

Group partition categories

Samuel Nyobe Likeng and Alistair Savage

Abstract. To every group G we associate a linear monoidal category $\mathcal{P}ar(G)$ that we call a *group partition category*. We give explicit bases for the morphism spaces and also an efficient presentation of the category in terms of generators and relations. We then define an embedding of $\mathcal{P}ar(G)$ into the group Heisenberg category associated to G . This embedding intertwines the natural actions of both categories on modules for wreath products of G . Finally, we prove that the additive Karoubi envelope of $\mathcal{P}ar(G)$ is equivalent to a wreath product interpolating category introduced by Knop, thereby giving a simple concrete description of that category.

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1. Introduction

The *partition category* is a \mathbb{k} -linear monoidal category, depending on a parameter d in the commutative ground ring \mathbb{k} , that encodes the homomorphism spaces between tensor powers of the permutation representation of all the finite symmetric groups in a uniform way. Its additive Karoubi envelope is the category $\underline{\text{Rep}}(S_d)$, introduced by Deligne in [4]. Deligne's category $\underline{\text{Rep}}(S_d)$ interpolates between categories of representations of symmetric groups in the sense that the category of representations of S_n is equivalent to the quotient of $\underline{\text{Rep}}(S_n)$ by the tensor ideal of negligible morphisms.

When working with linear monoidal categories in practice, it is useful to have two descriptions. First, one would like to have an explicit basis for each morphism space, together with an explicit rule for the tensor product and composition of elements of these bases. Second, one wants an efficient presentation of the category in terms of generators and relations. Such a presentation is particularly useful when working with categorical actions, since one can define the action of generators and check the relations. Both descriptions exist for the partition category. Bases for morphism spaces are given in terms of *partition diagrams* with simple rules for composition and tensor product. Additionally, there is an efficient presentation, which can be

summarized as the statement that the partition category is the free \mathbb{k} -linear symmetric monoidal category generated by a d -dimensional special commutative Frobenius object.

Deligne's original paper [4] has inspired a great deal of further research. Of particular importance for the current paper are the generalizations of Knop and Mori. In [10], Knop generalized Deligne's construction by embedding a regular category \mathcal{A} into a family of pseudo-abelian tensor categories $\mathcal{T}(\mathcal{A}, \delta)$, which are the additive Karoubi envelope of categories $\mathcal{T}^0(\mathcal{A}, \delta)$ depending on a degree function δ . Deligne's original construction corresponds to the case where \mathcal{A} is the category of finite boolean algebras (equivalently, the opposite of the category of finite sets). Knop's construction, which is inspired by the calculus of relations on \mathcal{A} , has the advantage of being very general, but the disadvantage of being rather abstract. In particular, Knop does not give a presentation of his categories in terms of generators and relations.

In [12], Mori generalized Deligne's construction in a somewhat different direction. For each $d \in \mathbb{k}$, Mori defines a 2-functor \mathcal{S}_d sending a tensor category \mathcal{C} to another tensor category $\mathcal{S}_d(\mathcal{C})$, which should be thought of as a sort of interpolating wreath product functor. Morphisms are described in terms of recollements and one has a presentation using the string diagram calculus for braided monoidal categories.

In the current paper, we are interested in a setting where the constructions of Knop and Mori are closely related. This occurs when \mathcal{A} is the category of finite boolean algebras with a locally free action of a finite group G , and when \mathcal{C} is the category of representations of G . With these choices, the categories defined by Knop and Mori can both be viewed as interpolating categories for the categories of representations of the wreath products $G^n \rtimes \mathfrak{S}_n$. In fact, Mori's interpolating category contains Knop's as a full subcategory; see [12, Remark 4.14] for a precise statement. These "wreath Deligne categories" and other variations have been further studied in [5, 6, 8, 15, 16].

Wreath products of groups and algebras appear in a surprising number of areas of mathematics, including vertex operators, the geometry of the Hilbert scheme, and categorification. In particular, to every Frobenius algebra (or, more generally, graded Frobenius superalgebra) A and choice of central charge $k \in \mathbb{Z}$, there is a *Frobenius Heisenberg category* $\mathcal{H}eis_k(A)$, introduced in [14, 17] and further studied in [2]. When $k = \pm 1$, this category encodes the representation theory of *all* the wreath product algebras $A^{\otimes n} \rtimes \mathfrak{S}_n$, $n \in \mathbb{N}$, *simultaneously*. (For other choices of k it encodes the representation theory of more general cyclotomic quotients of the affine wreath product algebras introduced in [18].) Generating objects of $\mathcal{H}eis_{\pm 1}(A)$ correspond to induction and restriction functors with respect to the natural embedding

$$A^{\otimes n} \rtimes \mathfrak{S}_n \hookrightarrow A^{\otimes(n+1)} \rtimes \mathfrak{S}_{n+1}.$$

In the current paper, we are interested in the case where the Frobenius algebra A is the group algebra of a finite group G . In this case, we call $\mathcal{H}eis(G) := \mathcal{H}eis_{-1}(\mathbb{k}G)$ the *group Heisenberg category*. When G is trivial, this category was first introduced

by Khovanov in [9]. In [13], the authors described a natural embedding of the partition category into Khovanov's Heisenberg category. This, in turn, induces an embedding of Deligne's interpolating category $\underline{\text{Rep}}(S_d)$ into the additive Karoubi envelope of the Heisenberg category.

The goal of the current paper is to give simple, explicit descriptions of wreath product analogues of partition categories and to relate these to group Heisenberg categories. First, to any group G we associate a G -partition category $\mathcal{P}ar(G)$ (Definition 3.6). The definition is given in terms of explicit G -partition diagrams, which form bases for the morphism spaces of the category. We then give an efficient presentation of $\mathcal{P}ar(G)$ in terms of generators and relations (Theorem 4.4). There is a natural categorical action of the G -partition category on tensor products of permutation representations of wreath products of G . This action can be described in terms of the generators (Theorem 5.1) or the bases of G -partition diagrams (Proposition 5.2). The action functor is full, and we give an explicit description of its kernel (Theorem 5.4). This gives a categorical analogue of a double centralizer property akin to Schur–Weyl duality, generalizing work of Bloss [1] who defined G -colored partition algebras which are isomorphic to the endomorphism algebras in $\mathcal{P}ar(G)$.

Next, we give an explicit embedding of $\mathcal{P}ar(G)$ into the group Heisenberg category $\mathcal{H}eis(G)$ (Theorem 7.1), generalizing the main result of [13]. This embedding intertwines the natural categorical actions of $\mathcal{P}ar(G)$ and $\mathcal{H}eis(G)$ on modules for wreath products (Theorem 8.1).

Finally, we prove (Theorem 9.5) that the group partition category $\mathcal{P}ar(G)$ is equivalent to Knop's category $\mathcal{T}^0(\mathcal{A}, \delta)$ when \mathcal{A} is the category of finite boolean algebras with a locally free G -action (equivalently, the opposite of the category of finite sets with a free G -action). Thus, one can view $\mathcal{P}ar(G)$ as a concrete, and very explicit, realization of the wreath product interpolating categories of Knop and Mori. In particular, the calculus of G -partition diagrams is significantly simpler than the previous constructions. (Although the latter are, of course, more general.) In addition, the presentation of $\mathcal{P}ar(G)$, in terms of generators and relations, given in the current paper is considerably more efficient than the presentation given by Mori in [12, Proposition 4.26]; see Remark 9.7 for further details.

The structure of the current paper is as follows. In Section 2 we recall some basic facts about wreath products. We define our main object of interest, the group partition category, in Section 3. We then give a presentation of $\mathcal{P}ar(G)$ in terms of generators and relations in Section 4. In Section 5 we define the natural categorical action of $\mathcal{P}ar(G)$. We recall the definition of the group Heisenberg category $\mathcal{H}eis(G)$ in Section 6 and then, in Section 7, we define the embedding of $\mathcal{P}ar(G)$ into $\mathcal{H}eis(G)$. In Section 8 we prove that this embedding intertwines the natural categorical actions of these categories on modules for wreath products. Finally, in Section 9 we relate $\mathcal{P}ar(G)$ to the constructions of Knop and Mori.

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2. Wreath products

Fix a commutative ground ring \mathbb{k} and a group G with identity element 1_G . We use an unadorned tensor product \otimes to denote the tensor product over \mathbb{k} . We will often define linear maps on tensor products by specifying the images of simple tensors; such maps are always extended by linearity.

For $n \geq 1$, the symmetric group \mathfrak{S}_n acts on G^n by permutation of the factors, where we *number factors from right to left*:

$$\pi \cdot (g_n, \dots, g_1) = (g_{\pi^{-1}(n)}, \dots, g_{\pi^{-1}(1)}).$$

The *wreath product group* $G_n := G^n \rtimes \mathfrak{S}_n$ has underlying set $G^n \times \mathfrak{S}_n$, and multiplication

$$(\mathbf{g}, \pi)(\mathbf{h}, \sigma) = (\mathbf{g}(\pi \cdot \mathbf{h}), \pi\sigma), \quad \mathbf{g}, \mathbf{h} \in G^n, \pi, \sigma \in \mathfrak{S}_n.$$

We identify G^n and \mathfrak{S}_n with the subgroups $G^n \times \{1_{\mathfrak{S}_n}\}$ and $\{1_G\} \times \mathfrak{S}_n$, respectively, of $G^n \rtimes \mathfrak{S}_n$. Hence, we write $\mathbf{g}\pi$ for (\mathbf{g}, π) .

Let $A := \mathbb{k}G$ be the group algebra of G . Then the group algebra $A_n := \mathbb{k}G_n$ is isomorphic to the *wreath product algebra* $A^{\otimes n} \rtimes \mathfrak{S}_n$, which is isomorphic to $A^{\otimes n} \otimes \mathbb{k}\mathfrak{S}_n$ as a \mathbb{k} -module, with multiplication given by

$$(\mathbf{a} \otimes \pi)(\mathbf{b} \otimes \sigma) = \mathbf{a}(\pi \cdot \mathbf{b}) \otimes \pi\sigma, \quad \mathbf{a}, \mathbf{b} \in A^{\otimes n}, \pi, \sigma \in \mathfrak{S}_n.$$

We adopt the convention that G_0 is the trivial group, so that $A_0 = \mathbb{k}$.

For $1 \leq i \leq n - 1$, let $s_i \in \mathfrak{S}_n$ be the simple transposition of i and $i + 1$. The elements

$$\pi_i := s_i s_{i+1} \cdots s_{n-1}, \quad i = 1, \dots, n,$$

form a complete set of left coset representatives of \mathfrak{S}_{n-1} in \mathfrak{S}_n . Here we adopt the convention that $\pi_n = 1_{\mathfrak{S}_n}$.

There is an injective group homomorphism

$$G_{n-1} \hookrightarrow G_n, \quad (g_{n-1}, \dots, g_1) \mapsto (1_G, g_{n-1}, \dots, g_1).$$

This induces an embedding of the wreath product algebra A_{n-1} into A_n . For $g \in G$ and $1 \leq i \leq n$, define

$$g^{(i)} := (\underbrace{1_G, \dots, 1_G}_{n-i \text{ entries}}, g, \underbrace{1_G, \dots, 1_G}_{i-1 \text{ entries}}) \in G^n.$$

Then the set

$$\{g^{(i)}\pi_i = \pi_i g^{(n)} : 1 \leq i \leq n, g \in G\} \tag{2.1}$$

is a complete set of left coset representatives of G_{n-1} in G_n . Hence, it is a basis for A_n as a right A_{n-1} -module.

Throughout the paper, we adopt the convention that

$$\mathbf{g} = (g_n, \dots, g_1) \quad \text{and} \quad \mathbf{h} = (h_n, \dots, h_1).$$

That is, g_i denotes the i th component of \mathbf{g} (counting from right to left, as usual), and similarly for h_i . This convention applies to all boldface letters denoting elements of G^n for some $n \in \mathbb{N}$.

Definition 2.1. The *permutation representation* of A_n is the \mathbb{k} -module $V = A^n$, with action given by

$$\mathbf{g}\pi \cdot (a_n, \dots, a_1) = (g_n a_{\pi^{-1}(n)}, \dots, g_1 a_{\pi^{-1}(1)}), \tag{2.2}$$

$\mathbf{g} \in G^n$, $\pi \in \mathfrak{S}_n$, $a_n, \dots, a_1 \in A$, extended by linearity. In other words, \mathfrak{S}_n acts by permuting the entries of elements of A^n , while G^n acts by componentwise multiplication.

For $i = 1, \dots, n$, define $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in V$, where the 1 appears in the i th position. Then (2.2) implies that

$$\mathbf{g}\pi \cdot (ae_i) = g_{\pi(i)}ae_{\pi(i)}, \quad \mathbf{g} \in G^n, \pi \in \mathfrak{S}_n, a \in A, 1 \leq i \leq n. \tag{2.3}$$

The set

$$\mathbf{B}_k := \{g_k e_{i_k} \otimes \dots \otimes g_1 e_{i_1} : g_1, \dots, g_k \in G, 1 \leq i_1, \dots, i_k \leq n\} \tag{2.4}$$

is a basis for $V^{\otimes k}$.

For the remainder of the paper, we write \otimes_n for \otimes_{A_n} .

Lemma 2.2. For any A_n -module W , we have an isomorphism of A_n -modules

$$A_n \otimes_{n-1} W \rightarrow V \otimes W, \quad \gamma \otimes w \mapsto \gamma e_n \otimes \gamma w, \quad \gamma \in G_n, w \in W,$$

with inverse

$$V \otimes W \rightarrow A_n \otimes_{n-1} W, \quad g e_i \otimes w \mapsto g^{(i)}\pi_i \otimes \pi_i^{-1}(g^{-1})^{(i)}w, \\ 1 \leq i \leq n, g \in G, w \in W.$$

Proof. The first map is well-defined since A_{n-1} acts trivially on e_n , and it is clearly a homomorphism of A_n -modules. It is straightforward to verify that the second map is the inverse of the first. □

Denote by $\mathbf{1}_n$ the trivial one-dimensional A_n -module. Let B denote A_n , considered as an (A_n, A_{n-1}) -bimodule and, for $k \geq 1$, define

$$B^k := \underbrace{B \otimes_{n-1} B \otimes_{n-1} \cdots \otimes_{n-1} B}_{k \text{ factors}} .$$

Corollary 2.3. *For $k \geq 1$, we have an isomorphism of A_n -modules*

$$\begin{aligned} \beta_k: V^{\otimes k} &\xrightarrow{\cong} B^k \otimes \mathbf{1}_n, \\ g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} &\mapsto g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \\ &\quad \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1, \end{aligned}$$

with inverse map

$$\begin{aligned} \beta_k^{-1}: B^k \otimes \mathbf{1}_n &\xrightarrow{\cong} V^{\otimes k}, \\ \gamma_k \otimes \cdots \otimes \gamma_1 \otimes 1 &\mapsto (\gamma_k e_n) \otimes (\gamma_k \gamma_{k-1} e_n) \otimes \cdots \otimes (\gamma_k \cdots \gamma_1 e_n), \quad \gamma_k, \dots, \gamma_1 \in G_n. \end{aligned}$$

3. Group partition categories

We continue to fix a group G and a commutative ring \mathbb{k} . For $k, l \in \mathbb{N} = \mathbb{Z}_{\geq 0}$, a *partition of type $\binom{l}{k}$* is a partition of the set

$$X_k^l := \{1, \dots, k, 1', \dots, l'\}.$$

A G -*partition of type $\binom{l}{k}$* is a pair (P, \mathbf{g}) , where P is a partition of type $\binom{l}{k}$ and

$$\mathbf{g} = (g_1, \dots, g_k, g_{1'}, \dots, g_{l'}) \in G^{X_k^l}.$$

We define a *part* of (P, \mathbf{g}) to be a part of the partition P .

