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A generalization of combinatorial identities for stable discrete series constants

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Abstract. This article is concerned with the constants that appear in Harish-Chandra's character formula for stable discrete series of real reductive groups, although it does not require any knowledge about real reductive groups or discrete series. In Harish-Chandra's work the only information we have about these constants is that they are uniquely determined by an inductive property. Later, Goresky–Kottwitz–MacPherson (1997) and Herb (2000) gave different formulas for these constants. In this article, we generalize these formulas to the case of arbitrary finite Coxeter groups (in this setting, discrete series no longer make sense), and give a direct proof that the two formulas agree. We actually prove a slightly more general identity that also implies the combinatorial identity underlying the discrete series character identities of Morel (2011). We deduce this identity from a general abstract theorem giving a way to calculate the alternating sum of the values of a valuation on the chambers of a Coxeter arrangement. We also introduce a ring structure on the set of valuations on polyhedral cones in Euclidean space with values in a fixed ring. This gives a theoretical framework for the valuation appearing in Goresky–Kottwitz–MacPherson's 1997 paper. In an appendix, we extend Herb's notion of 2-structures to pseudo-root systems.

1. Introduction

Although this paper deals exclusively with the combinatorics of real hyperplane arrangements and Coxeter complexes, it has its origin in the representation theory of real reductive groups and its connections with the cohomology of locally symmetric spaces, and in particular, of Shimura varieties. We start by explaining some of this background. This explanation can be safely skipped by the reader not interested in Shimura varieties.

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Let G be an algebraic group over $\mathbb{Q}$. To simplify the exposition, we assume that G is connected and semisimple. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and K be an open compact subgroup of $G(\mathbb{A}^\infty)$, where $\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles of $\mathbb{Q}$. We consider the double quotient

$$X_K = G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G(\mathbb{A}^\infty))/(K_\infty \times K).$$

This is a real analytic variety for K small enough, and the projective system $(X_K)_{K \subset G(\mathbb{A}^\infty)}$ has an action of $G(\mathbb{A}^\infty)$ by Hecke correspondences that induces an action of the Hecke algebra at level K on the cohomology of X_K for any reasonable cohomology theory.

We restrict our attention further to the case where the real Lie group $G(\mathbb{R})$ has a discrete series. This is the so-called "equal rank case" because it occurs if and only if the groups $G(\mathbb{R})$ and K_∞ have the same rank. Then the L^2-cohomology $H^*_{(2)}(X_K)$ is finite-dimensional, and Matsushima's formula, proved in this generality by Borel and Casselman [4], gives a description of this cohomology and of its Hecke algebra action in terms of discrete automorphic representations of G whose infinite component is a cohomological representation of $G(\mathbb{R})$, and in particular, either a discrete series or a special type of non-tempered representation.

Another cohomology of interest in this case is the intersection cohomology $IH^*(\overline{X}_K)$ of the minimal Satake compactification $\overline{X}_K$ of X_K. In order to study this cohomology, Goresky, Harder and MacPherson introduced in [12] a family of cohomology theories called "weighted cohomologies" and showed that the two middle weighted cohomologies agree with $IH^*(\overline{X}_K)$ if X_K has the structure of a complex algebraic variety. This result was later generalized by Saper in [30].

All the cohomology theories that we discussed have actions of the Hecke algebra, and the isomorphism of the previous paragraph is equivariant for this action. Zucker conjectured that there should be a Hecke-equivariant isomorphism between $H^*_{(2)}(X_K)$ and $IH(\overline{X}_K)$. This conjecture was proved by Looijenga [24], Looijenga–Rapoport [25] and Saper–Stern [31] if X_K has the structure of a complex algebraic variety and by Saper [30] in general. In particular, by comparing the formulas for the action of a Hecke operator on weighted cohomology (this was calculated by Goresky and MacPherson using topological methods in [14]) and on L^2-cohomology (this was calculated by Arthur using the Arthur–Selberg trace formula in [1]), one can obtain a formula for averaged discrete series characters of the group $G(\mathbb{R})$. One of the goals of the paper [13] of Goresky–Kottwitz–MacPherson was to prove this identity directly.

If moreover the space X_K is the set of complex points of a Shimura variety, then it descends to an algebraic variety over an explicit number field E known as the reflex field, as does the minimal Satake compactification, and so the intersection cohomology has a natural action of the absolute Galois group $\mathrm{Gal}(\overline{E}/E)$. We can further complicate the calculation by trying to calculate the trace on $IH^*(\overline{X}_K)$ of Hecke operators twisted by elements of the group $\mathrm{Gal}(\overline{E}/E)$, for example, powers of Frobenius maps. In the case where X_K is a Siegel modular variety, this was done by the second author in [28]. It requires a slightly different character identity for averaged discrete series characters of $G(\mathbb{R})$, also involving discrete series characters of the endoscopic groups of G, and whose relationship with the Goresky–Kottwitz–MacPherson identity was not clear.

For the specialists, we give a more detailed explanation of the relevance of our main results to cohomology calculations in Appendix C. Let us return here to a discussion of the current article.

In a previous article of the authors [7], we investigate the character identity of Morel [28]. In particular, we relate it to the geometry of the Coxeter complex of the symmetric group and give a simpler and more natural proof than the brute force calculation in the appendix of [28]. The goal of the present article is to generalize the approach of [7] and to prove a combinatorial identity (Theorem 4.2.2) that implies the character formulas of Goresky–Kottwitz–MacPherson [13] and of Morel [28]; see Sections 4.3 and 4.4. To obtain the character formula of [13] from our results, we need to use Herb's formula for averaged discrete series characters; see, for example, [17] and [19]. We also generalize, in Corollary 5.2.3 and Lemma 5.3.1, the geometric result of [7]; see Theorem 4.3 of that article. In fact, we prove an identity that holds not just for root systems that are generated by strongly orthogonal roots, but for all Coxeter systems with finite Coxeter group. The representation-theoretic interpretation of our identity in the general case is still unclear.

We now describe in more detail the different sections of the article.

In Section 2 we review some background material about real hyperplane arrangements and Coxeter arrangements.

In Section 3 we prove our first main theorem (Theorem 3.2.1) that concerns the calculation over the chambers T of a Coxeter arrangement $\mathscr{H}$ of the alternating sum of quantities $f(T)$, where f is a valuation defined on closed convex polyhedral cones. More precisely, Theorem 3.2.1 reduces this calculation to a similar calculation for simpler subarrangements of $\mathscr{H}$ and it is the main ingredient in the proof of our second main theorem (Theorem 4.2.2). The original proof of Theorem 4.2.2 used an induction argument similar to the ones used in the proofs of the character identities of [13, Theorem 3.1] and [19, Theorem 4.2], but we later realized that Theorem 4.2.2 was a particular case of the more general identity of Theorem 3.2.1.

In Section 4 we state and prove our second main theorem (Theorem 4.2.2). We first introduce in Section 4.1 our main geometric construction, which we call the *weighted complex*, that allows us to define the *weighted sum*; see Remark 4.1.6 for an explanation of these names. The weighted complex is the set of all the faces of a fixed hyperplane arrangement that are on the nonnegative side of an auxiliary hyperplane H_λ. It contains what is known as the *bounded complex* in the theory of affine oriented matroids, and coincides with it if H_λ is in general position. We state Theorem 4.2.2 in Section 4.2 and prove it in Section 4.5. The proof is straightforward: Using Corollary A.1.8, which generalizes [13, Proposition A.4], to reinterpret the weighted sum as an alternating sum on the chambers of the arrangement of the value of a particular valuation, we are able to show that Theorem 4.2.2 is a particular case of Theorem 3.2.1. In Sections 4.3 and 4.4 we explain how Theorem 4.2.2 implies the identities of [13, Theorem 3.1] and of [7, Theorem 6.4].

In Section 5 we study the geometric properties of the weighted complex. We prove in particular that, under a hypothesis about the dihedral angles between the hyperplanes of the arrangement (Condition (A) in Section 5.2, which always holds in the Coxeter case), the weighted complex is shellable; see Corollary 5.2.3, which generalizes Theorem 4.3 of [7]. We consider the case of Coxeter arrangements in Section 5.3. These geometric results were originally needed in the proof of Theorem 4.2.2, but the new proof via Theorem 3.2.1 allows us to circumvent them. We nevertheless decided to keep them in the article because we thought that they could be of independent interest.

In Section 6 we include concluding remarks.

We finish with three appendices. Each of the first two appendices can be read independently from the rest of the article (except that a proof in Appendix A uses Lemma 2.1.3). The goal of our Appendix A is to generalize [13, Proposition A.4], which is a key part in the proof of our main theorem. In Appendix A of their article [13], Goresky–Kottwitz–MacPherson show that a certain function, which they call $\psi_C(x, \lambda)$, is a *valuation* (see Definition A.1.3) on closed convex polyhedral cones, although they do not phrase it in these terms. We show that their function is a special case of a general construction that takes two valuations and produces a third one, and that this operation makes the set of valuations on closed convex polyhedral cones into a ring; see Theorem A.1.6 and its corollaries for the precise definition of this operation.

In Appendix B we review the theory of 2-structures, due to Herb; see for example Herb's review article [19]. We believe that this will be useful to the reader for a number of reasons. The proofs of the fundamental results of this theory are somewhat scattered in the literature and sometimes left as exercises. Furthermore, we needed to slightly adapt a number of results so that they continue to hold for Coxeter systems that do not necessarily arise from a (crystallographic) root system.

Finally, Appendix C is a continuation of the first part of the introduction, and is intended to give specialists more information about the way the weighted sum of Definition 4.1.5 and Theorem 4.2.2 appear in the calculation of the cohomology of locally symmetric varieties.

2. Hyperplane arrangements

2.1. Background material

We fix a finite-dimensional $\mathbb{R}$-vector space V with an inner product $(\cdot\,,\cdot)$. If $\alpha \in V$, we write

$$H_\alpha = \{x \in V : (\alpha, x) = 0\},$$
$$H_\alpha^+ = \{x \in V : (\alpha, x) > 0\}, \quad H_\alpha^- = \{x \in V : (\alpha, x) < 0\}.$$

We also denote by s_α the (orthogonal) reflection across the hyperplane H_α.

Let $(\alpha_e)_{e\in E}$ be a finite family of nonzero vectors in V. The corresponding *(central) hyperplane arrangement* is the family of hyperplanes $\mathscr{H} = (H_{\alpha_e})_{e\in E}$. Let V_0 be the intersection of all the hyperplanes, that is, $V_0 = \bigcap_{e\in E} H_{\alpha_e}$. We say that the arrangement $\mathscr{H}$ is *essential* if $V_0 = \{0\}$, which means that the family $(\alpha_e)_{e\in E}$ spans V.

Consider the map

$$s\colon V \to \{+,-,0\}^E$$

sending $x \in V$ to the family $(\mathrm{sign}((\alpha_e, x)))_{e\in E}$, where $\mathrm{sign}\colon \mathbb{R} \to \{+,-,0\}$ is the map sending positive numbers to $+$, negative numbers to $-$ and zero to 0.

Remark 2.1.1. The image of the map $s\colon V \to \{+,-,0\}^E$ is the set of covectors of an oriented matroid; see, for example, [3, Definition 4.1.1]. This is the oriented matroid corresponding to the hyperplane arrangement. In fact, some of our results extend to general oriented matroids. In this article we have chosen to concentrate on hyperplane arrangements to keep the exposition more concrete. In particular, we do not assume that the reader knows what an oriented matroid is.

