Defining relations for quantum symmetric pair coideals of Kac–Moody type

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Abstract. Classical symmetric pairs consist of a symmetrizable Kac–Moody algebra $\mathfrak{g}$, together with its subalgebra of fixed points under an involutive automorphism of the second kind. Quantum group analogs of this construction, known as quantum symmetric pairs, replace the fixed point Lie subalgebras by one-sided coideal subalgebras of the quantized enveloping algebra $U_q(\mathfrak{g})$. We provide a complete presentation by generators and relations for these quantum symmetric pair coideal subalgebras. These relations are of inhomogeneous q-Serre type and are valid without restrictions on the generalized Cartan matrix. We draw special attention to the split case, where the quantum symmetric pair coideal subalgebras are generalized q-Onsager algebras.

Contents

1. Introduction

A classical symmetric pair consists of a Lie algebra $\mathfrak{g}$ together with its subalgebra $\mathfrak{k}$ of fixed points under a Lie algebra involution θ. Quantum analogs of this construction, known as quantum symmetric pairs, have emerged in the beginning of the 1990s. They replace $\mathfrak{g}$ by its quantized universal enveloping algebra $U_q(\mathfrak{g})$ and $\mathfrak{k}$ by a one-sided coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$ of $U_q(\mathfrak{g})$, which is called a quantum symmetric pair (QSP) coideal subalgebra. The algebras $B_{\mathbf{c},\mathbf{s}}$ were first constructed by Noumi, Sugitani and Dijkhuizen [35–37] using methods of quantum integrability. A different approach, based on the Drinfeld–Jimbo presentation of $U_q(\mathfrak{g})$, was pursued by Letzter. She developed a comprehensive theory of quantum symmetric pairs for semisimple Lie algebras $\mathfrak{g}$ in an elaborate series of papers [29–31]. This has allowed to identify the zonal spherical functions on quantum symmetric spaces as Macdonald–Koornwinder polynomials [32]. This whole theory was later extended to symmetrizable Kac–Moody algebras $\mathfrak{g}$ by Kolb in [27], which treats the structure theory of the Kac–Moody QSP coideal subalgebras $B_{\mathbf{c},\mathbf{s}}$ in great detail.

   Over the years, it has become increasingly apparent that quantum symmetric pairs play a crucial role in quantum integrability, notably of the reflection equation [12,41]. The latter replaces the quantum Yang–Baxter equation when reflecting boundary conditions are imposed. Such boundaries break the quantum symmetry down to a coideal subalgebra of the quantum algebra which encodes the symmetries in the bulk of a quantum spin chain [18]. A universal solution for the reflection equation arises from the QSP coideal subalgebras $B_{\mathbf{c},\mathbf{s}}$ by means of a universal K-matrix, the analog of the universal R-matrix for $U_q(\mathfrak{g})$. The concept of a universal K-matrix has been introduced in [4] for $\mathfrak{g}$ of type A_{2r+1} and was extended to general Kac–Moody algebras $\mathfrak{g}$ in [3]. Both references aim to develop the theory of universal K-matrices in parallel to Lusztig's theory of universal R-matrices. More precisely, it was observed in [4] and [20] for specific types of QSP that $B_{\mathbf{c},\mathbf{s}}$ allows an intrinsic bar involution. The existence of such a bar involution in greater generality was established in [2]. Moreover, the universal K-matrix can be factorized in terms of a quasi K-matrix, which intertwines between the intrinsic bar involution and Lusztig's bar involution on $U_q(\mathfrak{g})$ [3,4].

   In this paper we will adopt the notational conventions of [27]. We will write $\mathfrak{g} = \mathfrak{g}(A)$ for the Kac–Moody algebra associated to a symmetrizable generalized Cartan matrix A of dimension n. We take I to be the set $\{0, \ldots, n-1\}$, such that we can write $A = (a_{ij})_{i,j \in I}$. We will use Kolb's definition of admissible pairs (X, τ), as will be repeated later in Definition 2.1, to parametrize the involutive automorphisms of $\mathfrak{g}$ of the second kind. To each such admissible pair one can associate a quantum symmetric pair and hence a coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$ of $U_q(\mathfrak{g})$, which also depends on a multiparameter $(\mathbf{c}, \mathbf{s})$.

   The QSP coideal subalgebras $B_{\mathbf{c},\mathbf{s}}$ can be presented in terms of generators and relations. The set of generators depends on the choice of admissible pair, but always

contains certain elements B_i, with $i \in I$. A set of defining relations which describe these algebras abstractly in terms of their generators was given in [27, Theorem 7.1] and will be repeated in the upcoming Theorem C. One of these relations states that

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_{q_i} B_i^{1-a_{ij}-m} B_j B_i^m \tag{1}$$

can be written as a lower-degree polynomial in B_i and B_j which depends on the entry a_{ij} of A. However, Kolb's theorem does not provide a precise form for this polynomial, which he denotes by $C_{ij}(\mathbf{c})$. In the quasi-split case, corresponding to admissible pairs with $X = \emptyset$, expressions for $C_{ij}(\mathbf{c})$ were obtained by Chen, Lu and Wang in [11] using q-binomial identities. Apart from this special case, expressions for $C_{ij}(\mathbf{c})$ were only known for a few possible values of a_{ij}, namely $a_{ij} \in \{0, -1, -2, -3\}$. These have been obtained in [27] and [2] by explicit calculations, which follow similar results in [31] for finite-dimensional $\mathfrak{g}$. It was suggested by Kolb that the same rationale could lead to expressions for $C_{ij}(\mathbf{c})$ valid for all a_{ij}, but this has not been made explicit before. This paper provides for the first time closed expressions for the polynomials $C_{ij}(\mathbf{c})$ valid beyond the quasi-split case, without restrictions on the Cartan matrix or the admissible pair. It thereby completes the presentation of the quantum symmetric pair coideal subalgebras by generators and relations.

Such a presentation is highly desirable in view of the representation theory of the algebras $B_{\mathbf{c},\mathbf{s}}$. This was already indicated in [2], where the definition of a new bar involution for quantum symmetric pairs was validated by showing that it respects the defining relations of $B_{\mathbf{c},\mathbf{s}}$. By the absence of such relations beyond the case $|a_{ij}| \le 3$, this could only be done for a limited class of Cartan matrices and admissible pairs. Our results allow to remove part of these restrictions, as we show in Corollary 4.8.

Our approach will be as follows. We will rewrite (1) as a complicated expression in $U_q(\mathfrak{g})^{\otimes 2}$, where one of the tensor components is acted upon with a projection operator. This leads to the upcoming expressions (39) and (42), which were essentially already contained in [27]. The main novelty of our approach lies in how we expand these expressions further. We will distinguish two cases, based on the behavior of i and j with respect to the admissible pair, each leading to a different expression for $C_{ij}(\mathbf{c})$. In Propositions 3.3 and 3.7 we will perform a binary distributive expansion to rewrite (1) as a polynomial which, in the first of these two cases, is of the form

$$\sum_{m,m'} \rho_{m,m'} \mathcal{Z}_i^{(1-a_{ij}-m-m')/2} B_i^m B_j B_i^{m'}, \tag{2}$$

whereas in the second case one finds

$$\sum_{m,m',t} \rho_{m,m',t} \mathcal{Z}_i^t B_i^m B_j B_i^{m'} \mathcal{Z}_i^{((1-a_{ij}-m-m')/2)-t}$$
$$+ \sum_{m,t} \sigma_{m,t} \mathcal{Z}_i^t \mathcal{W}_{ij} K_j \mathcal{Z}_i^{((-1-a_{ij}-m)/2)-t} B_i^m, \tag{3}$$

where the elements $\mathcal{Z}_i$, $\mathcal{W}_{ij}$ and K_j are well-defined in terms of the generators and where all sums are finite. The major difficulty lies in the determination of the coefficients $\rho_{m,m'}$, $\rho_{m,m',t}$, and $\sigma_{m,t}$, which we will refer to as the structure constants of the algebra $B_{\mathbf{c},\mathbf{s}}$. Initially, we will describe these in terms of monomials in $U_q(\mathfrak{g})$ acted upon with a projection operator. Closed expressions for the actions of these projection operators and hence for the structure constants are consequently derived in Theorems 3.13 and 3.19. It may not surprise that the formulae we obtain there turn out to be rather computationally extensive. Indeed, even the expressions obtained in [27] and [2] for small values of a_{ij}, as displayed in the upcoming Tables 1, 2 and 3, were already quite intricate. Nevertheless, our formulae contain nothing but finite sums and products, which can easily be carried out either by hand or by a computer.

In this paper, we will draw special attention to the QSP coideal subalgebras in the split case, corresponding to the trivial admissible pair $(\emptyset, \mathrm{id})$. These are known as generalized q-Onsager algebras. Their name has been derived from the algebra defined by Onsager in [38] as a tool towards his analytic solution of the planar Ising model in zero magnetic field. This algebra was presented in [19] and [39] as the infinite-dimensional Lie algebra with generators B_0 and B_1 subject to the Dolan–Grady relations

$$[B_0, [B_0, [B_0, B_1]]] = -4[B_0, B_1],$$
$$[B_1, [B_1, [B_1, B_0]]] = -4[B_1, B_0].$$

It has received much attention in special function theory and integrable lattice models [14, 22, 26]. It can be embedded in the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ as its subalgebra of fixed points under the Chevalley involution [40], and hence together with $\widehat{\mathfrak{sl}_2}$ it forms a (split) classical symmetric pair. The theory of quantum symmetric pairs thus offers a solid framework to deform the Onsager algebra to a quantum algebra. The resulting q-Onsager algebra [5, 7] is abstractly defined by the q-Dolan–Grady relations

$$[B_0, [B_0, [B_0, B_1]_q]_{q^{-1}}] = \rho[B_0, B_1],$$
$$[B_1, [B_1, [B_1, B_0]_q]_{q^{-1}}] = \rho[B_1, B_0], \tag{4}$$

where $[A, B]_q = qAB - q^{-1}BA$ is the q-commutator and ρ is a scalar depending on q. The q-Onsager algebra has become an important object of study in quantum integrability [5, 10, 28] and in connection with q-orthogonal polynomials [9] and Leonard pairs [24]. Upon adding a defining relation in its equitable presentation, the q-Onsager algebra is refined to the Askey–Wilson algebra [46], as was shown in [43]. A central extension of the latter, known as the universal Askey–Wilson algebra, was also identified as a quotient of the q-Onsager algebra [44]. This Askey–Wilson algebra provides an algebraic framework for the q-Askey scheme of orthogonal polynomials [8], see also [15, 16, 21] for some recent multivariate generalizations.

The left-hand side of (4) can be rewritten as

$$B_i^3 B_j - [3]_q B_i^2 B_j B_i + [3]_q B_i B_j B_i^2 - B_j B_i^3$$

for $i \neq j \in \{0, 1\}$. This coincides with the expression (1) for $n = 2$ and $a_{01} = a_{10} = -2$, i.e. for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. It is hence apparent that the q-Onsager algebra coincides with the quantum symmetric pair coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$ of $U_q(\widehat{\mathfrak{sl}_2})$ for the trivial admissible pair and a special choice of the parameters $\mathbf{c}, \mathbf{s}$.

Kac–Moody generalizations of the q-Onsager algebra were constructed by Baseilhac and Belliard in [6]. A presentation with generators and relations was given for affine Lie algebras $\mathfrak{g}$, again for a limited set of Cartan matrices. The relations we will derive in Theorem 3.13 extend this to symmetrizable Kac–Moody algebras without restrictions on the Cartan matrix. Moreover, we will use a recent result by Chen, Lu and Wang [11] to obtain alternative, transparent expressions of quantum Serre type for the defining relations of $B_{\mathbf{c},\mathbf{s}}$ in the quasi-split case, including the generalized q-Onsager algebras. More precisely, in Theorem 4.7 we will expand the ι-divided powers in [11] as polynomials in the generators B_i in order to find easier expressions for the structure constants $\rho_{m,m'}$ from (2). The advantage of expanding these ι-divided powers is two-fold. On the one hand, it follows from comparison with the earlier described projection technique that the relations arising from [11] hold even beyond the quasi-split case, provided the indices i and j satisfy the conditions of the aforementioned Case 1. On the other hand, the obtained expressions allow us to drop a crucial assumption in the development of the bar involution for QSP and the universal K-matrix in [2, 3], see Corollary 4.8. In addition, they make it possible to prove symmetry properties of the coefficients $\rho_{m,m'}$.

For $q = 1$, such inhomogeneous Serre relations for generalized Onsager algebras had already been obtained by Stokman in [42]. His classical generalized Onsager algebras extend those of [13, 45] to arbitrary root systems. The defining relations he provides, involve a set of coefficients which are defined in a recursive fashion. Our approach now allows to derive closed expressions for these coefficients and thus solve the recursion relations, by taking the limit $q \to 1$ of the analogous expressions in the quantum case. This will be performed in Theorem 4.15.

The paper is organized as follows. In Section 2 we recall the necessary prerequisites on quantum symmetric Kac–Moody pairs in the notation of [27]. We treat the classical symmetric pairs $(\mathfrak{g}, \boldsymbol{b})$ in Subsection 2.1 and their quantum analogs $(U_q(\mathfrak{g}), B_{\mathbf{c},\mathbf{s}})$ in Subsection 2.2. In Subsection 2.3 we state some of the results obtained in [27] based on the projection technique of [31], which we will need in what follows. The main body of work is contained in Section 3, where the missing defining relations for $B_{\mathbf{c},\mathbf{s}}$ will be derived. In Subsection 3.1 we will perform a binary distributive expansion to reduce the computation of the polynomials $C_{ij}(\mathbf{c})$ to an easier problem, namely determining the coefficients in (2) and (3) through the action of the counit and a certain projection operator on monomials in $U_q(\mathfrak{g})$. This problem will be solved in Subsections 3.2 and 3.3 treating Cases 1 and 2, respectively.

The principal results are presented in Theorems 3.13 and 3.19. To conclude, we will derive alternative and more accessible expressions for the polynomials $C_{ij}(\mathbf{c})$ in Case 1 based on the work [11] in Theorem 4.7. Finally, we turn our attention to the generalized q-Onsager algebras and their classical counterparts. We repeat the obtained relations applied to the split case and reconsider them in the limit $q \to 1$ to solve the recursion relations of [42] in Theorem 4.15.

2. Construction of the generators

Let us start by recalling some crucial concepts and notations introduced in [27].

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Let A be an indecomposable generalized Cartan matrix of dimension n and let us denote by I the set $\{0, 1, \ldots, n-1\}$. This means that $A = (a_{ij})_{i,j\in I}$ satisfies the properties:

(i) $a_{ii} = 2$, for all $i \in I$;

(ii) $a_{ij} \in \mathbb{Z}^-$, if $i \neq j \in I$;

(iii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, for any $i, j \in I$;

(iv) For every non-empty proper subset $I' \subset I$ there exist $i \in I'$, $j \in I \setminus I'$ such that $a_{ij} \neq 0$.

Moreover, we assume A to be symmetrizable, i.e. there exists a diagonal matrix $D = \mathrm{diag}(\epsilon_i : i \in I)$, with mutually coprime and nonzero entries $\epsilon_i \in \mathbb{N}$, such that DA is symmetric.

In Subsection 2.1, we will construct the classical symmetric pair $(\mathfrak{g}, \boldsymbol{b})$, where $\mathfrak{g} = \mathfrak{g}(A)$ is the Kac–Moody algebra associated to A. This construction will motivate the definition of the quantum symmetric pair $(U_q(\mathfrak{g}), B_{\mathbf{c},\mathbf{s}})$ inside the corresponding quantum group $U_q(\mathfrak{g})$, which will be given in Subsection 2.2.

2.1. The classical case. Let $\left(\mathfrak{h} = \mathfrak{h}(A), \Pi = \{\alpha_i : i \in I\}, \Pi^\vee = \{h_i : i \in I\}\right)$ be a minimal realization of A. This means that $\mathfrak{h}$ is a $\mathbb{K}$-vector space of dimension $2n - \mathrm{rank}(A)$ and that $\Pi^\vee$ and Π are linearly independent subsets of $\mathfrak{h}$ and its dual $\mathfrak{h}^*$, respectively, subject to $\alpha_j(h_i) = a_{ij}$ for any $i, j \in I$. Let $Q = \mathbb{Z}\Pi$ be the corresponding root lattice.

The Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated to A is the Lie algebra over $\mathbb{K}$ generated by $\mathfrak{h}$ and $2n$ Chevalley generators e_i, f_i with $i \in I$, with defining relations

$$[h, h'] = 0, \quad [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i, \quad [e_i, f_j] = \delta_{ij}h_i, \quad (5)$$

$$\left(\mathrm{ad}\, e_i\right)^{1-a_{ij}} e_j = \left(\mathrm{ad}\, f_i\right)^{1-a_{ij}} f_j = 0, \quad (6)$$

for all $i, j \in I$ and $h, h' \in \mathfrak{h}$. Here we denoted by ad the adjoint mapping

$$\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}_\mathfrak{g} \colon x \mapsto \mathrm{ad}\, x, \qquad \mathrm{ad}\, x \colon \mathfrak{g} \to \mathfrak{g} \colon y \mapsto [x, y]. \quad (7)$$

The derived Lie subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is generated by $\mathfrak{h}' = \sum_{i \in I} \mathbb{K}h_i$ and the elements e_i, f_i with $i \in I$.

As usual, we will write

$$\mathfrak{g}_\beta = \{x \in \mathfrak{g} : [h, x] = \beta(h)x, \forall h \in \mathfrak{h}\}$$

for any $\beta \in \mathfrak{h}^*$ and

$$\Phi = \{\beta \in \mathfrak{h}^* : \mathfrak{g}_\beta \neq \{0\}\}$$

for the corresponding root system.

For any $i \in I$ we denote by $r_i \in \mathrm{GL}(\mathfrak{h})$ the fundamental reflection which acts on $h \in \mathfrak{h}$ by

$$r_i(h) = h - \alpha_i(h)h_i.$$

The subgroup W of $\mathrm{GL}(\mathfrak{h})$ generated by all such r_i stands as the Weyl group of $\mathfrak{g}$. Via duality, W can also act on $\mathfrak{h}^*$ and hence in particular on Q, via

$$r_i(\alpha) = \alpha - \alpha(h_i)\alpha_i, \tag{8}$$

for any $\alpha \in \mathfrak{h}^*$.

Consider a subset $X \subseteq I$. Let $\mathfrak{g}_X$ be the corresponding Lie subalgebra of $\mathfrak{g}$, generated by the elements e_i, f_i and h_i with $i \in X$. Write $\Phi_X \subseteq \Phi$ for its root system and $\rho_X^\vee$ for half the sum of the positive coroots of Φ_X. We will write W_X for the parabolic subgroup of the Weyl group W associated to X, and w_X for its longest element. Finally, let us denote by $\mathrm{Aut}(A, X)$ the set of permutations σ of I subject to

$$\sigma(X) = X \quad \text{and} \quad a_{\sigma(i),\sigma(j)} = a_{ij}, \quad \forall i, j \in I.$$

Any $\sigma \in \mathrm{Aut}(A, X)$ extends to an automorphism of $\mathfrak{g}$ by taking

$$\sigma(e_i) = e_{\sigma(i)}, \quad \sigma(f_i) = f_{\sigma(i)}, \quad \sigma(h_i) = h_{\sigma(i)} \tag{9}$$

and defining the action of σ on $h \in \mathfrak{h} \setminus \mathfrak{h}'$ as described in [25, Section 4.19]. Similarly, $\sigma \in \mathrm{Aut}(A, X)$ extends to an automorphism of Q upon setting

$$\sigma(\alpha_i) = \alpha_{\sigma(i)}. \tag{10}$$

This terminology allows to repeat the definition of an admissible pair, as given in [27, Definition 2.3].

Definition 2.1. An admissible pair (X, τ) consists of a subset $X \subseteq I$ and an automorphism $\tau \in \mathrm{Aut}(A, X)$ subject to the following conditions:

(1) τ is an involution, i.e. $\tau^2 = \mathrm{id}$.

(2) The action of τ on X coincides with the corresponding action of $-w_X$, i.e. for any $j \in X$ one has $h_{\tau(j)} = -w_X(h_j)$ and $\alpha_{\tau(j)} = -w_X(\alpha_j)$, where we have used the interpretations of τ and w_X according to (8)–(10).

(3) For any $i \in I \setminus X$ satisfying $\tau(i) = i$, one has $\alpha_i(\rho_X^\vee) \in \mathbb{Z}$.

An important motivation for introducing admissible pairs is that they arise naturally as Kac–Moody generalizations of Satake diagrams [1]. Moreover, they parametrize the so-called involutive automorphisms of $\mathfrak{g}$ of the second kind [25, 33] up to conjugation by elements of $\mathrm{Aut}(\mathfrak{g})$, as was shown in [27, Theorem 2.7]. The automorphism $\theta(X, \tau)$ corresponding to an admissible pair (X, τ) can be constructed using the following four key concepts.

The first is the element $\tau \in \mathrm{Aut}(A, X)$, interpreted as an automorphism of $\mathfrak{g}$ according to (9).

Furthermore, we will need the Chevalley involution $\omega \in \mathrm{Aut}(\mathfrak{g})$ given by

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h) = -h, \tag{11}$$

for any $i \in I$ and $h \in \mathfrak{h}$.

Moreover, the longest element w_X of W_X can be lifted to an element m_X of the Kac–Moody group of $\mathfrak{g}'$, with corresponding automorphism $\mathrm{Ad}(m_X) \in \mathrm{Aut}(\mathfrak{g})$. For details we refer to [25, Section 1.3] and [27, Section 2].

Finally, one can define a group morphism $s(X, \tau) \colon Q \to \mathbb{K}^\times$ from the root lattice Q to the multiplicative group $\mathbb{K}^\times$, by

$$s(X, \tau)(\alpha_j) = \begin{cases} 1 & \text{if } j \in X \text{ or } \tau(j) = j, \\ i^{\alpha_j(2\rho_X^\vee)} & \text{if } j \in I \setminus X \text{ and } \tau(j) > j, \\ (-i)^{\alpha_j(2\rho_X^\vee)} & \text{if } j \in I \setminus X \text{ and } \tau(j) < j, \end{cases} \tag{12}$$

where $i \in \mathbb{K}$ is a square root of -1. The corresponding automorphism $\mathrm{Ad}(s(X, \tau)) \in \mathrm{Aut}(\mathfrak{g})$ is defined by

$$\mathrm{Ad}(s(X, \tau))(h) = h, \quad \mathrm{Ad}(s(X, \tau))(v) = s(X, \tau)(\alpha)v, \tag{13}$$

for all $h \in \mathfrak{h}$ and $v \in \mathfrak{g}_\alpha, \alpha \in \Phi$.

These four ingredients can now be combined to yield the following involutive automorphism $\theta(X, \tau)$.

Definition 2.2. To each admissible pair (X, τ) we associate the automorphism $\theta(X, \tau)$ of $\mathfrak{g}$ given by

$$\theta(X, \tau) = \mathrm{Ad}(s(X, \tau)) \circ \mathrm{Ad}(m_X) \circ \tau \circ \omega. \tag{14}$$

It is an involutive $\mathfrak{g}$-automorphism of the second kind by [27, Theorem 2.5].

Let us from now on fix an admissible pair (X, τ) and write θ for the above defined automorphism $\theta(X, \tau)$. Then θ gives rise to an algebra which will be of special interest in this paper.

Definition 2.3. We denote by $\boldsymbol{b} = \boldsymbol{b}(X, \tau)$ the subalgebra of $U(\mathfrak{g}')$ generated by the elements:

$$\begin{aligned} &f_i + \theta(f_i) && \text{with } i \in I \setminus X, \\ &e_i, f_i, h_i && \text{with } i \in X, \\ &h_i && \text{with } \theta(h_i) = h_i, i \in I. \end{aligned} \tag{15}$$

The couple $(\mathfrak{g}, \boldsymbol{b})$ stands as the (classical) symmetric pair associated to the admissible pair (X, τ).

In [27], the algebra $\boldsymbol{b}$ was denoted by $U(\mathfrak{k}')$. We have chosen to adopt this alternative notation, to emphasize that $\boldsymbol{b}$ is a classical counterpart of the quantum algebra $B_{\mathbf{c},\mathbf{s}}$, which we will define in Subsection 2.2. The defining relations of $\boldsymbol{b}$ will then follow as a limit $q \to 1$ of the quantum Serre relations for $B_{\mathbf{c},\mathbf{s}}$, which we will derive in Section 3.

2.2. The quantum case. Let q be an indeterminate, assumed not to be a root of unity in the field $\mathbb{K}$. We denote by $\mathbb{K}(q)$ the field of rational functions in q.

Recall the matrix $D = \mathrm{diag}(\epsilon_i : i \in I)$ we have introduced above. For each $i \in I$ we set $q_i = q^{\epsilon_i}$. For any $m \in \mathbb{N}$, we define the q_i-number $[m]_{q_i}$ and the q_i-factorial $[m]_{q_i}!$ as

$$[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, \quad [m]_{q_i}! = \prod_{\ell=1}^{m} [\ell]_{q_i},$$

with the convention that $[0]_{q_i}! = 1$. For $N, m \in \mathbb{N}$ with $N \geq m$, we define the q_i-binomial coefficient as

$$\begin{bmatrix} N \\ m \end{bmatrix}_{q_i} = \frac{[N]_{q_i}!}{[m]_{q_i}![N-m]_{q_i}!}.$$

Similarly to the classical case, one has

$$\begin{bmatrix} N \\ m \end{bmatrix}_{q_i} = \begin{bmatrix} N \\ N-m \end{bmatrix}_{q_i}. \tag{16}$$

We will often use the following polynomial in two non-commutative variables x and y, which we will refer to as the quantum Serre polynomial: for $i, j \in I$ we write

$$F_{ij}(x, y) = \sum_{\ell=0}^{1-a_{ij}} (-1)^{\ell} \begin{bmatrix} 1 - a_{ij} \\ \ell \end{bmatrix}_{q_i} x^{1-a_{ij}-\ell} y x^{\ell}. \tag{17}$$

Throughout the paper, we will perform calculations in the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$. In fact, it will suffice to work with its Hopf subalgebra $U_q(\mathfrak{g}')$, the associative $\mathbb{K}(q)$-algebra generated by $4n$ elements E_i, F_i, K_i, and K_i^{-1} with $i \in I$, subject to the relations

$$\begin{aligned}
K_i^{\pm 1} K_i^{\mp 1} &= 1, \quad [K_i^{\pm 1}, K_j^{\pm 1}] = 0, \\
K_i E_j &= q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \\
K_i^{-1} E_j &= q_i^{-a_{ij}} E_j K_i^{-1}, \quad K_i^{-1} F_j = q_i^{a_{ij}} F_j K_i^{-1},
\end{aligned} \tag{18}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{19}$$

$$F_{ij}(E_i, E_j) = F_{ij}(F_i, F_j) = 0, \tag{20}$$

for all $i, j \in I$. The relations (20) are referred to as the quantum Serre relations.