Let (P, \mathbf{g}) and (P, \mathbf{h}) be G -partitions of type $\binom{l}{k}$, with $P = \{P_1, \dots, P_r\}$. We say these G -partitions are *equivalent*, and we write $(P, \mathbf{g}) \sim (P, \mathbf{h})$, if there exist $t_1, \dots, t_r \in G$ such that, for each $i = 1, \dots, r$, we have

$$h_a = t_i g_a \quad \text{for every } a \in P_i.$$

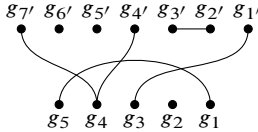
This clearly defines an equivalence relation on the set of G -partitions of type $\binom{l}{k}$. We let $[P, \mathbf{g}]$ denote the equivalence class of (P, \mathbf{g}) .

We depict the G -partition (P, \mathbf{g}) of type $\binom{l}{k}$ as a graph with l vertices in the top row, labeled $g_{1'}, \dots, g_{l'}$ from *right to left*, and k vertices in the bottom row, labeled g_1, \dots, g_k from *right to left*. (We will always number vertices from right to left.) We draw edges so that the parts of the partition are the connected components of the graph.

For example, the equivalence class of the G -partition (P, \mathbf{g}) of type $\binom{7}{5}$ with

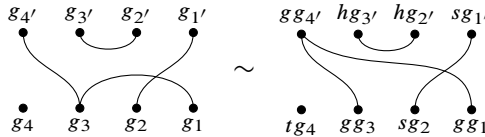
$$P = \{\{1, 5\}, \{2\}, \{3, 1'\}, \{4, 4', 7'\}, \{2', 3'\}, \{5'\}, \{6'\}\}$$

can be depicted as follows:



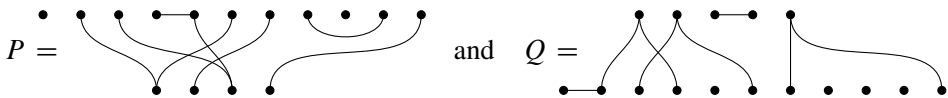
We call this a G -partition diagram. Forgetting the labels, we obtain a partition diagram for P . Note that different G -partition diagrams can correspond to the same G -partition since only the connected components of the graph are relevant, and similarly for partition diagrams. Two G -partition diagrams are equivalent if their graphs have the same connected components and the vertex labels of one are obtained from those of the other by, for each connected component, multiplying the labels in that component on the left by the same element of G .

Example 3.1. For $g, h, s, t \in G$, the following G -partition diagrams of type $\binom{4}{4}$ are equivalent:

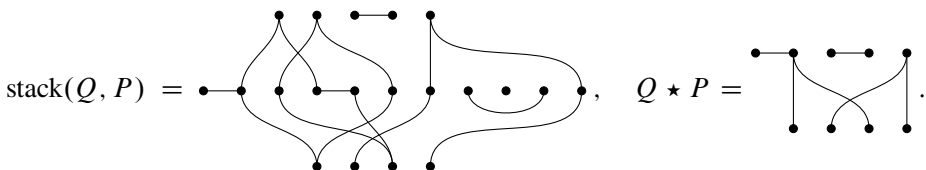


Suppose P is a partition of type $\binom{l}{k}$ and Q is a partition of type $\binom{m}{l}$. We can stack the partition diagram of Q on top of the partition diagram of P and identify the middle row of vertices to obtain a diagram $\text{stack}(Q, P)$ with three rows of vertices. We define $Q \star P$ to be the partition of type $\binom{k}{m}$ defined as follows: vertices are in the same part of $Q \star P$ if and only if the corresponding vertices in the top and bottom row of $\text{stack}(Q, P)$ are in the same connected component. We let $\alpha(Q, P)$ denote the number of connected components containing only vertices in the middle row of $\text{stack}(Q, P)$.

Example 3.2. If



then $\alpha(P, Q) = 2$, and



Suppose (P, \mathbf{g}) and (Q, \mathbf{h}) are G -partitions of types $\binom{l}{k}$ and $\binom{m}{l}$, respectively. We define $\text{stack}((Q, \mathbf{h}), (P, \mathbf{g}))$ to be the graph $\text{stack}(Q, P)$ with vertices labeled by elements of G as follows: vertices in the top and bottom rows are labeled as in the top and bottom rows of Q and P , respectively, while the i th vertex in the middle row is labeled by the product $g_{i'}h_i^{-1}$. (As usual, we label vertices from right to left.) We say that the pair $((Q, \mathbf{h}), (P, \mathbf{g}))$ is *compatible* if any two vertices in the middle row of $\text{stack}((Q, \mathbf{h}), (P, \mathbf{g}))$ that are in the same connected component of Q have the same label. If $((Q, \mathbf{h}), (P, \mathbf{g}))$ is compatible, we define $\mathbf{h} \star_{Q,P} \mathbf{g}$ to be the element $\mathbf{t} \in G^{X_k^m}$, where

- for $1 \leq i \leq m$, $t_{i'} = gh_{i'}$, where g is the common label of vertices in the middle row that are in the same connected component as the i th vertex of the top row of Q (i.e., the vertex labeled by $h_{i'}$), where we adopt the convention $g = 1_G$ if there are no such vertices;
- for $1 \leq i \leq k$, $t_i = g_i$.

Lemma 3.3. *Suppose that $((Q, \mathbf{h}), (P, \mathbf{g}))$ and $((Q, \mathbf{h}'), (P, \mathbf{g}'))$ are compatible, that $(Q, \mathbf{h}) \sim (Q, \mathbf{h}')$, and that $(P, \mathbf{g}) \sim (P, \mathbf{g}')$. Then,*

$$(Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}) \sim (Q \star P, \mathbf{h}' \star_{Q,P} \mathbf{g}').$$

Proof. By transitivity, it suffices to consider the case where $\mathbf{h} = \mathbf{h}'$ and the case where $\mathbf{g} = \mathbf{g}'$. Suppose $\mathbf{h} = \mathbf{h}'$ and consider a connected component Y of $\text{stack}(Q, P)$. Then Y is a union of some connected components of Q and some connected components P_1, \dots, P_r of P . (The case where Y is a single connected component of Q or a single connected component of P are straightforward, so we assume Y is a union of a positive number of connected components of P and a positive number of connected components of Q .) Since $(P, \mathbf{g}) \sim (P, \mathbf{g}')$, there exist $v_1, \dots, v_r \in G$ such that $g'_a = v_i g_a$ for all $a \in P_i, i \in \{1, 2, \dots, r\}$. Now, the fact that $((Q, \mathbf{h}), (P, \mathbf{g}))$ and $((Q, \mathbf{h}), (P, \mathbf{g}'))$ are compatible and that P_1, \dots, P_r are in the same connected component of $\text{stack}(Q, P)$ implies that

$$v_1 = v_2 = \dots = v_r.$$

Thus, if

$$\mathbf{t} = \mathbf{h} \star_{Q,P} \mathbf{g} \quad \text{and} \quad \mathbf{t}' = \mathbf{h} \star_{Q,P} \mathbf{g}',$$

we have $t'_a = v_1 t_a$ for all vertices $a \in Y$. Since this holds for each connected component Y of $\text{stack}(Q, P)$, we have

$$(Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}) \sim (Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}'),$$

as desired. The case where $\mathbf{g} = \mathbf{g}'$ is analogous. □

We say that the pair

$$([Q, \mathbf{h}], [P, \mathbf{g}])$$

is compatible if there exist representatives (Q, \mathbf{h}') and (P, \mathbf{g}') of the equivalence classes $[Q, \mathbf{h}]$ and $[P, \mathbf{g}]$ such that

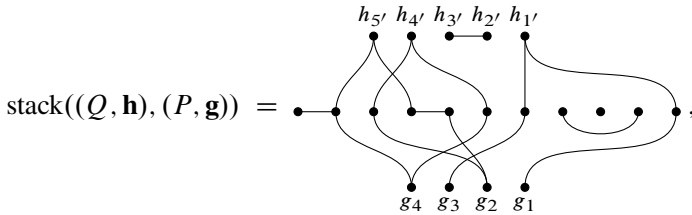
$$((Q, \mathbf{h}'), (P, \mathbf{g}'))$$

is compatible. Whenever we refer to a compatible pair $([Q, \mathbf{h}], [P, \mathbf{g}])$, we assume that $((Q, \mathbf{h}'), (P, \mathbf{g}'))$ is a compatible pair of representatives. By Lemma 3.3, we can define

$$[Q, \mathbf{h}] \star [P, \mathbf{g}] := [Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}] \tag{3.1}$$

for a compatible pair $((Q, \mathbf{h}'), (P, \mathbf{g}'))$, and this definition is independent of our choice of a compatible pair of representatives.

Example 3.4. If P and Q are as in Example 3.2, then



where the vertices in the middle row are labeled

$$g_1'h_1^{-1}, g_2'h_2^{-1}, g_3'h_3^{-1}, g_4'h_4^{-1}, g_5'h_5^{-1}, g_6'h_6^{-1}, \\ g_7'h_7^{-1}, g_8'h_8^{-1}, g_9'h_9^{-1}, g_{10}'h_{10}^{-1}, g_{11}'h_{11}^{-1}$$

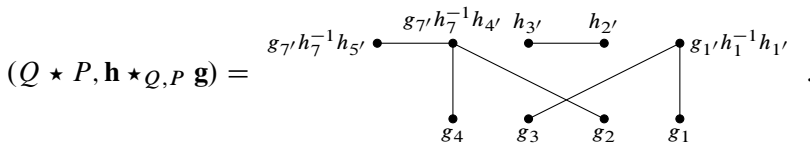
from right to left. The pair $((Q, \mathbf{h}'), (P, \mathbf{g}'))$ is compatible if and only if

$$g_1'h_1^{-1} = g_5'h_5^{-1}, \quad g_6'h_6^{-1} = g_9'h_9^{-1},$$

and

$$g_7'h_7^{-1} = g_8'h_8^{-1} = g_{10}'h_{10}^{-1} = g_{11}'h_{11}^{-1}.$$

If the pair is compatible, then



Convention 3.5. From now on we will consider equivalent G -partition diagrams to be equal. In other words, G -partition diagrams represent equivalence classes of G -partitions. We will also use the terms *part* (of a partition) and (*connected*) *component* (of the corresponding partition diagram) interchangeably.

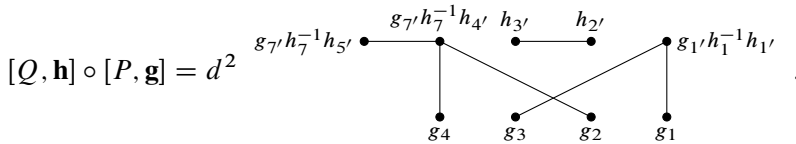
Recall that \mathbb{k} is a commutative ring, and fix $d \in \mathbb{k}$.

Definition 3.6. The G -partition category $\mathcal{P}ar(G, d)$ is the strict \mathbb{k} -linear monoidal category whose objects are nonnegative integers and, given two objects k, l in $\mathcal{P}ar(G, d)$, the morphisms from k to l are \mathbb{k} -linear combinations of equivalence classes of G -partitions of type $\binom{l}{k}$. The vertical composition is given by defining

$$[Q, \mathbf{h}] \circ [P, \mathbf{g}] = d^{\alpha(Q,P)} [Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}]$$

if $([Q, \mathbf{h}], [P, \mathbf{g}])$ is compatible, defining $[Q, \mathbf{h}] \circ [P, \mathbf{g}] = 0$ otherwise, and then extending by linearity. The tensor product is given on objects by $k \otimes l := k + l$, and on morphisms by horizontal juxtaposition of G -partition diagrams, extended by linearity. When we do not wish to make the group G explicit, we call $\mathcal{P}ar(G, d)$ a *group partition category*.

For example, if the pair $((Q, \mathbf{h}), (P, \mathbf{g}))$ of Example 3.4 is compatible, then



It is straightforward to verify that $\mathcal{P}ar(G, d)$ is, in fact, a category, i.e., the composition of morphisms is associative.

The category $\mathcal{P}ar(\{1\}, d)$ is the *partition category*; see [3, §2]. In fact, we have a faithful functor

$$\mathcal{P}ar(\{1\}, d) \rightarrow \mathcal{P}ar(G, d) \tag{3.2}$$

sending any partition diagram P to the corresponding G -partition diagram where all vertices are labeled by 1_G . (More generally, we have a faithful functor $\mathcal{P}ar(H, d) \rightarrow \mathcal{P}ar(G, d)$ for any subgroup H of G .) In what follows, we will identify a partition diagram P with its image under this functor. In other words, we write $[P]$ for $(P, \mathbf{1})$ and $[P, \mathbf{1}]$, respectively, where $\mathbf{1}$ is the identity element of $G^{X_k^l}$. An arbitrary equivalence class of G -partitions $[P, \mathbf{g}]: k \rightarrow l$ can be written in the form

$$[P, \mathbf{g}] = \begin{array}{c} g_{1'} \\ \vdots \\ g_{2'} g_{1'} \\ \vdots \end{array} \circ [P] \circ \begin{array}{c} \vdots \\ g_k \\ \vdots \\ g_2 \\ g_1 \end{array}, \tag{3.3}$$

where we adopt the convention that unlabeled vertices implicitly carry the label 1_G .

4. Presentation

In this section we give a presentation of group partition categories by generators and relations. We use the usual string diagram calculus for monoidal categories. We

denote the unit object of a monoidal category by $\mathbb{1}$ and the identity morphism of an object X by 1_X .