We denote by $\mathcal{L}(\mathscr{H})$ or just $\mathcal{L}$ the set of nonempty subsets of V of the form $C = s^{-1}(X)$, for a sign vector $X \in \{+,-,0\}^E$. The elements of $\mathcal{L}$ are called *faces* of the arrangement. The set $\mathcal{L}$ has a natural partial order given by $C \leq D$ if and only if $C \subseteq \overline{D}$. The relation $C \leq D$ is equivalent to the fact that for every $e \in E$ we have $s(C)_e = 0$ or $s(C)_e = s(D)_e$. The set $\mathcal{L}$ with this partial order is called the *face poset* of the arrangement. Note that V_0 is the minimal element of $\mathcal{L}$. When we adjoin a maximal element $\widehat{1}$ to the poset $\mathcal{L}$, we obtain a lattice $\mathcal{L} \cup \{\widehat{1}\}$ known as the *face lattice*. Note that under our convention faces other than V_0 are not closed subsets of V: for every $C \in \mathcal{L}$, the closure $\overline{C}$ is a closed convex polyhedral cone in V, and it is an intersection of closed half-spaces $\overline{H_{\alpha_e}^{\pm}}$. The poset $\mathcal{L}$ is graded with the rank of a face $C \in \mathcal{L}$ given by

$$\rho(C) = \dim(C) - \dim(V_0),$$

where we write $\dim(C)$ for $\dim(\mathrm{Span}(C))$.

We denote by $\mathcal{T}(\mathcal{H})$ or just $\mathcal{T}$ the set of maximal faces of $\mathcal{L}$. These elements are often called *chambers*, *regions* or *topes*, and are the connected components of

$$V - \bigcup_{e \in E} H_{\alpha_e}.$$

If $T \in \mathcal{T}$ then T is an open subset of V, and its closure is a closed convex polyhedral cone of dimension $\dim(V)$.

If $X, Y \in \{+, -, 0\}^E$, their *composition* $X \circ Y$ is the sign vector defined by

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise.} \end{cases}$$

If $C, D \in \mathcal{L}$ then $s(C) \circ s(D)$ is also the image of a face of $\mathcal{L}$, and we denote this face by $C \circ D$. This is the unique face of $\mathcal{L}$ that contains all vectors of V of the form $x + \varepsilon y$, with $x \in C$, $y \in D$ and $\varepsilon > 0$ sufficiently small (relative to x and y). Define the *separation set* of C and D to be the set

$$S(C, D) = \{e \in E : s(C)_e = -s(D)_e \neq 0\}.$$

This is the set of $e \in E$ such that C and D are on different sides of the hyperplane H_{α_e}.

Fix a chamber $B \in \mathcal{T}$. We can then define a partial order $\preceq_B$ on $\mathcal{T}$ by declaring that $T \preceq_B T'$ if and only if $S(B, T) \subseteq S(B, T')$. The resulting poset is called the *chamber poset with base chamber* B. We will denote it by $\mathcal{T}_B$. It is a poset with minimal element B and maximal element $-B$. When all the hyperplanes are distinct, this poset is also graded with the rank function $\rho(T) = |S(B, T)|$; see [3, Proposition 4.2.10].

If the choice of the base chamber B is understood, we write, for every face C of the arrangement,

$$(-1)^C = (-1)^{|S(B, C \circ B)|}.$$

We also consider the graph with vertex set $\mathcal{T}$, where two chambers $T, T' \in \mathcal{T}$ are connected by an edge if and only if $\overline{T} \cap \overline{T}'$ spans a hyperplane (necessarily one of the hyperplanes H_{α_e}). In this situation, we say that this hyperplane is a *wall* of the chambers T and T'. This graph is called the *chamber graph*. In the case when all the hyperplanes of the arrangement $\mathcal{H}$ are distinct, the distance between two chambers T and T' in this graph is $|S(T, T')|$; see [3, Proposition 4.2.3].

Consider the sphere $\mathbb{S}$ of center 0 and radius 1 in V/V_0. The intersections $\overline{C} \cap \mathbb{S}$, for $C \in \mathcal{L}$, form a regular cell decomposition $\Sigma(\mathcal{L})$ of $\mathbb{S}$, and we will identify $\mathcal{L}$ with the face poset of this regular cell decomposition.

Finally, we recall the definition of the star of a face in $\mathcal{L}$.

Definition 2.1.2. Let $C \in \mathcal{L}$. The *star* of C in $\mathcal{L}$ is $\{D \in \mathcal{L} : C \leq D\}$. Geometrically, it is the set of faces of $\mathcal{L}$ whose closure contains C. We will denote it by $\mathcal{L}_{\geq C}$.

Lemma 2.1.3. *Let $C \in \mathcal{L}$ and let $E(C) = \{e \in E : C \subset H_{\alpha_e}\}$. Consider the hyperplane arrangement $\mathcal{H}(C) = (H_{\alpha_e})_{e \in E(C)}$ and let $\mathcal{L}_{\mathcal{H}(C)}$ be its face poset. Then the following four statements hold:*

(i) *Each face D of $\mathcal{L}$ is contained in a unique face D' of $\mathcal{L}_{\mathcal{H}(C)}$, and the map $D \mapsto D'$ induces an isomorphism of posets $\iota_C : \mathcal{L}_{\geq C} \xrightarrow{\sim} \mathcal{L}_{\mathcal{H}(C)}$. In particular, it sends the chambers of $\mathcal{T} \cap \mathcal{L}_{\geq C}$ to the chambers of $\mathcal{L}_{\mathcal{H}(C)}$.*

(ii) *The isomorphism ι_C of* (i) *sends a face $D \geq C$ of $\mathcal{H}$ to the relative interior of the closed convex polyhedral cone $\overline{D} + \mathrm{Span}(C)$. Let $\mathcal{C}_C = \bigcap_{e \in E - E(C)} H_e^{\epsilon_e}$, where $\epsilon_e = s(C)_e$. The inverse of the isomorphism ι_C sends a face D' of $\mathcal{H}(C)$ to the intersection $D' \cap \mathcal{C}_C$.*

(iii) *If $D_1, D_2 \in \mathcal{L}_{\geq C}$ then the inclusion $S(D_1, D_2) \subset E(C)$ holds. In particular, we have the equality $S(D_1, D_2) = S(\iota_C(D_1), \iota_C(D_2))$, where the isomorphism ι_C is as in* (i).

(iv) *The isomorphism $\iota_C : \mathcal{L}_{\geq C} \xrightarrow{\sim} \mathcal{L}_{\mathcal{H}(C)}$ preserves composition and dimension, that is, for all $D, D' \in \mathcal{L}_{\geq C}$, the following identities hold:*

$$\iota_C(D \circ D') = \iota_C(D) \circ \iota_C(D'), \quad \dim(\iota_C(D)) = \dim(D).$$

In particular, $\mathcal{L}_{\geq C}$ is also isomorphic to the face poset of a regular cell decomposition of the unit sphere in $V / \bigcap_{e \in E(C)} H_{\alpha_e}$ that we denote by $\Sigma(\mathcal{L}_{\geq C})$.

Proof of Lemma 2.1.3. Statement (i) is clear.

We prove statement (ii). Let $D \in \mathcal{L}_{\geq C}$ and let $D' = \iota_C(D)$. As $C \leq D$, we have $s(D)_e = s(C)_e$ for every $e \in E - E(C)$, and so $D \subset \mathcal{C}_C$. As $D \subset D'$, we deduce that $D' \cap \mathcal{C}_C \supset D$. As D' is a face of $\mathcal{H}(C)$, it is an intersection

$$D' = \bigcap_{e \in E(C)} H_{\alpha_e}^{s_e},$$

with $s_e \in \{0, +, -\}$, the intersection $D' \cap \mathcal{C}_C$ is either empty or a face of $\mathcal{H}$. We have just proved that this intersection contains D, so it is not empty, and hence is equal to the face D of $\mathcal{H}$. It remains to prove that D' is the relative interior of $\overline{D} + \mathrm{Span}(C)$. We write

$$D = \bigcap_{e \in E_0} H_{\alpha_0} \cap \bigcap_{e \in E_+} H_{\alpha_e}^+ \cap \bigcap_{e \in E_-} H_{\alpha_e}^-,$$

with $E = E_0 \sqcup E_+ \sqcup E_-$. We then have $E_0 \subset E(C)$, and

$$D' = \bigcap_{e \in E_0} H_{\alpha_0} \cap \bigcap_{e \in E_+ \cap E(C)} H_{\alpha_e}^+ \cap \bigcap_{e \in E_- \cap E(C)} H_{\alpha_e}^-.$$

So it suffices to show that $\bar{D} + \mathrm{Span}(C) = K$, where

$$K = \bigcap_{e \in E_0} H_{\alpha_0} \cap \bigcap_{e \in E_+ \cap E(C)} \bar{H}^+_{\alpha_e} \cap \bigcap_{e \in E_- \cap E(C)} \bar{C} H^-_{\alpha_e}.$$

We clearly have $\bar{D} \subset K$ and $\mathrm{Span}(C) \subset K$, so $\bar{D} + \mathrm{Span}(C) \subset K$. Conversely, let $x \in K$ and let $y \in C$. Then $(\alpha_e, y) \neq 0$ for every $e \in E - E(C)$, so there exists $\lambda > 0$ such that $(\alpha_e, \lambda y) + (\alpha_e, x)$ has the same sign as (α_e, y) for every $e \in E - E(C)$. We then have $\lambda x + y \in \bar{D}$, and so $x \in \bar{D} + \mathrm{Span}(C)$.

We prove (iii). Let $D_1, D_2 \in \mathcal{L}_{\geq C}$, and let $e \in S(D_1, D_2)$. Suppose for example that $s(D_1)_e = +$ and $s(D_2)_e = -$. (The other case is similar.) Then $\bar{D}_1 \subset \overline{H^+_{\alpha_e}}$ and $\bar{D}_2 \subset \overline{H^-_{\alpha_e}}$, so

$$C \subset \bar{D}_1 \cap \bar{D}_2 \subset \overline{H^+_{\alpha_e}} \cap \overline{H^-_{\alpha_e}} = H_{\alpha_e},$$

which implies that $e \in E(C)$.

The first statement of (iv) follows easily from the definitions: the composition $D \circ D'$ is defined on the sign vectors of D and D', and the isomorphism ι_C just forgets the coordinates outside of $E(C)$ in these sign vectors.

We prove the second statement of (iv). Let $D \in \mathcal{L}_{\geq C}$, and let D' be the unique face of $\mathcal{L}_{\mathcal{H}(C)}$ containing D. We clearly have $\dim(D) \leq \dim(D')$. If $\dim(D') > \dim(D)$ then there exists $e \in E$ such that $D \subset H_{\alpha_e}$ and $D' \not\subset H_{\alpha_e}$. But $C \subset \bar{D}$, so this implies that $e \in E(C)$. As D' is not included in H_{α_e}, it must be contained in one of the open half-spaces $H^{\pm}_{\alpha_e}$, contradicting the fact that D' contains D. ■

Remark 2.1.4. Let $C \in \mathcal{L}$ and let $F' = \{e \in E : C \not\subset H_{\alpha_e}\}$. Then the set $\mathcal{T} \cap \mathcal{L}_{\geq C}$ is equal to $\{T \in \mathcal{T} : \forall e \in F' \ s(T)_e = s(C)_e\}$, so it is a T-convex subset of $\mathcal{T}$ in the sense of [3, Definition 4.2.5]; see [3, Proposition 4.2.6]. In other words, it contains every shortest path in the chamber graph between any two of its elements, so it is a lower order ideal in $\mathcal{T}_B$ for every choice of base chamber $B \in \mathcal{T} \cap \mathcal{L}_{\geq C}$.

2.2. Coxeter arrangements

Let (W, S) be a Coxeter system, that is, W is the group generated by the set S and the relations between the generators are of the form $(st)^{m_{s,t}} = 1$, where $m_{s,s} = 1$ and $m_{s,t} \geq 2$ for $s \neq t$; see [2, Section 1.1]. The corresponding Coxeter graph has vertex set S, and two generators s and t are connected with an edge if $m_{s,t} \geq 3$. If $m_{s,t} \geq 4$ it is customary to label the edge by the integer $m_{s,t}$.