Remark 1. The quantum group $U_q(\mathfrak{g}')$ can be considered a q-deformation of $\mathfrak{g}'$, upon viewing e_i and f_i as the limits of E_i and F_i respectively as q goes to 1, and identifying K_i with $q_i^{h_i}$. To view the quantum Serre relations (20) as q-deformations of the relations (6), it will be useful to introduce the q-commutators

$$\mathrm{ad}_{q_i^m} : U_q(\mathfrak{g}) \to \mathrm{Aut}(U_q(\mathfrak{g})) : x \mapsto \mathrm{ad}_{q_i^m}(x),$$
$$\mathrm{ad}_{q_i^m}(x) : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) : y \mapsto [x, y]_{q_i^m} = q_i^m xy - q_i^{-m} yx,$$

with $m \in \mathbb{Q}$. Notice that $\mathrm{ad}_{q_i^m}$ reduces to ad defined in (7) in the limit $q \to 1$, for any $m \in \mathbb{Q}$. It can easily be shown by induction that one has

$$\left(\overrightarrow{\prod_{m=(1-r)/2}^{(r-1)/2}} \mathrm{ad}_{q_i^m}(A) \right)(B) = \sum_{k=0}^{r} (-1)^k \begin{bmatrix} r \\ k \end{bmatrix}_{q_i} A^{r-k} B A^k \tag{21}$$

for any $A, B \in U_q(\mathfrak{g})$ and any $r \in \mathbb{N}$, which, upon substituting $r = 1 - a_{ij}$, becomes

$$\left(\overrightarrow{\prod_{m=a_{ij}/2}^{-a_{ij}/2}} \mathrm{ad}_{q_i^m}(A) \right)(B) = F_{ij}(A, B).$$

Hence, in the limit $q \to 1$, the expression $F_{ij}(A, B)$ reduces to

$$(\mathrm{ad}\, a)^{1-a_{ij}}(b),$$

where a and b are the specializations of A and B respectively, and so (20) indeed translates to (6). A detailed account on this notion of specialization, which is a formal way to implement this limiting process $q \to 1$, can be found in [17], [23, Sections 3.3 and 3.4] and [27, Section 10].

The quantum group $U_q(\mathfrak{g}')$ has the structure of a Hopf algebra, with the following expressions for the coproduct Δ, the counit ϵ and the antipode S:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1},$$
$$\epsilon(E_i) = 0, \qquad \epsilon(F_i) = 0, \qquad \epsilon(K_i^{\pm 1}) = 1,$$
$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}. \tag{22}$$

Now let us once more fix an admissible pair (X, τ). A quantum analog of the automorphism $\theta(X, \tau)$ defined in (14) can be built from five fundamental constituents, one of which is the mapping $\tau \in \operatorname{Aut}(A, X)$ viewed as an automorphism of $U_q(\mathfrak{g}')$ by

$$\tau(E_i) = E_{\tau(i)}, \quad \tau(F_i) = F_{\tau(i)}, \quad \tau(K_i^{\pm 1}) = K_{\tau(i)}^{\pm 1}.$$

Secondly, one can extend $\operatorname{Ad}(s(X, \tau)) \in \operatorname{Aut}(\mathfrak{g})$ to an automorphism of $U_q(\mathfrak{g})$ by

$$\operatorname{Ad}(s(X, \tau))(v) = s(X, \tau)(\alpha)v,$$

for all $v \in U_q(\mathfrak{g})_\alpha = \{u \in U_q(\mathfrak{g}) : K_i u = q^{(\alpha_i, \alpha)} u K_i, \forall i \in I\}, \alpha \in Q$. Here, we denote by $(\cdot, \cdot)$ the bilinear form on $\mathfrak{h}^*$ satisfying $(\alpha_i, \alpha_j) = \epsilon_i a_{ij}$.

Next, we will need a q-deformation of the Chevalley involution (11), which we will again denote by ω. It is given by

$$\omega(E_i) = -F_i, \quad \omega(F_i) = -E_i, \quad \omega(K_i) = K_i^{-1}$$

and classifies as a coalgebra antiautomorphism of $U_q(\mathfrak{g}')$.

To obtain a quantum analog of the element $\operatorname{Ad}(m_X)$ in (14) one needs the Lusztig automorphisms $T_i, i \in I$, which appeared in [34, Section 37.1] under the name $T''_{i,1}$. Let $w_X = r_{i_1} r_{i_2} \ldots r_{i_k}$ be a reduced expression for the longest element w_X of the parabolic subgroup W_X of W, then denote by T_{w_X} the corresponding automorphism $T_{w_X} = T_{i_1} T_{i_2} \ldots T_{i_k}$ of $U_q(\mathfrak{g}')$.

Finally, define another automorphism $\psi \colon U_q(\mathfrak{g}') \to U_q(\mathfrak{g}')$ by

$$\psi(E_i) = E_i K_i, \quad \psi(F_i) = K_i^{-1} F_i, \quad \psi(K_i) = K_i.$$

These are all the tools needed to q-deform $\theta(X, \tau)$.

Definition 2.4. To each admissible pair (X, τ) we associate the automorphism $\theta_q(X, \tau)$ of $U_q(\mathfrak{g}')$ given by

$$\theta_q(X, \tau) = \operatorname{Ad}(s(X, \tau)) \circ T_{w_X} \circ \psi \circ \tau \circ \omega. \tag{23}$$

Note that $\theta_q = \theta_q(X, \tau)$ is no longer involutive.

Finally, let us denote by Q^Θ the set $\{\alpha \in Q : -w_X \tau(\alpha) = \alpha\}$. Here, we interpret both $\tau \in \operatorname{Aut}(A, X)$ and $w_X \in W_X$ as automorphisms of Q according to (8) and (10). Moreover, if $\beta = \sum_{i \in I} m_i \alpha_i \in Q$, we will write K_β for $\prod_{i \in I} K_i^{m_i}$. This brings us to the definition of the quantum analog $B_{\mathbf{c}, \mathbf{s}}$ of the algebra $\boldsymbol{b}$ defined in (15).

Definition 2.5. For any vector

$$\mathbf{c} = (c_i)_{i \in I \setminus X} \in \left(\mathbb{K}(q)^\times \right)^{I \setminus X} \quad \text{and} \quad \mathbf{s} = (s_i)_{i \in I \setminus X} \in \mathbb{K}(q)^{I \setminus X},$$

we define $B_{\mathbf{c}, \mathbf{s}} = B_{\mathbf{c}, \mathbf{s}}(X, \tau)$ to be the subalgebra of $U_q(\mathfrak{g}')$ generated by the elements:

$$
\begin{aligned}
&B_i = F_i + c_i \theta_q(F_i K_i) K_i^{-1} + s_i K_i^{-1} && \text{with } i \in I \setminus X, \\
&E_i, F_i, K_i^{\pm 1} && \text{with } i \in X, \\
&K_\beta && \text{with } \beta \in Q^\Theta.
\end{aligned}
\tag{24}
$$

When applying the coproduct Δ described in (22) on the generators (24), one can make the following observation.

Proposition A ([27, Proposition 5.2]). *For any* $(\mathbf{c}, \mathbf{s}) \in (\mathbb{K}(q)^\times)^{I \setminus X} \times \mathbb{K}(q)^{I \setminus X}$, *the algebra* $B_{\mathbf{c},\mathbf{s}}$ *is a right coideal subalgebra of* $U_q(\mathfrak{g}')$, *i.e.* $\Delta(B_{\mathbf{c},\mathbf{s}}) \subset B_{\mathbf{c},\mathbf{s}} \otimes U_q(\mathfrak{g}')$.

Upon comparing (24) with (15) in the light of Remark 1, it is immediately clear that $B_{\mathbf{c},\mathbf{s}}$ is a q-deformation of the algebra $\boldsymbol{b}$ under certain conditions on the parameters c_i and s_i, and that it reduces to the latter under the specialization $q \to 1$. The precise conditions are described in the following theorem.

Theorem B ([27, Theorems 10.8, 10.11]). *Let* $\mathbf{c} = (c_i)_{i \in I \setminus X}$ *be a vector of parameters taking values in*

$$\mathcal{C} = \big\{ \mathbf{c} \in (\mathbb{K}(q)^\times)^{I \setminus X} : c_i = c_{\tau(i)} \text{ if } \tau(i) \neq i \text{ and } (\alpha_i, -w_X \tau(\alpha_i)) = 0 \big\}, \quad (25)$$

where τ *and* w_X *are again interpreted as automorphisms of* Q. *Let* $\mathbf{s} = (s_i)_{i \in I \setminus X}$ *be a vector of parameters with values in*

$$\mathcal{S} = \big\{ \mathbf{s} \in \mathbb{K}(q)^{I \setminus X} : s_i \neq 0 \Rightarrow \big(i \in I_{ns} \text{ and } a_{ji} \in -2\mathbb{N} \setminus \{0\}, \forall j \in I_{ns} \setminus \{i\} \big) \big\}, \quad (26)$$

where

$$I_{ns} = \big\{ i \in I \setminus X : \tau(i) = i \text{ and } a_{ij} = 0, \forall j \in X \big\}.$$

Moreover, let us assume that the vector $(\mathbf{c}, \mathbf{s})$ *is specializable, i.e.* $\lim_{q \to 1}(c_i) = 1$ *for any* $i \in I$ *and all* c_i, s_i *lie in the localization* $\mathbb{K}[q]_{(q-1)}$ *of the polynomial ring* $\mathbb{K}[q]$ *with respect to the ideal generated by* $q - 1$. *Then* $B_{\mathbf{c},\mathbf{s}}$ *reduces to the algebra* $\boldsymbol{b}$ *under the formal specialization* $q \to 1$ *and is maximal with this property.*

Although the assumption of specializability is required to obtain $\boldsymbol{b}$ as an exact limit of $B_{\mathbf{c},\mathbf{s}}$ for $q \to 1$, it is still commonly accepted to view $B_{\mathbf{c},\mathbf{s}}$ as a quantum analog of $\boldsymbol{b}$ even if the latter condition is not fulfilled. Hence Proposition A suggests the following terminology.

Definition 2.6. For $(\mathbf{c}, \mathbf{s}) \in \mathcal{C} \times \mathcal{S}$, the algebra $B_{\mathbf{c},\mathbf{s}}$ is called a quantum symmetric pair coideal subalgebra.

Throughout the rest of this paper, we will fix a vector of parameters $(\mathbf{c}, \mathbf{s}) \in \mathcal{C} \times \mathcal{S}$ and work with the corresponding quantum symmetric pair coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$.

2.3. The Letzter–Kolb projection technique. In this section, we repeat some of the results obtained by Kolb in [27], based on the techniques established by Letzter in [31, Section 7]. We will use these results in Section 3 to derive the defining relations of the quantum symmetric pair coideal subalgebras $B_{\mathbf{c},\mathbf{s}}$. For ease of notation, we will write $\mathfrak{M}_X^+$ and $U_\Theta^{0'}$ for the subalgebras of $U_q(\mathfrak{g}')$ generated by the sets

$$\{E_i : i \in X\} \quad \text{and} \quad \{K_\beta : \beta \in Q^\Theta\},$$

respectively, and set $B_j := F_j$ for $j \in X$. Let U^+, U^-, and $U^{0'}$ be the subalgebras of $U_q(\mathfrak{g}')$ generated by

$$\{E_i : i \in I\}, \quad \{F_i : i \in I\}, \quad \text{and} \quad \{K_i^{\pm 1} : i \in I\},$$

respectively. It was explained in [27, Section 5] that

$$B_{\mathbf{c,s}} \cap U^{0'} = U_\Theta^{0'}$$

for $(\mathbf{c}, \mathbf{s}) \in \mathcal{C} \times \mathcal{S}$. Hence, one can describe $B_{\mathbf{c,s}}$ as the subalgebra of $U_q(\mathfrak{g}')$ generated by

$$\{B_i : i \in I\} \cup \mathfrak{M}_X^+ \cup U_\Theta^{0'}.$$

Furthermore, for any $J \in I^m$, $m \in \mathbb{N}$, we will write B_J for the product

$$B_{j_1} B_{j_2} \dots B_{j_m} = \overrightarrow{\prod_{k=1}^{m}} B_{j_k}.$$

Let us also denote by $\mathfrak{J}_{i,j}$ the set of multi-indices given by

$$\mathfrak{J}_{i,j} = \{(\underbrace{i, i, \dots, i}_{s \text{ times}}) : s \leq 1 - a_{ij}\} \cup \{(\underbrace{i, \dots, i}_{\ell \text{ times}}, j, \underbrace{i, \dots, i}_{s-\ell \text{ times}}) : s \leq -a_{ij}, \ \ell \leq s\}.$$

With this notation, one can write down the following theorem.

Theorem C ([27, Theorem 7.1]). *For any distinct $i, j \in I$ there exist elements*

$$C_{ij}(\mathbf{c}) \in \sum_{J \in \mathfrak{J}_{i,j}} \mathfrak{M}_X^+ U_\Theta^{0'} B_J$$

depending on the parameter vector $\mathbf{c}$, *such that* $F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c})$, *or equivalently:* $F_{ij}(B_i, B_j)$ *can be expressed as a polynomial in* B_i *and* B_j *of smaller total degree with coefficients in* $\mathfrak{M}_X^+ U_\Theta^{0'}$, *possibly depending on* $\mathbf{c}$ *but not on* $\mathbf{s}$. *Moreover, the algebra* $B_{\mathbf{c,s}}$ *is abstractly defined by the relations*

$$F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c}) \qquad \text{for } i \neq j \in I, \tag{27}$$

$$[E_i, B_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \text{for } i \in X, j \in I, \tag{28}$$

$$K_\beta B_i = q^{-(\beta, \alpha_i)} B_i K_\beta \quad \text{for } \beta \in Q^\Theta, i \in I, \tag{29}$$

together with the relations

$$K_\beta K_{\beta'} = K_{\beta'} K_\beta \qquad \text{for } \beta, \beta' \in Q^\Theta,$$
$$F_{ij}(E_i, E_j) = 0 \qquad \text{for } i, j \in X,$$
$$K_\beta E_i = q^{(\beta, \alpha_i)} E_i K_\beta \quad \text{for } i \in X \text{ and } \beta \in Q^\Theta,$$

describing $\mathfrak{M}_X^+$ *and* $U_\Theta^{0'}$ *that follow from* (18) *and* (20).

Our main goal in this paper will be to find explicit expressions for these lower degree polynomials $C_{ij}(\mathbf{c})$, which, up to present, had not been written down in general. A few special cases had, however, already been treated by Kolb.

Theorem D ([27, Eq. (5.20), Theorem 7.3]). *For any $i, j \in I$ such that either $i \in X$ or $\tau(i) \notin \{i, j\}$, one has $F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c}) = 0$.*

Another case was treated by Balagović and Kolb in [2]. It requires us to introduce some more notation. We will denote by ad the left adjoint action of $U_q(\mathfrak{g})$ on itself: for every $x, u \in U_q(\mathfrak{g})$ one has

$$\mathrm{ad}(x)(u) = \sum x_{(1)} u S(x_{(2)}),$$

where we have used the Sweedler notation, i.e. $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. It is not to be confused with the adjoint map of the Kac–Moody algebra $\mathfrak{g}$, which we have introduced in (7) under the same notation. Recall also the notation T_{w_X} for the product of Lusztig automorphisms corresponding to a reduced expression of w_X.

Lemma E ([27, Eq. (4.4), Theorem 4.4]). *For any $i \in I \setminus X$ there exists a monomial*

$$Z_i^+ = E_{j_1} E_{j_2} \dots E_{j_r} \in \mathfrak{M}_X^+, \tag{30}$$

with $j_1, \dots, j_r \in X$, such that

$$T_{w_X}(E_i) = a_i \mathrm{ad}(Z_i^+)(E_i),$$

for some $a_i \in \mathbb{K}(q)$. Moreover, one has

$$\theta_q(F_i K_i) = -v_i \mathrm{ad}(Z_{\tau(i)}^+)(E_{\tau(i)}),$$

for some $v_i \in \mathbb{K}(q)^\times$.

For any $i \in I \setminus X$ we may now define

$$\mathcal{Z}_i = -v_i \mathrm{ad}(Z_{\tau(i)}^+)(K_{\tau(i)}^2) K_{\tau(i)}^{-1} K_i^{-1}, \tag{31}$$

where $Z_{\tau(i)}^+$ and v_i are as defined in Lemma E. It follows immediately from (30) and the expression (22) for $\Delta(E_j)$ that $\mathcal{Z}_i$ is a $\mathbb{K}(q)$-linear combination of elements of $\mathfrak{M}_X^+$, multiplied by $K_{\tau(i)} K_i^{-1}$. For any $i \in I \setminus X$ we have

$$K_{\tau(i)} K_i^{-1} \in B_{\mathbf{c},\mathbf{s}} \cap U^{0'} = U_\Theta^{0'}$$

by the requirement (2) in Definition 2.1, and hence, $\mathcal{Z}_i$ lies in $\mathfrak{M}_X^+ U_\Theta^{0'}$.

Furthermore, we will use the notation

$$(x; x)_m = \prod_{k=1}^m (1 - x^k).$$

This enables us to state the following theorem by Balagović and Kolb.

Theorem F ([2, Theorem 3.6]). *For any $i \in I \setminus X$ satisfying $\tau(i) = j \neq i$ one has*

$$C_{ij}(\mathbf{c}) = -\frac{1}{(q_i - q_i^{-1})^2}$$
$$\left(q_i^{a_{ij}-1}(q_i^2; q_i^2)_{1-a_{ij}} c_i B_i^{-a_{ij}} \mathcal{Z}_i + q_i (q_i^{-2}; q_i^{-2})_{1-a_{ij}} c_j B_i^{-a_{ij}} \mathcal{Z}_j \right).$$

By Theorems D and F, it only remains to compute $C_{ij}(\mathbf{c})$ in 2 cases, namely:

Case 1. $i \in I \setminus X$, $j \in I \setminus X$ and $\tau(i) = i$;

Case 2. $i \in I \setminus X$, $j \in X$ and $\tau(i) = i$.

These cases turn out to be remarkably complicated. In [27] and [2], explicit calculations have led to expressions for $C_{ij}(\mathbf{c})$ for $a_{ij} \in \{0, -1, -2, -3\}$ in Case 1 and for $a_{ij} \in \{0, -1, -2\}$ in Case 2. However, except for the case $X = \emptyset$, no attempt has been made to write down relations valid without restrictions on a_{ij}. In Section 3, we will derive such relations for both cases. As could be expected from the above mentioned calculations, these expressions will be rather intricate, but nevertheless easily computable, as they involve only finite sums and products in $\mathbb{K}(q)$.

The key tool to obtain such relations is the projection $P_{-\lambda_{ij}}$ introduced in [31]. The classical triangular decomposition for quantum groups can be deformed to

$$U_q(\mathfrak{g}') \cong U^+ \otimes U^{0'} \otimes S(U^-), \tag{32}$$

where the isomorphism is given by multiplication, and consequently

$$U_q(\mathfrak{g}') = \bigoplus_{\beta \in Q} U^+ K_\beta S(U^-). \tag{33}$$

Let

$$P_{-\lambda_{ij}} : U_q(\mathfrak{g}') \to U^+ K_{-\lambda_{ij}} S(U^-) \tag{34}$$

denote the corresponding projection with respect to the decomposition (33), where

$$\lambda_{ij} = (1 - a_{ij})\alpha_i + \alpha_j \in Q. \tag{35}$$

Then one can prove the following statements.

Lemma G ([27, Eq. (5.14)]). *$P_{-\lambda_{ij}}$ is a homomorphism of left $U_q(\mathfrak{g}')$-comodules:*

$$(\Delta \circ P_{-\lambda_{ij}})(v) = (\mathrm{id} \otimes P_{-\lambda_{ij}})\Delta(v),$$

for any $v \in U_q(\mathfrak{g}')$.

Proposition H ([27, Proposition 5.16]). *For any distinct $i, j \in I$ one has*

$$P_{-\lambda_{ij}}\big(F_{ij}(B_i, B_j)\big) = 0.$$

Combining Lemma G, Proposition H, and the fact that Δ is an algebra morphism, we find that

$$(\mathrm{id} \otimes P_{-\lambda_{ij}})\big(F_{ij}(\Delta(B_i), \Delta(B_j))\big) = (\mathrm{id} \otimes P_{-\lambda_{ij}})\Delta(F_{ij}(B_i, B_j))$$
$$= (\Delta \circ P_{-\lambda_{ij}})(F_{ij}(B_i, B_j)) = 0. \qquad (36)$$

Since $K_{-\lambda_{ij}}$ is invariant under $P_{-\lambda_{ij}}$ and sent to 1 by the counit ϵ, the expression (36) asserts

$$F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (37)$$
$$= (\mathrm{id} \otimes \epsilon)(\mathrm{id} \otimes P_{-\lambda_{ij}})\big(F_{ij}(B_i, B_j) \otimes K_{-\lambda_{ij}} - F_{ij}(\Delta(B_i), \Delta(B_j))\big),$$

where we identify $U_q(\mathfrak{g}')$ with $\mathbb{K}(q) \otimes U_q(\mathfrak{g}')$. Our main purpose in Section 3 will be to expand the right-hand side of (37) as a polynomial in B_J, $J \in \mathcal{J}_{i,j}$, with coefficients in $\mathcal{M}_X^+ U_\Theta^{0'}$. To do so, we will need an expression for the $\Delta(B_i)$ and $\Delta(B_j)$ in (37). These follow from the following lemma.

Lemma I ([27, Lemma 7.7]). *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and $j \in X$. Then, there exists an element $\mathcal{W}_{ij} \in \mathcal{M}_X^+$, independent of $\mathbf{c}$, such that*

$$\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}$$
$$+ c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1} + \Upsilon_i, \quad (38)$$

for some

$$\Upsilon_i \in \mathcal{M}_X^+ U_\Theta^{0'} \otimes \widehat{U_i^+} K_i^{-1},$$

where $\widehat{U_i^+} = \{u \in \mathcal{M}_X^+ E_i \mathcal{M}_X^+ : \exists \gamma \in Q, \gamma > \alpha_i, \gamma \neq \alpha_i + \alpha_j : u \in U_q(\mathfrak{g}')_\gamma\}$.

Note that the formulation of this lemma is somewhat stronger than the original one in [27], but one readily verifies the correctness of this extra restriction on Υ_i upon computing $\Delta(\mathrm{ad}(Z_i^+)(E_i))$.

Finally, let us note that the following relations follow immediately from (28)–(29).

Lemma J. *Let $i \in I \setminus X$ be such that $\tau(i) = i$, then for any $j \in I \setminus X$ one has*

$$[B_j, \mathcal{Z}_i] = 0,$$

whereas for $j \in X$ one has

$$B_i \mathcal{W}_{ij} K_j = q_i^{a_{ij}} \mathcal{W}_{ij} K_j B_i.$$

3. Quantum Serre relations for the algebras $B_{\mathbf{c},\mathbf{s}}$

We are now ready to derive closed expressions for the quantum Serre relations (27) by expanding the right-hand side of (37). Crucial in this respect is the presence of the

morphism $\mathrm{id} \otimes \epsilon$, which by (22) tells us that no term containing a nontrivial element of $U^+ U^-$ in the second tensor component will survive in (37). This allows us to eliminate some of the terms in (38).

We will first focus on Case 1.

Lemma 3.1. *Let $i, j \in I \setminus X$ be distinct such that $\tau(i) = i$. Then, one has*

$$F_{ij}(B_i, B_j) = \big(\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}})\big) \tag{39}$$
$$\big[F_{ij}(B_i, B_j) \otimes K_{-\lambda_{ij}} - F_{ij}\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}, B_j \otimes K_j^{-1}\big)\big].$$

Proof. First, let us note that the polynomial F_{ij} is of degree 1 and hence, linear in its second argument. Since $j \notin X$, the expression (38) for $\Delta(B_i)$ contains no factors E_j or F_j in its second tensor component. Since $\epsilon(E_j) = \epsilon(F_j) = 0$, the expression for $\Delta(B_j)$ obtained from Lemma I, together with the relation (37), asserts that

$$F_{ij}(B_i, B_j) = (\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}}))\big(F_{ij}(B_i, B_j) \otimes K_{-\lambda_{ij}} - F_{ij}(\Delta(B_i), B_j \otimes K_j^{-1})\big). \tag{40}$$

When expanding $\Delta(B_i)$ according to (38), there will be no contribution from the two latter terms

$$c_i \mathcal{W}_{ik} K_k \otimes (E_k E_i - q_i^{a_{ik}} E_i E_k) K_i^{-1} + \Upsilon_i, \tag{41}$$

with $k \in X$, since each term in (41) contains at least one factor $E_{k'}$, $k' \in X$, in its second tensor component, and again $\epsilon(E_{k'}) = 0$. Hence, $\Delta(B_i)$ in (40) can be replaced by $B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}$. $\square$

The same simplification can be performed for Case 2.

Lemma 3.2. *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and let $j \in X$. Then one has*

$$F_{ij}(B_i, B_j) = (\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}}))\big[F_{ij}(B_i, B_j) \otimes K_{-\lambda_{ij}}$$
$$- F_{ij}\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}, B_j \otimes K_j^{-1}\big)$$
$$- F_{ij}\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}$$
$$+ c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}, 1 \otimes F_j\big)\big]. \tag{42}$$

Proof. Since $j \in X$, we have $B_j = F_j$. Hence, it follows from (37), (22) and the linearity of F_{ij} in its second argument that

$$F_{ij}(B_i, B_j) = (\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}}))$$
$$\big(F_{ij}(B_i, B_j) \otimes K_{-\lambda_{ij}} - F_{ij}(\Delta(B_i), B_j \otimes K_j^{-1}) - F_{ij}(\Delta(B_i), 1 \otimes F_j)\big).$$

We will now expand $\Delta(B_i)$ using (38) with the given j. In the first occurrence of $\Delta(B_i)$, both $c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}$ and Υ_i will not contribute, since each of their terms contains at least one factor E_k with $k \neq i$ in the second tensor component, and $\epsilon(E_k) = 0$. For the second occurrence of $\Delta(B_i)$, the situation

is different. The term $c_i \, \mathcal{W}_{ij} \, K_j \otimes (E_j \, E_i - q_i^{a_{ij}} \, E_i \, E_j) K_i^{-1}$ will effectively contribute, since when expanding $F_{ij}(\Delta(B_i), 1 \otimes F_j)$, we may use the rule

$$F_j \, E_j = E_j \, F_j - \frac{K_j - K_j^{-1}}{q_j - q_j^{-1}}$$

and the last term in this expansion will turn out to be significant, as will be explained in what follows. The term Υ_i in $\Delta(B_i)$, however, will still not contribute. Indeed, each term in Υ_i contains either a factor E_j^2 or a factor E_k with $k \in X \setminus \{j\}$, and both $F_j \, E_j^2$ and $F_j \, E_k$ cannot be expanded to yield a non-vanishing term under ϵ. This proves the claim. $\square$

One observes immediately that the right-hand side of (42) equals the right-hand side of (39), added with a second term. In what follows, we will treat both terms separately and thereby obtain explicit expressions for each of the two cases.

3.1. Binary expansions. In this section, we will expand the right-hand sides of (39) and (42). We will first treat the right-hand side of (39), which occurs in (42) as well and which can, to a large extent, be rewritten irrespective of whether or not j lies in X. The second term, which appears only in (42), i.e. for $j \in X$, will be addressed afterwards.

Our main strategy will be to perform a "binary" distributive expansion, which requires summation over binary tuples $\boldsymbol{\ell} \in \{0, 1\}^N$, $N \in \mathbb{N}$. For any such tuple $\boldsymbol{\ell}$, we will use the notation

$$|\boldsymbol{\ell}| = \ell_1 + \ell_2 + \cdots + \ell_N, \quad |\boldsymbol{\ell}|_{r;s} = \begin{cases} \ell_r + \ell_{r+1} + \cdots + \ell_s & \text{if } r \leq s, \\ 0 & \text{otherwise.} \end{cases} \quad (43)$$

Throughout this paper, we will often meet finite sums and products over natural numbers. We will use the convention that a sum vanishes if its lower bound exceeds its upper bound or equivalently if it ranges over the empty set, whereas a product reduces to one in this situation. Otherwise stated, for any function a of r and any $M > N$ we take

$$\sum_{r=M}^{N} a(r) = \sum_{r \in \emptyset} a(r) = 0 \quad \text{and} \quad \prod_{r=M}^{N} a(r) = \prod_{r \in \emptyset} a(r) = 1.$$

Note also that in our convention 0 is a natural number, i.e. $0 \in \mathbb{N}$, and $0^0 = 1$.

