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Conical $SL(3)$ foams

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Abstract. In the unoriented $SL(3)$ foam theory, singular vertices are generic singularities of two-dimensional complexes. Singular vertices have neighborhoods homeomorphic to cones over the one-skeleton of the tetrahedron, viewed as a trivalent graph on the two-sphere. In this paper we consider foams with singular vertices with neighborhoods homeomorphic to cones over more general planar trivalent graphs. These graphs are subject to suitable conditions on their Kempe equivalence Tait coloring classes and include the dodecahedron graph. In this modification of the original homology theory it is straightforward to show that modules associated to the dodecahedron graph are free of rank 60, which is still an open problem for the original unoriented $SL(3)$ foam theory.

1. Introduction

P. Kronheimer and T. Mrowka [10] defined a functor $J^\sharp$ from the category of knotted trivalent graphs (KTG) in $\mathbb{R}^3$ and foams (with boundary) in $\mathbb{R}^3 \times [0,1]$ to the category of $\mathbf{k}$-vector spaces, where $\mathbf{k} = \mathbb{F}_2$ is the two-element field. They proved that if Γ is a bridge-less KTG embedded in the plane, then $J^\sharp(\Gamma)$ is nontrivial. Kronheimer and Mrowka proved that for such a KTG the inequality $\dim(J^\sharp(\Gamma)) \geq |\mathrm{Tait}(\Gamma)|$ holds and conjectured that it is an equality. Here $\mathrm{Tait}(\Gamma)$ is the set of Tait colorings of Γ, that is, 3-colorings of edges of Γ such that at each vertex the three colors are distinct. The number of Tait colorings of Γ is denoted $|\mathrm{Tait}(\Gamma)|$.

This conjecture would imply the four color theorem (4CT) (see [1]). In the same paper, for plane graphs $\Gamma \subset \mathbb{R}^2$, Kronheimer and Mrowka predicted the existence of a combinatorial analogue of $J^\sharp(\Gamma)$, denoted $J^\flat(\Gamma)$ in [10] and $\langle\Gamma\rangle$ here. The existence of $J^\flat(\Gamma) \cong \langle\Gamma\rangle$ was proved by the authors in [7].

State spaces $\langle\Gamma\rangle$, over all plane trivalent graphs Γ, extend to a functor from the category of foams with boundary to the category of $\mathbb{Z}$-graded modules over a suitable commutative ring. We denote this functor by $\langle\bullet\rangle$.

In fact, several versions of the functor $\langle\bullet\rangle$ are constructed in [7]. One of them takes values in the category of $\mathbb{Z}$-graded $\mathbf{k}$-vector spaces, another one – in the category of

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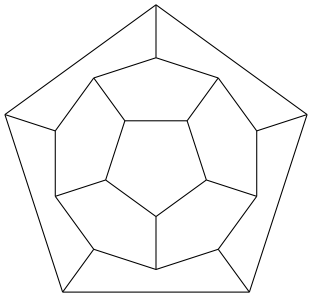


Figure 1. The dodecahedral graph G^d.

graded free $\mathbf{k}[E]$-modules, where $\deg E = 6$. Here $\mathbf{k}$ is a characteristic two field, which most of the time can be set to $\mathbb{F}_2$. The most general functor of this type defined in [7] takes values in the category of graded modules over the ring $R = \mathbf{k}[E_1, E_2, E_3]$ of symmetric polynomials in X_1, X_2, X_3, where E_1, E_2, E_3 are the elementary symmetric functions in these three variables and $\deg(X_i) = 2$, $i = 1, 2, 3$. In this paper we use the following three theories.

- The functor $\langle\bullet\rangle$ with state spaces – graded R-modules, with $R = \mathbf{k}[E_1, E_2, E_3]$ as above. This theory is based on the original foam evaluation in [7].
- The functor $\langle\bullet\rangle_0$ with state spaces – graded $\mathbf{k}$-vector spaces. It is defined by evaluating closed foams F to elements $\langle F\rangle_0$ of $\mathbb{F}_2 = \{0, 1\} \subset \mathbf{k}$ via the composition of the original evaluation $\langle F\rangle \in R$ and the homomorphism

$$R \to \mathbf{k} \cong R/(E_1, E_2, E_3).$$

  This evaluation just picks out the constant term in the evaluation $\langle F\rangle$ of a closed foam, viewed as a symmetric polynomial in X_1, X_2, X_3.
- The functor $\langle\bullet\rangle_E$ with state spaces – graded $\mathbf{k}[E]$-modules. This theory is defined via the evaluation taking closed foams F to elements $\langle F\rangle_E$ of $\mathbf{k}[E]$ via the composition of the original evaluation $\langle F\rangle \in R$ and the homomorphism $R \to \mathbf{k}[E]$ taking E_1, E_2 to 0 and E_3 to E.

  For each Γ, there are natural homomorphisms (base change homomorphisms)

$$\langle\Gamma\rangle \otimes_R \mathbf{k}[E] \to \langle\Gamma\rangle_E, \quad \langle\Gamma\rangle \otimes_R \mathbf{k} \to \langle\Gamma\rangle_0, \tag{1}$$

which induce natural transformations between the corresponding functors.

The smallest graph for which none of $\langle\bullet\rangle$, $\langle\bullet\rangle_E$, $\langle\bullet\rangle_0$, or $J^\sharp$ is fully understood is the dodecahedral graph depicted in Figure 1 and denoted G^d.

From [7, Proposition 4.18] and the behavior of the theory under the graded localization $R \to R[\mathcal{D}^{-1}]$, where $\mathcal{D} = E_3 - E_1E_2$ is the square root of the discriminant,

we know that $\langle G^d \rangle_E$ is a free graded $\mathbf{k}[E]$-module of rank 60 (the number of Tait colorings of G^d), but we do not know its graded rank, which is an element of $\mathbb{Z}_+[q, q^{-1}]$. The naive guess is that the graded rank of $\langle G^d \rangle_E$ is

$$10[3][2] = 10(q^{-2} + 1 + q^2)(q + q^{-1}).$$

Potentially, the rank or dimension of the state space of a graph Γ can drop upon changing the ground ring due to lack of commutativity between the base change operation on commutative rings and taking the quotient of the bilinear form on a free module by its kernel. The simplest example is given by the bilinear form (E) on a rank one free $\mathbf{k}[E]$-module V. The kernel is zero and the quotient by the kernel is V. Under the base change $\mathbf{k}[E] \to \mathbf{k}$ taking E to 0 bilinear form becomes (0), and the quotient by the kernel is the trivial $\mathbf{k}$-vector space. The latter is different from $V \otimes_{\mathbf{k}[E]} \mathbf{k} \cong \mathbf{k}$, showing a drop in dimension.

D. Boozer [4] wrote a program to compute approximations to the modules $\langle G^d \rangle$. His results show that $\dim \langle G^d \rangle_0 \geq 58$, which is, however, strictly smaller than 60, the number of Tait colorings of G^d. Boozer's program looks at several thousand foams bounded by G^d and computes the rank of the bilinear form used to define $\langle \bullet \rangle_0$ restricted to the subspace spanned by these foams. The rank is 58, not 60, indicating that the subspace spanned by these foams in $\langle G^d \rangle_0$ has dimension 58. An even stronger guess based on this data would be that $\langle G^d \rangle_0$ has dimension 58. Boozer's work [4] also suggests that the graded rank of $\langle G^d \rangle_E$ is

$$9q^{-3} + 20q^{-1} + 20q^1 + 11q^3,$$

rather than the naive guess given by formula (2) below, and that the natural map

$$\langle G^d \rangle_E \otimes_{\mathbf{k}[E]} \mathbf{k} \to \langle G^d \rangle_0$$

has a nontrivial kernel.

It follows from [7, 10] that $\langle \Gamma \rangle_0$ is naturally a subquotient of $J^\sharp(\Gamma)$ and

$$\dim(J^\sharp(\Gamma)) \geq \dim\big(\langle \Gamma \rangle_0\big).$$

Kronheimer and Mrowka [8,9] proved that $\dim J^\sharp(\Gamma) \geq |\mathrm{Tait}(\Gamma)|$ for any graph Γ, in particular,

$$\dim J^\sharp(G^d) \geq 60 = |\mathrm{Tait}(G^d)|.$$

This may suggest that $J^\sharp(G^d) \ncong \langle G^d \rangle_0$. At the same time, one can prove that $\mathrm{rk}\langle \Gamma \rangle_E = |\mathrm{Tait}(\Gamma)|$ for any Γ and, consequently, $\langle \Gamma \rangle_E$ is a free graded $\mathbf{k}[E]$-module of rank $|\mathrm{Tait}(\Gamma)|$. The graded rank of $\langle \Gamma \rangle_E$, an element of $\mathbb{Z}_+[q, q^{-1}]$, seems complicated to determine or understand, though, already for G^d.

In this paper we enhance the domain category of $\langle\bullet\rangle$ by extending the foams under consideration to *conical foams*. The category of foams Foam gets enlarged to the category Foam^c of conical foams that comes with evaluation functors $\langle\bullet\rangle^c$, $\langle\bullet\rangle_0^c$ and $\langle\bullet\rangle_E^c$ into categories of graded modules over rings R, $\mathbf{k}$ and $\mathbf{k}[E]$, correspondingly. The superscript c stands for *conical*. One advantage of this construction is that for the dodecahedral graph the resulting state spaces have the desired size:

Proposition 1.1. *The graded dimension or rank of vector spaces or modules $\langle G^d\rangle_0^c$, $\langle G^d\rangle^c$ and $\langle G^d\rangle_E^c$ is*

$$10[2][3] = 10(q+q^{-1})(q^2+1+q^{-2}). \tag{2}$$

Foams of a new kind introduced in this paper have singularities which are cones over planar trivalent graphs endowed with equivalence classes of Tait colorings of these graphs satisfying a technical condition. Our construction does not answer the original question about the difference in dimensions for $J^\sharp(G^d)$ and $\langle G^d\rangle_0$. Rather, it enlarges the state spaces $\langle\Gamma\rangle_0$ to $\langle\Gamma\rangle_0^c$ to achieve the desired dimension 60 for the dodecahedral graph.