There are three natural partial orders on the elements of the Coxeter group W. First the *strong Bruhat order* is defined by the following cover relation: $z \prec w$ if there exists $s \in S$ and $u \in W$ such that $(usu^{-1})z = w$ and $\ell(z) + 1 = \ell(w)$ where ℓ is the length function on W; see, for example, [2, Definition 2.1.1]. Next, we have the *right* (respectively, *left*) *weak Bruhat order*, where the cover relation is $z \prec w$ if there exists $s \in S$ such that $z \cdot s = w$ (respectively, $s \cdot z = w$) and $\ell(z) + 1 = \ell(w)$. The strong Bruhat order refines both the left and right weak Bruhat orders.

Let $V = \bigoplus_{s \in S} \mathbb{R} e_s$, with the symmetric bilinear form $(\cdot\,,\cdot)$ defined by

$$(e_s, e_t) = -\cos(\pi/m_{s,t}).$$

In particular, $(e_s, e_s) = 1$. The *canonical representation* of (W, S) is the representation of W on V given by

$$s(v) = v - 2 \cdot (e_s, v) \cdot e_s, \tag{2.1}$$

for every $s \in S$ and every $v \in V$. Note that this formula defines an orthogonal isomorphism of V for the symmetric bilinear form $(\cdot\,,\cdot)$. We refer the reader to [5, Chapitre V, § 4, № 8, Théorème 2, p. 98] for the next result.

Theorem 2.2.1. *Equation* (2.1) *defines a faithful representation of W on V, and the form $(\cdot\,,\cdot)$ is positive definite if and only if W is finite.*

From now on, we assume that W is finite, and we write

$$\Phi = \{w(e_s) : w \in W,\ s \in S\} \quad \text{and} \quad \Phi^+ = \Phi \cap \sum_{s \in S} \mathbb{R}_{\geq 0} e_s.$$

The set Φ is a *pseudo-root system*, its subset Φ^+ is a set of *positive pseudo-roots*, and the set $\Phi^- = -\Phi^+ = \Phi - \Phi^+$ is the corresponding set of *negative pseudo-roots*; see Definitions B.1.1 and B.1.4. Then $\mathscr{H} = (H_\alpha)_{\alpha \in \Phi^+}$ is an essential hyperplane arrangement on V. The set of chambers $\mathcal{T}$ of this arrangement is in canonical bijection with W: the unit element $1 \in W$ corresponds to the chamber

$$B = \bigcap_{\alpha \in \Phi^+} H_\alpha^+ = \bigcap_{s \in S} H_{e_s}^+,$$

and an arbitrary element w of W corresponds to the chamber $w(B)$.

More generally, a *parabolic subgroup* of W is a subgroup W_I generated by a subset I of S, and the left cosets of parabolic subgroups of W are called *standard cosets*. The *Coxeter complex* $\Sigma(W)$ of W is the set of standard cosets of W ordered by reverse inclusion. It is a simplicial complex, and we have an isomorphism of posets from $\Sigma(W)$ to the face poset $\mathcal{L}$ of $\mathscr{H}$ sending a standard coset wW_I to the cone

$$\{x \in V : \forall s \in I\ (x, w(e_s)) = 0 \text{ and } \forall s \in S - I\ (x, w(e_s)) > 0\}.$$

The fact that this is an isomorphism is proved in [5, Chapitre V, § 4, № 6, pp. 96–97], since the representation of W on $V^\vee$ is isomorphic to its canonical representation on V by Theorem 2.2.1. The fact that $\Sigma(W)$ is a simplicial complex then follows from [5, Chapitre V, § 3, № 3, Proposition 7, p. 85].

The definitions of B and of the isomorphism $\mathcal{T} \simeq W$ imply that, if $w, w' \in W$ and $T_w, T_{w'} \in \mathcal{T}$ are the corresponding chambers, then

$$\begin{aligned} S(T_w, T_{w'}) = \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^+ \text{ and } w'^{-1}(\alpha) \in \Phi^-\} \\ \cup \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^- \text{ and } w'^{-1}(\alpha) \in \Phi^+\}, \end{aligned}$$

and in particular

$$S(B, T_w) = \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^-\},$$

hence, by [2, Proposition 4.4.4],

$$(-1)^{T_w} = (-1)^{|S(B,T_w)|} = \det(w).$$

By [2, Propositions 3.1.3 and 4.4.6] this also implies that the isomorphism $\mathcal{T} \simeq W$ sends the partial order $\preceq_B$ to the right weak Bruhat order on W.

Definition 2.2.2. Let $\mathcal{H} = (H_{\alpha_e})_{e \in E}$ be a finite hyperplane arrangement on a finite-dimensional real inner product space V, with inner product denoted by $(\cdot\,,\cdot)$. We say that $\mathcal{H}$ is a *Coxeter arrangement* if $\alpha_e \notin \mathbb{R}\alpha_f$ for distinct $e, f \in E$ and if for every $e \in E$ the family of hyperplanes $\mathcal{H}$ is stable by the (orthogonal) reflection s_{α_e} across H_{α_e}.

Theorem 2.2.3. *The hyperplane arrangement associated to a Coxeter system with finite Coxeter group is a Coxeter arrangement. Conversely, suppose that $\mathcal{H}$ is a Coxeter arrangement on an inner product space V, and that there exists a chamber B of $\mathcal{H}$ that is on the positive side of each hyperplane in $\mathcal{H}$. Let W be the subgroup of* $\mathbf{GL}(V)$ *generated by the set $\{s_{\alpha_e} : e \in E\}$, let F be the set of $e \in E$ such that $\overline{B} \cap H_{\alpha_e}$ is a facet of $\overline{B}$ and let $S = \{s_{\alpha_f} : f \in F\}$. Then (W, S) is a Coxeter system, the group W is finite, and the hyperplane arrangement induced by $\mathcal{H}$ on $V/\bigcap_{e\in E} H_{\alpha_e}$ is isomorphic to the arrangement associated to the Coxeter system* (W, S).

Proof. The first statement is an immediate consequence of the definition of the arrangement associated to a Coxeter system. The second and fourth statements follow from [5, Chapitre V, § 3, № 2, Théorème 1, p. 74]. The statement that W is finite follows from [5, Chapitre V, § 3, № 7, Proposition 4, p. 80] and from the fact that the arrangement $\mathcal{H}$ is central. ∎

3. The abstract pizza quantity

3.1. 2-structures and signs

Let $\Phi \subset V$ be a pseudo-root system (see Definition B.1.1) with Coxeter group W (see Proposition B.1.6) and $\Phi^+ \subset \Phi$ be a system of positive pseudo-roots (see Definition B.1.4). Recall the definition of 2-structures from Section B.2: A 2-*structure* for Φ is a subset $\varphi \subseteq \Phi$ such that:

(a) φ is a pseudo-root system whose irreducible components are all of type A_1, B_2 or $I_2(2^k)$ with $k \geq 3$;

(b) for every $w \in W$ such that $w(\varphi \cap \Phi^+) = \varphi \cap \Phi^+$, we have $\det(w) = 1$.

Recall that $\mathcal{T}(\Phi)$ is the set of 2-structures for Φ. By Proposition B.2.4, the group W acts transitively on $\mathcal{T}(\Phi)$. In Definition B.2.8 we define the sign $\epsilon(\varphi) = \epsilon(\varphi, \Phi^+)$ of any 2-structure $\varphi \in \mathcal{T}(\Phi)$. If $\varphi \in \mathcal{T}(\Phi)$, we write $\varphi^+ = \varphi \cap \Phi^+$.

We have the following proposition that extends [20, Theorem 5.3] to the case of Coxeter systems. Note that our proof is a simple adaptation of Herb's proof.

Proposition 3.1.1. *The sum of the signs of all* 2*-structures of a pseudo-root system is equal to* 1*, that is,*

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) = 1.$$

Proof. We prove the result by induction on $|\Phi|$. It is clear if $\Phi = \varnothing$, because then $\mathcal{T}(\Phi) = \{\varnothing\}$ and the sign of $\varnothing$ is 1. Suppose that $|\Phi| \geq 1$ and that we know the result for all pseudo-root systems of smaller cardinality. Let $\alpha \in \Phi$, and set $\Phi_\alpha = \alpha^\perp \cap \Phi$; this is a pseudo-root system with positive system $\alpha^\perp \cap \Phi^+$.

Let $\mathcal{T}'' = \{\varphi \in \mathcal{T}(\Phi) : s_\alpha(\varphi) = \varphi\}$. By statement (0) of Lemma B.2.11, we have

$$\mathcal{T}'' = \{\varphi \in \mathcal{T}(\Phi) : \alpha \in \varphi\}.$$

If $\varphi \notin \mathcal{T}''$, then $\varphi^+ \subset \Phi^+ - \{\alpha\}$, so $s_\alpha(\varphi^+) \subset \Phi^+$ by Lemma 4.4.3 of [2], hence $\epsilon(s_\alpha(\varphi)) = -\epsilon(\varphi)$ by Lemma B.2.10. This implies that

$$\sum_{\varphi \in \mathcal{T}(\Phi) - \mathcal{T}''} \epsilon(\varphi) = 0.$$

We define subsets $\mathcal{T}_1''$ and $\mathcal{T}_2''$ of $\mathcal{T}''$ by

$$\begin{aligned} \mathcal{T}_1'' &= \{\varphi \in \mathcal{T}(\Phi) : \varphi \cap \Phi_\alpha \in \mathcal{T}(\Phi_\alpha)\}, \\ \mathcal{T}_2'' &= \mathcal{T}'' - \mathcal{T}_1''. \end{aligned}$$

By (3) of Lemma B.2.11, there exists an involution ι of $\mathcal{T}_2''$ such that, for every $\varphi \in \mathcal{T}_2''$, we have that $\iota(\varphi) \cap \Phi_\alpha = \varphi \cap \Phi_\alpha$ and $\epsilon(\iota(\varphi)) = -\epsilon(\varphi)$. This implies that

$$\sum_{\varphi \in \mathcal{T}_2''} \epsilon(\varphi) = 0,$$

and so

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) = \sum_{\varphi \in \mathcal{T}_1''} \epsilon(\varphi).$$

Finally, by (1) and (2) of Lemma B.2.11, the map $\varphi \mapsto \varphi \cap \Phi_\alpha$ induces a bijection from $\mathcal{T}_1''$ to $\mathcal{T}(\Phi_\alpha)$, and we have $\epsilon(\varphi) = \epsilon(\varphi \cap \Phi_\alpha)$ for every $\varphi \in \mathcal{T}_1''$. Hence, we obtain

$$\sum_{\varphi \in \mathcal{T}_1''} \epsilon(\varphi) = \sum_{\varphi_0 \in \mathcal{T}(\Phi_\alpha)} \epsilon(\varphi_0),$$

and this last sum is equal to 1 by the induction hypothesis. ■

Remark 3.1.2. As we are using the definition of the sign of a 2-structure from [18], our formula looks a bit different from the one of [20, Theorem 5.3]. This is explained in [20, Remark 5.1], and we generalize the comparison between the two definitions of the sign in Corollary 3.1.3 below.