Proposition 3.3. *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and let $j \in I$ be distinct from i. Then one has*

$$
\begin{aligned}
\big(\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}})\big)\big[& F_{ij}(B_i, B_j) \otimes K_{-\lambda_{ij}} \\
& - F_{ij}\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}, B_j \otimes K_j^{-1}\big)\big]
\end{aligned}
$$

$$
= \sum_{k=0}^{1-a_{ij}} \sum_{\substack{\boldsymbol{\ell} \in \{0,1\}^{1-a_{ij}} \\ |\boldsymbol{\ell}| \neq 1-a_{ij}}} \sum_{\boldsymbol{s} \in \{0,1\}^{1-a_{ij}-|\boldsymbol{\ell}|}} (-1)^{k+1} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (\epsilon \circ P_{-\lambda_{ij}})\big(\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})}\big)
$$

$$
(c_i \mathcal{Z}_i)^{\sum_{r=1}^{1-a_{ij}-k}(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})} B_i^{|\boldsymbol{\ell}|_{1;1-a_{ij}-k}} B_j B_i^{|\boldsymbol{\ell}|_{2-a_{ij}-k;1-a_{ij}}}
$$

$$
(c_i \mathcal{Z}_i)^{\sum_{r=2-a_{ij}-k}^{1-a_{ij}}(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}, \quad (44)
$$

where

$$
\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})} = \left(\overrightarrow{\prod_{r=1}^{1-a_{ij}-k} \mathfrak{T}_{\boldsymbol{\ell},\boldsymbol{s},r}^i} \right) K_j^{-1} \left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}} \mathfrak{T}_{\boldsymbol{\ell},\boldsymbol{s},r}^i} \right), \quad (45)
$$

with

$$
\mathfrak{T}_{\boldsymbol{\ell},\boldsymbol{s},r}^i = K_i^{-\ell_r} F_i^{(1-\ell_r)s_{r-|\boldsymbol{\ell}|_{1;r}}} (E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}. \quad (46)
$$

Proof. By the definition (17) of F_{ij}, we have

$$
\begin{aligned}
& - F_{ij}\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}, B_j \otimes K_j^{-1}\big) \\
& = \sum_{k=0}^{1-a_{ij}} (-1)^{k+1} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} \big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\big)^{1-a_{ij}-k} \\
& \qquad\qquad \big(B_j \otimes K_j^{-1}\big)\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\big)^k.
\end{aligned}
$$

The term $\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\big)^{1-a_{ij}-k}$ can be expanded distributively as

$$
\sum_{\boldsymbol{\ell} \in \{0,1\}^{1-a_{ij}-k}} \overrightarrow{\prod_{r=1}^{1-a_{ij}-k}} \big(B_i \otimes K_i^{-1}\big)^{\ell_r} \big(1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\big)^{1-\ell_r},
$$

and for each $\ell \in \{0,1\}^{1-a_{ij}-k}$ one has

$$\prod_{r=1}^{\overrightarrow{1-a_{ij}-k}} \left(B_i \otimes K_i^{-1}\right)^{\ell_r} \left(1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\right)^{1-\ell_r}$$

$$= \sum_{s \in \{0,1\}^{1-a_{ij}-k-|\ell|_{1;1-a_{ij}-k}}} \prod_{r=1}^{\overrightarrow{1-a_{ij}-k}} \left(B_i \otimes K_i^{-1}\right)^{\ell_r} \left(1 \otimes F_i\right)^{(1-\ell_r)s_{r-|\ell|_{1;r}}}$$

$$\left(c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\right)^{(1-\ell_r)(1-s_{r-|\ell|_{1;r}})}.$$

The rationale of this expansion is that for $\ell_r = 1$, we get the contribution of $B_i \otimes K_i^{-1}$, for $\ell_r = 0$ and $s_{r-|\ell|_{1;r}} = 1$ we find $1 \otimes F_i$, whereas for $\ell_r = 0$ and $s_{r-|\ell|_{1;r}} = 0$ we have $c_i \mathcal{Z}_i \otimes E_i K_i^{-1}$. Note that the indexation of the s-variables was chosen in such a way that there is only a summation over these variables in case the corresponding $\ell_r = 0$. Indeed, if we were to write s_r instead of $s_{r-|\ell|_{1;r}}$ and sum over all $s_1, \ldots, s_{1-a_{ij}-k} \in \{0,1\}$, then the terms corresponding to $\ell_r = 1$ would contribute twice.

Since B_i commutes with $\mathcal{Z}_i$ by Lemma J, we obtain

$$\left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\right)^{1-a_{ij}-k}$$

$$= \sum_{\ell \in \{0,1\}^{1-a_{ij}-k}} \sum_{s \in \{0,1\}^{1-a_{ij}-k-|\ell|_{1;1-a_{ij}-k}}}$$

$$(c_i \mathcal{Z}_i)^{\sum_{r=1}^{1-a_{ij}-k}(1-\ell_r)(1-s_{r-|\ell|_{1;r}})} B_i^{|\ell|_{1;1-a_{ij}-k}} \otimes \prod_{r=1}^{\overrightarrow{1-a_{ij}-k}} \mathcal{T}_{\ell,s,r}^i,$$

with $\mathcal{T}_{\ell,s,r}^i$ as in (46). Performing a similar expansion for the term

$$\left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\right)^k,$$

we find that $-F_{ij}\left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}, B_j \otimes K_j^{-1}\right)$ is equal to

$$\sum_{k=0}^{1-a_{ij}} \sum_{\ell \in \{0,1\}^{1-a_{ij}}} \sum_{s \in \{0,1\}^{1-a_{ij}-|\ell|}} (-1)^{k+1} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i}$$

$$\left((c_i \mathcal{Z}_i)^{\sum_{r=1}^{1-a_{ij}-k}(1-\ell_r)(1-s_{r-|\ell|_{1;r}})} B_i^{|\ell|_{1;1-a_{ij}-k}} B_j B_i^{|\ell|_{2-a_{ij}-k;1-a_{ij}}}\right.$$

$$\left.(c_i \mathcal{Z}_i)^{\sum_{r=2-a_{ij}-k}^{1-a_{ij}}(1-\ell_r)(1-s_{r-|\ell|_{1;r}})}\right) \otimes \mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})},$$

with $\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})}$ as in (45).

It remains only to observe that the term corresponding to $|\boldsymbol{\ell}| = 1 - a_{ij}$, i.e. $\boldsymbol{\ell} = (1, 1, \ldots, 1)$, yields

$$\sum_{k=0}^{1-a_{ij}} (-1)^{k+1} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} B_i^{1-a_{ij}-k} B_j B_i^k \otimes K_i^{-(1-a_{ij})} K_j^{-1}$$

$$= -F_{ij}(B_i, B_j) \otimes K_{-\lambda_{ij}}. \qquad \square$$

Many of the s in the sum in (44) will have a vanishing contribution. One can make the following observation.

Lemma 3.4. *Let* $i \in I \setminus X$ *be such that* $\tau(i) = i$ *and let* $j \in I$ *be distinct from* i. *Let* $\boldsymbol{\ell} \in \{0, 1\}^{1-a_{ij}}$ *with* $|\boldsymbol{\ell}| \neq 1 - a_{ij}$, $s \in \{0, 1\}^{1-a_{ij}-|\boldsymbol{\ell}|}$ *and* $k \in \{0, \ldots, 1 - a_{ij}\}$. *Then one has*

$$(\epsilon \circ P_{-\lambda_{ij}})\big(\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}\big) = 0$$

if one of the following conditions is fulfilled:

(a) $a_{ij} + |\boldsymbol{\ell}|$ *is even;*

(b) $|s| \neq \frac{1-a_{ij}-|\boldsymbol{\ell}|}{2}$;

(c) *There exists* $p \in \{1, \ldots, 1 - a_{ij} - |\boldsymbol{\ell}|\}$ *such that* $|s|_{1;p} < \frac{p}{2}$.

Proof. To acquire the action of $\epsilon \circ P_{-\lambda_{ij}}$ on $\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}$, we will write $\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}$ in a standard ordering, namely as a $\mathbb{K}(q)$-linear combination of elements of the form

$$E_i^{N_1} F_i^{N_2} K_i^{N_3} K_j^{-1},$$

with $N_1, N_2 \in \mathbb{N}$ and $N_3 \in \mathbb{Z}$. We may do so by applying the $U_q(\mathfrak{g}')$-relations (18)–(19). Each such element will be projected to either itself or 0 by $P_{-\lambda_{ij}}$. But when applying ϵ, such a term can only survive if

$$N_1 = N_2 = 0,$$

by (22). Suppose now s is such that $\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}$ contains an unequal number of factors F_i and E_i. Then each term in its standard ordering will still contain an unequal number of factors F_i and E_i, as follows from (19). Hence, the standard ordering will consist of terms $E_i^{N_1} F_i^{N_2} K_i^{N_3} K_j^{-1}$ with either N_1 or N_2 non-zero, which will be killed by ϵ. Thus, we must have an equal number of factors F_i and E_i in $\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}$. This number must then be half the total number of factors in $\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}$ with $\ell_r = 0$, i.e.

$$|s| = \frac{1 - a_{ij} - |\boldsymbol{\ell}|}{2}.$$

If $a_{ij} + |\boldsymbol{\ell}|$ is even, then the total number of factors in $\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}$ with $\ell_r = 0$ will be odd. Hence, the number of factors F_i and E_i in $\mathfrak{p}_{\boldsymbol{\ell},s,k}^{(i,j,a_{ij})}$ will always be unequal, for any s.

Finally, suppose $p \in \{1, \ldots, 1 - a_{ij} - |\boldsymbol{\ell}|\}$ is such that $|\boldsymbol{s}|_{1;p} < p/2$. This means that up to position p, the number of factors E_i will exceed the number of factors F_i. As the difference between these numbers is not altered by the relation (19), this means that the standard ordering of the corresponding term will consist only of terms $E_i^{N_1} F_i^{N_2} K_i^{N_3} K_j^{-1}$ with $N_1 \geq 1$, which are again killed by ϵ. $\qquad\square$

This result will help us to simplify the notation used in Proposition 3.3. Indeed, by Condition (b) in Lemma 3.4, we know that for any $(\boldsymbol{\ell}, \boldsymbol{s}, k)$ contributing non-trivially to (44), we have

$$\sum_{r=1}^{1-a_{ij}} (1 - \ell_r)(1 - s_{r-|\boldsymbol{\ell}|_{1;r}}) = 1 - a_{ij} - |\boldsymbol{\ell}| - |\boldsymbol{s}| = \frac{1 - a_{ij} - |\boldsymbol{\ell}|}{2}, \qquad (47)$$

and hence also

$$\sum_{r=1}^{1-a_{ij}-k} (1 - \ell_r)(1 - s_{r-|\boldsymbol{\ell}|_{1;r}})$$
$$= 1 - a_{ij} - k - |\boldsymbol{\ell}|_{1;1-a_{ij}-k} - |\boldsymbol{s}|_{1;1-a_{ij}-k-|\boldsymbol{\ell}|_{1;1-a_{ij}-k}},$$
$$\sum_{r=2-a_{ij}-k}^{1-a_{ij}} (1 - \ell_r)(1 - s_{r-|\boldsymbol{\ell}|_{1;r}}) = \frac{1 - a_{ij} - |\boldsymbol{\ell}|}{2} - \sum_{r=1}^{1-a_{ij}-k} (1 - \ell_r)(1 - s_{r-|\boldsymbol{\ell}|_{1;r}}).$$
$$(48)$$

Moreover, we will need the notion of the even and an odd part of an integer number $d \in \mathbb{Z}$, denoted by d_e and d_p respectively, and defined as

$$d_e = \left\lfloor \frac{d}{2} \right\rfloor = \begin{cases} \dfrac{d}{2} & \text{for } d \text{ even,} \\ \dfrac{d-1}{2} & \text{for } d \text{ odd,} \end{cases} \qquad\qquad d_p = \begin{cases} 0 & \text{for } d \text{ even,} \\ 1 & \text{for } d \text{ odd.} \end{cases} \qquad (49)$$

Note that for any $d \in \mathbb{Z}$ one has $d = 2d_e + d_p$.

This will now help us to rewrite $C_{ij}(\mathbf{c})$ for Case 1.

Corollary 3.5 (Case 1). *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and let $j \in I \setminus X$ be distinct from i. Then one has*

$$F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c}) = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \rho_{m,m'}^{(i,j,a_{ij})} \, \mathcal{Z}_i^{\frac{1-a_{ij}-m-m'}{2}} B_i^m B_j B_i^{m'}, \quad (50)$$

where

$$\rho_{m,m'}^{(i,j,a_{ij})} = (a_{ij} + m + m')_p c_i^{\frac{1-a_{ij}-m-m'}{2}}$$

$$\sum_{k=m'}^{1-a_{ij}-m} \sum_{\ell \in \mathfrak{L}_{m,m',k}} \sum_{s \in \mathcal{S}_{m,m'}} (-1)^{k+1} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (\epsilon \circ P_{-\lambda_{ij}})(\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})}), \quad (51)$$

with $\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})}$ *as in* (45) *and*

$$\mathfrak{L}_{m,m',k} = \{\ell \in \{0,1\}^{1-a_{ij}} : |\ell|_{1;1-a_{ij}-k} = m \text{ and } |\ell|_{2-a_{ij}-k;1-a_{ij}} = m'\},$$

$$\mathcal{S}_{m,m'} = \Big\{ s \in \{0,1\}^{1-a_{ij}-m-m'} : |s| = \tfrac{1-a_{ij}-m-m'}{2} \text{ and } |s|_{1;p} \geq \tfrac{p}{2}, \quad (52)$$

$$\forall p \in \{1,\ldots,1-a_{ij}-m-m'\}\Big\}.$$

Proof. Upon combining Lemma 3.1, Proposition 3.3, the equation (47) and the fact that $[B_i, \mathcal{Z}_i] = [B_j, \mathcal{Z}_i] = 0$ by Lemma J, one finds that $F_{ij}(B_i, B_j)$ is equal to

$$\sum_{k=0}^{1-a_{ij}} \sum_{\substack{\ell \in \{0,1\}^{1-a_{ij}} \\ |\ell| \neq 1-a_{ij}}} \sum_{s \in \{0,1\}^{1-a_{ij}-|\ell|}} (-1)^{k+1} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i}$$

$$(\epsilon \circ P_{-\lambda_{ij}})(\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})})(c_i \mathcal{Z}_i)^{\frac{1-a_{ij}-|\ell|}{2}} B_i^{|\ell|_{1;1-a_{ij}-k}} B_j B_i^{|\ell|_{2-a_{ij}-k;1-a_{ij}}}.$$

We can restrict the sum over ℓ to one over $\mathfrak{L}_{m,m',k}$, by setting

$$m = |\ell|_{1;1-a_{ij}-k}, \quad m' = |\ell|_{2-a_{ij}-k;1-a_{ij}}. \quad (53)$$

This requires an additional summation over m and m'. A priori, we have

$$m + m' = |\ell| \leq -a_{ij},$$

but if $m + m' = -a_{ij}$, then $(\epsilon \circ P_{-\lambda_{ij}})(\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})})$ will vanish for any s, by Condition (a) of Lemma 3.4. This explains the presence of $(a_{ij} + m + m')_p$ in (51) and the fact that in the sum in (50) we restrict to

$$m + m' \leq -1 - a_{ij}.$$

Note also that the requirements (53) imply that

$$1 - a_{ij} - k \geq m \quad \text{and} \quad k \geq m'.$$

Similarly, the sum over s may be restricted to $\mathcal{S}_{m,m'}$ by Conditions (b) and (c) of Lemma 3.4. $\qquad\square$

For Case 2, the first line of the right-hand side of (42) is identical to the right-hand side of (39), and hence, the first part of $C_{ij}(\mathbf{c})$ can be expanded as above. Nevertheless, we have to take into account that in this case $\mathcal{Z}_i$ and B_j no longer commute, which affects our notation.

Corollary 3.6 (Case 2). *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and let $j \in X$. Then one has*

$$
F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c})
$$
$$
= \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \sum_{t=0}^{(1-a_{ij}-m-m')/2} \rho_{m,m',t}^{(i,j,a_{ij})} \mathcal{Z}_i^t B_i^m B_j B_i^{m'} \mathcal{Z}_i^{((1-a_{ij}-m-m')/2)-t}
$$
$$
+ \left(\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}}) \right) \left(- F_{ij} \left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1} \right.\right.
$$
$$
\left.\left. + c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}, 1 \otimes F_j) \right) \right), \quad (54)
$$

where

$$
\rho_{m,m',t}^{(i,j,a_{ij})} = (a_{ij} + m + m')_p \, c_i^{(1-a_{ij}-m-m')/2}
$$
$$
\sum_{k=m'}^{1-a_{ij}-m} \sum_{\boldsymbol{\ell} \in \mathfrak{L}_{m,m',k}} \sum_{\boldsymbol{s} \in \mathcal{S}_{m,m',k,t}} (-1)^{k+1} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (\epsilon \circ P_{-\lambda_{ij}}) \left(\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})} \right), \quad (55)
$$

with $\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})}$ as in (45), $\mathfrak{L}_{m,m',k}$ as in (52) and

$$
\mathcal{S}_{m,m',k,t} = \left\{ \boldsymbol{s} \in \{0,1\}^{1-a_{ij}-m-m'} : |\boldsymbol{s}| = \frac{1-a_{ij}-m-m'}{2}, \right.
$$
$$
|\boldsymbol{s}|_{1;1-a_{ij}-k-m} = 1 - a_{ij} - k - m - t \text{ and } |\boldsymbol{s}|_{1;p} \geq \frac{p}{2},
$$
$$
\left. \forall p \in \{1, \ldots, 1 - a_{ij} - m - m'\} \right\}. \quad (56)
$$

Proof. Upon combining Lemma 3.2, Proposition 3.3 and the equations (47) and (48), we obtain

$$
F_{ij}(B_i, B_j) = \sum_{k=0}^{1-a_{ij}} \sum_{\substack{\boldsymbol{\ell} \in \{0,1\}^{1-a_{ij}} \\ |\boldsymbol{\ell}| \neq 1-a_{ij}}} \sum_{\boldsymbol{s} \in \{0,1\}^{1-a_{ij}-|\boldsymbol{\ell}|}} \left((-1)^{k+1} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} c_i^{(1-a_{ij}-|\boldsymbol{\ell}|)/2} \right.
$$
$$
(\epsilon \circ P_{-\lambda_{ij}}) \left(\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})} \right) \mathcal{Z}_i^{t_{\boldsymbol{\ell},\boldsymbol{s},k}} B_i^{|\boldsymbol{\ell}|_{1;1-a_{ij}-k}} B_j B_i^{|\boldsymbol{\ell}|_{2-a_{ij}-k;1-a_{ij}}} \mathcal{Z}_i^{((1-a_{ij}-|\boldsymbol{\ell}|)/2)-t_{\boldsymbol{\ell},\boldsymbol{s},k}} \right)
$$
$$
+ \left(\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}}) \right) \left(- F_{ij} \left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1} \right.\right.
$$
$$
\left.\left. + c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}, 1 \otimes F_j) \right) \right),
$$

with, by (48),

$$t_{\boldsymbol{\ell},\boldsymbol{s},k} = 1 - a_{ij} - k - |\boldsymbol{\ell}|_{1;1-a_{ij}-k} - |\boldsymbol{s}|_{1;1-a_{ij}-k-|\boldsymbol{\ell}|_{1;1-a_{ij}-k}}. \qquad (57)$$

The sum over $\boldsymbol{\ell}$ can be restricted to $\mathfrak{L}_{m,m',k}$, with an additional summation over m, m', just like in the proof of Corollary 3.5. Setting $t_{\boldsymbol{\ell},\boldsymbol{s},k}$ equal to a parameter t, over which we sum as well, determines the condition

$$|\boldsymbol{s}|_{1;1-a_{ij}-k-m} = 1 - a_{ij} - k - m - t,$$

as follows from (57). This restriction, together with Conditions (b) and (c) of Lemma 3.4, determines the definition of $\mathcal{S}_{m,m',k,t}$. $\qquad\square$

We will now perform a similar binary expansion for the last line of (54).

Proposition 3.7. *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and let $j \in X$. Then one has*

$$\big(\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}})\big)\big(- F_{ij}\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}$$
$$+ c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}, 1 \otimes F_j\big)\big)$$

$$= \sum_{k=1}^{1-a_{ij}} \sum_{d=0}^{k-1} \sum_{\boldsymbol{\ell}\in\{0,1\}^{-a_{ij}}} \sum_{\boldsymbol{s}\in\{0,1\}^{-a_{ij}-|\boldsymbol{\ell}|}} \bigg((-1)^{k+1} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_i^{a_{ij}|\boldsymbol{\ell}|_{1;1-a_{ij}-k+d}}$$

$$(\epsilon \circ P_{-\lambda_{ij}})(\mathfrak{r}_{\boldsymbol{\ell},\boldsymbol{s},k,d}^{(i,j,a_{ij})})(c_i \mathcal{Z}_i)^{\sum_{r=1}^{1-a_{ij}-k+d}(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}$$

$$(c_i \mathcal{W}_{ij} K_j)(c_i \mathcal{Z}_i)^{\sum_{r=2-a_{ij}-k+d}^{-a_{ij}}(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})} B_i^{|\boldsymbol{\ell}|}\bigg), \quad (58)$$

where

$$\mathfrak{r}_{\boldsymbol{\ell},\boldsymbol{s},k,d}^{(i,j,a_{ij})} = \left(\overrightarrow{\prod_{r=1}^{1-a_{ij}-k}} \mathfrak{T}_{\boldsymbol{\ell},\boldsymbol{s},r}^i\right) F_j \left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mathfrak{T}_{\boldsymbol{\ell},\boldsymbol{s},r}^i\right)$$

$$(E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1} \left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}} \mathfrak{T}_{\boldsymbol{\ell},\boldsymbol{s},r}^i\right), \quad (59)$$

with $\mathfrak{T}_{\boldsymbol{\ell},\boldsymbol{s},r}^i$ as in (46).

Proof. By the definition (17) of F_{ij}, the left-hand side of (58) can be written as

$$\big(\mathrm{id} \otimes (\epsilon \circ P_{-\lambda_{ij}})\big)\bigg(\sum_{k=0}^{1-a_{ij}} (-1)^{k+1} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i}$$

$$\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\big)^{1-a_{ij}-k} (1 \otimes F_j)$$

$$\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1} + c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}\big)^k\bigg).$$

In the term

$$\left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\right)^{1-a_{ij}-k}$$

preceding $1 \otimes F_j$, the term $c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}$ does not need to be taken into account. Indeed, the standard ordering of the expansion with respect to this term would consist of terms

$$E_i^{N_1} E_j^{M} E_i^{N_2} F_i^{N_3} F_j F_i^{N_4} K_i^{N_5}, \tag{60}$$

with $M \geq 1$. But of course each such term vanishes under ϵ. In the term

$$\left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1} + c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}\right)^{k}$$

succeeding $1 \otimes F_j$, it does need to be taken into account. More precisely, in the whole sum we obtain when expanding the k-th power, each term must contain exactly one factor $c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}$, such that we may use the rule

$$F_j E_j = E_j F_j - \frac{K_j - K_j^{-1}}{q_j - q_j^{-1}}$$

to obtain a non-zero contribution. Indeed, if we were to take more than one such factor, then we would end up with a normal ordering consisting of terms of the form (60) with $M \geq 1$ and

$$E_i^{N_1} E_j^{M} E_i^{N_2} F_i^{N_3} K_i^{N_4} K_j^{N_5}$$

with $M \geq 1$, which again disappear under ϵ, whereas if we were to take 0 such factors, then we would find

$$E_i^{N_1} F_i^{N_2} F_j F_i^{N_3} K_i^{N_4},$$

in the normal ordering, which also yields 0 under ϵ by the presence of F_j. This also explains why we can replace

$$\left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1} + c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}\right)^{k}$$

by

$$\sum_{d=0}^{k-1} \left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\right)^{d} \left(c_i \mathcal{W}_{ij} K_j \otimes (E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}\right)$$
$$\left(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1}\right)^{k-d-1}.$$

The claim now follows upon expanding binarily the powers of

$$B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_i K_i^{-1},$$

as in the proof of Proposition 3.3. Note that this time, we will need a total of

$$1 - a_{ij} - k + d + (k - d - 1) = -a_{ij}$$

variables ℓ_r. Observe that we have used Lemma J to obtain the factor $q_i^{a_{ij}|\ell|_{1;1-a_{ij}-k+d}}$. □

Once more, many of the s in the sum in (58) will not contribute. In analogy to Lemma 3.4, one can formulate the following result.

Lemma 3.8. *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and let $j \in X$. Let $\ell \in \{0, 1\}^{-a_{ij}}$, $s \in \{0, 1\}^{-a_{ij}-|\ell|}$, $k \in \{1, \ldots, 1 - a_{ij}\}$ and $d \in \{0, \ldots, k - 1\}$. Then one has*

$$(\epsilon \circ P_{-\lambda_{ij}})(\mathfrak{r}_{\ell,s,k,d}^{(i,j,a_{ij})}) = 0$$

if one of the Conditions (a), (b), (c) *from Lemma 3.4 is fulfilled, or in case we have:*

(d) *There exists*

$$p \in \{1 - a_{ij} - k + d - |\ell|_{1;1-a_{ij}-k+d}, \ldots, -a_{ij} - |\ell|\}$$

  such that $|s|_{1;p} = \frac{p}{2}$.

(e) $|s|_{1;1-a_{ij}-k+d-|\ell|_{1;1-a_{ij}-k+d}} = 0.$

Proof. As in the proof of Lemma 3.4, the requirement that $\mathfrak{r}_{\ell,s,k,d}^{(i,j,a_{ij})}$ must contain an equal number of factors F_i and E_i determines the conditions (a) and (b). Note that in this case, one comes to the number $(1 - a_{ij} - |\ell|)/2$ by considering the $-a_{ij} - |\ell|$ factors F_i or E_i arising from the $\mathfrak{T}_{\ell,s,r}^i$ in (59), together with the extra factor E_i in (59). The requirement that for each p, the number of factors F_i must exceed the number of factors E_i up to position p, determines in this case not only the condition (c), but also the extra conditions (d) and (e), again by the presence of $(E_j E_i - q_i^{a_{ij}} E_i E_j) K_i^{-1}$ in (59). □

As before, this means that we can determine

$$\sum_{r=1}^{-a_{ij}} (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}}) = -a_{ij} - |\ell| - |s| = \frac{-1 - a_{ij} - |\ell|}{2},$$

$$\sum_{r=1}^{1-a_{ij}-k+d} (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}}) = 1 - a_{ij} - k + d - |\ell|_{1;1-a_{ij}-k+d} \\ - |s|_{1;1-a_{ij}-k+d-|\ell|_{1;1-a_{ij}-k+d}},$$

$$\sum_{r=2-a_{ij}-k+d}^{-a_{ij}} (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}}) = \frac{-1 - a_{ij} - |\ell|}{2} \\ - \sum_{r=1}^{1-a_{ij}-k+d} (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}}). \quad (61)$$

Just like in the previous situation, this now leads to a complete description of $C_{ij}(\mathbf{c})$ in Case 2.