We do not know whether $\langle\Gamma\rangle^c$ is larger than $\langle\Gamma\rangle$ for some graph Γ. We also don't know the answer to the same question for the $\langle\Gamma\rangle_0^c$ versus $\langle\Gamma\rangle_0$ or $\langle\Gamma\rangle_E^c$ versus $\langle\Gamma\rangle_E$. The first potential example to further investigate is G^d.

In Section 2 the foam evaluation from [7] is extended to conical foams. We construct functors $\langle\bullet\rangle^c$, $\langle\bullet\rangle_0^c$ and $\langle\bullet\rangle_E^c$ from the category of conical foams Foam^c to categories of graded R-modules, $\mathbf{k}$-vector spaces and $\mathbf{k}[E]$-modules. In Section 3 we compute the state spaces of the dodecahedron and prove Proposition 1.1. In Section 4 we discuss the technical condition which appear in the definition of conical foams.

2. Conical foams

2.1. Tait colorings and Kempe moves

A *web* is a finite trivalent graph PL-embedded in $\mathbb{R}^2$. It may have multiple edges between a pair of vertices and loops. Furthermore, loops without vertices are allowed as well. A *Tait coloring* of a web $\Gamma = (V(\Gamma), E(\Gamma))$ is a map $c\colon E(\Gamma)\to I_3$ from the set of edges to the 3-element set $I_3 = \{1,2,3\}$ such that at each vertex the three adjacent edges are mapped to distinct elements in I_3. A vertex-less loop counts as an edge, an element of $E(\Gamma)$, and under c is also mapped to an element of I_3. The set of Tait colorings of a given web Γ is denoted by $\mathrm{Tait}(\Gamma)$. There is a natural action of S_3 on $\mathrm{Tait}(\Gamma)$ by permutations of colors.

If c is a coloring of Γ and i and j are two distinct elements of $\{1, 2, 3\}$, denote by $\Gamma_{ij}(c)$ the graph $(V(\Gamma), E_{ij}(\Gamma, c))$, where

$$E_{ij}(\Gamma, c) = \{e \in E(\Gamma) \mid c(e) \in \{i, j\}\}.$$

The graph $\Gamma_{ij}(c)$ is a bivalent graph and therefore a disjoint union of cycles of even lengths, possibly including length zero (when a circle of Γ is colored i or j). The number of such cycles is denoted $d_{ij}(\Gamma, c)$.

Let C be a connected component of $\Gamma_{ij}(c)$. Swapping the colors i, j of c along C provides a new coloring c' of Γ. One says that c and c' are related by a *Kempe move along* C. Two elements c and c' of $\mathrm{Tait}(\Gamma)$ are *Kempe equivalent* if there exists a finite sequence of Kempe moves transforming c into c', possibly for different pairs (i, j). Kempe equivalence is an equivalence relation, with equivalence classes called *Kempe classes*.

Proposition 2.1 ([5, Theorem 1]). *If a web Γ is bipartite then all Tait colorings are Kempe equivalent.*

A Kempe class κ is called *homogeneous* if the sum

$$d(\Gamma, c) := d_{12}(\Gamma, c) + d_{23}(\Gamma, c) + d_{13}(\Gamma, c) \tag{3}$$

is independent of the coloring $c \in \kappa$. This integer is called the *degree of* κ, for a homogeneous κ, and denoted by $d(\kappa)$ or $d(\Gamma, \kappa)$.

- A web is called *weakly homogeneous* if it admits at least one homogeneous Kempe class.
- A web Γ is called *homogeneous of degree* d if it is homogeneous and each Kempe class of Γ has the same degree d. For such webs, $d(\Gamma) := d$ is called the *Kempe-degree of* Γ.

Example 2.2. (1) The simplest homogeneous webs are the empty web, denoted by $\emptyset_1$ and the circle web, of degrees 0 and 2, respectively. The empty web has a unique Tait coloring and its Kempe class has degree 0. The circle web admits three Tait colorings that are all Kempe equivalent. This unique Kempe class κ is homogeneous and $d(\kappa) = 2$.

(2) The Θ-web is the web with two vertices and three edges connecting them. It admits six Tait colorings which are all Kempe equivalent. This unique Kempe class κ is homogeneous, with $d(\kappa) = 3$, so the Θ-web is homogeneous of Kempe-degree three.

(3) The tetrahedral web K_4 is the web obtained by embedding the complete graph on four vertices into the plane. It admits six Tait colorings which are all Kempe equivalent. The unique Kempe class κ is homogeneous and $d(\kappa) = 3$.

Figure 2. Two Tait colorings c and c' of the cube C_3. They are related by a Kempe move along the inner square. One has $d(C_3, c) = 2 + 2 + 2$ and $d(C_3, c') = 2 + 1 + 1$.

(4) The cubic web C_3 is the embedding of the 1-skeleton of the 3-dimensional cube in the plane. It admits 24 Tait colorings. Since it is bipartite, all its Tait colorings are Kempe equivalent. However, C_3 has Tait colorings c and c' with different degrees $d(C_3, c) = 6$ and $d(C_3, c') = 4$, see formula (3) and Figure 2. This implies that the cube graph C_3 is not weakly homogeneous.

(5) Section 3 investigates the dodecahedral graph G^d. The latter turns out to admit 60 Tait colorings which are partitioned into 10 Kempe classes, each homogeneous of degree three. Consequently, G^d is a homogeneous web of Kempe-degree three.

Define a *cone* over a web Γ in $\mathbb{R}^2$ by considering $\mathbb{R}^2$ as the boundary of $\mathbb{R}^3_+$. Choose a point p in the interior of $\mathbb{R}^3_+$ and form the cone $\mathrm{Cone}(\Gamma)$ over Γ by connecting p by straight intervals to all points of Γ. As a topological space,

$$\mathrm{Cone}(\Gamma) \cong \Gamma \times [0,1] \,/\, \Gamma \times \{0\},$$

and comes with an embedding into $\mathbb{R}^3$. The space $\mathrm{Cone}(\Gamma)$ may have a singularity at p which is more complicated than singular points of foams in [7]. The cone inherits from Γ the structure of a two-dimensional combinatorial CW-complex. The *pointed cone* $\mathrm{Cone}_\star(\Gamma)$ is the pair $(\mathrm{Cone}(\Gamma), p)$.

2.2. Foams

A *conical foam* is a finite 2-dimensional CW-complex F PL-embedded in $\mathbb{R}^3$ such that for any point p in F there exists a small 3-ball B centered in p such that $(F \cap B, p)$ is PL-homeomorphic to a pointed cone over a connected web $\Gamma(p)$ in $\mathbb{S}^2$. Note that the PL-isotopy type of $\Gamma(p) \subset \mathbb{S}^2$ is well-defined. The web $\Gamma(p)$ is the link of the vertex p of F.

A point p in F is *regular* if $\Gamma(p)$ is a circle, it is a *seam point* if $\Gamma(p)$ is a Θ-web, otherwise it is a *singular point*.

If for any singular point p of F, $\Gamma(p)$ is a tetrahedral graph K_4, then this definition of a foam is the same as in [7]. Such foams are called *regular* foams. The union of seams and singular points of F is denoted $s(F)$. The set $s(F)$ inherits from F

Figure 3. From left to right: the circle, the Θ-web, the tetrahedron and the cube.

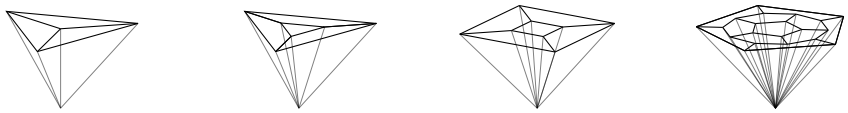


Figure 4. Examples of possible singular points appearing in conical foams.

a structure of finite 1-dimensional CW-complex except possibly for several circles in $s(F)$ that do not have singular points on them. We view $s(F)$ as a graph (possibly with multiple edges, loops, and circular edges without vertices). Vertices of $s(F)$ are the singular points of F. W call $s(F)$ the *seam graph* of F.

The connected components of $F \setminus s(F)$ are the *facets* of F. Denote by $f(F)$ the set of facets of F. A foam may carry a finite number of dots on its facets, which can freely float on a facet but are not allowed to move across seams into adjacent facets.

A conical foam F is *adorned* if for every singular point p of F there is a preferred Kempe class $\kappa(p)$ of $\Gamma(p)$. An adorned conical foam F is *homogeneous* if for any singular point p of F, $\kappa(p)$ is homogeneous.

2.3. Colorings

Let F be an adorned conical foam. An *admissible coloring* of F (or simply a *coloring*) is a map $c\colon f(F) \to \{1,2,3\}$ such that:

- For each seam of F, the three facets f_1, f_2 and f_3 adjacent to the seam carry different colors, that is, $\{c(f_1), c(f_2), c(f_3)\} = \{1,2,3\}$.
- For each singular point p of F, the Tait-coloring $c_{\Gamma(p)}$ of $\Gamma(p)$ induced by c is in the Kempe class $\kappa(p)$.

The set of admissible coloring of F is denoted $\mathrm{adm}(F)$.

Let c be an admissible coloring of F. For two distinct elements $i \neq j$ of $\{1,2,3\}$ denote by $\widetilde{F}_{ij}(c)$ the closure in F of the union of facets f such that $f(c) \in \{i,j\}$. This is a CW-complex embedded in $\mathbb{R}^3$. Near seam points and regular points contained in $\widetilde{F}_{ij}(c)$, the space $\widetilde{F}_{ij}(c)$ is locally PL-homeomorphic to a disk. Near a singular point p, $\widetilde{F}_{ij}(c)$ is PL-homeomorphic to a cone over a disjoint union of $d_{ij}(\Gamma(p), c_{\Gamma(p)})$ circles.

