if a standard computation $\mathcal{C}$ is very long in comparison with the lengths of the first and the last configuration, then it can be completely restored if one knows the history of $\mathcal{C}$, and the same is true for the long subcomputations of $\mathcal{C}$. This makes the auxiliary parameter $\sigma_\lambda(\Delta)$ useful for some estimates of areas of diagrams Δ. The second lemma is helpful for the proof of Lemma 9.23 in Section 9.

Lemma 4.26. *Let* $\mathcal{C}: W_0 \to \cdots \to W_t$ *be a reduced computation with standard base, where* $t \ge c_5 \max(\|W_0\|, \|W_t\|)$ *for sufficiently large constant* $c_5 = c_5(\mathbf{M})$. *Suppose the word* W_0 *is accepted. Then any subcomputation* $\mathcal{D}: W_r \to \cdots \to W_s$ *of* $\mathcal{C}$ *(or the inverse for* $\mathcal{D}$*) of length at least* $0.4t$ *contains one of the words*

$$(21^-)(1^-)(1^-1), \quad (12^-)(2^-)(2^-2), \quad (23^-)(3^-)(3^-3), \quad (34^-)(4^-)(4^-4)$$

in the step history.

Proof. If the block history of $\mathcal{C}$ is (F), we refer to Lemma 4.20 as follows.

Assume that Property (a) of that lemma holds. Then every step of the computation $\mathcal{D}$ has length at most $4c_4 \max(\|W_0\|, \|W_t\|)$ by Lemma 3.9 for Steps 1, 2, and 3 and by Lemma 3.6 (b) for other steps. Hence the number of steps in $\mathcal{D}$ has to be at least 10 since the length of its history is at least $0.4c_5 \max(\|W_0\|, \|W_t\|)$ and c_5 can be chosen large enough. It follows from Lemma 4.9 (1) that the step history of $\mathcal{D}$ contains subwords

$$(21^-)(1^-)(1^-1) \text{ and } (12^-)(2^-)(2^-2),$$

as required.

If Property (a) of Lemma 4.20 fails, then by Property (b), we have a sub-computation of length $> 0.98t$ with step history $((1^-)(1)(2^-)(2))^{\pm m}$, where every cycle with block history $((1^-)(1)(2^-)(2))^{\pm 1}$ has length $< t/10$. Then the subcomputation $\mathcal{D}$ of length $\geq 0.4t$ has to contain such a cycle, and so the step history of $\mathcal{D}$ contains $(12^-)(2^-)(2^-2)$, as required. Thus, we may assume that the block history of $\mathcal{C}$ is not (F).

If the block history of $\mathcal{D}$ contains a subword $(E)(F)(E)$, then the statement follows from Lemma 4.9 (where Lemma 4.5 eliminates the case $m = 0$). So the block history of $\mathcal{D}$ is a subword of $(F)(E)(F)$. By Lemma 4.22 (2), the length of the (E)-subcomputation of $\mathcal{D}$ is less than

$$t/100 + 3(\|W_0\| + \|W_t\|) \leq t/50.$$

So one of the (F)-subcomputations of $\mathcal{D}$ has length $> (0.4 - 0.02)t/2 = 0.19t$.

Case 1. Assume that there is a block (E) in the block history of $\mathcal{D}$, and without loss of generality, we may assume that the computation $\mathcal{D}$ has a subcomputation

$$\mathcal{D}' : W_r \to \cdots \to W_j$$

of type (F) with $j - r > 0.19t$ and the subcomputation of type (E) occurs after $\mathcal{D}'$ in $\mathcal{D}$. Proving by contradiction, we conclude that the step history of $\mathcal{D}'$ is (4^-) since a longer step history would provide us with the subword

$$(34^-)(4^-)(4^-4)$$

in the step history of $\mathcal{D}$.

Suppose the subcomputation

$$W_0 \to \cdots \to W_r$$

also has a block (E). Then $\mathcal{C}$ has a subcomputation with block history $(E)(F)(E)$. Let the subcomputation $\bar{\mathcal{C}}$ correspond to the middle block (F). Then the first and the last configurations of $\bar{\mathcal{C}}$ are admissible for some rules of type (E). Therefore by

Lemma 4.15 (1), one can construct an auxiliary computation $\widetilde{\mathcal{C}} = \mathcal{C}'\bar{\mathcal{C}}\mathcal{C}''$, where the first factor (the third one) starts (resp., ends) with Step (5). Then by Lemmas 4.9 and 4.19 for $\widetilde{\mathcal{C}}$, the subcomputation $\bar{\mathcal{C}}$ of $\mathcal{C}$ has a subcomputation of type (F) with step history

$$((1^-)(1)(2^-)(2))^{m_1}(3^-)(3)(4^-),$$

$|m_1| \geq 8k^3 \geq 1000$, and the subcomputation corresponding to the last Step (4^-) has length $> 0.19t$. It follows that at least 1000 control steps of the form (1^-) or (2^-) should have length $> 0.19t$ since their control S-machines have to check the big and small historical sectors too (and the length of the historical sectors are unchanged by the rules of

$$((1^-)(1)(2^-)(2))^{m_1}(3^-)(3)(4^-)).$$

We obtain a contradiction since $1000 \times 0.19t > t$.

Thus, the computation $\mathcal{E}: W_0 \to \cdots \to W_j$ is of type (F). Hence $\|W_j\| \leq \|W_0\|$, because only historical sectors can be unlocked in W_j, but neither control S-machine can decrease the lengths of these sectors in the computation $W_j \to \cdots \to W_0$ with block history (F) by Lemma 3.1, nor the computations of Steps 1, 2, and 3 can do this by Lemma 3.9.

If the step history of $\mathcal{E}$ ends with $(34^-)(4^-)(4^-4)$, then

$$\|W_0\| \geq \|W_j\| \geq 0.19t/4 > 0.04t$$

by Lemma 3.6, which contradicts to the assumption of the lemma. Hence the step history of $\mathcal{E}$ is (4^-), and so $0.19t \leq 4\|W_0\|$ by Lemma 3.6 (b), a contradiction again.

Case 2. The block history of $\mathcal{D}$ is (F). Since the block history of $\mathcal{C}$ is not (F) but a subword of $(F)(E)\ldots$, we conclude without loss of generality, that $\mathcal{C}$ begins with a maximal subcomputation $\mathcal{E}: W_0 \to \cdots \to W_u$ of type (F), where $r \leq u < t$. Then as in Case 1, we have $\|W_u\| \leq \|W_0\|$.

Now consider the options (a) and (b) provided by Lemma 4.20 for $\mathcal{E}$. The option (b) is eliminated exactly as in the second paragraph of the proof of the current lemma, where t can be replaced by $0.4t$ since $u \geq r - s \geq 0.4t$. Hence we have by (a) that every configuration W_j of $\mathcal{E}$ satisfies the inequality $\|W_j\| \leq c_4\|W_0\|$ since $\|W_u\| \leq \|W_0\|$. Then the length of every single step of $\mathcal{E}$ cannot exceed $4c_4\|W_0\|$ (see Lemma 3.6 (b) for control steps and 2.8 for Steps 1, 2, and 3). Here we have $4c_4 < c_5/10$ by the choice of c_5.

Since $s - r \geq 0.4t \geq 0.4c_5\|W_0\|$, we see that the length of the step history of $\mathcal{D}$ is at least 4. It follows from Lemma 4.9 that the step history of $\mathcal{D}$ contains one of the words mentioned in the formulation of Lemma 4.26. $\square$

Lemma 4.27. *Let a reduced computation $W_0 \to \cdots \to W_t$ start with an accepted word W_0, have standard base, and have step history of length 1. Assume that for some index j, we have $|W_j|_a > 3|W_0|_a$. Then there is a sector QQ' such that a state*

letter from Q or from Q' inserts a letter increasing the length of this sector after any transition of the subcomputation $W_j \to \cdots \to W_t$.

Proof. Let the step history be (1). Note that all big historical sectors of any configuration W_i have the same content (up to taking a copy) since the word W_0 is accepted. Assume that no rule of the subcomputation $\mathcal{D}: W_0 \to \cdots \to W_j$ increases the length of big historical sectors. Then by Lemma 2.8 (b) the length of the history of $\mathcal{D}$ does not exceed h, where h is the a-length of such sectors.

Every rule of the subcomputation $\mathcal{D}$ can change the length of any working sector at most by 1. (See Lemma 2.1 (3)). Hence if its length in W_0 is ℓ, its length in W_j is at most $\ell + h$. It follows that $|W_j|_a \leq 3|W_0|_a$, because the working sectors of $\mathbf{M_2}$ and its historical sectors alternate in the standard base. This contradicts to the assumption of the lemma.

Thus, there is a rule in the history of $\mathcal{D}$ increasing the length of a big historical sector QQ'. It has to insert a letter from $X_{i,\ell}$ from the left and a letter from $X_{i,r}$ from the right. Since the obtained word is not a word over one of these alphabets, Step 1 is not over, and the next rule has to increase the length of the sector again in the same manner since the computation is reduced. This procedure will repeat until one gets W_t. This proves the statement. The same proof works for Steps 2 and 3. (In the later case, one will consider small historical sectors.)

It follows from the definition of Step 4 (of Step 5) that every rule either increase or decrease the length of small (resp., of big) history sectors. If any rule increases it, then all the next rules will increase the lengths of these sectors too. Hence the argument of the previous paragraph works for Steps 4 and 5 as well.

For the control Steps $1^- - 5^-$, the statement of the lemma follows from Lemma 3.1 (1): if we have a transition of a primitive S-machine, where the control state letter increases the length of a sector, then it will keep increasing it in any reduced computation. $\square$

5. Groups and diagrams

5.1. The groups. Every S-machine can be simulated by finitely presented group (see [22] and also [17, 19]). Here we apply such a construction to the S-machine $\mathbf{M}$. To simplify formulas, it is convenient to change the notation. From now on we shall denote by N the length of the standard base of $\mathbf{M}$.

Thus the set of state letters is $Q = \sqcup_{i=0}^{N} Q_i$, where $Q_N = Q_0 = \{t\}$, $Y = \sqcup_{i=1}^{N} Y_i$, and Θ is the set of rules of the S-machine $\mathbf{M}$.

The finite set of generators of the group M consists of *q-letters* corresponding to the states Q, *a-letters* corresponding to the tape letters from Y, and *θ-letters* corresponding to the rules from the positive part Θ^+ of Θ (the same letter as for the S-machine).

The relations of the group M correspond to the rules of the S-machine $\mathbf{M}$. Recall that the cyclic S-machine $\mathbf{M}$ satisfies Property (1) of Lemma 2.1, and so every rule $\theta \in \Theta^+$ of it has the form

$$\theta : [U_0 \to V_0, \ldots U_N \to V_N],$$

where $U_0 \equiv U_N$ and $V_0 \equiv V_N$. For every such rule θ, we introduce the following relation of the group M.

$$U_i \theta_{i+1} = \theta_i V_i, \quad \theta_j a = a \theta_j, \qquad i, j = 0, \ldots, N \tag{5.1}$$

for all $a \in Y_j(\theta)$. (Here $\theta_N \equiv \theta_0$.) The first type of relations will be called (θ, q)-*relations*, the second type - (θ, a)-*relations*.

Finally, the required group G is given by the generators and relations of the group M and by one more additional relation, namely the *hub*-relation

$$(W_M)^L = 1, \tag{5.2}$$

where W_M is the accept word (of length N) of the S-machine $\mathbf{M}$ and the exponent L is a large enough integer. (It depends on $\mathbf{M}$ and will be made more precise later.) The corresponding cells in van Kampen diagrams looks like hubs in the net of q-bands (see pictures in [15, 20, 22]).

5.2. Van Kampen diagrams. Recall that a van Kampen *diagram* Δ over a presentation $P = \langle A \mid \mathcal{R} \rangle$ (or just over the group P) is a finite oriented connected and simply–connected planar 2–complex endowed with a labeling function

$$\text{Lab} : E(\Delta) \to A^{\pm 1},$$

where $E(\Delta)$ denotes the set of oriented edges of Δ, such that $\text{Lab}(e^{-1}) \equiv \text{Lab}(e)^{-1}$. Given a cell (that is a 2-cell) Π of Δ, we denote by $\partial \Pi$ the boundary of Π; similarly, $\partial \Delta$ denotes the boundary of Δ. The labels of $\partial \Pi$ and $\partial \Delta$ are defined up to cyclic permutations. An additional requirement is that the label of any cell Π of Δ is equal to (a cyclic permutation of) a word $R^{\pm 1}$, where $R \in \mathcal{R}$. The label and the combinatorial length $\|\mathbf{p}\|$ of a path $\mathbf{p}$ are defined as for Cayley graphs.

The van Kampen Lemma states that a word W over the alphabet $A^{\pm 1}$ represents the identity in the group P if and only if there exists a diagram Δ over P such that $\text{Lab}(\partial \Delta) \equiv W$, in particular, the combinatorial perimeter $\|\partial \Delta\|$ of Δ equals $\|W\|$. ([11, Ch. 5, Theorem 1.1]; our formulation is closer to Lemma 11.1 of [13]). The word W representing 1 in P is freely equal to a product of conjugates to the words from $R^{\pm 1}$. The minimal number of factors in such products is called the *area* of the word W. The *area* of a diagram Δ is the number of cells in it. By van Kampen Lemma, $\text{Area}(W)$ is equal to the area of a diagram having the smallest number of cells among all diagrams with boundary label $\text{Lab}(\partial \Delta) \equiv W$.

We will study diagrams over the groups M and G. The edges labeled by state letters (= q-letters) will be called q-edges, the edges labeled by tape letters (= a-letters) will be called a-edges, and the edges labeled by θ-letters are θ-edges.

We denote by $|\mathbf{p}|_{\mathbf{a}}$, ($|\mathbf{p}|_{\boldsymbol{\theta}}$, $|\mathbf{p}|_{\mathbf{q}}$) the a-length (resp., the θ-length, the q-length) of a path/word $\mathbf{p}$, i.e. the number of a-edges/letters (the number of θ-edges/letters, the number of q-edges/letters) in $\mathbf{p}$.

The cells corresponding to relation (5.2) are called *hubs*, the cells corresponding to (θ, q)-relations are called (θ, q)-*cells*, and the cells are called (θ, a)-*cells* if they correspond to (θ, a)-relations.

A van Kampen diagram is *reduced*, if it does not contain two cells (= closed 2-cells) that have a common edge e such that the boundary labels of these two cells are equal if one reads them starting with e (if such pairs of cells exist, they can be removed to obtain a diagram of smaller area and with the same boundary label). To study (van Kampen) diagrams over the group G we shall use their simpler subdiagrams such as bands and trapezia, as in [2, 15, 22], etc. Here we repeat one more necessary definition.

Definition 5.1. Let $\mathcal{Z}$ be a subset of the set of letters in the set of generators of the group M. A $\mathcal{Z}$-band $\mathcal{B}$ is a sequence of cells $\pi_1, \ldots, \pi_n$ in a reduced van Kampen diagram Δ such that:

- Every two consecutive cells π_i and π_{i+1} in this sequence have a common boundary edge $\mathbf{e}_i$ labeled by a letter from $\mathcal{Z}^{\pm 1}$.

- Each cell π_i, $i = 1, \ldots, n$ has exactly two $\mathcal{Z}$-edges in the boundary $\partial \pi_i$, $\mathbf{e}_{i-1}^{-1}$ and $\mathbf{e}_i$ (i.e. edges labeled by a letter from $\mathcal{Z}^{\pm 1}$) with the requirement that either both $\text{Lab}(e_{i-1})$ and $\text{Lab}(e_i)$ are positive letters or both are negative ones.

- If $n = 0$, then $\mathcal{B}$ is just a $\mathcal{Z}$-edge.

The counter-clockwise boundary of the subdiagram formed by the cells $\pi_1, \ldots, \pi_n$ of $\mathcal{B}$ has the factorization $\mathbf{e}^{-1} \mathbf{q}_1 \mathbf{f} \mathbf{q}_2^{-1}$ where $\mathbf{e} = \mathbf{e}_0$ is a $\mathcal{Z}$-edge of π_1 and $\mathbf{f} = \mathbf{e}_n$ is an $\mathcal{Z}$-edge of π_n. We call $\mathbf{q}_1$ the *bottom* of $\mathcal{B}$ and $\mathbf{q}_2$ the *top* of $\mathcal{B}$, denoted $\mathbf{bot}(\mathcal{B})$ and $\mathbf{top}(\mathcal{B})$. Top/bottom paths and their inverses are also called the *sides* of the band. The $\mathcal{Z}$-edges $\mathbf{e}$ and $\mathbf{f}$ are called the *start* and *end* edges of the band. If $n \geq 1$ but $\mathbf{e} = \mathbf{f}$, then the $\mathcal{Z}$-band is called a $\mathcal{Z}$-*annulus*.

We will consider q-bands, where $\mathcal{Z}$ is one of the sets Q_i of state letters for the S-machine $\mathbf{M}$, θ-bands for every $\theta \in \Theta$, and a-bands, where $\mathcal{Z} = \{a\} \subseteq Y$. The convention is that a-bands do not contain (θ, q)-cells, and so they consist of (θ, a)-cells only.

Remark 5.2. To construct the top (or bottom) path of a band $\mathcal{B}$, at the beginning one can just form a product $\mathbf{x}_1 \ldots \mathbf{x}_n$ of the top paths $\mathbf{x}_i$-s of the cells $\pi_1, \ldots, \pi_n$ (where each π_i is a $\mathcal{Z}$-bands of length 1). No θ-letter is being canceled in the word $W \equiv \text{Lab}(\mathbf{x}_1) \ldots \text{Lab}(\mathbf{x}_n)$ if $\mathcal{B}$ is a q- or a-band since otherwise two neighbor cells

of the band would make the diagram non-reduced. For similar reason, there are no cancellations of q-letters in W if $\mathcal{B}$ is a θ-band

If $\mathcal{B}$ is a θ-band then a few cancellations of a-letters (but not q-letters) are possible in W. (This can happen if one of π_i, π_{i+1} is a (θ, q)-cell and another one is a (θ, a)-cell.) We will always assume that the top/bottom label of a θ-band is a reduced form of the word W. This property is easy to achieve: by folding edges with the same labels having the same initial vertex, one can make the boundary label of a subdiagram in a van Kampen diagram reduced (e.g. see [13] or [22]).

If the path $(\mathbf{e}^{-1}\mathbf{q}_1\mathbf{f})^{\pm 1}$ or the path $(\mathbf{f}\mathbf{q}_2^{-1}\mathbf{e}^{-1})^{\pm 1}$ is the subpath of the boundary path of Δ then the band is called a *rim* band of Δ. We shall call a $\mathcal{Z}$-band *maximal* if it is not contained in any other $\mathcal{Z}$-band. Counting the number of maximal $\mathcal{Z}$-bands in a diagram we will not distinguish the bands with boundaries $\mathbf{e}^{-1}\mathbf{q}_1\mathbf{f}\mathbf{q}_2^{-1}$ and $\mathbf{f}\mathbf{q}_2^{-1}\mathbf{e}^{-1}\mathbf{q}_1$, and so every $\mathcal{Z}$-edge belongs to a unique maximal $\mathcal{Z}$-band.

We say that a $\mathcal{Z}_1$-band and a $\mathcal{Z}_2$-band *cross* if they have a common cell and $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset$.

Sometimes we specify the types of bands as follows. A q-band corresponding to one of the letters t of the base is called a t-*band*.

The papers [2, 15, 18] contain the proof of the following lemma in a more general setting. (In contrast to [15, Lemma 6.1] and [18, Lemma 3.11], we have no x-cells here.)

Lemma 5.3. *A reduced van Kampen diagram Δ over M has no q-annuli, no θ-annuli, and no a-annuli. Every θ-band of Δ shares at most one cell with any q-band and with any a-band.* □

If $W \equiv x_1 \ldots x_n$ is a word in an alphabet X, X' is another alphabet, and $\phi \colon X \to X' \cup \{1\}$ (where 1 is the empty word) is a map, then $\phi(W) \equiv \phi(x_1) \ldots \phi(x_n)$ is called the *projection* of W onto X'. We shall consider the projections of words in the generators of M onto Θ (all θ-letters map to the corresponding element of Θ, all other letters map to 1), and the projection onto the alphabet $\{Q_0 \sqcup \cdots \sqcup Q_{N-1}\}$ (every q-letter maps to the corresponding Q_i, all other letters map to 1).

Definition 5.4. The projection of the label of a side of a q-band onto the alphabet Θ is called the *history* of the band. The step history of this projection is the *step history* of the q-band. The projection of the label of a side of a θ-band onto the alphabet $\{Q_0, \ldots, Q_{N-1}\}$ is called the *base* of the band, i.e. the base of a θ-band is equal to the base of the label of its top or bottom.

As for words, we may use representatives of Q_j-s in base words.

Definition 5.5. Let Δ be a reduced diagram over M, which has boundary path of the form $\mathbf{p}_1^{-1}\mathbf{q}_1\mathbf{p}_2\mathbf{q}_2^{-1}$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are sides of q-bands, and $\mathbf{q}_1$, $\mathbf{q}_2$ are maximal parts of the sides of θ-bands such that $\mathrm{Lab}(\mathbf{q}_1)$, $\mathrm{Lab}(\mathbf{q}_2)$ start and end with q-letters.

Then Δ is called a *trapezium*. The path $\mathbf{q}_1$ is called the *bottom*, the path $\mathbf{q}_2$ is called the *top* of the trapezium, the paths $\mathbf{p}_1$ and $\mathbf{p}_2$ are called the *left and right sides*

of the trapezium. The history (step history) of the q-band whose side is $\mathbf{p}_2$ is called the *history* (resp., step history) of the trapezium; the length of the history is called the *height* of the trapezium. The base of $\mathrm{Lab}(\mathbf{q}_1)$ is called the *base* of the trapezium.
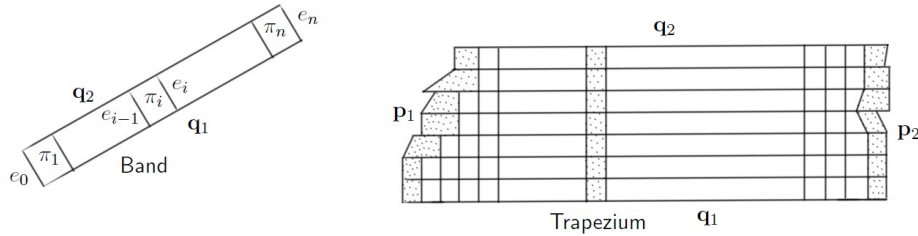

Figure 2. Band and trapezium.

Remark 5.6. Notice that the top (bottom) side of a θ-band $\mathcal{T}$ does not necessarily coincide with the top (bottom) side $\mathbf{q}_2$ (side $\mathbf{q}_1$) of the corresponding trapezium of height 1, and $\mathbf{q}_2$ ($\mathbf{q}_1$) is obtained from $\mathbf{top}(\mathcal{T})$ (resp., $\mathbf{bot}(\mathcal{T})$) by trimming the first and the last a-edges if these paths start and/or end with a-edges. We shall denote the *trimmed* top and bottom sides of $\mathcal{T}$ by $\mathbf{ttop}(\mathcal{T})$ and $\mathbf{tbot}(\mathcal{T})$. By definition, for arbitrary θ-band $\mathcal{T}$, $\mathbf{ttop}(\mathcal{T})$ is obtained by such a trimming only if $\mathcal{T}$ starts and/or ends with a (θ, q)-cell; otherwise $\mathbf{ttop}(\mathcal{T}) = \mathbf{top}(\mathcal{T})$. The definition of $\mathbf{tbot}(\mathcal{T})$ is similar.

By Lemma 5.3, any trapezium Δ of height $h \geq 1$ can be decomposed into θ-bands $\mathcal{T}_1, \ldots, \mathcal{T}_h$ connecting the left and the right sides of the trapezium. The word written on the trimmed top side of one of the bands $\mathcal{T}_i$ is the same as the word written on the trimmed bottom side of $\mathcal{T}_{i+1}$, $i = 1, \ldots, h$. Moreover, the following lemma claims that every trapezium simulates the work of $\mathbf{M}$. It summarizes the assertions of Lemmas 6.1, 6.3, 6.9, and 6.16 from [18]. For the formulation (1) below, it is important that $\mathbf{M}$ is an S-machine. The analog of this statement is false for Turing machines. (See [17] for a discussion.)

Lemma 5.7. (1) *Let Δ be a trapezium with history $\theta_1 \ldots \theta_d$ ($d \geq 1$). Assume that Δ has consecutive maximal θ-bands $\mathcal{T}_1, \ldots \mathcal{T}_d$, and the words U_j and V_j are the trimmed bottom and the trimmed top labels of $\mathcal{T}_j$ ($j = 1, \ldots, d$). Then the history of Δ is a reduced word, U_j, V_j are admissible words for M, and*

$$V_1 \equiv U_1 \cdot \theta_1, U_2 \equiv V_1, \ldots, U_d \equiv V_{d-1}, V_d \equiv U_d \cdot \theta_d.$$

(2) *For every reduced computation $U \to \cdots \to U \cdot H \equiv V$ of $\mathbf{M}$ with $\|H\| \geq 1$ there exists a trapezium Δ with bottom label U, top label V, and with history H.*

Using Lemma 5.7, one can immediately derive properties of trapezia from the properties of computations obtained earlier.

If $H' \equiv \theta_i \dots \theta_j$ is a subword of the history $\theta_1 \dots \theta_d$ from Lemma 5.7 (1), then the bands $\mathcal{T}_i, \dots, \mathcal{T}_j$ form a subtrapezium Δ' of the trapezium Δ. This subtrapezium is uniquely defined by the subword H' (more precisely, by the occurrence of H' in the word $\theta_1 \dots \theta_d$), and Δ' is called the H'-*part* of Δ.

We say that a trapezium Δ is *standard* if the base of Δ is the standard base **B** of **M** or $\mathbf{B}^{-1}$, and the step history of Δ (or the inverse one) contains one of the words

$$(21^-)(1^-)(1^-1), \ (12^-)(2^-)(2^-2), \ (23^-)(3^-)(3^-3),$$
$$(34^-)(4^-)(4^-4), \ (45^-)(5^-)(5^-5).$$

Remark 5.8. By Lemmas 5.7 and 4.10 (2), given the history H, one can reconstruct the entire standard trapezium Δ.

Definition 5.9. We say that a trapezium Γ is *big* if:

(1) the base of Δ or the inverse word has the form xvx, where xv a cyclic shift of the L-s power of the standard base;

(2) the diagram Γ contains a standard trapezium.

Lemma 5.10. *Let Δ be a trapezium whose base is xvx, where x occurs in v exactly $L-1$ times and other letters occur $< L$ times each (where L is as in (5.2)). Then either Δ is big or the length of a side of every θ-band of Δ does not exceed $c_5(\|W\| + \|W'\|)$, where W, W' are the labels of its top and bottom, respectively.*

Proof. The diagram Δ is covered by L subtrapezia Γ_i $(i = 1, \dots, L)$ with bases xv_ix.