Definition 4.1. Let $\mathcal{P}ar(G)$ be the strict \mathbb{k} -linear monoidal category with one generating object V , where we denote

$$| := 1_V,$$

and generating morphisms

$$\begin{aligned} \frown : V \otimes V &\rightarrow V, & \smile : V &\rightarrow V \otimes V, & \times : V \otimes V &\rightarrow V \otimes V, \\ \circ : \mathbb{1} &\rightarrow V, & \ominus : V &\rightarrow \mathbb{1}, & g \bullet : V &\rightarrow V, \quad g \in G, \end{aligned}$$

subject to the following relations:

$$\begin{aligned} \text{cup with pin} = |, & \quad \text{cup with open dot} = | = \text{cup with pin}, & \quad \text{cup with open dot} = \text{cup with pin}, & \quad (4.1) \end{aligned}$$

$$\text{crossing} = | |, \quad \text{crossing} = \text{crossing}, \quad (4.2)$$

$$\text{crossing with open dot} = \circ |, \quad \text{crossing with open dot} = \ominus |, \quad \text{crossing with open dot} = \text{crossing}, \quad \text{crossing with open dot} = \text{crossing}, \quad (4.3)$$

$$\text{cup with open dot} = \text{cup}, \quad g \bullet \text{ cup } h = \delta_{g,h} g \bullet, \quad (4.4)$$

$$g \bullet h \bullet = gh \bullet, \quad 1 \bullet = |, \quad g \bullet \text{ crossing} = \text{crossing } g \bullet, \quad g \bullet \text{ cup } = g \bullet \text{ cup } g \bullet, \quad g \bullet \text{ cup } = \circ. \quad (4.5)$$

Remark 4.2. The relations (4.1) are equivalent to the statement that $(V, \frown, \circ, \smile, \ominus)$ is a Frobenius object (see, for example, [11, Proposition 2.3.24]). Relations (4.2) and (4.3) and the third relation in (4.5) are precisely the statement that \times equips $\mathcal{P}ar(G)$ with the structure of a symmetric monoidal category (see, for example, [11, §1.3.27, §1.4.35]). Then the first relation in (4.4) is the statement that the Frobenius object V is commutative. When $g = h = 1_G$, the second relation in (4.4) is the statement that the Frobenius object V is special.

We refer to the morphisms $g \bullet$ as *tokens*, and the open dots in \ominus and \circ as *pins*. We call \frown a *merge*, \smile a *split*, and \times a *crossing*. Define *cups* and *caps* by

$$\cup := \smile \quad \text{and} \quad \cap := \frown.$$

Proposition 4.3. *The following relations hold in $\mathcal{P}ar(G)$:*

$$\begin{array}{c} \diagdown \diagup = | \circlearrowleft, \quad \diagup \diagdown = | \circlearrowright, \quad \begin{array}{c} \diagdown \\ \diagup \end{array} = |, \quad \begin{array}{c} \diagdown \\ \diagup \end{array} \circlearrowleft = \begin{array}{c} \diagdown \\ \diagup \end{array}, \end{array} \quad (4.6)$$

$$\begin{array}{c} \cup \\ \cap \end{array} = | = \begin{array}{c} \cup \\ \cap \end{array}, \quad (4.7)$$

$$\begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array}, \quad (4.8)$$

$$\begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array}, \quad \begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} \cup \\ \cap \end{array}, \quad (4.9)$$

$$g \downarrow \cup = \cup \downarrow g^{-1}, \quad g \downarrow \cap = \cap \downarrow g^{-1}, \quad \begin{array}{c} \diagdown \\ \diagup \end{array} g = \begin{array}{c} g \\ \diagdown \\ \diagup \end{array}, \quad g \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagdown \\ \diagup \\ g \end{array}, \quad (4.10)$$

$$\begin{array}{c} g \\ \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagdown \\ \diagup \\ g \end{array} g^{-1}, \quad \begin{array}{c} \diagdown \\ \diagup \\ g \end{array} = \begin{array}{c} g^{-1} \\ \diagdown \\ \diagup \end{array}, \quad \begin{array}{c} g \\ \diagdown \\ \diagup \end{array} = g \begin{array}{c} \diagdown \\ \diagup \end{array}, \quad (4.11)$$

$$\begin{array}{c} \diagdown \\ \diagup \\ g \end{array} = \begin{array}{c} g \\ \diagdown \\ \diagup \end{array} g^{-1}, \quad \begin{array}{c} \diagdown \\ \diagup \\ g \end{array} = \begin{array}{c} g^{-1} \\ \diagdown \\ \diagup \end{array}.$$

Proof. The first two relations in (4.6) follow from the first two relations in (4.3) by composing on the top and bottom, respectively, with the crossing and using (4.2). Then the third relation in (4.6) follows from the first relation in (4.1) using the first relation in (4.4) and the first relation in (4.6). (The fourth relation in (4.6) will be proved below.)

The relations (4.7) follow from the fourth and fifth equalities in (4.1) after placing pins on the merges and splits. The first equality in (4.8) follows from the fifth equality in (4.1) after placing a pin on the bottom-left of both diagrams involved in the equality. The remaining equalities in (4.8) are proved similarly.

Starting with the third relation in (4.3), adding a pin to the top-right strand, a crossing to the two rightmost strands at the bottom, and using the second relation in (4.3) and the first relation in (4.2), we obtain the relation

$$\begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}.$$

Adding a strand on the left (i.e., tensoring on the left with 1_V), then adding a cup to the two leftmost bottom strands and using (4.7), yields the first equality in (4.9). The second equality in (4.9) is proved similarly. The remaining relations in (4.9) follow from placing pins at the top of the morphisms in first relation in (4.1) and the third relation in (4.6) and from placing pins at the bottom of the morphisms in the second two equalities in (4.1).

Now the fourth relation in (4.6) follows from rotating the first relation in (4.4) using the cups and caps, together with (4.7)–(4.9).

To prove the first relation in (4.10), we compute

$$g \downarrow \cup = \begin{array}{c} g \\ \diagdown \\ \diagup \end{array} = \begin{array}{c} g \\ \diagdown \\ \diagup \\ g \end{array} g^{-1} = \begin{array}{c} \diagdown \\ \diagup \\ g \end{array} g^{-1} = \begin{array}{c} \diagdown \\ \diagup \\ g \end{array} = \begin{array}{c} \diagdown \\ \diagup \\ g \end{array} = \begin{array}{c} \diagdown \\ \diagup \\ g \end{array} = \cup \downarrow g^{-1}.$$

Then we prove the second relation in (4.10) as follows:

$$g \curvearrowright \stackrel{(4.7)}{=} \curvearrowright g = \curvearrowright^{g^{-1}} \stackrel{(4.7)}{=} \curvearrowright^{g^{-1}} .$$

The third relation in (4.10) is obtained from the third relation in (4.5) by composing on the top and bottom with a crossing and then using (4.2). The fourth relation in (4.10) is obtained by starting with the third equality in (4.9), attaching a token labeled g to the bottom of the strands, then using the second relation in (4.10) and the fifth relation in (4.5).

Finally, the relations in (4.11) are obtained from the fourth relation in (4.5) by attaching the appropriate cups and caps, then using (4.8) and (4.10). \square

The relation (4.7) implies that the object V is self-dual. It follows from Proposition 4.3 that the cups and caps endow $\mathcal{P}ar(G)$ with the structure of a strict pivotal category: we have an isomorphism of strict monoidal categories

$$*: \mathcal{P}ar(G) \rightarrow (\mathcal{P}ar(G)^{op})^{rev} ,$$

where op denotes the opposite category and rev denotes the reversed category (switching the order of the tensor product). This isomorphism is the identity on objects and, for a general morphism f represented by a single string diagram, the morphism f^* is given by rotating the diagram through 180° .

Moreover, we have that morphisms are invariant under isotopy, except that we must use (4.10) when we slide tokens over cups and caps. In addition, it follows from the first two relations in (4.3) and (4.6) that

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} = \begin{array}{c} | \\ \circ \end{array} . \tag{4.12}$$

In other words, the morphism $\begin{array}{c} \circ \\ | \\ \circ \end{array}$ is strictly central.

Theorem 4.4. *Let $d \in \mathbb{k}$. As a \mathbb{k} -linear monoidal category, $\mathcal{P}ar(G, d)$ is isomorphic to the quotient of $\mathcal{P}ar(G)$ by the relation*

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} = d . \tag{4.13}$$

Proof. Let $\mathcal{P}ar'(G, d)$ denote the quotient of $\mathcal{P}ar(G)$ by the additional relation (4.13). We define a functor

$$F: \mathcal{P}ar'(G, d) \rightarrow \mathcal{P}ar(G, d)$$

as follows: On objects, define $F(V^{\otimes k}) = k, k \in \mathbb{N}$. We define F on the generating morphisms by

$$\begin{array}{c} \text{Y-junction} \mapsto \text{Y-junction with dots}, \quad \text{X-junction} \mapsto \text{X-junction with dots}, \quad \text{Crossing} \mapsto \text{Crossing with dots}, \\ \circ \mapsto \bullet, \quad \circ \mapsto \bullet, \quad g \bullet \mapsto \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}^{g^{-1}} \end{array} ,$$

where, by convention, unlabeled vertices in G -partition diagrams carry the label 1_G . Note that the image of \downarrow is the unique G -partition diagram of type $\binom{1}{0}$ (i.e., the vertex there is in the top row), while the image of \uparrow is the unique G -partition diagram of type $\binom{0}{1}$. It is straightforward to verify that the relations (4.1)–(4.5) and (4.13) are preserved by F , so that F is well defined.

Since F is clearly bijective on objects, it remains to show that it is full and faithful. Since Theorem 4.4 is known to hold in the case where G is the trivial group (see [3, Theorem 2.1], or [13, Proposition 2.1] for a diagrammatic treatment), it follows from the existence of the functor (3.2) that any partition diagram $[P] = [P, \mathbf{1}]$ is in the image of F . Thus, by (3.3) so is an arbitrary equivalence class $[P, \mathbf{g}]$ of G -partitions. Hence, F is full.

It remains to prove that F is faithful. To do this, it suffices to show that

$$\dim \text{Hom}_{\mathcal{P}ar'(G,d)}(\mathbb{V}^{\otimes k}, \mathbb{V}^{\otimes l}) \leq \dim \text{Hom}_{\mathcal{P}ar(G,d)}(k, l) \quad \text{for all } k, l \in \mathbb{N}.$$

We do this by showing that every morphism of $\mathcal{P}ar'(G, d)$ obtained from the generators by composition and tensor product can be reduced to a scalar multiple of a standard form, with the standard forms being in natural bijection with the number of G -partition diagrams.

We first introduce *star diagrams*

$$S_a^b \in \text{Hom}_{\mathcal{P}ar'(G,d)}(\mathbb{V}^{\otimes a}, \mathbb{V}^{\otimes b})$$

for $(a, b) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ as follows. Define

$$S_1^0 := \uparrow, \quad S_0^1 := \downarrow, \quad S_1^1 := |.$$

Then define general star diagrams recursively by

$$\begin{aligned} S_a^{b+1} &:= (1_{\mathbb{V}^{\otimes(b-1)}} \otimes \downarrow) \circ S_a^b \quad \text{for } b \geq 1, \\ S_{a+1}^b &:= S_a^b \circ (1_{\mathbb{V}^{\otimes(a-1)}} \otimes \uparrow) \quad \text{for } a \geq 1. \end{aligned}$$

For example, we have

$$S_2^0 = \downarrow \circ \downarrow = \cap, \quad S_0^3 = \uparrow \circ \uparrow \circ \uparrow, \quad S_3^4 = \downarrow \circ \downarrow \circ \downarrow \circ \downarrow.$$

Every permutation $\pi \in \mathfrak{S}_k$ gives rise to a partition of type $\binom{k}{k}$ with parts $\{i, \pi(i)'\}$, $1 \leq i \leq k$. Fixing a reduced decomposition for π induces a decomposition of the corresponding partition diagram as a composition of tensor products of the generator \times and identity morphisms. We fix such a decomposition for each permutation,

writing D_π for the corresponding element of $\mathcal{P}ar'(G, d)$. For example, if we choose the reduced decomposition $s_1 s_2 s_1$ for the permutation $(1\ 3) \in \mathfrak{S}_3$, we have

$$D_{(1\ 3)} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}.$$

Now, fix a representative (P, \mathbf{g}) of each equivalence class $[P, \mathbf{g}]$ of G -partitions. Then, for each such representative, fix a *standard decomposition*

$$[P, \mathbf{g}] = \begin{array}{c} g_{l'} \\ \bullet \\ \vdots \\ \bullet \end{array} \cdots \begin{array}{c} g_{2'} g_{1'} \\ \bullet \\ \vdots \\ \bullet \end{array} \circ F(D_\pi) \circ F(S) \circ F(D_\sigma) \circ \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \cdots \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}, \quad (4.14)$$

$g_k \quad g_2 \quad g_1$

where $\pi \in \mathfrak{S}_l$, $\sigma \in \mathfrak{S}_k$, and S is a tensor product of star diagrams. The existence of such a standard decomposition follows from (3.3), together with the existence of a decomposition of the form $[P] = F(D_\pi) \circ F(S) \circ F(D_\sigma)$ for any partition diagram P . (Precisely, S is a tensor product of star diagrams, one for each connected component in P , and then π and σ are permutations such that $F(D_\pi)$ and $F(D_\sigma)$ connect the top and bottom vertices of P to the corresponding legs of S .) Then define

$$y_{P, \mathbf{g}} := \left(g_{l'}^{-1} \bullet \cdots g_{2'}^{-1} \bullet g_{1'}^{-1} \bullet \right) \circ D_\pi \circ S \circ D_\sigma \circ \left(g_k \bullet \cdots g_{2'} \bullet g_{1'} \bullet \right). \quad (4.15)$$

Hence, $F(y_{P, \mathbf{g}}) = [P, \mathbf{g}]$

To complete the proof that F is faithful, it remains to show that any morphism in $\mathcal{P}ar'(G, d)$ that is obtained from the generators by tensor product and composition is equal to a scalar multiple of $y_{P, \mathbf{g}}$ for some chosen representative (P, \mathbf{g}) . As noted above, Theorem 4.4 holds for the partition category, which is the case where $G = \{1\}$ is the trivial group. Now, if we ignore tokens, the relations (4.1)–(4.4) and (4.13) correspond to the relations in the $G = \{1\}$ case, except for the fact that the second relation in (4.4) gives zero when $g \neq h$. It follows that every morphism in $\mathcal{P}ar'(G, d)$ obtained from the generators by tensor product and composition is equal to a (potentially zero) scalar multiple of a morphism obtained from some $D_\pi \circ S \circ D_\sigma$ by adding tokens (since, ignoring tokens, this can be done in the partition category). Then, since the string diagram for $D_\pi \circ S \circ D_\sigma$ is a tree (i.e., contains no cycles), one can use relations (4.5), (4.10), and (4.11) to move all tokens to the ends of strings and combine them into a single token at each endpoint. This yields a diagram of the form (4.15), except that the tokens may not correspond to our chosen representative of the equivalence class of G -partitions. However, we then use the relations (4.5), (4.10), and (4.11) to adjust the tokens at the endpoints so that we obtain the chosen representative. □

In the context of Remark 4.2, (4.13) is the statement that the Frobenius object V has dimension d . From now on, we will identify $\mathcal{P}ar(G, d)$ with the quotient of $\mathcal{P}ar(G)$ by the relation (4.13) via the isomorphism of Theorem 4.4. In particular, we will identify the object k of $\mathcal{P}ar(G, d)$ with the object $V^{\otimes k}$ of $\mathcal{P}ar(G)$, for $k \in \mathbb{N}$.