To *smooth out* $\widetilde{F}_{ij}(c)$ *at* p means to replace this cone by a collection of embedded disks $d_{ij}(\Gamma(p), c_{\Gamma(p)})$ by separating the disks that meet at the vertex p. An example of

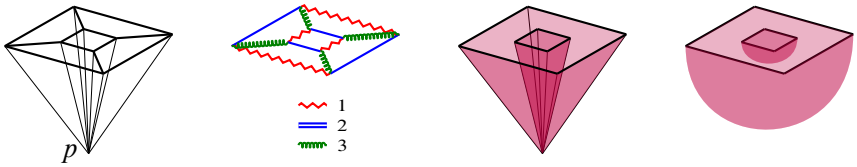


Figure 5. Smoothing out $\widetilde{F}_{12}(c)$ at a singular point p. From left to right: a neighborhood U of singular point p; the coloring induced by c on its link $\Gamma(p)$; the space $\widetilde{F}_{12}(c) \cap U$; the surface $F_{12}(c) \cap U$. Note, though, that graph $\Gamma(p)$ is not even weakly homogeneous.

this operation is given in Figure 5. Note that if $d_{ij}(\Gamma(p), c_{\Gamma(p)}) = 1$, there is only one disk near p and this procedure does not change the neighborhood of p. The *bicolored surface* $F_{ij}(c)$ is obtained by smoothing out $\widetilde{F}_{ij}(c)$ at every singular point of F. It is a surface embedded in $\mathbb{R}^3$ and therefore has even Euler characteristic.

Lemma 2.3. *For an homogeneous adorned conical foam F and its coloring c the following relation holds:*

$$\begin{aligned}\chi(F_{12}(c)) + \chi(F_{13}(c)) + \chi(F_{23}(c)) \\ = 3\chi(s(F)) + 2\sum_{f\in f(F)} \chi(f) + \sum_{p\in p(F)} (d(\kappa(p)) - 3).\end{aligned} \tag{4}$$

In particular, this quantity is independent from the admissible coloring c.

Proof. Let us first prove that

$$\chi(\widetilde{F}_{12}(c)) + \chi(\widetilde{F}_{13}(c)) + \chi(\widetilde{F}_{23}(c)) = 3\chi(s(F)) + 2\sum_{f\in f(F)} \chi(f). \tag{5}$$

Here $\widetilde{F}_{12}(c)$ is a finite CW-complex which can be constructed starting with $s(F)$ and attaching to it 1- and 2-colored facets of F. Hence,

$$\chi(\widetilde{F}_{12}(c)) = \chi(s(F)) + \sum_{f\in f_1(F,c)} \chi(f) + \sum_{f\in f_2(F,c)} \chi(f),$$

where $f_i(F, c)$ denotes the set of i-colored facets of F with the coloring c. The same holds for $\widetilde{F}_{13}(c)$ and $\widetilde{F}_{23}(c)$. Summing over these identities, one obtains (5).

Let p be a singular point of F. Smoothing out $\widetilde{F}_{ij}(c)$ at p amounts to removing a point in $\widetilde{F}_{ij}$ and replacing it with $d_{ij}(\Gamma(p), c_{\Gamma(p)})$ points: it increases the Euler characteristic by $d_{ij}(\Gamma(p), c_{\Gamma(p)}) - 1$. This proves (4). ∎

The *degree* $d(F)$ of an homogeneous adorned foam F is given by the following formula:

$$d(F) = 2\#\{\text{dots on } F\} - 3\chi(s(F)) - 2\sum_{f\in f(F)}\chi(f) - \sum_{p\in p(F)}(d(\kappa(p)) - 3). \quad (6)$$

2.4. Foam evaluation

Foam evaluation was introduced in [15] and adapted by the authors [7] to the unoriented $SL(3)$ case. In this section we briefly review this construction in the generalized framework of homogeneous adorned conical foams. Let

$$R' = \mathbf{k}[X_1, X_2, X_3] \quad (7)$$

be the graded ring of polynomials in X_1, X_2, X_3, with each X_i in degree 2. Denote by E_1, E_2 and E_3 the elementary symmetric polynomials in X_1, X_2, X_3. They have degrees 2, 4 and 6 respectively and generate the subring of S_3-invariants

$$R := (R')^{S_3} \simeq \mathbf{k}[E_1, E_2, E_3]. \quad (8)$$

Let

$$R'' := R'\left[\frac{1}{X_i + X_j}\right]_{1\le i<j\le 3}. \quad (9)$$

Finally, for $i \neq j$, denote by $R''_{ij} \subset R''$ the graded ring

$$R'\left[\frac{1}{(X_i + X_k)(X_j + X_k)}\right],$$

for $\{i, j, k\} = \{1, 2, 3\}$. One has $R' = R''_{12} \cap R''_{13} \cap R''_{23}$. There are inclusions of rings

$$R \subset R' \subset R''_{ij} \subset R''. \quad (10)$$

From now on a *conical foam* means a homogeneous adorned conical foam. Let F be a conical foam and c an admissible coloring of F.

Define

$$Q(F, c) = (X_1 + X_2)^{\frac{\chi(F_{12}(c))}{2}}(X_2 + X_3)^{\frac{\chi(F_{23}(c))}{2}}(X_1 + X_3)^{\frac{\chi(F_{13}(c))}{2}}, \quad (11)$$

$$P(F, c) = \prod_{f\in f(F)} X_{c(f)}^{n(f)}, \quad (12)$$

where $n(f)$ denotes the number of dots on the facet f. The *evaluation* of the colored conical foam (F, c) is the rational function $\langle F, c\rangle$ given by

$$\langle F, c\rangle := \frac{P(F, c)}{Q(F, c)} \in R''. \quad (13)$$

The *evaluation* of the conical foam F is defined by

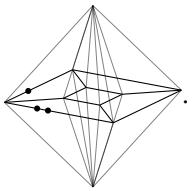
$$\langle F\rangle = \sum_{c\in \mathrm{adm}(F)} \langle F, c\rangle. \tag{14}$$

Proposition 2.4 ([7, Theorem 2.17]). *For any conical foam F the evaluation $\langle F\rangle \in R$ is a symmetric polynomial in X_1, X_2 and X_3. It is homogeneous of degree $d(F)$.*

The proof in [7] extends to conical foams in a straightforward way. The nontrivial point is to show that $\langle F\rangle$ is a polynomial. For this purpose, one shows that $\langle F\rangle$ belongs to R''_{ij} for all pairs $\{i, j\} \in \{1, 2, 3\}$. The above evaluation coincides with the one in [7] for the usual foams.

2.5. Trying to extend foam evaluation to non-homogeneous foams

There are two problems with extending our conical foam evaluation to foams F with non-homogeneous singular vertices. In principle we can use formulas (13) and (14) to define $\langle F\rangle$. However, with this definition $\langle F\rangle$ may not be homogeneous. Second, the evaluation may have nontrivial denominators and be a rational function rather than a polynomial. For instance, consider

$F :=$  .

This non-homogeneous conical foam F consists of two cones over the square web glued along their common boundary with two dots on one facet, one dot on an adjacent facet, and no dots on the other facets. Since the square web has only one Kempe class of colorings, F can be considered an adorned conical foam. A straightforward computation shows that foam evaluation $\langle F\rangle$ is given by the following formula:

$$\langle F\rangle = \frac{1 + X_1^2 + X_2^3 + X_3^2 + X_1X_2 + X_1X_3 + X_2X_3}{(X_1 + X_2)(X_1 + X_3)(X_2 + X_3)},$$

which is neither a polynomial, nor a homogeneous rational function.

Here is what goes wrong with evaluations for such more general conical foams. Suppose one is doing, say, a $(1, 2)$-Kempe move on an $\mathbb{S}^2$-component of $F_{12}(c)$ for some coloring c of F (these are the components that contribute to the denominators in $\langle F, c\rangle$). To ensure cancellation with $x_1 + x_2$ in the denominator, one needs the balancing relation

$$\chi(F_{13}(c)) + \chi(F_{23}(c)) = \chi(F_{13}(c')) + \chi(F_{23}(c')) \tag{15}$$

on the Euler characteristics of the bicolored surfaces for c and c', see [7, Lemmas 2.12 and 2.18]. This relation, in general, fails to hold if we do not impose homogeneity requirement (3) from Section 2 on the Kempe classes and restrict to such classes only. The quantity $d(\Gamma, c)$ in (3) describes the contribution of the vertex which is a cone over Γ to the sum of Euler characteristics

$$\chi(F_{12}(c)) + \chi(F_{13}(c)) + \chi(F_{23}(c)),$$

see also Lemma 2.3, and this quantity needs to be invariant under Kempe moves to have a homogeneous and integral (polynomial) evaluation. For c, c' related by a $(1,2)$-Kempe move, $F_{12}(c) \cong F_{12}(c')$, so the invariance reduces to that in (15).

2.6. State spaces

Using conical foam evaluation we construct three functors $\langle\bullet\rangle^c$, $\langle\bullet\rangle^c_E$ and $\langle\bullet\rangle^c_0$ from the category Foam^c described below to categories R-mod, $\mathbf{k}[E]$-mod and $\mathbf{k}$-vect. The ground field $\mathbf{k}$ can be taken to be $\mathbb{F}_2$, but occasionally it is useful to take an arbitrary characteristic two field as $\mathbf{k}$.