Assume that the the step history of Δ (or inverse step history) contains one of the subwords

$$(21^-)(1^-)(1^-1), \ (12^-)(2^-)(2^-2), \ (23^-)(3^-)(3^-3), \ (34^-)(4^-)(4^-4).$$

Then by Lemma 4.5 (2) (and 5.7), the base of Δ has the form $(xu)^L x$, where xu is a cyclic shift of the standard base (or the inverse one) and the diagrams Γ_i-s $(i = 1, \dots, L)$ are just copies of each other. Since Δ contains a standard subtrapezia, it is big.

Now, under the assumption that the step history has no subwords mentioned in the previous paragraph, it suffices to bound the the length of a side of every θ-band of arbitrary Γ_i by $\leq c_4(\|V\|_a + \|V'\|)$, where V and V' are the labels of the top and the bottom of Γ_i.

Assume that the word xv_ix has a proper subword yuy, where u has no letters y, and any other letter occurs in u at most once. Then the word yuy is faulty since v_i has no letters x. By Lemma 4.13, we have

$$|U_j|_a \leq C \max \left(|U_0|_a, |U_t|_a\right)$$

for every configuration U_j of the computation given by Lemma 5.7 restricted to the base yuy. Since $c_4 > C$, it suffices to obtain the desired estimate for the computation

whose base is obtained by deleting the subword yu from xv_ix. Hence inducting on the length of the base of Γ_i, one may assume that it has no proper subwords yuy, and so the base of Γ_i is revolving. Now the required upper estimate for Γ_i follows from Lemma 4.25 (see (1) and (2c) there). ☐

5.3. Parameters. The following constants will be used for the proofs in this paper.

$$\lambda^{-1}, \; N \ll c_0 \ll \cdots \ll c_5 \ll L_0 \ll L \ll K \ll J$$
$$\ll \delta^{-1} \ll c_6 \ll c_7 \ll N_1 \ll N_2 \ll N_3 \ll N_4. \quad (5.3)$$

For each of the inequalities of this paper, one can find the highest constant (with respect to the order $\ll$) involved in the inequality and see that for fixed lower constants, the inequality is correct as soon as the value of the highest one is sufficiently large. This principle makes the system of all inequalities used in this paper consistent.

6. Diagrams without hubs

6.1. A modified length function. Let us modify the length function on the words and paths. The standard length of a word (a path) will be called the *combinatorial length* of it. From now on we use the word *"length"* for the modified length. We set the length of every q-letter equal 1, and the length of every a-letter equal to a small enough number δ so that

$$J\delta < 1. \quad (6.1)$$

We also set to 1 the length of every word of length ≤ 2 which contains exactly one θ-letter and no q-letters (such words are called (θ, a)-*syllables*). The length of a decomposition of an arbitrary word in a product of letters and (θ, a)-syllables is the sum of the lengths of the factors. *The length $|w|$ of a word w is the smallest length of such decompositions. The length $|\mathbf{p}|$ of a path* in a diagram is the length of its label. The *perimeter $|\partial\Delta|$* of a van Kampen diagram is similarly defined by cyclic decompositions of the boundary $\partial\Delta$.

The next statement follows from the above definitions and from the property of (θ, q)-relations and their cyclic shifts: The subword between two q-letters (between θ-letters) in arbitrary (θ, q)-relation is a syllable (has at most one q-letter and at most two a-letters).

Lemma 6.1. *Let $\mathbf{s}$ be a path in a diagram Δ having c θ-edges and d a-edges. Then:*

(a) $|\mathbf{s}| \geq \max(c, c + (d - c)\delta)$.

(b) $|\mathbf{s}| = c$ *if $\mathbf{s}$ is a top or a bottom of a q-band.*

(c) *for any product $\mathbf{s} = \mathbf{s_1}\mathbf{s_2}$ of two paths in a diagram, we have*

$$|\mathbf{s_1}| + |\mathbf{s_2}| \geq |\mathbf{s}| \geq |\mathbf{s_1}| + |\mathbf{s_2}| - \delta. \quad (6.2)$$

(d) *Let $\mathcal{T}$ be a θ-band with base of length l_b. Let l_a be the number of a-edges in the top path $\mathbf{topp}(\mathcal{T})$. Then the length of $\mathcal{T}$ (i.e. the number of cells in $\mathcal{T}$) is between $l_a - l_b$ and $l_a + 3l_b$.* □

Lemma 6.2. *Let Δ be a van Kampen diagram whose rim θ-band $\mathcal{T}$ has base with at most K letters. Denote by Δ' the subdiagram $\Delta \backslash \mathcal{T}$. Then $|\partial \Delta| - |\partial \Delta'| > 1$.*

Proof. Let $\mathbf{s}$ be the top side of $\mathcal{T}$ and $\mathbf{s} \subset \partial \Delta$. Note that the difference between the number of a-edges in the bottom $\mathbf{s}'$ of $\mathcal{T}$ and the number of a-edges in s cannot be greater than $2K$, because every (θ, q)-relator has at most two a-letters. Hence $|\mathbf{s}'| - |\mathbf{s}| \leq 2K\delta$. However, Δ' is obtained by cutting off $\mathcal{T}$ along $\mathbf{s}'$, and its boundary contains two θ-edges fewer than Δ. Hence we have $|\mathbf{s_0}| - |\mathbf{s_0'}| \geq 2 - 2\delta$ for the complements $\mathbf{s_0}$ and $\mathbf{s'_0}$ of $\mathbf{s}$ and $\mathbf{s}'$, respectively, in the boundaries $\partial \Delta$ and $\partial \Delta'$. Finally,

$$|\partial \Delta| - |\partial \Delta'| \geq 2 - 2\delta - 2K\delta - 4\delta > 1$$

by (6.1) and (6.2). □

We call a base word w *tight* if:

(1) for some letter x the word w has the form $uxvx$, where the letter x does not occur in u and x occurs in v exactly $L - 1$ times;

(2) every proper prefix w' of w does not satisfy Property (1).

Lemma 6.3. *If a base w of a θ-band has no tight prefixes, then $\|w\| \leq K_0$, where $K_0 = 2LN$.*

Proof. The hub base includes every base letter L times. Hence every word in this group alphabet of length $\geq K_0 + 1$ includes one of the letters $L + 1$ times. □

From now on we shall fix a constant K such that

$$K > 2K_0 = 4LN. \tag{6.3}$$

Definition 6.4. We say that a reduced diagram Γ is a *comb* if it has a maximal q-band $\mathcal{Q}$ (the *handle* of the comb), such that:

(C_1) $\mathbf{bott}(\mathcal{Q})$ is a part of $\partial \Gamma$, and every maximal θ-band of Γ ends at a cell in $\mathcal{Q}$.

If in addition the following properties hold:

(C_2) one of the maximal θ-bands $\mathcal{T}$ in Γ has a tight base (if one reads the base towards the handle); and

(C_3) other maximal θ-bands in Γ have tight bases or bases without tight prefixes,

then the comb is called *tight*.

The number of cells in the handle $\mathcal{Q}$ is the *height* of the comb, and the maximal length of the bases of the θ-bands of a comb is called the *basic width* of the comb.

Notice that every trapezium is a comb.
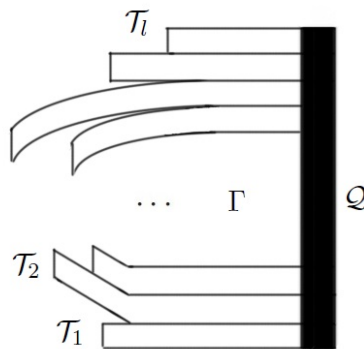
Figure 3. Comb.

Lemma 6.5 ([19, Lemma 4.10]). *Let l and b be the height and the basic width of a comb Γ and let $\mathcal{T}_1, \ldots \mathcal{T}_l$ be consecutive θ-bands of Γ (as in Figure 3). We can assume that* $\mathbf{bot}(\mathcal{T}_1)$ *and* $\mathbf{top}(\mathcal{T}_l)$ *are contained in $\partial\Gamma$. Denote by $\alpha = |\partial\Gamma|_a$ the number of a-edges in the boundary of Γ, and by α_1 the number of a-edges on* $\mathbf{bot}(\mathcal{T}_1)$. *Then $\alpha + 2lb \geq 2\alpha_1$, and the area of Γ does not exceed $c_0 bl^2 + 2\alpha l$ for some constant $c_0 = c_0(\mathbf{M})$.*

We say that a subdiagram Γ of a diagram Δ is a *subcomb* of Δ if Γ is a comb, the handle of Γ divides Δ in two parts, and Γ is one of these parts.

Lemma 6.6. *Let Δ be a reduced diagram over G with non-zero area, where every rim θ-band has base of length at least K. Assume that:*

(1) *Δ is a diagram over the group M; or*

(2) *Δ has a subcomb of basic width at least K_0.*

Then there exists a maximal q-band $\mathcal{Q}$ dividing Δ in two parts, where one of the parts is a tight subcomb with handle $\mathcal{Q}$.

Proof. (1) Let $\mathcal{T}_0$ be a rim band of Δ (Fig 1). Its base w is of length at least K, and therefore w has disjoint prefix and suffix of length K_0 since $K > 2K_0$ by (6.3). The prefix of this base word must have its own tight subprefix w_1, by Lemma 6.3 and the definition of tight words. A q-edge of $\mathcal{T}_0$ corresponding to the last q-letter of w_1 is the start edge of a maximal q-band $\mathcal{Q}'$ which bounds a subdiagram Γ' containing a band $\mathcal{T}$ (a subband of $\mathcal{T}_0$) satisfying Property (C_2). It is useful to note that a minimal suffix w_2 of w, such that w_2^{-1} is tight, allows us to construct another band $\mathcal{Q}''$ and a subdiagram Γ'' which satisfies (C_2) and has no cells in common with Γ'.

Thus, there are $\mathcal{Q}$ and Γ satisfying (C_2). Let us choose such a pair with minimal Area(Γ). Assume that there is a θ-band in Γ which does not cross $\mathcal{Q}$. Then there must exist a rim band $\mathcal{T}_1$ which does not cross $\mathcal{Q}$ in Γ. Hence one can apply the

construction from the previous paragraph to $\mathcal{T}_1$ and construct two bands $\mathcal{Q}_1$ and $\mathcal{Q}_2$ and two disjoint subdiagrams Γ_1 and Γ_2 satisfying the requirement (C_2) for Γ. Since Γ_1 and Γ_2 are disjoint, one of them, say Γ_1, is inside Γ. But the area of Γ_1 is smaller than the area of Γ, and we come to a contradiction. Hence Γ is a comb and condition (C_1) is satisfied.

Assume that the base of a maximal θ-band $\mathcal{T}$ of Γ has a tight proper prefix (we may assume that $\mathcal{T}$ terminates on $\mathcal{Q}$), and again one obtain a q-band $\mathcal{Q}'$ in Γ, which provides us with a smaller subdiagram Γ' of Δ, satisfying (C_2), a contradiction. Hence Γ satisfies Property (C_3) as well.

(2) The proof is shorter since a comb is given in the very beginning.   □
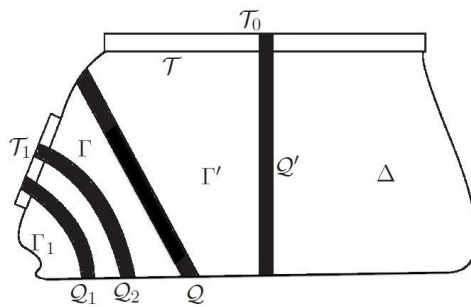
Figure 4. Lemma 6.6.

6.2. Mixture on the boundaries of diagrams. We will need a parameter of diagrams introduced in [16]. It was called mixture.

Let O be a circle with two-colored finite set of points (or vertices) on it, more precisely, let any vertex of this finite set be either black or white. We call O a *necklace* with black and white *beads* on it. We want to introduce the *mixture* of this finite set of beads.

Assume that there are n white beads and n' black ones on O. We define sets $\mathbf{P}_j$ of ordered pairs of distinct white beads as follows. A pair (o_1, o_2) $(o_1 \neq o_2)$ belongs to the set $\mathbf{P}_j$ if the simple arc of O drawn from o_1 to o_2 in clockwise direction has at least j black beads. We denote by $\mu_J(O)$ the sum $\sum_{j=1}^{J} \#\mathbf{P}_j$ (the J-*mixture* on O). Below similar sets for another necklace O' are denoted by $\mathbf{P}'_J$. In this subsection, $J \geq 1$, but later on it will be a fixed large enough number J from the list (5.3).

Lemma 6.7 ([16, Lemma 6.1]). (a) $\mu_J(O) \leq J(n^2 - n)$.

(b) *Suppose a necklace O' is obtained from O after removal of a white bead v. Then $\#\mathbf{P}'_j \leq \#\mathbf{P}_j$ for every j, and $\mu_J(O) - Jn < \mu_J(O') \leq \mu_J(O)$.*

(c) *Suppose a necklace O' is obtained from O after removal of a black bead v. Then $\#\mathbf{P}'_j \leq \#\mathbf{P}_j$ for every j, and $\mu_J(O') \leq \mu_J(O)$.*

(d) *Assume that there are three black beads v_1, v_2, v_3 of a necklace O, such that the clockwise arc v_1–v_3 contains v_2 and has at most J black beads (excluding v_1 and v_3), and the arcs v_1–v_2 and v_2–v_3 have m_1 and m_2 white beads, respectively. If O' is obtained from O by removal of v_2, then $\mu_J(O') \leq \mu_J(O) - m_1m_2$.* $\quad\square$

For any diagram Δ over G, we introduce the following invariant $\mu(\Delta) = \mu_J(\partial\Delta)$ depending on the boundary only. To define it, we consider the boundary $\partial(\Delta)$, as a *necklace*, i.e. we consider a circle O with $\|\partial\Delta\|$ edges labeled as the boundary path of Δ. By definition, the white beads are the mid-points of the θ-edges of O and black beads are the mid-points of the q-edges O. Then, by definition, the *mixture* on $\partial\Delta$ is $\mu(\Delta) = \mu_J(O)$.

6.3. Quadratic upper bound for quasi-areas of diagrams over M.

The Dehn function of the group M is greater that the required function $F(n) = n^2 f(n)^3$. For example, it is cubic if $f(n) = $ const. However we are going to find the Dehn function of G, and first we want to bound the areas of the words vanishing in M with respect to the presentation of G. For this goal we artificially introduce the concept of *G-area*. The G-area of a big trapezia can be much less that the real area of it in M. This concept will be justified at the end of this paper, where some big trapezia are replaced by diagrams with hubs, but having lesser areas.

Definition 6.8. The G-area $\mathrm{Area}_G(\Gamma)$ of a big trapezium Γ is, by definition, the minimum of the half of its area (i.e. the number of cells) and the product

$$c_5 h \big(\|\mathbf{top}(\Gamma)\| + \|\mathbf{bot}(\Gamma)\| \big),$$

where h is the height of Γ and c_5 is the constant from (5.3).

To define the G-area of a diagram Δ over M, we consider a family $\mathbf{S}$ of big subtrapezia (i.e. subdiagrams, which are trapezia) and single cells of Δ such that every cell of Δ belongs to a member Σ of this family, and if a cell Π belongs to different Σ_1 and Σ_2 from Σ, then both Σ_1 and Σ_2 are big subtrapezia of Δ with bases xv_1x, xv_2x, and Π is a (θ, x)-cell. (In the later case, the intersection $\Sigma_1 \cap \Sigma_2$ must be an x-band.) There is such a family "covering" Δ, e.g. just the family of all cells of Δ.

The G-area of $\mathbf{S}$ is the sum of G-areas of all big trapezia from $\mathbf{S}$ plus the number of single cells from $\mathbf{S}$ (i.e. the G-area of a cell Π is $\mathrm{Area}(\Pi) = 1$). Finally, the *G-area* $\mathrm{Area}_G(\Delta)$ is the minimum of thea G-areas of all "coverings" $\mathbf{S}$ as above.

It follows from the definition that $\mathrm{Area}_G(\Delta) \leq \mathrm{Area}(\Delta)$ since the G-area of a big trapezium does not exceed a half of its area.

Lemma 6.9. *Let Δ be a reduced diagram, and every cell π of Δ belongs in one of subdiagrams $\Delta_1, \ldots, \Delta_m$, where any intersection $\Delta_i \cap \Delta_j$ either has no cells or it is a q-band, Then $\mathrm{Area}_G(\Delta) \leq \sum_{i=1}^m \mathrm{Area}_G(\Delta_i)$.*

Proof. Consider the families $\mathbf{S}_1, \ldots, \mathbf{S}_m$ given by the definition of G-areas for the diagrams $\Delta_1, \ldots, \Delta_m$. Then the set $\mathbf{S} = \mathbf{S}_1 \cup \cdots \cup \mathbf{S}_m$ "covers" the entire Δ according to the above definition. This implies the required inequality for G-areas. $\square$

We will show that for some constant N_2 and N_1 the G-area of any reduced diagram Δ over M with perimeter n does not exceed $N_2 n^2 + N_1 \mu(\Delta)$. (Using the quadratic upper bound for $\mu(\Delta)$ from Lemma 6.7 (a), one deduces that the G-area is bounded by $N'n^2$ for some constant N'.) Roughly speaking, we are doing the following. We use induction on the perimeter of the diagram. First we remove rim θ-bands (those with one side and both ends on the boundary of the diagram) with short bases. This operation decreases the perimeter and preserves the sign of $N_2 n^2 + N_1 \mu(\Delta) - \mathrm{Area}_G(\Delta)$, so we can assume that the diagram does not have such bands. Then we use Lemma 6.6 and find a tight comb inside the diagram with a handle $\mathcal{C}$. We also find a long enough q-band $\mathcal{C}'$ that is close to $\mathcal{C}$. We use a surgery which amounts to removing a part of the diagram between $\mathcal{C}'$ and $\mathcal{C}$ and then gluing the two remaining parts of Δ together. The main difficulty is to show that, as a result of this surgery, the perimeter decreases and the measure and the mixture change in such a way that the expression $N_2 n^2 + N_1 \mu(\Delta) - \mathrm{Area}_G(\Delta)$ does not change its sign. In the proof, we need to consider several cases depending on the shape of the subdiagram between $\mathcal{C}'$ and $\mathcal{C}$. Note that neither $N_2 n^2$ nor $N_1 \mu(\Delta)$ nor $\mathrm{Area}_G(\Delta)$ alone behave in the appropriate way as a result of the surgery, but the expression $N_2 n^2 + N_1 \mu(\Delta) - \mathrm{Area}_G(\Delta)$ behaves as needed.

Remark 6.10. We introduced the surgery and used induction mentioned above in [19, Lemma 6.2] to obtain a worse upper bound $n^2 \log n$ for the area. But there were neither mixture, nor G (just M), nor G-area in [19], and a different definition for length $|*|$ was used there. Besides, we will use an auxiliary function $\Phi(x)$ in the proof to be able to repeat in part our argument later, for diagrams over G. So we shall prove Lemma 6.16 anew to obtain the better estimate.

So, N_1 and N_2 are big enough constants from the list (5.3). Here "big enough" means that they satisfy the inequalities used in the proof of Lemma 6.16 (such that as (6.13), (6.14), (6.15), (6.26), (6.30), (6.32), (6.34), (6.39)). Each of them has the form $N_i > *$ $(i = 1, 2)$, where the right-hand side $*$ does not depend on N_i (but depends on the constants introduced earlier). Since the number of inequalities is finite, the right choice of N_1, N_2 is possible.

Let $\Phi(x)$ be an arbitrary function defined for real $x \geq 0$ such that $\Phi(x) = x^2 \phi(x)$ for a non-decreasing function $\phi(x) > 0$ with $\phi(1) \geq 1$ and

$$\Phi(x) - \Phi(x - y) \geq xy\phi(x) \quad \text{for } 0 \leq y \leq x. \tag{6.4}$$

Remark 6.11. For this section, it suffices to take quadratic $\Phi(x)$ and $\phi(x) = \mathrm{const.}$, but to estimate the G-area of diagrams with hubs, we will take the functions $\Phi(x) = F(x)$ and $\phi(x) = g(x)$, satisfying inequality (6.4) by Lemma 4.18.

We are going to prove that the G-area of a reduced diagram Δ over M does not exceed $N_2 \Phi(n) + N_1 \phi(n) \mu(\Delta)$, where $n = |\partial \Delta|$. Arguing by contradiction in the remaining part of this section, we consider a *counter-example* Δ with minimal perimeter n. Of course, its G-area is positive, and, by Lemma 5.3, we have at least two θ-edges on the boundary $\partial \Delta$, and so $n \geq 2$.

If Γ is a comb with handle $\mathcal{C}$ and $\mathcal{B}$ is another maximal q-band in Γ, then $\mathcal{B}$ cuts up Γ in two parts, where the part that does not contain $\mathcal{C}$ is a comb Γ' with handle $\mathcal{B}$. It follows from the definition of comb, that every maximal θ-band of Γ crossing $\mathcal{B}$ connects $\mathcal{B}$ with $\mathcal{C}$. If $\mathcal{B}$ and $\mathcal{C}$ can be connected by a θ-band containing no (θ, q)-cells, then Γ' is called the *derivative subcomb* of Γ. Note that no maximal θ-band of Γ can cross the handles of two derivative subcombs.

Lemma 6.12. (1) *The diagram Δ has no two disjoint subcombs Γ_1 and Γ_2 of basic widths at most K with handles $\mathcal{B}_1$ and $\mathcal{B}_2$ such that some ends of these handles are connected by a subpath $\mathbf{x}$ of the boundary path of Δ, where $\mathbf{x}$ has at most N q-edges.*

(2) *The boundary of every subcomb Γ with basic width $s \leq K$ has $2s$ q-edges.*
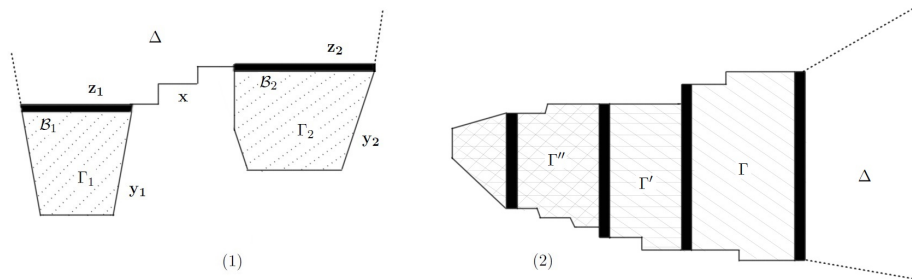
Figure 5. Lemma 6.12.

Proof. We will prove Statements (1) and (2) using simultaneous induction on $A = \text{Area}(\Gamma_1) + \text{Area}(\Gamma_2)$ (resp., on $A = \text{Area}(\Gamma)$). Arguing by contradiction, we consider a counter-example with minimal A.

(1) Since the area of Γ_i ($i = 1, 2$) is less than A, we may use Statement (2), and so we have at most $2K$ q edges in $\partial \Gamma_i$.

Let h_1 and h_2 be the lengths of the handles $\mathcal{B}_1$ and $\mathcal{B}_2$ of Γ_1 and Γ_2, resp. Without loss of generality, we assume that $h_1 \leq h_2$. Denote by $\mathbf{y}_i \mathbf{z}_i$ the boundaries of Γ_i ($i = 1, 2$), where $\mathbf{y}_i$ is the part of $\partial \Delta$ and $\mathbf{z}_i$ is the side of the handle of Γ_i (so $\mathbf{y}_1 \mathbf{x} \mathbf{y}_2$ is the part of the boundary path of Δ, see Fig. 5 (1)). Then each of the θ-edges $\mathbf{e}$ of $\mathbf{y}_1$ is separated in $\partial \Delta$ from every θ-edge $\mathbf{f}$ of $\mathbf{y}_2$ by less than $4K + N < J$ q-edges. Hence every such pair $(\mathbf{e}, \mathbf{f})$ (or the pair of white beads on these edges) makes a contribution to $\mu(\Delta)$.

Let Δ' be the diagram obtained by deleting the subdiagram Γ_1 from Δ. When passing from $\partial\Delta$ to $\partial\Delta'$, one replaces the θ-edges from $\mathbf{y}_1$ by the θ-edge of $\mathbf{z}_1$ belonging to the same maximal θ-band. The same is true for white beads.

But each of the $h_1 h_2$ pairs in the corresponding set P' of white beads is separated in $\partial\Delta'$ by less number of black beads than the pair defined by Δ. Indeed, since the handle of Γ_1 is removed when one replaces $\partial\Delta$ by $\partial\Delta'$, two black bead at the ends of this handle are removed, and therefore

$$\mu(\Delta) - \mu(\Delta') \geq h_1 h_2 \tag{6.5}$$

by Lemma 6.7 (d).

Let α be the number of a-edges in $\partial\Gamma_1$. It follows from Lemma 6.5 that the area, and so the G-area of Γ_1, does not exceed $C_1(h_1)^2 + 2\alpha h_1$, where $C_1 = c_0 K$.

Remark 6.13. The constants C_1, C_2, C_{12}, C_3 are not included in the list (5.3) since their values chosen here make sense only in the present subsection.

Since the boundary of Δ' has at least two q-edges fewer than Δ and $|\mathbf{z}_1| = h_1 \leq |\mathbf{y}_1|$, we have $|\partial\Delta'| \leq |\partial\Delta| - 2$. Moreover, we have from Lemma 6.1 (a) and Lemma 5.3 that

$$|\partial\Delta| - |\partial\Delta'| \geq \gamma = \max\left(2, \delta(\alpha - 2h_1)\right), \tag{6.6}$$

because the top/bottom of $\mathcal{B}_1$ has at most h_1 a-edges.

This inequality, inequality (6.5), and the inductive assumption related to Δ', imply that the G-area of Δ' is not greater than

$$N_2\Phi(n-\gamma) + N_1\phi(n)\mu(\Delta) - N_1\phi(n)h_1 h_2.$$

Adding the G-area of Γ_1 and using inequality (6.4), we see that by Lemma 6.9, the G-area of Δ does not exceed

$$N_2\Phi(n) - N_2\gamma n + N_1\phi(n)\mu(\Delta) - N_1\phi(n)h_1 h_2 + C_1 h_1^2 + 2\alpha h_1.$$

Since $h_1 \leq h_2$ and $\phi(n) \geq 1$, this will contradict the choice of the counter-example Δ when we prove that

$$-N_2\gamma n - N_1 h_1^2 + C_1 h_1^2 + 2\alpha h_1 < 0. \tag{6.7}$$

If $\alpha \leq 4h_1$, then inequality (6.7) follows from the inequalities $\gamma \geq 2$ and

$$N_1 \geq C_1 + 8. \tag{6.8}$$

Assume that $\alpha > 4h_1$. Then by (6.6), we have $\gamma \geq \frac{1}{2}\delta\alpha$ and $N_2\gamma n > 2\alpha h_1$ since $n \geq 2h_1$ by Lemma 5.3, and

$$N_2 > 2\delta^{-1}. \tag{6.9}$$

Since $N_1 h_1^2 > C_1 h_1^2$ by (6.8), the inequality (6.7) follows.