Corollary 3.1.3. *Let $\varphi \in \mathcal{T}(\Phi)$, and*

$$W(\varphi, \Phi^+) = \{w \in W : w(\varphi^+) \subset \Phi^+\}, \quad W_1(\varphi, \Phi^+) = \{w \in W : w(\varphi^+) \subset \varphi^+\}.$$

Then the sign $\epsilon(\varphi, \Phi^+)$ is given by

$$\epsilon(\varphi, \Phi^+) = \frac{1}{|W_1(\varphi, \Phi^+)|} \sum_{w \in W(\varphi, \Phi^+)} \det(w).$$

Proof. By Corollary B.2.5, we have a bijection

$$W(\varphi, \Phi^+)/W_1(\varphi, \Phi^+) \to \mathcal{T}(\Phi), \quad w \mapsto w(\varphi).$$

By Proposition 3.1.1 and Lemma B.2.10, we obtain

$$\begin{aligned} 1 &= \frac{1}{|W_1(\varphi, \Phi^+)|} \sum_{w \in W(\varphi, \Phi^+)} \epsilon(w(\varphi), \Phi^+) \\ &= \epsilon(\varphi, \Phi^+) \frac{1}{|W_1(\varphi, \Phi^+)|} \sum_{w \in W(\varphi, \Phi^+)} \det(w). \end{aligned}$$ ■

We consider the hyperplane arrangement $\mathcal{H} = (H_\alpha)_{\alpha \in \Phi^+}$ corresponding to Φ, with base chamber $B = \bigcap_{\alpha \in \Phi^+} H_\alpha^+$. For every 2-structure $\varphi \in \mathcal{T}(\Phi)$, we denote by $\mathcal{H}_\varphi$ the hyperplane arrangement $(H_\alpha)_{\alpha \in \varphi^+}$, with base chamber $B_\varphi = \bigcap_{\alpha \in \varphi^+} H_\alpha^+$. If T is a chamber of $\mathcal{H}$, we denote by $Z_\varphi(T)$ the unique chamber of $\mathcal{H}_\varphi$ containing T; as $\varphi^+ \subset \Phi^+$, we have $Z_\varphi(B) = B_\varphi$.

Corollary 3.1.4. *For every chamber T of $\mathcal{H}$, we have*

$$(-1)^T = \sum_{\varphi \in \mathcal{T}(\Phi)} (-1)^{Z_\varphi(T)} \epsilon(\varphi).$$

Recall that $(-1)^T = (-1)^{|S(B,T)|}$ for every $T \in \mathfrak{T}(\mathcal{H})$, and similarly for $T \in \mathfrak{T}(\mathcal{H}_\varphi)$.

Proof of Corollary 3.1.4. For every $\varphi \in \mathcal{T}(\Phi)$, we denote the Coxeter group of φ by $W(\varphi)$. We also use the notation of Lemma B.2.10. Let w be the unique element of W such that $T = w^{-1}(B)$. Let $\varphi \in \mathcal{T}(\Phi)$. Then $\varphi \cap w(\Phi^+)$ is a system of positive pseudo-roots in φ, so there exists a unique $v \in W(\varphi)$ such that $v(\varphi^+) = \varphi \cap w(\Phi^+)$; we write $v = v_\varphi(w)$. As $T = \{x \in V : \forall\alpha \in w(\Phi^+)\ (x,\alpha) > 0\}$, we have

$$\begin{aligned} Z_\varphi(T) &= \{x \in V : \forall\alpha \in w(\Phi^+) \cap \varphi\ \ (x,\alpha) > 0\} \\ &= \{x \in V : \forall\alpha \in v_\varphi(w)(\varphi)\ \ (x,\alpha) > 0\}, \end{aligned}$$

and so $v_\varphi(w)$ is the element of $W(\varphi)$ corresponding to $Z_\varphi(T)$ by the bijection from $W(\varphi)$ to the set of chambers of $\mathcal{H}_\varphi$ sending v to $v^{-1}(Z_\varphi(B))$.

For a 2-structure $\varphi \in \mathcal{T}(\Phi)$ we have that

$$w^{-1}v_\varphi(w)(\varphi^+) = w^{-1}(\varphi \cap w(\Phi^+)) \subset \Phi^+,$$

so by Lemma B.2.10 (and the fact that $v_\varphi(w)(\varphi) = \varphi$), we obtain that

$$\epsilon(w^{-1}(\varphi)) = \det(w^{-1}v_\varphi(w))\epsilon(\varphi).$$

Hence, we have

$$\begin{aligned} \sum_{\varphi\in\mathcal{T}(\Phi)} (-1)^{Z_\varphi(T)}\epsilon(\varphi) &= \sum_{\varphi\in\mathcal{T}(\Phi)} \det(v_\varphi(w))\epsilon(\varphi) \\ &= \det(w)\cdot \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(w^{-1}(\varphi)) \\ &= \det(w)\cdot \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi), \end{aligned}$$

where in the last step we used that the map $\varphi \mapsto w^{-1}(\varphi)$ on the set $\mathcal{T}(\Phi)$ is bijective. The result now follows by Proposition 3.1.1. ∎

3.2. Calculating the abstract pizza quantity with 2-structures

We use the notation of Appendix A. In particular, if K is a closed convex polyhedral cone in V, we denote the set of its closed faces by $\mathcal{F}(K)$ (we include K itself in the set of its faces). The *dimension* $\dim K$ of K is by definition the dimension of its span $\mathrm{Span}(K)$, and the *relative interior* $\mathring{K}$ of K is the interior of K in $\mathrm{Span}(K)$. We say that K is *degenerate* if $\mathrm{Span}(K)$ is strictly included in V, equivalently, if K has empty interior.

Let $\mathcal{H}$ be a central hyperplane arrangement on V with fixed base chamber B. Let $\mathcal{C}_{\mathcal{H}}(V)$ be the set of closed convex polyhedral cones in V that are intersections of closed half-spaces bounded by hyperplanes H where $H \in \mathcal{H}$. Denote the free abelian group on $\mathcal{C}_{\mathcal{H}}(V)$ by $\bigoplus_{K \in \mathcal{C}_{\mathcal{H}}(V)} \mathbb{Z}[K]$ and let $K_{\mathcal{H}}(V)$ be its quotient by the relations

$$[K] + [K'] = [K \cup K'] + [K \cap K']$$

for all $K, K' \in \mathcal{C}_{\mathcal{H}}(V)$ such that $K \cup K' \in \mathcal{C}_{\mathcal{H}}(V)$. For $K \in \mathcal{C}_{\mathcal{H}}(V)$, we still denote the image of K in $K_{\mathcal{H}}(V)$ by $[K]$. For the relative interior $\mathring{K}$ we also define a class $[\mathring{K}] \in K_{\mathcal{H}}(V)$ by

$$[\mathring{K}] = (-1)^{\dim K} \sum_{F \in \mathcal{F}(K)} (-1)^{\dim F} [F].$$

We then have

$$[K] = \sum_{F \in \mathcal{F}(K)} [\mathring{F}]$$

by [13, formula (A.4), p. 543].

Recall that $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ are the set of faces and chambers of the arrangement $\mathcal{H}$ as in Section 2.1. Each $C \in \mathcal{L}(\mathcal{H})$ is the relative interior of its closure and, if $T \in \mathcal{T}(\mathcal{H})$, then $\mathcal{F}(\overline{T}) = \{\overline{C} \ : \ C \in \mathcal{L}(\mathcal{H}), \ C \leq T\}$. We have $V = \coprod_{C \in \mathcal{L}(\mathcal{H})} C$, and the family $([C])_{C \in \mathcal{L}(\mathcal{H})}$ is a $\mathbb{Z}$-basis of $K_{\mathcal{H}}(V)$. As in Section 2.1, the *sign* of a face $C \in \mathcal{F}(\mathcal{H})$ is defined by $(-1)^C = (-1)^{|S(B, C \circ B)|}$.

We consider the following quantity:

$$\Pi(\mathcal{H}) = \sum_{C \in \mathcal{L}(\mathcal{H})} (-1)^C [C] \in K_{\mathcal{H}}(V).$$

Let A be an abelian group. We say that a function $\nu \colon \mathcal{C}_{\mathcal{H}}(V) \to A$ is a *valuation* on $\mathcal{C}_{\mathcal{H}}(V)$ if, for all $K, K' \in \mathcal{C}_{\mathcal{H}}(V)$ such that $K \cup K' \in \mathcal{C}_{\mathcal{H}}(V)$, we have

$$\nu(K \cup K') + \nu(K \cap K') = \nu(K) + \nu(K').$$

Such a valuation ν defines a morphism of abelian groups $K_{\mathcal{H}}(V) \to A$ sending $[K]$ to $\nu(K)$ for every $K \in \mathcal{C}_{\mathcal{H}}(V)$, and we still denote this morphism by $\nu \colon K_{\mathcal{H}}(V) \to A$. We set

$$\Pi(\mathcal{H}, \nu) = \nu(\Pi(\mathcal{H})) \in A.$$

If ν vanishes on degenerate cones, then we have

$$\Pi(\mathcal{H}, \nu) = \sum_{T \in \mathcal{T}(\mathcal{H})} (-1)^T \nu(T) = \sum_{T \in \mathcal{T}(\mathcal{H})} (-1)^T \nu(\overline{T}) \in A. \tag{3.1}$$

The first main theorem of this article is the following. For Coxeter arrangements we can express the quantity $\Pi(\mathcal{H})$ in terms of the quantities $\Pi(\mathcal{H}_\varphi)$ for the arrangements $\mathcal{H}_\varphi$ associated to the 2-structures of the arrangement.

Theorem 3.2.1. *Let $\Phi \subset V$ be a pseudo-root system. Choose a system of positive pseudo-roots $\Phi^+ \subset \Phi$ and let $\mathcal{H}$ be the hyperplane arrangement $(H_\alpha)_{\alpha\in\Phi^+}$ on V.*

(i) *We have the identity*

$$\Pi(\mathcal{H}) = \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi)\Pi(\mathcal{H}_\varphi)$$

in the quotient $K_{\mathcal{H}}(V)$, where $\mathcal{H}_\varphi$ is as before the arrangement $(H_\alpha)_{\alpha\in\varphi\cap\Phi^+}$ for every $\varphi \in \mathcal{T}(\Phi)$.

(ii) *If $\nu\colon \mathcal{C}_{\mathcal{H}}(V) \to A$ is a valuation, we have*

$$\Pi(\mathcal{H},\nu) = \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi)\Pi(\mathcal{H}_\varphi,\nu).$$

If $\varphi \in \mathcal{T}(\Phi)$ then the faces of $\mathcal{H}_\varphi$ are relative interiors of elements of $\mathcal{C}_{\mathcal{H}}(V)$, so $\Pi(\mathcal{H}_\varphi)$ makes sense as an element of $K_{\mathcal{H}}(V)$.

Remark 3.2.2. This theorem is useful in the following situation. Suppose that we have a function f on closed convex polyhedral cones and that we wish to calculate the alternating sum over the chambers T of a hyperplane arrangement $\mathcal{H}$ of the values $f(\overline{T})$. If $\mathcal{H}$ is a Coxeter arrangement and the function f is a valuation that vanishes on cones contained in hyperplanes of $\mathcal{H}$, then the theorem says that we can reduce the problem to a similar calculation for very simple subarrangements of $\mathcal{H}$ that are products of rank 1 and rank 2 Coxeter arrangements.

Here are two situations when we wish to calculate alternating sums of $f(\overline{T})$ for such a valuation f:

(a) The weighed sums of Section 4. These sums appear in the calculation of weighted cohomology of locally symmetric spaces and Shimura varieties; see Appendix C for additional details and references. We want to relate them to stable discrete series constants to get a spectral description of that cohomology.

(b) The pizza problem; see, for example, the paper [9]. In this setting, we fix a measurable subset K of V with finite volume, and the function f sends a cone C to the volume of $C \cap K$. We are interested in "the pizza quantity", that is, the alternating sum of the volumes $f(K \cap \overline{T})$. In particular, we would like to know when this alternating sum vanishes, which is to say that the "pizza" K has been evenly divided among the two participants, $+$ and $-$. This problem is approached by analytic methods in [9]. Theorems 1.1 and 1.2 in [9] give general sufficient conditions to guarantee that the pizza quantity vanishes. Using Theorem 3.2.1 we can give a dissection proof; see [8].