Remark 4.5. Suppose we define $\mathcal{P}ar(G, d)$ as in Definition 3.6, but over the ring $\mathbb{k}[d]$, so that d is an indeterminate. It then follows from Theorem 4.4 that $\mathcal{P}ar(G, d)$ is isomorphic to $\mathcal{P}ar(G)$ as a \mathbb{k} -linear monoidal category. Under this isomorphism, $d1_{\mathbb{1}}$ corresponds to \circlearrowleft .

For $k \in \mathbb{N}$, we define the G -partition algebra

$$P_k(G, d) := \text{End}_{\mathcal{P}ar(G, d)}(V^{\otimes k}). \tag{4.16}$$

When we do not wish to make G explicit, we call these *group partition algebras*. These algebras appeared in [1], where they are called *G -colored partition algebras*. A diagrammatic description of these algebras, different from that of the current paper, is given [1, §6.2].

5. Categorical action

In this section we assume that the group G is finite of order $|G|$. We define a categorical action of the G -partition category on the category of modules for the wreath product groups $G_n = G^n \rtimes \mathfrak{S}_n$. We first describe this action by giving the action of the generators, and then describe the action of an arbitrary G -partition diagram. Recall that $A_n = \mathbb{k}G_n$ is the group algebra of G_n , and so we can naturally identify A_n -modules and representations of G_n .

Theorem 5.1. *For $n \in \mathbb{N}$, we have a strong monoidal \mathbb{k} -linear functor*

$$\Phi_n: \mathcal{P}ar(G, n|G) \rightarrow A_n\text{-mod}$$

given as follows. On objects, Φ_n is determined by $\Phi_n(V) = V$. On generating morphisms, Φ_n is given by

$$\begin{aligned} \Phi_n(\text{cup}): V \otimes V &\rightarrow V, & ge_i \otimes he_j &\mapsto \delta_{g,h} \delta_{i,j} ge_i, \\ \Phi_n(\text{cap}): V &\rightarrow V \otimes V, & ge_i &\mapsto ge_i \otimes ge_i, \\ \Phi_n(\text{cross}): V \otimes V &\rightarrow V, & v \otimes w &\mapsto w \otimes v, \\ \Phi_n(\text{id}): \mathbf{1}_n &\rightarrow V, & 1 &\mapsto \sum_{g \in G} \sum_{i=1}^n ge_i, \\ \Phi_n(\text{dot}): V &\rightarrow \mathbf{1}_n, & ge_i &\mapsto 1, \\ \Phi_n(g \bullet): V &\rightarrow V, & he_i &\mapsto hg^{-1}e_i, \end{aligned}$$

for $g, h \in G, 1 \leq i, j \leq n, v, w \in V$.

Proof. We must show that the action preserves the relations (4.1)–(4.5).

Relations (4.1): To prove the first three equalities in (4.1), we compute

$$\begin{aligned}\Phi_n(\bigwedge) \circ \Phi_n(| \otimes \circlearrowleft)(ge_i) &= \sum_{h \in G} \sum_{j=1}^n \Phi_n(\bigwedge)(ge_i \otimes he_j) \\ &= \sum_{h \in G} \sum_{j=1}^n \delta_{g,h} \delta_{i,j} ge_i = ge_i, \\ \Phi_n(| \otimes \circlearrowright) \circ \Phi_n(\bigvee)(ge_i) &= \Phi_n(| \otimes \circlearrowright)(ge_i \otimes ge_i) \\ &= ge_i = \Phi_n(\circlearrowleft \otimes |) \circ \Phi_n(\bigvee)(ge_i).\end{aligned}$$

To prove the third relation, we compute

$$\begin{aligned}\Phi_n(\bigvee) \circ \Phi_n(\bigwedge)(ge_i \otimes he_j) &= \delta_{g,h} \delta_{i,j} \Phi_n(\bigvee)(ge_i) \\ &= \delta_{g,h} \delta_{i,j} (ge_i \otimes ge_i), \\ \Phi_n(\bigwedge \otimes |) \circ \Phi_n(| \otimes \bigvee)(ge_i \otimes he_j) &= \Phi_n(\bigwedge \otimes |)(ge_i \otimes he_j \otimes he_j) \\ &= \delta_{g,h} \delta_{i,j} (ge_i \otimes ge_i), \\ \Phi_n(| \otimes \bigwedge) \circ \Phi_n(\bigvee \otimes |)(ge_i \otimes he_j) &= \Phi_n(| \otimes \bigwedge)(ge_i \otimes ge_i \otimes he_j) \\ &= \delta_{g,h} \delta_{i,j} (ge_i \otimes ge_i),\end{aligned}$$

concluding that the three maps are identical.

Relations (4.2): The relations (4.2) are straightforward.

Relations (4.3): To prove the first relation in (4.3), we compute

$$\Phi_n(\bigotimes) \circ \Phi_n(| \otimes \circlearrowleft)(ge_i) = \sum_{h \in G} \sum_{j=1}^n \Phi_n(\bigotimes)(ge_i \otimes he_j) = \sum_{h \in G} \sum_{j=1}^n (he_j \otimes ge_i),$$

and

$$\Phi_n(\circlearrowleft \otimes |)(ge_i) = \sum_{h \in G} \sum_{j=1}^n (he_j \otimes ge_i).$$

Similarly, to prove the second relation, we calculate

$$\begin{aligned}\Phi_n(| \otimes \circlearrowright) \circ \Phi_n(\bigotimes)(ge_i \otimes he_j) &= \Phi_n(| \otimes \circlearrowright)(he_j \otimes ge_i) \\ &= he_j = \Phi_n(\circlearrowleft \otimes |)(ge_i \otimes he_j).\end{aligned}$$

The proof of the last two equalities in (4.3) are similar; we only check the last one. We have

$$\begin{aligned}\Phi_n(| \otimes \bigotimes) \circ \Phi_n(\bigotimes \otimes |) \circ \Phi_n(| \otimes \bigvee)(ge_i \otimes he_j) \\ = \Phi_n(| \otimes \bigotimes) \circ \Phi_n(\bigotimes \otimes |)(ge_i \otimes he_j \otimes he_j) \\ = \Phi_n(| \otimes \bigotimes)(he_j \otimes ge_i \otimes he_j) = he_j \otimes he_j \otimes ge_i\end{aligned}$$

and

$$\Phi_n(\swarrow \otimes |) \circ \Phi_n(\searrow \times)(ge_i \otimes he_j) = \Phi_n(\swarrow \otimes |)(he_j \otimes ge_i) = he_j \otimes he_j \otimes ge_i.$$

Relations (4.4): To prove the first relation in (4.4), we compute

$$\begin{aligned} \Phi_n(\swarrow \wedge) \circ \Phi_n(\searrow \times)(ge_i \otimes he_j) &= \Phi_n(\swarrow \wedge)(he_j \otimes ge_i) \\ &= \delta_{g,h} \delta_{i,j} ge_i = \Phi_n(\swarrow \wedge)(ge_i \otimes he_j). \end{aligned}$$

For the second relation, we have

$$\begin{aligned} \Phi_n(\swarrow \wedge) \circ \Phi_n(g \downarrow \otimes h \downarrow) \circ \Phi_n(\swarrow \vee)(ke_i) &= \Phi_n(\swarrow \wedge) \circ \Phi_n(g \downarrow \otimes h \downarrow)(ke_i \otimes ke_i) \\ &= \Phi_n(\swarrow \wedge)(kg^{-1}e_i \otimes kh^{-1}e_i) = \delta_{g,h}(kg^{-1}e_i) = \delta_{g,h} \Phi_n(g \downarrow)(ke_i). \end{aligned}$$

Relations (4.5): The relations (4.5) are straightforward to verify. □

If (P, \mathbf{g}) is a G -partition of type $\binom{l}{k}$, then $\Phi_n([P, \mathbf{g}]) \in \text{Hom}_{G_n}(V^k, V^l)$ is uniquely described by its matrix coefficients:

$$\begin{aligned} \Phi_n([P, \mathbf{g}])(h_k e_{i_k} \otimes \cdots \otimes h_1 e_{i_1}) &= \sum_{\substack{h_{1'}, \dots, h_{l'} \in G \\ 1 \leq i_{1'}, \dots, i_{l'} \leq n}} M(P, \mathbf{g})_{h_1, \dots, h_k, i_1, \dots, i_k}^{h_{1'}, \dots, h_{l'}, i_{1'}, \dots, i_{l'}} h_{l'} e_{i_{l'}} \otimes \cdots \otimes h_{1'} e_{i_{1'}}. \end{aligned} \tag{5.1}$$

The matrix $M(P, \mathbf{g})$ depends only on the equivalence class $[P, \mathbf{g}]$ of (P, \mathbf{g}) .

Proposition 5.2. *Suppose (P, \mathbf{g}) is a G -partition of type $\binom{l}{k}$. Then,*

$$M(P, \mathbf{g})_{h_1, \dots, h_k, i_1, \dots, i_k}^{h_{1'}, \dots, h_{l'}, i_{1'}, \dots, i_{l'}} = 1$$

if $i_a = i_b$ and $h_a g_a^{-1} = h_b g_b^{-1}$ for all $a, b \in X_k^l$ in the same part of P . Otherwise,

$$M(P, \mathbf{g})_{h_1, \dots, h_k, i_1, \dots, i_k}^{h_{1'}, \dots, h_{l'}, i_{1'}, \dots, i_{l'}} = 0.$$

Proof. This follows from a straightforward computation using the definition of Φ_n in Theorem 5.1 and the isomorphism described in Theorem 4.4, writing each component of P as a composition of tokens, merges, splits, and crossings as in (4.14). □

Recall the basis \mathbf{B}_k for $V^{\otimes k}$ from (2.4). Given a G -partition (P, \mathbf{g}) of type $\binom{0}{k}$, let $O_{P, \mathbf{g}}$ denote the set of all $h_k e_{i_k} \otimes \cdots \otimes h_1 e_{i_1} \in \mathbf{B}_k$ such that

- $i_a = i_b$ if and only if a, b are in the same part of the partition P , and
- $h_a g_a^{-1} = h_b g_b^{-1}$ for all a, b in the same part of the partition P .

Then we have $O_{P,\mathbf{g}} = O_{P,\mathbf{h}}$ if and only if $(P, \mathbf{g}) \sim (P, \mathbf{h})$, and so we can define

$$O_{[P,\mathbf{g}]} := O_{P,\mathbf{g}}.$$

The G_n -orbits of \mathbf{B}_k are the $O_{[P,\mathbf{g}]}$ with $[P, \mathbf{g}]: V^{\otimes k} \rightarrow \mathbb{1}$ having at most n parts (i.e., P is a partition of $\{1, \dots, k\}$ having at most n parts).

For $[P, \mathbf{g}]: V^{\otimes k} \rightarrow \mathbb{1}$, let $f_{[P,\mathbf{g}]}: V^{\otimes k} \rightarrow \mathbb{k}$ be the \mathbb{k} -linear map determined on the basis \mathbf{B}_k by

$$f_{[P,\mathbf{g}]}(h_k e_{i_k} \otimes \cdots \otimes h_1 e_{i_1}) = \begin{cases} 1 & \text{if } h_k e_{i_k} \otimes \cdots \otimes h_1 e_{i_1} \in O_{[P,\mathbf{g}]}, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma is now immediate.

Lemma 5.3. *The set of all $f_{[P,\mathbf{g}]}$ with $[P, \mathbf{g}]: V^{\otimes k} \rightarrow \mathbb{1}$ having at most n parts is a basis for $\text{Hom}_{G_n}(V^{\otimes k}, \mathbf{1}_n)$.*

For partitions P, Q of X_k^l , we write $Q \geq P$ when Q is coarser than P . Thus, $Q \geq P$ if and only if every part of P is a subset of some part of Q . It follows from Proposition 5.2 that

$$\Phi_n([P, \mathbf{g}]) = \sum_{Q \geq P} f_{[Q,\mathbf{g}]} \quad \text{for all } [P, \mathbf{g}]: V^{\otimes k} \rightarrow \mathbb{1}. \tag{5.2}$$

Thus, if we define a new basis

$$\{x_{[P,\mathbf{g}]} : [P, \mathbf{g}]: V^{\otimes k} \rightarrow V^{\otimes l}\}$$

of $\text{Hom}_{\text{Par}(G,n|G)}(V^{\otimes k}, V^{\otimes l})$ recursively by

$$x_{[P,\mathbf{g}]} = [P, \mathbf{g}] - \sum_{Q > P} x_{[Q,\mathbf{g}]}, \tag{5.3}$$

then a straightforward argument by induction shows that

$$\Phi_n(x_{[P,\mathbf{g}]}) = f_{[P,\mathbf{g}]} \quad \text{for any } G\text{-partition } (P, \mathbf{g}) \text{ of type } \binom{0}{k}. \tag{5.4}$$

In particular, $\Phi_n(x_{[P,\mathbf{g}]}) = 0$ if $[P, \mathbf{g}]: V^{\otimes k} \rightarrow \mathbb{1}$ has more than n parts.

Theorem 5.4. (1) *The functor Φ_n is full.*

(2) *The kernel of the induced map*

$$\text{Hom}_{\text{Par}(G,n|G)}(V^{\otimes k}, V^{\otimes l}) \rightarrow \text{Hom}_{G_n}(V^{\otimes k}, V^{\otimes l})$$

is the span of all $x_{[P,\mathbf{g}]}$ with $[P, \mathbf{g}]: V^{\otimes k} \rightarrow V^{\otimes l}$ having more than n parts. In particular, this map is an isomorphism if and only if $k + l \leq n$.