(2) If there are at least two derivative subcombs of Γ, then one can find two of them satisfying the assumptions of Statement (1) (moreover, with $|x|_q = 0$), and $\mathrm{Area}(\Gamma_1) + \mathrm{Area}(\Gamma_2) < \mathrm{Area}(\Gamma) = A$, a contradiction. Therefore there is a most one derivative subcomb Γ' in Γ (Fig. 5 (2)). In turn, Γ' has at most one derivative subcomb Γ'', and so one. It follows that there are no maximal q-bands in Γ except for the handles of $\Gamma', \Gamma'', \ldots$. Since the basic width of Γ is s, we have s maximal q-bands in Γ, and the lemma is proved. $\qquad\square$

Lemma 6.14. *There are no pair of subcombs Γ and Γ' in Δ with handles $\mathcal{X}$ and $\mathcal{X}'$ of length ℓ and ℓ' such that Γ' is a subcomb of Γ, the basic width of Γ does not exceed K_0 and $\ell' \leq \ell/2$.*
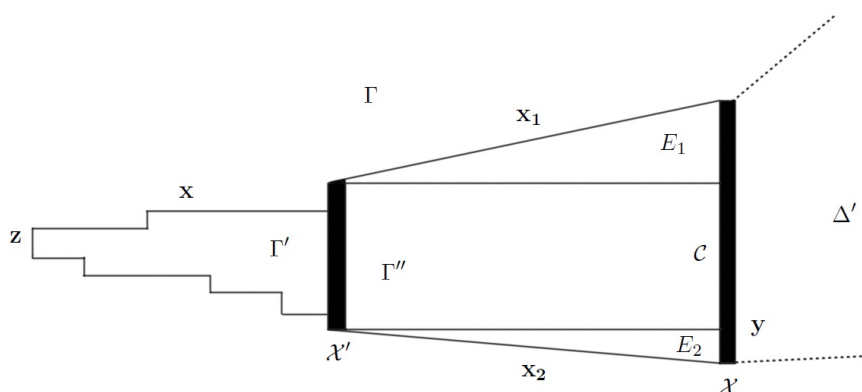
Figure 6. Lemma 6.14.

Proof. Proving by contradiction, one can choose Γ' so that ℓ' is minimal for all subcombs in Γ and so Γ' has no proper subcombs, i.e. its basic width is 1 (Fig. 6). It follows from Lemma 6.5 that for $\alpha = |\Gamma'|_a$, we have

$$\mathrm{Area}_G(\Gamma') \leq \mathrm{Area}(\Gamma') \leq c_0(l')^2 + 2\alpha l'. \qquad (6.10)$$

Let Δ' be the diagram obtained after removing the subdiagram Γ' from Δ. The following inequality is the analog of (6.6) (where h_1 is replaced by ℓ')

$$|\partial\Delta| - |\partial\Delta'| \geq \gamma = \max\big(2, \delta(\alpha - 2l')\big). \qquad (6.11)$$

The q-band $\mathcal{X}$ contains a subband $\mathcal{C}$ of length l'. Moreover one can choose $\mathcal{C}$ so that all maximal θ-bands of Γ crossing the handle $\mathcal{X}'$ of Γ', start from $\mathcal{C}$. These θ-bands form a comb Γ'' contained in Γ, and in turn, Γ'' contains Γ'. The two parts of the complement $\mathcal{X}\backslash\mathcal{C}$ are the handles of two subcombs E_1 and E_2 formed by

maximal θ-bands of Γ, which do not cross $\mathcal{X}'$. Let the length of these two handles be ℓ_1 and ℓ_2, respectively, and so we have $\ell_1 + \ell_2 = l - l' > l'$. ($E_1$ or E_2 can be empty; then ℓ_1 or ℓ_2 equals 0.)

It will be convenient to assume that Γ is drawn from the left of the vertical handle $\mathcal{X}$. Denote by $\mathbf{yz}$ the boundary path of Γ, where $\mathbf{y}$ is the right side of the band $\mathcal{X}$. Thus, there are l_1 (resp., l_2) θ-edges on the common subpath $\mathbf{x}_1$ (subpath $\mathbf{x}_2$) of $\mathbf{z}$ and ∂E_1 (and ∂E_2).

By Lemma 6.12 (2), the path $\mathbf{z}$ contains at most $2K_0$ q-edges, because the basic width of Γ is at most K_0.

Consider the factorization $\mathbf{z} = \mathbf{x}_2\mathbf{x}\mathbf{x}_1$, where $\mathbf{x}$ is a subpath of $\partial\Gamma'$. It follows that between every white bead on $\mathbf{x}_1$ (i.e. the middle point of the θ-edges on $\mathbf{x}_1$) and a white bead on $\mathbf{x}$ we have at most $2K_0$ black beads (i.e. the middle points of the q-edges of the path $\mathbf{x}$). Since J is greater than $2K_0$, every pair of white beads, where one bead belongs to $\mathbf{x}$ and another one belongs to $\mathbf{x}_1$ (or, similarly, to $\mathbf{x}_2$) contributes 1 to $\mu(\Delta)$. Let P denote the set of such pairs. By the definition of E_1 and E_2, we have $\#P = l'(\ell_1 + \ell_2) = l'(l - l') > (l')^2$.

When passing from $\partial\Delta$ to $\partial\Delta'$, one replaces the left-most θ-edges of every maximal θ-band from Γ' with the right-most θ-edges lying on the right side of $\mathcal{X}'$. The same is true for white beads. But each of the $l'(l - l')$ pairs in the corresponding set P' of white beads is separated in $\partial\Delta'$ by less number of black beads since the q-band $\mathcal{X}'$ is removed. Therefore every pair from P' gives less by 1 contribution to the mixture, as it follows from the definition of mixture. Hence $\mu(\Delta) - \mu(\Delta') \geq l'(l - l') \geq (l')^2$. This inequality, inequality (6.11), and the inductive assumption related to Δ', imply that the G-area of Δ' is not greater than

$$N_2\Phi(n - \gamma) + N_1\phi(n)\mu(\Delta) - N_1\phi(n)(\ell')^2.$$

Adding the G-area of Γ' (6.10) and applying inequality (6.4), we see that the G-area of Δ does not exceed

$$N_2\Phi(n) + N_1\phi(n)\mu(\Delta) - N_2\gamma n - N_1\phi(n)(l')^2 + c_0(l')^2 + 2\alpha l'.$$

This will contradict the choice of the counter-example Δ when we prove that

$$- N_2\gamma n - N_1(l')^2 + c_0(l')^2 + 2\alpha l' < 0, \tag{6.12}$$

because $\phi(n) \geq 1$. Consider two cases:

(a) Let $\alpha \leq 4l'$. Then inequality (6.12) follows from the inequalities $\gamma \geq 2$ and

$$N_1 \geq c_0 + 8. \tag{6.13}$$

(b) Assume that $\alpha > 4l'$. Then by (6.11) we have $\gamma \geq \frac{1}{2}\delta\alpha$ and $N_2\gamma n > 2\alpha l'$ since $n \geq 2l \geq 4l'$ by Lemma 5.3, and

$$N_2 > \delta^{-1}. \tag{6.14}$$

Since $N_1(l')^2 > c_0(l')^2$ by (6.13), the inequality (6.12) follows.

Thus, the lemma is proved by contradiction.   □

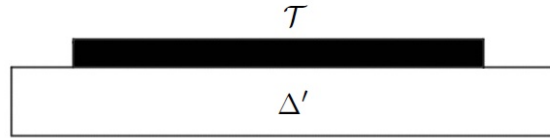**Lemma 6.15.** Δ *has no rim θ-band whose base has $s \leq K$ letters.*

Figure 7. Rim θ-band.

Proof. Assume by contradiction that such a rim θ-band $\mathcal{T}$ exists, and $\mathbf{top}(\mathcal{T})$ belongs in $\partial(\Delta)$ (Fig. 7). When deleting $\mathcal{T}$, we obtain, by Lemma 6.2, a diagram Δ' with $|\partial\Delta'| \leq n - 1$. Since $\mathbf{top}(\mathcal{T})$ lies on $\partial\Delta$, we have from the definition of the length, that the number of a-edges in $\mathbf{top}(\mathcal{T})$ is less than $\delta^{-1}(n - s)$. By Lemma 5.1, the length of $\mathcal{T}$ is at most $3s + \delta^{-1}(n - s) < \delta^{-1}n$. Thus, by applying the inductive hypothesis to Δ', we have that G-area of Δ is not greater than

$$N_2\Phi(n - 1) + N_1\phi(n)\mu(\Delta) + \delta^{-1}n$$

because $\mu(\Delta') \leq \mu(\Delta)$ by Lemma 6.7 (b). But the first term of this sum does not exceed $N_2\Phi(n) - N_2n$ by (6.4), and so the entire sum is bounded by

$$N_2\Phi(n) + N_1\phi(n)\mu(\Delta)$$

provided

$$N_2 \geq \delta^{-1}. \tag{6.15}$$

This contradicts to the choice of Δ, and the lemma is proved.   □

Lemma 6.16. *The G-area of a reduced diagram Δ over M does not exceed*

$$N_2\Phi(n) + N_1\phi(n)\mu(\Delta),$$

where $n = |\partial\Delta|$.

Proof. We continue studying the hypothetical minimal counter-example Δ. By Lemma 6.15, now we can apply Lemma 6.6 (1). By that lemma, there exists a tight subcomb $\Gamma \subset \Delta$. Let $\mathcal{T}$ be a θ-band of Γ with a tight base.

The basic width of Γ is less than K_0 by Lemma 6.3. Since the base of Γ is tight, it is equal to $uxvx$ for some x, where the last occurrence of x corresponds to the handle $\mathcal{Q}$ of Γ, the word u does not contain x, and v has exactly $L - 1$ occurrences of x. Let $\mathcal{Q}'$ be the maximal x-band of Γ crossing $\mathcal{T}$ at the cell corresponding to the first occurrence of x in $uxvx$ (Fig. 8 (a)).
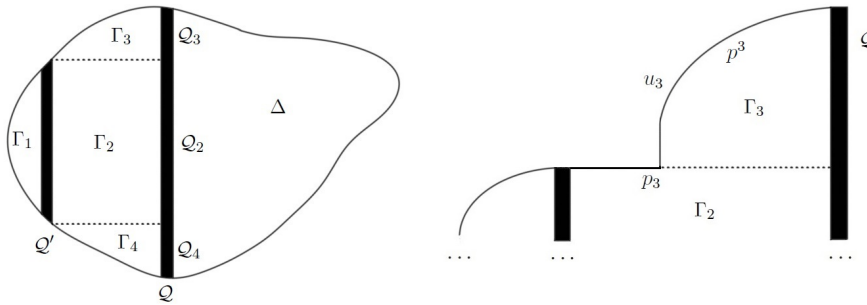
Figure 8. Lemma 6.16.

We consider the smallest subdiagram Γ' of Δ containing all the θ-bands of Γ crossing the x-band $\mathcal{Q}'$. It is a comb with handle $\mathcal{Q}_2 \subset \mathcal{Q}$. The comb Γ' is covered by a trapezium Γ_2 placed between $\mathcal{Q}'$ and $\mathcal{Q}$, and the comb Γ_1 with handle $\mathcal{Q}'$. The band $\mathcal{Q}'$ belongs to both Γ_1 and Γ_2. The remaining part of Γ is a disjoint union of two combs Γ_3 and Γ_4 whose handles $\mathcal{Q}_3$ and $\mathcal{Q}_4$ contain the cells of $\mathcal{Q}$ that do not belong to the trapezium Γ_2. The handle of Γ is the composition of handles $\mathcal{Q}_3$, $\mathcal{Q}_2$, $\mathcal{Q}_4$ of Γ_3, Γ' and Γ_4 in that order.

Let the lengths of $\mathcal{Q}_3$ and $\mathcal{Q}_4$ be l_3 and l_4, respectively. Let l' be the length of the handle of Γ'. Then by Lemma 6.14, we have

$$l' > l/2 \quad \text{and} \quad l = l' + l_3 + l_4. \tag{6.16}$$

For $i \in \{3, 4\}$ and $\alpha_i = |\partial \Gamma_i|_a$, Lemma 6.5 gives inequalities

$$A_i \le C_1 l_i^2 + 2\alpha_i l_i, \tag{6.17}$$

where A_i is the G-area of Γ_i. (We take into account that G-area cannot exceed area.)

Let $\mathbf{p}_3, \mathbf{p}_4$ be the top and the bottom of the trapezium Γ_2. Here $\mathbf{p}_3^{-1}$ (resp., $\mathbf{p}_4^{-1}$) shares some initial edges with $\partial \Gamma_3$ (with $\partial \Gamma_4$), the rest of these paths belong to the boundary of Δ. We denote by d_3 the number of a-edges of $\mathbf{p}_3$ and by d_3' the number of the a edges of $\mathbf{p}_3$ which do not belong to Γ_3. Similarly, we introduce d_4 and d_4'.

Let A_2 be the G-area of Γ_2. Then by Lemma 5.10 and the definition of the G-area for big trapezia, we have

$$A_2 \le C_2 l'(d_3 + d_4 + 1) \tag{6.18}$$

for some constant $C_2 < \delta^{-1}$, because the basic width of Γ_2 is less than K.

Now we observe that the handle $\mathcal{Q}_2$ of Γ' is a copy of $\mathcal{Q}'$ because both maximal q-bands of the trapezium Γ_2 correspond to the same basic letter x.

This makes the following surgery possible. The diagram Δ is covered by two subdiagrams: Γ and another subdiagram Δ_1, having only the band $\mathcal{Q}$ in common. We construct a new auxiliary diagram by attaching Γ_1 to Δ_1 with identification of the of the band $\mathcal{Q}'$ of Γ_1 and the band $\mathcal{Q}_2$. We denote the constructed diagram by Δ_0.

Δ_0 is a reduced diagram because every pair of its cells having a common edge, has a copy either in Γ_1 or in Δ_1. Now we need the auxiliary:

Lemma 6.17. *The G-area A_0 of Δ_0 is at least the sum of the G-areas of Γ_1 and Δ_1 minus l'.*

Proof. Consider a minimal "covering" **S** of Δ_0 from the definition of G-area, and assume that there is a big trapezium $E \in \mathbf{S}$, such that neither Γ_1 nor Δ_1 contains it. Then E has a base ywy, where $(yw)^{\pm 1}$ is a cyclic permutation of the Lth power of the standard base, and the first y-band of E is in Γ_1, but it is not a subband of $\mathcal{Q}'$.

Since the history H of the big trapezium E is a subhistory of the history of Γ_2, we conclude that Γ_2 is a big trapezium itself, and therefore $(xv)^{\pm 1}$ is an Lth power of a cyclic shift of the standard base (or of the inverse of it). Since the first y occurs in $uxvx$ before the first x it follows that we have the $(L+1)$-th occurrence of y before the last occurrence of x in the word $uxvx$. But this contradicts to the definition of tight comb Γ.

Hence every big trapezium from **S** entirely belongs either to Γ_1 or to Δ_1. Therefore one can obtain "coverings" $\mathbf{S}'$ and $\mathbf{S}''$ of these two diagrams if (1) every Σ from **S** is assigned either to $\mathbf{S}'$ or to $\mathbf{S}''$ and then (2) one add at most l' single cells since the common band $\mathcal{Q}'$ in Δ_0 should be covered twice in disjoint diagrams Γ_1 and Δ_1. These construction complete the proof of the lemma. $\square$

By Lemma 6.9, the G-area of Δ does not exceed the sum of G-areas of the five subdiagrams $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, and Δ_1. But the direct estimate of each of these values is not efficient. Therefore we will use Lemma 6.17 to bound the G-area of the auxiliary diagram Δ_0 built of two pieces Γ_1 and Δ_1.

It follows from our constructions and Lemmas 6.9 and 6.17, that

$$\mathrm{Area}_G(\Delta) \leq A_2 + A_3 + A_4 + A_0 + l'. \tag{6.19}$$

Now we continue proving Lemma 6.16.

Let $\mathbf{p}^3$ be the segment of the boundary $\partial\Gamma_3$ that joins $\mathcal{Q}$ and Γ_2 along the boundary of Δ (Fig. 8 (b)). It follows from the definition of d_3, d_3', l_3, and α_3, that the number of a-edges lying on $\mathbf{p}^3$ is at least $\alpha_3 - (d_3 - d_3') - l_3$.

Let $\mathbf{u}_3$ be the part of $\partial\Delta$ that contains $\mathbf{p}^3$ and connects $\mathcal{Q}$ with $\mathcal{Q}'$. It has l_3 θ-edges. Hence we have, by Lemma 6.1, that at least

$$|\mathbf{u}_3| \geq \max\left(l_3, l_3 + \delta\left(|p^3|_a - l_3\right)\right) \geq \max\left(l_3, l_3 + \delta\left(\alpha_3 - (d_3 - d_3') - 2l_3\right)\right).$$

Since $\mathbf{u}_3$ includes a subpath of length d_3' having no θ-edges, we also have by Lemma 6.1 (c) that $|\mathbf{u}_3| \geq l_3 + \delta(d_3' - 1)$.

One can similarly define $\mathbf{p}^4$ and $\mathbf{u}_4$ for Γ_4. When passing from $\partial\Delta$ to $\partial\Delta_0$ we replace the end edges of $\mathcal{Q}'$, $\mathbf{u}_3$ and $\mathbf{u}_4$ by two subpaths of $\partial\mathcal{Q}$ having lengths l_3 and l_4. Let $n_0 = |\partial\Delta_0|$. Then it follows from the previous paragraph that

$$n - n_0 \geq 2 + \delta\big(\max\big(0, d_3' - 1, \alpha_3 - (d_3 - d_3') - 2l_3\big)$$
$$+ \max\big(0, d_4' - 1, \alpha_4 - (d_4 - d_4') - 2l_4\big)\big). \quad (6.20)$$

In particular, $n_0 \leq n - 2$. By the inductive hypothesis,

$$A_0 \leq N_2\Phi(n_0) + N_1\phi(n_0)\mu(\Delta_0). \quad (6.21)$$

We note that the mixture $\mu(\Delta_0)$ of Δ_0 is not greater than $\mu(\Delta) - l'(l - l')$. Indeed, by Lemma 6.14 (2), one can use the same trick as in Lemma 6.14 as follows. For every pair of white beads, where one bead corresponds to a θ-band of Γ_2 and another one to a θ-band of Γ_3 or Γ_4, the contribution of this pair to $\mu(\Delta_0)$ is less than the contribution to Δ. It remains to count the number of such pairs: $l'(l_3 + l_4) = l(l - l')$.

Therefore, by inequalities (6.21) and (6.4), the G-area of Δ is not greater than

$$N_2\Phi(n) + N_1\phi(n)\mu(\Delta) - N_2n(n - n_0) - N_1\phi(n)l'(l - l') + A_2 + A_3 + A_4 + l'. \quad (6.22)$$

In view of inequalities (6.18), (6.17), and (6.19), to obtain the desired contradiction, it suffices to prove that

$$N_2n(n - n_0) + N_1l'(l - l') \geq C_{12}l'(d_3 + d_4 + 1) + C_{12}(l_3^2 + l_4^2)$$
$$+ 2\alpha_3l_3 + 2\alpha_4l_4 + l', \quad (6.23)$$

where $C_{12} = \max(C_1, C_2)$.

Since $l = l' + l_3 + l_4$, it suffices to prove that

$$N_2n(n - n_0) + N_1l'(l - l') \geq C_3l'(d_3 + d_4 + 1) + C_3(l_3^2 + l_4^2) + 2\alpha_3l_3 + 2\alpha_4l_4, \quad (6.24)$$

where $C_3 = C_{12} + 1$.

Note that we can assume that

$$C_3 \gg 1. \quad (6.25)$$

First we can choose N_1 big enough so that

$$N_1l'(l - l')/3 \geq C_3(l_3 + l_4)^2 \geq C_3(l_3^2 + l_4^2).$$

Indeed, by (6.16), we obtain

$$\frac{N_1}{3}l'(l - l') \geq \frac{N_1}{3}(l_3 + l_4)(l_3 + l_4),$$

so it is enough to assume that

$$N_1 > 3C_3. \quad (6.26)$$

We also have that

$$\frac{N_2}{2}n(n - n_0) \geq C_3 l',$$ (6.27)

because $n - n_0 \geq 2$, $n \geq 2l'$,and $N_2 \geq C_3$ by (6.26).

It remains to prove that

$$\frac{N_2}{2}n(n - n_0) + \frac{2N_1}{3}l'(l - l') > C_3 l'(d_3 + d_4) + 2\alpha_3 l_3 + 2\alpha_4 l_4.$$ (6.28)

We assume without loss of generality that $\alpha_3 \geq \alpha_4$, and consider two cases:

(a) Suppose $\alpha_3 \leq 2C_3(l - l')$. Since $d_i \leq \alpha_i + d_i'$ for $i = 3, 4$, by inequality (6.20), we have

$$d_3 + d_4 \leq \alpha_3 + \alpha_4 + d_3' + d_4' < 4C_3(l - l') + \delta^{-1}(n - n_0) + 2 - 2\delta^{-1}$$
$$< 4C_3(l - l') + \delta^{-1}(n - n_0).$$

Therefore,

$$\frac{N_1}{3}l'(l - l') + \frac{N_2}{2}n(n - n_0) \geq 4C_3^2 l'(l - l') + C_3\delta^{-1}(n - n_0)l'$$
$$> C_3 l'(d_3 + d_4)$$ (6.29)

since we can assume that

$$N_1 > 12C_3^2, \quad N_2/2 > C_3\delta^{-1}.$$ (6.30)

We also have by (6.16):

$$\frac{N_1}{3}l'(l - l') \geq \frac{N_1}{3}(l_3 + l_4)(l_3 + l_4)$$
$$\geq \frac{N_1}{3}\frac{\alpha_3 + \alpha_4}{4C_3}(l_3 + l_4) > 2\alpha_3 l_3 + 2\alpha_4 l_4$$ (6.31)

since we can assume that

$$N_1 > 24C_3.$$ (6.32)

The sum of inequalities (6.29) and (6.31) gives us the desired inequality (6.28).

(b) Assume now that $\alpha_3 > 2C_3(l - l')$. Then, applying Lemma 6.5 to the comb Γ_3, we obtain

$$d_3 - d_3' < \frac{1}{2}\alpha_3 + K_0 l_3 \leq \frac{5}{6}\alpha_3$$ (6.33)

since $l_3 \leq l - l' < \frac{\alpha_3}{2C_3}$ and

$$C_3 > 3K_0.$$ (6.34)

We also have

$$d_4 - d_4' < \frac{1}{2}\alpha_4 + K_0 l_4 \leq \frac{5}{6}\alpha_3.$$

These two inequalities and inequality (6.20) lead to

$$d_3 + d_4 \leq \frac{5}{3}\alpha_3 + \delta^{-1}(n - n_0). \tag{6.35}$$

It follows from (6.33) that

$$\alpha_3 - (d_3 - d_3') - 2l_3 \geq \frac{1}{6}\alpha_3 - \frac{2}{2C_3}\alpha_3 \geq \frac{1}{7}\alpha_3,$$

since $l_3 \leq l - l' < \frac{\alpha_3}{2C_3}$ and $C_3 > 42$ by (6.25). Therefore, by (6.20),

$$n - n_0 \geq \frac{1}{7}\delta\alpha_3. \tag{6.36}$$

Thus, by (6.35),

$$d_3 + d_4 < 13\delta^{-1}(n - n_0). \tag{6.37}$$

Since $2l' < n$ and $n - n_0 \geq 2$, inequality (6.37) implies

$$\frac{N_2}{3}n(n - n_0) > C_3 l'(d_3 + d_4), \tag{6.38}$$

because we can assume that

$$N_2 \gg C_3\delta^{-1} \tag{6.39}$$

($N_2 > 21C_3\delta^{-1}$ is enough).

Inequalities (6.36), (6.39), $\alpha_3 \geq \alpha_4$, and $4(l_3 + l_4) \leq n$ give us

$$\frac{N_2}{6}n(n - n_0) \geq \frac{7}{2}C_3\delta^{-1}(n - n_0)n \geq 2\alpha_3(l_3 + l_4) \geq 2\alpha_3 l_3 + 2\alpha_4 l_4. \tag{6.40}$$

The inequality (6.28) follows now from inequalities (6.38), and (6.40). $\qquad\square$

7. Minimal diagrams over G

Given a reduced diagram Δ over the group G, one can construct a planar graph whose vertices are the hubs of this diagram plus one improper vertex outside Δ, and the edges are maximal t-bands of Δ.

Let us consider two hubs Π_1 and Π_2 in a reduced diagram, connected by two neighbor t-bands $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$, where there are no other hubs between these t-bands. These bands, together with parts of $\partial\Pi_1$ and $\partial\Pi_2$, bound either a subdiagram having no cells, or a trapezium Ψ of height ≥ 1 (Fig. 9). The former case is impossible since in this case the hubs have a common t-edge and, the diagram is not reduced since all cyclic permutations of the word $(W_M)^L$ starting with t are equal. We want to show that the latter case is not possible either if the diagram Δ is chosen with minimal number of hubs among the diagrams with the same boundary label.
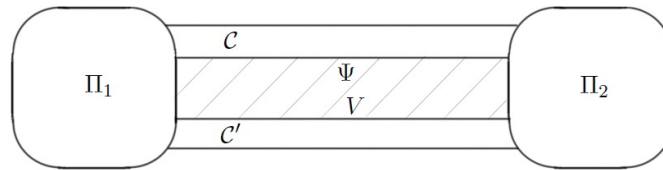
Figure 9. Cancellation of two hubs.

Indeed, both the top and the bottom labels of Ψ are equal to the word $(W_M t)^{\pm 1}$, and removing $\mathcal{C}_{i+1}$ from Ψ, we get a subdiagram Ψ' with top/bottom label $(W_M)^{\pm 1}$ and the same label V of its sides. It follows that W_M and V commute in the group M. Hence the word $U \equiv (W_M)^{1-L} V (W_M)^{L-1} V^{-1}$ is equal to 1 in M. But U is the boundary label of a subdiagram Γ containing Ψ' and both Π_1 and Π_2. Hence one can replace Γ with a diagram over M, decreasing the nubmer of hubs in Δ, a contradiction.

If W is an **M**-accepted word, then the word $(W)^L$ is equal to 1 in G. To see this, one can glue up L copies $\Delta_1, \ldots, \Delta_L$ ot the trapezia corresponding to the accepting computation of W, identifying the right side of each Δ_i and the left side of Δ_{i+1} (indices are taken modulo L). The obtained annulus has inner boundary labeled by the hub word $(W_M)^L$, and so the hole can be glued up by a hub cell.

As in [22] and [15], we will increase the set of relations of G by adding the (infinite) set of *disk relators* $(W)^L$ for every accepted word W. So we will consider diagrams with *disks*, where every disk cell (or just *disk*) is labeled by such a word $(W)^{\pm L}$. (In particular, the hub is a disk.)

Again, if two disks are connected by two t-bands and there are no other disks between these t-bands, then one can reduce the number of disks in the diagram. For this aid, it suffices to replace the disks with hubs and the cells corresponding to the defining relations of M, and apply the trick exploited above.

Definition 7.1. We will call a reduced diagram Δ *minimal* if:

(1) the number of disks is minimal for the diagrams with the same boundary label; and

(2) Δ has minimal number of (θ, t)-cells among the diagrams with the same boundary label and with minimal number of disks.

Clearly, a subdiagram of a minimal diagram is minimal itself.