When f is a valuation that does not vanish on cones contained in hyperplanes of $\mathcal{H}$, we have to decide how to count the contributions of lower-dimensional faces of $\mathcal{H}$. One possibility is given in Theorem 3.2.1, and another in Corollary 3.2.4. In both cases, if $\mathcal{H}$ is a Coxeter arrangement, then we can again reduce the calculation to the case of simpler subarrangements of $\mathcal{H}$. This is not needed in situation (a), but in situation (b) it allows us to obtain versions of the pizza theorem that hold for all the intrinsic volumes; see [8] for this.

We will provide a proof of Theorem 3.2.1 in Section 3.3. First we state and prove a corollary. For $\mathcal{H}$ a central hyperplane arrangement on V with a fixed base chamber, we define

$$P(\mathcal{H}) = \sum_{T \in \mathfrak{T}(\mathcal{H})} (-1)^T [\overline{T}] \in K_{\mathcal{H}}(V),$$
$$P_0(\mathcal{H}) = \sum_{T \in \mathfrak{T}(\mathcal{H})} (-1)^T [T] \in K_{\mathcal{H}}(V).$$

Analogous to the pizza quantity defined in Section 2 of [9], we call $P(\mathcal{H})$ the *abstract pizza quantity* of the arrangement $\mathcal{H}$.

Lemma 3.2.3. *For $\mathcal{H}$ a central hyperplane arrangement on V, we have*

$$P_0(\mathcal{H}) = P(\mathcal{H}).$$

Proof. If $T \in \mathfrak{T}(\mathcal{H})$ then we have

$$[\overline{T}] = [T] + \sum_{\substack{F \in \mathfrak{L}(\mathcal{H}) \\ F < T}} [F].$$

Summing over all chambers T of $\mathcal{H}$ yields

$$\begin{aligned} \sum_{T \in \mathfrak{T}(\mathcal{H})} (-1)^T [\overline{T}] &= \sum_{T \in \mathfrak{T}(\mathcal{H})} (-1)^T \Bigg([T] + \sum_{\substack{F \in \mathfrak{L}(\mathcal{H}) \\ F < T}} [F] \Bigg) \\ &= \sum_{T \in \mathfrak{T}(\mathcal{H})} (-1)^T [T] + \sum_{F \in \mathfrak{L}(\mathcal{H}) - \mathfrak{T}(\mathcal{H})} [F] \sum_{\substack{T \in \mathfrak{T}(\mathcal{H}) \\ T > F}} (-1)^T. \end{aligned}$$

The last inner sum is equal to zero, yielding the result. ∎

Corollary 3.2.4. *If Φ and $\mathcal{H}$ are as in Theorem 3.2.1, we have*

$$P(\mathcal{H}) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) P(\mathcal{H}_\varphi).$$

Proof. By Lemma 3.2.3, it suffices to prove that

$$P_0(\mathcal{H}) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) P_0(\mathcal{H}_\varphi),$$

where $P_0(\mathcal{H})=\sum_{T\in\mathfrak{T}(\mathcal{H})}(-1)^T[T]$ and $P_0(\mathcal{H}_\varphi)$ is defined similarly. For every $x\in V$ and every $\varphi\in\mathcal{T}(\Phi)$, let $a_\varphi(x)$ equal $(-1)^Z$ if there exists a chamber Z of φ such that $x\in Z$, and 0 otherwise. We need to show that $\sum_{\varphi\in\mathcal{T}(\Phi)}\epsilon(\varphi)a_\varphi(x)$ is equal to $(-1)^T$ if there exists a chamber T of $\mathcal{H}$ such that $x\in T$, and otherwise is zero.

Consider the valuation $\nu\colon\mathcal{C}_{\mathcal{H}}(V)\to K_{\mathcal{H}}(V)$ sending a cone $C\in\mathcal{C}_{\mathcal{H}}(V)$ to

$$\sum_{T\in\mathfrak{T}(\mathcal{H}),\,T\subset C}[T]\in K_{\mathcal{H}}(V).$$

This valuation corresponds to the endomorphism of $K_{\mathcal{H}}(V)$ sending the class of T to itself if $T\in\mathfrak{T}(\mathcal{H})$, and the class of F to 0 if $F\in\mathcal{F}(\mathcal{H})-\mathfrak{T}(\mathcal{H})$. The valuation ν vanishes on degenerate cones, so by statement (ii) of Theorem 3.2.1 and equation (3.1), we have that if T is a chamber of $\mathcal{H}$ and $x\in T$, then

$$\sum_{\varphi\in\mathcal{T}(\Phi)}\epsilon(\varphi)a_\varphi(x)=(-1)^T.$$

Now let $x\in V-\bigcup_{T\in\mathfrak{T}(\mathcal{H})}T$, and let F be the unique face of $\mathcal{H}$ such that $x\in F$. For each $\varphi\in\mathcal{T}(\Phi)$, there is at most one chamber of φ that contains x. We denote by X the set of pairs (φ,Z), where $\varphi\in\mathcal{T}(\Phi)$ and Z is a chamber of φ such that $x\in Z$. As F is not a chamber, there exists $e\in E$ such that $F\subset H_e$. We denote by s the orthogonal reflection in the hyperplane H_e. As $s(x)=x$, we can make s act on X by sending (φ,Z) to $(s(\varphi),s(Z))$. This is a fixed-point free involution. Indeed, if φ is a 2-structure such that $s(\varphi)=\varphi$, then $e\in\varphi$ by statement (0) of Lemma B.2.11, so H_e is a hyperplane of $\mathcal{H}_\varphi$, which is impossible because x is both in H_e and in a chamber of φ. To prove that

$$\sum_{\varphi\in\mathcal{T}(\Phi)}\epsilon(\varphi)a_\varphi(x)=\sum_{(\varphi,Z)\in X}\epsilon(\varphi)(-1)^Z$$

is equal to 0, it suffices to show that for every $(\varphi,Z)\in X$, we have

$$\epsilon(\varphi)(-1)^Z=-\epsilon(s(\varphi))(-1)^{s(Z)}.$$

After applying an element of W to the entire setup, we may assume without loss of generality that x is in the base chamber of $\mathcal{H}$. Then Z, respectively, $s(Z)$, is the base chamber of φ, respectively, $s(\varphi)$, so $(-1)^Z=(-1)^{s(Z)}=1$. Also, as the reflection s sends the base chamber of φ to that of $s(\varphi)$, we have $s(\varphi^+)\subset\Phi^+$, and so $\epsilon(s(\varphi))=-\epsilon(\varphi)$ by Lemma B.2.10. ∎

3.3. Proof of Theorem 3.2.1

In this subsection we prove Theorem 3.2.1. We begin by stating and proving a lemma.

Lemma 3.3.1. *Let $\mathcal{H}$ be a central hyperplane arrangement on V, let C be a face of $\mathcal{H}$ and let $x \in V$. We denote by C_0 the unique face of $\mathcal{H}$ containing x. Then for $D \leq C$ a face of $\mathcal{H}$ the following three conditions are equivalent:*

(a) $x \in \overline{C} + \mathrm{Span}(D) = \overline{C} + \mathrm{Span}(\overline{D})$;

(b) $\psi_x(\overline{D}^{\perp,\overline{C}}) = 1$, *where* $\overline{D}^{\perp,\overline{C}} = \overline{D}^{\perp} \cap \overline{C}^*$ *as in Section* A.1;

(c) $D \circ C_0 \leq C$.

Moreover, if C_0 is a chamber then these conditions can only hold if C is also a chamber, and they are equivalent to the following condition:

(d) $D \circ C_0 = C$.

Proof. We first note that $\mathrm{Span}(\overline{D}) = \mathrm{Span}(D)$ because $\mathrm{Span}(D)$ is a finite-dimensional subspace of V, hence it is closed and so contains $\overline{D}$. This explains the equality in condition (a).

To prove that conditions (a) and (b) are equivalent, we note that by the definition of the valuation ψ_x in Lemma A.1.10 we have $\psi_x(\overline{D}^{\perp,\overline{C}}) = 1$ if and only if $x \in (\overline{D}^{\perp,\overline{C}})^*$. As $\overline{D}^{\perp,\overline{C}} = \overline{D}^{\perp} \cap \overline{C}^*$ by definition, we have $(\overline{D}^{\perp,\overline{C}})^* = \mathrm{Span}(\overline{D}) + \overline{C}$, so condition (b) is equivalent to the fact that $x \in \mathrm{Span}(\overline{D}) + \overline{C}$, which is condition (a).

We prove that (c) implies (a). Let $y \in D$. If $\varepsilon > 0$ then $y + \varepsilon x \in D \circ C_0$, so $y + \varepsilon x \in \overline{C}$ by (c). Thus,

$$x = \frac{1}{\varepsilon}(y + \varepsilon x - y) \in \overline{C} + \mathrm{Span}(D),$$

which is condition (a).

We prove that (a) implies (c). By condition (a) we can write $x = x_1 + x_2$, with $x_1 \in \overline{C}$ and $x_2 \in \mathrm{Span}(D)$. Let $y \in D$. If $\varepsilon > 0$ is small enough, then $y + \varepsilon x_2 \in D$, so

$$y + \varepsilon x = (y + \varepsilon x_2) + \varepsilon x_1 \in \overline{C}.$$

As $y + \varepsilon x \in D \circ C_0$ for $\varepsilon > 0$ small enough, this shows that $D \circ C_0 \subset \overline{C}$, that is, $D \circ C_0 \leq C$, which is condition (a).

Finally, suppose that C_0 is a chamber. Then $D \circ C_0$ is a chamber, so condition (c) can only hold if C is also a chamber, and it is equivalent to condition (d) because chambers are maximal faces. ∎

Proof of Theorem 3.2.1. We first prove statement (ii) of Theorem 3.2.1 for a valuation $\nu: \mathcal{C}_{\mathcal{H}}(V) \to A$ that vanishes on degenerate cones. For every $\varphi \in \mathcal{T}(\Phi)$, we have by equation (3.1):

$$\Pi(\mathcal{H}_\varphi, \nu) = \sum_{Z \in \mathcal{T}(\mathcal{H}_\varphi)} (-1)^Z \nu(Z) = \sum_{Z \in \mathcal{T}(\mathcal{H}_\varphi)} (-1)^Z \sum_{T \in \mathcal{T}(\mathcal{H}),\, T \subset Z} \nu(T).$$

Hence,

$$\sum_{\varphi\in\mathcal{T}(\Phi)}\epsilon(\varphi)\Pi(\mathcal{H}_\varphi,\nu)=\sum_{\varphi\in\mathcal{T}(\Phi)}\epsilon(\varphi)\sum_{Z\in\mathfrak{T}(\mathcal{H}_\varphi)}(-1)^Z\sum_{T\in\mathfrak{T}(\mathcal{H}),\,T\subset Z}\nu(T)$$
$$=\sum_{T\in\mathfrak{T}(\mathcal{H})}\nu(T)\sum_{\varphi\in\mathcal{T}(\Phi)}\epsilon(\varphi)(-1)^{Z_\varphi(T)}.$$

As $\sum_{\varphi\in\mathcal{T}(\Phi)}\epsilon(\varphi)(-1)^{Z_\varphi(T)}=(-1)^T$ for every $T\in\mathfrak{T}(\mathcal{H})$ by Corollary 3.1.4, the statement follows.