Corollary 3.9 (Case 2). *Let $i \in I \setminus X$ be such that $\tau(i) = i$ and let $j \in X$. Then one has*

$$
F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c})
$$

$$
= \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \sum_{t=0}^{(1-a_{ij}-m-m')/2} \rho_{m,m',t}^{(i,j,a_{ij})} \mathcal{Z}_i^t B_i^m B_j B_i^{m'} \mathcal{Z}_i^{((1-a_{ij}-m-m')/2)-t}
$$

$$
+ \sum_{m=0}^{-1-a_{ij}} \sum_{t=0}^{(-1-a_{ij}-m)/2} \sigma_{m,t}^{(i,j,a_{ij})} \mathcal{Z}_i^t \mathcal{W}_{ij} K_j \mathcal{Z}_i^{((-1-a_{ij}-m)/2)-t} B_i^m, \quad (62)
$$

with $\rho_{m,m',t}^{(i,j,a_{ij})}$ as obtained in (55), and where

$$
\sigma_{m,t}^{(i,j,a_{ij})} = (a_{ij} + m)_p c_i^{(1-a_{ij}-m)/2} \sum_{k=1}^{1-a_{ij}} \sum_{d=0}^{k-1} \sum_{m'=0}^{m} \sum_{\boldsymbol{\ell} \in \mathcal{L}'_{m,m',k,d}} \sum_{\boldsymbol{s} \in \mathcal{S}'_{m,m',k,t,d}}
$$

$$
(-1)^{k+1} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} q_i^{m' a_{ij}} (\epsilon \circ P_{-\lambda_{ij}})(\mathfrak{r}_{\boldsymbol{\ell},\boldsymbol{s},k,d}^{(i,j,a_{ij})}), \quad (63)
$$

with $\mathfrak{r}_{\boldsymbol{\ell},\boldsymbol{s},k,d}^{(i,j,a_{ij})}$ as in (59) and

$$
\mathcal{L}'_{m,m',k,d} = \{ \boldsymbol{\ell} \in \{0,1\}^{-a_{ij}} : |\boldsymbol{\ell}| = m \text{ and } |\boldsymbol{\ell}|_{1;1-a_{ij}-k+d} = m' \},
$$

$$
\mathcal{S}'_{m,m',k,t,d} = \left\{ \boldsymbol{s} \in \{0,1\}^{-a_{ij}-m} : |\boldsymbol{s}| = \frac{1-a_{ij}-m}{2}, \ |\boldsymbol{s}|_{1;p} \geq \frac{p+\delta^{(p,k,d,m')}}{2}, \right. \quad (64)
$$

$$
\forall p \in \{1,\ldots,-a_{ij}-m\} \text{ and } |\boldsymbol{s}|_{1;1-a_{ij}-k-m'+d} = 1 - a_{ij} - k - m' - t + d \neq 0 \bigg\},
$$

where

$$
\delta^{(p,k,d,m')} = \begin{cases} 0 & \text{for } p < 1 - a_{ij} - k + d - m', \\ 1 & \text{for } p \geq 1 - a_{ij} - k + d - m'. \end{cases}
$$

Proof. This follows from Corollary 3.6, Proposition 3.7 and the equations (61) in exactly the same fashion as we have derived Corollaries 3.5 and 3.6, i.e. upon setting

$$
m = |\boldsymbol{\ell}|, \quad m' = |\boldsymbol{\ell}|_{1;1-a_{ij}-k+d}, \quad t = 1 - a_{ij} - k - m' + d - |\boldsymbol{s}|_{1;1-a_{ij}-k-m'+d}.
$$

Again, $|\boldsymbol{\ell}| = m$ cannot equal $-a_{ij}$, since then $a_{ij} + m$ would be even, which is excluded by Condition (a) in Lemma 3.4. So m runs from 0 to $-1 - a_{ij}$. It follows immediately that m' runs from 0 to m. The conditions in Lemma 3.8 determine the definition of $\mathcal{S}'_{m,m',k,t,d}$. $\square$

The relations we have obtained in Corollaries 3.5 and 3.9 comply with the explicit calculations performed in [27] and [2] by Balagović and Kolb. They also obtained explicit values for the structure constants for a limited set of possible a_{ij}: they computed $\rho_{m,m'}^{(i,j,a_{ij})}$ for $a_{ij} \in \{-1,-2,-3\}$ and $\rho_{m,m',t}^{(i,j,a_{ij})}$ and $\sigma_{m,t}^{(i,j,a_{ij})}$ for $a_{ij} \in \{-1,-2\}$. These values are displayed below.

m \ m'	0
0	$c_i q_i$

(a) $a_{ij} = -1$.

m \ m'	0	1
0	0	$-\rho_{1,0}^{(i,j,-2)}$
1	$c_i q_i (q_i + q_i^{-1})^2$	

(b) $a_{ij} = -2$.

m \ m'	0	1	2
0	$-c_i^2 q_i^2 [3]_{q_i}^2$	0	$\rho_{2,0}^{(i,j,-3)}$
1	0	$-c_i q_i (q_i^2 + 3 + q_i^{-2})[4]_{q_i}$	
2	$c_i q_i (1 + [3]_{q_i}^2)$		

(c) $a_{ij} = -3$.

Table 1. Structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ for $a_{ij} \in \{-1,-2,-3\}$.

The main purpose of this paper will be to find expressions in $\mathbb{K}(q)$ for the structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$, $\rho_{m,m',t}^{(i,j,a_{ij})}$, and $\sigma_{m,t}^{(i,j,a_{ij})}$, valid without any restrictions on a_{ij}. By Corollaries 3.5, 3.6, and 3.9, this amounts to deriving how $\epsilon \circ P_{-\lambda_{ij}}$ acts on $\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})}$ and $\mathfrak{r}_{\ell,s,k,d}^{(i,j,a_{ij})}$. This computation will be performed in the next two subsections.

3.2. Case 1: $\tau(i) = i \in I \setminus X$ and $j \in I \setminus X$. Let us now fix $i \in I \setminus X$ such that $\tau(i) = i$ and $j \in I$ distinct from i. A priori, we don't specify whether or not $j \in X$. Let us also fix $m, m' \in \mathbb{N}$ such that $a_{ij} + m + m'$ is odd and $m + m' \leq -1 - a_{ij}$, $k \in \mathbb{N}$ such that $m' \leq k \leq 1 - a_{ij} - m$, $t \in \{0, \ldots, (1 - a_{ij} - m - m')/2\}$, $\ell \in \mathfrak{L}_{m,m',k}$ and $s \in \mathcal{S}_{m,m',k,t}$. Note that this automatically implies that $s \in \mathcal{S}_{m,m'}$, by (52) and (56). Hence, by (51) and (55) it suffices to compute the action of $\epsilon \circ P_{-\lambda_{ij}}$ on $\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})}$, defined in (45), in order to obtain the full polynomial $C_{ij}(\mathbf{c})$ for Case 1, as well as the first of the two parts of this polynomial for Case 2. This computation will now be performed.

Let us introduce the notation $\widehat{P}_N^i$, with $N \in \mathbb{Z}$, for the projection operator

$$\widehat{P}_N^i \colon U_q(\mathfrak{g}') \to U^+ K_i^N S(U^-) \tag{65}$$

with respect to the decomposition (33). Let us also renormalize the element E_i as

$$\widetilde{E}_i = (q_i - q_i^{-1}) E_i. \tag{66}$$

(m, m') ╲ t	0	1
$(0, 0)$	$\dfrac{c_i q_i^2}{q_i - q_i^{-1}}$	$\dfrac{-c_i}{q_i - q_i^{-1}}$

(a) $a_{ij} = -1$.

(m, m') ╲ t	0	1
$(0, 1)$	$-c_i q_i^2 \dfrac{q_i^2 + 2}{q_i - q_i^{-1}}$	$c_i \dfrac{[3]_{q_i}}{q_i - q_i^{-1}}$
$(1, 0)$	$c_i q_i^2 \dfrac{[3]_{q_i}}{q_i - q_i^{-1}}$	$-c_i \dfrac{q_i^{-2} + 2}{q_i - q_i^{-1}}$

(b) $a_{ij} = -2$.

Table 2. Structure constants $\rho_{m,m',t}^{(i,j,a_{ij})}$ for $a_{ij} \in \{-1, -2\}$.

m ╲ t	0
0	$c_i \dfrac{q_i + q_i^{-1}}{q_j - q_j^{-1}}$

m ╲ t	0
1	$-c_i q_i^{-2} [3]_{q_i} \dfrac{(q_i - q_i^{-1})(q_i + q_i^{-1})^2}{q_j - q_j^{-1}}$

(a) $a_{ij} = -1$.   (b) $a_{ij} = -2$.

Table 3. Structure constants $\sigma_{m,t}^{(i,j,a_{ij})}$ for $a_{ij} \in \{-1, -2\}$.

Then we can state the following result.

Proposition 3.10. *For i, j, m, m', k, ℓ and s as fixed before, one has*

$$(\epsilon \circ P_{-\lambda_{ij}})\big(\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})}\big) = \left(\frac{q_i^2}{q_i - q_i^{-1}}\right)^{(1-a_{ij}-m-m')/2} q_i^{\beta_{\ell,s,k}} \big(\epsilon \circ \widehat{P}_{-(1-a_{ij}-m-m')/2}^i\big)(Y_{\ell,s}),$$

where

$$Y_{\ell,s} = \prod_{r=1}^{\overrightarrow{1-a_{ij}}} F_i^{(1-\ell_r)s_{r-|\ell|_{1;r}}} \widetilde{E}_i^{(1-\ell_r)(1-s_{r-|\ell|_{1;r}})}, \tag{67}$$

$$\beta_{\ell,s,k} = -a_{ij}\zeta_{\ell,s}^{(1-a_{ij}-k)} - 2\sum_{r=1}^{1-a_{ij}} \zeta_{\ell,s}^{(r-1)}\big(\ell_r + (1-\ell_r)(1 - s_{r-|\ell|_{1;r}})\big), \tag{68}$$

$$\zeta_{\ell,s}^{(r)} = 2|s|_{1;r-|\ell|_{1;r}} + |\ell|_{1;r} - r. \tag{69}$$

Proof. As argued in the proof of Lemma 3.4, the total number of factors F_i and E_i in $\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})}$ must be equal and must yield

$$\#(\text{factors } E_i) = \#(\text{factors } F_i) = |s| = \frac{1 - a_{ij} - m - m'}{2}. \tag{70}$$

When shifting the factor K_j^{-1} through the second term between brackets in (45) using (18), we will induce a factor $q_i^{-a_{ij}x}$, with

$$x = \#(\text{factors } E_i \text{ succeeding } K_j^{-1}) - \#(\text{factors } F_i \text{ succeeding } K_j^{-1})$$
$$= \big(\#(\text{factors } E_i) - \#(\text{factors } E_i \text{ preceding } K_j^{-1})\big)$$
$$\qquad - \big(\#(\text{factors } F_i) - \#(\text{factors } F_i \text{ preceding } K_j^{-1})\big). \tag{71}$$

By (70) this is reduced to

$$x = \#(\text{factors } F_i \text{ preceding } K_j^{-1}) - \#(\text{factors } E_i \text{ preceding } K_j^{-1})$$
$$= \#(\text{factors } F_i \text{ preceding } K_j^{-1}) - \big[\#(\text{factors preceding } K_j^{-1})$$
$$\qquad - \#(\text{factors } K_i^{-1} \text{ preceding } K_j^{-1}) - \#(\text{factors } F_i \text{ preceding } K_j^{-1})\big]$$
$$= |s|_{1;1-a_{ij}-k-|\ell|_{1;1-a_{ij}-k}}$$
$$\qquad - \big[(1 - a_{ij} - k) - |\ell|_{1;1-a_{ij}-k} - |s|_{1;1-a_{ij}-k-|\ell|_{1;1-a_{ij}-k}}\big]$$
$$= \zeta_{\ell,s}^{(1-a_{ij}-k)}. \tag{72}$$

So we have

$$\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})} = q_i^{-a_{ij}\zeta_{\ell,s}^{(1-a_{ij}-k)}}$$
$$\left(\overrightarrow{\prod_{r=1}^{1-a_{ij}}} K_i^{-\ell_r} F_i^{(1-\ell_r)s_r-|\ell|_{1;r}} (E_i K_i^{-1})^{(1-\ell_r)(1-s_r-|\ell|_{1;r})}\right) K_j^{-1}.$$

We will perform the same shifting process for the factors $K_i^{-\ell_r}$ with $\ell_r = 1$. For each such r this will induce a factor $q_i^{-2x'}$ with

$$x' = \#(\text{factors } E_i \text{ succeeding } K_i^{-\ell_r}) - \#(\text{factors } F_i \text{ succeeding } K_i^{-\ell_r}).$$

Applying the same reasoning as in (71)–(72), we obtain $x' = \zeta_{\ell,s}^{(r-1)}$, such that one can write

$$\mathfrak{p}_{\ell,s,k}^{(i,j,a_{ij})} = q_i^{-a_{ij}\zeta_{\ell,s}^{(1-a_{ij}-k)} - 2\sum_{r=1}^{1-a_{ij}} \zeta_{\ell,s}^{(r-1)}\ell_r}$$
$$\left(\overrightarrow{\prod_{r=1}^{1-a_{ij}}} F_i^{(1-\ell_r)s_r-|\ell|_{1;r}} (E_i K_i^{-1})^{(1-\ell_r)(1-s_r-|\ell|_{1;r})}\right) K_i^{-m-m'} K_j^{-1}.$$

Finally, we will do the same for the K_i^{-1} occurring in a factor $(E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}$ with $\ell_r = 0$ and $s_{r-|\boldsymbol{\ell}|_{1;r}} = 0$. This will give rise to a factor $q_i^{-2x''}$ with

$$x'' = \#\big(\text{factors } E_i \text{ succeeding } (E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}\big)$$
$$- \#\big(\text{factors } F_i \text{ succeeding } (E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}\big).$$

The same reasoning now shows that x'' yields

$$\#\big(\text{factors } F_i \text{ preceding } (E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}\big)$$
$$- \#\big(\text{factors } E_i \text{ preceding } (E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}\big) - 1 = \zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(r-1)} - 1,$$

where the extra -1 comes from the E_i inside $(E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}$. The total power of q_i we can put in front hence becomes

$$-a_{ij}\zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(1-a_{ij}-k)} - 2 \sum_{r=1}^{1-a_{ij}} \zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(r-1)}\big(\ell_r + (1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})\big)$$
$$+ 2 \sum_{r=1}^{1-a_{ij}} (1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}}) = \beta_{\boldsymbol{\ell},\boldsymbol{s},k} + (1-a_{ij}-m-m'),$$

where we have applied (47), and with $\beta_{\boldsymbol{\ell},\boldsymbol{s},k}$ as in (68).

Finally, we will perform the renormalization $\tilde{E}_i = (q_i - q_i^{-1})E_i$, which, again taking into account the formula (70), leads to

$$\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})} = \left(\frac{q_i^2}{q_i - q_i^{-1}}\right)^{(1-a_{ij}-m-m')/2} q_i^{\beta_{\boldsymbol{\ell},\boldsymbol{s},k}}$$
$$\left(\overrightarrow{\prod_{r=1}^{1-a_{ij}}} F_i^{(1-\ell_r)s_{r-|\boldsymbol{\ell}|_{1;r}}} \tilde{E}_i^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}\right) K_i^{-(1-a_{ij}+m+m')/2} K_j^{-1}. \quad (73)$$

It now follows from (22), (34), (35), and (65) that

$$(\epsilon \circ P_{-\lambda_{ij}})\big[Y_{\boldsymbol{\ell},\boldsymbol{s}} K_i^{-(1-a_{ij}+m+m')/2} K_j^{-1}\big] = \big(\epsilon \circ \hat{P}_{-(1-a_{ij}-m-m')/2}^i\big)\big[Y_{\boldsymbol{\ell},\boldsymbol{s}}\big].$$

Together with (73), this yields the anticipated result. $\qquad\square$

We have now reduced the computation of $(\epsilon \circ P_{-\lambda_{ij}})\big(\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})}\big)$ to a simpler problem, namely computing how $\epsilon \circ \hat{P}_{-(1-a_{ij}-m-m')/2}^i$ acts on a product of an equal number of factors F_i and $\tilde{E}_i$, which is balanced in the sense that up to each position in the product, the number of factors F_i exceeds or equals the number of factors $\tilde{E}_i$, as imposed by Condition (c) of Lemma 3.4. This action can be deduced from the following lemma.

We will need the notation $(N)_{q_i^2}$ for the modified q_i^2-number

$$(N)_{q_i^2} = \frac{1 - q_i^{2N}}{1 - q_i^2}. \tag{74}$$

Note that it relates to the ordinary q_i-number as

$$(N)_{q_i^2} = q_i^{N-1}[N]_{q_i}.$$

Lemma 3.11. *Let $M \in \mathbb{N}$ be such that $M \geq 1$. Let $Y \in U_q(\mathfrak{g}')$ be a product of M factors F_i and M factors $\widetilde{E}_i$, appearing in any order, but with F_i as the first factor. Let $N \in \mathbb{N}$ be maximal such that the first N factors of Y are F_i, such that we can write $Y = F_i^N \widetilde{E}_i X$, for some $X \in U_q(\mathfrak{g}')$. Then we have*

$$\left(\epsilon \circ \widehat{P}^i_{-M}\right)(Y) = (N)_{q_i^2} q_i^{-2N+2}\left(\epsilon \circ \widehat{P}^i_{-(M-1)}\right)(F_i^{N-1} X). \tag{75}$$

Proof. We will prove this by induction on N. Our strategy will be to rewrite Y in its standard ordering, i.e. as a $\mathbb{K}(q)$-linear combination of terms of the form $\widetilde{E}_i^{m_1} F_i^{m_2} K_i^{m_3}$, and then observe that for any $M' \in \mathbb{Z}$ one has

$$\widehat{P}^i_{-M'}\left(\widetilde{E}_i^{m_1} F_i^{m_2} K_i^{m_3}\right) = \begin{cases} \widetilde{E}_i^{m_1} F_i^{m_2} K_i^{m_3} & \text{if } m_3 - m_2 = -M', \\ 0 & \text{otherwise}, \end{cases}$$

by (65) and the definition (22) of the antipode. Hence, again by (22), we have

$$(\epsilon \circ \widehat{P}^i_{-M'})\left(\widetilde{E}_i^{m_1} F_i^{m_2} K_i^{m_3}\right) = \begin{cases} 1 & \text{if } m_1 = m_2 = 0 \text{ and } m_3 = -M', \\ 0 & \text{otherwise}. \end{cases}$$

Otherwise stated, the action of $\epsilon \circ \widehat{P}^i_{-M'}$ on Y equals the coefficient of $K_i^{-M'}$ in its standard ordering.

For $N = 1$, we may apply (19) and (66) to obtain

$$Y = F_i \widetilde{E}_i X = \widetilde{E}_i F_i X - K_i X + K_i^{-1} X.$$

The first term will have a standard ordering consisting of terms $\widetilde{E}_i^{m_1} F_i^{m_2} K_i^{m_3}$ with $m_1 \geq 1$, which will all be killed by ϵ. For the second term, observe that X contains $M - 1$ factors F_i and the same number of factors $\widetilde{E}_i$, since $N = 1$. Each factor F_i, when taken together with a factor $\widetilde{E}_i$, can contribute at most one factor K_i^{-1} by (19). Hence, the lowest possible power of K_i occurring in the normal ordering of $K_i X$ will be

$$1 - (M - 1) = -M + 2 > -M.$$

Hence, the second term will not contribute either. For the third term, we have $K_i^{-1} X = X K_i^{-1}$, since X contains an equal number of factors F_i and $\widetilde{E}_i$. So we have

$$\left(\epsilon \circ \widehat{P}^i_{-M}\right)(Y) = \left(\epsilon \circ \widehat{P}^i_{-M}\right)(X K_i^{-1}) = \left(\epsilon \circ \widehat{P}^i_{-(M-1)}\right)(X),$$

in agreement with (75).

Now suppose the claim has been proven for $N - 1 \geq 1$, then we have

$$Y = F_i^N \tilde{E}_i X = F_i^{N-1} \tilde{E}_i X' - F_i^{N-1} K_i X + F_i^{N-1} K_i^{-1} X,$$

where $X' = F_i X$. As before, the second term will not contribute: the coefficient of K_i^{-M} in its standard ordering will vanish, as the lowest power of K_i that can occur will again be $-M + 2$. Consider now the third term in this sum. When shifting K_i^{-1} through X, we will induce a factor q_i^{-2x}, where

$$
\begin{aligned}
x &= \#(\text{factors } \tilde{E}_i \text{ in } X) - \#(\text{factors } F_i \text{ in } X) \\
&= \big(\#(\text{factors } \tilde{E}_i \text{ in } Y) - 1\big) - \big(\#(\text{factors } F_i \text{ in } Y) - N\big) \\
&= (M - 1) - (M - N) = N - 1,
\end{aligned}
$$

such that

$$F_i^{N-1} K_i^{-1} X = q_i^{-2N+2} F_i^{N-1} X K_i^{-1}.$$

So, we have

$$\big(\epsilon \circ \hat{P}^i_{-M}\big)(Y) = \big(\epsilon \circ \hat{P}^i_{-M}\big)\big(F_i^{N-1} \tilde{E}_i X'\big) + q_i^{-2N+2}\big(\epsilon \circ \hat{P}^i_{-(M-1)}\big)\big(F_i^{N-1} X\big).$$

Note that $F_i^{N-1} \tilde{E}_i X'$ still contains M factors F_i and M factors $\tilde{E}_i$, and has F_i as its first factor, since $N - 1 \geq 1$. Hence, we may apply the induction hypothesis to write

$$\big(\epsilon \circ \hat{P}^i_{-M}\big)\big(F_i^{N-1} \tilde{E}_i X'\big) = (N-1)_{q_i^2} q_i^{-2N+4} \big(\epsilon \circ \hat{P}^i_{-(M-1)}\big)\big(F_i^{N-2} X'\big).$$

The statement now follows from $F_i^{N-2} X' = F_i^{N-1} X$ and upon observing that

$$(N - 1)_{q_i^2} q_i^{-2N+4} + q_i^{-2N+2} = (N)_{q_i^2} q_i^{-2N+2}. \qquad \square$$

Let once more $Y, X \in U_q(\mathfrak{g}')$ and $M, N \in \mathbb{N}$ be as in the statement of Lemma 3.11. As already observed, the element $F_i^{N-1} X$ is again of the type described in Lemma 3.11: it is a product of $M - 1$ factors F_i and the same number of factors $\tilde{E}_i$, and has F_i as its first factor, provided X has F_i as its first factor or $N - 1 \geq 1$. If $N' \geq N - 1$ is the maximal number such that the first N' factors of $F_i^{N-1} X$ are F_i, then we may write

$$F_i^{N-1} X = F_i^{N'} \tilde{E}_i X',$$

for some $X' \in U_q(\mathfrak{g}')$. Consequently, Lemma 3.11 asserts

$$\big(\epsilon \circ \hat{P}^i_{-(M-1)}\big)\big(F_i^{N-1} X\big) = (N')_{q_i^2} q_i^{-2N'+2}\big(\epsilon \circ \hat{P}^i_{-(M-2)}\big)\big(F_i^{N'-1} X'\big),$$

and thus,

$$\big(\epsilon \circ \hat{P}^i_{-M}\big)(Y) = (N)_{q_i^2} q_i^{-2N+2} (N')_{q_i^2} q_i^{-2N'+2}\big(\epsilon \circ \hat{P}^i_{-(M-2)}\big)\big(F_i^{N'-1} X'\big).$$

This process will only terminate if at some position p in the product, the number of factors $\widetilde{E}_i$ preceding p exceeds the number of factors F_i preceding p. In that case, we would at some point be left with $N' = 1$ and a corresponding X' starting with $\widetilde{E}_i$ instead of F_i.

Let us now assume that this is not the case, i.e. up to each position p in Y, the number of factors F_i preceding p exceeds or equals the number of factors $\widetilde{E}_i$ preceding p. Then this process of applying Lemma 3.11 consecutively will continue until we have applied it M times and we have reached $N' = 1$ and $X' = 1$, and of course $(\epsilon \circ \widehat{P}_0^i)(1) = 1$. Each factor $\widetilde{E}_i$ can now be assigned a level, which is the exponent N of F_i that will occur in front of $\widetilde{E}_i$ at the moment this factor is cancelled when applying the formula (75) in this consecutive process. Then our reasoning in fact asserts

$$\left(\epsilon \circ \widehat{P}_{-M}^i\right)(Y) = \prod_{\text{factors } \widetilde{E}_i} \left(\text{level}(\widetilde{E}_i)\right)_{q_i^2} q_i^{-2\,\text{level}(\widetilde{E}_i)+2},$$

where the product runs over all factors $\widetilde{E}_i$ in Y. Now note that each application of the formula (75) cancels one factor F_i and one factor $\widetilde{E}_i$. Hence, each F_i is in fact coupled to exactly one factor $\widetilde{E}_i$. Thus instead of running over all $\widetilde{E}_i$ in Y, we might as well run over all factors F_i in Y and assign to each F_i a level, which equals the level of the $\widetilde{E}_i$ to which it is coupled. We find

$$\left(\epsilon \circ \widehat{P}_{-M}^i\right)(Y) = \prod_{\text{factors } F_i} \left(\text{level}(F_i)\right)_{q_i^2} q_i^{-2\,\text{level}(F_i)+2}. \tag{76}$$

Now say the element Y contains a factor $\widetilde{E}_i$ at position p in the product, which, in the process above, is coupled to a factor F_i at position r, with of course $r < p$. From the definition, it follows that the level of the $\widetilde{E}_i$ at position p is the total number of factors F_i preceding it, minus the number of factors $\widetilde{E}_i$ preceding it, again since each application of (75) kills one F_i and one $\widetilde{E}_i$. So

$$\begin{aligned}
\text{level}(F_i \text{ at position } r) &= \text{level}(\widetilde{E}_i \text{ at position } p) \tag{77}\\
&= \#(\text{factors } F_i \text{ preceding } p) - \#(\text{factors } \widetilde{E}_i \text{ preceding } p)\\
&= \#(\text{factors } F_i \text{ preceding } r) + 1 + \#(\text{factors } F_i \text{ between } r+1 \text{ and } p-1)\\
&\quad - \left(\#(\text{factors } \widetilde{E}_i \text{ preceding } r) + \#(\text{factors } \widetilde{E}_i \text{ between } r+1 \text{ and } p-1)\right),
\end{aligned}$$

where the $+1$ comes from the F_i at position r itself. Moreover, we have that

$$\#(\text{factors } F_i \text{ between } r+1 \text{ and } p-1) = \#(\text{factors } \widetilde{E}_i \text{ between } r+1 \text{ and } p-1). \tag{78}$$

Indeed, suppose not, then after coupling all possible $\widetilde{E}_i$ between positions $r+1$ and $p-1$ with an F_i, there would either still be F_i's left. Hence, position p would be

coupled to some position $r' > r$, or else there would still be $\widetilde{E}_i$ left, so position r would be coupled to $p' < p$. Inserting (78) into (77), we obtain

$$\text{level}(F_i \text{ at position } r) = \#(\text{factors } F_i \text{ preceding } r) + 1 - \#(\text{factors } \widetilde{E}_i \text{ preceding } r).$$
(79)

Let us now return to the statement of Proposition 3.10. The element

$$Y_{\boldsymbol{\ell},\boldsymbol{s}} = \prod_{r=1}^{\overrightarrow{1-a_{ij}}} F_i^{(1-\ell_r)s_{r-|\boldsymbol{\ell}|_{1;r}}} \widetilde{E}_i^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}$$

is a product of an equal number of factors F_i and $\widetilde{E}_i$, namely

$$\#(\text{factors } F_i) = |\boldsymbol{s}| = \frac{1 - a_{ij} - m - m'}{2},$$

$$\#(\text{factors } \widetilde{E}_i) = \sum_{r=1}^{1-a_{ij}} (1 - \ell_r)(1 - s_{r-|\boldsymbol{\ell}|_{1;r}}) = \frac{1 - a_{ij} - m - m'}{2},$$

where we have applied (47). Moreover, at each position p in $Y_{\boldsymbol{\ell},\boldsymbol{s}}$, the number of factors F_i preceding p, i.e. $|\boldsymbol{s}|_{1;p}$, exceeds or equals the number of factors $\widetilde{E}_i$ preceding p, i.e. $p - |\boldsymbol{s}|_{1;p}$, by Condition (c) of Lemma 3.4. Hence, the formula (76) is applicable.