Proof. We have a commutative diagram

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathcal{P}ar(G,n|G)}(V^{\otimes k}, V^{\otimes l}) & \longrightarrow & \mathrm{Hom}_{\mathcal{P}ar(G,n|G)}(V^{\otimes(k+l)}, \mathbb{1}) \\
 \Phi_n \downarrow & & \downarrow \Phi_n \\
 \mathrm{Hom}_{G_n}(V^{\otimes k}, V^{\otimes l}) & \longrightarrow & \mathrm{Hom}_{G_n}(V^{\otimes(k+l)}, \mathbf{1}_n),
 \end{array}$$

where the horizontal maps are adjunction isomorphisms arising from the fact that V is a self-dual object in $\mathcal{P}ar(G, n|G)$ and that V is a self-dual object in the category of A_n -modules. (We refer the reader to the proof of [3, Theorem 2.3] for more details of these adjunctions in the special case $G = \{1\}$. The argument is the same in the case of general G .) It follows from Lemma 5.3 and (5.4) that the right-hand vertical map is surjective and its kernel is the span of all $x_{[P, \mathbf{g}]}$ with $[P, \mathbf{g}]: V^{\otimes(k+l)} \rightarrow \mathbb{1}$ having more than n parts. Since the adjunction isomorphisms preserve the number of parts of G -partitions, as well as the partial order on G -partitions, the result follows. \square

When $G = \{1\}$ is the trivial group, Theorem 5.4 reduces to [3, Theorem 2.3]. In general, Theorem 5.4 is a categorical generalization of the double centralizer property [1, Theorem 6.6]. More precisely, recall the G -partition algebras from (4.16). The functor Φ_n induces an algebra homomorphism

$$P_k(G, n|G) \rightarrow \mathrm{End}_{G_n}(V^{\otimes k}).$$

Theorem 5.4 implies that this homomorphism is surjective, and is an isomorphism when $n \geq 2k$. When the characteristic of \mathbb{k} does not divide $n!|G| = |G_n|$, so that A_n is semisimple, the Double Centralizer theorem implies that A_n generates

$$\mathrm{End}_{P_k(G,n|G)}(V^{\otimes k}).$$

Hence, G_n and $P_k(G, n|G)$ generate the centralizers of each other in $\mathrm{End}_{\mathbb{k}}(V^{\otimes k})$.

6. The group Heisenberg category

In this section we recall a special case of the Frobenius Heisenberg category. We are interested in the special case of central charge -1 , where this category was first defined in [14]. Furthermore, we will specialize to the case where the Frobenius algebra is the group algebra of a finite group G . We follow the presentation in [17], referring the reader to that paper for proofs of the statements made here.

Definition 6.1. The *group Heisenberg category* $\mathcal{H}eis(G)$ associated to the finite group G is the strict \mathbb{k} -linear monoidal category generated by two objects \uparrow, \downarrow , and morphisms

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \nearrow \\ \searrow \end{array} : \uparrow\uparrow \rightarrow \uparrow\uparrow, & g \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array} : \uparrow \rightarrow \uparrow, & g \in G, \\
 \cup : \mathbb{1} \rightarrow \uparrow\downarrow, & \cap : \downarrow\uparrow \rightarrow \mathbb{1}, & \cup : \mathbb{1} \rightarrow \uparrow\downarrow, & \cap : \downarrow\uparrow \rightarrow \mathbb{1},
 \end{array}
 \end{array}$$

subject to the relations

$$\begin{array}{c} \begin{array}{ccc} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \uparrow \uparrow \\ \downarrow \downarrow \end{array}, & \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}, \\ \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} gh, & \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} 1 = \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array}, & \begin{array}{c} \diagdown \diagup \\ \bullet \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ \bullet \\ \diagdown \diagup \end{array}, \end{array} \end{array} \tag{6.1}$$

$$\begin{array}{c} \begin{array}{c} \uparrow \\ \cup \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array}, \quad \begin{array}{c} \downarrow \\ \cap \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array}, \end{array} \tag{6.2}$$

$$\begin{array}{c} \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \uparrow \uparrow \\ \downarrow \downarrow \end{array}, \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \uparrow \uparrow \\ \downarrow \downarrow \end{array} - \sum_{g \in G} \begin{array}{c} \cup \\ \bullet \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \bullet \\ \cup \end{array} g^{-1}, \quad \begin{array}{c} \uparrow \\ \cap \end{array} = 0, \quad \begin{array}{c} \circ \\ \bullet \\ \circ \end{array} g = \delta_{g,1} 1_{\mathbb{1}}. \end{array} \tag{6.3}$$

Here the left and right crossings are defined by

$$\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} := \begin{array}{c} \diagdown \diagup \\ \cup \end{array}, \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} := \begin{array}{c} \diagup \diagdown \\ \cap \end{array}.$$

The objects \uparrow and \downarrow are both left and right dual to each other. Furthermore, the cups and caps endow $\mathcal{H}eis(G)$ with the structure of a strict pivotal category, meaning that morphisms are invariant under isotopy. We define downwards crossings and downward tokens by

$$\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} := \begin{array}{c} \diagdown \diagup \\ \cup \end{array} = \begin{array}{c} \diagup \diagdown \\ \cap \end{array}, \quad \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g := \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g \cup = \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g \cap. \end{array} \tag{6.4}$$

It follows that,

$$\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \quad \text{for all possible orientations of the strands,} \tag{6.5}$$

and we that tokens slide over cups and caps:

$$\begin{array}{c} \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} \cup = \cup \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} g, \quad \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} \cap = \cap \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g, \\ \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g \cup = \cup \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g, \quad \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} g \cap = \cap \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \end{array} g. \end{array} \tag{6.6}$$

(One should compare this to the first two relations in (4.10).) Because of this, we will sometimes place tokens at the critical point of cups and caps, since there is no ambiguity. In addition it follows from (6.1), (6.2), and (6.6) that

$$\begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g, \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g = \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \quad \text{for all possible orientations of the strands.} \tag{6.7}$$

We will use (6.6) and (6.7) frequently without mention. Note how tokens multiply on downward strands (using (6.1), (6.4), and (6.6)):

$$\begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} g \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} h = \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} hg. \tag{6.8}$$

7. The embedding functor

In this section we define an explicit embedding of the group partition category into the group Heisenberg category. We assume throughout this section that G is a finite group.

Theorem 7.1. *There is a faithful strict \mathbb{k} -linear monoidal functor*

$$\Psi: \mathcal{P}ar(G) \rightarrow \mathcal{H}eis(G)$$

defined on objects by $V \mapsto \uparrow \otimes \downarrow$ and on generating morphisms by

$$\begin{aligned} \text{Y-junction} &\mapsto \text{Y-junction with curl}, & \text{Y-junction} &\mapsto \text{Y-junction with curl}, & \text{X-junction} &\mapsto \text{X-junction} + \sum_{g \in G} g^{-1} \text{X-junction with dots}, \\ \text{cup} &\mapsto \text{cup}, & \text{cap} &\mapsto \text{cap}, & g \bullet &\mapsto g \bullet \uparrow \downarrow g^{-1}, \quad g \in G. \end{aligned}$$

The proof of Theorem 7.1 will occupy the remainder of this section. We break the proof into two parts, first showing that Ψ is well defined, and then that it is faithful.

Proposition 7.2. *The functor Ψ is well defined.*

Proof. It suffices to show that the images of the generating morphisms of the partition category $\mathcal{P}ar(G)$ satisfy relations (4.1)–(4.5).

Relations (4.1): These relations are easy to check using isotopy invariance in $\mathcal{H}eis(G)$.

Relations (4.2): Using the third relation in (6.3), which says that left curls are equal to zero, we have

$$\begin{aligned} \Psi(\text{X}) \circ \Psi(\text{X}) &= \text{X-junction} + \sum_{g, h \in G} h^{-1} g^{-1} \text{X-junction with dots} \\ &\stackrel{(6.3)}{=} \text{X-junction} + \sum_{g \in G} \text{X-junction with dots} \stackrel{(6.3)}{=} \Psi(\text{||}), \end{aligned}$$

proving the first relation in (4.2). To prove the second relation in (4.2), we compute

$$\begin{aligned} \Psi(\text{||} \otimes \text{X}) \circ \Psi(\text{X} \otimes \text{||}) &= \text{X-junction} + \sum_{g \in G} g^{-1} \text{X-junction with dots} \\ &+ \sum_{g \in G} g^{-1} \text{X-junction with dots} + \sum_{g, h \in G} g^{-1} \text{X-junction with dots} \quad (7.1) \end{aligned}$$

Therefore, using the fact that left curls are zero, we compute

$$\begin{aligned}
 & \Psi(|\otimes \times|) \circ \Psi(\times \otimes |) \circ \Psi(|\otimes \times|) \\
 & \stackrel{(6.3)}{=} \stackrel{(6.7)}{=} \left(\text{Diagram 1} \right) + \sum_{g \in G} g^{-1} \left(\text{Diagram 2} \right) \\
 & + \sum_{g \in G} g^{-1} \left(\text{Diagram 3} \right) + \sum_{g \in G} g^{-1} \left(\text{Diagram 4} \right) \\
 & + \sum_{g, h, t \in G} g^{-1} \left(\text{Diagram 5} \right) \left(ht \right)^{-1} \\
 & \stackrel{(6.1)}{=} \stackrel{(6.3)}{=} \left(\text{Diagram 6} \right) + \sum_{g \in G} g^{-1} \left(\text{Diagram 7} \right) + \sum_{g \in G} g^{-1} \left(\text{Diagram 8} \right) \\
 & + \sum_{g \in G} g^{-1} \left(\text{Diagram 9} \right) + \sum_{g, h \in G} g^{-1} \left(\text{Diagram 10} \right) .
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \Psi(\times \otimes |) \circ \Psi_n(|\otimes \times|) \circ \Psi(\times \otimes |) \\
 & = \left(\text{Diagram 11} \right) + \sum_{g \in G} g^{-1} \left(\text{Diagram 12} \right) + \sum_{g \in G} g^{-1} \left(\text{Diagram 13} \right) \\
 & + \sum_{g \in G} g^{-1} \left(\text{Diagram 14} \right) + \sum_{g, h \in G} g^{-1} \left(\text{Diagram 15} \right) .
 \end{aligned}$$

We then use (6.5) to see that the expressions are equal.

Relations (4.3): We will check the first and the fourth relations, since the proofs of the second and third are analogous. For the first relation we compute:

$$\Psi(\times \otimes |) \circ \Psi(|\otimes \circ) \stackrel{(6.1)}{=} \left(\text{Diagram 16} \right) + \sum_{g \in G} g^{-1} \left(\text{Diagram 17} \right) \stackrel{(6.1)}{=} \stackrel{(6.3)}{=} \left(\text{Diagram 18} \right) = \Psi(\circ \otimes |) .$$

For the fourth relation, we use (7.1) and the fact that left curls are zero to compute

$$\begin{aligned} & \Psi(|\otimes \otimes| \circ \Psi(\otimes \otimes)) \circ \Psi_n(|\otimes \vee|) \\ &= \text{[Diagram: A complex braiding of four strands with arrows]} + \sum_{g,h \in G} g^{-1} \cdot \text{[Diagram: A vertical strand with a dot and two curls labeled } h^{-1}g \text{ and } g^{-1}h \text{]} \cdot h^{-1} \\ & \stackrel{(6.1)}{=} \text{[Diagram: A braiding of four strands with arrows]} + \sum_{g \in G} g^{-1} \cdot \text{[Diagram: A vertical strand with a dot and two curls labeled } g \text{ and } g \text{]} \cdot g^{-1} \\ & \stackrel{(6.3)}{=} \text{[Diagram: A braiding of four strands with arrows]} + \sum_{g \in G} g^{-1} \cdot \text{[Diagram: A vertical strand with a dot and two curls labeled } g \text{ and } g \text{]} \cdot g^{-1} = \Psi(\vee |) \circ \Psi(\otimes \otimes). \end{aligned}$$

Relations (4.4): To check the first relation in (4.4), we use the fact that left curls are zero to see that

$$\Phi_n(\vee) \circ \Phi_n(\otimes) = \sum_{g \in G} g^{-1} \cdot \text{[Diagram: A vertical strand with a dot and two curls labeled } g \text{ and } g \text{]} \cdot g^{-1} \stackrel{(6.3)}{=} \text{[Diagram: A braiding of two strands]} = \Phi_n(\vee).$$

For the second relation in (4.4), we compute

$$\begin{aligned} \Phi_n(\vee) \circ \Phi_n(g \bullet | | h \bullet |) \circ \Phi_n(\vee) & \stackrel{(6.1)}{=} \text{[Diagram: A vertical strand with a dot and two curls labeled } g^{-1}h \text{ and } h^{-1} \text{]} \\ & \stackrel{(6.3)}{=} \delta_{g,h} g \bullet | | g^{-1} = \delta_{g,h} \Phi_n(g \bullet |). \end{aligned}$$

Relations (4.5): For the third relation in (4.5), we compute

$$\begin{aligned} \Psi(\otimes) \circ \Psi(g \bullet | |) &= \text{[Diagram: A braiding of two strands with dots labeled } g \text{ and } g^{-1} \text{]} + \sum_{h \in G} h^{-1} g \bullet | | h^{-1} \\ &= \text{[Diagram: A braiding of two strands with dots labeled } g \text{ and } g^{-1} \text{]} + \sum_{t \in G} t^{-1} \cdot \text{[Diagram: A vertical strand with a dot and two curls labeled } gt \text{ and } t \text{]} \cdot t^{-1} g^{-1} = \Psi(| | g \bullet) \circ \Psi(\otimes), \end{aligned}$$

where we let $t = g^{-1}h$. The other relations in (4.5) are straightforward to verify. □

We now wish to show that Ψ is faithful. Our approach is inspired by that of [13, Appendix A], which deals with the case where G is the trivial group.

In what follows, we will identify a permutation $\pi \in \mathfrak{S}_k$ with the partition of type $\binom{k}{k}$ with parts $\{i, \pi(i)'\}$, $1 \leq i \leq k$. Recall that, for $1 \leq i < j \leq k$, the pair (i, j) is an

inversion in $\pi \in \mathfrak{S}_k$ if $\pi(i) > \pi(j)$. Suppose Q is a partition of type $\binom{l}{k}$. We say that a permutation $P \in \mathfrak{S}_l$ is a *left shuffle* for Q if there is no inversion (i, j) in P such that vertices i' and j' lie in the same connected component of Q . Intuitively, P is a left shuffle for Q if it does not change the relative order of vertices in each component. Similarly, we say that a permutation $P \in \mathfrak{S}_k$ is a *right shuffle* for Q if there is no inversion (i, j) in P^{-1} such that vertices i and j lie in the same component of Q . For example, if

$$P = \begin{array}{c} \bullet & \bullet & \bullet \\ \diagdown & \diagup & \\ \bullet & \bullet & \bullet \end{array} \quad \text{and} \quad Q = \begin{array}{c} \bullet & \bullet & \bullet \\ \square & & \\ \bullet & \bullet & \bullet \end{array},$$

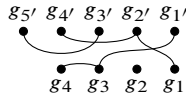
then P is a left shuffle for Q but not a right shuffle for Q .