A *conical foam with boundary* is a finite 2-dimensional CW-complex F properly PL-embedded in $\Sigma \times [0;1]$ (for $\Sigma = \mathbb{R}^2$ or $\mathbb{S}^2$), such that there exist $\varepsilon > 0$ and two webs Γ_0 and Γ_1 such that

$$\Sigma \times [0;\varepsilon) \cap F = \Gamma_0 \times [0;\varepsilon) \quad \text{and} \quad \Sigma \times (1-\varepsilon;1] \cap F = \Gamma_1 \times (1-\varepsilon;1]$$

and for any point p in $F \cap \Sigma \times (0;1)$ there exists a small 3-ball B centered in p such that $(F \cap B, p)$ is PL-homeomorphic to a pointed cone over a connected web $\Gamma(p)$ in $\mathbb{S}^2$. Conical foams with boundary carry dots just like conical foams. The concepts of homogeneous and adorned foams extend mutatis mutandis to conical foams with boundary.

The category Foam^c has webs (either in $\mathbb{R}^2$ or $\mathbb{S}^2$) as objects. If Γ_0 and Γ_1 are two webs, then $\mathrm{Hom}_{\mathsf{Foam}^c}(\Gamma_0, \Gamma_1)$ consists of homogeneous adorned conical foams U with boundary properly embedded in $\mathbb{R}^2 \times [0;1]$ (respectively, in $\mathbb{S}^2 \times [0;1]$) such that

$$\partial U = U \cap \{0,1\} = \Gamma_0 \times \{0\} \cup \Gamma_1 \times \{1\}.$$

Isotopic rel boundary conical foams define the same morphism in Foam^c. Composition is given by concatenation.

The category Foam is a subcategory of Foam^c that consists of foams in the usual sense, as in [7]. It has the same objects as Foam^c, and a morphism F in Foam^c is a morphism in Foam if for every singular point p of F, $\Gamma(p)$ is the tetrahedral graph K_4. Note that since K_4 has only one Kempe class of Tait colorings, there is only one way to adorn singular point p. Such singular points are standard singular points on a foam

as defined in [7], and the category Foam is the original category of unoriented $SL(3)$ foams in [7].

Functors $\langle\bullet\rangle^c$, $\langle\bullet\rangle^c_E$ and $\langle\bullet\rangle^c_0$ are defined via universal construction, see [3, 6], just as in [7]. Let S be a graded unital $\mathbf{k}$-algebra and $\phi\colon R \to S$ a homomorphism of graded unital commutative $\mathbf{k}$-algebras. The homomorphisms considered here are:

- $\mathrm{id}_R\colon R \to R$,
- $\phi_E\colon R \to \mathbf{k}[E]$ given by $\phi_E(E_i) = \delta_{i\,3}E$,
- $\phi_0\colon R \to \mathbf{k}$ given by $\phi_0(E_i) = 0$.

For a web Γ, we define $W_S(\Gamma)$ to be the free graded S-module generated by $\mathrm{Hom}_{\mathsf{Foam}^c}(\emptyset_1, \Gamma)$. Consider the symmetric bilinear form $(\,,\,)_\phi$ given on generators of $W_S(\Gamma)$ by:

$$(F, G)_\phi := \phi\big(\langle \overline{G} \circ F\rangle\big) \in S,$$

where $\overline{G} \in \mathrm{Hom}(\Gamma, \emptyset_1)$ is the mirror image of G with respect to the plane $\mathbb{R}^2 \times \{\frac{1}{2}\}$ (respectively, $\mathbb{S}^2 \times \{\frac{1}{2}\}$). Define

$$\langle\Gamma\rangle^c_\phi := W_S(\Gamma)/\ker((\,,\,)_\phi) \tag{16}$$

as the quotient of $W_S(\Gamma)$ by the kernel of $(\,,\,)_\phi$. These quotients extend to a functor

$$\langle\bullet\rangle^c_\phi\colon \mathsf{Foam}^c \to S\text{-}\mathsf{mod} \tag{17}$$

in a canonical way: if $F \in \mathrm{Hom}_{\mathsf{Foam}^c}(\Gamma, \Gamma')$, $\langle F\rangle^c_\phi$ is the S-linear map which maps the equivalence class of $H \in \mathrm{Hom}_{\mathsf{Foam}^c}(\emptyset_1, \Gamma)$ to the equivalence class of $F \circ H \in \mathrm{Hom}_{\mathsf{Foam}^c}(\emptyset_1, \Gamma')$. Since the homomorphism assigned to a conical foam F has degree $d(F)$, one can use the category of graded modules and homogeneous module homomorphisms, denoted S-mod here, as the target category. This category is not pre-additive, since the sum of two homogeneous homomorphisms of different degrees is not homogeneous.

Functors $\langle\bullet\rangle^c$, $\langle\bullet\rangle^c_E$ and $\langle\bullet\rangle^c_0$ correspond to the choices $\phi = \mathrm{id}_R$, $\phi = \phi_E$ and $\phi = \phi_0$, respectively, see above. Functors $\langle\bullet\rangle$, $\langle\bullet\rangle_E$ and $\langle\bullet\rangle_0$ and, more generally, $\langle\bullet\rangle_\phi$ are defined in [7] in exactly the same manner but using the category Foam instead of Foam^c.

A closed conical foam F is an endomorphism of the empty web $\emptyset_1$. $\langle F\rangle^c_\phi$ is then an endomorphism of $\langle\emptyset_1\rangle^c_\phi \cong S$, hence an element of S, and

$$\langle F\rangle^c_\phi = \phi\big(\langle F\rangle\big).$$

Proposition 2.5. *For every web Γ and every graded ring homomorphism $\phi\colon R \to S$, the graded S-module $\langle\Gamma\rangle_\phi$ is a subquotient of $\langle\Gamma\rangle^c_\phi$.*

Proof. This follows from the general fact that if W is an S-module endowed with a bilinear form and $V \subseteq W$ is a submodule, then $V/\ker(\,,\,)_V$ is a subquotient of $W/\ker(\,,\,)$. Indeed, there is the following diagram:

$$W/\ker(\,,\,) \hookleftarrow V/(\ker(\,,\,) \cap V) \twoheadrightarrow V/\ker(\,,\,)_{|V}. \qquad \blacksquare$$

We can consider an intermediate theory $\langle\bullet\rangle^i$ that mediates between $\langle\bullet\rangle$ and $\langle\bullet\rangle^c$. Given a web Γ, define $\langle\Gamma\rangle^i$ as the subspace of $\langle\Gamma\rangle^c$ spanned by the usual foams in Foam with boundary Γ and not by conical foams with more general singularities. This collection of subspaces $\langle\Gamma\rangle^i$, over all Γ, extends to a functor from Foam to the category of graded modules over R. The space $\langle\Gamma\rangle$ is naturally a quotient of $\langle\Gamma\rangle^i$. The inclusion and the quotient together constitute a diagram

$$\langle\Gamma\rangle^c \hookleftarrow \langle\Gamma\rangle^i \twoheadrightarrow \langle\Gamma\rangle \tag{18}$$

that commutes with maps induced by foams. Given a foam F with $\partial_0(F) = \Gamma_0$ and $\partial_1(F) = \Gamma_1$, there is a commutative diagram

$$\begin{array}{ccccc} \langle\Gamma_0\rangle^c & \hookleftarrow & \langle\Gamma_0\rangle^i & \twoheadrightarrow & \langle\Gamma_0\rangle \\ \downarrow{\scriptstyle \langle F\rangle^c} & & \downarrow{\scriptstyle \langle F\rangle^c} & & \downarrow{\scriptstyle \langle F\rangle} \\ \langle\Gamma_1\rangle^c & \hookleftarrow & \langle\Gamma_1\rangle^i & \twoheadrightarrow & \langle\Gamma_1\rangle. \end{array} \tag{19}$$

Consequently, there are natural transformations of functors

$$\langle\bullet\rangle^c \hookleftarrow \langle\bullet\rangle^i \twoheadrightarrow \langle\bullet\rangle, \tag{20}$$

where these three flavors of $\langle\bullet\rangle$ are viewed as functors on the category Foam. Note that $\langle\bullet\rangle^c$ is a functor on the larger category Foam^c.

We do not know a graph where even one of the maps in (18) can be shown not to be an isomorphism. It is an open problem to compute the spaces and maps in (18) for the dodecahedral graph G^d. Note however that Proposition 3.2 computes $\langle G^d\rangle^c$.

Many properties of functors $\langle\bullet\rangle_\phi$ proved in [7] hold (with the same proofs) for functors $\langle\bullet\rangle^c_\phi$, including [7, Propositions 3.10 to 3.16].

Unoriented $SL(3)$ foams are remarkable in that we do not yet know a way to determine the state space of an arbitrary unoriented $SL(3)$ web. Oriented $SL(3)$ foams and webs are much better understood, and a basis in the state space of any oriented $SL(3)$ web can be constructed recursively, see [6,12–14] and [11], where $SL(3)$ webs (Kuperberg webs) were introduced.

3. State space of the dodecahedron

In this section S is a graded ring and $\phi\colon R \to S$ is a graded ring homomorphism. If M is a $\mathbb{Z}$-graded S-module, $M\{q^i\}$ denotes the graded S-module M with the grading shifted up by i. In other words, $(M\{q^i\})_j = M_{j-i}$. If $P = \sum_{i\in\mathbb{Z}} a_i q^i$ is a Laurent polynomial in q with nonnegative integer coefficients, $M\{P\}$ denotes the S-module

$$\bigoplus_{i\in\mathbb{Z}} M\{q^i\}^{a_i}.$$

The state space $\langle \emptyset_1\rangle^c_\phi$ of the empty web $\emptyset_1$ is a graded free S-module with the generator in degree zero represented by the empty foam $\emptyset_2$,

$$\langle \emptyset_1\rangle^c_\phi \cong S\langle \emptyset_2\rangle_\phi \cong S. \tag{21}$$

3.1. Theta-web

Before inspecting state spaces of the dodecahedron, let us inspect state spaces of the Θ-web, which are well-understood without any need for conical foams.