Thus, no two disks of a minimal diagram are connected by two t-bands, such that the subdiagram bounded by them contains no other disks. This property makes the disk graph of a reduced diagram hyperbolic, in a sense, if the degree L of every proper vertex (= disk) is high ($L \gg 1$). Below we give a more precise formulation

(proved for diagrams with such a disk graph, in particular, in [22, Lemma 11.4] and in [15, Lemma 3.2]).

Lemma 7.2. *If a a minimal diagram contains at least one disk, then there is a disk Π in Δ such that $L-3$ consecutive maximal t-bands $\mathcal{B}_1, \ldots, \mathcal{B}_{L-3}$ start on $\partial\Pi$, end on the boundary $\partial\Delta$, and for any $i \in [1, L-4]$, there are no disks in the subdiagram Γ_i bounded by $\mathcal{B}_i$, $\mathcal{B}_{i+1}$, $\partial\Pi$, and $\partial\Delta$ (Fig. 10).* □
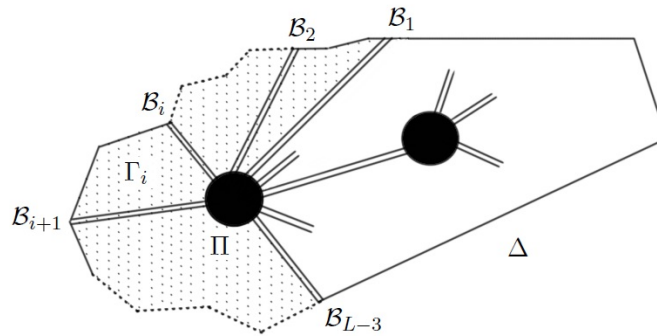
Figure 10. Lemma 7.2.

A maximal q-band starting on a disk of a diagram is called a *spoke*.

Lemma 7.2 implies by induction on the number of hubs:

Lemma 7.3 (see [16, Lemma 5.19]). *If a reduced diagram Δ has $m \geq 1$ hubs, then the number of spokes of Δ ending on the boundary $\partial\Delta$, and therefore the number of q-edges in the boundary path of Δ, is greater than $mLN/2$.* □

Recall the following transformation for diagrams with disks, exploited earlier in [15, 22]. Assume there is a disk Π and a θ-band $\mathcal{T}$ subsequently crossing some spokes $\mathcal{B}_1, \ldots, \mathcal{B}_k$ which start (say, counter-clockwise) from Π. Assume that $k \geq 2$ and there are no other cells between Π and the bottom of $\mathcal{T}$, and so there is a subdiagram Γ formed by Π and $\mathcal{T}$.

We describe the *transposition* (band moving construction in the terms of [22]) of the disk and the band as follows. We have a word $V \equiv tW_1t\ldots tW_{k-1}t$ written on the top of the subband $\mathcal{T}'$ of $\mathcal{T}$, that starts on $\mathcal{B}_1$ and ends on $\mathcal{B}_k$. The bottom $\mathbf{q}_2$ of $\mathcal{T}'$ is the subpath of the boundary path $\mathbf{q}_2\mathbf{q}_3$ of Π (Fig. 11).

Note that $W_1 \equiv W_2 \equiv \cdots \equiv W_{k-1}$ and tW_1 is an accepted word by Lemma 5.7. Therefore one can construct a new disk $\bar{\Pi}$ with boundary label $(tW_1)^L$ and boundary $\mathbf{s}_1\mathbf{s}_2$, where $\mathrm{Lab}(\mathbf{s}_1) \equiv V$. Also one construct an auxiliary band $\mathcal{T}''$ with top label $W_1^{-1}t^{-1}\ldots t^{-1}W_1^{-1}$, where the number of occurrences of t^{-1} is $L-k$, and attach it to $\mathbf{s}_2^{-1}$, which has the same label. Finally we replace the subband $\mathcal{T}'$ by $\mathcal{T}''$

(and make cancellations in the new θ-band $\bar{\tilde{\mathcal{T}}}$ if any appear). The new diagram $\bar{\Gamma}$ formed by $\bar{\Pi}$ and $\tilde{\mathcal{T}}$ has the same boundary label as Γ.
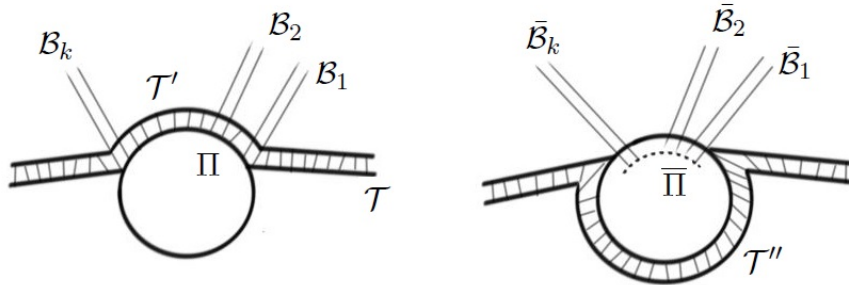
Figure 11. Transposition of a θ-band and a disk.

Remark 7.4. After the transposition, the first (θ, t)-cells of t-spokes $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are removed and the total number of common (θ, t)-cells of the new spokes $\bar{\mathcal{B}}_1, \ldots, \bar{\mathcal{B}}_k$ of $\bar{\Pi}$ and $\tilde{\mathcal{T}}$ is less than the number of common (θ, t)-cells of $\mathcal{B}_1, \ldots, \mathcal{B}_k$ and $\mathcal{T}$ at least by k. In particular, if $k > L - k$, then the number of (θ, t)-cells in $\bar{\Gamma}$ is less than the number of (θ, t)-cells in Γ. This observatiom implies

Lemma 7.5. *Let Δ be a minimal diagram.*

(1) *Assume that a θ-band $\mathcal{T}_0$ crosses k t-spokes $\mathcal{B}_1, \ldots, \mathcal{B}_k$ starting on a disk Π, and there are no disks in the subdiagram Δ_0, bounded by these spokes, by $\mathcal{T}_0$ and by Π. Then $k \leq L/2$.*

(2) *Assume that there are two disjoint θ-bands $\mathcal{T}$ and $\mathcal{S}$ whose bottoms are parts of the boundary of a disk Π and these bands correspond to the same rule θ (if their histories are read towards the disk). Suppose $\mathcal{T}$ crosses $k \geq 2$ t-spokes starting on $\partial\Pi$ and $\mathcal{S}$ crosses $\ell \geq 2$ t-spokes starting on $\partial\Pi$. Then $k + \ell \leq L/2$.*

(3) *Δ contains no θ-annuli.*

(4) *A θ-band cannot cross a maximal q-band (in particular, a spoke) twice.*

Proof. (1) Since every cell, except for disks, belongs to a maximal θ-band, it follows from Lemma 5.3 that there is a θ-band $\mathcal{T}$ such that $\mathcal{T}$ crosses all $\mathcal{B}_1, \ldots, \mathcal{B}_k$ and Δ_0 has no cells between $\mathcal{T}$ and Π. If $k > L/2$, then by Remark 7.4, the transposition of Π and $\mathcal{T}$ would decrease the number of (θ, t)-cells in Δ, a contradiction, since Δ is a minimal diagram.

(2) As above, let us transpose $\mathcal{T}$ and Π. This operation removes k (θ, t)-cells but add $L - k$ new (θ, t)-cells in $\bar{\mathcal{T}}$. However ℓ (θ, t)-cells of $\mathcal{S}$ and ℓ (θ, t)-cells of $\bar{\mathcal{T}}$ will form mirror pairs, and so after cancellations one will have at most $L - k - 2\ell$ new

(θ, t)-cells. This number is less than k if $k + \ell > L/2$ contrary to the minimality of the original diagram. Therefore $k + \ell \leq L/2$.

(3) Proving by contradiction, consider the subdiagram Δ' bounded by a θ-annulus. It has to contain disks by Lemma 5.3. Hence it must contain spokes $\mathcal{B}_1, \ldots, \mathcal{B}_{L-3}$ introduced in Lemma 7.2. But this contradits to item (1) of the lemma since $L - 3 > L/2$.

(4) The argument of item (3) works if there is a subdiagram Δ' of Δ bounded by an q-band and a θ-band. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The transposition transformation will be used for extrusion of disks from quasi-trapezia. The definition of a *quasi-trapezium* sounds as the definition of trapezium, but quasi-trapezia may contain disks. (So a quasi-trapezium without disks is a trapezium.)

Lemma 7.6. *Let a minimal diagram Γ be a quasi-trapezium with standard factorization of the boundary as $\mathbf{p}_1^{-1}\mathbf{q}_1\mathbf{p}_2\mathbf{q}_2^{-1}$. Then there is a diagram Γ' such that:*

(1) *the boundary of Γ' is $(\mathbf{p}_1')^{-1}\mathbf{q}_1'\mathbf{p}_2'(\mathbf{q}_2')^{-1}$, where*

$$\mathrm{Lab}(\mathbf{p'}_j) \equiv \mathrm{Lab}(\mathbf{p}_j) \quad and \quad \mathrm{Lab}(\mathbf{q'}_j) \equiv \mathrm{Lab}(\mathbf{q}_j)$$

   for $j = 1, 2$;

(2) *the numbers of hubs and (θ, q)-cells in Γ' are the same as in Γ;*

(3) *the vertices $(\mathbf{p'}_1)_-$ and $(\mathbf{p'}_2)_-$ (the vertices $(\mathbf{p'}_1)_+$ and $(\mathbf{p'}_2)_+$) are connected by a simple path $\mathbf{s}_1$ (by $\mathbf{s}_2$, resp.) such that we have three subdiagrams $\Gamma_1, \Gamma_2, \Gamma_3$ of Γ', where Γ_2 is a trapezium with standard factorization of the boundary $(\mathbf{p}_1')^{-1}\mathbf{s}_1\mathbf{p}_2'\mathbf{s}_2^{-1}$ and all cells of the subdiagrams Γ_1 and Γ_3 with boundaries $\mathbf{q}_1'\mathbf{s}_1^{-1}$ and $\mathbf{s}_2(\mathbf{q}_1')^{-1}$ are disks;*

(4) *all maximal θ-bands of Γ and all maximal θ-bands of Γ_2 have the same number ot (θ, t)-cells (equal for Γ and Γ_2).*

Proof. Every maximal θ-band of Γ must connect an edge of $\mathbf{p}_1$ with an edge of $\mathbf{p}_2$; this follows from Lemma 7.5 (3). Hence we can enumerate these bands from bottom to top: $\mathcal{T}_1, \ldots, \mathcal{T}_h$, where $h = |\mathbf{p}_1| = |\mathbf{p}_2|$.

If Γ has a disk, then by Lemma 7.2, there is a disk Π such that at least $L - 3$ t-spokes of it end on $\mathbf{q}_1$ and $\mathbf{q}_2$, and there are no disks between the spokes ending on $\mathbf{q}_1$ (and on $\mathbf{q}_2$). By Lemma 7.5 (2), at least $L - 3 - L/2 \geq 2$ of these spokes must end on $\mathbf{q}_1$ (resp., on $\mathbf{q}_2$).

If Π lies between $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$, then the number of its t-spokes crossing $\mathcal{T}_j$ (crossing $\mathcal{T}_{j+1}$) is at least 2. So one can make a transposition of Π with each of these two θ-bands. So we can move the disk toward $\mathbf{q}_1$ (or toward $\mathbf{q}_2$) until the disk is extruded from the quasi-trapezium. (We use that if k t-spokes $\mathcal{B}_1, \ldots, \mathcal{B}_k$

of Π end on $\mathbf{q}_1$, then after transposition, we again have k t-spokes $\bar{\mathcal{B}}_1, \ldots, \bar{\mathcal{B}}_k$ of $\bar{\Pi}$ ending on $\mathbf{q}_1$; see the notation of Remark 7.4.)

No pair $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$ corresponds to two mutual inverse letters of the history. This is clear if there are no discs between these θ-bands. If there is a disk, then this is impossible too by Lemma 7.6 (2) since one could choose a disk Π as in the previous paragraph. So the projection of the label of $\mathbf{p}_1$ on the history is reduced.

Let us choose i such that the number m of (θ, t)-cells in $\mathcal{T}_i$ is minimal. It follows that Γ has at least hm (θ, t)-cells.

If the disk Π lies above $\mathcal{T}_i$, we will move it upwards using transpositions. So after a number of transpositions all such (modified) disks will be placed above the θ-band number h and form the subdiagram Γ_1. Similarly we can form Γ_3 moving other disks downwards.

In the obtaining diagram Γ_2 lying between Γ_1 and Γ_3, every θ-band is reduced by the definition of transpositions. The neighbor maximal θ-band of Γ_2 cannot correspond to mutual inverse letters of the history since the labels of $\mathbf{p}_1$ and $\mathbf{p}'_1$ are equal. It follows that the diagram Γ_2 (without disks) is a reduced diagram, and so it is a trapezium of height h.

The θ-band $\mathcal{T}_i$ did not participate in the transpositions. Therefore it is a maximal θ-band of Γ_2. Hence the trapezium Γ_2 contains exactly mh (θ, t)-cells, which does not exceed the number of (θ, t)-cells in Γ. In fact these two numbers are equal since Γ is a minimal diagram. So every maximal θ-band of Γ and every maximal θ-band of Γ_2 has m (θ, t)-cells. $\qquad\square$

We say that a history H is *standard* if there is a standard trapezium with history H.

Definition 7.7. Suppose we have a disk Π with boundary label $(tW)^L$ and $\mathcal{B}$ be a t-spoke starting on Π. Suppose there is a subband $\mathcal{C}$ of $\mathcal{B}$, which also starts on Π and has a standard history H, for which the word tW is H-admissible. Then we call the t-band $\mathcal{C}$ a *shaft*.

For a constant $\lambda \in [0; 1/2)$ we also define a stronger concept of λ-shaft at Π as follows. A shaft $\mathcal{C}$ with history H is a λ-shaft if for every factorization of the history $H \equiv H_1 H_2 H_3$, where $\|H_1\| + \|H_3\| < \lambda \|H\|$, the middle part H_2 is still a standard history. (So a shaft is a 0-shaft).

Lemma 7.8. *Let Π be a disk in a minimal diagram Δ and $\mathcal{C}$ be a λ-shaft at Π with history H. Then $\mathcal{C}$ has no factorizations $\mathcal{C} = \mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3$ such that:*

(a) *the sum of lengths of $\mathcal{C}_1$ and $\mathcal{C}_3$ do not exceed $\lambda \|H\|$; and*

(b) *Δ has a quasi-trapezium Γ such that top (or bottom) label of Γ has $L + 1$ occurrences of t and $\mathcal{C}_2$ starts on the bottom and ends on the top of Γ.*

Proof. Proving by contradiction, we first replace Γ by a trapezium Γ' according to Lemma 7.6. The transpositions used for this goal do not affect neither Π nor $\mathcal{C}$

since $\mathcal{C}$ crosses all the maximal θ-bands of Γ. Also one can replace Γ' by a trapezium with shorter base and so we assume that the base of it starts and ends with letter t.

For the beginning, we assume that $\mathcal{C}$ is a shaft (i.e. $\lambda = 0$). Then it follows from the definition of shaft and Remark 5.8 that **bot**(Γ') is labeled by $(tW)^L t$, where $(tW)^L$ is the boundary label of Π. One can remove the first or the last maximal t-band from Γ' and obtain a subtrapezium Γ'' whose boundary label coincides with the label of $\partial\Pi$ (up to cyclic permutation), and $\partial\Gamma''$ shares a t-edge with $\partial\Pi$ (Fig. 12 with $\lambda = 0$). It follows that the subdiagram $\Delta' = \Pi \cup \Gamma''$ has boundary label freely equal to Lab(**top**(Γ'')). However Lab(**top**$(\Gamma'')) \equiv (tW')^L$, where $tW' = (tW) \cdot H$ by Lemma 5.7, and so there is a disk Π' with boundary label $(tW')^L$. Therefore the subdiagram Δ' can be replaced by a single disk. So we decrease the number of (θ, t)-cells contrary to the minimality of Δ.
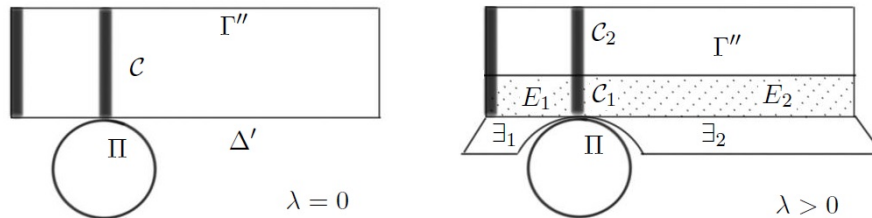
Figure 12. Lemma 7.8.

Now we consider the general case, where $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3$. As above, we replace Γ by a trapezium Γ' and obtain a trapezium Γ'' after removing of one t-band in Γ'. To obtain a contradiction, it suffices to consider the diagram $\Delta' = \Pi \cup \mathcal{C}_1\mathcal{C}_2 \cup \Gamma''$ (forgetting of the complement of Δ' in Δ) and find another diagram Δ'' with one disk and fewer (θ, t)-cells such that Lab$(\partial\Delta'') = $ Lab$(\partial\Delta')$ in the free group.

Since both histories H and H_2 (and so $H_1 H_2$) are standard, one can enlarge Γ'' and construct a trapezium Γ''' with history $H_1 H_2$. (The added parts E_1 and E_2 are dashed in Figure 12 with $\lambda > 0$). Note that we add $< \lambda \|H\| L$ new (θ, t)-cells since every maximal θ-band of Γ''' has L such cells. As in case $\lambda = 0$, this trapezium Γ''' and the disk Π can be replaced by one disk Π'. However to obtain the boundary label equal to Lab$(\partial\Delta')$, we should attach the mirror copies $\exists_1$ and $\exists_2$ of E_1 and E_2 to Π'. The obtained diagram Δ'' has at most $\lambda \|H_1\| L$ (θ, t)-cells, while Δ' has at least $\|H_2\| L \geq (1 - \lambda)\|H\|$ (θ, t)-cells. Since $\lambda < 1 - \lambda$, we have the desired contradiction. $\qquad\square$

Lemma 7.8 will be used to obtain a linear bound, in terms of perimeter $|\partial\Delta|$, for the sum of lengths $\sigma = \sigma_\lambda(\Delta)$ of all λ-shafts in a minimal diagram Δ, which make

possible to exploit σ_λ as an inductive parameter along with $|\partial\Delta|$. One more tool needed to linearly bound σ_λ, is a combinatorial proposition of two finite systems of disjoint segments on Euclidean plane proved in the next section.

8. Designs in topological disk

By $\mathcal{D}$, we denote Euclidean closed disk of radius 1. Let $\mathbf{T}$ be a finite set of disjoint chords (plain lines in Fig. 13) and $\mathbf{Q}$ a finite set of disjoint simple curves in $\mathcal{D}$ (dotted lines in Fig. 13). One may think of a curve as a non-oriented broken line, i.e. it is built from finitely many finite segments. To distinguish the elements from $\mathbf{T}$ and $\mathbf{Q}$, we will say that the elements of $\mathbf{Q}$ are *arcs*.

We shall assume that the arcs belong to the open disk D^o, an arc may cross a chord transversally at most once, and the intersection point cannot coincide with one of the two ends of an arc.

Under these assumptions, we shall say that the pair $(\mathbf{T}, \mathbf{Q})$ is a *design*.

By definition, the *length* $|C|$ of an arc C is the number of the chords crossing C. The term *subarc* will be used in natural way; obviously one has $|D| \leq |C|$ if D is a subarc of an arc C.

We say that an arc C_1 is *parallel* to an arc C_2 and write $C_1 \parallel C_2$ if every chord (from $\mathbf{T}$) crossing C_1 also crosses C_2. So the relation $\parallel$ is transitive. (The arc of length 2 is parallel to the arc of length 5 in Fig. 13.)
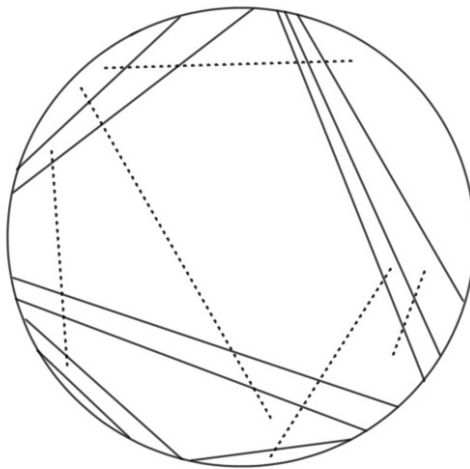
Figure 13. Design.

Definition 8.1. Given $\lambda \in (0\,;1)$ and an integer $n \geq 1$, Property $P(\lambda, n)$ of a design says that for any n different arcs $C_1, \ldots, C_n$, there exist no subarcs $D_1, \ldots, D_n$, respectively, such that

$$|D_i| > (1 - \lambda)|C_i|$$

for every $i = 1, \ldots, n$ and $D_1 \parallel D_2 \parallel \cdots \parallel D_n$.

By definition, the length $\ell(\mathbf{Q})$ of the set of arcs $\mathbf{Q}$ is defined by the equality

$$\ell(\mathbf{Q}) = \sum_{C \in \mathbf{Q}} |C|. \tag{8.1}$$

The number of chords will be denoted by $\#\mathbf{T}$. The goal of this subsection is to prove the following:

Theorem 8.2. *There is a constant $c = c(\lambda, n)$ such that for any design $(\mathbf{T}, \mathbf{Q})$ with Property $P(\lambda, n)$, we have*

$$\ell(\mathbf{Q}) \leq c\,(\#\mathbf{T}). \tag{8.2}$$

To prove Theorem 8.2, we may assume that $\mathbf{Q}$ has no arcs of length 0 and that every chord is crossed by an arc. Also we may assume that $\#\mathbf{T} > 1$ since otherwise all the arcs are parallel, and Property $P(\lambda, n)$ implies that the number of arcs is at most $n - 1$; therefore one can take $c = n - 1$.

Every chord T divides the disk $\mathcal{D}$ in two half-disks. If one of these half-disks contains no other chords, we call the chord T *peripheral* and denote the *peripheral half-disk* (without chord) by O_T.

An arc D is called an *extension* of an arc C if C is a subarc of D. (An extension need not be an element of $\mathbf{Q}$.) We will consider only extensions of $C \in \mathbf{Q}$ such that replacing C by D we again obtain a design $(\mathbf{T}, \mathbf{Q}')$ (but Property $P(\lambda, n)$ can be violated for the new design).

An arc C of a design is called *maximal* if there exists no extension D of C with $|D| > |C|$.

Lemma 8.3. *Let $(\mathbf{T}, \mathbf{Q})$ be a design with $\#\mathbf{T} \geq 1$. Then all the arcs $C_1, C_2, \ldots$ from $\mathbf{Q}$ have maximal extensions $D_1, D_2, \ldots$ forming a set of arcs $\mathbf{Q}'$ such that the design $(\mathbf{T}, \mathbf{Q}')$ has the following property: for every arc D_i, its ends belong to two different peripheral half-disks.*

Proof. Since no arc can cross a chord twice and the set of chords is finite, there is a system of maximal arcs $D_1, D_2, \ldots$ such that every D_i is an extension of C_i. It suffices to prove that the ends of every D_i belong to peripheral half-disks.

Assume that we have an end o of an arc $D = D_i$ and o belongs to no peripheral half-disk. Let us choose the direction for D toward o, and assume that T is the last chord crossed by D. Let H be the half-disk defined by T, where the point o belongs to. The half-disk H is not peripheral, and so it contains a chord $T' \neq T$. None of such T' could be crossed by D because otherwise D had to cross the chord T at least

twice. We may assume that T' is the closest to o in the sense that one can connect o and T' by a path $\mathbf{p}$ inside H, $\mathbf{p}$ consequently intersect ℓ arcs $D_{i_1}, \ldots, D_{i_\ell}$ from the set $\{D_1, D_2, \ldots\}$, and the number ℓ is minimal.

If $\ell = 0$, then using $\mathbf{p}$ one could extend D so that the extension crosses T'. If $\ell > 0$ we come to a contradiction too. Indeed, let us consider a small neighborhood U of D_{i_1}, which contains neither points of other arcs, nor the boundary points of $\mathcal{D}$. If one can bypath D_{i_1} in U moving around it, thereby replacing $\mathbf{p}$ by a path $\mathbf{p}'$ having $\ell - 1$ intersections with arcs (see Fig. 14 (a)), we get a contradiction. Otherwise going around D_{i_1} in U clockwise or counter-clockwise one will cross an interval of some chord $T'' \neq T$ from H (Fig. 14 (b)). Then one can use U to connect o with the chord T'' and an extention of D crosses T'', a contradiction again. $\qquad\square$
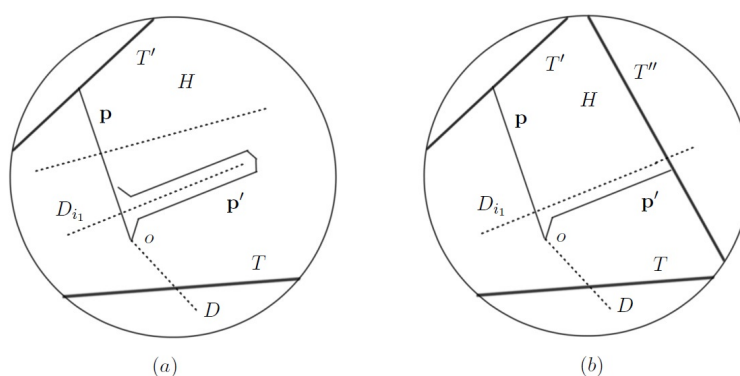
Figure 14. Lemma 8.3.

To continue the proof of Theorem 8.2, we modify the number #$\mathbf{T}$, taking every chord T with a weight $\nu(T)$. By definition:

$$\nu(T) = 1, \qquad \text{if } T \text{ crosses exactly one arc from } \mathbf{Q},$$
$$\nu(T) = 2, \qquad \text{if } T \text{ crosses exactly two arcs,}$$
$$\vdots$$
$$\nu(T) = 2n - 2, \quad \text{if } T \text{ crosses exactly } 2n - 2 \text{ arcs,}$$
$$\nu(T) = 2n - 1, \quad \text{if } T \text{ crosses } \textit{at least } 2n - 1 \text{ arcs.}$$

Clearly, we have

$$\#\mathbf{T} \leq \nu(\mathbf{T}) \stackrel{\text{def}}{=} \sum_{T \in \mathbf{T}} \nu(T) \leq (2n - 1)\#\mathbf{T}.$$

Therefore, instead of (8.2), it suffices to prove the following inequality

$$\ell(\mathbf{Q}) \leq d \ \nu(\mathbf{T}) \tag{8.3}$$

for some $d = d(\lambda, n) > 0$. We will prove (8.3) for any $d \geq \frac{1}{\lambda}$ by induction on the number of arcs in $\mathbf{Q}$. If there is only one arc, then it is nothing to prove since $\ell(\mathbf{Q}) \leq \#\mathbf{T}$ in this case. So we will assume that there are at least two arcs.

Assume that there is an arc C, which, after one choose a direction, can be factorized as $C_1 C_2 C_3$, where:

(a) $|C_1| + |C_3| \geq \lambda|C|$; and

(b) every chord crossing C_1 or C_3 has weight at most $2n - 2$.