Running over the factors F_i in $Y_{\boldsymbol{\ell},\boldsymbol{s}}$ amounts to running over $r \in \{1, \ldots, 1 - a_{ij}\}$ and checking for each r whether the element at position r is F_i, i.e. whether $\ell_r = 0$ and $s_{r-|\boldsymbol{\ell}|_{1;r}} = 1$. Thus, we have

$$\left(\epsilon \circ \widehat{P}^i_{-(1-a_{ij}-m-m')/2}\right)(Y_{\boldsymbol{\ell},\boldsymbol{s}})$$

$$= \prod_{r=1}^{1-a_{ij}} \left[\left(\text{level}(F_i \text{ at position } r)\right)_{q_i^2} q_i^{-2\,\text{level}(F_i \text{ at position } r)+2}\right]^{(1-\ell_r)s_{r-|\boldsymbol{\ell}|_{1;r}}}$$

$$= q_i^{-2\sum_{r=1}^{1-a_{ij}} (\text{level}(F_i \text{ at position } r)-1)(1-\ell_r)s_{r-|\boldsymbol{\ell}|_{1;r}}}$$

$$\prod_{r=1}^{1-a_{ij}} \left(\left(\text{level}(F_i \text{ at position } r)\right)_{q_i^2}\right)^{(1-\ell_r)s_{r-|\boldsymbol{\ell}|_{1;r}}}. \quad (80)$$

Applying the formula (79), we find

$$\text{level}(F_i \text{ at position } r) = |\boldsymbol{s}|_{1;r-1-|\boldsymbol{\ell}|_{1;r-1}} + 1 - \left(r - 1 - |\boldsymbol{\ell}|_{1;r-1} - |\boldsymbol{s}|_{1;r-1-|\boldsymbol{\ell}|_{1;r-1}}\right)$$
$$= \zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(r-1)} + 1.$$

Combining this with (80), we immediately obtain the following result.

Corollary 3.12. *For i, j, m, m', k, ℓ and s as fixed before, one has*

$$
\left(\epsilon \circ \widehat{P}^i_{-(1-a_{ij}-m-m')/2}\right)\left(\overrightarrow{\prod_{r=1}^{1-a_{ij}}} F_i^{(1-\ell_r)s_{r-|\ell|_{1;r}}} \widetilde{E}_i^{(1-\ell_r)(1-s_{r-|\ell|_{1;r}})}\right)
$$

$$
= q_i^{\gamma_{\ell,s,k}} \prod_{r=1}^{1-a_{ij}} \left(\left(\zeta_{\ell,s}^{(r-1)} + 1\right)_{q_i^2}\right)^{(1-\ell_r)s_{r-|\ell|_{1;r}}},
$$

where

$$
\gamma_{\ell,s,k} = -2 \sum_{r=1}^{1-a_{ij}} \zeta_{\ell,s}^{(r-1)}(1 - \ell_r)s_{r-|\ell|_{1;r}},
$$

with $\zeta_{\ell,s}^{(r)}$ as in (69).

Alternatively, one can also iterate over the factors $\widetilde{E}_i$ in $Y_{\ell,s}$ rather than the factors F_i, as initially established. Since (77) implies

$$
\text{level}(\widetilde{E}_i \text{ at position } p) = \zeta_{\ell,s}^{(p-1)},
$$

we also have

$$
\left(\epsilon \circ \widehat{P}^i_{-(1-a_{ij}-m-m')/2}\right)(Y_{\ell,s}) = \prod_{p=1}^{1-a_{ij}} \left[\left(\zeta_{\ell,s}^{(p-1)}\right)_{q_i^2} q_i^{-2\zeta_{\ell,s}^{(p-1)}+2}\right]^{(1-\ell_p)(1-s_{p-|\ell|_{1;p}})}.
$$

(81)

This formula will be of use in Subsection 3.3.

Corollaries 3.5 and 3.12 and Proposition 3.10 now lead to an explicit expression for the structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ for Case 1.

Theorem 3.13 (Case 1). *For any $i \in I \setminus X$ such that $\tau(i) = i$ and any $j \in I \setminus X$ distinct from i, one has*

$$
F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c}) = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \rho_{m,m'}^{(i,j,a_{ij})} \mathbb{Z}_i^{(1-a_{ij}-m-m')/2} B_i^m B_j B_i^{m'},
$$

(82)

where the structure constants are given by

$$
\rho_{m,m'}^{(i,j,a_{ij})} = (a_{ij} + m + m')_p \left(\frac{c_i q_i^2}{q_i - q_i^{-1}}\right)^{(1-a_{ij}-m-m')/2}
$$

$$
\sum_{k=m'}^{1-a_{ij}-m} \sum_{\ell \in \mathfrak{L}_{m,m',k}} \sum_{s \in \mathscr{S}_{m,m'}} (-1)^{k+1} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i}
$$

$$
q_i^{\theta_{\ell,s,k}} \prod_{r=1}^{1-a_{ij}} \left(\left(\zeta_{\ell,s}^{(r-1)} + 1\right)_{q_i^2}\right)^{(1-\ell_r)s_{r-|\ell|_{1;r}}}, \quad (83)
$$

where

$$\theta_{\boldsymbol{\ell},\boldsymbol{s},k} = -a_{ij}\zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(1-a_{ij}-k)} - 2\sum_{r=1}^{1-a_{ij}} \zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(r-1)},$$

with $\mathfrak{L}_{m,m',k}$ and $\mathfrak{S}_{m,m'}$ as in (52) and $\zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(r)}$ as in (69).

Proof. This follows upon combining Corollary 3.5, Proposition 3.10 and Corollary 3.12. Note that for each k, $\boldsymbol{\ell}$ and $\boldsymbol{s}$, the exponent of q_i becomes

$$\beta_{\boldsymbol{\ell},\boldsymbol{s},k} + \gamma_{\boldsymbol{\ell},\boldsymbol{s},k}$$

$$= -a_{ij}\zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(1-a_{ij}-k)} - 2\sum_{r=1}^{1-a_{ij}} \zeta_{\boldsymbol{\ell},\boldsymbol{s}}^{(r-1)}\big(\ell_r + (1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}}) + (1-\ell_r)s_{r-|\boldsymbol{\ell}|_{1;r}}\big)$$

$$= \theta_{\boldsymbol{\ell},\boldsymbol{s},k}. \qquad \square$$

Similarly, this leads us to an explicit expression for the structure constants $\rho_{m,m',t}^{(i,j,a_{ij})}$ for the first part of $C_{ij}(\mathbf{c})$ for Case 2.

Corollary 3.14. *Let $i \in I \setminus X$ be such that $\tau(i) = i$, $j \in X$ and m, m' and t as fixed before. Then the structure constants $\rho_{m,m',t}^{(i,j,a_{ij})}$ are obtained from the expression (83) upon replacing $\mathfrak{S}_{m,m'}$ by $\mathfrak{S}_{m,m',k,t}$ defined in (56).*

Proof. This follows upon comparing (51) with (55). $\qquad \square$

It can readily be checked that these expressions comply with the values computed in [27] and [2], as displayed in Tables 1 and 2.

3.3. Case 2: $\tau(i) = i \in I \setminus X$ and $j \in X$. Consequently, we will obtain the second part of the polynomial $C_{ij}(\mathbf{c})$ for Case 2, as described by the last line of (62). To this end, let us fix $i \in I \setminus X$ such that:

$$\tau(i) = i, \quad j \in X, \quad m \in \{0, \ldots, -1 - a_{ij}\}, \quad t \in \left\{0, \ldots, \frac{-1 - a_{ij} - m}{2}\right\},$$

$$k \in \{1, \ldots, 1 - a_{ij}\}, \quad d \in \{0, \ldots, k - 1\}, \quad m' \in \{0, \ldots, m\},$$

$$\boldsymbol{\ell} \in \mathfrak{L}'_{m,m',k,d} \quad \text{and} \quad \boldsymbol{s} \in \mathfrak{S}'_{m,m',k,t,d}.$$

By (63), the calculation of the structure constants $\sigma_{m,t}^{(i,j,a_{ij})}$ comes down to computing the action of $\epsilon \circ P_{-\lambda_{ij}}$ on $\mathfrak{r}_{\boldsymbol{\ell},\boldsymbol{s},k,d}^{(i,j,a_{ij})}$, defined in (59). This will be the subject of the present subsection.

As a first step, we will again shift all factors K_i^{-1} in $\mathfrak{r}_{\boldsymbol{\ell},\boldsymbol{s},k,d}^{(i,j,a_{ij})}$ to the back, as we have done for $\mathfrak{p}_{\boldsymbol{\ell},\boldsymbol{s},k}^{(i,j,a_{ij})}$ in Proposition 3.10. Recall the notation $\widetilde{E}_i = (q_i - q_i^{-1})E_i$ and let us write, as an extension of (65),

$$\widehat{P}_{N,M}^{i,j} : U_q(\mathfrak{g}') \to U^+ K_i^N K_j^M S(U^-), \tag{84}$$

for the projection operator with respect to the decomposition (33), where $M, N \in \mathbb{Z}$.

Proposition 3.15. *For $i, j, m, t, k, d, m', \ell$ and $\mathbf{s}$ as fixed before, we have*

$$(\epsilon \circ P_{-\lambda_{ij}})\big(\mathfrak{r}_{\ell,s,k,d}^{(i,j,a_{ij})}\big) = \frac{q_i^{\eta_{\ell,s,k,d,t,m'}}}{(q_i - q_i^{-1})^{(1-a_{ij}-m)/2}(q_j - q_j^{-1})}$$
$$\big(\epsilon \circ \widehat{P}_{-(1-a_{ij}-m)/2,-1}^{i,j}\big)\big(Y_{\ell,s,k,d}^{(0)} - q_i^{a_{ij}} Y_{\ell,s,k,d}^{(1)}\big),$$

where

$$Y_{\ell,s,k,d}^{(0)} = \left(\overrightarrow{\prod_{r=1}^{1-a_{ij}-k}} \mathfrak{v}_{\ell,s,r}^i\right) F_j \left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mathfrak{v}_{\ell,s,r}^i\right) \widetilde{E}_j \widetilde{E}_i \left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}} \mathfrak{v}_{\ell,s,r}^i\right),$$

(85)

$$Y_{\ell,s,k,d}^{(1)} = \left(\overrightarrow{\prod_{r=1}^{1-a_{ij}-k}} \mathfrak{v}_{\ell,s,r}^i\right) F_j \left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mathfrak{v}_{\ell,s,r}^i\right) \widetilde{E}_i \widetilde{E}_j \left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}} \mathfrak{v}_{\ell,s,r}^i\right),$$

(86)

$$\mathfrak{v}_{\ell,s,r}^i = F_i^{(1-\ell_r)s_{r-|\ell|_{1;r}}} \widetilde{E}_i^{(1-\ell_r)(1-s_{r-|\ell|_{1;r}})}, \tag{87}$$

$$\eta_{\ell,s,k,d,t,m'} = -2\zeta_{\ell,s}^{(1-a_{ij}-k+d)} - 2\sum_{r=1}^{-a_{ij}} \zeta_{\ell,s}^{(r-1)}\big(\ell_r + (1-\ell_r)(1 - s_{r-|\ell|_{1;r}})\big)$$
$$- a_{ij}\big(1 + a_{ij} + k + m' + t + |s|_{1;1-a_{ij}-k-|\ell|_{1;1-a_{ij}-k}}\big) - 2(m' + t),$$

(88)

with $\zeta_{\ell,s}^{(r)}$ as in (69).

Proof. Let us start by shifting the factor K_i^{-1} arising from $(E_j E_i - q_i^{a_{ij}} E_i E_j)K_i^{-1}$ in (59) to the back. Reasoning as in the proof of Proposition 3.10, this induces a factor q_i^{-2x}, where

$$x = \#(\text{factors } F_i \text{ preceding } K_i^{-1}) - \#(\text{factors } E_i \text{ preceding } K_i^{-1})$$
$$= |s|_{1;1-a_{ij}-k+d-|\ell|_{1;1-a_{ij}-k+d}}$$
$$\quad - \big(1 - a_{ij} - k + d - |\ell|_{1;1-a_{ij}-k+d} - |s|_{1;1-a_{ij}-k+d-|\ell|_{1;1-a_{ij}-k+d}}\big) - 1$$
$$= \zeta_{\ell,s}^{(1-a_{ij}-k+d)} - 1,$$

where the -1 comes from the factor E_i in $(E_j E_i - q_i^{a_{ij}} E_i E_j)K_i^{-1}$.

Now let us perform the same shifting for the factors $K_i^{-\ell_r}$ with $\ell_r = 1$, which leads to a factor $q_i^{x_r'}$, where this time x_r' depends on r. In general, we have

$$x_r' = -2\big(\#(\text{factors } F_i \text{ preceding } K_i^{-\ell_r}) - \#(\text{factors } E_i \text{ preceding } K_i^{-\ell_r})\big)$$
$$- a_{ij}\big(\#(\text{factors } F_j \text{ preceding } K_i^{-\ell_r}) - \#(\text{factors } E_j \text{ preceding } K_i^{-\ell_r})\big),$$

again since $\mathfrak{r}^{(i,j,a_{ij})}_{\ell,s,k,d}$ contains an equal number of factors F_i and E_i, and precisely 1 factor F_j and 1 factor E_j. For $r \in \{1, \ldots, 1 - a_{ij} - k\}$ we have

$$x'_r = -2\big(|s|_{r-1-|\ell|_{1;r-1}} - \big(r - 1 - |\ell|_{1;r-1} - |s|_{1;r-1-|\ell|_{1;r-1}}\big)\big) = -2\zeta^{(r-1)}_{\ell,s}.$$

For $r \in \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\}$ on the other hand, by the same reasoning this becomes

$$x'_r = -2\zeta^{(r-1)}_{\ell,s} - a_{ij},$$

whereas for $r \in \{2 - a_{ij} - k + d, \ldots, -a_{ij}\}$ one has

$$x'_r = -2\big(\zeta^{(r-1)}_{\ell,s} - 1\big),$$

where the -1 arises from the factor E_i in $E_j E_i - q_i^{a_{ij}} E_i E_j$.

Finally, this shifting process for the factor K_i^{-1} in $(E_i K_i^{-1})^{(1-\ell_r)(1-s_{r-|\ell|_{1;r}})}$ with $\ell_r = 0$ and $s_{r-|\ell|_{1;r}} = 0$ induces a factor $q_i^{x''_r}$, with, reasoning as above,

$$x''_r = \begin{cases} -2\big(\zeta^{(r-1)}_{\ell,s} - 1\big) & \text{for } r \in \{1, \ldots, 1 - a_{ij} - k\}, \\ -2\big(\zeta^{(r-1)}_{\ell,s} - 1\big) - a_{ij} & \text{for } r \in \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\}, \\ -2\big(\zeta^{(r-1)}_{\ell,s} - 2\big) & \text{for } r \in \{2 - a_{ij} - k + d, \ldots, -a_{ij}\}. \end{cases}$$

In total, this shifting gives rise to a factor q_i^{η}, with

$$\eta = -2\zeta^{(1-a_{ij}-k+d)}_{\ell,s} + 2 - 2 \sum_{r=1}^{-a_{ij}} \zeta^{(r-1)}_{\ell,s}\big(\ell_r + (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}})\big)$$

$$- a_{ij} \sum_{r=2-a_{ij}-k}^{1-a_{ij}-k+d} \big(\ell_r + (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}})\big)$$

$$+ 2 \sum_{r=2-a_{ij}-k+d}^{-a_{ij}} \big(\ell_r + (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}})\big) + 2 \sum_{r=1}^{-a_{ij}} (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}})$$

$$= -2\zeta^{(1-a_{ij}-k+d)}_{\ell,s} + 2 - 2 \sum_{r=1}^{-a_{ij}} \zeta^{(r-1)}_{\ell,s}\big(\ell_r + (1 - \ell_r)(1 - s_{r-|\ell|_{1;r}})\big)$$

$$- a_{ij}\big(m' + t - (1 - a_{ij} - k) + |s|_{1;1-a_{ij}-k-|\ell|_{1;1-a_{ij}-k}}\big)$$

$$+ 2\left(\frac{1 - a_{ij} + m}{2} - m' - t - 1\right) + (-1 - a_{ij} - m),$$

in agreement with (88), where we have used (61) and the definition (64) of $\mathcal{S}'_{m,m',k,t,d}$.

Finally, the renormalization (66) gives rise to a factor

$$(q_i - q_i^{-1})^{-(1-a_{ij}-m)/2}(q_j - q_j^{-1})^{-1},$$

since by (61) we have

$$\#(\text{factors } E_i) = \#(\text{factors } F_i) = |s| = \frac{1 - a_{ij} - m}{2}$$

and, of course, $\#(\text{factors } E_j) = \#(\text{factors } F_j) = 1$. So we find

$$\mathfrak{r}_{\ell,s,k,d}^{(i,j,a_{ij})} = \frac{q_i^{\eta_{\ell,s,k,d,t,m'}}}{(q_i - q_i^{-1})^{(1-a_{ij}-m)/2}(q_j - q_j^{-1})}\big(Y_{\ell,s,k,d}^{(0)} - q_i^{a_{ij}} Y_{\ell,s,k,d}^{(1)}\big)K_i^{-(1-a_{ij}+m)/2},$$

which yields the claim by (22), (34), (35), and (84). $\qquad\square$

We have hence reduced our problem to computing how $\epsilon \circ \hat{P}_{-(1-a_{ij}-m)/2,-1}^{i,j}$ acts on $Y_{\ell,s,k,d}^{(0)} - q_i^{a_{ij}} Y_{\ell,s,k,d}^{(1)}$. Each of the latter terms is a product of an equal number of factors F_i and $\widetilde{E}_i$ and precisely one factor F_j and $\widetilde{E}_j$, which is balanced in the sense that up to each position in the product, the number of factors F_i exceeds or equals the number of factors $\widetilde{E}_i$, and that the factor F_j precedes the factor $\widetilde{E}_j$. The presence of F_j and $\widetilde{E}_j$ now complicates matters substantially in comparison to the situation in Case 1, because F_i does not commute with F_j and similarly for $\widetilde{E}_i$ and $\widetilde{E}_j$. We will need to derive an analog of Lemma 3.11 which takes into account the presence of these factors.

Recall the notation $(N)_{q_i^2}$ for the modified q_i^2-number (74) and let us also define

$$\alpha_N = (N)_{q_i^2} q_i^{-2N+2}, \tag{89}$$

$$\gamma_{M,N} = (N - M)_{q_i^2} q_i^{-a_{ij}-2N+2}, \tag{90}$$

for $M, N \in \mathbb{N}$. Write also $\alpha_N = 0$ for $N < 0$. Then one can prove the following result.

Lemma 3.16. *Let $M \in \mathbb{N}$ be such that $M \geq 1$. Let $Y \in U_q(\mathfrak{g}')$ be a product of M factors F_i, M factors $\widetilde{E}_i$, 1 factor F_j and 1 factor $\widetilde{E}_j$, appearing in any order but with F_i as its first N_0 factors, for some $N_0 \in \mathbb{N}$, followed by a factor F_j. Let $N_1 \in \mathbb{N}$ be maximal such that the first N_1 factors of Y succeeding F_j are F_i, so that we can write $Y = F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i X$, for some $X \in U_q(\mathfrak{g}')$. Then we have*

$$\big(\epsilon \circ \hat{P}_{-M,-1}^{i,j}\big)(Y) = \alpha_{N_0}\big(\epsilon \circ \hat{P}_{-(M-1),-1}^{i,j}\big)\big(F_i^{N_0-1} F_j F_i^{N_1} X\big)$$
$$+ \gamma_{N_0,N_0+N_1}\big(\epsilon \circ \hat{P}_{-(M-1),-1}^{i,j}\big)\big(F_i^{N_0} F_j F_i^{N_1-1} X\big).$$

Proof. We prove this by induction on N_1. As before, our strategy will be to write Y in its standard ordering, i.e. as a $\mathbb{K}(q)$-linear combination of

$$\widetilde{E}_i^{m_1} \widetilde{E}_j^{\delta} \widetilde{E}_i^{m_2} F_i^{m_3} F_j^{\delta} F_i^{m_4} K_i^{m_5} K_j^{\delta'},$$

with $m_1, \ldots, m_4 \in \mathbb{N}$, $m_5 \in \mathbb{Z}$, $\delta \in \{0, 1\}$ and $\delta' \in \{-1, 0, 1\}$, and then observe that, for any $M' \in \mathbb{Z}$, one has

$$
\begin{aligned}
&\left(\epsilon \circ \widehat{P}^{i,j}_{-M',-1}\right)\left(\widetilde{E}_i^{m_1} \widetilde{E}_j^{\delta} \widetilde{E}_i^{m_2} F_i^{m_3} F_j^{\delta} F_i^{m_4} K_i^{m_5} K_j^{\delta'}\right) \\
&\quad = \begin{cases} 1 & \text{if } m_1 = m_2 = m_3 = m_4 = \delta = 0, m_5 = -M', \text{ and } \delta' = -1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{91}
$$

by (84) and (22). Hence, $\epsilon \circ \widehat{P}^{i,j}_{-M',-1}$ in fact projects Y onto the coefficient of $K_i^{-M'} K_j^{-1}$ in its standard ordering.

For $N_1 = 0$, we may write

$$
Y = F_i^{N_0} \widetilde{E}_i F_j X = F_i^{N_0} \widetilde{E}_i X'
$$

since F_j and $\widetilde{E}_i$ commute. A straightforward generalization of Lemma 3.11 then asserts

$$
\left(\epsilon \circ \widehat{P}^{i,j}_{-M,-1}\right)(Y) = (N_0)_{q_i^2} q_i^{-2N_0+2} \left(\epsilon \circ \widehat{P}^{i,j}_{-(M-1),-1}\right)\left(F_i^{N_0-1} X'\right),
$$

which yields the claim since $X' = F_j X$ and by the definition (89) of α_{N_0} and the fact that $\gamma_{N_0, N_0} = 0$.

Suppose now the claim has been proven for $N_1 - 1 \geq 0$. Note that by (19) and (66) we have

$$
Y = F_i^{N_0} F_j F_i^{N_1-1} \widetilde{E}_i X' - F_i^{N_0} F_j F_i^{N_1-1} K_i X + F_i^{N_0} F_j F_i^{N_1-1} K_i^{-1} X,
$$

with $X' = F_i X$.

The second term will not contribute, since its standard ordering cannot contain a multiple of $K_i^{-M} K_j^{-1}$. Indeed, this term contains $M - 1$ factors F_i and the same number of factors $\widetilde{E}_i$, and each factor F_i can only contribute one factor K_i^{-1} to the normal ordering upon combining it with a factor $\widetilde{E}_i$, by (19). Hence, the lowest possible power of K_i occurring in the standard ordering of this term will be

$$
-(M-1) + 1 > -M.
$$

The third term contains again as many F_i as $\widetilde{E}_i$ and can whence be rewritten as

$$
q_i^x F_i^{N_0} F_j F_i^{N_1-1} X K_i^{-1},
$$

with

$$
\begin{aligned}
x &= -2\big(\#(\text{factors } F_i \text{ preceding } K_i^{-1}) - \#(\text{factors } \widetilde{E}_i \text{ preceding } K_i^{-1})\big) \\
&\quad - a_{ij}\big(\#(\text{factors } F_j \text{ preceding } K_i^{-1}) - \#(\text{factors } \widetilde{E}_j \text{ preceding } K_i^{-1})\big) \\
&= -2(N_0 + N_1 - 1) - a_{ij}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\big(\epsilon \circ \widehat{P}^{i,j}_{-M,-1}\big)(Y) &= \big(\epsilon \circ \widehat{P}^{i,j}_{-M,-1}\big)\big(F_i^{N_0} F_j F_i^{N_1-1} \widetilde{E}_i X'\big) \\
&\quad + q_i^{-2(N_0+N_1-1)-a_{ij}} \big(\epsilon \circ \widehat{P}^{i,j}_{-(M-1),-1}\big)\big(F_i^{N_0} F_j F_i^{N_1-1} X\big).
\end{aligned}
$$

As $F_i^{N_0} F_j F_i^{N_1-1} \widetilde{E}_i X'$ still contains M factors F_i and the same number of factors $\widetilde{E}_i$, and meets all other requirements of the statement as well, we may apply the induction hypothesis to write

$$
\begin{aligned}
\big(\epsilon \circ \widehat{P}^{i,j}_{-M,-1}\big)\big(F_i^{N_0} F_j F_i^{N_1-1} \widetilde{E}_i X'\big) &= \alpha_{N_0}\big(\epsilon \circ \widehat{P}^{i,j}_{-(M-1),-1}\big)\big(F_i^{N_0-1} F_j F_i^{N_1-1} X'\big) \\
&\quad + \gamma_{N_0,N_0+N_1-1}\big(\epsilon \circ \widehat{P}^{i,j}_{-(M-1),-1}\big)\big(F_i^{N_0} F_j F_i^{N_1-2} X'\big).
\end{aligned}
$$

The statement now follows from $X' = F_i X$ and the fact that

$$
\gamma_{N_0,N_0+N_1-1} + q_i^{-2(N_0+N_1-1)-a_{ij}} = \gamma_{N_0,N_0+N_1}. \qquad \square
$$

The formula obtained in Lemma 3.11 can easily be iterated, since its right-hand side consists of only one term, leading to a product iteration of the form (76). The formula obtained in Lemma 3.16, however, is much more complicated, since its right-hand side consists of two different terms, each containing a projection operator and the counit ϵ. One iteration of Lemma 3.16 hence leads to a right-hand side containing three terms. Indeed, if $Y = F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i F_i^{N_2} \widetilde{E}_i X$ is of the type described in Lemma 3.16, then

$$
\begin{aligned}
\big(\epsilon \circ \widehat{P}^{i,j}_{-M,-1}\big)(Y) &= \alpha_{N_0}\alpha_{N_0-1}\big(\epsilon \circ \widehat{P}^{i,j}_{-(M-2),-1}\big)\big(F_i^{N_0-2} F_j F_i^{N_1+N_2} X\big) \\
&\quad + \alpha_{N_0}\big(\gamma_{N_0-1,N_0+N_1+N_2-1} + \gamma_{N_0,N_0+N_1}\big) \\
&\qquad\qquad \big(\epsilon \circ \widehat{P}^{i,j}_{-(M-2),-1}\big)\big(F_i^{N_0-1} F_j F_i^{N_1+N_2-1} X\big) \\
&\quad + \gamma_{N_0,N_0+N_1}\gamma_{N_0,N_0+N_1+N_2-1}\big(\epsilon \circ \widehat{P}^{i,j}_{-(M-2),-1}\big)\big(F_i^{N_0} F_j F_i^{N_1+N_2-2} X\big).
\end{aligned}
$$

A second iteration will then lead to four terms in the right-hand side and so on. Meanwhile, the occurring coefficients become increasingly intricate at each further iteration. To describe the full outcome after T iterations, for any $T \in \mathbb{N}$, let us introduce the notation

$$
c^{(b)}_{a,N} = \sum_{p_1 \leq p_2 \leq \cdots \leq p_{b-a}=0}^{a} \prod_{r=1}^{b-a} \gamma_{N_0-a+p_r,|N|_{0;b-p_r-r+1}-(b-p_r-r)}, \qquad (92)
$$

where $a < b \in \mathbb{N}$,

$$
N = (N_0, N_1, \ldots, N_{b+1}) \in \mathbb{N}^{b+2} \quad \text{and} \quad |N|_{0;p} = N_0 + N_1 + \cdots + N_p.
$$

We also set $c^{(a)}_{a,N} = 1$.