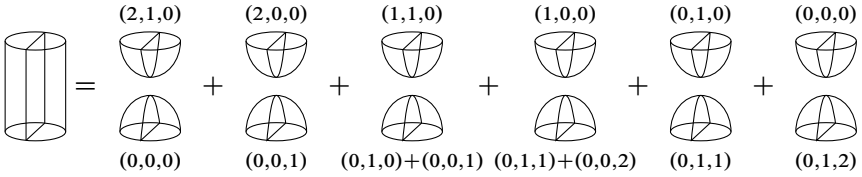
Proposition 3.1. *Let Γ be the Θ-web. Then the following isomorphism holds:*

$$\langle \Gamma\rangle^c_\phi \simeq \langle \emptyset_1\rangle^c_\phi\{(q^{-1}+q)(q^{-2}+1+q^2)\} \simeq S\{[2][3]\}.$$

The above notations use quantum $[n] = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$.

This proposition follows directly from [7, Propositions 3.12 and 3.13], but it is instructive to give a "one-step" proof.

Proof of Proposition 3.1. Let us first show that foam evaluation satisfies the following local identity:



(22)

where the triples of integers on the top and bottom encode the number of dots on each of the three facets. Note that on the right-hand side of the identity, two middle terms are each a sum of two foams.

It is convenient to introduce some notations at this point: let us fix a web Γ, v a vertex of Γ and denote e_1, e_2 and e_3 the three edges of Γ adjacent to v. For $i = 1, \ldots, 6$ define x_i and y'_i as the foams $\Gamma \times [0, 1]$ with dots on the facets $e_1 \times [0, 1]$, $e_2 \times [0, 1]$ and $e_3 \times [0, 1]$ given by Table 1.

Number of dots on:	$e_1 \times [0,1]$	$e_2 \times [0,1]$	$e_3 \times [0,1]$
x_1	2	1	0
x_2	2	0	0
x_3	1	1	0
x_4	1	0	0
x_5	0	1	0
x_6	0	0	0
y'_1	0	0	0
y'_2	0	0	1
y'_3	0	1	0
y'_4	0	0	2
y'_5	0	1	1
y'_6	0	1	2

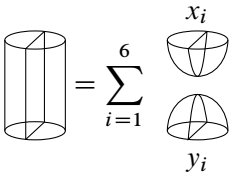
Table 1. Definition of x_i and y'_i.

Finally, for $i = 1, \ldots, 6$ define

$$\begin{cases} y_i := y'_i & \text{if } i \neq 3, 4, \\ y_3 := y'_2 + y'_3, & \\ y_4 := y'_3 + y'_4. & \end{cases} \tag{23}$$

The foams x_i and y_i are endomorphisms of Γ in Foam and Foam^c. In particular, $x_6 = y_1 = \mathrm{id}_\Gamma$.

Choosing v to be the top vertex of the Θ-foam and the edges e_1, e_2 and e_3 going from left to right, we can rewrite (22):

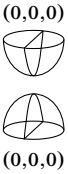


(24)

Local identity (22) (or (24)) says that no matter how the foams in the equation are completed to closed foams by an outside conical foam H, the corresponding linear relation on the evaluations holds.

Denote by F the closed conical foam given by gluing H onto the foam on the left-hand side of (24). Denote by G^i for $i \in \{1, \ldots, 6\}$ the six closed conical foams

given by gluing H to the six foams on the right-hand side of (24), going from left to right. Two of these six foams are actually sums of conical foams. Denote by G the conical foam which is identical to G_i's but without the dots that the latter carry in the region shown in (24):

$G :=$ 

The foam G can be viewed as the composition of two Θ-*half-foams*. These are two foams with Θ-web as the boundary and a single singular seam. They are reflections of each other about a horizontal plane. Their composition in the opposite order is the usual Θ-foam.

Conical foams $G^1, \dots, G^6$ and G are identical, except for the distribution of dots. In particular, their sets of colorings are canonically isomorphic. The set $\mathrm{adm}(G)$ can be partitioned into two subsets: colorings which restrict to the same colorings of the two Θ-webs on the top and bottom of the identity and the ones which restrict to different colorings on the Θ-webs on the top and bottom of the identity. The former is canonically isomorphic to $\mathrm{adm}(F)$, the latter is denoted $\mathrm{adm}(G)'$, so that one may write

$$\mathrm{adm}(G) \cong \mathrm{adm}(F) \sqcup \mathrm{adm}(G)'.$$

If c is an element of $\mathrm{adm}(F) \sqcup \mathrm{adm}(G)'$ we view it as a coloring of G and G_i as well.

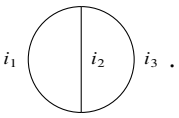
Let us prove that

$$\langle F, c\rangle = \sum_{i=1}^{6} \langle G^i, c\rangle \quad \text{for all } c \in \mathrm{adm}(F) \tag{25}$$

and

$$0 = \sum_{i=1}^{6} \langle G^i, c\rangle \quad \text{for all } c \in \mathrm{adm}(G)'. \tag{26}$$

To show (25), fix $c \in \mathrm{adm}(F)$. The top and bottom Θ-webs are colored by c as follows, where (i_1, i_2, i_3) is a permutation of $(1, 2, 3)$:



One has $\chi(F_{ij}(c)) = \chi(G_{ij}(c)) - 2$ for all pairs (i, j). Hence,

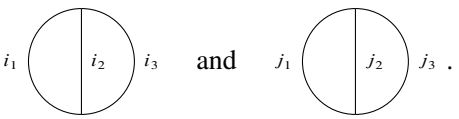
$$\langle F, c\rangle = \langle G, c\rangle (X_{i_1} + X_{i_2})(X_{i_1} + X_{i_3})(X_{i_2} + X_{i_3}).$$

Furthermore,

$$\begin{aligned}\sum_{i=1}^{6}\langle G_i, c\rangle &= \langle G, c\rangle\big(X_{i_1}^2 X_{i_2} + X_{i_2}^2 X_{i_3} + X_{i_1} X_{i_2}(X_{i_2} + X_{i_3}) \\ &\qquad + X_{i_1}(X_{i_2} X_{i_3} + X_{i_3}^2) + X_{i_2}^2 X_{i_3} + X_{i_2} X_{i_3}^2\big) \\ &= \langle G, c\rangle\big(X_{i_1}^2 X_{i_2} + X_{i_2}^2 X_{i_3} + X_{i_1} X_{i_2}^2 + X_{i_1} X_{i_3}^2 + X_{i_2}^2 X_{i_3} + X_{i_2} X_{i_3}^2\big) \\ &= \langle G, c\rangle(X_{i_1} + X_{X_{i_2}})(X_{i_1} + X_{i_3})(X_{i_2} + X_{i_3}) \\ &= \langle F, c\rangle,\end{aligned}$$

which proves (25).

To show (26), fix $c \in \mathrm{adm}(G)'$. Denote the induced Tait colorings of the top and bottom Θ-web boundaries of G as follows:



Then,

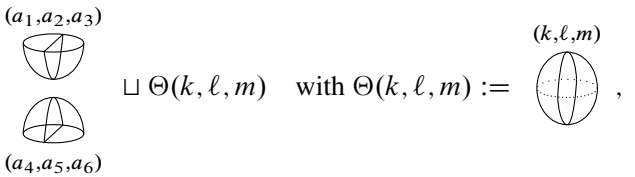
$$\begin{aligned}\sum_{i=1}^{6}\langle G_i, c\rangle &= \langle G, c\rangle\big(X_{i_1}^2 X_{i_2} + X_{i_1}^2 X_{j_3} + X_{i_1} X_{i_2}(X_{j_2} + X_{j_3}) \\ &\qquad + X_{i_1}(X_{j_2} X_{j_3} + X_{j_3}^2) + X_{i_2} X_{j_2} X_{i_3} + X_{j_2} X_{j_3}^2\big) \\ &= \langle G, c\rangle(X_{i_1} + X_{j_2})(X_{i_1} + X_{j_3})(X_{i_2} + X_{j_3}) \\ &= 0.\end{aligned}$$

For the last equality, observe that $(i_1, i_2, i_3) \neq (j_1, j_2, j_3)$ and, consequently, one of the three linear in X terms on the right-hand side is zero.

We next show that the terms on the right-hand side of (22) are pairwise orthogonal idempotents, as follows. Denote by P_i for $i = 1, \ldots, 6$ the foams with boundary on the right-hand side of (22). Any two of them, say P_i and P_j, compose by stacking one onto the other and gluing along the Θ-web, resulting in the foam $P_i P_j$. The foam evaluation satisfies the following local relation:

$$P_i P_j = \delta_{ij} P_i. \tag{27}$$

Note that $P_i P_j$ is always a sum of foams of the form



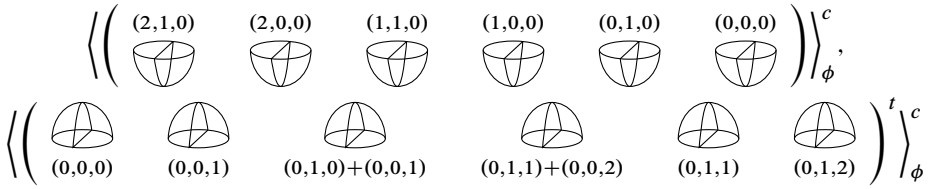
where k, ℓ and m denote the number of dots on the left, right and middle facets of the Θ-foam $\Theta(k, \ell, m)$. Hence, (27) is a direct consequence of the multiplicativity of foam

evaluation with respect to disjoint union and formulas for the evaluation of closed Θ-foams. One can easily check each of the 36 cases using the following evaluation rules

$$\langle \Theta(k,\ell,m)\rangle = \begin{cases} 1 & \text{if } \{k,\ell,m\} = \{0,1,2\}, \\ 0 & \text{if } \#\{k,\ell,m\} \leq 2, \end{cases}$$

R-linearity of closed foam evaluation, and various symmetries, to shorten the computation.