Let us remove C from $\mathbf{Q}$. We obtain a new design $(\mathbf{T}', Q')$. Here $\mathbf{T}'$ has the same chords as $\mathbf{T}$, but their weights change. Obviously, Property $P(\lambda, n)$ holds for the design $(\mathbf{T}', \mathbf{Q}')$. Hence by inductive hypothesis, the inequalty $\ell(\mathbf{Q}') \leq d \ \nu(\mathbf{T}')$ is true.

Also we have $\ell(\mathbf{Q}) = \ell(\mathbf{Q}') + |C|$ and

$$\nu(\mathbf{T}) \geq \nu(\mathbf{T}') + |C_1| + |C_3| \geq \nu(\mathbf{T}') + \lambda|C|$$

since all chords crossing C_1 and C_3 loss their weight by 1. It follows that

$$\ell(\mathbf{Q}) = \ell(\mathbf{Q}') + |C| \leq d \ \nu(\mathbf{T}') + |C| \leq d\big(\nu(\mathbf{T}') + \lambda|C|\big) \leq d \ \nu(\mathbf{T}),$$

as desired, since $d \geq \lambda^{-1}$.

It remains to obtain a contradiction assuming that no arc $C \in \mathbf{Q}$ has a factorization with Properties (a) and (b). In other words, every arc $C \in \mathbf{Q}$ has a subarc D maximal with respect to the following properties:

(A) $|D| > (1 - \lambda)|C|$; and

(B) the first and the last chords crossing D have weight $2n - 1$.

We denote by $(\bar{\mathbf{T}}, \bar{\mathbf{Q}})$ the design obtained after the transition $C \to D$ for every arc C. Here we assume that $\bar{\mathbf{T}}$ contains the chords from $\mathbf{T}$, which cross some arcs from $\bar{\mathbf{Q}}$. Observe that all chords of weight $2n - 1$ from $\mathbf{T}$ have the same weight in $\bar{\mathbf{T}}$, as it follows from the definition of $\bar{\mathbf{Q}}$. (We do not claim Property $P(\lambda, n)$ for $(\bar{\mathbf{T}}, \bar{\mathbf{Q}})$.)

Let $\bar{T}$ be a peripheral chord from $\bar{\mathbf{T}}$. It is crossed by (at least) $2n - 1$ arcs $D_1, \ldots, D_{2n-1}$ since $\bar{T}$ is the first/last chord crossing the arcs.

By Lemma 8.3, there are maximal extensions $\tilde{D}_1, \ldots, \tilde{D}_{2n-1}$ of $D_1, \ldots, D_{2n-1}$, respectively. Moreover, such extensions can be constructed for *every* peripheral chord, and the two ends of every extension must belong to different peripheral half-disks of the design $(\bar{\mathbf{T}}, \bar{\mathbf{Q}})$.

Suppose one can choose n extension, say $\tilde{D}_1, \ldots, \tilde{D}_n$ starting in $O_{\bar{T}}$ and ending in the same half-disk $O_{T'}$ ($T' \in \bar{\mathbf{T}}$). Then every chord of $\bar{\mathbf{T}}$ crossing $\tilde{D}_i$ has to cross $\tilde{D}_j$ for $1 \leq i, j \leq n$. The same is true for the chords of $\mathbf{T}$ disposed between $\bar{T}$ and T'. Since $\tilde{D}_i$ starts with D_i and $\tilde{D}_j$ starts with D_j, the inequality $|D_i| \leq |D_j|$ implies that every chord of $\mathbf{T}$ crossing D_i has to cross D_j too. Therefore assuming that

$|D_1| \leq \cdots \leq |D_n|$, we have $D_1 \parallel \cdots \parallel D_n$. However this violates Property $P(\lambda, n)$ for the arcs $C_1, \ldots C_n$ since $|D_i| > (1 - \lambda)|C_i|$ for every $i = 1, \ldots, n$.

Thus, there are no n arcs among $\widetilde{D}_1, \ldots, \widetilde{D}_{2n-1}$ connecting the half-disk $O_{\bar{T}}$ with the same peripheral half-disk. We see that for every peripheral half-disk $O_{\bar{T}}$ one can find three arcs, say $\widetilde{D}_i, \widetilde{D}_j, \widetilde{D}_k$ connecting $O_{\bar{T}}$ with three *different* peripheral half-disks.

Now let us choose one vertex in every peripheral half-disk (e.g, on the boundary of the disk $\mathcal{D}$) and connect it with three other vertices using the triples of arcs obtained in the previous paragraph. We obtain an outerplanar graph with at least four vertices, where every vertex has degree at least 3. However there exist no such graphs ([10, Corollary 11.9]). The obtained contradiction completes the proof of Theorem 8.2. □

Remark 8.4. One may allow the ends of arcs to belong to the boundary of $\mathcal{D}$, and then the same inequality (8.2) holds since one can cut off the ends of every arc C preserving the length of C.

Let us have a parameter $\lambda \in [0, 1/2)$. For every t-spoke $\mathcal{B}$ of a minimal diagram Δ, we choose the λ-shaft of maximal length in it (if a λ-shaft exists). If $\mathcal{B}$ connects two disks Π_1 and Π_2, then there can be two maximal λ-shafts: at Π_1 and at Π_2. We denote by $\sigma_\lambda(\Delta)$ the sum of lengths of all λ-shafts in this family.

Lemma 8.5. *There is a constant $c = c(\lambda)$ such that $\sigma_\lambda(\Delta) \leq c|\partial\Delta|$ for every minimal diagram Δ over the group G.*

Proof. Let us associate the following design with Δ. We say that the middle lines of the maximal θ-bands (they cross θ-edges of the bands in the middle points) are the chords and the middle lines of the maximal λ-shafts are the arcs. Here we use two disjoint middle lines if two maximal λ-shafts share a (θ, t)-cell. By Lemma 7.5 (3), (4), we obtain a design, indeed.

Observe that the length $|C|$ of an arc is the number of cells in the λ-shaft and $\#\mathbf{T} \leq |\partial\Delta|/2$ since every maximal θ-band has two θ-edges on $\partial\Delta$.

Thus, by Theorem 8.2, it suffices to show that the constructed design satisfies the condition $P(\lambda, n)$, where n does not depend on Δ.

Let $n = 2L + 1$. If Property $P(\lambda, n)$ is violated, then we have n maximal λ-shafts $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and a subband $\mathcal{D}$ of $\mathcal{C}_1$, such that $|\mathcal{D}| > (1 - \lambda)|\mathcal{C}_1|$, and every maximal θ-band crossing $\mathcal{D}$ must cross each of $\mathcal{C}_2, \ldots, \mathcal{C}_n$. (Here $|\mathcal{B}|$ is the length of a t-band $\mathcal{B}$.) It follows that each of these θ-bands crosses at least $L + 1$ maximal t-bands. (See Lemma 7.5 (3), (4). Here we take into account that the same t-spoke can generate two arcs in the design.) Hence using the λ-shaft $\mathcal{C}_1$ one can construct a quasi-trapezium of height $|\mathcal{D}|$, which contradicts the statement of Lemma 7.8. □

9. Upper bound for G-areas of diagrams over the group G

By definition, the *G-area of a disk* Π is just the minimum of areas of diagrams over the presentation ((5.1), (5.2)) of G having the same boundary label as Π.

Lemma 9.1. *There is a constant c_6 such that the G-area of any disk does not exceed $c_6 F(|\partial \Pi|)$.*

Proof. The disk Π can be built of a hub and L standard accepting trapezia over M. By Lemma 4.23, and the definition of the functions $f(n)$, $g(n)$, there are such trapezia of height

$$O\big(\|\partial \Pi\|/L\big)g\big(\|\partial \Pi\|/L\big) = O\big(|\partial \Pi|\big)g\big(|\partial \Pi|\big).$$

The step history of these trapezia has length

$$O\big(f\big(|\partial \Pi|\big)^3\big) = O\big(|\partial \Pi|\big)$$

by Lemma 4.23. Therefore the length of every θ-band in it is $O(|\partial \Pi|)$ by Lemmas 4.17 and 6.1 (a), (d). The statement of the lemma follows. $\qquad\square$

By definition, the G-area of a minimal diagram Δ over G is the sum of G-areas of its disks plus the G-area of the complement. For the complement, as in Subsection 6.3, we consider a family **S** of big subtrapezia and single cells of Δ such that every cell of Δ belongs to a member Σ of this family, and if a cell Π belongs to different Σ_1 and Σ_2 from Σ, then both Σ_1 and Σ_2 are big subtrapezia of Δ with bases $xv_1 x$, $xv_2 x$, and Π is an (θ, x)-cell.) Hence the statement of Lemma 6.9 holds for minimal diagrams over G as well.

We want to prove that for big enough constants N_3 and N_4,

$$\text{Area}(\Delta) \le N_4 F\big(n + \sigma_\lambda(\Delta)\big) + N_3 \mu(\Delta)g(n)$$

for every minimal diagram Δ with perimeter n. For this goal, we will argue by contradiction in this section and study a *counter-example* with minimal $n + \sigma_\lambda(\Delta)$.

Lemma 9.2. *The diagram Δ has no rim θ-bands with base of length at most K.*

Proof. The functions $F(x)$ and $g(x)$ satisfy the definition given for $\Phi(x)$ and $\phi(x)$, and the inequality (6.4) by Lemma 4.18. Hence the proof of Lemma 6.15 works for the minimal counter-example over G. It suffices to replace N_2 and N_1 by N_4 and N_3, replace n by $n + \sigma_\lambda(\Delta)$, and notice that the value of σ_λ does nor increase when passing from Δ to Δ'. $\qquad\square$

By Lemma 6.16, Δ has at least one disk. Using Lemma 7.2, we fix a disk Π in Δ such that $L - 3$ consecutive maximal t-bands $\mathcal{B}_1, \dots, \mathcal{B}_{L-3}$ start on $\partial \Delta$, end on the boundary $\partial \Pi$, and for any $i \in [1, L-4]$, there are no disks in the subdiagram bounded by $\mathcal{B}_i$, $\mathcal{B}_{i+1}$, $\partial \Pi$, and $\partial \Delta$. (See Fig. 10.)

We denote by $\Psi = \mathrm{cl}(\Pi, \mathcal{B}_1, \mathcal{B}_{L-3})$ the subdiagram without disks bounded by the spokes $\mathcal{B}_1$, $\mathcal{B}_{L-3}$ (and including them) and by subpaths of the boundaries of Δ and Π, and call this subdiagram a *clove*. Similarly one can define the cloves $\Psi_{ij} = \mathrm{cl}(\Pi, \mathcal{B}_i, \mathcal{B}_j)$ if $1 \leq i < j \leq L - 3$.

Lemma 9.3. *The clove $\Psi = \mathrm{cl}(\Pi, \mathcal{B}_1, \mathcal{B}_{L-3})$ has no subcombs of basic width at least K_0.*

Proof. Proving by contradiction, we may assume that there is a tight subcomb Γ by Lemma 6.6 (2). Then contradiction appears exactly as in the proofs of Lemmas 6.12–6.16, since Lemma 4.18 allows us to define $\Phi(x) = F(x)$. It suffices to replace N_2 and N_1 with N_4 and N_3, replace n with $n + \sigma_\lambda(\Delta)$, and notice that the value of σ_λ does not increase when passing from Δ to Δ_0 since no t-band of Γ is a part of a spoke. $\square$

The statements of auxiliary Lemmas 6.12, 6.14, and 6.21 holds as well for the minimal counter-example over G. Below we use the following analog of Lemma 6.12:

Lemma 9.4. (1) *The counter-example Δ has no two disjoint subcombs Γ_1 and Γ_2 of basic widths at most K with handles $\mathcal{C}_1$ and $\mathcal{C}_2$ such that some ends of these handles are connected by a subpath $\mathbf{x}$ of the boundary path of Δ with $|\mathbf{x}|_q \leq N$.*

(2) *The boundary of every subcomb Γ with basic width $s \leq K$ has $2s$ q-edges.* $\square$

Lemma 9.5. (1) *Every maximal θ-band of Ψ crosses either $\mathcal{B}_1$ or $\mathcal{B}_{L-1}$.*

(2) *There exists r, $L/2 - 3 \leq r \leq L/2$, such that the θ-bands of Ψ crossing $\mathcal{B}_{L-3}$ do not cross $\mathcal{B}_r$, and the θ-bands of Ψ crossing $\mathcal{B}_1$ do not cross $\mathcal{B}_{r+1}$.*

Proof. (1) If the claim were wrong, then one could find a rim θ-band $\mathcal{T}$ in Ψ, which crosses neither $\mathcal{B}_1$ nor $\mathcal{B}_{L-3}$. By Lemma 9.2, the basic width of $\mathcal{T}$ is greater than K. Since (1) a disk has LN spokes, (2) no q-band of Ψ intersects $\mathcal{T}$ twice by Lemma 5.3 (3), $\mathcal{T}$ has at least K q-cells, and (4) $K > 2K_0 + LN$, there exists a maximal q-band $\mathcal{C}'$ such that a subdiagram Γ' separated from Ψ by $\mathcal{C}'$ contains no edges of the spokes of Π and the part of $\mathcal{T}$ belonging to Γ' has at least K_0 q-cells (Fig. 15).

If Γ' is not a comb, and so a maximal θ-band of it does not cross $\mathcal{C}'$, then Γ' must contain another rim band $\mathcal{T}'$ having at least K q-cells. This makes possible to find a subdiagram Γ'' of Γ' such that a part of $\mathcal{T}'$ is a rim band of Γ'' containing at least K_0 q-cells, and Γ'' does not contain $\mathcal{C}'$. Since $\mathrm{Area}(\Gamma') > \mathrm{Area}(\Gamma'') > \cdots$, such a procedure must stop. Hence, for some i, we obtain a subcomb $\Gamma^{(i)}$ of basic width $\geq K_0$, contrary to Lemma 9.3.

(2) Assume there is a maximal θ-band $\mathcal{T}$ of Ψ crossing the spoke $\mathcal{B}_1$. Then assume that $\mathcal{T}$ is the closest to the disk Π, i.e. the intersection of $\mathcal{T}$ and $\mathcal{B}_1$ is the first cell of the spoke $\mathcal{B}_1$. If $\mathcal{B}_1, \ldots, \mathcal{B}_r$ are all the spokes crossed by $\mathcal{T}$, then $r \leq L/2$ by Lemma 6.3. Since the band $\mathcal{T}$ does not cross the spoke $\mathcal{B}_{r+1}$, no other θ-band of Ψ

crossing $\mathcal{B}_1$ can cross $\mathcal{B}_{r+1}$ and no θ-band crossing the spoke $\mathcal{B}_{L-3}$ can cross $\mathcal{B}_r$. The same argument shows that $r + 1 \geq L/2 - 2$ if there is a θ-band of Ψ crossing the spoke $\mathcal{B}_{L-3}$. $\qquad\square$
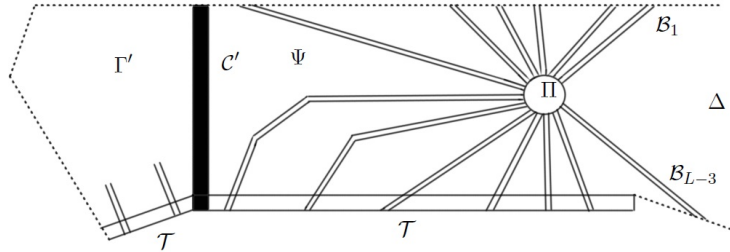
Figure 15. Lemma 9.5.

For the clove $\Psi = \mathrm{cl}(\pi, \mathcal{B}_1, \mathcal{B}_{L-3})$ in Δ, we denote by $\mathbf{p} = \mathbf{p}(\Psi)$ the common subpath of $\partial\Psi$ and $\partial\Delta$ starting with the t-edge of $\mathcal{B}_1$ and ending with the t-edge of $\mathcal{B}_{L-3}$. Similarly we define the (outer) path $\mathbf{p}_{ij} = \mathbf{p}(\Psi)_{ij}$ for every smaller clove Ψ_{ij}.

Lemma 9.6. *Every path $\mathbf{p}_{i,i+1}$ $(i = 1, \dots, L - 4)$ has less than $3K_0$ q-edges.*

Proof. Let a maximal q-band $\mathcal{C}$ of Ψ start on $\mathbf{p}_{i,i+1}$ and do not end on Π. Then it has to end on $\mathbf{p}_{i,i+1}$ too.

If Γ is the subdiagram without disks separated by $\mathcal{C}$, then every maximal θ-band of Γ has to cross the q-band $\mathcal{C}$ since its extension in Ψ must cross either $\mathcal{B}_1$ or $\mathcal{B}_{L-3}$ by Lemma 9.5. Therefore Γ is a comb with handle $\mathcal{C}$.

Consider the q-bands of this kind defining maximal subcombs $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ in $\Psi_{i,i+1}$. The basic width of each of them is less than K_0 by Lemma 9.3. Therefore, $k \leq 1$ since otherwise one can get two subcombs contradicting to Lemma 9.4 (1), because there are at most $N + 1$ maximal q-bands starting on $\partial\Pi$ in $\Psi_{i,i+1}$. By Lemma 9.4 (2), such a subcomb has at most $2K_0$ q-edges in the boundary. Hence there are at most $2K_0 + N < 3K_0$ q-edges in the path $\mathbf{p}_{i,i+1}$. $\qquad\square$

We denote by $\overline{\Delta}$ the subdiagram formed by Π and Ψ, and denote by $\overline{\mathbf{p}}$ the path

$$\mathbf{top}(\mathcal{B}_1)\mathbf{u}^{-1}\mathbf{bot}(\mathcal{B})_{L-3}^{-1},$$

where $\mathbf{u}$ is a subpath of $\partial\Pi$, such that $\overline{\mathbf{p}}$ separates $\overline{\Delta}$ from the remaining subdiagram Ψ' of Δ (Fig. 16).
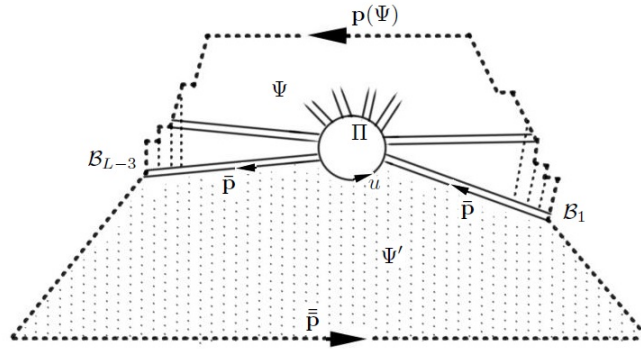
Figure 16. Boundaries of Ψ and Ψ'.

Similarly we define subdiagrams $\overline{\Delta}_{ij}$, paths

$$\overline{\mathbf{p}}_{i,j} = \mathbf{top}(\mathcal{B}_i)\mathbf{u}_{ij}^{-1}\mathbf{bot}(\mathcal{B})_j^{-1},$$

where $\mathbf{u}_{ij}$ is a subpath of $\partial\Pi$, and the subdiagrams Ψ'_{ij}.

We denote by $H_1, \dots, H_{L-3}$ the histories of the spokes $\mathcal{B}_1, \dots, \mathcal{B}_{L-3}$ (read starting from the disk Π) and by $h_1, \dots, h_{L-3}$ their lengths, i.e. the numbers of (θ, t)-cells. By Lemma 9.5, these lengths non-increase and then non-decrease as follows:

$$h_1 \geq h_2 \geq \cdots \geq h_r, \quad h_{r+1} \leq \cdots \leq h_{L-3} \qquad (L/2 - 3 \leq r \leq L/2), \qquad (9.1)$$

and therefore H_{i+1} is a prefix of H_i (H_j is a prefix H_{j+1}) for $i = 1, \dots, r-1$ (resp., for $j = r+1, \dots, L-4$).

Recall that the boundary label of $\partial\Pi$ has the form W^L, i.e. it is the Lth power of an accepted word W.

Lemma 9.7. *We have the following inequalities*

$$|\overline{\mathbf{p}}_{ij}| \leq h_i + h_j + (L - j + i)|W| - 1$$

and, if $i \leq r$ and $j \geq r + 1$, then

$$|\mathbf{p}_{ij}| \geq |\mathbf{p}_{ij}|_\theta + |\mathbf{p}_{ij}|_q \geq h_i + h_j + (j - i)N + 1.$$

Proof. The first iequality follows from Lemma 6.1 (b) since the path $\mathbf{u}_{ij}$ has $L - j + i - 1$ t-edges. To prove the second inequality, we observe that the path $|\mathbf{p}_{ij}|$ has $(j - i)N + 1$ q-edges and it has $h_i + h_j$ θ-edges by Lemma 9.5. $\qquad\square$

The large constants L and L_0 are chosen so that

$$L_0^3 \leq L. \qquad (9.2)$$

Lemma 9.8. *If $j - i > L/2$, then we have*

$$\mu(\Delta) - \mu(\Psi'_{ij}) > -2Jn(h_i + h_j) \geq -2Jn|\mathbf{p}_{ij}|.$$

Proof. The number of q-edges in the path $\bar{\mathbf{p}}_{ij}$ (or in the path $\mathbf{u}_{ij}$) does not exceed the similar number for $\mathbf{p}_{ij}$ provided $j - i \geq L/2$. Therefore any two white beads o, o' of the necklace on $\partial\Delta$, provided they both do not belong to $\mathbf{p}_{ij}$, are separated by at least the same number of black beads in the necklace for Δ as in the necklace for Ψ'_{ij} (either the clockwise arc o–o' includes $\mathbf{p}_{ij}$ or not). So such a pair contributes to $\mu(\Delta)$ at least the amount it contributes to $\mu(\Psi'_{ij})$. Thus, to estimate $\mu(\Delta) - \mu(\Psi'_{ij})$ from below, it suffices to consider the contribution to $\mu(\Psi')$ for the pairs o, o', where one of the two beads lies on $\mathbf{p}_{ij}$. The number of such (unordered) pairs is bounded by $n(h_i + h_j)$ by Lemma 9.5. Taking into account the definition of μ of diagrams and inequalities (9.1), we get the required inequality. □

Lemma 9.9. *If $j - i > L/2$, then the following inequality holds:*

$$|\mathbf{p}_{ij}| < (1 + \varepsilon)|\bar{\mathbf{p}}_{ij}|,$$

where $\varepsilon = N_4^{-1/2}$. Moreover, we have

$$|\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) < (1 + \varepsilon)|\bar{\mathbf{p}}_{ij}|.$$

Proof. It suffices to prove the second statement. Let d be the difference

$$|\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) - |\bar{\mathbf{p}}_{ij}|$$

and assume by contradiction that $d \geq \varepsilon|\bar{\mathbf{p}}_{ij}|$. Then

$$d \geq |\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) - \varepsilon^{-1}d,$$

whence

$$d \geq (1 + \varepsilon^{-1})^{-1}\big(|\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij})\big) \geq \frac{\varepsilon}{2}\big(|\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij})\big) \geq \frac{\varepsilon y}{2}, \qquad (9.3)$$

where by definition, $y = |\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij})$.

We have

$$\big(|\partial\Delta| + \sigma_\lambda(\Delta)\big) - \big(|\partial\Psi'_{ij}| + \sigma_\lambda(\Psi'_{ij})\big) \geq d > 0,$$

because $|\partial\Delta| - |\partial\Psi'_{ij}| \geq |\mathbf{p}_{ij}| - |\bar{\mathbf{p}}_{ij}|$ and $\sigma_\lambda(\bar{\Delta}_{ij}) + \sigma_\lambda(\bar{\Psi}'_{ij}) \leq \sigma_\lambda(\Delta)$. Therefore for $x = n + \sigma_\lambda(\Delta)$, we obtain from the minimality of the counter-example Δ that

$$\begin{aligned} \text{Area}_G(\Psi'_{ij}) &\leq N_4 F(x - d) + N_3\mu(\Psi'_{ij})g(n) \\ &\leq N_4 F(x) - N_4 F(x)x^{-1}d + N_3\mu(\Delta)g(n) + 2N_3 Jn|\mathbf{p}_{ij}|g(n) \\ &\leq N_4 F(x) + N_3\mu(\Delta)g(n) - N_4 F(x)x^{-1}d + 2N_3 Jnyg(n) \quad (9.4) \end{aligned}$$

by Lemma 9.8, inequality $\sigma_\lambda(\Psi'_{ij})) \leq \sigma_\lambda(\Delta)$, and Lemma 4.18. By Lemma 9.7, $|\bar{\mathbf{p}}_{ij}| < |\mathbf{p}_{ij}| + |\partial\Pi|$, and so the perimeter $|\partial\Psi_{ij}|$ is less than $2|\mathbf{p}_{ij}| + |\partial\Pi|$. Since

$$|\partial\Pi| \leq L|\bar{\mathbf{p}}_{ij}| < L(|\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij})),$$

we have

$$|\partial\Psi_{ij}| < (2 + L)|\mathbf{p}_{ij}| + L\sigma_\lambda(\bar{\Delta}_{ij}) \leq (L + 2)y. \tag{9.5}$$

By the inequalities $N_2 > N_1$, (9.5), Lemmas 6.16 and 6.7 (a), the G-area of Ψ_{ij} does not exceed

$$N_2(2 + L)^2 y^2 + N_1\mu(\Psi_{ij}) \leq N_2(J + 1)(2 + L)^2 y^2. \tag{9.6}$$

By Lemma 9.1, the G-area of Π does not exceed

$$c_6 F(|\partial\Pi|) \leq c_6 F((L + 2)y),$$

and by definition of the functions f and F, there is a constant $c_7 = c_7(L)$ such that $\text{Area}_G(\Pi) \leq c_7 F(y)$.

This estimate and (9.6) give the inequality

$$\text{Area}_G(\bar{\Delta}_{ij}) \leq N_2(J + 1)(2 + L)^2 y^2 + c_7 F(y),$$

and we obtain with (9.4) that

$$\begin{aligned}\text{Area}_G(\Delta) \leq{}& N_4 F(x) + N_3\mu(\Delta)g(n) - N_4 F(x)x^{-1}d \\ &+ 2N_3 Jnyg(n) + N_2(J + 1)(2 + L)^2 y^2 + c_7 F(y).\end{aligned}$$

To obtain the desired contradiction, it suffices to show that here, the number $T = N_4 F(x)x^{-1}d/3$ is greater than each of the last three summands. Recall that

$$F(x)x^{-1} = xg(x) \geq ng(n), \quad d > \varepsilon y/2 \text{ by (9.3)}, \quad \varepsilon = N_4^{-1/2},$$

and so

$$T > 2N_3 Jnyg(n)$$

if N_4 is large enough in comparison with N_3 and other constants chosen earlier. Also we have

$$T > N_2(J + 1)(2 + L)^2 y^2,$$

because $x = n + \sigma_\lambda(\Delta) > |\mathbf{p}_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) = y$, and so

$$F(x)x^{-1}d > xg(x)\varepsilon y/2 \geq \varepsilon y^2/2.$$

Finally, $T > c_7 F(y)$ since

$$F(x)x^{-1}d > xg(x)\varepsilon y/2 \geq y^2 g(y)\varepsilon/2 = \varepsilon F(y)/2. \qquad \square$$

For every path $\mathbf{p}_{i,i+1}$ we will fix a shortest path $\mathbf{q}_{i,i+1}$ homotopic to $\mathbf{p}_{i,i+1}$ in the subdiagram Ψ_{ij}, such that the first and the last t-edges of $\mathbf{q}_{i,i+1}$ coincide with the first and the last t-edges of $\mathbf{p}_{i,i+1}$. For $j > i + 1$ the path $\mathbf{q}_{i,j}$ is formed by $\mathbf{q}_{i,i+1}, \ldots, \mathbf{q}_{j-1,j}$. The following lemma is similar to the second part of Lemma 9.7.