We say a partition diagram is *tensor-planar* if it is a tensor product (horizontal juxtaposition) of partition diagrams consisting of a single connected component. Note that every tensor-planar partition diagram is planar (i.e., can be represented as a graph without edge crossings inside of the rectangle formed by its vertices) but the converse is false.

Every equivalence class $[P, \mathbf{g}]$ of G -partitions can be factored as a product

$$[P, \mathbf{g}] = \begin{array}{c} g_{1'} \\ \bullet \\ \vdots \\ \bullet \end{array} \cdots \begin{array}{c} g_{2'} g_{1'} \\ \bullet \\ \vdots \\ \bullet \end{array} \circ [P_1] \circ [P_2] \circ [P_3] \circ \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \cdots \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}, \quad (7.2)$$

where P_2 is tensor-planar, P_1 is a left shuffle for P_2 , and P_3 is a right shuffle for P_2 . (See (4.14).) The number of connected components in P is equal to the number of connected components in P_2 . For example, the G -partition diagram



has four connected components and decomposition

$$\begin{array}{c} g_{5'} g_{4'} g_{3'} g_{2'} g_{1'} \\ \bullet \bullet \bullet \bullet \bullet \\ \curvearrowright \quad \curvearrowright \quad \curvearrowright \\ \bullet \bullet \bullet \bullet \\ g_4 g_3 g_2 g_1 \end{array} = \begin{array}{c} g_{5'} g_{4'} g_{3'} g_{2'} g_{1'} \\ \bullet \bullet \bullet \bullet \bullet \\ \vdots \\ \bullet \end{array} \circ [P_1] \circ [P_2] \circ [P_3] \circ \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array},$$

where

$$P_1 = \begin{array}{c} \bullet & \bullet & \bullet \\ \diagdown & \diagup & \\ \bullet & \bullet & \bullet \end{array}, \quad P_2 = \begin{array}{c} \bullet & \bullet & \bullet & \bullet & \bullet \\ \curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} = \bullet \otimes \curvearrowright \otimes \curvearrowright \otimes \curvearrowright, \\ P_3 = \begin{array}{c} \bullet & \bullet & \bullet \\ \diagdown & \diagup & \\ \bullet & \bullet & \bullet \end{array}.$$

For $n, k, l \in \mathbb{N}$, let

$$\text{Hom}_{\text{Par}(G)}^{\leq n} (V^{\otimes k}, V^{\otimes l})$$

denote the subspace of $\text{Hom}_{\text{Par}(G)}(\mathbb{V}^{\otimes k}, \mathbb{V}^{\otimes l})$ spanned by G -partition diagrams with at most n connected components. Composition respects the corresponding filtration on morphism spaces since the composite of a partition with n connected components and a partition with m connected components has at most $n + m$ connected components.

Recall the bases of the morphism spaces of $\mathcal{H}eis(G)$ given in [2, Theorem 7.2]. (The category $\mathcal{H}eis(G)$ is $\mathcal{H}eis_{-1}(\mathbb{k}G)$ in the notation of [2].) For any such basis element f in

$$\text{Hom}_{\mathcal{H}eis(G)}((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l}),$$

define the *block number* of f to be number of distinct closed (possibly intersecting) loops in the diagram

$$\curvearrowright^{\otimes l} \circ f \circ \curvearrowleft^{\otimes k}.$$

For $n \in \mathbb{N}$, let

$$\text{Hom}_{\mathcal{H}eis(G)}^{\leq n}((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l})$$

denote the subspace of $\text{Hom}_{\mathcal{H}eis(G)}((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l})$ spanned by basis elements with block number at most n . Composition respects the resulting filtration on morphism spaces.

The image under Ψ of tensor-planar partition diagrams (writing the image in terms of the aforementioned bases of the morphism spaces of $\mathcal{H}eis(G)$) is particularly simple to describe. Since each tensor-planar partition diagram is a tensor product of single connected components, consider the case of a single connected component. Then, for example, we have

$$\Psi(\text{arc}) = \text{loop} \quad \text{and} \quad \Psi(\text{square}) = \text{planar diagram}.$$

The general case is analogous. (In fact, the images of all planar partition diagrams are similarly easy to describe.) In particular, if P is a tensor-planar partition diagram with n connected components, then $\Psi(P)$ is a planar diagram with block number n .

For $i = 1, \dots, k - 1$, consider the morphism

$$\begin{aligned} \chi_i &:= \Psi(1_V^{\otimes(k-i-1)} \otimes (\text{cross} - \text{square}) \otimes 1_V^{\otimes(i-1)}) \\ &= \uparrow \downarrow \cdots \uparrow \downarrow \text{cross} \uparrow \downarrow \cdots \uparrow \downarrow \in \text{End}_{\mathcal{H}eis(G)}((\uparrow\downarrow)^{\otimes k}). \end{aligned} \tag{7.3}$$

For a permutation partition diagram $P: k \rightarrow k$, let $T(P)$ be the morphism in $\mathcal{H}eis(G)$ defined as follows: Write $D = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_r}$ as a reduced word in simple transpositions and let

$$T(P) = \chi_{i_1} \circ \chi_{i_2} \circ \cdots \circ \chi_{i_r}.$$

It follows from the braid relations (6.5) that $T(P)$ is independent of the choice of reduced word for P .

Lemma 7.3. *Suppose (P, \mathbf{g}) is a G -partition diagram with decomposition (7.2). Then*

$$\Psi([P, \mathbf{g}]) - \begin{array}{c} g_1' \quad g_2' g_1' \\ \vdots \quad \vdots \quad \vdots \\ \cdots \quad \vdots \quad \vdots \end{array} \circ [T(P_1)] \circ [P_2] \circ [T(P_3)] \circ \begin{array}{c} \vdots \quad \vdots \quad \vdots \\ g_k \quad g_2 \quad g_1 \end{array} \in \text{Hom}_{\mathcal{H}eis(G)}^{\leq n-1} ((\uparrow\downarrow)^{\otimes k}, (\uparrow\downarrow)^{\otimes l}).$$

Proof. The case where $\mathbf{g} = \mathbf{1}$ is [13, Proposition A.1]. Since composition with G -partition diagrams of the form

$$\begin{array}{c} g_1' \quad g_2' g_1' \\ \vdots \quad \vdots \quad \vdots \\ \cdots \quad \vdots \quad \vdots \end{array} \quad \text{and} \quad \begin{array}{c} \vdots \quad \vdots \quad \vdots \\ g_k \quad g_2 \quad g_1 \end{array}$$

does not change the block number, the general case follows. □

Proof of Theorem 7.1. Since Ψ is well defined by Proposition 7.2, it remains to show it is faithful. As in Remark 4.5, we view $\mathcal{P}ar(G)$ as $\mathcal{P}ar(G, d)$ over the ring $\mathbb{k}[d]$. Note that the image of Ψ is contained in the full monoidal subcategory $\mathcal{H}eis_{\uparrow\downarrow}(G)$ of $\mathcal{H}eis(G)$ generated by the object $\uparrow \otimes \downarrow$. It follows from the defining relations of $\mathcal{H}eis(G)$ that

$$\begin{array}{c} \circlearrowright \\ \uparrow \downarrow \end{array} = \begin{array}{c} \uparrow \downarrow \\ \circlearrowleft \end{array}.$$

In other words, the clockwise bubble is strictly central in $\mathcal{H}eis_{\uparrow\downarrow}(G)$. Let $\mathcal{H}eis_{\uparrow\downarrow}(G, d)$ be the quotient of $\mathcal{H}eis_{\uparrow\downarrow}(G)$, defined over $\mathbb{k}[d]$, by the additional relation

$$\begin{array}{c} \circlearrowright \\ \uparrow \downarrow \end{array} = d1_{\mathbb{1}}.$$

Then, as in Remark 4.5, $\mathcal{H}eis_{\uparrow\downarrow}(G)$, defined over \mathbb{k} , is isomorphic as a \mathbb{k} -linear category to $\mathcal{H}eis_{\uparrow\downarrow}(G, d)$. In other words, we can view the clockwise bubble of $\mathcal{H}eis_{\uparrow\downarrow}(G)$ and the morphism \circlearrowright of $\mathcal{P}ar(G)$, both of which are strictly central, as elements of the ground ring.

Now, it is clear that, in the setting of Lemma 7.3,

$$\begin{array}{c} g_1' \quad g_2' g_1' \\ \vdots \quad \vdots \quad \vdots \\ \cdots \quad \vdots \quad \vdots \end{array} \circ [T(P_1)] \circ [P_2] \circ [T(P_3)] \circ \begin{array}{c} \vdots \quad \vdots \quad \vdots \\ g_k \quad g_2 \quad g_1 \end{array} \tag{7.4}$$

is uniquely determined by $[P, \mathbf{g}]$. Indeed, P is the partition diagram obtained from

$$T(P_1) \circ \Psi(P_2) \circ T(P_3)$$

by replacing each pair $\uparrow\downarrow$ by a vertex and each strand by an edge. Furthermore, the diagrams of the form (7.4) are linearly independent by [2, Theorem 7.2]. The result then follows by a standard triangularity argument. \square

8. Compatibility of categorical actions

We continue to assume that G is a finite group. The group Heisenberg category acts naturally on the direct sum of the categories $A_n\text{-mod}$, $n \in \mathbb{N}$. In this section, we recall this action and show that it is compatible with the embedding of the group partition category into the group Heisenberg category and the action of the group partition category described in Theorem 5.1.

For $0 \leq m, k \leq n$, let ${}_k(n)_m$ denote A_n , considered as an (A_k, A_m) -bimodule. We will omit the subscript k or m when $k = n$ or $m = n$, respectively. We denote tensor product of such bimodules by juxtaposition. For instance $(n)_{n-1}(n)$ denotes $A_n \otimes_{A_{n-1}} A_n$, considered as an (A_n, A_n) -bimodule, where we write \otimes_m for the tensor product over A_m . As explained in [14, §7], we have a strong \mathbb{k} -linear monoidal functor

$$\Theta: \mathcal{H}eis(G) \rightarrow \prod_{m \in \mathbb{N}} \left(\bigoplus_{n \in \mathbb{N}} (A_n, A_m)\text{-bimod} \right)$$

given by

$$\begin{aligned} \Theta(\curvearrowright) &= ((n)_{n-2} \rightarrow (n)_{n-2}, x \mapsto x s_{n-1})_{n \geq 2}, \\ \Theta(\cup) &= ((n-1) \rightarrow_{n-1} (n)_{n-1}, x \mapsto x)_{n \geq 1}, \\ \Theta(\cap) &= ((n)_{n-1}(n) \rightarrow (n), x \otimes y \mapsto xy)_{n \geq 1}, \\ \Theta(\cup) &= ((n) \rightarrow (n)_{n-1}(n), x \mapsto x \sum_{1 \leq i \leq n, h \in G} h^{(i)} \pi_i \otimes \pi_i^{-1} (h^{-1})^{(i)})_{n \geq 1}, \\ \Theta(\cap) &= \left({}_{n-1}(n)_{n-1} \rightarrow (n-1), \right. \\ &\quad \left. \mathbf{g}\pi \mapsto \begin{cases} (g_{n-1}, g_{n-2}, \dots, g_1)\pi & \text{if } \pi \in \mathfrak{S}_{n-1}, g_n = 1_G \\ 0 & \text{otherwise} \end{cases} \right)_{n \geq 1}, \\ \Theta(\uparrow \mathbf{g}) &= ((n)_{n-1} \rightarrow (n)_{n-1}, x \mapsto x (g^{-1})^{(n)})_{n \geq 1}, \\ \Theta(\curvearrowleft) &= \left({}_{n-1}(n)_{n-1} \rightarrow (n-1)_{n-2}(n-1), \right. \\ &\quad \left. \mathbf{g}\pi \mapsto \begin{cases} (g_{n-1}, \dots, g_1)\sigma_1 \otimes g_n^{(n-1)}\sigma_2 & \text{if } \pi = \sigma_1 s_{n-1} \sigma_2 \text{ for } \sigma_1, \sigma_2 \in \mathfrak{S}_{n-1}, \\ 0 & \text{if } \pi \in \mathfrak{S}_{n-1} \end{cases} \right)_{n \geq 2}, \\ \Theta(\curvearrowright) &= ((n-1)_{n-2}(n-1) \rightarrow_{n-1} (n)_{n-1}, x \otimes y \mapsto x s_{n-1} y)_{n \geq 2}, \end{aligned}$$

$$\Theta(\text{cross}) = ({}_{n-2}(n) \rightarrow {}_{n-2}(n), x \mapsto s_{n-1}x)_{n \geq 2},$$

$$\Theta(\text{g down}) = ({}_{n-1}(n) \rightarrow {}_{n-1}(n), x \mapsto (g^{-1})^{(n)}x)_{n \geq 1}.$$

As noted in the proof of Theorem 7.1, the image of Ψ lies in the full monoidal subcategory $\mathcal{H}eis_{\uparrow\downarrow}(G)$ of $\mathcal{H}eis(G)$ generated by $\uparrow \otimes \downarrow$. For $n \in \mathbb{N}$, consider the composition

$$\Omega_n: \mathcal{H}eis_{\uparrow\downarrow}(G) \xrightarrow{\Theta} \bigoplus_{m \in \mathbb{N}} (A_m, A_m)\text{-bimod} \xrightarrow{-\otimes_{A_n} \mathbf{1}_n} \bigoplus_{m \in \mathbb{N}} A_m\text{-mod},$$

where we declare $M \otimes_{A_n} \mathbf{1}_n = 0$ for $M \in (A_m, A_m)\text{-bimod}$ with $m \neq n$. The functor Ω_n is \mathbb{k} -linear, but no longer monoidal.

Theorem 8.1. *Consider the functors:*

$$\begin{array}{ccc} \text{Par}(G) & \xrightarrow{\Psi} & \mathcal{H}eis_{\uparrow\downarrow}(G) \\ & \searrow \Phi_n & \downarrow \Omega_n \\ & & A_n\text{-mod}. \end{array} \tag{8.1}$$

The isomorphisms $\beta_k, k \in \mathbb{N}$, defined in Corollary 2.3, give a natural isomorphism of functors $\Omega_n \circ \Psi \cong \Phi_n$.