Proposition 3.17. *Let $M \in \mathbb{N}$ be such that $M \geq 1$. Let $Y \in U_q(\mathfrak{g}')$ be a product of M factors F_i, M factors $\widetilde{E}_i$, 1 factor F_j and 1 factor $\widetilde{E}_j$, of the form*

$$Y = F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i F_i^{N_2} \widetilde{E}_i \ldots F_i^{N_T} \widetilde{E}_i \widetilde{E}_j X, \tag{93}$$

for some $X \in U_q(\mathfrak{g}')$, where $\boldsymbol{N} = (N_0, N_1, \ldots, N_T) \in \mathbb{N}^{T+1}$ and $T \geq 1$. Then we have

$$\big(\epsilon \circ \widehat{P}_{-M,-1}^{i,j}\big)(Y) = \upsilon_{\boldsymbol{N}} \big(\epsilon \circ \widehat{P}_{-(M-T)}^{i}\big)\big(F_i^{|\boldsymbol{N}|_{0;T}-T} X\big),$$

with

$$\upsilon_{\boldsymbol{N}} = \sum_{u=\max(0,T-|\boldsymbol{N}|_{1;T}-1)}^{T-1} q_i^{-a_{ij}(N_0-u)} c_{u,\boldsymbol{N}}^{(T-1)} \big(q_i^{a_{ij}} \alpha_{N_0-u} + \gamma_{N_0-u,|\boldsymbol{N}|_{0;T}-(T-1)}\big) \bigg(\prod_{r=0}^{u-1} \alpha_{N_0-r}\bigg).$$

Proof. We will prove this by induction on T. For $T = 1$, we have

$$Y = F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i \widetilde{E}_j X,$$

and so its follows from Lemma 3.16 that

$$\big(\epsilon \circ \widehat{P}_{-M,-1}^{i,j}\big)(Y) = \alpha_{N_0} \big(\epsilon \circ \widehat{P}_{-(M-1),-1}^{i,j}\big)\big(F_i^{N_0-1} F_j \widetilde{E}_j F_i^{N_1} X\big)$$
$$+ \gamma_{N_0,N_0+N_1} \big(\epsilon \circ \widehat{P}_{-(M-1),-1}^{i,j}\big)\big(F_i^{N_0} F_j \widetilde{E}_j F_i^{N_1-1} X\big),$$

where we have used the fact that F_i and $\widetilde{E}_j$ commute. When rewriting $F_j \widetilde{E}_j$ in its standard ordering via

$$F_j \widetilde{E}_j = \widetilde{E}_j F_j - K_j + K_j^{-1},$$

only the last term will contribute by (91), so we may replace $F_j \widetilde{E}_j$ by K_j^{-1} in the equation above. Since both $F_i^{N_0-1} K_j^{-1} F_i^{N_1} X$ and $F_i^{N_0} K_j^{-1} F_i^{N_1-1} X$ contain as many F_i as $\widetilde{E}_i$, it is evident that

$$F_i^{N_0-1} K_j^{-1} F_i^{N_1} X = q_i^{-a_{ij}(N_0-1)} F_i^{N_0+N_1-1} X K_j^{-1},$$
$$F_i^{N_0} K_j^{-1} F_i^{N_1-1} X = q_i^{-a_{ij} N_0} F_i^{N_0+N_1-1} X K_j^{-1}.$$

Hence, it follows from (65) and (84) that

$$\big(\epsilon \circ \widehat{P}_{-M,-1}^{i,j}\big)(Y) = q_i^{-a_{ij} N_0} \big(q_i^{a_{ij}} \alpha_{N_0} + \gamma_{N_0,N_0+N_1}\big)\big(\epsilon \circ \widehat{P}_{-(M-1)}^{i}\big)\big(F_i^{N_0+N_1-1} X\big),$$

which agrees with the claim since $c_{0,(N_0,N_1)}^{(0)} = 1$ and $T - |\boldsymbol{N}|_{1;T} - 1 \leq 0$ for $T = 1$.

Suppose now the claim has been proven for $T \geq 1$ and set

$$Y = F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i F_i^{N_2} \widetilde{E}_i \ldots F_i^{N_T} \widetilde{E}_i F_i^{N_{T+1}} \widetilde{E}_i \widetilde{E}_j X.$$

Then Lemma 3.16 asserts

$$\big(\epsilon \circ \widehat{P}_{-M,-1}^{i,j}\big)(Y) = \alpha_{N_0} \big(\epsilon \circ \widehat{P}_{-(M-1),-1}^{i,j}\big)(Y') + \gamma_{N_0,N_0+N_1} \big(\epsilon \circ \widehat{P}_{-(M-1),-1}^{i,j}\big)(Y''),$$

with

$$Y' = F_i^{N_0-1} F_j F_i^{N_1+N_2} \widetilde{E}_i F_i^{N_3} \widetilde{E}_i \ldots F_i^{N_T+1} \widetilde{E}_i \widetilde{E}_j X,$$
$$Y'' = F_i^{N_0} F_j F_i^{N_1+N_2-1} \widetilde{E}_i F_i^{N_3} \widetilde{E}_i \ldots F_i^{N_T+1} \widetilde{E}_i \widetilde{E}_j X.$$

Both Y' and Y'' satisfy the requirements of the statement: they each contain $M - 1$ factors F_i, the same number of factors $\widetilde{E}_i$, 1 factor F_j and 1 factor $\widetilde{E}_j$, and they are of the form (93) with

$$N' = (N_0 - 1, N_1 + N_2, N_3, \ldots, N_{T+1})$$
and
$$N'' = (N_0, N_1 + N_2 - 1, N_3, \ldots, N_{T+1}),$$

respectively. Both $N_0 - 1$ and $N_1 + N_2 - 1$ might become negative, but in this case the corresponding coefficients α_{N_0} and γ_{N_0,N_0+N_1} will vanish. We may hence assume that $N', N'' \in \mathbb{N}^{T+1}$ and apply the induction hypothesis to obtain

$$\big(\epsilon \circ \widehat{P}^{i,j}_{-M,-1}\big)(Y) = \Theta_{N',N''}\big(\epsilon \circ \widehat{P}^i_{-(M-T-1)}\big)\big(F_i^{|N|_{0;T+1}-(T+1)} X\big),$$

where $\Theta_{N',N''}$ is given by

$$\alpha_{N_0}\left[\sum_{u=\max(0,T-|N|_{1;T+1}-1)}^{T-1} q_i^{-a_{ij}(N_0-1-u)} c_{u,N'}^{(T-1)} \right.$$
$$\left. \big(q_i^{a_{ij}} \alpha_{N_0-1-u} + \gamma_{N_0-1-u,|N|_{0;T+1}-T}\big)\Big(\prod_{r=0}^{u-1} \alpha_{N_0-1-r}\Big) \right]$$
$$+ \gamma_{N_0,N_0+N_1}\left[\sum_{u=\max(0,T-|N|_{1;T+1})}^{T-1} q_i^{-a_{ij}(N_0-u)} c_{u,N''}^{(T-1)} \right.$$
$$\left. \big(q_i^{a_{ij}} \alpha_{N_0-u} + \gamma_{N_0-u,|N|_{0;T+1}-T}\big)\Big(\prod_{r=0}^{u-1} \alpha_{N_0-r}\Big) \right], \quad (94)$$

where we have used the fact that $|N'|_{0;T} - T = |N''|_{0;T} - T = |N|_{0;T+1} - (T+1)$. It now suffices to show that

$$\Theta_{N',N''} = \left[\sum_{u=\max(0,T-|N|_{1;T+1})}^{T} q_i^{-a_{ij}(N_0-u)} c_{u,N}^{(T)} \right.$$
$$\left. \big(q_i^{a_{ij}} \alpha_{N_0-u} + \gamma_{N_0-u,|N|_{0;T+1}-T}\big)\Big(\prod_{r=0}^{u-1} \alpha_{N_0-r}\Big) \right]. \quad (95)$$

Upon replacing the summation index u in the first line in (94) by $u' = u + 1$, which we thereafter rename to u again, this term becomes

$$
\alpha_{N_0}\Bigg[\sum_{u=\max(1,T-|\boldsymbol{N}|_{1;T+1})}^{T} q_i^{-a_{ij}(N_0-u)} c_{u-1,\boldsymbol{N}'}^{(T-1)} \\
\left(q_i^{a_{ij}} \alpha_{N_0-u} + \gamma_{N_0-u,|\boldsymbol{N}|_{0;T+1}-T} \right) \left(\prod_{r=0}^{u-2} \alpha_{N_0-1-r} \right) \Bigg] \quad (96)
$$

and it is immediate that $\alpha_{N_0}\big(\prod_{r=0}^{u-2}\alpha_{N_0-1-r}\big) = \prod_{r=0}^{u-1}\alpha_{N_0-r}$. Replacing the first line of (94) by (96) and separating the term corresponding to $u = 0$ in the second line and the one with $u = T$ in the first line, we find that $\Theta_{\boldsymbol{N}',\boldsymbol{N}''}$ is equal to

$$
q_i^{-a_{ij}N_0} c_{0,\boldsymbol{N}''}^{(T-1)} \gamma_{N_0,N_0+N_1} \big(q_i^{a_{ij}} \alpha_{N_0} + \gamma_{N_0,|\boldsymbol{N}|_{0;T+1}-T} \big) \nu_{\boldsymbol{N},T}
$$
$$
+ \sum_{u=\max(1,T-|\boldsymbol{N}|_{1;T+1})}^{T-1} \Bigg[q_i^{-a_{ij}(N_0-u)} \big(c_{u-1,\boldsymbol{N}'}^{(T-1)} + \gamma_{N_0,N_0+N_1} c_{u,\boldsymbol{N}''}^{(T-1)} \big)
$$
$$
\big(q_i^{a_{ij}} \alpha_{N_0-u} + \gamma_{N_0-u,|\boldsymbol{N}|_{0;T+1}-T} \big) \left(\prod_{r=0}^{u-1} \alpha_{N_0-r} \right) \Bigg]
$$
$$
+ q_i^{-a_{ij}(N_0-T)} c_{T-1,\boldsymbol{N}'}^{(T-1)} \big(q_i^{a_{ij}} \alpha_{N_0-T} + \gamma_{N_0-T,|\boldsymbol{N}|_{0;T+1}-T} \big) \left(\prod_{r=0}^{T-1} \alpha_{N_0-r} \right), \quad (97)
$$

with

$$
\nu_{\boldsymbol{N},T} = \begin{cases} 1 & \text{if } T - |\boldsymbol{N}|_{1;T+1} \le 0, \\ 0 & \text{otherwise.} \end{cases}
$$

By definition of $c_{a,\boldsymbol{N}}^{(b)}$ we have that $c_{T-1,\boldsymbol{N}'}^{(T-1)} = c_{T,\boldsymbol{N}}^{(T)} = 1$, such that the last line in (97) agrees with the term in the right-hand side of (95) corresponding to $u = T$. Hence, it suffices to prove the following two claims:

$$
c_{0,\boldsymbol{N}}^{(T)} = c_{0,\boldsymbol{N}''}^{(T-1)} \gamma_{N_0,N_0+N_1}, \quad (98)
$$
$$
c_{u,\boldsymbol{N}}^{(T)} = c_{u-1,\boldsymbol{N}'}^{(T-1)} + \gamma_{N_0,N_0+N_1} c_{u,\boldsymbol{N}''}^{(T-1)}, \quad (99)
$$

for all $u \in \{\max(1, T - |\boldsymbol{N}|_{1;T+1}), \dots, T - 1\}$.

It follows immediately from (92) that one has

$$
c_{0,\boldsymbol{N}''}^{(T-1)} \gamma_{N_0,N_0+N_1} = \left(\prod_{r=1}^{T-1} \gamma_{N_0,|\boldsymbol{N}''|_{0;T-r}-(T-r-1)} \right) \gamma_{N_0,N_0+N_1}.
$$

The definition of N'' asserts that

$$|N''|_{0;T-r} = |N|_{0;T-r+1} - 1$$

for any $r \in \{1, \dots, T-1\}$, and hence

$$c_{0,N''}^{(T-1)} \gamma_{N_0,N_0+N_1} = \prod_{r=1}^{T} \gamma_{N_0,|N|_{0;T-r+1}-(T-r)} = c_{0,N}^{(T)},$$

which proves (98).

Now let $u \in \{\max(1, T - |N|_{1;T+1}), \dots, T-1\}$ be fixed. By (92) we have

$$c_{u-1,N'}^{(T-1)} = \sum_{p_1 \le p_2 \le \cdots \le p_{T-u}=0}^{u-1} \prod_{r=1}^{T-u} \gamma_{N_0-u+p_r,|N'|_{0;T-p_r-r}-(T-p_r-r-1)},$$

where we have used the fact that $N_0' = N_0 - 1$. Now since for every occurring r one has $r \le T - u$ and $p_r \le u - 1$, we have that $T - p_r - r \ge 1$. Hence,

$$|N'|_{0;T-p_r-r} = |N|_{0;T-p_r-r+1} - 1,$$

such that

$$c_{u-1,N'}^{(T-1)} = \sum_{p_1 \le p_2 \le \cdots \le p_{T-u}=0}^{u-1} \prod_{r=1}^{T-u} \gamma_{N_0-u+p_r,|N|_{0;T+1-p_r-r}-(T-p_r-r)}. \qquad (100)$$

It is evident that one has

$$\{(p_1, \dots, p_{T-u-1}, p_{T-u}) \in \mathbb{N}^{T-u} : 0 \le p_1 \le \cdots \le p_{T-u-1} \le p_{T-u} \le u\}$$
$$= \{(p_1, \dots, p_{T-u-1}, p_{T-u}) \in \mathbb{N}^{T-u} : 0 \le p_1 \le \cdots \le p_{T-u-1} \le p_{T-u} \le u-1\}$$
$$\cup \{(p_1, \dots, p_{T-u-1}, u) \in \mathbb{N}^{T-u} : 0 \le p_1 \le \cdots \le p_{T-u-1} \le u\}.$$

Hence, (100) implies

$$c_{u-1,N'}^{(T-1)} = \sum_{p_1 \le p_2 \le \cdots \le p_{T-u}=0}^{u} \prod_{r=1}^{T-u} \gamma_{N_0-u+p_r,|N|_{0;T+1-p_r-r}-(T-p_r-r)} \qquad (101)$$

$$- \gamma_{N_0,|N|_{0;1}} \sum_{p_1 \le p_2 \le \cdots \le p_{T-u-1}=0}^{u} \prod_{r=1}^{T-u-1} \gamma_{N_0-u+p_r,|N|_{0;T+1-p_r-r}-(T-p_r-r)},$$

where in the last line we have separated the factor in the product corresponding to $r = T - u$, since here we have set $p_{T-u} = u$. One immediately recognizes the first line as $c_{u,N}^{(T)}$ and, moreover, one has

$$c_{u,N''}^{(T-1)} = \sum_{p_1 \le p_2 \le \cdots \le p_{T-u-1}=0}^{u} \prod_{r=1}^{T-u-1} \gamma_{N_0-u+p_r,|N''|_{0;T-p_r-r}-(T-p_r-r-1)}. \qquad (102)$$

Again every $T - p_r - r \geq 1$, and hence

$$|N''|_{0;T-p_r-r} = |N|_{0;T+1-p_r-r} - 1,$$

such that (102) coincides with the sum in the last line in (101). We conclude that

$$c_{u-1,N'}^{(T-1)} = c_{u,N}^{(T)} - \gamma_{N_0,N_0+N_1} c_{u,N''}^{(T-1)},$$

and so we have shown (99). This concludes the proof.   □

The question now remains how one can apply Proposition 3.17 to compute the action of $\epsilon \circ \hat{P}^{i,j}_{-(1-a_{ij}-m)/2,-1}$ on $Y^{(0)}_{\boldsymbol{\ell},\boldsymbol{s},k,d} - q_i^{a_{ij}} Y^{(1)}_{\boldsymbol{\ell},\boldsymbol{s},k,d}$, as defined in (85)–(86). This will be addressed in the following proposition.

Proposition 3.18. *Let $i, j, m, t, k, d, m', \boldsymbol{\ell}$ and $\boldsymbol{s}$ be as fixed before and let $\lambda \in \{0, 1\}$, then one has*

$$\left(\epsilon \circ \hat{P}^{i,j}_{-(1-a_{ij}-m)/2,-1}\right)\left(Y^{(\lambda)}_{\boldsymbol{\ell},\boldsymbol{s},k,d}\right) = q_i^{-a_{ij}N_0}\left(\prod_{r \in \mathfrak{R}_{k,d}} \left(\alpha_{\zeta^{(r-1)}_{\boldsymbol{\ell},\boldsymbol{s}} - \nu_{r,k}}\right)^{(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})}\right)$$

$$\left(\alpha_{\zeta^{(1-a_{ij}-k+d)}_{\boldsymbol{\ell},\boldsymbol{s}}}\right)^{1-\lambda}\left(\sum_{u=\max(0,\xi_\lambda)}^{T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+\lambda-1}\left[q_i^{a_{ij}u}c_{u,N^{(\lambda)}}^{(T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+\lambda-1)}\right.\right.$$

$$\left.\left.\left(q_i^{a_{ij}}\alpha_{N_0-u} + \gamma_{N_0-u,|N|_{0;T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+\lambda-(T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+\lambda-1)}}\right)\left(\prod_{r=0}^{u-1}\alpha_{N_0-r}\right)\right]\right)^{1-\delta_{T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+\lambda,0}},$$

where $N^{(\lambda)} = (N_0, N_1, \ldots, N_{T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+\lambda})$, with

$$T_{\boldsymbol{\ell},\boldsymbol{s},k,t} = \zeta^{(1-a_{ij}-k)}_{\boldsymbol{\ell},\boldsymbol{s}} + t - |\boldsymbol{s}|_{1;1-a_{ij}-k-|\boldsymbol{\ell}|_{1;1-a_{ij}-k}}, \tag{103}$$

$$N_0 = \zeta^{(1-a_{ij}-k)}_{\boldsymbol{\ell},\boldsymbol{s}}, \tag{104}$$

$$|N|_{1;b} = r_b + a_{ij} + k - b - 1 - |\boldsymbol{\ell}|_{2-a_{ij}-k;r_b}, \tag{105}$$

$$r_b = \sum_{r=2-a_{ij}-k}^{1-a_{ij}-k+d} r(1-\ell_r)(1-s_{r-|\boldsymbol{\ell}|_{1;r}})$$
$$\delta_{r+a_{ij}+k-b-1-|\boldsymbol{\ell}|_{2-a_{ij}-k;r},|\boldsymbol{s}|_{2-a_{ij}-k-|\boldsymbol{\ell}|_{1;2-a_{ij}-k};r-|\boldsymbol{\ell}|_{1;r}} \tag{106}$$

for any $b \in \{1, \ldots, T_{\boldsymbol{\ell},\boldsymbol{s},k,t}\}$, and

$$|N|_{0;T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+1} = \zeta^{(1-a_{ij}-k+d)}_{\boldsymbol{\ell},\boldsymbol{s}} + T_{\boldsymbol{\ell},\boldsymbol{s},k,t}. \tag{107}$$

Moreover,

$$\xi_\lambda = T_{\boldsymbol{\ell},\boldsymbol{s},k,t} - |N|_{1;T_{\boldsymbol{\ell},\boldsymbol{s},k,t}+\lambda} + \lambda - 1,$$

$$\Re_{k,d} = \{1,\ldots,-a_{ij}\} \setminus \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\},$$

$$v_{r,k} = \begin{cases} 0 & \text{if } r \leq 1 - a_{ij} - k, \\ 1 & \text{if } r > 1 - a_{ij} - k. \end{cases} \tag{108}$$

Proof. Following the same reasoning that led us to the formula (81), we find that $\left(\epsilon \circ \widehat{P}^{i,j}_{-(1-a_{ij}-m)/2,-1}\right)\!\left(Y^{(\lambda)}_{\ell,s,k,d}\right)$ is equal to

$$\left(\prod_{r=1}^{1-a_{ij}-k} \alpha_{\zeta^{(r-1)}_{\ell,s}}^{(1-\ell_r)(1-s_{r-|\ell|_{1;r}})} \right)\!\left(\epsilon \circ \widehat{P}^{i,j}_{-x,-1}\right)$$

$$\left[F_i^{\zeta^{(1-a_{ij}-k)}_{\ell,s}} F_j \left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d} \mho^i_{\ell,s,r}} \right) \widetilde{E}_i^\lambda \widetilde{E}_j \widetilde{E}_i^{1-\lambda} \left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}} \mho^i_{\ell,s,r}} \right) \right],$$

where

$$x = \frac{1 - a_{ij} - m}{2} - \#(\text{factors } \widetilde{E}_i \text{ preceding } F_j)$$

$$= \zeta^{(1-a_{ij}-k)}_{\ell,s} - |s|_{2-a_{ij}-k-|\ell|_{1;2-a_{ij}-k};-a_{ij}-m}.$$

To proceed, we will need to write the term between square brackets in the form

$$F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i F_i^{N_2} \widetilde{E}_i \ldots F_i^{N_T} \widetilde{E}_i \widetilde{E}_j X,$$

for some $N_0,\ldots,N_T \in \mathbb{N}$, $T \in \mathbb{N}$, $X \in U_q(\mathfrak{g}')$. It is immediately clear that N_0 agrees with (104). Furthermore, let T be the total number of factors $\widetilde{E}_i$ in $\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mho^i_{\ell,s,r}$ and let us define

$$r_1 < r_2 < \cdots < r_T \in \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\}$$

such that

$$\mho^i_{\ell,s,r_b} = \widetilde{E}_i$$

for all $b \in \{1,\ldots,T\}$. This amounts to saying that $r_1,\ldots,r_T$ are the positions of the factors $\widetilde{E}_i$ in this product. Then for any b one has

$$|N|_{1;b} = \#(\text{elements } r \in \{2 - a_{ij} - k, \ldots, r_b\} \text{ such that } \mho^i_{\ell,s,r} = F_i)$$

$$= \#(\text{elements } r \in \{2 - a_{ij} - k, \ldots, r_b\} \text{ such that } \ell_r = 0 \text{ and } s_{r-|\ell|_{1;r}} = 1)$$

$$= \#(\text{elements } r \in \{2 - a_{ij} - k, \ldots, r_b\})$$

$$\quad - \#(\text{elements } r \in \{2 - a_{ij} - k, \ldots, r_b\} \text{ such that } \ell_r = 1)$$

$$\quad - \#(\text{elements } r \in \{2 - a_{ij} - k, \ldots, r_b\} \text{ such that } \mho^i_{\ell,s,r} = \widetilde{E}_i)$$

$$= (r_b - (1 - a_{ij} - k)) - |\ell|_{2-a_{ij}-k;r_b} - b.$$

Note also that this number is equal to

$$|N|_{1;b} = |s|_{2-a_{ij}-k-|\ell|_{1;2-a_{ij}-k};r_b-|\ell|_{1;r_b}}.$$

Hence, for any $b \in \{1, \ldots, T\}$, the element r_b can be found as the unique

$$r \in \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\}$$

such that $\ell_r = 0$, $s_{r-|\ell|_{1;r}} = 0$ and

$$|s|_{2-a_{ij}-k-|\ell|_{1;2-a_{ij}-k};r-|\ell|_{1;r}} = (r - (1 - a_{ij} - k)) - |\ell|_{2-a_{ij}-k;r} - b.$$

This agrees with (105)–(106). The total number T of factors $\widetilde{E}_i$ in $\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mathcal{V}_{\ell,s,r}^i$ can be found as

$$
\begin{aligned}
T &= \#(r \in \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\}) \\
&\quad - \#(r \in \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\} \text{ such that } \ell_r = 1) \\
&\quad - \#(r \in \{2 - a_{ij} - k, \ldots, 1 - a_{ij} - k + d\} \text{ such that } \ell_r = 0 \text{ and } s_{r-|\ell|_{1;r}} = 1) \\
&= d - |\ell|_{2-a_{ij}-k;1-a_{ij}-k+d} - \sum_{r=2-a_{ij}-k}^{1-a_{ij}-k+d} (1 - \ell_r)s_{r-|\ell|_{1;r}} \\
&= |\ell|_{1;1-a_{ij}-k} + |s|_{1;1-a_{ij}-k-|\ell|_{1;1-a_{ij}-k}} + a_{ij} + k + t - 1,
\end{aligned}
$$

in agreement with (103).