Lemma 9.10. *If $i \leq r$ and $j \geq r + 1$, then*

$$|\mathbf{q}_{ij}| \geq |\mathbf{q}_{ij}|_\theta + |\mathbf{q}_{ij}|_q \geq h_i + h_j + (j - i)N + 1. \qquad \square$$

Let Ψ_{ij}^0 (let Ψ^0, Δ^0) be the subdiagram of Ψ_{ij} (of Ψ, of Δ) obtained after replacement of the subpath $\mathbf{p}_{ij}$ (of $\mathbf{p}$) by $\mathbf{q}_{ij}$ (by $\mathbf{q} = \mathbf{q}_{1,L-3}$, resp.) in the boundary.

Lemma 9.11. (1) *The subdiagram Ψ_{ij}^0 has no maximal q-bands except for the q-spokes starting from $\partial\Pi$.*

(2) *Every θ-band of Ψ_{ij}^0 is crossed by the path $\mathbf{q}_{ij}$ at most once.*

Proof. Assume there is a q-band $\mathcal{Q}$ of Ψ_{ij}^0 starting and ending on $\mathbf{q}_{ij}$. Then $j = i + 1$ and $\mathbf{q}_{i,i+1} = \mathbf{uevfw}$, where $\mathcal{Q}$ starts with the q-edge $\mathbf{e}$ and ends with the q-edge $\mathbf{f}$. Let $\mathcal{Q}$ have length ℓ. Then $|v| \geq \ell$ since every maximal θ-band of $\Psi_{i,i+1}^0$ crossing $\mathcal{Q}$ has to end on the subpath $\mathbf{v}$. So one has $|\mathbf{evf}| = \ell + 2$, and replacing the subpath $\mathbf{evf}$ by a side of $\mathcal{Q}$ of length ℓ one replaces the path $\mathbf{q}_{i,i+1}$ with a shorter homotopic path by Lemma 6.1. This contradicts to the choice of $\mathbf{q}_{i,i+1}$, and so the first statement is proved. The prove of the second statement is similar. $\qquad \square$

It follows from Lemma 9.5 that between the spokes $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$ ($1 \leq j \leq r-1$), there is a trapezium Γ_j of height h_{j+1} with the side t-bands $\mathcal{B}_{j+1}$ and $\mathcal{B}'_j$, where $\mathcal{B}'_j$ is the beginning of length h_{j+1} of the t-spoke $\mathcal{B}_j$. Similarly, we have trapezia Γ_j for $r + 1 \leq j \leq L - 4$. By Lemma 9.11 (2), every trapezium Γ_j is contained in both $\Psi_{j,j+1}$ and $\Psi_{j,j+1}^0$. The bottoms $\mathbf{y}_j$ of all trapezia Γ_j belong to $\partial\Pi$ and have the same label Wt. We will use $\mathbf{z}_j$ for the tops of these trapezia. Since Γ_j and Γ_{j-1} ($2 \leq j \leq r - 1$) have the same bottom labels and the history H_j is a prefix of H_{j-1}, by Lemma 5.7, h_j different θ-bands of Γ_{j-1} form the copy Γ'_j of the trapezium Γ_j with top and bottom paths $\mathbf{z}'_j$ and $\mathbf{y}'_j = \mathbf{y}_{j-1}$.

We denote by E_j (by E_j^0) the comb formed by the maximal θ-bands of $\Psi_{j,j+1}$ (of $\Psi_{j,j+1}^0$, resp.) crossing the t-spoke $\mathcal{B}_j$ but not crossing $\mathcal{B}_{j+1}$ ($1 \leq j \leq r - 1$); See Fig. 17. Its handle $\mathcal{C}_j$ of height $h_j - h_{j+1}$ is contained in $\mathcal{B}_j$. The boundary ∂E_j (resp., ∂E_j^0) consists of the side of this handle, the path $\mathbf{z}_j$ and the path $\mathbf{p}_{j,j+1}$ (the path $\mathbf{q}_{j,j+1}$, respectively).

Assume that a maximal a-band $\mathcal{A}$ of E_j^0 ($2 \leq j \leq r - 1$) starts on the path $\mathbf{z}_j$ and ends on a side a-edge of a maximal q-band $\mathcal{C}$ of E_j^0. Then $\mathcal{A}$, a part of $\mathcal{C}$ and a part $\mathbf{z}$ of $\mathbf{z}_j$ bound a comb ∇.
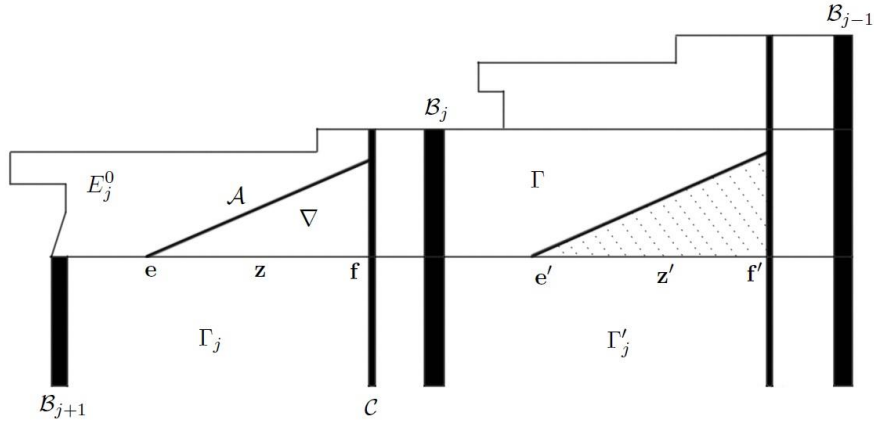
Figure 17. Lemma 9.12.

Lemma 9.12. *There is a copy of the comb ∇ in the trapezium $\Gamma = \Gamma_{j-1} \backslash \Gamma'_j$.*

Proof. The subpath $\mathbf{z}$ of $\mathbf{z}_j$ starts with an a-edge $\mathbf{e}$ and ends with a q-edge $\mathbf{f}$. There is a copy $\mathbf{z}'$ of $\mathbf{z}$ in $\mathbf{z}'_j$ starting with $\mathbf{e}'$ and ending with $\mathbf{f}'$. Note that the θ-cells π and π' attached to $\mathbf{f}$ and to $\mathbf{f}'$ in ∇ and in Γ are copies of each other since they correspond to the same letter of the history. Now moving from $\mathbf{f}$ to $\mathbf{e}$, we see that the whole maximal θ-band $\mathcal{T}_1$ of ∇ containing π has a copy in Γ. Similarly we obtain a copy of the next maximal θ-band $\mathcal{T}_2$ of ∇, and so on. $\qquad\square$

Lemma 9.13. *At most N a-bands starting on the path $\mathbf{y}_j$ can end on the (θ, q)-cells of the same θ-band. This property holds for the a-bands starting on $\mathbf{z}_j$ too.*

Proof. We will prove the second claim only since the proof of the first one is similar. Assume that the a-bands $\mathcal{A}_1, \ldots, \mathcal{A}_s$ start from $\mathbf{z}_j$ and end on some (θ, q)-cells of a θ-band $\mathcal{T}$. Let $\mathcal{T}_0$ be the minimal subband of $\mathcal{T}$, where the a-bands $\mathcal{A}_2, \ldots, \mathcal{A}_{s-1}$ end and $\bar{\mathbf{z}}_j$ be the minimal subpath of $\mathbf{z}_j$, where they start. Then by Lemma 5.3, every maximal q-band starting on $\bar{\mathbf{z}}_j$ has to cross the band $\mathcal{T}_0$ and vice versa. Hence the base of $\mathcal{T}_0$ is a subbase of the standard base (or of its inverse). Since every rule of $\mathbf{M}$ can change at most $N - 2$ a-letters in a word with standard base, all (θ, q)-cells of $\mathcal{T}_0$ have at most $N - 2$ a-edges, and the statement of the lemma follows. $\qquad\square$

Without loss of generality, we assume that

$$h = h_{L_0+1} \geq h_{L-L_0-3}. \tag{9.7}$$

Lemma 9.14. *If $h \leq L_0^2 |W|_a$, then the number of trapezia Γ_j with the properties*

$$|\mathbf{z}_j|_a \geq |W|_a / c_5 N$$

for $j \in [L_0 + 1, r - 1]$ or $j \in [r + 1, L - L_0 - 5]$, is less than $L/5$.

Proof. Consider Γ_j as in the assumption of the lemma with $j \in [L_0 + 1, r - 1]$. The subcomb E_j^0 has at most N maximal q-bands by Lemma 9.11. So there are at most N maximal a-bands starting on $\mathbf{z}_j$ and ending on each of the θ-bands of E_j^0. Proving by contradiction, we have at least $L|W|_a/5c_5N$ such a-bands for all $j \in S$, where S the set of integers in $[L_0 + 1, r - 1] \cup [r + 1, L - L_0 - 5]$; denote this set of a-bands by $\mathbf{A}$. But the number of maximal θ-bands in all such subcombs E_j^0 does not exceed $2h$. Therefore at least $L|W|_a/5c_5N - 2hN$ bands from $\mathbf{A}$ end on the subpaths $\mathbf{q}_{j,j+1}$ for $j \in S$. Therefore by Lemmas 9.10 and 6.1, we have

$$
\begin{aligned}
|\mathbf{p}_{L_0+1,L-L_0-5}| &\geq |\mathbf{q}_{L_0+1,L-L_0-5}| \\
&\geq h_{L_0+1} + h_{L-L_0-5} + LN/2 + \delta\big(L|W|_a/5c_5N - 2hN\big) \\
&\geq h_{L_0+1} + h_{L-L_0-5} + LN/2 + \delta L|W|_a/10c_5N \qquad (9.8)
\end{aligned}
$$

since

$$
2hN \leq 2L_0^2 N|W|_a < L_0^3|W|_a/10c_5N \leq L|W|_a/10c_5N
$$

by the choice of L_0 and L (9.2).

Also by Lemma 9.7, we have

$$
\begin{aligned}
|\bar{\mathbf{p}}_{L_0+1,L-L_0-5}| &\leq h_{L_0+1} + h_{L-L_0-5} + 3L_0N + 3L_0\delta|W|_a \\
&\leq h_{L_0+1} + h_{L-L_0-5} + 3L_0N + \delta L|W|_a/20c_5N, \qquad (9.9)
\end{aligned}
$$

because by the choice of L, $3L_0 < L/20c_5N$. Since

$$
h_{L_0+1} + h_{L-L_0-5} \leq 2h \leq 2L_0^2|W|_a < L|W|_a,
$$

L is chosen after c_5N, and $\varepsilon = N_4^{-1/2}$ is chosen after L, the inequality

$$
\frac{|\mathbf{p}_{L_0+1,L-L_0-5}|}{|\bar{\mathbf{p}}_{L_0+1,L-L_0-5}|} \geq 1 + \frac{\delta}{20c_5N} > 1 + \varepsilon
$$

follows from (9.8) and (9.9), contrary to Lemma 9.9. The lemma is proved by contradiction. $\square$

Lemma 9.15. *If $h \leq L_0^2|W_a|$, then the histories H_1 and H_{L-3} have different first letters.*

Proof. Let $\mathcal{T}$ and $\mathcal{S}$ be the maximal θ-bands of Ψ crossing $\mathcal{B}_1$ and $\mathcal{B}_{L-3}$, respectively, and the closest to the disk Π. Let them cross k and ℓ spokes of Π, respectively. By Lemma 9.14,

$$
k + \ell > L - L/5 - 3L_0 > 2L/3,
$$

and also $k, \ell \geq 2$ since $L/2 - 3 \leq r \leq L/2$. It follows from Lemma 7.5 that the first letters of H_1 and H_{L-3} are different. $\square$

Lemma 9.16. *We have $h > L_0^2 |W|_a$.*

Proof. If this inequality is wrong, then by Lemma 9.14, there are at least

$$L - L/5 - 3L_0 > 0.7L$$

trapezia Γ_j with $|\mathbf{z}_j|_a < |W|_a/c_5 N$, and one can choose two such trapesia Γ_k and Γ_ℓ such that $k < r$, $\ell \geq r + 1$ and $\ell - k > 0.6L$. Since H_{k+1} (resp., H_ℓ) is a prefix of H_1 (of H_{L-3}), it follows from Lemma 9.15 that the first letters of H_{k+1} and H_ℓ are different.

Since the bottoms of Γ_k and Γ_ℓ (which belong to $\partial\Delta$) have the same label, one can construct an auxiliary trapezium E identifying the bottom of a copy of Γ_k and the bottom of a mirror copy of Γ_ℓ. The history of E is $H_\ell^{-1} H_{k+1}$, which is a reduced word since the first letters of H_k and H_ℓ are different, i.e. E is a trapezium indeed by Lemma 5.7.

The top and the bottom of E have a-lengths less than $|W|_a/c_5 N$. Without loss of generality, one may assume that $h_{k+1} \geq h_\ell$, and so $h_{k+1} \geq t/2$, where t is the height of E.

Note that the difference of a-lengths $|W|_a - |W|_a/c_5 N > |W|_a/2$, and so

$$h_{k+1}, h_\ell > |W|_a/2N \tag{9.10}$$

since the difference of a-lengths for the top and the bottom of every maximal θ-band of E does not exceed N. Therefore $t > |W|_a/N$, and the computation corresponding E satisfies the assumption of Lemma 4.26.

So for every factorization $H'H''H'''$ of the history of Γ_k, where

$$\|H'\| + \|H''\| \leq \lambda \|H'H''H'''\|,$$

we have $\|H''\| > 0.4t$, since $\lambda < 1/5$. Therefore by Lemma 4.26, the spoke $\mathcal{B}_{k+1}$ is a λ-shaft.

Using Lemma 9.7, we obtain:

$$|\mathbf{p}_{k+1,\ell}| + \sigma_\lambda\big(\overline{\Delta}_{k+1,\ell}\big) \geq h_{k+1} + h_\ell + 0.6LN + h_{k+1}. \tag{9.11}$$

By inequality (9.10), we have $\delta L |W|_a \leq 2LN\delta h_{k+1} < h_{k+1}$ by the choice of δ and by Lemma 9.7,

$$|\overline{\mathbf{p}}_{k+1,\ell}| \leq h_{k+1} + h_\ell + 0.4LN + 0.4L\delta|W|_a \leq h_{k+1} + h_\ell + h_{k+1}/2 + 0.4LN. \tag{9.12}$$

The right-hand side of the inequality (9.11) divided by the right-hand side of (9.12) is greater than 1.1 (because $h_{k+1} \geq h_\ell$), which contradicts Lemma 9.9. Thus, the lemma is proved. $\qquad\square$

Lemma 9.17. *We have $h_i > \delta^{-1}$ for every $i = 1, \dots, L_0$.*

Proof. By inequalities (9.7) and (9.1), we have $h_i \geq h_{L-L_0-3}$. Proving by contradiction, we obtain $|W|_a < h_i \leq \delta^{-1}$ by Lemma 9.16. Then,

$$|\bar{\mathbf{p}}_{i,L-L_0-3}| < h_i + h_{L-L_0-3} + 3L_0(N + \delta^{-1}\delta) \leq h_i + h_{L-L_0-3} + 4L_0 N$$

by Lemma 9.7, and

$$|\mathbf{p}_{i,L-L_0-3}| \geq h_i + h_{L-L_0-3} + LN/2.$$

Since $h_i + h_{L-L_0-3} \leq 2\delta^{-1}$ and $4L_0 N < LN/4$, we see that

$$\frac{|\mathbf{p}_{i,L-L_0-3}|}{|\bar{\mathbf{p}}_{i,L-L_0-3}|} > 1 + \delta > 1 + \varepsilon$$

contrary to Lemma 9.9. The lemma is proved by contradiction.   □

Lemma 9.18. *None of the spokes $\mathcal{B}_1, \dots, \mathcal{B}_{L_0}$ contains a λ-shaft at Π of length at least δh.*

Proof. On the one hand, by Lemmas 9.7 and 9.16,

$$
\begin{aligned}
|\bar{\mathbf{p}}_{L_0+1,L-L_0-3}| &< h_{L_0+1} + h_{L-L_0-3} + 3L_0(N + \delta|W|_a) \\
&< h_{L_0+1} + h_{L-L_0-3} + 3L_0(N + \delta L_0^{-2}h). \quad (9.13)
\end{aligned}
$$

On the other hand, by Lemma 9.7,

$$|\mathbf{p}_{L_0+1,L-L_0-3}| > h_{L_0+1} + h_{L-L_0-3} + (L - 3L_0)N. \quad (9.14)$$

If the statement of the lemma were wrong, then we would have $\sigma_\lambda(\bar{\Delta}) \geq \delta h$, and inequalities (9.13) and (9.14) would imply that

$$
\begin{aligned}
|\mathbf{p}_{L_0+1,L-L_0-3}| - |\bar{\mathbf{p}}_{L_0+1,L-L_0-3}| + \sigma_\lambda(\bar{\Delta}) &\geq (L - 6L_0)N - 3L_0^{-1}\delta h + \delta h \\
&\geq LN/2 + \delta h/2.
\end{aligned}
$$

The right-hand side of the last inequality divided by the right-hand side of (9.13) is greater than $\varepsilon = N_4^{-1/2}$, because $h \geq h_{L_0+1}, h_{L-L_0-3}$, which would contradict to Lemma 9.9. Thus, the lemma is proved.   □

Lemma 9.19. *For every $j \in [1, L_0 - 1]$, we have $|\mathbf{z}_j|_a > h_{j+1}/c_5$.*

Proof. If $|\mathbf{z}_j|_a \leq h_{j+1}/c_5$, then the computation $\mathcal{C}: W_0 \to \cdots \to W_t$ corresponding to the trapezium Γ_j satisfies the assumption of Lemma 4.26, since

$$t = h_{j+1} > c_5|W_t|_a = c_5|\mathbf{z}_j|_a$$

and by Lemma 9.16,

$$t = h_{j+1} \geq L_0^2 |W_0|_a \geq c_5 |W|_a$$

since $L_0 > c_5$. Hence $\mathcal{B}_{j+1}$ is a λ-shaft by Lemma 4.26 since $\lambda < 1/2$. We obtain a contradiction with Lemma 9.18 since $\delta h \leq h \leq h_{j+1}$. Thus, the lemma is proved. □

Lemma 9.20. *For every* $j \in [1, L_0 - 1]$, *we have* $h_{j+1} < (1 - (1/10c_5N)) h_j$.

Proof. By Lemma 9.19, we have $|\mathbf{z}_j|_a \geq h_{j+1}/c_5$. Let us assume that

$$h_{j+1} \geq \left(1 - \frac{1}{10c_5N}\right) h_j,$$

that is the handle $\mathcal{C}_j$ of E_j has height at most $h_j/10c_5N$. By Lemma 9.13, at most $h_j/10c_5$ maximal a-bands of E_j starting on $\mathbf{z}_j$ can end on the (θ, q)-cells of E_j. Hence at least

$$|\mathbf{z}_j|_a - h_j/10c_5 \geq |\mathbf{z}_j|_a - 2h_{j+1}/10c_5 \geq 0.8h_{j+1}/c_5 > 0.7h_j/c_5$$

of them have to end on the path $\mathbf{p}_{j,j+1}$.

The path $\mathbf{p}_{j,j+1}$ has at most $h_j/10c_5N$ θ-edges. Hence by Lemma 6.1,

$$|\mathbf{p}_{j,j+1}| \geq h_j - h_{j+1} + \delta(0.7h_j/c_5 - h_j/10c_5N) \geq h_j - h_{j+1} + 0.6\delta h_j/c_5,$$

and therefore by Lemma 9.7,

$$|\mathbf{p}_{j,L-L_0-3}| \geq LN/2 + h_j + h_{L-L_0-3} + 0.6\delta h_j/c_5.$$

On the other hand by Lemma 9.7, we have

$$\begin{aligned}|\overline{\mathbf{p}}_{j,L-L_0-3}| &\leq h_j + h_{L-L_0-3} + 3NL_0 + 3L_0\delta |W|_a \\ &\leq h_j + h_{L-L_0-3} + 3NL_0 + 3L_0^{-1}\delta h_{j+1}\end{aligned}$$

by Lemma 9.16 and inequality $h \leq h_{j+1}$. Hence,

$$\frac{|\mathbf{p}_{j,L-L_0-3}|}{|\overline{\mathbf{p}}_{j,L-L_0-3}|} \geq (1 + \delta/10c_5)$$

since $h_{L-L_0-3} \leq h_{L_0+1} \leq h_{j+1} \leq h_j$ and $L_0 \gg c_5$. We have a contradiction with Lemma 9.9 since $\delta/10c_5 > \varepsilon$. The lemma is proved by contradiction. □

The proof of the next lemma is similar.

Lemma 9.21. *For every* $j \in [2, L_0 - 1]$, *we have* $|\mathbf{z}_j|_a \leq 2Nh_j$.

Proof. Assume that $|\mathbf{z}_j|_a \geq 2Nh_j$. By Lemma 9.13, at most Nh_j maximal a-bands of E_j starting on $\mathbf{z}_j$ can end on the (θ, q)-cells of E_j. Hence at least $|\mathbf{z}_j|_a - Nh_j \geq Nh_j$ of them has to end on the path $\mathbf{p}_{j,j+1}$. The path $\mathbf{p}_{j,j+1}$ has at most h_j θ-edges. Hence by Lemma 6.1,

$$|\mathbf{p}_{j,j+1}| \geq h_j - h_{j+1} + \delta(Nh_j - h_j) = h_j - h_{j+1} + \delta(N-1)h_j,$$

and therefore by Lemma 9.7,

$$|\mathbf{p}_{j,L-L_0-3}| \geq LN/2 + h_j + h_{L-L_0-3} + \delta(N-1)h_j.$$

On the other hand by Lemmas 9.7 and 9.16, we have

$$|\bar{\mathbf{p}}_{j,L-L_0-3}| \leq h_j + h_{L-L_0-3} + 3NL_0 + 3L_0\delta|W|_a$$
$$\leq h_j + h_{L-L_0-3} + 3NL_0 + \frac{3\delta h_j}{L_0},$$

because $h \leq h_j$. Since $h_j \geq h \geq h_{L-L_0-3}$, we have

$$\frac{|\mathbf{p}_{j,L-L_0-3}|}{|\bar{\mathbf{p}}_{j,L-L_0-3}|} \geq 1 + \varepsilon,$$

a contradiction by Lemma 9.9.   □

Lemma 9.22. *There is no $i \in [2, L_0 - 3]$ such that the histories*

$$H_{i-1} = H_i H' = H_{i+1} H'' H' = H_{i+2} H''' H'' H'$$

and the computation $\mathcal{C}$ with history H_i corresponding to the trapezium Γ_{i-1} satisfy the following condition:

The history $H''' H'' H'$ has only one step, and for the subcomputation $\mathcal{D}$ with this history, there is a sectors $Q'Q$ such that a state letter from Q or from Q' (∗) inserts a letter increasing the length of this sector after every transition of $\mathcal{D}$.

Proof. Recall that the standard base of $\mathbf{M}$ is built of the standard base B of $\mathbf{M}_4$ and its inverse copy $(B')^{-1}$ (plus letter t). Due to this mirror symmetry of the standard base, we have mirror symmetry for any accepting computation, in particular, for $\mathcal{C}$ and $\mathcal{D}$. Therefore proving by contradiction, we may assume that the a-letters are inserted from the left of Q.

Let $\mathcal{Q}$ be the maximal q-spoke of the subdiagram $E_i^0 \subset \Gamma_i$ corresponding to the base letter Q. If $\mathcal{Q}'$ is the neighbor from the left q-spoke for $\mathcal{Q}$ (the spokes are directed from the disk Π), then the subpath $\mathbf{x}$ of $\mathbf{z}_i$ between these two q-spokes has at least $h_{i+1} - h_{i+2} = \|H'''\|$ a-letters. Indeed, Γ_i contains a copy Γ'_{i+1} of Γ_{i+1}, the bottom of the trapezium $\Gamma_i \setminus \Gamma'_{i+1}$ is the copy $\mathbf{z}'_{i+1}$ of $\mathbf{z}_{i+1}$ and the top of it iz $\mathbf{z}_i$,

and so the subcomputation with history H''' has already increased the length of the $Q'Q$-sector. Thus, by Lemmas 9.20, 9.16, and the choice of $L_0 > 100c_5 N$, we have

$$|\mathbf{x}|_a \geq h_{i+1} - h_{i+2} \geq \frac{1}{10c_5 N} h_{i+1} \geq 10 L_0 |W|_a. \tag{9.15}$$

Note that an a-band $\mathcal{A}$ starting on $\mathbf{x}$ cannot end on a (θ, q)-cell from $\mathcal{Q}$. Indeed, otherwise by Lemma 9.12, there is a copy of this configuration in the diagram Γ_{i-1}, i.e. the copy of $\mathcal{A}$ ends on the copy of $\mathcal{Q}$ contrary the assumption that the rules of computation with history $H'''H''H'$ do not delete a-letters.

Let us consider the comb bounded by $\mathcal{Q}$, $\mathcal{Q}'$, $\mathbf{x}$ and the boundary path of Δ^0 (without the cells from $\mathcal{Q}'$). If the lengths of $\mathcal{Q}$ and $\mathcal{Q}'$ are s and s', respectively, then there are $|\mathbf{x}| + s$ maximal a-bands starting on $\mathbf{x}$ and $\mathcal{Q}$ and ending either on $\mathcal{Q}'$ or on $\partial \Delta^0$ since the comb has no maximal q-bands by Lemma 9.11. At most $s' < s$ of these a-bands can end on $\mathcal{Q}'$. Therefore at least $|\mathbf{x}| + s - s'$ of them end on the segment of the boundary path of Δ^0 lying between the ends of $\mathcal{Q}'$ and $\mathcal{Q}$.