Proof. Since the β_k are isomorphisms of A_n -modules it suffices to show that they determine a natural transformation between the given functors. Therefore, following the argument used in the proof of [13, Theorem 5.1] we need to check elements of the form

$$1_{V^{\otimes k}} \otimes x \otimes 1_{V^{\otimes j}}, \quad k, j \in \mathbb{N}, \quad x \in \{ \text{h down}, \text{cross}, \text{Y}, \text{X}, \text{circle}, \text{circle with dot} : h \in G \}.$$

Tokens: For $h \in G$,

$$\beta_k^{-1} \circ (\Omega_n \circ \Psi(1_{V^{\otimes(k-j-1)}} \otimes \text{h down} \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes k}$$

is the A_n -module map given by

$$\begin{aligned} & g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \\ \mapsto & g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ \mapsto & g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} g_1^{(i_j)} \pi_{i_j} (h^{-1})^{(n)} \\ & \otimes h^{(n)} \pi_{i_j}^{-1} (g_j^{-1})^{(i_j)} g_1^{i_{j-1}} \pi_{i_{j-1}} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ \mapsto & g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g_k^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} (g_j h^{-1})^{(i_j)} \pi_{i_j} \\ & \otimes \pi_{i_j}^{-1} (h g_j^{-1})^{(i_j)} g_1^{i_{j-1}} \pi_{i_{j-1}} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \end{aligned}$$

$$\mapsto g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes g_j h^{-1} e_{i_j} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1}.$$

This is precisely the map $\Phi_n(1_{V^{\otimes(k-j-1)}} \otimes h \downarrow \otimes 1_{V^{\otimes(j-1)}})$.

Merge: For $g, h \in G$ and $1 \leq i, j \leq n$,

$$\pi_i^{-1}(g^{-1})^{(i)} h^{(j)} \pi_j = (g^{-1})^{(n)} h^{(\pi_i^{-1}(j))} \pi_i^{-1} \pi_j. \quad (8.2)$$

We have $\pi_i^{-1} \pi_j \in \mathfrak{S}_{n-1}$ if and only if $i = j$, in which case (8.2) is equal to $(g^{-1}h)^{(n)}$. Thus, the composition

$$\beta_{k-1}^{-1} \circ (\Omega_n \circ \Psi(1_{V^{\otimes(k-j-1)}} \otimes \frown \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k-1)}$$

is the A_n -module map given by

$$\begin{aligned} & g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \\ \mapsto & g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} g_j^{(i_j)} \pi_{i_j} \\ & \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ \mapsto & \delta_{i_j, i_{j+1}} \delta_{g_j, g_{j+1}} g_k e_{i_k} \otimes \cdots \otimes g_{j+2} e_{i_{j+2}} \otimes g_j e_{i_j} \otimes \cdots \otimes g_1 e_{i_1}. \end{aligned}$$

This is precisely the map $\Phi_n(1_{V^{\otimes(k-j-1)}} \otimes \frown \otimes 1_{V^{\otimes(j-1)}})$.

Split: The composition

$$\beta_{k+1}^{-1} \circ (\Omega_n \circ \Psi(1_{V^{\otimes(k-j)}} \otimes \vee \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k+1)}$$

is the A_n -module map given by

$$\begin{aligned} & g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \\ \mapsto & g_k^{(i_k)} \pi_{i_k} \otimes \pi_{i_k}^{-1} (g^{-1})^{(i_k)} g_{k-1}^{(i_{k-1})} \pi_{i_{k-1}} \otimes \cdots \otimes \pi_{i_{j+1}}^{-1} (g_{j+1}^{-1})^{(i_{j+1})} g_j^{(i_j)} \pi_{i_j} \otimes 1 \\ & \otimes \pi_{i_j}^{-1} (g_j^{-1})^{(i_j)} g_{j-1}^{(i_{j-1})} \otimes \cdots \otimes \pi_{i_2}^{-1} (g_2^{-1})^{(i_2)} g_1^{(i_1)} \pi_{i_1} \otimes 1 \\ \mapsto & g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes g_j e_{i_j} \otimes g_j e_{i_j} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1}. \end{aligned}$$

This is precisely the map $\Phi_n(1_{V^{\otimes(k-j)}} \otimes \vee \otimes 1_{V^{\otimes(j-1)}})$.

Unit pin: The composition

$$\beta_{k+1}^{-1} \circ (\Omega_n \circ \Psi(1_{V^{\otimes(k-j)}} \otimes \circ \otimes 1_{V^{\otimes(j)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k+1)}$$

is the map

$$g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \mapsto \sum_{h \in G} \sum_{i=1}^n g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes h e_i \otimes g_j e_{i_j} \otimes \cdots \otimes g_1 e_{i_1},$$

which is equal to the map $\Phi_n(1_{V^{\otimes(k-j)}} \otimes \circ \otimes 1_{V^{\otimes(j)}})$.

Count pin: The composition

$$\beta_{k-1}^{-1} \circ (\Omega_n \circ \Psi(1_{V^{\otimes(k-j)}} \otimes \uparrow \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k: V^{\otimes k} \rightarrow V^{\otimes(k-1)}$$

is the map

$$g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} \mapsto g_k e_{i_k} \otimes \cdots \otimes g_{j+1} e_{i_{j+1}} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1},$$

which is equal to the map $\Phi_n(1_{V^{\otimes(k-j)}} \otimes \uparrow \otimes 1_{V^{\otimes(j-1)}})$.

Crossing: Define the elements $f, f' \in \text{End}_{\mathcal{H}eis}(\uparrow \downarrow \uparrow \downarrow)$ by

$$f = \begin{array}{c} \curvearrowright \quad \curvearrowleft \\ \curvearrowleft \quad \curvearrowright \end{array}, \quad f' = \sum_{g \in G} g^{-1} \begin{array}{c} \curvearrowright \quad \curvearrowleft \\ \bullet \quad \bullet \\ \curvearrowleft \quad \curvearrowright \\ \bullet \quad \bullet \\ g^{-1} \end{array}. \quad (8.3)$$

Note that

$$f = f_3 \circ f_2 \circ f_1,$$

where

$$f_1 = \uparrow \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \downarrow, \quad f_2 = \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array}, \quad f_3 = \uparrow \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \downarrow.$$

Suppose $i, j \in \{1, \dots, n\}$ and $x, y \in A_n$. We first compute the action of $\Theta(f)$ and $\Theta(f')$ on

$$\begin{aligned} \alpha &= x g_1^{(i)} \pi_i \otimes \pi_i^{-1} (g_1^{-1})^{(i)} g_2^{(j)} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} y \\ &= x g_1^{(i)} \pi_i \otimes (g_1^{-1})^{(n)} g_2^{(\pi_i^{-1}(j))} \pi_i^{-1} \pi_j \otimes \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-1} (n)_{n-1} (n), \end{aligned}$$

where $x, y \in A_n$. If $i = j$, then $\pi_i^{-1} \pi_j = 1_{\mathfrak{S}_n}$, and so $\Omega_n(f_1)(\alpha) = 0$. Now suppose $i < j$ so that

$$\pi_i^{-1} \pi_j = s_{n-1} \cdots s_i s_j \cdots s_{n-1} = s_{j-1} \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_i.$$

Thus,

$$\begin{aligned} \Theta(f_1)(\alpha) &= x g_1^{(i)} \pi_i g_2^{(\pi_i^{-1}(j))} s_{j-1} \cdots s_{n-2} \otimes (g_1^{-1})^{(n-1)} s_{n-2} \\ &\quad \cdots s_i \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-2} (n). \end{aligned}$$

Hence,

$$\Theta(f_2 \circ f_1)(\alpha) = x g_1^{(i)} \pi_i g_2^{(\pi_i^{-1}(j))} \pi_{j-1} \otimes (g_1^{-1})^{(n)} \pi_i^{-1} \pi_j^{-1} (g_2^{-1})^{(j)} y \in (n)_{n-2} (n),$$

and so

$$\Theta(f)(\alpha) = x g_1^{(i)} \pi_i g_2^{(\pi_i^{-1}(j))} \pi_{j-1} \otimes s_{n-1} \otimes (g_1^{-1})^{(n)} \pi_i^{-1} \pi_j^{-1} (g_2^{-1})^{(j)} y$$

$$\begin{aligned}
 &= xg_2^{(j)}\pi_j g_1^{(i)}s_i \cdots s_{n-2} \otimes s_{n-1} \otimes s_{n-2} \cdots s_{j-1}\pi_i^{-1}(g_2^{-1})^{(j)}(g_1^{-1})^{(i)}y \\
 &= xg_2^{(j)}\pi_j \otimes g_1^{(i)}\pi_i s_{n-2} \cdots s_{j-1}(g_2^{-1})^{(j-1)} \otimes \pi_i^{-1}(g_1^{-1})^{(i)}y \\
 &= xg_2^{(j)}\pi_j \otimes \pi_j^{-1}g_1^{(i)}\pi_i(g_2^{-1})^{(j-1)} \otimes \pi_i^{-1}(g_1^{-1})^{(i)}y \\
 &= xg_2^{(j)}\pi_j \otimes \pi_j^{-1}(g_2^{-1})^{(j)}g_1^{(i)}\pi_i \otimes \pi_i^{-1}(g_1^{-1})^{(i)}y.
 \end{aligned}$$

The case $i > j$ is similar, giving

$$\Theta(f)(\alpha) = \begin{cases} 0 & \text{if } i = j, \\ xg_2^{(j)}\pi_j \otimes \pi_j^{-1}(g_2^{-1})^{(j)}g_1^{(i)}\pi_i \otimes \pi_i^{-1}(g_1^{-1})^{(i)}y & \text{if } i \neq j. \end{cases}$$

We also compute that

$$\Theta(f')(\alpha) = \begin{cases} xg_2^{(j)}\pi_j \otimes \pi_j^{-1}(g_2^{-1})^{(j)}g_1^{(i)}\pi_i \otimes \pi_i^{-1}(g_1^{-1})^{(i)}y & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus, for all $i, j \in \{1, \dots, n\}$, we have

$$\Theta(f + f')(\alpha) = xg_2^{(j)}\pi_j \otimes \pi_j^{-1}(g_2^{-1})^{(j)}g_1^{(i)}\pi_i \otimes \pi_i^{-1}(g_1^{-1})^{(i)}y.$$

Therefore, we have that

$$\begin{aligned}
 &\beta_k^{-1} \circ (\Omega_n \circ \Psi(1_{V^{\otimes(k-j-1)}} \otimes \bigotimes \otimes 1_{V^{\otimes(j-1)}})) \circ \beta_k \\
 &= \beta_k^{-1} \circ (\Omega_n(1_{\uparrow\downarrow}^{\otimes(k-j-1)} \otimes (f + f') \otimes 1_{\uparrow\downarrow}^{\otimes(j-1)})) \circ \beta_k
 \end{aligned}$$

is the map

$$\begin{aligned}
 g_k e_{i_k} \otimes \cdots \otimes g_1 e_{i_1} &\mapsto g_k e_{i_k} \otimes \cdots \otimes g_{j+2} e_{i_{j+2}} \\
 &\quad \otimes g_j e_{i_j} \otimes g_{j+1} e_{i_{j+1}} \otimes g_{j-1} e_{i_{j-1}} \otimes \cdots \otimes g_1 e_{i_1},
 \end{aligned}$$

which is precisely the map $\Phi_n(1_{V^{\otimes(k-j-1)}} \otimes \bigotimes \otimes 1_{V^{\otimes(j-1)}})$. □

9. Interpolating categories

We assume throughout this section that G is a finite group. In [10], Knop generalized the work [4] of Deligne by embedding a regular category \mathcal{A} into a family of pseudo-abelian tensor categories $\mathcal{T}(\mathcal{A}, \delta)$, which are the additive Karoubi envelope of categories $\mathcal{T}^0(\mathcal{A}, \delta)$ depending on a degree function δ . Deligne’s original construction corresponds to the case where \mathcal{A} is the category of finite boolean algebras.

As we now explain, the group partition category $\mathcal{P}ar(G, d)$ is equivalent to $\mathcal{T}^0(\mathcal{A}, \delta)$, where \mathcal{A} is the category of finite boolean algebras with a locally free

G -action and δ is a degree function depending on d . In this way, $\mathcal{P}ar(G, d)$ can be viewed as a concrete realization (including explicit bases of morphisms spaces) of the category $\mathcal{T}^0(\mathcal{A}, \delta)$, whose definition is rather abstract. Moreover, Theorem 4.4 can be viewed as giving an efficient presentation of Knop’s category. On the other hand, the equivalence of $\mathcal{P}ar(G, d)$ and $\mathcal{T}^0(\mathcal{A}, \delta)$ allows us to deduce from Knop’s work several important properties of $\mathcal{P}ar(G, d)$.

For an arbitrary finite set X , let $\text{Pow}(X)$ denote the power set of X . For $Y \subseteq X$, let $\neg Y = X - Y$ denote its complement. The 5-tuple $(\text{Pow}(X), \cap, \cup, \neg, \emptyset, X)$ is an example of a *finite boolean algebra*. In what follows, we simply denote this boolean algebra by $\text{Pow}(X)$. It turns out that every finite boolean algebra is isomorphic to one of this form. In fact, the category $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg$ of finite boolean algebras is equivalent to the opposite of the category $\mathcal{F}in\mathcal{S}et$ of finite sets. To a map $f: X \rightarrow Y$ of finite sets, the corresponding homomorphism of boolean algebras is the map

$$\text{Pow}(Y) \rightarrow \text{Pow}(X), \quad Z \mapsto f^{-1}(Z).$$

We refer the reader to [7, Chapter 15] for details.

By definition, an action of a group G on the boolean algebra $\text{Pow}(X)$ is a group homomorphism from G to the automorphism group of $\text{Pow}(X)$ in $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg$. It follows from the axioms of a boolean algebra that this action is uniquely determined by the action of G on singletons or, equivalently, by a G -action on the set X . In this way, the category $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)$ of finite boolean algebras with G -actions (with morphisms being homomorphisms of boolean algebras that intertwine the G -actions) is equivalent to the opposite of the category of finite G -sets.

We say that a G -action on a boolean algebra is *locally free* if every element of the boolean algebra is a union of elements on which G acts freely. In the case of the finite boolean algebra $\text{Pow}(X)$, this is equivalent to the condition that G acts freely on the singletons. (Note that, since X is finite, this forces the group G to be finite.) Hence the category $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$ of finite boolean algebras with locally free G -actions is equivalent to the opposite of the category of finite sets with free G -action:

$$\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}} \simeq \mathcal{F}in\mathcal{S}et(G)_{\text{free}}^{\text{op}}. \tag{9.1}$$

The category $\mathcal{F}in\mathcal{B}ool\mathcal{A}lg(G)_{\text{lf}}$ is regular, exact, and Malcev, using the definitions of these concepts given in [10].