We now prove statement (i) of Theorem 3.2.1. Fix a point x in the base chamber B of $\mathcal{H}$. For every closed convex polyhedral cone $K\subset V$, let $\mathcal{F}_x(K)$ be the set of closed faces F of K such that $x\in K+\operatorname{Span}(F)$. Consider the function $\psi\colon\mathcal{C}_\mathcal{H}(V)\to K_\mathcal{H}(V)$ defined by

$$\psi(K)=\sum_{F\in\mathcal{F}_x(K)}(-1)^{\dim F}[F].$$

This is the $\star$-product in the sense of Corollary A.1.8 (see also Remark A.1.9) of the valuations $\mathcal{C}_\mathcal{H}(V)\to K_\mathcal{H}(V)$, $K\mapsto[K]$ and $\psi_x\colon\mathcal{C}(V^\vee)\to\mathbb{Z}$, where $V^\vee$ is the dual of V and ψ_x is the valuation of Lemma A.1.10. More explicitly, for $K\subset V^\vee$ a nonempty closed convex polyhedral cone, we have $\psi_x(K)=1$ if and only if $x\in K^*$. Indeed, with the notation of that definition, we have $(F^{\perp,K})^*=K+\operatorname{Span}(F)$ for every $K\in\mathcal{C}_\mathcal{H}(V)$ and every closed face F of K. By Corollary A.1.8, the function ψ is a valuation, so it induces a morphism $\psi\colon K_\mathcal{H}(V)\to K_\mathcal{H}(V)$. Moreover, the valuation ψ vanishes on degenerate cones in $\mathcal{C}_\mathcal{H}(V)$. Indeed, if $K\in\mathcal{C}_\mathcal{H}(V)$ is contained in a hyperplane H of $\mathcal{H}$, then $K+\operatorname{Span}(F)\subset H$ for every $F\in\mathcal{F}(K)$, so $\mathcal{F}_x(K)=\varnothing$ because x is not on any hyperplane of $\mathcal{H}$. Hence, we can apply statement (i) that we just proved to ψ.

Let $\mathcal{H}'$ be a subarrangement of $\mathcal{H}$, and let B' be the unique chamber of $\mathcal{H}'$ containing B. If $T\in\mathfrak{T}(\mathcal{H}')$, we write

$$\mathcal{F}_0(T)=\{C\in\mathcal{L}(\mathcal{H}'):C\le T\text{ and }C\circ B'=T\}.$$

As $x\in B\subset B'$, we have $\mathcal{F}_x(\overline{T})=\{\overline{C}:C\in\mathcal{F}_0(T)\}$ by Lemma 3.3.1. We deduce that

$$\Pi(\mathcal{H}',\psi)=\sum_{T\in\mathfrak{T}(\mathcal{H}')}(-1)^T\psi(\overline{T})$$
$$=\sum_{T\in\mathfrak{T}(\mathcal{H}')}(-1)^T\sum_{F\in\mathcal{F}_x(T)}(-1)^{\dim F}[F]$$
$$=\sum_{T\in\mathfrak{T}(\mathcal{H}')}(-1)^T\sum_{C\in\mathcal{L}(\mathcal{H}'),\,C\circ B'=T}(-1)^{\dim C}[\overline{C}]$$
$$=\sum_{C\in\mathcal{L}(\mathcal{H}')}(-1)^C(-1)^{\dim C}[\overline{C}].$$

Using statement (i) for the valuation ψ, we obtain that

$$\sum_{C \in \mathcal{L}(\mathcal{H})} (-1)^C (-1)^{\dim C} [\overline{C}] = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \sum_{C \in \mathcal{L}(\mathcal{H}_\varphi)} (-1)^C (-1)^{\dim C} [\overline{C}]. \quad (3.2)$$

By the top of page 544 of [13], there exists an endomorphism of $K_{\mathcal{H}}(V)$ sending $[K]$ to $(-1)^{\dim K}[\mathring{K}]$, for every $K \in \mathcal{C}_{\mathcal{H}}(V)$. Applying this endomorphism to the identity (3.2) yields statement (i).

Finally, the general case of statement (ii) immediately follows from applying the morphism $\nu \colon K_{\mathcal{H}}(V) \to A$ to both sides of the identity of statement (i). ∎

4. The weighted sum

4.1. The weighted complex and the weighted sum

We return to the situation of Section 2.1. In particular, we fix a finite-dimensional real inner product space V and a central hyperplane arrangement $\mathcal{H} = (H_{\alpha_e})_{e \in E}$ on V, and we denote by $\mathcal{L}$ and $\mathcal{T}$ the sets of faces and chambers of $\mathcal{H}$.

Definition 4.1.1. Let $\lambda \in V$. We consider the following subset of the face poset $\mathcal{L}$:

$$\mathcal{L}_\lambda = \{C \in \mathcal{L} : C \subseteq \overline{H_\lambda^+}\}.$$

In other words, the set $\mathcal{L}_\lambda$ is the collection of faces on the nonnegative side of the hyperplane H_λ. More generally, if C_0 is a fixed face of $\mathcal{L}$, we also consider the intersection

$$\mathcal{L}_{\lambda, \geq C_0} = \mathcal{L}_\lambda \cap \mathcal{L}_{\geq C_0}.$$

Remark 4.1.2 (see [3, Section 4.5] for definitions.). If $\lambda \neq 0$ then the hyperplane arrangement $\{H_\lambda\} \cup \{H_{\alpha_e} : e \in E\}$ defines an affine oriented matroid with distinguished hyperplane H_λ. If H_λ is in general position relative to the H_{α_e}, that is, if λ is not in the span of any family $(\alpha_e)_{e \in F}$ for $|F| \leq \dim(V) - 1$, then $\mathcal{L}_\lambda$ coincides with the bounded complex of this affine oriented matroid. In general, $\mathcal{L}_\lambda$ is larger.

The basic properties of the subsets $\mathcal{L}_\lambda$ and $\mathcal{L}_{\lambda, \geq C_0}$ are given in the following proposition.

Proposition 4.1.3. *The following two statements hold:*

(i) *For a fixed face C_0 of $\mathcal{L}$ the set $\mathcal{L}_{\lambda, \geq C_0}$ is a lower order ideal in $\mathcal{L}_{\geq C_0}$.*

(ii) *Let $C \in \mathcal{L}_\lambda$. Then there exists $T \in \mathcal{T} \cap \mathcal{L}_\lambda$ such that $C \leq T$.*

Proof. It suffices to prove (i) when C_0 is the minimal face of $\mathcal{L}$. Let $C, D \in \mathcal{L}$ such that $C \leq D$ and $D \in \mathcal{L}_\lambda$. The hypothesis implies that $C \subset \overline{D}$ and $D \subset \overline{H_\lambda^+}$. As $\overline{H_\lambda^+}$ is closed, this immediately gives $C \subset \overline{H_\lambda^+}$, hence $C \in \mathcal{L}_\lambda$.

To show (ii) let $D_1, D_2, \dots, D_p$ be the chambers of $\mathcal{T}$ that are larger than C with respect to the partial order $\leq$. If one of them is contained in $\overline{H_\lambda^+}$, then we are done. Otherwise, for every $1 \leq i \leq p$, we can find a point $x_i \in D_i$ such that $(\lambda, x_i) < 0$. Let $\mathcal{H}'$ be the subarrangement of $\mathcal{H}$ where we remove all the hyperplanes of $\mathcal{H}$ that contain the cone C. In the arrangement $\mathcal{H}'$ all the points x_i are contained in the same chamber C'. In particular, the convex hull P of the points $x_1, x_2, \dots, x_p$ is contained in C'. The convex hull P intersects the linear span of the cone C in a point x. Since all the points x_i are in the open half-space H_λ^-, so is the point x, that is, $(\lambda, x) < 0$. By inserting the hyperplanes of $\mathcal{H}$ that contain the cone C back in the arrangement $\mathcal{H}'$, we subdivide the region C' into regions $C_1, C_2, \dots, C_p$. But the point x belongs to the closure of each region C_i, thus the point x belongs to the cone C. This is a contradiction since C is contained in the half-space $\overline{H_\lambda^+}$, so $(\lambda, x) \geq 0$. ■

Definition 4.1.4. The subcomplex of the cell decomposition $\Sigma(\mathcal{L})$, respectively, $\Sigma(\mathcal{L}_{\geq C_0})$, whose face poset is the lower order ideal $\mathcal{L}_\lambda$, respectively, $\mathcal{L}_{\lambda, \geq C_0}$, is called the *weighted complex* and denoted by $\Sigma(\mathcal{L}_\lambda)$, respectively, $\Sigma(\mathcal{L}_{\lambda, \geq C_0})$. By Proposition 4.1.3 it is pure of the same dimension as $\Sigma(\mathcal{L}_\lambda)$, respectively, $\Sigma(\mathcal{L}_{\lambda, \geq C_0})$.

We are interested in the following quantity.

Definition 4.1.5. Let λ be a vector in V and B a chamber of $\mathcal{H}$, that is, $B \in \mathcal{T}$. The *weighted sum* is defined to be

$$\psi_{\mathcal{H}}(B, \lambda) = \sum_{D \in \mathcal{L}_\lambda} (-1)^{\dim(D)} \cdot (-1)^{|S(B, D \circ B)|}. \tag{4.1}$$

More generally, if C is a face of the arrangement $\mathcal{H}$, that is, $C \in \mathcal{L}$, and if B is a chamber whose closure contains the face C, that is, $B \in \mathcal{T} \cap \mathcal{L}_{\geq C}$, we define the weighted sum to be

$$\psi_{\mathcal{H}/C}(B, \lambda) = \sum_{D \in \mathcal{L}_{\lambda, \geq C}} (-1)^{\dim(D)} \cdot (-1)^{|S(B, D \circ B)|}. \tag{4.2}$$

Remark 4.1.6. This definition seems very arbitrary and mysterious. We try to give some context for it in Appendix C. Without getting into too much detail here, the weighted sum (in the situation of Example 4.2.1) plays a role in the calculation of the trace of Hecke operators on the weighted cohomology of locally symmetric varieties that is very similar to the role played by stable discrete constants (see, for example, [13, pp. 493, 498–500]) in the calculation of the trace of Hecke operators on L^2 cohomology of these varieties. It is because of this that we chose the names “weighted complex” and “weighted sum”. In fact, in that situation we only really need the “absolute” version where C is the minimal face of $\mathcal{H}$. The “relative” version, where C is not minimal anymore, appears when the locally symmetric variety is a Shimura variety defined over a number field E and we are considering the trace of a Hecke operator multiplied by an element of the absolute Galois group of E.

Remark 4.1.7. In our previous paper (see [7, equation (6.1)]), we used the notation $S(\lambda)$ to denote what turns out to be a particular case of the sum in equation (4.2) in the type B Coxeter case; see equation (4.3) in Section 4.4 for the precise relation between the two. In this paper, we decided to follow the notation of [13] in order to avoid overuse of the letter S.

We state the following lemma. It reduces the calculation of $\psi_{\mathcal{H}/C}(B,\lambda)$ to the case of an essential arrangement.