Since this segment has $s - s'$ θ-edges, its length is at least $s - s' + \delta |\mathbf{x}|_a$ by Lemma 6.1. This inequality, Lemma 9.10 and inequality (9.15) imply

$$|\mathbf{p}_{i,L-L_0-3}| \geq |\mathbf{q}_{i,L-L_0-3}|$$
$$\geq LN/2 + h_i + h_{L-L_0-3} + \frac{\delta}{10c_5 N} h_{i+1}$$
$$\geq LN/2 + h_i + h_{L-L_0-3} + 10\delta L_0 |W|_a.$$

Therefore,

$$|\mathbf{p}_{i,L-L_0-3}| - \frac{7\delta}{100c_5 N} h_{i+1} > 3L_0 N + h_i + h_{L-L_0-3} + 3\delta L_0 |W|_a$$
$$\geq |\bar{\mathbf{p}}_{i,L-L_0-3}|, \tag{9.16}$$

by Lemma 9.7, and since Δ is a minimal counter-example, we obtain by the definition of $F(x)$, $g(x)$ and inequality (6.4) that

$$\text{Area}_G \left(\Psi'_{i+1,L-L_0-3} \right)$$
$$\leq N_4 F \left(n + \sigma_\lambda(\Delta) - \frac{7\delta}{100c_5 N} h_{i+1} \right) + N_3 g(n) \mu \left(\Psi'_{i,L-L_0-3} \right)$$
$$\leq N_4 F \left(n + \sigma_\lambda(\Delta) \right) - N_4 \frac{7\delta n}{100c_5 N} h_{i+1} g(n) + N_3 g(n) \mu \left(\Psi'_{i,L-L_0-3} \right). \tag{9.17}$$

By Lemma 9.16, $|W|_a \leq L_0^{-2} h_i$, and by Lemma 9.17, $h_i > \delta^{-1} > 100 L_0 N$, whence

$$|\bar{\mathbf{p}}_{i,L-L_0-3}| \leq 2h_i + 3L_0 N + 3\delta L_0 |W|_a \leq \left(2 + 0.03 + \frac{3\delta}{L_0} \right) h_i \leq 2.1 h_i,$$

and by Lemma 9.9, we have

$$|\mathbf{p}_{i,L-L_0-3}| \leq (1 + \varepsilon)|\bar{\mathbf{p}}_{i,L-L_0-3}| < 2.2h_i. \tag{9.18}$$

By Lemmas 6.16 and 6.7 (a) and inequalities (9.18) and (9.16), the G-area of $\Psi_{i,L-L_0-3}$ does not exceed

$$N_2\big(2|\mathbf{p}_{i,L-L_0-3}|\big)^2 + N_1\mu(\Psi_{i,L-L_0-3}) \leq N_2(4J + 4)|\mathbf{p}_{i,L-L_0-3}|^2$$
$$\leq 5N_2(4J + 4)h_i^2. \tag{9.19}$$

By Lemma 9.1, the G-area of Π is bounded by $c_6 F(|\partial\Pi|)$. The inequalities (9.16) and (9.18) imply the inequality

$$|\partial\Pi| < L|\bar{\mathbf{p}}_{i,L-L_0-3}| < L|\mathbf{p}_{i,L-L_0-3}| < 3Lh_i.$$

Therefore one may assume that the constant c_7 is chosen so that

$$\text{Area}_G(\Pi) < c_6 F\big(|\partial\Pi|\big) < c_7 F(h_i) = c_7 h_i^2 g(h_i) \leq c_7 h_i^2 g(n). \tag{9.20}$$

(Recall that $h_i \leq n/2$ here since h_i is the number θ-bands crossing $\mathcal{B}_i$; they start and end on $\partial\Delta$.) It follows from (9.19) and (9.20) that

$$\text{Area}_G\big(\bar{\Delta}_{i,L-L_0-3}\big) \leq 5N_2(4J + 4)h_i^2 + c_7 h_i^2 g(n). \tag{9.21}$$
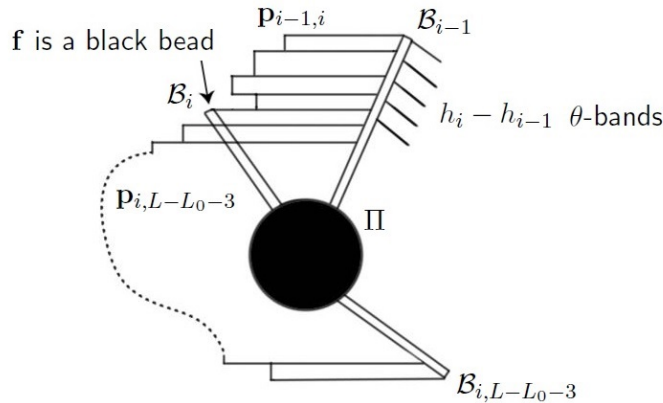
Figure 18. $\mu(\Psi'_{i+1,L-L_0-3}) - \mu(\Psi'_{i,L-L_0-3})$.

We need an estimate for $\mu(\Psi'_{i+1,L-L_0-3}) - \mu(\Psi'_{i,L-L_0-3})$ now. To obtain it, we observe that by Lemma 9.5, the common q-edge $\mathbf{f}$ of the spoke $\mathcal{B}_i$ and $\partial\Delta$ separates at least $h_{i-1} - h_i = m_1$ θ-edges of the path $\mathbf{p}_{i-1,i}$ and m_2 ones lying on $\mathbf{p}_{i,L-L_0-3}$,

where $m_2 = h_i + h_{i,L-L_0-3}$ (see Fig. 18). Since the number of q-edges of $\mathbf{p}$ is less than $3K_0L < J$ by Lemma 9.6, one decreases $\mu(\Psi'_{i+1,L-L_0-3})$ at least by m_1m_2 when erasing the black bead on $\mathbf{f}$ in the necklace on $\partial\Psi'_{i+1,L-L_0-3}$ by Lemma 6.7 (d,b,c). Hence,

$$
\begin{aligned}
\mu(\Psi'_{i+1,L-L_0-3}) - \mu(\Psi'_{i,L-L_0-3}) &\geq m_1m_2 \\
&= (h_{i-1} - h_i)(h_i + h_{L-L_0-3}) \\
&\geq \frac{1}{c_5N}h_{i-1}(h_i + h_{L-L_0-3})
\end{aligned}
$$

by Lemma 9.20. This inequality and Lemma 9.8 applied to $\Psi'_{i+1,L-L_0-3}$, imply

$$
\begin{aligned}
\mu(\Delta) &- \mu(\Psi'_{i,L-L_0-3}) \\
&\geq -2Jn(h_{i+1} + h_{L-L_0-3}) + \frac{1}{10c_5N}h_{i-1}(h_i + h_{L-L_0-3}).
\end{aligned}
$$

Note that $(h_{i+1} + h_{L-L_0-3}) \leq 2h_{i+1}$ by (9.1) and (9.7). Hence,

$$
\begin{aligned}
N_3\mu(\Delta) &- N_3\mu(\Psi'_{i,L-L_0-3}) \\
&\geq -4N_3Jnh_{i+1} + \frac{N_3}{10c_5N}h_{i-1}(h_i + h_{L-L_0-3}). \quad (9.22)
\end{aligned}
$$

It follows from (9.21, 9.22, 9.17) that

$$
\begin{aligned}
\mathrm{Area}_G(\Delta) &\leq \mathrm{Area}_G(\Psi'_{i,L-L_0-3}) + \mathrm{Area}_G(\overline{\Delta}_{i,L-L_0-3}) \\
&\leq N_4F(n + \sigma_\lambda(\Delta)) - N_4\frac{7\delta n}{100c_5N}h_{i+1}g(n) \\
&\quad + N_3g(n)\mu(\Psi'_{i,L-L_0-3}) + 5N_2(4J+4)h_i^2 + c_7h_i^2g(n) \\
&\leq N_4F(n + \sigma_\lambda(\Delta)) + N_3\mu(\Delta)g(n) - N_4\frac{7\delta n}{100c_5N}h_{i+1}g(n) \\
&\quad - \frac{N_3}{10c_5N}h_{i-1}(h_i + h_{L-L_0-3})g(n) \\
&\quad + 4N_3Jnh_{i+1}g(n) + 5N_2(4J+4)h_i^2 + c_7h_i^2g(n).
\end{aligned}
$$

We come to a contradiction since we obtain inequality

$$
\mathrm{Area}_G(\Delta) \leq N_4F(n + \sigma_\lambda(\Delta)) + N_3\mu(\Delta)g(n + \sigma_\lambda(\Delta)),
$$

because

$$
N_4\frac{7\delta}{100c_5N} > 4N_3J \quad \text{and} \quad \frac{N_3}{10c_5N} > 5N_2(4J+4) + c_7. \qquad \square
$$

Lemma 9.23. *There exists no counter-example Δ, and therefore*

$$\mathrm{Area}_G(\Delta) \le N_4 F\big(n + \sigma_\lambda(\Delta)\big) + N_3 \mu(\Delta) g(n)$$

for any minimal diagram Δ with $|\partial\Delta| = n$.

Proof. Recall that for $j = 1, \ldots, L_0 - 1$, we have

$$h_{j+1} < \Big(1 - \frac{1}{10c_5 N}\Big) h_j$$

by Lemma 9.20, and by Lemmas 9.19 and 9.21, we have inequalities $|\mathbf{z}_j|_a \ge h_{j+1}/c_5$ and $|\mathbf{z}_k|_a \le 2Nh_k$ for $2 \le k \le L_0 - 1$. One can choose an integer $\rho = \rho(\mathbf{M})$ (it depends on the S-machine $\mathbf{M}$ only as c_5 and N) so that

$$\Big(1 - \frac{1}{10c_5 N}\Big)^\rho < \frac{1}{6Nc_5},$$

and so $h_{j+1} > 6Nc_5 h_k$ if $k - j - 1 \ge \rho$. Hence,

$$|\mathbf{z}_j|_a \ge h_{j+1}/c_5 \ge 6Nh_k > 3|\mathbf{z}_k|_a.$$

If L_0 is large enough, say $L_0 > 2000\rho$, one can obtain 1000 indices

$$j_1 < j_2 < \cdots < j_{1000} < L_0$$

such that for $i = 2, \ldots, 1000$, one obtains inequalities

$$|\mathbf{z}_{j_{i-1}}|_a > 3|\mathbf{z}_{j_i}|_a \quad and \quad h_{j_{i-1}} \ge h_{j_{i-1}+1} > 6c_5 Nh_{j_i}. \tag{9.23}$$

Let $\mathcal{C}\colon W \equiv W_0 \to \cdots \to W_t$ be the computation corresponding to the trapezium Γ_{j_2}. Since it contains the copy Γ'_{j_2+1} of Γ_{j_2+1}, which in turn contains a copy of Γ_{j_2+2} and so on, we have some configurations $W(k)$ in $\mathcal{C}$ ($k = 1, \ldots, 999$), that are the labels of some $\mathbf{z}_{i_k}$ and $|W(k+1)|_a > 3|W(k)|_a$ for $k = 1, \ldots, 998$. If for some k we were obtain one-step subcomputation

$$W(k) \to \cdots \to W(k+4),$$

then the statement of Lemma 4.27 would give a subcomputation

$$W(k+1) \to \cdots \to W(k+4)$$

contradicting to the statement of Lemma 9.22. Hence no five consecutive words $W(k)$-s are configuration of a one-step subcomputation, and so the number of steps in $W(1) \to \cdots \to W(999)$ in at least 100.

It follows now from Lemmas 4.9 and 4.7 that the step history of $\Gamma_{j_2}\backslash\Gamma$, where Γ is the copy of Γ_{L_0} in Γ_{j_2}, has a subword

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}.$$

Without loss of generality we assume that the exponent is $+1$. Therefore the history H_{j_2+1} of Γ_{j_2} can be decomposed as $H'H''H'''$, where H'' has step history $(12^-)(2^-)(2\text{-}2)$, $\|H'\| \geq \|H''\|$ by Lemma 4.14 and $\|H'\| \geq h$ since H_{L_0} is a prefix of H'.

Since $h_{j_1+1} > 2h_{j_2}$ by (9.23), the history H_{j_1+1} of Γ_{j_1} has a prefix $H'H''H^*$, where $\|H^*\| = \|H'\| \geq \|H''\|$, and so the t-spoke $\mathcal{B}_{j_1+1}$ has a t-subband $\mathcal{C}$ starting with $\partial\Pi$ and having the history $H'H''H^*$.

For any factorization $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3$ with $\|\mathcal{C}_1\| + \|\mathcal{C}_2\| \leq \|\mathcal{C}\|/3$, the history of $\mathcal{C}_2$ contains the subhistory H'', since $\|H^*\| = \|H'\| \geq \|H''\|$. It follows that $\mathcal{C}$ is a λ-shaft, because $\lambda < 1/3$. The shaft has length at least $\|H'\| \geq h$ contrary to Lemma 9.18. We come to the final contradiction in this section. $\square$

10. Proof of Theorem 1.3

10.1. Dehn function of the group G.

Lemma 10.1. *For every big trapezia Δ, there is a diagram $\widetilde{\Delta}$ over G with the same boundary label, such that the area of $\widetilde{\Delta}$ does not exceed $2\,\mathrm{Area}_G(\Delta)$.*

Proof. Consider the computation $\mathcal{C}: V_0 \to \cdots \to V_t$ corresponding to Δ. According to Definition 6.8, one may assume that

$$\mathrm{Area}_G(\Delta) = c_5 h\big(\|V_0\| + \|V_t\|\big)$$

since otherwise $\widetilde{\Delta} = \Delta$.

Δ is covered by L trapezia $\Delta_1, \ldots, \Delta_L$ with base xvx, where xv (or the inverse word) is a cyclic shift ot the standard base of $\mathbf{M}$. By Remark 5.8 all $\Delta_1, \ldots, \Delta_L$ are copies of each other. Let us apply Lemma 4.25 to any of them, say to Δ_1, whose top and bottom have labels W_0 and W_t. If we have Property (1) of that lemma, then the area of Δ_1 does not exceed $c_4 h(\|W_0\| + \|W_t\|)$ since every maximal θ-band of Δ_1 has at most $c_4(\|W_0\| + \|W_t\|)$ cells in this case. Hence area of Δ does not exceed

$$Lc_4 h\big(\|W_0\| + \|W_t\|\big) \leq 2c_4 h\big(\|V_0\| + \|V_t\|\big) < c_5 h\big(\|V_0\| + \|V_t\|\big) = \mathrm{Area}_G(\Delta),$$

i.e. $\widetilde{\Delta} = \Delta$ in this case too.

Hence one may assume that Property (2) of Lemma 4.25 holds for Δ_1. By that Lemma, items (b,d), the corresponding cyclic shifts W_0' and W_t' are accepted, and there is a factorization of $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3$ (we use the same letter for the computations corresponding to Δ, to Δ_1, and for the revolving computation with standard base), where

$$\mathcal{C}_1: W_0' \to \cdots \to W_{n_1}', \quad \mathcal{C}_2: W_{n_1}' \to \cdots \to W_{n_1+n_2}'$$

and $\qquad \mathcal{C}_3: W_{n_1+n_2}' \to \cdots \to W_{n_1+n_2+n_3}' \quad (n_1 + n_2 + n_3 = t),$

where

$$\max\left(\|W'_{n_1}\|, \|W'_{n_1+n_2}\|\right) \le \max\left(\|W_0\|, \|W_t\|\right)$$

and for each $\mathcal{C}_i$, either:

(d1) $\|W'_j\| \le c_4 \max\left(\|W'_0\|, \|W'_t\|\right)$, for every configuration W_j of $\mathcal{C}_i$; or

(d2) there are accepting computations for the first and the last configuration of $\mathcal{C}_i$ with histories H'_i and H''_i such that $\|H'_i\| + \|H''_i\| < n_i$.

So Δ is built of at most three trapezia, where $\Delta(i)$ ($i = 1, 2, 3$) corresponds to the computations $\mathcal{C}_i$. Since their tops and bottoms have lengths at most $\max(\|V\|_0, \|V_t\|)$, it suffices to estimate the area of $\widetilde{\Delta}(i)$ for $i = 1, 2, 3$. Again, we have $\widetilde{\Delta}(i) = \Delta(i)$ in the case (d1).

Assume that we have Property (d2) for $\mathcal{C}_i$. Denote by U_1 and U_2 the first and the last configurations of $\mathcal{C}_i$ with standard base. By (d2),

$$\|U_1\|, \|U_2\| \le \max\left(\|W_0\|, \|W_t\|\right).$$

By Property (d2), there is an accepting computations $\mathcal{D}$ of length $\le n_i$ for U_1, and we may assume that $\mathcal{D}$ is the shortest such computation. Then case (b) of Lemma 4.20 gives a contradiction of the form $\ell < \ell/100$ for the length ℓ of the computation $\mathcal{D}$. Hence we should have case (a), and so every configuration of $\mathcal{D}$ has length at most $c_4\|U_1\|$. If Γ is the trapezium corresponding to $\mathcal{D}$ with bottom (top) label U_1 (resp., W_M), then the lengths of its θ-bands are less than $2c_4\|U_1\|$ by Lemma 6.1 and therefore the area of Γ is less than $2n_i c_4\|U_1\| - 1$. Therefore L copies of Γ can be attached to an auxiliary hub so that one gets an auxiliary disk Π_1 of area

$$\le 2Ln_i c_4\|U_1\| \le 3c_4 n_i \|V(1)\|,$$

where $V(1)t$ is the label of the bottom of $\Delta(i)$ up to cyclic permutations. Thus, the word $V(1)$ is equal to the boundary label of Π_1. Similarly, one can construct a disk of area

$$\le 3c_4 n_i \|V(2)\|$$

for the top of $\Delta(2)$.

Denote by Δ_- the diagram Δ without maximal rim x-band. So Δ_- has the boundary $\mathbf{p}_1\mathbf{q}_1\mathbf{p}_2^{-1}\mathbf{q}_2^{-1}$, where $\mathrm{Lab}(\mathbf{p}_1)$ and $\mathrm{Lab}(\mathbf{p}_2)$ are the boundary labels of the disks Π_1 and Π_2 (up to cyclic permutations) and $\mathrm{Lab}(\mathbf{q})_1 \equiv \mathrm{Lab}(\mathbf{q})_2$ since two x-bands with the same history have the same boundary labels.

If we attach disks Π_1 and Π_2 along their boundaries to the top and the bottom of Δ_-, we obtain a diagram, whose boundary label is trivial in the free group. Hence there is a diagram E with two disks whose boundary label is equal to the boundary label of Δ_- and the area is less than

$$\le 3c_4 n_i (\|V(1)\| + \|V(2)\|).$$

If we attach one x-band of length n_i to E, we construct the required diagram $\widetilde{\Delta}(i)$ of area at most

$$\leq 3c_4 n_i \big(\|V(1)\| + \|V(2)\|\big) + n_i < c_5 n_i \big(\|V(1)\| + \|V(2)\|\big). \qquad \square$$

The proof of Lemma 10.1 shows that the area of a minimal diagram with some boundary label V can be much greater than the area of V, which is equal to the minimal number of cells in all diagrams with boundary label V over the presentation (5.1)–(5.2). So to obtain the lower bound for the Dehn function of G, we prove in the next lemma that these two areas "almost equal" for the words having no θ-letters.

Lemma 10.2. *Let $(tW)^L = 1$ in G, where the reduced word W has no θ-letters and no letters $t^{\pm 1}$. Then there exists a reduced diagram Δ over the presentation (5.1)–(5.2) such that it has exactly one hub, has boundary label $V \equiv (tW)^L$ and $\mathrm{Area}(\Delta) \leq 2\,\mathrm{Area}(V)$.*

Proof. Let Δ_0 be a diagram over the presentation (5.1)–(5.2) of G with boundary label $(tW)^L$, where

$$\mathrm{Area}(\Delta_0) = \mathrm{Area}(tW)^L.$$

We say that Γ is a disk subdiagram of Δ_0 if it has reduced boundary, has exactly one hub and every θ-cell of Γ (if any) belongs in a θ-annulus surrounding this hub. The diagram Δ_0 can be covered by a family of subdiagrams Γ, where each Γ is either a disk subdiagram or a θ-cell and different subdiagram of this covering $\mathbf{S}_0$ have no cells in common. Let $A(\mathbf{S}_0)$ be the sum of the areas of all disk subdiagram from $\mathbf{S}_0$ plus doubled number of the single θ-cells from $\mathbf{S}_0$. By $A(\Delta_0)$ we denote the minimum of the numbers $A(\mathbf{S}_0)$ over all such coverings $\mathbf{S}_0$. Clearly,

$$A(\Delta_0) \leq 2\,\mathrm{Area}(\Delta_0) = 2\,\mathrm{Area}(V),$$

and so it suffices to prove that a reduced diagram Δ with boundary label $(tW)^L$ and minimal possible value of $A(\Delta)$ has exactly one disk, because

$$\mathrm{Area}(\Delta) \leq A(\Delta) \leq A(\Delta_0).$$

Below we fix the covering $\mathbf{S}$ of Δ such that $A(\Delta) = A(\mathbf{S})$. Note that every θ-annulus of Δ surrounds at least one disk by Lemma 5.3 since the diagram Δ is reduced.

By induction on the number of θ-annuli in a disk subdiagram $\Gamma \in \mathbf{S}$, we see that the boundary label of Γ has the form $(tU)^L$. Therefore there is only one cyclic shift of the word $(tU)^L$ starting with tU. Note that there are no two distinct disk subdiagrams Γ_1 and Γ_2 in $\mathbf{S}$ whose boundaries share at least two t-edges, provided there are no other disk subdiagrams between Γ_1 and Γ_2, because such pair of subdiagrams could be canceled out, which would decrease the value of $A(\Delta)$.

Assume that $\mathbf{S}$ has at least one single θ-cell. Let $\mathcal{T}$ be the maximal θ-band of Δ containing this cell. It has to be a θ-annulus, where every cell is a member of $\mathbf{S}$

since $\mathcal{T}$ can end neither on $\partial\Delta$ nor on the boundary of a disk subdiagram from $\mathbf{S}$. So one can choose a minimal θ-annulus $\mathcal{T}$ whose cells do not belong to the disk subdiagrams from the family $\mathbf{S}$, and $\mathcal{T}$ surrounds a subdiagram E having no single θ-cells from $\mathbf{S}$.

The reduced diagram E must contain disk subdiagrams by Lemma 5.3. Hence as in Lemma 7.2, we have a disk graph, where there are no two different edges connecting neighbor disk subdiagrams in E (and crossing the t-edges on the boundaries of these subdiagrams) provided there are no other disk subdiagrams between these two edges of the disk graph. Hence there is a disk subdiagram Γ in E sharing a boundary subpath $\mathbf{q}$ with a side of $\mathcal{T}$, where $\mathrm{Lab}(\mathbf{q}) = (tU)^{L-4}t$. After the transposition of $\mathcal{T}$ and Γ we can obtain a new disk subdiagram Γ' with

$$\mathrm{Area}(\Gamma') \leq \mathrm{Area}(\Gamma) + L(u+1),$$

where u is the number of cells between two neighbor (θ, t)-cells in $\mathcal{T}$. However, the transposition removes $(L-4)u + L - 3$ cells from $\mathcal{T}$ and add at most $4u+3$ new cells. Since

$$(L-4)u + L - 3 - 4u - 3 > \frac{L}{2}(u+1),$$

we have a new diagram and new covering $\mathbf{S}'$ after the transposition, where $A(\mathbf{S}') < A(\mathbf{S})$, because the single θ-cell is taken with coefficient 2 in the above definition of $A(\cdot)$; a contradiction.

Thus, the covering $\mathbf{S}$ has no single (θ, q)-cells. Then the standard argument implies that $\mathbf{S}$ has at most one disk subdiagram (see Lemma 7.3). The diagram Δ cannot be a diagram over M since all t-letters occur in the boundary label $(tV)^L$ with exponent $+1$, and so the t-edges of $\partial\Delta$ cannot be connected in Δ by a t-band. Thus, the number of hubs of Δ is 1. $\qquad\square$

Lemma 10.3. *The Dehn function $d(n)$ of the group G is equivalent to $F(n)$.*

Proof. To obtain the upper bound for $d(n)$ (with respect to the finite presentation of G given in Section 5), it suffices, for every word W vanishing in G with $\|W\| \leq n$, to find a diagram over G of area $O(F(n))$ with boundary label W. Since $|W| \leq \|W\|$, van Kampen's lemma and Lemma 9.16 provide us with a minimal diagram Δ such that

$$\mathrm{Area}_G(\Delta) \leq N_4 F\big(n + \sigma_\lambda(\Delta)\big) + N_3 \mu(\Delta) g(n)$$

for some constants N_3 and N_4 depending on the presentation of G. By Lemmas 8.5, 6.7 (a), and the definition of $\mu(\Delta)$, the right-hand side does not exceed

$$N_4 F((1+c)n) + N_3 J n^2 g(n).$$

Since $F(O(n)) = O(F(n))$ and $n^2 g(n) = F(n)$, we conclude that

$$\mathrm{Area}_G(\Delta) \leq C_0 F(n)$$

for some constant C_0.

Recall that in the definition of G-area, the subdiagrams, which are big trapezia $\Gamma, \Gamma', \ldots$, can have common cells in their rim q-bands only. By Lemma 10.1, any big trapezia Γ from this list with top $\mathbf{p}_1$ and bottom $\mathbf{p}_2$ can be replaced by a diagram $\widetilde{\Gamma}$ with (combinatorial) area at most $2 \operatorname{Area}_G(\Gamma)$. When replacing all big trapezia $\Gamma, \Gamma', \ldots$ in this way, we should add q-bands for the possible intersection of big trapezia, but for every Γ of height h, we add at most $2h$ new cells. So the area of the modified diagram E is at most

$$3 \operatorname{Area}_G(\Delta) \leq 3C_0 F(n).$$

Hence the required diagram is found for given word W.

To obtain the lower bound for $d(n)$, we will use the series of $\mathbf{M}$-accepted words $V(n)$ of (combinatorial) length $\Theta(n)$ constructed in the proof of Lemma 4.23. Since $V(n)^L = 1$ in G, it will be suffice to bound from below the areas of the diagrams $\Delta(n)$ given by Lemma 10.2: $\Delta(n)$ has boundary label $V(n)^L$, exactly one disk and the area equal to $\operatorname{Area}(V(n)^L)$ up to a multiplicative factor from the segment $[1, 2]$.

A q-band starting on the hub π of $\Delta(n)$ cannot end on it since all occurrences of a particular q-letter in the hub relation have the same exponent. Hence the spokes of π end on LN q-edges of $\partial\Delta$. Hence Δ has L trapezia corresponding to an accepting reduced computation $\mathcal{C}$ for $V(n)$, and it suffices to get a lower bound for the area of one trapezium Γ.

By Remark 4.24, Γ has at least $\Theta(ng(n))$ maximal θ-bands of length at least $\Theta(n)$. Therefore the area of Γ is at least $\Theta(F(n))$. Since $\|\partial(\Delta)\| = \Theta(n)$, the Dehn function $d(n)$ is bounded from below by a function equivalent to $F(n)$, as required. $\square$

10.2. Supercubic Dehn functions.

Here we show that for the Dehn functions $F(n)$ obtained earlier, one can construct a finitely presented group with Dehn function $nF(n)$. This will complete the proof of Theorem 1.3. For this goal we modify the control S-machines used in the definition of the S-machine $\mathbf{M}$. The unnecessary extra-control will just slow down the work of $\mathbf{M}$. The construction resembles the one from Subsection 4.2 of [16]. We will modify only the S-machine $\mathbf{P}$ defined in Subsection 3.1. The copies of the auxiliary primitive S-machine $Z(A)$ will work between the applications of the (copies of) the rules of $\mathbf{P}$.