Knop’s definition of the category $\mathcal{T}^0(\mathcal{A}, \delta)$ involves the diagram [10, (3.2)]:

$$\begin{array}{ccccc}
 & & r \times_y s & & \\
 & \swarrow & \downarrow & \searrow & \\
 & r & s \circ r & s & \\
 \swarrow & & \downarrow & & \searrow \\
 x & & y & & z,
 \end{array} \tag{9.2}$$

where $x, y, z \in \mathcal{A}$, r is a subobject of $x \times y$, s is a subobject of $y \times z$, and $s \circ r$ is the image of the natural surjective map $r \times_y s \rightarrow x \times z$. To relate Knop's construction to the G -partition category, we consider the diagram (9.2) in the case where $\mathcal{A} = \mathit{FinBoolAlg}(G)_{\text{lf}}$.

First note that the product $\text{Pow}(X) \times \text{Pow}(Y)$ is isomorphic to $\text{Pow}(X \sqcup Y)$. Our next goal is to interpret the subobjects r, s in (9.2) as G -partition diagrams.

Every finite free G -set is isomorphic to one of the form $X \times G$, where X is a finite set (indexing the G -orbits), with G action given by

$$g \cdot (x, h) = (x, hg^{-1}), \quad x \in X, g, h \in G.$$

Thus, by (9.1), every element of $\mathit{FinBoolAlg}(G)_{\text{lf}}$ is isomorphic to one of the form $\text{Pow}(X \times G)$. Moreover, every element is, in fact, isomorphic to

$$\text{Pow}(\{1, 2, \dots, r\} \times G)$$

for some $r \in \mathbb{N}$. (We adopt the convention that $\{1, 2, \dots, r\} = \emptyset$ when $r = 0$.)

Define the natural projection map $p_X: X \times G \rightarrow X$. For a morphism

$$\varphi: \text{Pow}(\{1, 2, \dots, r\} \times G) \rightarrow \text{Pow}(X \times G),$$

define

$$P_i^\varphi := p_X \circ \varphi(\{(i, 1_G)\}) \subseteq X, \quad 1 \leq i \leq r,$$

$$\lambda^\varphi := \bigcup_{i=1}^r \varphi(\{(i, 1_G)\}) \in G^X,$$

and set $\vec{P}^\varphi = (P_1^\varphi, \dots, P_r^\varphi)$. Here we use the formal definition of an element of G^X , the set of functions $X \rightarrow G$, as a subset of $X \times G$. Let

$$\text{Par}_r(X) := \{(P_1, \dots, P_r) \in \text{Pow}(X)^r : \bigcup_{i=1}^r P_i = X, P_i \neq \emptyset, P_i \cap P_j = \emptyset \text{ for all } 1 \leq i, j \leq r\}.$$

In other words, $\text{Par}_r(X)$ is the set of all r -tuples of nonempty disjoint sets whose union is X . For $x, y \in \mathit{FinBoolAlg}(G)_{\text{lf}}$, let $\text{Mon}(x, y)$ denote the set of monomorphisms $x \rightarrow y$ in $\mathit{FinBoolAlg}(G)_{\text{lf}}$.

Lemma 9.1. *The map*

$$\text{Mon}(\text{Pow}(\{1, 2, \dots, r\} \times G), \text{Pow}(X \times G)) \rightarrow \text{Par}_r(X) \times G^X, \quad \varphi \mapsto (\vec{P}^\varphi, \lambda^\varphi), \tag{9.3}$$

is a bijection.

Proof. For $(\vec{P}, \lambda) \in \text{Par}_r(X) \times G^X$, define

$$\varphi'_{\vec{P}, \lambda}: \{1, 2, \dots, r\} \times G \rightarrow \text{Pow}(X \times G)$$

by

$$\varphi'_{\vec{P},\lambda}(i, g) = \{(x, \lambda(x)g) : x \in P_i\} \subseteq X \times G.$$

This induces a map

$$\varphi_{\vec{P},\lambda}: \text{Pow}(\{1, 2, \dots, r\} \times G) \rightarrow \text{Pow}(X \times G).$$

It is straightforward to verify that the map $(\vec{P}, \lambda) \mapsto \varphi_{\vec{P},\lambda}$ is inverse to (9.3). □

The proof of the following lemma is straightforward.

Lemma 9.2. *The automorphism group of $\text{Pow}(\{1, 2, \dots, r\} \times G)$ in $\text{FinBoolAlg}(G)_{\text{lf}}$ is the wreath product $G_r = G^r \rtimes \mathfrak{S}_r$, where the action is determined by its action on elements of $\{1, 2, \dots, r\} \times G$ as follows:*

$$(\mathbf{g}, \pi) \cdot (i, h) := (\pi(i), g_{\pi(i)}h), \quad \pi \in \mathfrak{S}_r, i = \{1, 2, \dots, r\}, h \in G, \mathbf{g} \in G^r. \tag{9.4}$$

We also have an action of G_r on $\text{Pow}_r(X) \times G^X$ given by

$$(\mathbf{g}, \pi) \cdot ((P_1, \dots, P_r), (h_x)_{x \in X}) := ((P_{\pi^{-1}(1)}, \dots, P_{\pi^{-1}(r)}), (g_{\pi(i_x)}h_x)_{x \in X}), \tag{9.5}$$

where $i_x \in \{1, 2, \dots, r\}$ is determined by $x \in P_{i_x}$. The following lemma is also a straightforward verification.

Lemma 9.3. *The bijection (9.3) intertwines the actions (9.4) and (9.5).*

Let $\text{Par}(X)$ denote the set of partitions of X . For $\vec{P} = (P_1, \dots, P_r) \in \text{Par}_r(X)$, let $P = \{P_1, \dots, P_r\} \in \text{Par}(X)$ denote the corresponding partition of X . For $P = \{P_1, \dots, P_r\} \in \text{Par}(X)$, define an equivalence relation \sim_P on G^X as follows: $(g_x)_{x \in X} \sim_P (h_x)_{x \in X}$ if and only if there exist $t_1, \dots, t_r \in G$ such that $g_x = t_i h_x$ for all $x \in P_i$.

Corollary 9.4. *The subobjects of $\text{Pow}(X \times G)$ in the category $\text{FinBoolAlg}(G)_{\text{lf}}$ are naturally enumerated by the set*

$$\bigsqcup_{P \in \text{Par}(X)} G^X / \sim_P. \tag{9.6}$$

Now consider the diagram (9.2) with

$$x = \text{Pow}(\{1, 2, \dots, k\} \times G), \quad y = \text{Pow}(\{1, 2, \dots, l\} \times G),$$

and

$$z = \text{Pow}(\{1, 2, \dots, m\} \times G).$$

When $X = X_k^l \cong \{1, 2, \dots, k\} \sqcup \{1, 2, \dots, l\}$, the set (9.6) can be naturally identified with the equivalence classes of G -partitions of type $\binom{l}{k}$. Thus we can view r and s as equivalence classes of G -partitions $[P, \mathbf{g}]$ and $[Q, \mathbf{h}]$, respectively. We then leave it to the reader to verify that $r \times_y s$ exists if and only if the pair $([Q, \mathbf{g}], [P, \mathbf{h}])$

is compatible. If this pair is compatible, then $r \times_y s$ is the equivalence class of stack $((Q, \mathbf{h}), (P, \mathbf{g}))$ and $s \circ r = [Q \star P, \mathbf{h} \star_{Q,P} \mathbf{g}]$. Thus, we have the following result.

Theorem 9.5. *The G -partition category $\mathcal{P}ar(G, d)$ is equivalent to the category $\mathcal{T}^0(\mathcal{A}, \delta)$ defined in [10, Definition 3.2], where $\mathcal{A} = \mathit{FinBoolAlg}(G)_{\text{lf}}$ and δ is the degree function of [10, (8.15)] with the t there equal to d .*

Proof. This follows from the above discussion and [10, Example 2, p. 596]. □

Let $\text{Kar}(\mathcal{P}ar(G, d))$ be the additive Karoubi envelope (also known as the pseudo-abelian completion) of $\mathcal{P}ar(G, d)$. Let $\mathcal{N}(G, d)$ be the tensor radical (also known as the tensor ideal of negligible morphisms) of $\text{Kar}(\mathcal{P}ar(G, d))$.

Corollary 9.6. *Suppose \mathbb{k} is a field of characteristic zero.*

- (1) *The category $\text{Kar}(\mathcal{P}ar(G, d))/\mathcal{N}(G, d)$ is a semisimple (hence, abelian) category.*
- (2) *We have $\mathcal{N}(G, d) = 0$ if and only if $d \notin \mathbb{N}|G|$.*
- (3) *If $\mathcal{N}(G, d) = 0$, then the simple objects of $\text{Kar}(\mathcal{P}ar(G, d))$ are naturally parameterized by the set of N -tuples of Young diagrams, where N is the number of isomorphism classes of simple G -modules.*
- (4) *If $d = n|G|$, then $\text{Kar}(\mathcal{P}ar(G, d))/\mathcal{N}(G, d)$ is equivalent to the category of $\mathbb{k}G_n$ -modules.*

Proof. (1) This follows from [10, Theorem 6.1 (i)].

(2) This follows from [10, Example 2, p. 596].

(3) By parts (iii) and (iv) of [10, Theorem 6.1], the simple objects of $\text{Kar}(\mathcal{P}ar(G, d))$ are in bijection with the simple modules of the automorphism groups of objects of $\mathit{FinBoolAlg}(G)_{\text{lf}}$ which, by Lemma 9.2, are precisely the wreath products $G_n, n \in \mathbb{N}$. The statement then follows from the classification of irreducible modules of wreath product groups. (See, for example, [14, Proposition 4.3].)

(4) This is explained in [10, Example 2, p. 606]. □

Remark 9.7. Deligne’s construction has also been generalized by Mori [12], who defined, for each $d \in \mathbb{k}$, a 2-functor \mathcal{S}_d sending a tensor category \mathcal{C} to another tensor category $\mathcal{S}_d(\mathcal{C})$, which should be thought of as a sort of interpolating wreath product functor. When \mathcal{C} is the category $G\text{-mod}$ of G -modules, $\mathcal{S}_t(G\text{-mod})$ can also be thought of as a family of interpolating categories for modules of the wreath products $G_n, n \in \mathbb{N}$. Mori’s interpolating category contains Knop’s as a full subcategory; see [12, Remark 4.14]. Mori gives a presentation of his categories, the relations of which can be found in [12, Proposition 4.26]. The presentation of Definition 4.1 is considerably more efficient. For example, $\mathcal{P}ar(G)$ has just

one generating object, whereas Mori's category (before taking the additive Karoubi envelope) has a generator for each representation of G . In addition, the presentation of [12] includes as generating morphisms *all* morphisms in the category $G\text{-mod}$, whereas the presentation of Definition 4.1 only includes a morphism for each element of the group (the tokens).

References

- [1] M. Bloss, G -colored partition algebras as centralizer algebras of wreath products. *J. Algebra* **265** (2003), no. 2, 690–710 [Zbl 1028.20007](#) [MR 1987025](#)
- [2] J. Brundan, A. Savage, and B. Webster, Foundations of Frobenius Heisenberg categories. *J. Algebra* **578** (2021), 115–185 [Zbl 07335722](#) [MR 4234799](#)
- [3] J. Comes, Jellyfish partition categories. *Algebr. Represent. Theory* **23** (2020), no. 2, 327–347 [Zbl 1462.18004](#) [MR 4097319](#)
- [4] P. Deligne, La catégorie des représentations du groupe symétrique S_t , lorsque t n'est pas un entier naturel. In *Algebraic groups and homogeneous spaces*, pp. 209–273, Tata Inst. Fund. Res. Stud. Math. 19, Tata Inst. Fund. Res., Mumbai, 2007 [Zbl 1165.20300](#) [MR 2348906](#)
- [5] P. Etingof, Representation theory in complex rank, I. *Transform. Groups* **19** (2014), no. 2, 359–381 [Zbl 1336.20011](#) [MR 3200430](#)
- [6] A. Freslon and A. Skalski, Wreath products of finite groups by quantum groups. *J. Noncommut. Geom.* **12** (2018), no. 1, 29–68 [Zbl 06866125](#) [MR 3782053](#)
- [7] S. Givant and P. Halmos, *Introduction to Boolean algebras*. Undergrad. Texts Math., Springer, New York, 2009 [Zbl 1168.06001](#) [MR 2466574](#)
- [8] N. Harman, Generators for the representation rings of certain wreath products. *J. Algebra* **445** (2016), 125–135 [Zbl 1366.20005](#) [MR 3418050](#)
- [9] M. Khovanov, Heisenberg algebra and a graphical calculus. *Fund. Math.* **225** (2014), no. 1, 169–210 [Zbl 1304.18019](#) [MR 3205569](#)
- [10] F. Knop, Tensor envelopes of regular categories. *Adv. Math.* **214** (2007), no. 2, 571–617 [Zbl 1127.18004](#) [MR 2349713](#)
- [11] J. Kock, *Frobenius algebras and 2D topological quantum field theories*. London Math. Soc. Stud. Texts 59, Cambridge Univ. Press, Cambridge, 2004 [Zbl 1046.57001](#) [MR 2037238](#)
- [12] M. Mori, On representation categories of wreath products in non-integral rank. *Adv. Math.* **231** (2012), no. 1, 1–42 [Zbl 1248.18007](#) [MR 2935382](#)
- [13] S. Nyobe Likeng and A. Savage, Embedding Deligne's category $\text{Rep}(S_t)$ in the Heisenberg category (with an appendix with Christopher Ryba). *Quantum Topol.* **12** (2021), no. 2, 211–242 [Zbl 07377316](#) [MR 4261657](#)
- [14] D. Rosso and A. Savage, A general approach to Heisenberg categorification via wreath product algebras. *Math. Z.* **286** (2017), no. 1-2, 603–655 [Zbl 1366.18006](#) [MR 3648512](#)
- [15] C. Ryba, The structure of the Grothendieck rings of wreath product Deligne categories and their generalisations. *Int. Math. Res. Not. IMRN* (2021), no. 16, 12420–12462 [MR 4300230](#)

- [16] C. Ryba, Stable Grothendieck rings of wreath product categories. *J. Algebraic Combin.* **49** (2019), no. 3, 267–307 [Zbl 1431.16034](#) [MR 3945269](#)
- [17] A. Savage, Frobenius Heisenberg categorification. *Algebr. Comb.* **2** (2019), no. 5, 937–967 [Zbl 1459.18008](#) [MR 4023572](#)
- [18] A. Savage, Affine wreath product algebras. *Int. Math. Res. Not. IMRN* (2020), no. 10, 2977–3041 [Zbl 07217942](#) [MR 4098632](#)

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