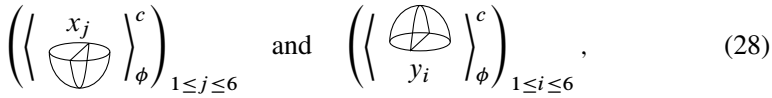
From the orthogonality relations (27) and taking degrees into account, one deduces that the matrices



give mutually inverse isomorphisms between the state space $\langle\Gamma\rangle_\phi^c$ and

$$\langle \emptyset_1\rangle_\phi^c\{(q^{-1}+q)(q^{-2}+1+q^2)\} \cong S\{[2][3]\},$$

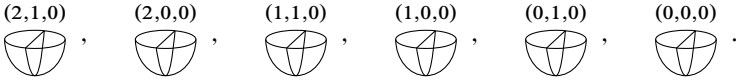
see also (21). In the definition of the matrices, the functor $\langle F\rangle_\phi^c$ is meant to be applied to each entry. Using x_i's and y_i's these matrices can be rewritten:



(28)

seen as line and column matrices, respectively. ∎

From identity (22), one obtains that the state space of the Θ-web Γ is a free S-module with a basis of foams



Due to the Θ-web Γ admitting a symmetry axis which does not contain vertices of Γ, the state space $\langle\Gamma\rangle$ is a Frobenius algebra over S rather than just an S-module. (The Frobenius algebra structures comes from the symmetry axis and from the fact that the functor $\langle\cdot\rangle$ is constructed by the universal construction, as explained in [15, Section 4.2].) The multiplicative structure can be seen as coming from the foam depicted in Figure 6.

The state space is a quotient of $S[Y_1, Y_2, Y_3]$, where

$$Y_1 = \overset{(1,0,0)}{\quad}, \quad Y_2 = \overset{(0,1,0)}{\quad}, \quad Y_3 = \overset{(0,0,1)}{\quad}.$$

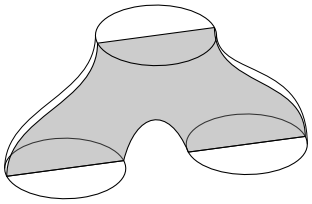


Figure 6. This pair of seamed pants gives the multiplicative structure on the state space of the Θ-web.

In the case $S = R = \mathbf{k}[E_1, E_2, E_3]$ and $\phi = \mathrm{id}_R$, one can show that

$$\langle\Gamma\rangle^c \simeq R[Y_1, Y_2, Y_3] \Big/ \begin{pmatrix} Y_1 + Y_2 + Y_3 - E_1, \\ Y_1Y_2 + Y_2Y_3 + Y_1Y_3 - E_2, \\ Y_1Y_2Y_3 - E_3 \end{pmatrix} \simeq \mathbf{k}[Y_1, Y_2, Y_3] \simeq R'.$$

The state space $\langle\Gamma\rangle^c$ is a free R-module of rank six and can be identified with the $GL(3)$-equivariant cohomology of the full flag variety of $\mathbb{C}^3$ with coefficients in $\mathbf{k}$. The ground ring R is isomorphic to the $GL(3)$-equivariant cohomology of a point with the same coefficient field.

3.2. Tait colorings of the dodecahedron

The dodecahedron graph G^d has 60 Tait colorings. A quick analysis indicates that for any Tait coloring c of G^d, the graphs $G^d_{12}(c)$, $G^d_{13}(c)$ and $G^d_{23}(c)$ are connected and therefore constitute three Hamiltonian cycles of G^d. Conversely, any Hamiltonian cycle can be used to construct six colorings by coloring alternatively the edges belonging to this cycle and using the third color for the remaining edges. Hence we have a 6:3 correspondence between Tait colorings and Hamiltonian cycles of G^d, so that G^d has 30 Hamiltonian cycles.

Let us investigate quickly the various symmetry at play. Since $G^d_{12}(c)$, $G^d_{13}(c)$ and $G^d_{23}(c)$ are connected for any coloring c of G^d, Kempe equivalence classes coincide with the equivalence classes for the permutation action of S_3 on Tait colorings. For more details about such webs, see Section 4.

The group A_5 of rotations of the regular dodecahedron has order 60 [2, Chapter 8] and acts with two orbits on the set of Hamiltonian cycles. The two orbits consist of 15 "clockwise" and 15 "counterclockwise" Hamiltonian cycles, respectively. Figure 7 shows one such cycle. It's determined by a pair of opposite edges in the dodecahedron graph (there are 15 such pairs, since there are 30 edges), and the direction of the 'twist' around the edge in a pair, either clockwise or counterclockwise, as the Hamiltonian cycle exits that edge. In Figure 7 the cycle goes clockwise.

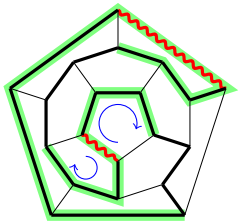


Figure 7. A clockwise Hamiltonian cycle in G^d.

The stabilizer subgroup of a cycle determined by a pair of opposite edges and a direction is $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset A_5$, generated by 180° rotation around the line through the center of the two edges and by the 180° rotation around the line parallel to the two opposite edges and through the center of the dodecahedron. Indeed the four elements of this subgroups are precisely the rotations that preserve the pair of edges and they happen to preserve the direction.

The group $A_5 \times \mathbb{Z}_2$ of all isometries of the dodecahedron (and not just rotations) acts transitively on the set of Hamiltonian cycles. Stabilizer subgroups for this action are the same as those for A_5.

3.3. State space

Proposition 3.2. *The following isomorphism of graded R-modules holds:*

$$\langle G^d \rangle^c \cong 10(q + q^{-1})(q^2 + 1 + q^{-2})\langle \emptyset_1 \rangle^c \cong R\{10[2][3]\}.$$

In particular, $\langle G^d \rangle^c$ is a free R-module of graded rank 10[2][3].

This decomposition holds for any homomorphism $\phi\colon R \to S$ of graded commutative rings as well, and

$$\langle G^d \rangle^c_\phi \cong S\{10[2][3]\}.$$

Proof. The proof uses a decomposition of the identity foam on G^d into a sum of 60 pairwise orthogonal idempotents.

Recall that in the unoriented $SL(3)$ theory the state space of the Θ-web is a free rank six module over R, the state space of the empty graph. This decomposition can be written via a relation similar to the neck-cutting relation for the identity foam on the circle graph. The latter has three terms on the right-hand side [7] and mimics the formula

$$a = \sum_{i=1}^{n} x_i \otimes \varepsilon_A(y_i a), \tag{29}$$

where A is a commutative Frobenius algebra over a ground ring R with dual bases $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ relative to the trace map $\varepsilon_A\colon A \to R$.

Note that this is not specific to the Θ-web. Indeed, such a decomposition in the unoriented $SL(3)$ theory [7] for the state space $\langle\Gamma\rangle$ of a web Γ always exists if $\langle\Gamma\rangle$ is a free graded module over R (necessarily of finite rank), with the number of terms equal to the rank of the module.

Recall that G^d is homogeneous of Kempe-degree 3 and has 10 different Kempe equivalence classes. We denote these classes by κ_j, where $1 \le j \le 10$, in some order.

Denote by $C_+ \subset \mathbb{R}^3_+$ the conical foam which is the cone over G^d. This is a conical foam with a single singular point p_+. The bottom boundary of C_+ is G^d and the top boundary is empty. Denote by C_- the mirror image of C_+ with respect to reflection about the horizontal plane and p_- the singular point of C_-.

Both C_+ and C_- are unadorned conical foams with boundary with a single singular point each. We think of C_+ as a singular cap and C_- as a singular cup. Composing these unadorned conical foams along the common boundary G^d results in a closed unadorned conical foam C_+C_- with two singular points (see Figure 8).

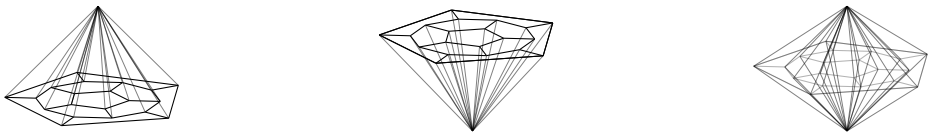


Figure 8. From left to right, conical foams C_+, C_- and C_+C_-.

For each Kempe class κ of G^d, denote by $C_+(\kappa)$ (respectively $C_-(\kappa)$) the homogeneous adorned conical foam with boundary C_+ (respectively, C_-), where p_+ (respectively, p_-) is mapped to κ.

The closed homogeneous adorned foam $C_+(\kappa)C_-(\kappa')$ is given by composition of these two conical foams along the common boundary G_d. It admits a Tait coloring if and only if $\kappa = \kappa'$. When this is the case, there are exactly six Tait colorings of $C_+(\kappa)C_-(\kappa)$. Furthermore, together they constitute a single S_3 orbit on the set of Tait colorings of this homogeneous adorned conical foam.

The unadorned conical foam with boundary C_-C_+ is the disjoint union of C_- and C_+, and it induces an endomorphism of G^d. It has two singular points p_- and p_+. For $j = 1, \dots, 10$, denote by P_j adorned conical foam with boundary $C_-(\kappa_j)C_+(\kappa_j)$. This is obtained by mapping both p_- and p_+ to κ_j. Conical foam P_j yields an endomorphism of the state space $\langle G^d\rangle$.