With these notations one may now write

$$
F_i^{\zeta_{\ell,s}^{(1-a_{ij}-k)}} F_j \left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mathcal{V}_{\ell,s,r}^i \right) \widetilde{E}_j \widetilde{E}_i
$$
$$
= F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i F_i^{N_2} \widetilde{E}_i \ldots F_i^{N_{T_{\ell,s,k,t}}} \widetilde{E}_i \widetilde{E}_j F_i^{N_{T_{\ell,s,k,t+1}}} \widetilde{E}_i,
$$

for some $N_{T_{\ell,s,k,t}+1} \in \mathbb{N}$, where we have used the fact that $[F_i, \widetilde{E}_j] = 0$. The analogous term with $\lambda = 1$ becomes

$$
F_i^{\zeta_{\ell,s}^{(1-a_{ij}-k)}} F_j \left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mathcal{V}_{\ell,s,r}^i \right) \widetilde{E}_i \widetilde{E}_j
$$
$$
= F_i^{N_0} F_j F_i^{N_1} \widetilde{E}_i F_i^{N_2} \widetilde{E}_i \ldots F_i^{N_{T_{\ell,s,k,t}}} \widetilde{E}_i F_i^{N_{T_{\ell,s,k,t+1}}} \widetilde{E}_i \widetilde{E}_j,
$$

for the same unknown $N_{T_{\ell,s,k,t}+1} \in \mathbb{N}$. By Proposition 3.17 we thus have, for $\lambda = 0$

$$\left(\epsilon \circ \hat{P}^{i,j}_{-x,-1}\right)\left[F_i^{\zeta^{(1-a_{ij}-k)}_{\ell,s}} F_j\left(\overrightarrow{\prod_{r=2-a_{ij}-k}^{1-a_{ij}-k+d}} \mathcal{v}^i_{\ell,s,r}\right)\widetilde{E}_j\widetilde{E}_i\left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}} \mathcal{v}^i_{\ell,s,r}\right)\right]$$

$$=\left[\sum_{u=\max(0,\xi_0)}^{T_{\ell,s,k,t}-1} q_i^{-a_{ij}(N_0-u)} c_{u,\boldsymbol{N}^{(0)}}^{(T_{\ell,s,k,t}-1)}\left(q_i^{a_{ij}}\alpha_{N_0-u}+\gamma_{N_0-u,|\boldsymbol{N}|_{0;T_{\ell,s,k,t}}-(T_{\ell,s,k,t}-1)}\right)\right.$$

$$\left.\left(\prod_{r=0}^{u-1}\alpha_{N_0-r}\right)\right]^{1-\delta_{T_{\ell,s,k,t},0}}\left(q_i^{-a_{ij}N_0}\right)^{\delta_{T_{\ell,s,k,t},0}}$$

$$\left(\epsilon\circ\hat{P}^i_{-(x-T_{\ell,s,k,t})}\right)\left(F_i^{|\boldsymbol{N}|_{0;T_{\ell,s,k,t}+1}-T_{\ell,s,k,t}}\widetilde{E}_i\left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}}\mathcal{v}^i_{\ell,s,r}\right)\right).$$

Here, we have observed that Proposition 3.17 is only applicable for $T \geq 1$, which explains the power $\delta_{T_{\ell,s,k,t},0}$. The analogous term with $\widetilde{E}_j\widetilde{E}_i$ replaced by $\widetilde{E}_i\widetilde{E}_j$ becomes

$$\left[\sum_{u=\max(0,\xi_1)}^{T_{\ell,s,k,t}} q_i^{-a_{ij}(N_0-u)} c_{u,\boldsymbol{N}^{(1)}}^{(T_{\ell,s,k,t})}\left(q_i^{a_{ij}}\alpha_{N_0-u}+\gamma_{N_0-u,|\boldsymbol{N}|_{0;T_{\ell,s,k,t}+1}-T_{\ell,s,k,t}}\right)\right.$$

$$\left.\left(\prod_{r=0}^{u-1}\alpha_{N_0-r}\right)\right]$$

$$\left(\epsilon\circ\hat{P}^i_{-(x-T_{\ell,s,k,t}-1)}\right)\left(F_i^{|\boldsymbol{N}|_{0;T_{\ell,s,k,t}+1}-T_{\ell,s,k,t}-1}\left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}}\mathcal{v}^i_{\ell,s,r}\right)\right).$$

Note also that we have

$$|\boldsymbol{N}|_{0;T_{\ell,s,k,t}+1}-T_{\ell,s,k,t}=\#(\text{factors } F_i \text{ in } Y^{(0)}_{\ell,s,k,d}\text{ preceding }\widetilde{E}_j\widetilde{E}_i)$$
$$-\#(\text{factors }\widetilde{E}_i\text{ in }Y^{(0)}_{\ell,s,k,d}\text{ preceding }\widetilde{E}_j\widetilde{E}_i)$$
$$=\zeta^{(1-a_{ij}-k+d)}_{\ell,s},$$

which determines the unknown $N_{T_{\ell,s,k,t}+1}$, in agreement with (107). By Lemma 3.11 this also implies

$$\left(\epsilon\circ\hat{P}^i_{-(x-T_{\ell,s,k,t})}\right)\left(F_i^{|\boldsymbol{N}|_{0;T_{\ell,s,k,t}+1}-T_{\ell,s,k,t}}\widetilde{E}_i\left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}}\mathcal{v}^i_{\ell,s,r}\right)\right)$$

$$=\alpha_{\zeta^{(1-a_{ij}-k+d)}_{\ell,s}}\left(\epsilon\circ\hat{P}^i_{-(x-T_{\ell,s,k,t}-1)}\right)\left(F_i^{\zeta^{(1-a_{ij}-k+d)}_{\ell,s}-1}\left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}}\mathcal{v}^i_{\ell,s,r}\right)\right).$$

It now remains only to apply the formula (81) to find

$$
\left(\epsilon \circ \widehat{P}^i_{-(x - T_{\boldsymbol{\ell},\boldsymbol{s},k,t} - 1)}\right) \left(F_i^{\zeta_{\boldsymbol{\ell},s}^{(1-a_{ij}-k+d)}-1} \left(\overrightarrow{\prod_{r=2-a_{ij}-k+d}^{-a_{ij}}} \mathcal{U}^i_{\boldsymbol{\ell},\boldsymbol{s},r} \right) \right)
$$

$$
= \prod_{r=2-a_{ij}-k+d}^{-a_{ij}} \left(\alpha_{\zeta_{\boldsymbol{\ell},s}^{(r-1)}-1} \right)^{(1-\ell_r)(1-s_r - |\boldsymbol{\ell}|_{1;r})}. \qquad \square
$$

With this result, we now have all necessary tools in hand to write down the polynomial $C_{ij}(\mathbf{c})$ for Case 2.

Theorem 3.19 (Case 2). *For any $i \in I \setminus X$ such that $\tau(i) = i$ and any $j \in X$, one has*

$$
F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c})
$$

$$
= \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \sum_{t=0}^{(1-a_{ij}-m-m')/2} \rho_{m,m',t}^{(i,j,a_{ij})} \mathcal{Z}_i^t B_i^m B_j B_i^{m'} \mathcal{Z}_i^{((1-a_{ij}-m-m')/2)-t}
$$

$$
+ \sum_{m=0}^{-1-a_{ij}} \sum_{t=0}^{(-1-a_{ij}-m)/2} \sigma_{m,t}^{(i,j,a_{ij})} \mathcal{Z}_i^t \mathcal{W}_{ij} K_j \mathcal{Z}_i^{((-1-a_{ij}-m)/2)-t} B_i^m, \quad (109)
$$

with $\rho_{m,m',t}^{(i,j,a_{ij})}$ as obtained in Corollary 3.14 and

$$
\sigma_{m,t}^{(i,j,a_{ij})} = (a_{ij} + m)_p\, c_i^{(1-a_{ij}-m)/2}
$$

$$
\sum_{k=1}^{1-a_{ij}} \sum_{d=0}^{k-1} \sum_{m'=0}^{m} \sum_{\boldsymbol{\ell} \in \mathscr{L}'_{m,m',k,d}} \sum_{\boldsymbol{s} \in \mathscr{S}'_{m,m',k,t,d}} (-1)^{k+1} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i}
$$

$$
\frac{q_i^{\kappa_{\boldsymbol{\ell},\boldsymbol{s},k,t,d,m'}}}{(q_i - q_i^{-1})^{(1-a_{ij}-m)/2}(q_j - q_j^{-1})} \left(\prod_{r \in \mathbb{R}_{k,d}} \left(\alpha_{\zeta_{\boldsymbol{\ell},s}^{(r-1)}-v_{r,k}} \right)^{(1-\ell_r)(1-s_r-|\boldsymbol{\ell}|_{1;r})} \right)
$$

$$
\left[\sum_{u=\max(0,\xi_0)}^{T_{\boldsymbol{\ell},\boldsymbol{s},k,t}-1} q_i^{a_{ij}u} \omega_{N^{(0)},u} + \sum_{u=\max(0,\xi_1)}^{T_{\boldsymbol{\ell},\boldsymbol{s},k,t}} q_i^{a_{ij}u} \omega_{N^{(1)},u} \right]^{1-\delta_{T_{\boldsymbol{\ell},\boldsymbol{s},k,t},0}}
$$

$$
\left[\alpha_{\zeta_{\boldsymbol{\ell},s}^{(1-a_{ij}-k+d)}} - q_i^{a_{ij}} \left(q_i^{a_{ij}} \alpha_{N_0} + \gamma_{N_0,N_0+N_1} \right) \right]^{\delta_{T_{\boldsymbol{\ell},\boldsymbol{s},k,t},0}},
$$

where

$$
\omega_{N^{(0)},u} = \alpha_{\zeta_{\boldsymbol{\ell},s}^{(1-a_{ij}-k+d)}} c_{u,N^{(0)}}^{(T_{\boldsymbol{\ell},\boldsymbol{s},k,t}-1)}
$$

$$
\left(q_i^{a_{ij}} \alpha_{N_0-u} + \gamma_{N_0-u,|N|_{0;T_{\boldsymbol{\ell},\boldsymbol{s},k,t}}-(T_{\boldsymbol{\ell},\boldsymbol{s},k,t}-1)} \right) \left(\prod_{r=0}^{u-1} \alpha_{N_0-r} \right),
$$

$$\omega_{N^{(1)},u} = -q_i^{a_{ij}} c_{u,N^{(1)}}^{(T_{\ell,s,k,t})}$$

$$\left(q_i^{a_{ij}} \alpha_{N_0-u} + \gamma_{N_0-u,|N|_{0;T_{\ell,s,k,t}+1}-T_{\ell,s,k,t}}\right)\left(\prod_{r=0}^{u-1} \alpha_{N_0-r}\right),$$

$$\kappa_{\ell,s,k,t,d,m'} = -2\sum_{r=1}^{-a_{ij}} \zeta_{\ell,s}^{(r-1)}\big(\ell_r + (1-\ell_r)(1-s_{r-|\ell|_{1;r}})\big)$$

$$- a_{ij}(N_0 - |N|_{1;T_{\ell,s,k,t}+1} - m' + d) + 2(k+t-d-1).$$

Here we have used the notations (64), (69), (89), (90), (92), (103)–(105), (107), *and* (108).

Proof. This follows upon combining Corollary 3.9 and Propositions 3.15 and 3.18, after expanding $\eta_{\ell,s,k,d,t,m'}$ using (64) and observing that

$$|s|_{2-a_{ij}-k-|\ell|_{1;2-a_{ij}-k};1-a_{ij}-k-m'+d} = |N|_{1;T_{\ell,s,k,t}+1}. \qquad \square$$

These expressions for the structure constants $\sigma_{m,t}^{(i,j,a_{ij})}$ comply with the values computed in [27], as displayed in Table 3. Moreover, Theorems 3.13 and 3.19 and Corollary 3.14 make it possible to compute the structure constants for higher values of $|a_{ij}|$. For example, it follows from Theorem 3.13 that for $a_{ij} = -4$ one has

$$F_{ij}(B_i, B_j) = \rho_{0,1}^{(i,j,-4)} Z_i^2 B_j B_i + \rho_{1,0}^{(i,j,-4)} Z_i^2 B_i B_j + \rho_{0,3}^{(i,j,-4)} Z_i B_j B_i^3$$
$$+ \rho_{3,0}^{(i,j,-4)} Z_i B_i^3 B_j + \rho_{1,2}^{(i,j,-4)} Z_i B_i B_j B_i^2 + \rho_{2,1}^{(i,j,-4)} Z_i B_i^2 B_j B_i,$$

if $i, j \in I \setminus X$ are distinct such that $\tau(i) = i$, where the structure constants $\rho_{m,m'}^{(i,j,-4)}$ are given in Table 4. Similarly, for $a_{ij} = -3$, $i \in I \setminus X$ with $\tau(i) = i$ and $j \in X$ one has

$$F_{ij}(B_i, B_j) = \rho_{0,0,0}^{(i,j,-3)} B_j Z_i^2 + \rho_{0,0,1}^{(i,j,-3)} Z_i B_j Z_i + \rho_{0,0,2}^{(i,j,-3)} Z_i^2 B_j$$
$$+ \rho_{0,2,0}^{(i,j,-3)} B_j B_i^2 Z_i + \rho_{0,2,1}^{(i,j,-3)} Z_i B_j B_i^2 + \rho_{1,1,0}^{(i,j,-3)} B_i B_j B_i Z_i$$
$$+ \rho_{1,1,1}^{(i,j,-3)} Z_i B_i B_j B_i + \rho_{2,0,0}^{(i,j,-3)} B_i^2 B_j Z_i + \rho_{2,0,1}^{(i,j,-3)} Z_i B_i^2 B_j$$
$$+ \sigma_{0,0}^{(i,j,-3)} W_{ij} K_j Z_i + \sigma_{0,1}^{(i,j,-3)} Z_i W_{ij} K_j + \sigma_{2,0}^{(i,j,-3)} W_{ij} K_j B_i^2,$$

with $\rho_{m,m',t}^{(i,j,-3)}$ and $\sigma_{m,t}^{(i,j,-3)}$ as in Tables 5 and 6.

Theorems 3.13 and 3.19, together with the previously obtained Theorems C, D, and F now yield a complete set of defining relations for the quantum symmetric pair coideal subalgebras $B_{c,s}$.

m \ m'	0	1	2	3
0	0	$c_i^2 q_i^2 [2]_{q_i}^2 [4]_{q_i^2}^2$	0	$c_i q_i \left([2]_{q_i}^2 + [4]_{q_i}^2\right)$
1	$-\rho_{0,1}^{(i,j,-4)}$	0	$c_i q_i [2]_{q_i}^2 [3]_{q_i} [5]_{q_i}$	
2	0	$-\rho_{1,2}^{(i,j,-4)}$		
3	$-\rho_{0,3}^{(i,j,-4)}$			

Table 4. Structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ for $a_{ij} = -4$.

(m,m') \ t	0	1	2
$(0,0)$	$-c_i^2 q_i^6 \dfrac{[3]_{q_i}}{(q_i - q_i^{-1})^2}$	$c_i^2 q_i^2 \dfrac{[3]_{q_i}(q_i^2 + q_i^{-2})}{(q_i - q_i^{-1})^2}$	$-c_i^2 q_i^{-2} \dfrac{[3]_{q_i}}{(q_i - q_i^{-1})^2}$
$(0,2)$	$c_i q_i^2 \dfrac{2 + q_i^2 [2]_{q_i}^2}{q_i - q_i^{-1}}$	$-c_i \dfrac{[3]_{q_i}(q_i^2 + q_i^{-2})}{q_i - q_i^{-1}}$	
$(1,1)$	$-c_i q_i^2 \dfrac{[4]_{q_i}(q_i^2 + 2)}{q_i - q_i^{-1}}$	$c_i \dfrac{[4]_{q_i}(q_i^{-2} + 2)}{q_i - q_i^{-1}}$	
$(2,0)$	$c_i q_i^2 \dfrac{[3]_{q_i}(q_i^2 + q_i^{-2})}{q_i - q_i^{-1}}$	$-c_i \dfrac{2 + q_i^{-2}[2]_{q_i}^2}{q_i - q_i^{-1}}$	

Table 5. Structure constants $\rho_{m,m',t}^{(i,j,a_{ij})}$ for $a_{ij} = -3$.

m \ t	0	1
0	$-c_i^2 q_i^2 \dfrac{[3]_{q_i}[4]_{q_i}}{(q_i - q_i^{-1})(q_j - q_j^{-1})}$	$c_i^2 q_i^{-4} \dfrac{[3]_{q_i}[4]_{q_i}}{(q_i - q_i^{-1})(q_j - q_j^{-1})}$
2	$c_i q_i^{-5} [2]_{q_i} [3]_{q_i}^2 [4]_{q_i} \dfrac{(q_i - q_i^{-1})^2}{q_j - q_j^{-1}}$	

Table 6. Structure constants $\sigma_{m,t}^{(i,j,a_{ij})}$ for $a_{ij} = -3$.

4. Alternative expressions for Case 1

In this section, we will derive alternative expressions for the polynomial $C_{ij}(\mathbf{c})$ in Case 1. We will start from a result by Chen, Lu and Wang. In [11] these authors provide defining relations of q-Serre type for what they call ι-quantum groups, which are in fact quasi-split quantum symmetric pair coideal subalgebras, coinciding with the algebras $B_{\mathbf{c},\mathbf{s}}$ in the special case $X = \emptyset$. These correspond to Satake diagrams without black nodes. Since $\mathcal{Z}_i = -1$ in this situation, the polynomial $C_{ij}(\mathbf{c})$ will be given by

$$F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c})$$
$$= \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} (-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} B_i^m B_j B_i^{m'}, \quad (110)$$

if $\tau(i) = i$, as follows from Corollary 3.5, and where the structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ were obtained in Theorem 3.13. In this section we will derive equivalent expressions for $\rho_{m,m'}^{(i,j,a_{ij})}$, based on the results of [11]. These expressions will also be valid beyond the quasi-split case. Indeed, our result (51) shows that $\rho_{m,m'}^{(i,j,a_{ij})}$ is independent of X and can be obtained solely from the $U_q(\mathfrak{g}')$-relations (18)–(20). Hence, the expressions for $\rho_{m,m'}^{(i,j,a_{ij})}$ we will derive in this section will be valid not only for $X = \emptyset$, but for any admissible pair (X, τ) provided we restrict to Case 1, i.e. $\tau(i) = i \in I \setminus X$ and $j \in I \setminus X$ distinct from i.

Before we can state the result from [11], we need to introduce the following notation.

Definition 4.1 ([11, Formulae (3.2)–(3.3)]). For any $i \in I$ and $m \in \mathbb{N}$ one defines the ι-divided powers of B_i as the elements

$$B_{i,0}^{(m)} = \frac{B_i^{m_p}}{[m]_{q_i}!} \prod_{k=1}^{m_e} \left(B_i^2 + q_i c_i [2(k-1+m_p)]_{q_i}^2 \right), \quad (111)$$

$$B_{i,1}^{(m)} = \frac{B_i^{m_p}}{[m]_{q_i}!} \prod_{k=1}^{m_e} \left(B_i^2 + q_i c_i [2k-1]_{q_i}^2 \right), \quad (112)$$

where we have again used the notation (49).

Using Lusztig's theory of modified quantum groups [34, Section 23.1] and a class of intricate q-binomial identities, Chen, Lu, and Wang were able to prove a result, which, translated to our notations, can be formulated as follows.

Theorem K ([11, Theorem 3.1]). *Consider the quantum symmetric pair coideal algebra $B_{\mathbf{c},\mathbf{s}}$ corresponding to an admissible pair $(X = \emptyset, \tau)$. For any $i \in I$*

satisfying $\tau(i) = i$ *and any* $j \in I$ *distinct from* i, *the ι-divided powers satisfy the relations*

$$\sum_{m=0}^{1-a_{ij}} (-1)^m B_{i,(a_{ij})_p}^{(m)} B_j B_{i,0}^{(1-a_{ij}-m)} = 0 \qquad (113)$$

and

$$\sum_{m=0}^{1-a_{ij}} (-1)^m B_{i,1-(a_{ij})_p}^{(m)} B_j B_{i,1}^{(1-a_{ij}-m)} = 0. \qquad (114)$$

Remark 2. It was suggested in [11, Remark 3.4] that these relations should remain valid for $X \neq \emptyset$ under the restriction that $\tau(i) = w_X(i) = i$.

The relations (113) and (114) can be rewritten in the form $F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c})$, where $C_{ij}(\mathbf{c})$ is an explicit polynomial in $\sum_{J \in \mathfrak{I}_{i,j}} \mathbb{K}(q) B_J$. The computation of this polynomial will be the subject of the following subsection.

4.1. Quantum Serre relations from ι-divided powers. In the present section, we will rewrite the relations (113) and (114) as inhomogeneous quantum Serre relations, so as to derive two new expressions for the structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ for Case 1. Let us start by introducing the following notation.

Definition 4.2. Let $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{-1\}$, $s \in \{0, 1\}$ and $i \in I$. We will denote by $\alpha_{k,N}^{(s,i)}$ the following elements of $\mathbb{K}(q)$:

$$\alpha_{k,N}^{(s,i)} = \begin{cases} \sum_{\substack{\ell_1,\ell_2,\dots,\ell_k=1-s \\ \ell_1<\ell_2<\dots<\ell_k}}^{N} [2\ell_1+s]_{q_i}^2 [2\ell_2+s]_{q_i}^2 \dots [2\ell_k+s]_{q_i}^2 & \text{for } 1 \leq k \leq N+s \\ 1 & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

These $\alpha_{k,N}^{(s,i)}$ arise as coefficients when expanding the ι-divided powers from Definition 4.1 as polynomials in B_i.

Lemma 4.3. *For any* $i \in I$, $s \in \{0, 1\}$ *and* $r \in \mathbb{N}$, *one can write*

$$B_{i,s}^{(r)} = \frac{1}{[r]_{q_i}!} \sum_{k=0}^{r_e} (q_i c_i)^k \alpha_{k,r_e+r_p(1-s)-1}^{(s,i)} B_i^{r-2k}.$$

Proof. Expanding (112) distributively, it is clear that

$$B_{i,1}^{(r)} = \begin{cases} \dfrac{1}{[r]_{q_i}!} \sum_{k=0}^{r/2} (q_i c_i)^k \alpha_{k,(r/2)-1}^{(1,i)} B_i^{r-2k} & \text{for } r \text{ even,} \\ \dfrac{1}{[r]_{q_i}!} \sum_{k=0}^{(r-1)/2} (q_i c_i)^k \alpha_{k,((r-1)/2)-1}^{(1,i)} B_i^{r-2k} & \text{for } r \text{ odd,} \end{cases}$$

in agreement with the proposed formula. Similarly, for r odd one has

$$B_{i,0}^{(r)} = \frac{1}{[r]_{q_i}!} \sum_{k=0}^{(r-1)/2} (q_i c_i)^k \alpha_{k,(r-1)/2}^{(0,i)} B_i^{r-2k},$$

whereas for r even, we have

$$B_{i,0}^{(r)} = \frac{1}{[r]_{q_i}!} \sum_{k=0}^{\frac{r}{2}} (q_i c_i)^k \gamma_{k,r}^{(i)} B_i^{r-2k},$$

where

$$\gamma_{k,r}^{(i)} = \sum_{\substack{\ell_1,\ell_2,\dots,\ell_k=0 \\ \ell_1<\cdots<\ell_k}}^{(r/2)-1} [2\ell_1]_{q_i}^2 [2\ell_2]_{q_i}^2 \dots [2\ell_k]_{q_i}^2.$$

But of course, since $[0]_{q_i} = 0$, we have that

$$\gamma_{k,r}^{(i)} = \sum_{\substack{\ell_1,\ell_2,\dots,\ell_k=1 \\ \ell_1<\cdots<\ell_k}}^{(r/2)-1} [2\ell_1]_{q_i}^2 [2\ell_2]_{q_i}^2 \dots [2\ell_k]_{q_i}^2 = \alpha_{k,(r/2)-1}^{(0,i)},$$

which again agrees with the statement of the lemma. $\qquad \square$

In the upcoming proofs, we will often be required to switch the order of summation in a particular kind of nested sums. Below, we propose a general strategy for this resummation.

Lemma 4.4. *Let f be any function of three discrete variables k, ℓ and m, and let N be any natural number, then one has*

$$\sum_{m=0}^{N} \sum_{k=0}^{m} \sum_{\ell=0}^{N-m} f(k,\ell,m) = \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} \sum_{m=0}^{\ell} f(m,\ell-m,m+k).$$

Proof. We will derive this identity in several steps, which are explained below:

$$\sum_{m=0}^{N} \sum_{k=0}^{m} \sum_{\ell=0}^{N-m} f(k,\ell,m) \overset{(1)}{=} \sum_{m=0}^{N} \sum_{k=0}^{m} \sum_{\ell=0}^{N-m} f(m-k, N-m-\ell, m)$$

$$\overset{(2)}{=} \sum_{k=0}^{N} \sum_{m=k}^{N} \sum_{\ell=0}^{N-m} f(m-k, N-m-\ell, m)$$

$$\overset{(3)}{=} \sum_{k=0}^{N} \sum_{m=0}^{N-k} \sum_{\ell=0}^{m} f(N-m-k, m-\ell, N-m)$$

$$\stackrel{(4)}{=} \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} \sum_{m=\ell}^{N-k} f(N-m-k, m-\ell, N-m)$$

$$\stackrel{(5)}{=} \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} \sum_{m=0}^{\ell} f(m, \ell-m, m+k).$$

(1) Replace k by the new summation index $k' = m - k$, replace ℓ by $\ell' = N - m - \ell$ and rename $k' \to k$, $\ell' \to \ell$.

(2) Switch the summation order of the sums over m and k.

(3) Replace m by the new summation index $m' = N - m$ and then rename $m' \to m$.

(4) Switch the summation order of the sums over m and ℓ.

(5) Replace ℓ by the new summation index $\ell' = N - k - \ell$, replace m by $m' = N - m - k$ and rename $\ell' \to \ell$, $m' \to m$.

$$\square$$

We will now rewrite the results of Theorem K using Lemma 4.3.

Proposition 4.5. *Let $\tau \in \mathrm{Aut}(A, \emptyset)$ be such that $(X = \emptyset, \tau)$ is a quasi-split admissible pair, let $B_{\mathbf{c},\mathbf{s}}$ be the corresponding quantum symmetric pair coideal subalgebra and let $i, j \in I$ be such that $\tau(i) = i$ and $j \neq i$. Then the relations* (113)–(114) *can equivalently be expressed as*

$$F_{ij}(B_i, B_j) = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} (a_{ij} + m + m')_p (-1)^{a_{ij}+m}$$
$$(q_i c_i)^{(1-a_{ij}-m-m')/2} \Theta_{m,m'}^{(0,i,a_{ij})} B_i^m B_j B_i^{m'}, \quad (115)$$

where

$$\Theta_{m,m'}^{(0,i,a_{ij})} = \sum_{r=0}^{(1-a_{ij}-m-m')/2} \begin{bmatrix} 1-a_{ij} \\ m+2r \end{bmatrix}_{q_i} \alpha_{r,r+m_e+m_p-1}^{(0,i)}$$
$$\alpha_{((1-a_{ij}-m-m')/2)-r,-(a_{ij})_e-(a_{ij})_p-r-m_e-m_p}^{((a_{ij})_p,i)}, \quad (116)$$

with $\alpha_{k,N}^{(s,i)}$ as in Definition 4.2.

Proof. Let us consider the case where a_{ij} is even. We will start by splitting the sum over m in (113) into a sum over m even and one over m odd:

$$\sum_{m=0}^{-(a_{ij}/2)} B_{i,0}^{(2m)} B_j B_{i,0}^{(1-a_{ij}-2m)} - \sum_{m=0}^{-(a_{ij}/2)} B_{i,0}^{(2m+1)} B_j B_{i,0}^{(-a_{ij}-2m)} = 0.$$

Substituting the expressions for $B_{i,0}^{(r)}$ obtained in Lemma 4.3, we find

$$\sum_{m=0}^{-(a_{ij}/2)} \sum_{k=0}^{m} \sum_{\ell=0}^{-(a_{ij}/2)-m} \frac{(q_i c_i)^{k+\ell}}{[2m]_{q_i}![1-a_{ij}-2m]_{q_i}!}$$
$$\alpha_{k,m-1}^{(0,i)} \alpha_{\ell,-(a_{ij}/2)-m}^{(0,i)} B_i^{2m-2k} B_j B_i^{1-a_{ij}-2m-2\ell}$$

$$-\sum_{m=0}^{-(a_{ij}/2)} \sum_{k=0}^{m} \sum_{\ell=0}^{-(a_{ij}/2)-m} \frac{(q_i c_i)^{k+\ell}}{[2m+1]_{q_i}![-a_{ij}-2m]_{q_i}!}$$
$$\alpha_{k,m}^{(0,i)} \alpha_{\ell,-(a_{ij}/2)-m-1}^{(0,i)} B_i^{2m-2k+1} B_j B_i^{-a_{ij}-2m-2\ell} = 0.$$

Multiplying both sides with $[1-a_{ij}]_{q_i}!$ and applying Lemma 4.4, this becomes

$$\sum_{k=0}^{-(a_{ij}/2)} \sum_{\ell=0}^{-(a_{ij}/2)-k} (q_i c_i)^{\ell} \left(\sum_{m=0}^{\ell} \begin{bmatrix} 1-a_{ij} \\ 2m+2k \end{bmatrix}_{q_i} \alpha_{m,m+k-1}^{(0,i)} \alpha_{\ell-m,-(a_{ij}/2)-m-k}^{(0,i)} \right)$$
$$B_i^{2k} B_j B_i^{1-a_{ij}-2k-2\ell}$$

$$-\sum_{k=0}^{-(a_{ij}/2)} \sum_{\ell=0}^{-(a_{ij}/2)-k} (q_i c_i)^{\ell} \left(\sum_{m=0}^{\ell} \begin{bmatrix} 1-a_{ij} \\ 2m+2k+1 \end{bmatrix}_{q_i} \alpha_{m,m+k}^{(0,i)} \alpha_{\ell-m,-(a_{ij}/2)-m-k-1}^{(0,i)} \right)$$
$$B_i^{2k+1} B_j B_i^{-a_{ij}-2k-2\ell} = 0.$$

Referring to the notation (116), we may write the terms between brackets above as

$$\Theta_{2k,1-a_{ij}-2k-2\ell}^{(0,i,a_{ij})} \quad \text{and} \quad \Theta_{2k+1,-a_{ij}-2k-2\ell}^{(0,i,a_{ij})},$$

respectively. Replacing then $2k$ by m in the first sum and $2k+1$ by m in the second, this becomes

$$\sum_{\substack{m=0 \\ m \text{ even}}}^{1-a_{ij}} \sum_{\ell=0}^{(1-a_{ij}-m)_e} (q_i c_i)^{\ell} \Theta_{m,1-a_{ij}-m-2\ell}^{(0,i,a_{ij})} B_i^m B_j B_i^{1-a_{ij}-m-2\ell}$$

$$-\sum_{\substack{m=0 \\ m \text{ odd}}}^{1-a_{ij}} \sum_{\ell=0}^{(1-a_{ij}-m)_e} (q_i c_i)^{\ell} \Theta_{m,1-a_{ij}-m-2\ell}^{(0,i,a_{ij})} B_i^m B_j B_i^{1-a_{ij}-m-2\ell}$$

$$= \sum_{m=0}^{1-a_{ij}} \sum_{\ell=0}^{(1-a_{ij}-m)_e} (-1)^m (q_i c_i)^{\ell} \Theta_{m,1-a_{ij}-m-2\ell}^{(0,i,a_{ij})} B_i^m B_j B_i^{1-a_{ij}-m-2\ell} = 0. \quad (117)$$

Now observe that the term corresponding to $\ell = 0$ can be written as

$$
\sum_{m=0}^{1-a_{ij}} (-1)^m \Theta_{m,1-a_{ij}-m}^{(0,i,a_{ij})} B_i^m B_j B_i^{1-a_{ij}-m}
$$
$$
= (-1)^{1+a_{ij}} \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} B_i^{1-a_{ij}-m} B_j B_i^m
$$
$$
= (-1)^{1+a_{ij}} F_{ij}(B_i, B_j),
$$

where we have replaced m by the new summation index $m' = 1 - a_{ij} - m$ for the first equality, which we have thereafter renamed to m again, and where we have used the fact that

$$
\Theta_{1-a_{ij}-m,m}^{(0,i,a_{ij})} = \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i}
$$

by (16). Consequently, when separating the term corresponding to $\ell = 0$ in (117), we obtain

$$
F_{ij}(B_i, B_j) = (-1)^{a_{ij}} \sum_{m=0}^{1-a_{ij}} \sum_{\ell=1}^{(1-a_{ij}-m)_e} (-1)^m (q_i c_i)^\ell
$$
$$
\Theta_{m,1-a_{ij}-m-2\ell}^{(0,i,a_{ij})} B_i^m B_j B_i^{1-a_{ij}-m-2\ell}.
$$

Now observe that when m equals $-a_{ij}$ or $1 - a_{ij}$, the range of the second summation index ℓ is empty. Hence, the sum over m runs in fact from 0 to $-1 - a_{ij}$. Moreover, we may replace ℓ by the new summation index $m' = 1 - a_{ij} - m - 2\ell$, which runs over $\{0, 2, \ldots, -1 - a_{ij} - m\}$ if $1 - a_{ij} - m$ is even and over $\{1, 3, \ldots, -1 - a_{ij} - m\}$ if $1 - a_{ij} - m$ is odd, hence over $\{0, 1, \ldots, -1 - a_{ij} - m\}$ after multiplying the summand with $(a_{ij} + m + m')_p$. This leads us to

$$
F_{ij}(B_i, B_j) = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} (a_{ij} + m + m')_p (-1)^{a_{ij}+m} (q_i c_i)^{(1-a_{ij}-m-m')/2}
$$
$$
\Theta_{m,m'}^{(0,i,a_{ij})} B_i^m B_j B_i^{m'},
$$

as was to be proven. The statement for a_{ij} odd follows analogously, starting from the relation (114). $\qquad\square$

In a similar fashion, one can combine the relation (113) for a_{ij} odd with the relation (114) for a_{ij} even. This gives rise to the following expressions.