For every set of letters A, let A', A'', and A''' be disjoint copies of A, the maps $a \mapsto a', a \mapsto a''$ and $a \mapsto a'''$ identify A with A', A'' and A''', respectively. Let $\overleftarrow{Z}$ be the S-machines with tape alphabet $A' \sqcup A'' \sqcup A'''$, state alphabet $\{L\} \sqcup K \sqcup P \sqcup \{R\}$, with the following positive rules.

$$\chi_1(a) = \left[L \xrightarrow{\ell} L, k(1) \to k(1), p(1) \to (a'')^{-1} p(1) a''', R \to R\right], \quad a \in A;$$

$$\chi_2 := \left[L \xrightarrow{\ell} L, k(1) \xrightarrow{\ell} k(2), p(1) \to p(2), R \to R\right];$$

$$\chi_3(a) = \left[L \xrightarrow{\ell} L, k(2) \to k(2), p(2) \to a'' p(2)(a''')^{-1}, R \to R \right];$$

$$\chi_4 = \left[L \xrightarrow{\ell} L, k(2) \to k(3), p(2) \xrightarrow{\ell} p(3), R \to R \right].$$

The rules of $\overrightarrow{Z}$ are similar, but the moving base letter is K, while the sector PR is locked.

To define the composition $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ we insert the base of $\overleftarrow{Z}$ (and $\overrightarrow{Z}$) between every two consecutive state letters of $\mathbf{P}$. In this subsection, we assume that $\mathbf{P}$ has the standard base $Q_0 Q_1 \ldots Q_N$ (and forget more detailed earlier notation).

For every $i = 1, \ldots, N$, we make copies Y_i', Y_i'', and Y_i''' of the alphabet Y_i of $\mathbf{P}$ ($i = 1, \ldots, N$). Let Θ be the set of positive commands of $\mathbf{P}$. The set of state letters of $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ is

$$S_0 \sqcup K_1 \sqcup P_1 \sqcup S_1 \sqcup K_2 \sqcup P_2 \sqcup \cdots \sqcup P_N \cup S_N,$$

where

$$P_i = \left\{ p^{(i)}, p^{(i,1)}, p^{(i,0)}, p^{(\theta,i)}(1), p^{(\theta,i)}(2), p^{(\theta,i)}(3) \mid \theta \in \Theta \right\},$$

K_i is defined similarly for $i = 1, \ldots, N$, $S_i = Q_i \sqcup (Q_i \times \Theta)$. Thus the state letters L and R of the copies of the S-machines $\overrightarrow{Z}$ and $\overleftarrow{Z}$ are identified with the corresponding S-letters. We shall call the state letters from P_i-s and K_i-s the *p-letters*, and the other state letters (i.e. the copies of the state letters of $\mathbf{P}$), the *basic state letters*.

The set of tape letters of $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ is

$$Y = Y_1 \sqcup \cdots \sqcup Y_{3N} = Y_1' \sqcup Y_1'' \sqcup Y_1''' \sqcup Y_2' \sqcup Y_2'' \sqcup Y_2''' \sqcup \cdots \sqcup Y_N' \sqcup Y_N'' \sqcup Y_N'''.$$

Assume θ is a positive $\mathbf{P}$-rule of the form

$$\left[s_0 u_1 \to s_0' u_1', v_1 s_1 u_2 \to v_1' s_1' u_2', \ldots, v_N s_N \to v_N' s_N' \right],$$

where $s_i, s_i' \in Q_i$, and v_i-s, u_i-s are words in Y. Then this rule is replaced in $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ by positive

$$\bar{\theta} = \begin{bmatrix} s^{(\theta,0)} \xrightarrow{\ell} (s')^{(\theta,0)}, & k^{(\theta,1)}(3)u_1 \to k^{(\theta,1)}(1)u_1', & v_1 p^{(\theta,1)}(3) \xrightarrow{\ell} v_1' p^{(\theta,1)}(1), \\ s^{(\theta,1)} \xrightarrow{\ell} (s')^{(\theta,1)}, & k^{(\theta,2)}(3)u_2 \to k^{(\theta,2)}(1)u_2', & v_2 p^{(\theta,2)}(3) \xrightarrow{\ell} v_2' p^{(\theta,2)}(1), \ldots, \end{bmatrix}$$

with $Y_{3i-2}(\bar{\theta}) = \emptyset$, $Y_{3i-1}(\bar{\theta}) = Y_i''(\theta)$, and $Y_{3i}(\bar{\theta}) = Y_i'''(\theta)$.

Now we want to describe the alternating work of the auxiliary S-machines $\overleftarrow{Z}^{(\theta,i)}$ and $\overrightarrow{Z}^{(\theta,i)}$. Normally each of them is switched on exactly once in the frame of the rule θ, but the sequence of their turning on depends on θ. First, we need the following

transition rule $\chi_-(\theta)$. This rule adds θ to all state letters and turns all $k^{(j)}$ and $p^{(j)}$ into $k^{(\theta,j)}(1)$ and $p^{(\theta,j)}(1)$:

$$\left[s_i \xrightarrow{\ell} s^{(\theta,i)}, k^{(j)} \to k^{(\theta,j)}(1), p^{(j)} \xrightarrow{\ell} p^{(\theta,j)}(1), i = 0, \ldots, N, j = 1, \ldots, N \right].$$

Then the S-machines $\overrightarrow{Z}^{(\theta,1)}, \ldots, \overrightarrow{Z}^{(\theta,N)}$ and $\overleftarrow{Z}^{(\theta,1)}, \ldots, \overleftarrow{Z}^{(\theta,N)}$ are switched on in a specific order (defined below) after the rule $\chi_-(\theta)$ is applied. So the state letters $k^{(\theta,j)}(1)$, $p^{(\theta,j)}(1)$ ($j = 1, \ldots, N$) successively turn into $k^{(\theta,j)}(3)$, $p^{(\theta,j)}(3)$, find themselves just after s_{i-1}- and before s_i-letters, respectively, and the rule $\bar{\theta}$ can be applicable.

After an application of $\bar{\theta}$, the S-machines

$$\overrightarrow{Z}^{(\theta,1)}, \ldots, \overrightarrow{Z}^{(\theta,N)} \quad \text{and} \quad \overleftarrow{Z}^{(\theta,1)}, \ldots, \overleftarrow{Z}^{(\theta,N)}$$

are switched on again in the following order.

Assume that the rule θ is a rule of a primitive S-machine $\mathcal{P}$. (Recall that **P** is composed from primitive machines.) The S-machine $\mathcal{P}$ can work in several sectors. (For example, θ can be the control rule checking all the big historical sectors simultaneously.) Let i be the minimal index such that S_i has a control running state letter of $\mathcal{P}$. Then the rule $\bar{\theta}$ first switches on the S-machines $\overleftarrow{Z}^{(\theta,j)}$ for $j = i$ and simultaneuosly for all other j-s, where S_j is also has a control running state letter of $\mathcal{P}$. The last rule χ_4 of this S-machine switches on the S-machine $\overleftarrow{Z}^{(\theta,i-1)}$ and similar S-machines in similar sectors (e.g. in all small historical sectors if the sectors $S_{i-2}S_{i-1}$ is a small historical sector). The next S-machine is $\overleftarrow{Z}^{(\theta,i-2)}$, if it did not work ealier, and so on. Then the S-machines $\overrightarrow{Z}^{(\theta,i+1)}, \ldots$ subsequently work, except for the sectors, where the auxiliary Z-machines worked earlier.

The same S-machines work after the application of the rule $\chi_-(\theta)$ but they are switched on in the inverse order.

Finally, the transition rule $\chi_+(\theta)$ removes the index θ from all state letters, and turns all $p^{(\theta,j)}(3)$ into $p^{(j)}$:

$$\left[s^{(\theta,i)} \xrightarrow{\ell} s_i, k^{(\theta,j)}(3) \to k^{(j)}, p^{(\theta,j)}(3) \to p^{(j)}, i = 0, \ldots, N, j = 1, \ldots, N \right].$$

For every admissible word W of **P** with standard base, let $\iota(W)$ be the admissible word of **P** $\circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ obtained by inserting state letters $k^{(i)}$ and $p^{(i)}$ next to the right of each s_{i-1} and next to the left of each s_i, $i \leq N$.

Assume that $W \to W \cdot \theta$ is a computation of the S-machine **P** with standard base and a positive rule θ. Then, by the definition of **P** $\circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$, we have the canonically defined reduced computation

$$\cdots \to \iota(W) \to \iota(W) \cdot \bar{\theta} \to \cdots$$

starting and ending with words whose state letters have no θ-indices and all other words do have θ-indices. The computation of $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ with these properties is unique by Lemma 3.1 (3) since the S-machines $\overrightarrow{Z}$ and $\overleftarrow{Z}$ are primitive. Thus the following claim is true.

Lemma 10.4 (similar to [16, Lemma 4.24]). *For every computation $W \rightarrow W \cdot \theta$ of the S-machine $\mathbf{P}$ with standard base and a positive rule θ, there is a unique reduced $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ -computation*

$$\cdots \rightarrow \iota(W) \rightarrow \iota(W) \cdot \overline{\theta} \rightarrow \cdots$$

starting and ending with words whose state letters have no θ-indices and all other words have θ-indices. The history of this computation starts with $\chi_-(\theta)$ and ends with $\chi_+(\theta)$.

For every admissible word W of $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ with the standard base, let $\pi(W)$ be the word obtained by removing state k- and p-letters, θ-indices of state letters, and the indices that distinguish a-letters from the left and from the right of k- and p-letters. After possible cancellations of a-letters, we obtain an admissible word of $\mathbf{P}$. Note that we have

Given a computation $\mathcal{C}$ of the S-machine $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ with standard base and history H involving a $\overline{\theta}$-rule , we define the *projection* $\pi(\mathcal{C})$ of it, which is a computation of $\mathbf{P}$. To obtain it, one removes all transitions given by χ-rules and replaces the configurations by their projections. Note that this operation makes sense since χ-rules do not change the projection of the word onto a word in Y_i. The projection $\pi(H)$ of the history is defined in obvious way: one forgets the χ-rules and removes bars over θ-rules.

Lemma 10.5 (see [16, Lemma 4.28]). *If $\mathcal{C}$ is a reduced computation of $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ with standard base, then $\pi(\mathcal{C})$ is a reduced computation of $\mathbf{P}$.*

Proof. Assume that we have a subword $\overline{\theta} H' \overline{\theta}^{-1}$ in the history H of $\mathcal{C}$, where H' has no rules of the form $\overline{\theta}$. If H' is non-empty, then one obtains a contradiction by Lemma 3.1 (4) applied to the work of the primitive S-machines $\overrightarrow{Z}^{(\theta,i)}$ and $\overleftarrow{Z}^{(\theta,i)}$. Hence $\pi(H)$ is a reduced history. $\square$

Below we change some formulations of Subsection 3.1 as applied to $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$.

Lemma 10.6 (duplicate of Lemma 3.1). *Let $\mathcal{C} \colon C_0 \rightarrow \cdots \rightarrow C_t$ be a reduced computation of the S-machine $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ with the standard base and with $t \geq 1$. Then the following properties hold:*

(1) *If $|C_i|_a > |C_{i-1}|_a$ for some $i = 1, \ldots, t-1$, then*

$$|C_i|_a \leq |C_{i+1}|_a \leq |C_{i+2}|_a \leq \cdots.$$

(2) $|C_i|_a \leq \max(|C_0|_a, |C_t|_a)$ *for every* $i = 0, 1, \ldots t$.

(3) *Assume that the words C_0 and C_t have a-letters only from the subalphabets Y_i'' ($i = 1, \ldots, N$) and that for the primitive S-machines $\mathcal{P}$ forming $\mathbf{P}$, all their subwords in $\pi(C_0)$ and $\pi(C_t)$ look like in Lemma 3.1 (3), i.e. as $q^1 u p^1 q^2$ and $q^1 v p^2 q^2$ for some words u, v in the notation of Lemma 3.1. Then a-words in the corresponding sectors of C_0 and C_t are equal, $|C_i|_a = |C_0|_a$ for every $i = 0, \ldots, t$ and $t = \Theta(s^2)$, where $s = |C_0|_a$.*

(4) *Assume that the words C_0 and C_t have a-letters only from the subalphabets Y_i'' ($i = 1, \ldots, N$) and that for the primitive S-machines $\mathcal{P}$ forming $\mathbf{P}$, all their subwords in $\pi(C_0)$ and $\pi(C_t)$ look like in Lemma 3.1 (4). Then it is not possible that the configurations C_0 and C_t have the same set of state letters.*

(5) *If C_0 (or C_t) satisfies the assumptions of item (3), then*

$$|C_i|_a \geq |C_0|_a$$

(respectively, $|C_i|_a \geq |C_t|_a$) for every $i = 0, \ldots, t$.

Proof. Let us start with Property (1). If

$$\mathcal{C}_i = \mathcal{C}_{i-1} \chi_1^{(\theta,i)}(a)^{\pm 1}$$

(or $\mathcal{C}_i = \mathcal{C}_{i-1} \chi_2^{(\theta,i)}(a)^{\pm 1}$), then $p(1)^{(\theta,i)}$ inserts letters from both sides and the next rule of the computation must be again $(\chi_1^{(\theta,i)})^{\pm 1}(c)$ for some c. It again must increase the length of the configuration by two, and so on.

If $\mathcal{C}_i = \mathcal{C}_{i-1}\bar{\theta}$ for some $\bar{\theta}$-rule, then the transition $\pi(C_{i-1}) \to \pi(\mathcal{C}_i)$ increases the length by Lemmas 10.5 and 3.1 (1). The work of $\overrightarrow{Z}^{(\theta,i)}, \overleftarrow{Z}^{(\theta,i)}$ cannot decrease configuration length by Lemma 3.1 (5) for these primitive S-machines. Therefore Statement (1) is true and Statement (2) is also true since one can choose a shortest C_j and consider the subcomputation

$$C_j \to \cdots \to C_t$$

and inverse subcomputation

$$C_j \to \cdots \to C_0.$$

To prove equalities $|C_i|_a = |C_0|_a$ in Statement (3), one just apply Lemma 3.1 (3) first, to $\pi(\mathcal{C})$ and second, to the maximal subcomputations of Z-machines. Since the length of $\pi(\mathcal{C})$ is $2k + 1$ by that Lemma, and if $k > 0$ the maximal subcomputations of Z-machines work with configurations of lengths $k - 1, k - 2, \ldots, 1$, we obtain the later claim of (3) by summation.

Property (4) follows from Lemma 3.1 (4) applied to the computation $\pi(\mathcal{C})$ and the maximal subcomputations of Z-machines. Property (5) follows from the projection argument as in Lemma 3.1 (5). □

Lemma 10.7 (duplicate of Lemma 3.3). *If $C_0 \to \cdots \to C_t$ is a reduced computation of $\mathbf{P}$ with base $S_{i-1}K_i K_i^{-1} S_{i-1}^{-1}$ or $S_i^{-1} P_i^{-1} P_i S_i$ and C_0 has a-letters from the alphabet Y_i'' only. Then,*

$$|C_i|_a \geq |C_0|_a$$

for every $i = 0, \ldots, t$.

Proof. The statement follows from the projection argument 3.2 as in Lemma 3.3. $\square$

Let us call the constructed S-machine $\mathbf{P} \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ *biprimitive*.

To define the modified S-machine $\mathbf{M}_4'$ we insert two more base letters in each pair $\mathcal{R}_{i-1}$ and $\mathcal{P}_i$ of the standard base, i.e. this base has the subwords $\mathcal{Q}_{i-1}\mathcal{R}_{i-1}\mathcal{R}_{i-1}'\mathcal{P}_i'\mathcal{P}_i\mathcal{Q}_i$; now the first big historical sector is $\mathcal{R}_0'\mathcal{P}_1'$ (instead of $\mathcal{R}_0\mathcal{P}_1$). In the definition of Step 1^-, we now replace the rules of the primitive control S-machines with the rules of the corresponding biprimitive machines. By definition, at all other steps the control S-machines are just primitive, i.e. the sectors $\mathcal{R}_{i-1}\mathcal{R}_{i-1}'$ and $\mathcal{P}_i'\mathcal{P}_i$ are locked, there are no χ-rules, and two base letters of the locked sectors work as one letter of a primitive S-machine.

Thus, only at Step 1^-, the biprimitive S-machine $\mathbf{P}'$ works.

Lemma 10.8 (duplicate of Lemma 3.6). *Let $\mathcal{C}: W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{P}'$ with standard base. Then:*

(a) *$|W_j|_a \leq \max(|W_0|_a, |W_t|_a)$ for every configuration of $\mathcal{C}$; moreover, $|W_0|_a \leq \cdots \leq |W_t|_a$ if every control P- and R-letter neighbors some Q-letter and P'-letter (R'-letter) neighbors a P-letter (resp., an R-letter) in the word W_0;*

(b) *we have $t = O(\|W_0\|^2 + \|W_t\|^2)$, moreover, $t = O(\|W_t\|^2)$ if every p-letters and control letters has a neighbor in the word W_0 as in item (a).*

Proof. (a) The first property is given by Lemma 10.6 (2). Under the additional assumption for control letters and p-letters, W_0 is the shortest configuration by the projection argument.

(b) Let us say that a subcomputation without $\overline{\theta}$-rules in the history is a Z-subcomputation. If the computation is a Z-subcomputation, then the statement (b) follows from Lemma 3.6 (b) for the (composition of) primitive S-machines. Otherwise we chose a $\overline{\theta}$-transition $W_r \to W_{r+1}$ with minimal

$$\min\left(\|W_r\|, \|W_{r+1}\|\right).$$

Without loss of generality, one may assume that this minimum is $\|W_r\|$. Lemma 3.6 (a) for $\pi(\mathcal{C})$ and for Z-subcomputations implies that $\|W\|_r \leq \|W_i\|$ for every i, and therefore it suffices to bound the histories of length-non-decreasing computations

$$\mathcal{C}': W_r \to \cdots \to W_t \quad \text{and} \quad \mathcal{C}'': W_r \to \cdots \to W_0.$$

The length of $\pi(\mathcal{C}')$ is at most $2\|W_t\|$ by Lemma 3.6 (b) for the S-machine **P**. By the same lemma for maximal Z-subcomputations, the length of every maximal Z-subcomputation is at most $2\|W_t\|$ too. Since the number of such maximal subcomputations is at most $2\|W_t\|$, we have that $t - r = O(\|W_t\|)^2$. Similarly, we obtain $r = O(\|W_0\|)^2$, and the the first estimate of (b) is obtained. For the second estimate, one can choose $r = 0$, since the whole computation $\mathcal{C}$ is length-non decreasing by Lemma 10.6 (5), (1). □

We see that Lemmas 10.6 (3) and 10.8 (b) provide us with quadratic estimate of the computation time for the biprimitive S-machine against the linear time for S-primitive machines. This have a few consequences mentioned below.

For the same function $f(x)$ recognized by the original Turing machine and $F(x) = x^2 g(x)$, we define the function $g(x)$ to be equivalent to $x f(x)^3$ now.

The extra sectors of the biprimitive S-machine do not affect the work of all other steps except for 1^- since they are locked therein.

The formulation of Lemma 4.5(1) is unchanged but the proof is now based on Lemma 10.6 (4). For the step history $(21^-)(1^-)(1^-1)$ in the formulation of Lemma 4.5(2), we have now

$$\|H\| = O\big(\|W_0\|^2\big),$$

which follows from Lemma 10.8 (b). The estimates of $\|H\|$ for other step histories mentioned in Lemma 4.5(1) remain unchanged. Also we add an item to the formulation of Lemma 4.5:

(3) *Let the history of a reduced computation*

$$\mathcal{C}: W_0 \to \cdots \to W_t$$

with standard base have a subword $\bar{\theta}_1 H \bar{\theta}_2$, where $\bar{\theta}_1$ and $\bar{\theta}_2$ are $\bar{\theta}$-rules of the S-machine **P'** *and H has only χ-rules of* **P'**. *Then all configurations of $\mathcal{C}$ are uniquely determined by H,*

$$|W_1|_a = \cdots = |W_{t-1}|_a \quad and \quad \|W_j\| = \Theta\big(\|H\|\big)$$

for $j \leq t$.

The proof of this statement follows the proof of Lemma 4.5 (2), but now one refers to Lemma 10.4 instead of Lemma 3.1 (3).

To the assumption of Lemma 4.10, we add:

"or the history of the computation $\mathcal{C}$ has a subword $\theta_1 H \theta_2$, where θ_1 and θ_2 are $\bar{\theta}$-rules of the S-machine **P'** *and H has only χ-rules of* **P'**".

In the proof of Lemma 4.10 (2), one should use the following property. The computation of the biprimitive S-machine does not change the length of configurations by

Lemma 10.8 (a). Hence the reduced computation of it is canonical by Lemma 10.4, and the history restores the tape words.

Consider the single Step 1^- in item 2 of the proof of Lemma 4.13. If we have the work of primitive S-machines only, then the proof is unchanged. If there is a $\bar{\theta}$-rule of the biprimitive S-machine in the computation $\mathcal{C}$, we condider the projection $\pi(\mathcal{C})$, where the a-lengths of all configurations are at most

$$C\big(|W_0|_a + |W_t|_a\big)$$

by Lemma 4.13. It remains to consider maximal subcomputation $W_r \to \cdots \to W_s$ of Z-machines, where

$$|W_r|_a, |W_s|_a \leq C\big(|W_0|_a + |W_t|\big)_a.$$

By Lemma 3.6 (a) we have

$$|W_i|_a \leq \max\big(|W_r|_a, |W_s|_a\big) \leq C\big(|W_0|_a + |W_t|_a\big)$$

for $r \leq i \leq s$, as required.

In item 3 of the proof of Lemma 4.13, the base letters of the first big historical sector should be replaced by their dashed duplicates.

The estimate $\|H\| \leq c_2\|W_0\|$ of Lemma 4.16 changes now by the quadratic estimate $\|H\| \leq c_2\|W_0\|^2$ due to the application of Lemma 10.8 (b). Respectively, the upper bound

$$c_3(k^3 + 1)\big(\|W_0\| + k^3\big)$$

of Lemma 4.17 is now replaced by

$$c_3(k^3 + 1)\big(\|W_0\|^2 + k^3\big).$$

The formulation of Lemma 4.20 does not change since comparing the lengths of histories, we now increase all of them. In particular, we have now in equation (4.3) that

$$2w^2 + 2(w - 1)^2 + \cdots + 2(w - (m-1))^2 \geq |m|w^2/2,$$

we have that the difference of lengths of subcomputations $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ does not exceed $10w$, we have $\|H_0\| = O(w^2)$ and obtain

$$\|H'H''\| \leq c_2(k^3 + 1)O(w^2),$$

which leads to the same estimates for $\|H_0\|$ and $\|H_t\|$ since the constant c_4 can be chosen large enough.

The change of linear estimates by quadratic ones in the proof of Lemma 4.21 just sharpens the required inequalities.

Lemma 4.23 claims now that the generalized time function $T'(n)$ of the S-machine M' is equivalent to $n^2 f^3(n)$. Thus, we multiply the generalized time function

of $\mathbf{M}$ by n. This is sufficient for the upper bound. Indeed the replacements of the form $\|W_j\| \rightarrow \|W_j\|^2$ in the proof of the modified Lemma 4.23 can multiply the generalized time function at most by n since it is shown there that $\|W_j\| = O(n)$ for every configuration W_j. The lower bound obtained in the original Lemma 4.23 must also be multiplied by n now. Indeed, the time of the constructed subcomputations with step history

$$((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1}$$

will be at least $\Theta(n^2)$ now (instead of $\Theta(n)$) since by Lemma 10.6 (3), we have such lower bound for the Step 1^-.

Similar replacement should be made in Remark 4.24: There are $\Theta(n^2 f(n)^3)$ configurations of length at least $\Theta(n)$ for any computation accepting the word $V(n)$.

The statement of Lemma 4.26 is modified now by adding the words:

"*or the history of $\mathcal{C}$ has a subword $\bar{\theta}_1 H \bar{\theta}_2$, where $\bar{\theta}_1$ and $\bar{\theta}_2$ are $\bar{\theta}$-rules of the S-machine $\mathbf{P}'$ and H has only χ-rules of $\mathbf{P}'$*".

If the step history of $\mathcal{C}$ is (F) and Property (a) of Lemma 4.20 holds we have the same proof as in the original Lemma 4.26 since the length of a subcomputation of Step 1^- is at most

$$5 \max \left(\|W_0\|_a, \|W_t\|_a \right)$$

provided the history of $\mathcal{D}$ has no subwords $\bar{\theta}_1 H \bar{\theta}_2$. Indeed, in this case, the history of any subcomputation of Step 1^- is a subword of $H'\bar{\theta}H''$, where H' and H'' have only χ-rules of $\mathbf{P}'$, and one can bound each of $\|H'\|$ and $\|H''\|$ by $2c_4(\|W_0\|_a, \|W_t\|_a)$ applying Lemma 3.6 (b).

The formulation of Lemma 4.27 remains unchanged for all steps except for Step 1^-. If the step is 1^-, we add the assumption that *the history has no $\bar{\theta}$-rules of the S-machine $\mathbf{P}'$*. Since the biprimitive S-machine $\mathbf{P}'$ works as a primitive one in the later case, the proof does not change.

The set of *standard trapezia* is enlarged now, namely, a trapezium with standard base having a subword $\bar{\theta}_1 H \bar{\theta}_2$ in the history, where $\bar{\theta}_1$ and $\bar{\theta}_2$ are $\bar{\theta}$-rules of the S-machine $\mathbf{P}'$ and H has only χ-rules of $\mathbf{P}'$, is also standard by definition.

The above modification of Lemma 4.27 changes the proof of Lemma 9.23 as follows. One obtains a contradiction if for some k, there is one-step subcomputation

$$W(k) \rightarrow \cdots \rightarrow W(k+4),$$

where the step differs from 1^- *or the step is 1^- and the history of this subcomputation has no $\bar{\theta}$-rules of the S-machine $\mathbf{P}'$*. Hence either every subcomputation

$$W(k) \rightarrow \cdots \rightarrow W(k+8)$$

has at least two steps, and so the computation

$$W(1) \rightarrow \cdots \rightarrow W(999)$$

has at least 100 steps, which leads to a contradiction as in the original proof of Lemma 9.23, or for some k, one obtains at least two $\bar{\theta}$-rules in the history of

$$W(k) \rightarrow \cdots \rightarrow W(k+8),$$

i.e. the history of this subcomputation has a subword $\bar{\theta}_1 H \bar{\theta}_2$, where $\bar{\theta}_1$ and $\bar{\theta}_2$ are $\bar{\theta}$-rules of the S-machine $\mathbf{P}'$ and H has only χ-rules of $\mathbf{P}'$. In the later case one should use the modified Lemma 4.10 (instead of the original Lemma 4.10) to complete the proof.

Lemma 9.23 proves Theorem 1.3 for $s = 2$. In the present subsection, we have modified the main S-machine and the corresponding groups M and G so that the modified Lemma 9.23 provides us with the statement of Theorem 1.3 for $s = 3$. One can make further modifications, which similarly slow down the work of the previously modified S-machines. This will give the proofs of Theorem 1.3 for $s = 4, 5, \ldots$. However we can leave the details to the reader taken into account that for $s \geq 4$, Corollary 1.4 is obtained in [22]. Thus, the proof of Theorem 1.3 is complete.

Subject index

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