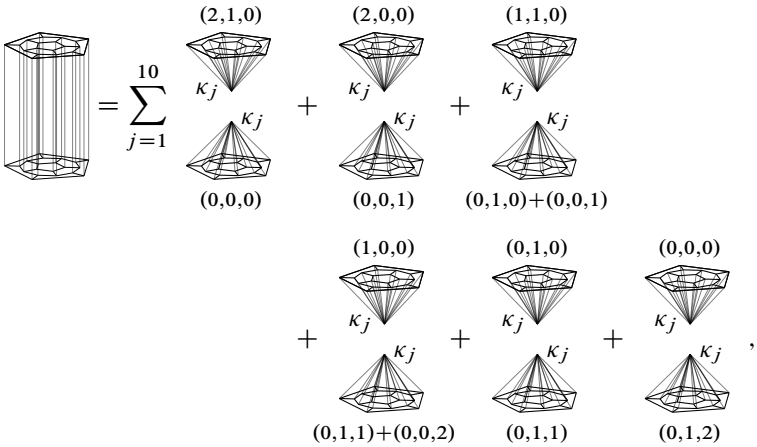
Choose a vertex v of G^d and denote its adjacent edges by e_1, e_2 and e_3. We use foams x_i's and y_i's ($1 \le i \le 6$) introduced in the proof of Proposition 3.1. In our context they are endomorphisms of web G^d which differ from $G^d \times [0, 1]$ only by their dot distributions which are given by (23) and Table 1.

We claim that foam evaluation satisfies the following local relation:

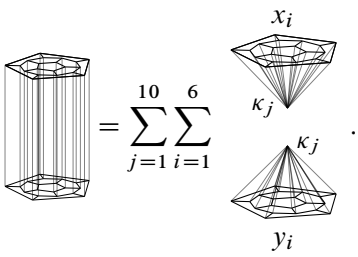
$$G^d \times [0,1] = \sum_{j=1}^{10} \sum_{i=1}^{6} x_i \, P_j \, y_i, \tag{30}$$

and furthermore, the $6 \times 10 = 60$ terms on the right-hand side of the equation, when seen as endomorphisms of $\langle G^d \rangle$, are pairwise orthogonal idempotents.

Diagrammatically, this reads as



or as



(31)

Proof of this claim is identical to that of (22) except taking into account that G^d has ten distinct Kempe classes while the Θ-foam has only one. We sum over these ten Kempe classes.

Proving identity (30) is equivalent to proving that for any conical foam F bounding $G^d \sqcup (-G^d)$, the foam evaluation satisfies

$$\langle (G^d \times [0,1]) \cup F \rangle = \sum_{j=1}^{10} \sum_{i=1}^{6} \langle x_i \, P_j \, y_i \cup F \rangle. \tag{32}$$

Fix a Kempe class κ_{j_0} with $1 \leq j_0 \leq 10$. The sets of admissible colorings of the conical foams $(x_i \, P_{j_0} \, y_i \cup F)_{1 \leq i \leq 6}$ appearing in the sum are in natural bijections with

$\mathrm{adm}(P_{j_0} \cup F)$. Let c be an element of $\mathrm{adm}(P_{j_0} \cup F)$. It induces colorings of $C_+(\kappa_{j_0})$ and of $C_-(\kappa_{j_0})$ and therefore two colorings c_+ and c_- of G^d. Note that c_+ and c_- are both in the Kempe class κ_{j_0} but might not be equal.

If $c_+ = c_-$, the coloring c induces an admissible coloring of $(G^d \times [0,1]) \cup F$, still denoted c. The same computation as the one proving (25) gives

$$\big\langle \big(G^d \times [0,1]\big) \cup F, c \big\rangle = \sum_{i=1}^{6} \langle x_i P_{j_0} y_i \cup F, c \rangle.$$

Note, furthermore, that any coloring of $(G^d \times [0,1]) \cup F$ is obtained in this manner (for arbitrary j_0).

If, on the contrary, $c_+ \neq c_-$, the same computation as the one proving (26) gives:

$$\sum_{i=1}^{6} \langle x_i P_{j_0} y_i \cup F, c \rangle = 0.$$

Summing over all Kempe classes and all colorings of $P_{j_0} \cup F$, we obtain identity (30).

It remains to show that the 60 terms on the right-hand side of are pairwise orthogonal idempotents.

Orthogonality of elements involving different Kempe classes follows from the absence of admissible colorings of the conical foam obtained by concatenating these elements. Idempotent property and orthogonality for elements involving the same Kempe class follow from the computation proving identity (27). ∎

Remark 3.3. From Proposition 2.5, it follows that $\langle G^d \rangle_E$ is a subquotient of $\langle G^d \rangle_E^c$. The latter module is free of graded rank

$$10q^{-3} + 20q^{-1} + 20q + 10q^3$$

by Proposition 3.2. Boozer's computations [4] suggest that the former is free of graded rank

$$9q^{-3} + 20q^{-1} + 20q + 11q^3.$$

This is not contradictory. For instance, it may happen that an element x of degree 3 in $\langle G^d \rangle_E$ is equal to $E \cdot y$ for an element y of degree -3 which is in $\langle G^d \rangle_E^c$ but not in $\langle G^d \rangle_E$.

The reader may have noticed that proofs of Propositions 3.1 and 3.2 are virtually the same. What matters is that both the Θ-web and the dodecahedral graph G^d are homogeneous of Kempe-degree 3. On can adapt the argument to obtain the following proposition.

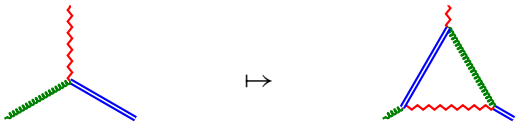


Figure 9. Bijection between colorings of Γ and Γ' in Proposition 4.2.

Proposition 3.4. *Let Γ be an homogeneous web of Kempe-degree 3. The following isomorphism of graded R-modules holds:*

$$\langle\Gamma\rangle^c \cong \frac{|\mathrm{Tait}(\Gamma)|}{6}(q+q^{-1})(q^2+1+q^{-2})\langle\emptyset_1\rangle^c \cong R\left\{\frac{|\mathrm{Tait}(\Gamma)|}{6}[2][3]\right\}.$$

In particular, $\langle\Gamma\rangle^c$ is a free R-module of graded rank $\frac{|\mathrm{Tait}(\Gamma)|}{6}[2][3]$.

4. Homogeneous webs

Kempe equivalence is at the heart of the definition of weakly homogeneous, homogeneous webs and conical foams. Proposition 3.4 suggests that homogeneous webs of degree 3 are particularly interesting. However, to the best of our knowledge, very little is known about them.

In this section we consider homogeneous webs and raise some questions related to them. It is natural here to work with webs in $\mathbb{S}^2$ rather than in $\mathbb{R}^2$. We introduce the notion of a *Kempe-small web* which is closely related to that of homogeneous web of degree 3, but tailored to include the circle and the empty web.

Definition 4.1. For webs Γ and Γ' a map $\varphi\colon \mathrm{Tait}(\Gamma) \to \mathrm{Tait}(\Gamma')$ *intertwines Kempe equivalence* if two colorings c_1 and c_2 of Γ are Kempe equivalent if and only if $\varphi(c_1)$ and $\varphi(c_2)$ are Kempe equivalent.

In particular, for such a map, all the pre-images of a given coloring of Γ' are Kempe equivalent.

The operation of converting a small neighborhood of a vertex of a web Γ into a triangular region with three vertices and three outgoing regions is called *blowing up a vertex* of Γ.

Proposition 4.2. *If Γ' is obtained from Γ by blowing up a vertex, then there exists a canonical bijection between the colorings of Γ and Γ'. This bijection preserves the*

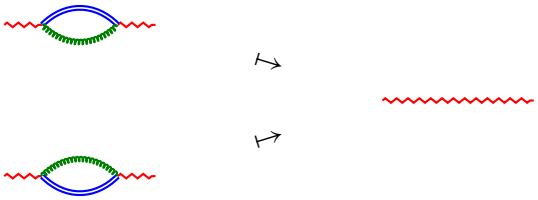


Figure 10. 2-to-1 map between colorings of Γ and of Γ' in Proposition 4.3.

degree and intertwines Kempe equivalence. In particular, Γ is weakly homogeneous (respectively, homogeneous) if and only if Γ' is.

Proof. The bijection between colorings is given in Figure 9. Let c be a coloring of Γ and denote by c the corresponding coloring of Γ'. For $1 \leq i < j \leq 3$ there is a canonical bijection between connected components of $\Gamma_{ij}(c)$ and $\Gamma'_{ij}(c)$ (see Section 2.1 for this notation). Kempe moves along components of $\Gamma_{ij}(c)$ correspond to Kempe moves along components of $\Gamma'_{ij}(c)$. ∎

Proposition 4.3. *Let Γ and Γ' be two webs which are identical except in a small disk where they are related as follows:*

Then there exists a canonical 2-to-1 map from the set of colorings of Γ to that of Γ'. This map decreases the degree of a coloring by 1 and intertwines Kempe equivalence. In particular, Γ is weakly homogeneous (respectively, homogeneous) if and only if Γ' is.

Proof. The 2-to-1 map is given in Figure 10. Denote i, j and k the three elements of $\{1, 2, 3\}$. Let c be a coloring of Γ' and denote by c_1 and c_2 the corresponding colorings of Γ. Let c associate i to the distinguished edge of Γ'. Colorings c_1 and c_2 are related by a Kempe move. Besides this Kempe move, all Kempe moves that one can do on c_1 correspond canonically to Kempe moves on c (and vice versa). Moreover, one has

$$\begin{aligned} d_{ij}(\Gamma, c) &= d_{ij}(\Gamma', c_1) = d_{ij}(\Gamma', c_2), \\ d_{ik}(\Gamma, c) &= d_{ik}(\Gamma', c_1) = d_{ik}(\Gamma', c_2), \\ d_{jk}(\Gamma, c) + 1 &= d_{jk}(\Gamma', c_1) = d_{jk}(\Gamma', c_2). \end{aligned}$$ ∎

Among homogeneous webs, one class seems of peculiar interest: the webs for which Kempe classes coincide with orbits of the action of S_3 on the set of colorings.