Lemma 4.1.8. *Let V_0 be the intersection of all the hyperplanes of $\mathcal{H}$, that is, $V_0 = \bigcap_{e\in E} H_{\alpha_e}$. Let π denote the projection $V \to V/V_0$. Let $\mathcal{H}/V_0$ be the hyperplane arrangement $(H_{\alpha_e}/V_0)_{e\in E}$ on V/V_0. Note that π induces an isomorphism between the face poset of $\mathcal{H}$ and $\mathcal{H}/V_0$. Let $C \in \mathcal{L}$, let B be a chamber of $\mathcal{H}$ such that $B \in \mathcal{L}_{\geq C}$ and let $\lambda \in V$. Then the following identity holds:*

$$\psi_{\mathcal{H}/C}(B,\lambda) = \begin{cases} (-1)^{\dim(V_0)} \cdot \psi_{(\mathcal{H}/V_0)/\pi(C)}(\pi(B),\pi(\lambda)) & \text{if } \lambda \in V_0^{\perp}, \\ 0 & \text{if } \lambda \notin V_0^{\perp}. \end{cases}$$

Proof. Note that $\lambda \in V_0^{\perp}$ if and only if $V_0 \subset H_\lambda$. If $\lambda \notin V_0^{\perp}$, then the linear functional $(\lambda,\cdot)$ takes both positive and negative values on V_0. As $V_0 \subset \overline{D}$ for every $D \in \mathcal{L}$, this linear functional also takes both positive and negative values on D, so $D \notin \mathcal{L}_\lambda$. This shows that $\mathcal{L}_\lambda = \varnothing$ if $\lambda \notin V_0^{\perp}$ and gives the second case. Now suppose that $\lambda \in V_0^{\perp}$. Then $V_0 \subset H_\lambda$, and it is easy to see that $D \in \mathcal{L}_\lambda$, respectively, $D \in \mathcal{L}_{\geq C}$, if and only if $\pi(D) \subset \overline{H^{+}_{\pi(\lambda)}}$, respectively, $\pi(D) \geq \pi(C)$, and that

$$\dim(\pi(D)) = \dim(D) - \dim(V_0).$$

This yields the first case. ■

Suppose that $V = V_1 \times \cdots \times V_r$ with the V_i mutually orthogonal subspaces of V and that $\mathcal{H}$ also decomposes as a product $\mathcal{H}_1 \times \cdots \times \mathcal{H}_r$. By this, we mean that there is a decomposition $E = E_1 \sqcup \cdots \sqcup E_r$ such that, for $1 \leq i \leq r$ and every $e \in E_i$, we have $\alpha_e \in V_i$. The arrangement $\mathcal{H}_i = (V_i \cap H_{\alpha_e})_{e\in E_i}$ is a hyperplane arrangement on the subspace V_i, and each hyperplane of $\mathcal{H}$ is of the form $H \times \prod_{j\neq i} V_j$, where $1 \leq i \leq r$ and H is one of the hyperplanes of $\mathcal{H}_i$.

Let $\mathcal{L}_i$ be the face poset of $\mathcal{H}_i$ for $1 \leq i \leq r$. Then the faces of $\mathcal{L}$ are exactly the products $C_1 \times \cdots \times C_r$ where $C_i \in \mathcal{L}_i$, and the order on $\mathcal{L}$ is the product order. In particular, C is a chamber in $\mathcal{L}$ if and only if all the C_i are chambers in $\mathcal{L}_i$.

Lemma 4.1.9. *Assume that the arrangement $\mathcal{H}$ factors as $\mathcal{H}_1 \times \cdots \times \mathcal{H}_r$ as described in the two previous paragraphs. Let $C = C_1 \times \cdots \times C_r$ be a face in $\mathcal{L}$, and let $B = B_1 \times \cdots \times B_r$ be a chamber in $\mathcal{L}_{\geq C}$. Finally, let $\lambda \in V$. Then*

$$\psi_{\mathcal{H}/C}(B, \lambda) = \prod_{i=1}^{r} \psi_{\mathcal{H}_i/C_i}(B_i, \lambda_i),$$

where, for $1 \leq i \leq r$, λ_i is the orthogonal projection of λ on V_i.

Proof. The expression for $\psi_{\mathcal{H}/C}(B, \lambda)$ follows from the fact that $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_r$ as posets once we prove the following statement: Let $D = D_1 \times \cdots \times D_r \in \mathcal{L}$, with $D_i \in \mathcal{L}_i$. Then $D \in \mathcal{L}_\lambda$ if and only if $D_i \in \mathcal{L}_{i,\lambda_i}$ for $1 \leq i \leq r$.

We prove this last fact. Note that $\lambda = (\lambda_1, \ldots, \lambda_r)$ in $V_1 \times \cdots \times V_r = V$ because the V_i are pairwise orthogonal. If $D_i \in \mathcal{L}_{i,\lambda_i}$ for every $1 \leq i \leq r$ then for every $x = (x_1, \ldots, x_r) \in V$, we have

$$(\lambda, x) = \sum_{i=1}^{r} (\lambda_i, x_i) \geq 0.$$

Conversely, suppose that $D \in \mathcal{L}_\lambda$. Let $x_j \in D_j$ for $1 \leq j \leq r$. As all the D_j are cones, for every $\varepsilon > 0$ the element εx_j is in D_j. Fix $1 \leq i \leq r$ and consider the element $x_\varepsilon = (\varepsilon x_1, \ldots, \varepsilon x_{i-1}, x_i, \varepsilon x_{i+1}, \ldots, \varepsilon x_r)$. Then x_ε is in D. Thus, we have the inequality

$$0 \leq (\lambda, x_\varepsilon) = (\lambda_i, x_i) + \varepsilon \sum_{j \neq i} (\lambda_j, x_j).$$

Letting ε tend to 0, we obtain $(\lambda_i, x_i) \geq 0$, and hence $D_i \in \mathcal{L}_{i,\lambda_i}$. ∎

4.2. Calculating the weighted sum for some arrangements with many symmetries

We continue to use the notation of Sections 2.1 and 2.2. Suppose that $E = E^{(1)} \sqcup E^{(2)}$, with $\mathcal{H}^{(1)} = (H_{\alpha_e})_{e \in E^{(1)}}$ a Coxeter arrangement whose Coxeter group W stabilizes $\mathcal{H}^{(2)} = (H_{\alpha_e})_{e \in E^{(2)}}$, and that

$$C = \Bigg(\bigcap_{e \in E^{(1)}} H_{\alpha_e}\Bigg) \cap \Bigg(\bigcap_{e \in E^{(2)}} H_{\alpha_e}^{+}\Bigg),$$

so that $E(C) = \{e \in E : C \subset H_{\alpha_e}\} = E^{(1)}$. To simplify some of the notation, without loss of generality we may assume that the vectors α_e, where $e \in E$, are all unit vectors. We assume that there exists a chamber B of $\mathcal{H}$ that is on the positive side of every hyperplane of $\mathcal{H}$ (not just $\mathcal{H}^{(1)}$); in particular, we have $C \leq B$. This also defines a chamber of the arrangement $\mathcal{H}^{(1)}$, and we denote by (W, S) the associated Coxeter system, as in Theorem 2.2.3.

The set $\Phi = \{\pm\alpha_e : e \in E^{(1)}\}$ is a normalized pseudo-root system (see Definition B.1.1), the subset $\Phi^+ = \{\alpha_e : e \in E^{(1)}\}$ is a system of positive pseudo-roots in Φ (see Definition B.1.4), and (W, S) is the corresponding Coxeter system; see Proposition B.1.6.

Our main example of such arrangements is the following.

Example 4.2.1. Let (W, S) be a Coxeter system, let V be the canonical representation of W, and let $\mathcal{H} = (H_\alpha)_{\alpha\in\Phi^+}$ be the associated hyperplane arrangement on V as in Section 2.2. Let I be a subset of S, set

$$\Phi^{(1)} = \Phi^+ \cap \left(\sum_{\alpha\in I} \mathbb{R}\alpha\right) \quad \text{and} \quad \Phi^{(2)} = \Phi^+ - \Phi^{(1)}.$$

Then $\mathcal{H}^{(1)} = (H_\alpha)_{\alpha\in\Phi^{(1)}}$ is a Coxeter arrangement with associated Coxeter system (W_I, I), where W_I is the subgroup of W generated by I, and W_I preserves the arrangement $\mathcal{H}^{(2)} = (H_\alpha)_{\alpha\in\Phi^{(2)}}$. If

$$C = \left(\bigcap_{\alpha\in\Phi^{(1)}} H_\alpha\right) \cap \left(\bigcap_{\alpha\in\Phi^{(2)}} H_\alpha^+\right)$$

as before, then the chamber B corresponding to $1 \in W$ is in $\mathcal{L}_{\geq C}$.

We use again the notion of 2-structures for Φ; see Section 3.1. If $\varphi \in \mathcal{T}(\Phi)$ we write $\varphi^+ = \varphi \cap \Phi^+$, and we denote by $\mathcal{H}_\varphi$ the hyperplane arrangement $(H_\alpha)_{\alpha\in\varphi^+ \sqcup E^{(2)}}$ and by B_φ, respectively, C_φ, the unique chamber of $\mathcal{H}_\varphi$ containing B, respectively C. By the choice of B, the chamber B_φ is also the unique chamber on the positive side of every hyperplane in $\mathcal{H}_\varphi$.

Theorem 4.2.2. *Let $\mathcal{H} = \mathcal{H}^{(1)} \sqcup \mathcal{H}^{(2)}$ be an arrangement in V with base chamber B. Assume that the subarrangement $\mathcal{H}^{(1)}$ is a Coxeter arrangement with pseudo root system Φ and its Coxeter group stabilizes $\mathcal{H}$. Let C be the intersection of the base chamber of $\mathcal{H}^{(2)}$ and the hyperplanes in $\mathcal{H}^{(1)}$. Then for every $\lambda \in V$ we have*

$$\psi_{\mathcal{H}/C}(B, \lambda) = \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \lambda).$$

This is the second main theorem of this article. It will be proved in Section 4.5; see Definition 4.1.5 for the description of the terms and Remark 4.1.6 for an explanation of their significance. Theorem 4.2.2 states that the weighted sum for a Coxeter arrangement can be expressed as an alternating sum of weighted sums for much simpler subarrangements (the 2-structures) that are direct products of rank one and rank two Coxeter arrangements. As weighted sums for these subarrangements can be calculated directly (see Corollary 4.3.1 and Proposition 4.3.2 for the case where C is the minimal face), this gives a way to calculate the weighted sum for the original Coxeter arrangement, and thus, as explained in Appendix C, to relate weighted cohomology of locally symmetric varieties to the spectral side of the Arthur–Selberg trace formula.

We first give some applications of Theorem 4.2.2.

4.3. First application: the case of Coxeter arrangements

We specialize Theorem 4.2.2 to the case where $\mathscr{H} = \mathscr{H}^{(1)}$ is a Coxeter arrangement. In particular, C is the minimal face of $\mathcal{L}$, so $\psi_{\mathscr{H}/C}(B,\lambda) = \psi_{\mathscr{H}}(B,\lambda)$ for every $\lambda \in V$.

Let $\varphi \in \mathcal{T}(\Phi)$, and let $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$ be the decomposition of φ into irreducible pseudo-root systems. Let $V_{i,\varphi} = \mathrm{Span}(\varphi_i)$ for $1 \le i \le r$. Then

$$V = V_{0,\varphi} \times V_{1,\varphi} \times \cdots \times V_{r,\varphi},$$

where $V_{0,\varphi} = \varphi^{\perp}$. The dimension of $V_{0,\varphi}$ is equal to $\dim(V) - \mathrm{rank}(\varphi)$, so it is independent of φ by Proposition B.2.4. Let $\mathscr{H}_{i,\varphi}$ be the hyperplane arrangement given by $\varphi_i \cap \Phi^+$ on $V_{i,\varphi}$ where $1 \le i \le r$. For a fixed index i let $B_{i,\varphi}$ be the chamber of the arrangement $\mathscr{H}_{i,\varphi}$ that is on the positive side of every hyperplane, and let $\lambda_{i,\varphi}$ be the orthogonal projection of λ on $V_{i,\varphi}$.

Combining Theorem 4.2.2 with Lemmas 4.1.8 and 4.1.9, we obtain the following corollary.

Corollary 4.3.1. *For every $\lambda \in V$, we have*

$$\psi_{\mathscr{H}}(B,\lambda) = (-1)^{\dim(V)-R} \cdot \sum_{\substack{\varphi\in\mathcal{T}(\Phi)\\ \lambda\in\mathrm{Span}(\varphi)}} \epsilon(\varphi) \cdot \prod_{i=1}^{r} \psi_{\mathscr{H}_{i,\varphi}}(B_{i,\varphi},\lambda_{i,\varphi}),$$

where R is the rank of any $\varphi \in \mathcal{T}(\Phi)$.