Proposition 4.6. *Let $\tau \in \mathrm{Aut}(A, \emptyset)$ be such that $(X = \emptyset, \tau)$ is a quasi-split admissible pair, let $B_{\mathbf{c},\mathbf{s}}$ be the corresponding quantum symmetric pair coideal*

subalgebra and let $i, j \in I$ be such that $\tau(i) = i$ and $j \neq i$. The relations (113)–(114) can equivalently be expressed as

$$F_{ij}(B_i, B_j) = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} (a_{ij} + m + m')_p (-1)^{a_{ij}+m} (q_i c_i)^{(1-a_{ij}-m-m')/2}$$
$$\Theta_{m,m'}^{(1,i,a_{ij})} B_i^m B_j B_i^{m'}, \quad (118)$$

where

$$\Theta_{m,m'}^{(1,i,a_{ij})} = \sum_{r=0}^{(1-a_{ij}-m-m')/2} \begin{bmatrix} 1-a_{ij} \\ m+2r \end{bmatrix}_{q_i} \alpha_{r,r+m_e-1}^{(1,i)} \alpha_{((1-a_{ij}-m-m')/2)-r,-(a_{ij})_e-r-m_e-1}^{(1-(a_{ij})_p,i)}. \quad (119)$$

Comparing the relations (115) and (118) with (110), we obtain alternative expressions for the structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$. As explained in the introduction of Section 4, these will not only be valid for the quasi-split case, but for any admissible pair, provided we restrict to Case 1. Hence, from now on we may assume (X, τ) to be an arbitrary admissible pair and consider the corresponding quantum symmetric pair coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$. The relations (115) and (118) then allow us to prove the following.

Theorem 4.7. *For any distinct $i, j \in I \setminus X$ such that $\tau(i) = i$, one has*

$$F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c}) = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \rho_{m,m'}^{(i,j,a_{ij})} \mathcal{Z}_i^{(1-a_{ij}-m-m')/2} B_i^m B_j B_i^{m'}, \quad (120)$$

where the structure constants are given by

$$\rho_{m,m'}^{(i,j,a_{ij})} = (a_{ij} + m + m')_p (-1)^{a_{ij}+m} (-q_i c_i)^{(1-a_{ij}-m-m')/2} \Theta_{m,m'}^{(s,i,a_{ij})} \quad (121)$$

with $s \in \{0, 1\}$ and where we have used the notations (116) and (119).

Proof. We will use the same strategy as in the proof of [27, Proposition 6.1]. Assume first that $X = \emptyset$. If we write

$$\omega_{m,m'}^{(i,j,a_{ij})} = (a_{ij} + m + m')_p (-1)^{a_{ij}+m} (q_i c_i)^{(1-a_{ij}-m-m')/2} \Theta_{m,m'}^{(s,i,a_{ij})},$$

with $s \in \{0, 1\}$, then comparison of (115) and (118) with (110) yields

$$\sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \left((-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} - \omega_{m,m'}^{(i,j,a_{ij})} \right) B_i^m B_j B_i^{m'} = 0. \quad (122)$$

Separating the term F_i in each

$$B_i = F_i + c_i \theta_q(F_i K_i) K_i^{-1} + s_i K_i^{-1} = F_i - c_i E_{\tau(i)} K_i^{-1} + s_i K_i^{-1}$$

and the F_j in B_j, the relation (122) asserts $\mathfrak{F}_{i,j} + \mathfrak{D}_{i,j} = 0$, where

$$\mathfrak{F}_{i,j} = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \left((-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} - \omega_{m,m'}^{(i,j,a_{ij})} \right) F_i^m F_j F_i^{m'}$$

and $\mathfrak{D}_{i,j}$ lies in the set $\mathcal{E}_{i,j}$ of $\mathbb{K}(q)$-linear combinations of monomials in $U_q(\mathfrak{g}')$ containing at most $-1-a_{ij}$ factors F_i, and either one factor K_j^{-1}, or one factor F_j and at least one factor K_i^{-1}. Since the $U_q(\mathfrak{g}')$-relations (18)–(20) imply $\mathcal{E}_{i,j} \cap U^- = \{0\}$, both $\mathfrak{D}_{i,j}$ and $\mathfrak{F}_{i,j}$ must vanish. The assertion $\mathfrak{F}_{i,j} = 0$ is a polynomial equation of degree 1 in F_j and at most of degree $-1-a_{ij}$ in F_i. But such a polynomial must have trivial coefficients, since the lowest degree $\mathbb{K}(q)$-linear combination of F_j and powers of F_i with non-trivial coefficients that vanishes, is precisely the quantum Serre polynomial $F_{ij}(F_i, F_j)$, which is of degree $1 - a_{ij}$ in F_i. Hence, we find

$$(-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} = \omega_{m,m'}^{(i,j,a_{ij})},$$

for any m, m' with $m + m' \leq -1 - a_{ij}$. This holds for the special case $X = \emptyset$, and since $\rho_{m,m'}^{(i,j,a_{ij})}$ is independent of X as follows from Theorem 3.13, this establishes the same relation for admissible pairs with $X \neq \emptyset$. $\qquad\square$

Remark 3. It follows from Propositions 4.5 and 4.6 that $\Theta_{m,m'}^{(0,i,a_{ij})} = \Theta_{m,m'}^{(1,i,a_{ij})}$ for any $m, m' \in \mathbb{N}$ with $m + m' \leq -1 - a_{ij}$ and any distinct $i, j \in I \setminus X$ with $\tau(i) = i$. Hence, the expressions (116) and (119) must be equal, which determines a non-trivial identity of q-binomial type.

Recall from [2, Section 3.3] that the algebras $B_{\mathbf{c,s}}$ allow an intrinsic bar involution under certain conditions on the generalized Cartan matrix and the admissible pair. More precisely, this was expressed as follows.

Theorem L ([2, Theorem 3.11]). *Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix and (X, τ) be an admissible pair satisfying the following conditions:*

(i) *If $i \in I \setminus X$, $\tau(i) = i$ and $j \in X$, then $a_{ij} \in \{0, -1, -2\}$.*

(ii) *If $i \in I \setminus X$, $\tau(i) = i$ and $i \neq j \in I \setminus X$, then $a_{ij} \in \{0, -1, -2, -3\}$.*

The following statements are equivalent:

(1) *There exists a $\mathbb{K}$-algebra automorphism $\bar{}$ of $B_{\mathbf{c,s}}$, which restricts to the bar involution of $U_q(\mathfrak{g})$ on $\mathbb{M}_X U_\Theta^{0'}$ and satisfies $\bar{B}_i = B_i$ for all $i \in I \setminus X$.*

(2) *Let $i \in I \setminus X$ be such that $\tau(i) \neq i$ or $a_{ij} \neq 0$ for at least one $j \in I \setminus \{i\}$, and let us denote by $\bar{}$ the bar involution on $U_q(\mathfrak{g})$. Then $\mathbf{c} \in (\mathbb{K}(q)^\times)^{I \setminus X}$ is such that*

$$\overline{c_i \mathcal{Z}_i} = q_i^{a_{i,\tau(i)}} c_{\tau(i)} \mathcal{Z}_{\tau(i)}.$$

The main advantage of the expressions (121) for the structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ is that they allow to drop condition (ii) in Theorem L.

Corollary 4.8. *Theorem L still holds if the condition (ii) is not satisfied.*

Proof. Let $i \in I \setminus X$ be such that $\tau(i) = i$, let $j \in I \setminus X$ be different from i, and let $a_{ij} \in \mathbb{Z}^-$ be arbitrary. By the proof of [2, Theorem 3.11], it suffices to show that the right-hand side of the relation (120) is preserved under the bar involution if and only if (2) holds. Since $\bar{q} = q^{-1}$, hence

$$\overline{\Theta_{m,m'}^{(s,i,a_{ij})}} = \Theta_{m,m'}^{(s,i,a_{ij})}$$

for $s \in \{0, 1\}$. It follows immediately from (121) that

$$\overline{\rho_{m,m'}^{(i,j,a_{ij})} \mathcal{Z}_i^{(1-a_{ij}-m-m')/2}} = q_i^{-(1-a_{ij}-m-m')} \rho_{m,m'}^{(i,j,a_{ij})} \left(\frac{\overline{c_i \mathcal{Z}_i}}{c_i} \right)^{(1-a_{ij}-m-m')/2},$$

which is equal to $\rho_{m,m'}^{(i,j,a_{ij})} \mathcal{Z}_i^{(1-a_{ij}-m-m')/2}$ if and only if $\overline{c_i \mathcal{Z}_i} = q_i^2 c_i \mathcal{Z}_i$.   □

Consequently, the condition (ii) can also be removed from the list of assumptions needed to develop the theory of universal K-matrices in [3, Section 5.4].

To conclude, we will show that the structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ exhibit certain symmetry properties, as suggested by the values in Table 1. In practical calculations, this significantly reduces the number of couples (m, m') for which the structure constants must be computed.

Proposition 4.9. *The structure constants $\rho_{m,m'}^{(i,j,a_{ij})}$ are symmetric in m and m' if a_{ij} is odd and antisymmetric if a_{ij} is even. In other words:*

$$\rho_{m,m'}^{(i,j,a_{ij})} = (-1)^{1-a_{ij}} \rho_{m',m}^{(i,j,a_{ij})}.$$

Proof. We will treat the case a_{ij} odd, which is the most subtle case in some sense. The statement is trivial for $m + m'$ odd, since in this case $\rho_{m,m'}^{(i,j,a_{ij})}$ will vanish, because of the factor $(a_{ij} + m + m')_p$ in (121). So we may assume $m + m'$ to be even. Let us start by observing that $\rho_{m',m}^{(i,j,a_{ij})}$ yields

$$(-1)^{a_{ij}+m'}(-q_i c_i)^{(1-a_{ij}-m-m')/2} \sum_{r=0}^{(1-a_{ij}-m-m')/2} \begin{bmatrix} 1-a_{ij} \\ m'+2r \end{bmatrix}_{q_i} \alpha_{r,r+m'_e+m'_p-1}^{(0,i)}$$
$$\alpha_{((1-a_{ij}-m-m')/2)-r,((-a_{ij}-1)/2)-r-m'_e-m'_p}^{(1,i)}$$

by (121) with $s = 0$. Since $m + m'$ is even, we have $(-1)^{m'} = (-1)^m$. Moreover, we can rewrite the sum above using a new summation index

$$r' = \frac{1 - a_{ij} - m - m'}{2} - r,$$

which we thereafter rename to r again. This way, $\rho_{m',m}^{(i,j,a_{ij})}$ becomes

$$(-1)^{a_{ij}+m}(-q_i c_i)^{(1-a_{ij}-m-m')/2} \sum_{r=0}^{(1-a_{ij}-m-m')/2} \begin{bmatrix} 1-a_{ij} \\ m+2r \end{bmatrix}_{q_i}$$

$$\alpha_{((1-a_{ij}-m-m')/2)-r,((1-a_{ij}-m-m')/2)-r+m'_e+m'_p-1}^{(0,i)}\alpha_{r,r+((m+m')/2)-1-m'_e-m'_p}^{(1,i)},$$

where we have used the property (16) of the q_i-binomial symbol. Next, since m' and m have the same parity, we find

$$m'_e + m'_p = \begin{cases} \dfrac{m'}{2} & \text{if } m \text{ is even,} \\ \dfrac{m'+1}{2} & \text{if } m \text{ is odd} \end{cases} = \dfrac{m'+m_p}{2},$$

and so

$$\frac{1-a_{ij}-m-m'}{2} - r + m'_e + m'_p = \frac{1-a_{ij}}{2} - r - \left(\frac{m-m_p}{2}\right)$$

$$= \frac{1-a_{ij}}{2} - r - m_e = -(a_{ij})_e - r - m_e.$$

Thus, we obtain

$$\rho_{m',m}^{(i,j,a_{ij})} = (-1)^{a_{ij}+m}(-q_i c_i)^{(1-a_{ij}-m-m')/2}$$

$$\sum_{r=0}^{(1-a_{ij}-m-m')/2} \begin{bmatrix} 1-a_{ij} \\ m+2r \end{bmatrix}_{q_i} \alpha_{((1-a_{ij}-m-m')/2)-r,-(a_{ij})_e-r-m_e-1}^{(0,i)}\alpha_{r,r+m_e-1}^{(1,i)},$$

which precisely equals $\rho_{m,m'}^{(i,j,a_{ij})}$ according to (121) with $s = 1$. This proves the symmetry.

For a_{ij} even, the proof goes along the same lines, starting from (121) with either $s = 0$ or $s = 1$. $\qquad\square$

4.2. Generalized q-Onsager algebras and their classical counterparts.

A special class of quantum symmetric pair coideal subalgebras is known under the name generalized q-Onsager algebras. They coincide with the algebras $B_{\mathbf{c},\mathbf{s}}$ in the split case, i.e. for the trivial admissible pair ($X = \emptyset, \tau = \text{id}$), corresponding to Satake diagrams without black nodes and with the trivial diagram involution. In this case we have $\theta_q(F_i K_i) = -E_i$ by Lemma E and, moreover, $Q^\Theta = \{0\}$ since $w_X = \text{id}$. Hence, we may formulate the following definition.

Definition 4.10. The generalized q-Onsager algebra $\mathcal{O}_q(\mathfrak{g})$ associated to the Kac–Moody algebra $\mathfrak{g}$ is the subalgebra of $U_q(\mathfrak{g}')$ generated by the elements

$$B_i = F_i - c_i E_i K_i^{-1} + s_i K_i^{-1}, \tag{123}$$

with $i \in I$, and where $(\mathbf{c}, \mathbf{s})$ takes values in the set $\mathcal{C} \times \mathcal{S}$ defined in (25) and (26). By Theorem C, Corollary 3.5 and the fact that in this case $\mathcal{Z}_i = -1$, it is abstractly defined by the relations

$$F_{ij}(B_i, B_j) = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} (-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} B_i^m B_j B_i^{m'}, \quad (124)$$

for $i \neq j \in I$, with $\rho_{m,m'}^{(i,j,a_{ij})}$ as obtained in (83) or equivalently in (121).

In the special case $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$, this algebra coincides with the q-Onsager algebra [5], which is typically described as generated by two elements B_0 and B_1 subject to the q-Dolan–Grady relations

$$\begin{aligned} [B_0, [B_0, [B_0, B_1]_q]_{q^{-1}}] &= -c_0 q (q + q^{-1})^2 [B_0, B_1], \\ [B_1, [B_1, [B_1, B_0]_q]_{q^{-1}}] &= -c_1 q (q + q^{-1})^2 [B_1, B_0], \end{aligned} \quad (125)$$

for certain $c_0, c_1 \in \mathbb{K}(q)$, where $[\cdot, \cdot]_q$ denotes the q-commutator, defined by

$$[A, B]_q = qAB - q^{-1}BA.$$

Its generalization $\mathcal{O}_q(\mathfrak{g})$ to other Kac–Moody algebras $\mathfrak{g}$ was introduced in [6], where defining relations were presented for the cases $a_{ij} \in \{0, -1, -2, -3, -4\}$ under some additional restrictions on a_{ji}. The expressions (124) we have derived in this paper extend those relations to arbitrary Cartan matrices. By Remark 1, we may equivalently write them as

$$\left(\prod_{m=(a_{ij}/2)}^{\overrightarrow{-(a_{ij}/2)}} \mathrm{ad}_{q_i^m}(B_i) \right)(B_j)$$

$$= \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} (-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} B_i^m B_j B_i^{m'}, \quad (126)$$

which, by the presence of nested q-commutators, can be considered relations of q-Dolan–Grady type.

To conclude, we will consider the limit of the generalized q-Onsager algebra $\mathcal{O}_q(\mathfrak{g})$ under the specialization $q \to 1$ described in Remark 1, which is precisely the algebra $\boldsymbol{b} = \boldsymbol{b}(X, \tau)$ from Definition 2.3 in the special case $X = \emptyset$ and $\tau = \mathrm{id}$. It follows immediately that in this case $\mathrm{Ad}(s(X, \tau)) = \mathrm{Ad}(m_X) = \mathrm{id}$, and hence the automorphism $\theta(X, \tau)$ coincides with the classical Chevalley involution ω defined in (11). Moreover, Definition 2.3 asserts that we may state the following.

Definition 4.11. The (classical) generalized Onsager algebra is the Lie subalgebra $\mathcal{O}(\mathfrak{g})$ of the Kac–Moody algebra $\mathfrak{g}$ generated by the elements

$$b_i = f_i + \omega(f_i) = f_i - e_i,$$

with $i \in I$. By [42, Lemma 2.2], $\mathcal{O}(\mathfrak{g})$ is the fixed point Lie subalgebra of $\mathfrak{g}$ under the Chevalley involution ω.

The algebras $\mathcal{O}(\mathfrak{g})$ were studied by Stokman in [42], where a complete set of defining relations of inhomogeneous Serre type or Dolan–Grady type was given. To describe these relations, we will need the following recursively defined coefficients.

Definition 4.12. Let i, j be distinct elements of I and $r \in \mathbb{N}$ arbitrary. For any $s \in \mathbb{N}$ satisfying $s \leq r$ we define $c_s^{ij}[r]$ through the recursion relation

$$c_s^{ij}[r] = c_{s-1}^{ij}[r-1] + (r-1)c_s^{ij}[r-2], \tag{127}$$

for $r \geq 2$, with the convention that $c_{-1}^{ij}[r] = 0$ for any r, and with boundary conditions $c_r^{ij}[r] = 1$ for $r \geq 0$ and $c_{r-1}^{ij}[r] = 0$ for $r \geq 1$.

The relation (127) coincides with [42, Formula (2.4)] upon setting $r = 1 - a_{ij}$, as we will do in the following theorem.

Theorem M ([42, Proposition 2.4, Theorem 2.7]). *The algebra $\mathcal{O}(\mathfrak{g})$ is abstractly defined by the inhomogeneous Serre relations*

$$\sum_{s=0}^{1-a_{ij}} (-1)^{s+1} c_s^{ij}[1-a_{ij}](\operatorname{ad} b_i)^s b_j = 0, \tag{128}$$

for any distinct i, $j \in I$.

Note that the relations (128) differ from those given in [42] by a factor $(-1)^{s+1}$. This is caused by the fact that the generators used in [42] differ from ours by a sign as well, but of course this does not alter the algebra under consideration.

It follows from Theorem B that the generators B_i of the generalized q-Onsager algebra $\mathcal{O}_q(\mathfrak{g})$ reduce to the generators b_i of $\mathcal{O}(\mathfrak{g})$ under the specialization $q \to 1$, provided the parameters $\mathbf{c} \in \mathcal{C}$ are specializable and $\mathbf{s} = \mathbf{0}$. Consequently, the same holds true for the defining relations of the q-deformed and classical Onsager algebras. It will hence be possible to derive closed expressions for the recursively defined coefficients $c_s^{ij}[1-a_{ij}]$ in (128) from the previously obtained equation (126). We begin with a straightforward identity.

Lemma 4.13. *For any $A, B \in U(\mathfrak{g})$ and $r \in \mathbb{N}$, one has*

$$(\operatorname{ad} A)^r(B) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} A^{r-k} B A^k.$$

Proof. This follows immediately from the equation (21) in the limit $q \to 1$. $\qquad \square$

This identity allows us to expand the nested commutators in the relation (128).

Lemma 4.14. *The inhomogeneous Serre relations* (128) *defining the algebra* $\mathcal{O}(\mathfrak{g})$ *can be rewritten as*

$$(\operatorname{ad} b_i)^{1-a_{ij}} b_j = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} (-1)^{a_{ij}+m} \binom{m+m'}{m'} c_{m+m'}^{ij}[1-a_{ij}] \, b_i^m b_j b_i^{m'}.$$

$$(129)$$

Proof. It follows immediately from Lemma 4.13 and the fact that $c_{1-a_{ij}}^{ij}[1-a_{ij}] = 1$ and $c_{-a_{ij}}^{ij}[1-a_{ij}] = 0$ that the relations (128) can be rewritten as

$$(\operatorname{ad} b_i)^{1-a_{ij}} b_j = (-1)^{a_{ij}} \sum_{s=0}^{-1-a_{ij}} (-1)^s c_s^{ij}[1-a_{ij}](\operatorname{ad} b_i)^s b_j$$

$$= \sum_{s=0}^{-1-a_{ij}} \sum_{m'=0}^{s} (-1)^{a_{ij}+s+m'} \binom{s}{m'} c_s^{ij}[1-a_{ij}] \, b_i^{s-m'} b_j b_i^{m'}.$$

The claim now follows upon changing the order of summation, replacing s by the new summation index $m = s - m'$ and observing that

$$\big\{(m, m') : m \in \{0, \ldots, -1 - a_{ij}\}, \ m' \in \{0, \ldots, -1 - a_{ij} - m\}\big\}$$
$$= \big\{(m, m') : m \in \{0, \ldots, -1 - a_{ij} - m'\}, \ m' \in \{0, \ldots, -1 - a_{ij}\}\big\}.$$

$\square$

An alternative set of defining relations for the generalized Onsager algebras $\mathcal{O}(\mathfrak{g})$ can be found by taking the limit $q \to 1$ of the $\mathcal{O}_q(\mathfrak{g})$-relations (124). Comparison of both types of relations leads to closed expressions for the recursively defined coefficients $c_s^{ij}[r]$.

Theorem 4.15. *For any distinct* $i, j \in I$ *and any* $r, s \in \mathbb{N}$ *with* $s \leq r$ *we have*

$$c_s^{ij}[r] = (r-s+1)_p \sum_{\substack{\ell_1, \ldots, \ell_{(r-s)/2} = r_p \\ \ell_1 < \cdots < \ell_{(r-s)/2}}}^{r_e+r_p-1} (2\ell_1 + 1 - r_p)^2 (2\ell_2 + 1 - r_p)^2 \ldots (2\ell_{(r-s)/2} + 1 - r_p)^2,$$

$$(130)$$

or, equivalently,

$$c_s^{ij}[r] = (r - s + 1)_p \sum_{m=0}^{(r-s)/2} \Bigg[\binom{r}{2m} \Bigg(\prod_{k=0}^{m} (2k-1)^2 \Bigg)$$

$$\sum_{\substack{\ell_1, \ldots, \ell_{((r-s)/2)-m} = 1 - r_p \\ \ell_1 < \cdots < \ell_{((r-s)/2)-m}}}^{r_e-m-1} (2\ell_1 + r_p)^2 \ldots (2\ell_{((r-s)/2)-m} + r_p)^2 \Bigg], \quad (131)$$

where the sum in (130) *and* (131) *should be read as* 1 *if* $s = r$, *respectively, if* $m = (r - s)/2$, *and where we have used the notation* (49).

Proof. By the above observations, in the limit $q \to 1$ the relation (126) becomes

$$(\operatorname{ad} b_i)^{1-a_{ij}} b_j = \sum_{m=0}^{-1-a_{ij}} \sum_{m'=0}^{-1-a_{ij}-m} \left[\lim_{q \to 1} \left((-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} \right) \right] b_i^m b_j b_i^{m'}.$$

Upon comparison with (129), and following the same reasoning as in the proof of [27, Proposition 6.1], it follows that

$$(-1)^{a_{ij}+m} \binom{m + m'}{m'} c_{m+m'}^{ij}[1 - a_{ij}] = \lim_{q \to 1} \left((-1)^{(1-a_{ij}-m-m')/2} \rho_{m,m'}^{(i,j,a_{ij})} \right),$$

for any $m, m' \in \mathbb{N}$ with $m + m' \leq -1 - a_{ij}$. For $m = 0$, upon using the expression (121) with $s = 0$, this becomes

$$\begin{aligned}
&(-1)^{a_{ij}} c_{m'}^{ij}[1 - a_{ij}] \\
&= (a_{ij} + m')_p \lim_{q \to 1} \left((-1)^{a_{ij}} (q_i c_i)^{(1-a_{ij}-m')/2} \alpha_{(1-a_{ij}-m')/2,-(a_{ij})_e-(a_{ij})_p}^{((a_{ij})_p,i)} \right),
\end{aligned}$$

by (116), where we have used the fact that $\alpha_{r,r-1}^{(0,i)} = \delta_{r,0}$. The expression (130) now follows upon setting $r = 1 - a_{ij}$, renaming m' to s and using Definition 4.2 and the assumption of specializability of **c**. Equation (131) follows similarly from (121) with $s = 1$. $\qquad\square$

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