Figure 11. The disjoint union of Γ_1 and Γ_2 and one of their vertex-connected sums along v_1 and v_2.

Definition 4.4. A web Γ is called *Kempe-small* if $\mathrm{Tait}(\Gamma) \neq \emptyset$ and any Tait coloring c of Γ has the property that its (i, j)-subcoloring $\Gamma_{ij}(c)$ is connected for all $1 \leq i < j \leq 3$ (the empty graph is considered connected).

Examples of Kempe-small webs include the empty web, the circle, the Θ-web, the tetrahedral web (both shown in Figure 3) and the dodecahedral web, see Figure 1. Note that a Kempe-small web is necessarily connected. A Kempe-small web which is neither the empty web nor the circle is homogeneous of Kempe-degree 3. Conversely, homogeneous webs of Kempe-degree 3 are Kempe-small.

Remark 4.5. There is no obstruction to define *Kempe-smallness*, *weak homogeneity* or *homogeneity* for abstract trivalent graphs.

The next statement follows from Proposition 4.2.

Proposition 4.6. *If Γ' is obtained from Γ by blowing up a vertex, then one of Γ, Γ' is Kempe-small if and only if the other one is.*

For $i = 1, 2$, consider a web Γ_i on the 2-sphere $\mathbb{S}^2$ and a vertex v_i of Γ_i. A *vertex-connected sum of* Γ_1 *and* Γ_2 *along* (v_1, v_2) is a web Γ in $\mathbb{S}^2$ obtained by removing small disk neighborhoods of v_1, v_2 and gluing the complements of this neighborhoods together to get a trivalent graph embedded in $\mathbb{S}^2$. There are six ways to form a vertex-connected sum along v_1, v_2 due to the six ways to match the three legs of Γ_1 at v_1 with the three legs of Γ_2 at v_2. Three out of these six ways require reversing the orientation of one of $\mathbb{S}^2$'s.

The Θ-web is a neutral element for the vertex-connected sum: vertex-connected sum of Γ_1 and Θ is isomorphic to Γ_1. Performing a vertex-connected sum with the tetrahedral web amounts to blowing up a vertex.

Proposition 4.7. *A vertex-connected sum of two webs Γ' and Γ'' is Kempe-small if and only if both Γ' and Γ'' are Kempe-small.*

Proof. Denote by Γ the vertex-connected sum of Γ' and Γ''. Suppose that Γ is Kempe-small. Any coloring of Γ assigns three different colors to the three distinguished edges $\{e_1, e_2, e_3\}$ of Γ. Indeed, for any such coloring, the number of edges in $\{e_1, e_2, e_3\}$ that are in $\Gamma_{12}(c)$, (respectively, $\Gamma_{13}(c)$, $\Gamma_{23}(c)$) is even since these graphs are collec-

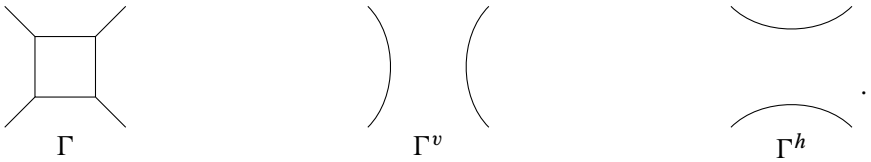
Figure 12. Possible colorings of the square.

tions of cycles. This is only possible if $\{e_1, e_2, e_3\}$ are assigned different colors. This implies that Γ' and Γ'' both have admissible colorings: any coloring of Γ induces colorings of Γ' and Γ''. Let c' and c'' be colorings of Γ' and Γ'' respectively. Applying an S_3-symmetry to c'', if necessary, one can associate a coloring c of Γ to c', c''. Since Γ is Kempe-small, $\Gamma_{ij}(c)$ is connected for any $\{i, j\} \subset \{1, 2, 3\}$. This implies that $\Gamma'_{ij}(c')$ and $\Gamma''_{ij}(c'')$ are connected for any $\{i, j\} \subset \{1, 2, 3\}$ and finally that Γ' and Γ'' are Kempe-small.

Conversely, suppose that Γ' and Γ'' are Kempe-small. Let c be a coloring of Γ (this always exists since one can construct such a coloring using c' and c'' as stated in the previous paragraph). Since c assigns three different colors to the three distinguished edges, c induces colorings c' and c'' of Γ' and Γ''. Finally, for all pairs $\{i, j\} \subset \{1, 2, 3\}$, $\Gamma_{ij}(c)$ is the connected sum of $\Gamma'_{ij}(c')$ and $\Gamma''_{ij}(c'')$ and is therefore connected. ∎

The proof of the following proposition uses planarity of webs. We don't know if an analogous statement holds without planarity.

Proposition 4.8. *Let Γ, Γ_v and Γ_h be three webs which are identical except in a disk B where they are related as follows:*



Then Γ is Kempe-small if and only if one of the following holds:

- *both Γ^v and Γ^h are Kempe-small,*
- *Γ^v is Kempe-small and Γ^h admits no coloring,*
- *Γ^h is Kempe-small and Γ^v admits no coloring.*

Proof. Suppose first that Γ is Kempe-small and let us prove that Γ^v and Γ^h fall into one of these three cases. Let c be a coloring of Γ. Up to an S_3-symmetry, we can suppose that c has one of the three types shown in Figure 12. This implies that either Γ^v, Γ^h, or both admit a Tait coloring. Hence it remains to show that if Γ^h (respectively, Γ^v) admits a coloring then it is Kempe-small.

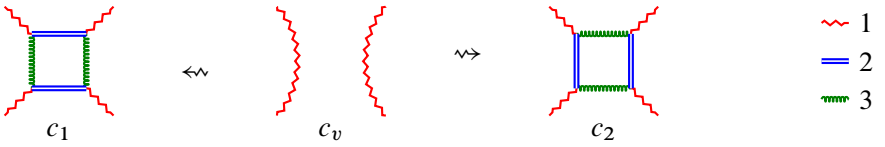


Figure 13. Coloring c_v of Γ^v induces two colorings c_1 and c_2 of Γ.

Figure 14. Colorings c_1 and c_2 of Γ: planarity of Γ implies that one of $\Gamma_{12}(c_1)$ or $\Gamma_{12}(c_2)$ is not connected.

Without loss of generality, one can assume that Γ^v admits a coloring c_v. Suppose that c_v assigns the same color to the two edges of Γ^v intersecting the disk B. Up to an S_3-symmetry, we can suppose that these edges are colored by 1. There are two Kempe-equivalent colorings c_1 and c_2 of Γ which are induced by c_v but differ in B, see Figure 13. Since Γ is Kempe-small, $\Gamma_{12}(c_1)$ is connected. This in turns implies that $\Gamma_{12}(c_2)$ is not connected, see Figure 14), and contradicts Γ being Kempe-small. Hence, c_v assigns different colors to the two edges in B. There is a unique coloring of c of Γ which coincides with c_v except in B. Since Γ is Kempe-small, $\Gamma_{12}(c)$, $\Gamma_{13}(c)$ and $\Gamma_{23}(c)$, are Hamiltonian cycles in Γ, and $\Gamma^v_{12}(c_v)$, $\Gamma^v_{13}(c_v)$, $\Gamma^v_{23}(c_v)$ are connected.

Conversely, suppose that Γ^v and Γ^h satisfy one of the three conditions listed in the proposition. In particular one of Γ^v, Γ^h admits a coloring and so does Γ. Let c be a coloring of Γ. Suppose that it assigns the same color to the four edges pointing out of the square. Up to an S_3-symmetry, one may suppose that this color is 1. The coloring c induces colorings c_v and c_h of Γ^v and Γ^h respectively, see Figure 15. Hence, we are in the case where Γ^v and Γ^h are Kempe-small. Graphs $\Gamma^h_{12}(c)$ and $\Gamma^v_{12}(c)$ are embedded in $\mathbb{R}^2$ and related by a saddle move. Hence one of them is not connected. This is not possible with webs Γ^v and Γ^h being Kempe-small, thus not all the edges of the square have the same color in c. Up to an S_3-symmetry, there are only two possibilities for c in B:

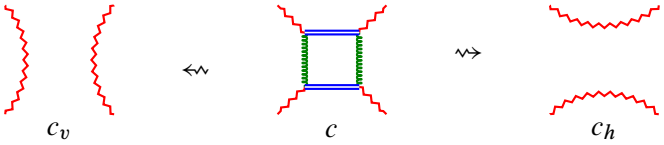


Figure 15. Coloring c of Γ induces coloring c_v of Γ_v and coloring c_v of Γ_v.

In the first case, c induces a coloring c_v of Γ^v, hence Γ^v is Kempe-small. Connectedness of $\Gamma_{12}(c)$, $\Gamma_{13}(c)$ and $\Gamma_{23}(c)$ follows from that of $\Gamma^v_{12}(c_v)$, $\Gamma^v_{13}(c_v)$ and $\Gamma^v_{23}(c_v)$.

In the second case, c induces a coloring c_h of Γ^h, hence Γ^h is Kempe-small. Connectedness of $\Gamma_{12}(c)$, $\Gamma_{13}(c)$ and $\Gamma_{23}(c)$ follows from that of $\Gamma^h_{12}(c_h)$, $\Gamma^h_{13}(c_h)$ and $\Gamma^h_{23}(c_h)$. ∎

Beyond the smallest examples of a circle and the theta-web, Kempe-small webs can be built inductively from the tetrahedral and dodecahedral webs by blowing up vertices and forming vertex-connected sums. We don't know other examples of Kempe-small webs.

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