To finish the calculation of $\psi_{\mathscr{H}}(B,\lambda)$ in this case, we use the following proposition, whose proof is a straightforward calculation.

Proposition 4.3.2. *In types A_1, $B_2 = I_2(4)$ and $I_2(2^k)$ for $k \ge 3$, the function ψ is given by the following expressions:*

(1) *Type A_1: Suppose that $V = \mathbb{R}e_1$ and that $\Phi^+ = \{e_1\}$. Then ψ is given by*

$$\psi_{\mathscr{H}}(B,ce_1) = \begin{cases} 0 & \text{if } c > 0,\\ 1 & \text{if } c = 0,\\ 2 & \text{if } c < 0.\end{cases}$$

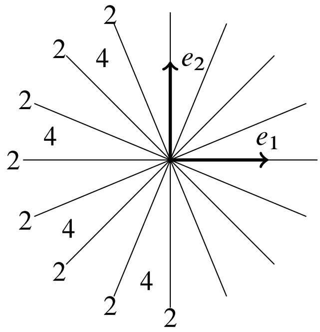

Figure 1. The function $\psi_{\mathcal{H}}(B, \lambda)$ in the dihedral pseudo-root system $I_2(8)$. The origin is assigned the value 1 and the unmarked faces are assigned the value 0.

(2) *Type $I_2(2^k)$, where $k \geq 2$: Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with the usual inner product. For every $v \in V - \{0\}$, let $\theta(v) \in [0, 2\pi)$ be the angle from e_1 to v. Suppose that Φ is the set of unit vectors that have an angle of $r\pi/2^k$ with e_1, where $r \in \mathbb{Z}$, and that B is the set of nonzero vectors $v \in V$ such that*

$$0 < \theta(v) < \pi/2^k.$$

Then ψ is given by

$$\psi_{\mathcal{H}}(B, \lambda) = \begin{cases} 1 & \text{if } \lambda = 0, \\ 2 & \text{if } \lambda \neq 0 \text{ and } \theta(\lambda) = r\pi/2^k \text{ with } 2^{k-1}+1 \leq r \leq 3 \cdot 2^{k-1}, \\ 4 & \text{if } \lambda \neq 0 \text{ and } r\pi/2^k < \theta(\lambda) < (r+1)\pi/2^k \text{ with } r \text{ odd} \\ & \text{and } 2^{k-1} + 1 \leq r \leq 3 \cdot 2^{k-1} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.3.3. If (W, S) arises from a root system Φ and -1 is an element of W (or, equivalently, the root system is generated by strongly orthogonal roots), then Goresky–Kottwitz–MacPherson [13, Theorem 3.1] and Herb [19, Theorem 4.2] give two different expressions for the coefficients appearing in the formula for the averaged discrete series characters of a real reductive group with root system Φ. Corollary 4.3.1 asserts the equality of these two formulas. In general, although there no longer exist discrete series in this setting, the formulas of Goresky–Kottwitz–MacPherson and Herb still make sense, and Corollary 4.3.1 says that they are still equal. Also, Corollary 4.3.1 implies that $\psi_{\mathcal{H}}(B, \lambda) = 0$ if λ is not in the span of any 2-structure for Φ, so it implies [13, Theorem 5.3]. It is not clear whether this is an easier proof than the one given in [13].

4.4. Second application: the type A identity involving ordered set partitions

We now show how to deduce [7, Theorem 6.4][1] from Theorem 4.2.2. We take $V = \mathbb{R}^n$ with the usual inner product, and we denote by $(e_1, \dots, e_n)$ the standard basis of V. We consider the hyperplane arrangement $\mathcal{H}$ of type B_n on V, that is, $\mathcal{H} = (H_\alpha)_{\alpha\in\Phi_B^+}$, where

$$\Phi_B^+ = \{e_i \pm e_j : 1 \le i < j \le n\} \cup \{e_1, \dots, e_n\}.$$

We write $\Phi_B^+ = \Phi^{(1)} \sqcup \Phi^{(2)}$, where $\Phi^{(1)} = \{e_i - e_j : 1 \le i < j \le n\}$, and we denote by $\mathcal{H} = \mathcal{H}^{(1)} \sqcup \mathcal{H}^{(2)}$ the corresponding decomposition of $\mathcal{H}$. The arrangement $\mathcal{H}^{(1)}$ is a Coxeter arrangement of type A_{n-1}, and we denote by $\Phi = \Phi^{(1)} \cup (-\Phi^{(1)})$ the associated root system. Let C be the intersection $C = (\bigcap_{\alpha\in\Phi^{(1)}} H_\alpha) \cap (\bigcap_{\alpha\in\Phi^{(2)}} H_\alpha^+)$. Then C is the open ray $\mathbb{R}_{>0} \cdot (e_1 + e_2 + \dots + e_n)$.

Recall that $\mathcal{L}$ is the face poset of $\mathcal{H}$. We will now give a description of $\mathcal{L}$ in terms of signed ordered partitions; see also [10, Section 5] for this description. A *signed block* is a nonempty subset $\widetilde{B}$ of $\{\pm 1, \dots, \pm n\}$ such that, for every $i \in \{1, \dots, n\}$, at most one of $\pm i$ is in $\widetilde{B}$. We then denote by B the subset of $\{1, \dots, n\}$ defined by $B = \{|i| : i \in \widetilde{B}\}$. A *signed ordered partition* of a subset I of $\{1, \dots, n\}$ is a list $(\widetilde{B}_1, \dots, \widetilde{B}_r)$ of signed blocks such that $(B_1, \dots, B_r)$ is an ordered partition of I.

We consider the poset $\Pi_n^{\mathrm{ord},B}$ whose elements are pairs $\pi = (\widetilde{\pi}, Z)$, where $Z \subseteq \{1, \dots, n\}$ and $\widetilde{\pi}$ is a signed ordered partition of $\{1, \dots, n\} - Z$, and the cover relation is given by the following two rules:

$$((\widetilde{B}_1, \dots, \widetilde{B}_r), Z) \prec ((\widetilde{B}_1, \dots, \widetilde{B}_{r-1}), B_r \cup Z),$$
$$((\widetilde{B}_1, \dots, \widetilde{B}_r), Z) \prec ((\widetilde{B}_1, \dots, \widetilde{B}_{i-1}, \widetilde{B}_i \cup \widetilde{B}_{i+1}, \widetilde{B}_{i+2}, \dots, \widetilde{B}_r), Z).$$

The set Z is usually called the *zero block* of π.

Let $\pi = (\widetilde{\pi}, Z)$ be an element of $\Pi_n^{\mathrm{ord},B}$, with $\widetilde{\pi} = (\widetilde{B}_1, \dots, \widetilde{B}_r)$. We define the cone C_π to be the set of $(x_1, \dots, x_n) \in V$ such that (with the convention that $x_{-i} = -x_i$ for $1 \le i \le n$):

(i) if $Z = \{i_1, \dots, i_m\}$ then the equalities $x_{i_1} = \dots = x_{i_m} = 0$ hold;

(ii) for every block $\widetilde{B} = \{i_1, \dots, i_m\}$ in $\widetilde{\pi}$, the equalities and inequality $x_{i_1} = \dots = x_{i_m} > 0$ hold;

(iii) for every two consecutive blocks $\widetilde{B}_s$ and $\widetilde{B}_{s+1}$ in $\widetilde{\pi}$ with $i \in \widetilde{B}_s$ and $j \in \widetilde{B}_{s+1}$, the inequality $|x_i| > |x_j|$ holds.

It is easy to see that the map $\varphi\colon \Pi_n^{\mathrm{ord},B} \to \mathcal{L}$ sending π to C_π is a bijection, and that it induces an order-reversing isomorphism between the poset $\Pi_n^{\mathrm{ord},B}$ and the

[1] This is a reformulation of [28, Proposition A.4].

face poset $\mathcal{L}$. The inverse image of the ray $C = \mathbb{R}_{>0} \cdot (e_1 + e_2 + \cdots + e_n)$ by this bijection is the element $\pi_0 = ((\{1, \ldots, n\}), \varnothing)$ of $\Pi_n^{\mathrm{ord},B}$, so the elements of $\mathcal{L}_{\geq C}$ correspond exactly to the (unsigned) ordered partitions of $\{1, \ldots, n\}$. In other words, the bijection φ induces an order-reversing isomorphism between the poset Π_n^{ord} of ordered partitions of $\{1, \ldots, n\}$ defined in [7, Section 2] and the poset $\mathcal{L}_{\geq C}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. For a signed block $\widetilde{B}$, we set

$$\lambda_{\widetilde{B}} = \sum_{i \in B} \lambda_i,$$

with the convention that $\lambda_{-i} = -\lambda_i$ for $1 \leq i \leq n$. Define the subset $\Pi_n^{\mathrm{ord},B}(\lambda)$ of $\Pi_n^{\mathrm{ord},B}$ by

$$\Pi_n^{\mathrm{ord},B}(\lambda) = \left\{((\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_r), Z) \in \Pi_n^{\mathrm{ord},B} \ : \ \sum_{i=1}^{s} \lambda_{\widetilde{B}_i} \geq 0 \text{ for } 1 \leq s \leq r\right\}.$$

Then an element π of $\Pi_n^{\mathrm{ord},B}$ is in $\Pi_n^{\mathrm{ord},B}(\lambda)$ if and only if C_π is in $\mathcal{L}_\lambda$. Moreover, the subset $\mathcal{L}_{\lambda,\geq C}$ corresponds to the set $\Pi_n^{\mathrm{ord}}(\lambda)$ of ordered partitions $(B_1, \ldots, B_r)$ of $\{1, \ldots, n\}$ such that, for every $1 \leq s \leq r$, we have $\sum_{i=1}^{s} \lambda_{B_i} \geq 0$. This is almost the set $\mathcal{P}(\lambda)$ of [7, Section 3]; the only difference is that the inequalities defining $\mathcal{P}(\lambda)$ are strict. We can give the following identity relating these two sets: For every $\varepsilon \in \mathbb{R}$, let $\lambda_\varepsilon = (\lambda_1 - \varepsilon, \ldots, \lambda_n - \varepsilon)$. Then if $\varepsilon > 0$ is sufficiently small, we have $\Pi_n^{\mathrm{ord}}(\lambda_\varepsilon) = \mathcal{P}(\lambda)$.

Let B be the unique chamber of $\mathcal{L}$ that is on the positive side of every hyperplane, that is, $B = \{x_1 > x_2 > \cdots > x_n > 0\}$. As we already observed, $\mathcal{L}_{\geq C}$ is isomorphic to the face poset of the arrangement $\mathcal{H}^{(1)}$, which is a Coxeter arrangement of type A_{n-1}. The unique chamber of this arrangement containing B corresponds to the identity element in the symmetric group $\mathfrak{S}_n$. It then follows from Proposition 5.1.4 that the function $f_B \colon \mathcal{L}_{\geq C} \to \mathcal{T} \cap \mathcal{L}_{\geq C}$ sending $C' \in \mathcal{L}_{\geq C}$ to $C \circ B$ corresponds via $\varphi \colon \Pi_n^{\mathrm{ord}} \xrightarrow{\sim} \mathcal{L}_{\geq C}$ to the function $f \colon \Pi_n^{\mathrm{ord}} \to \mathfrak{S}_n$ of [7, Section 4]. We obtain the equality:

$$\psi_{\mathcal{H}/C}(B, \lambda) = \sum_{\pi \in \Pi_n^{\mathrm{ord}}(\lambda)} (-1)^{|\pi|} \cdot (-1)^{f(\pi)},$$

where $|\pi|$ denotes the number of blocks of the ordered partition $\pi = (B_1, \ldots, B_r)$, in other words, $|\pi| = r$. Let $\overline{\lambda}$ denote the reverse of λ, that is, $\overline{\lambda} = (\lambda_n, \ldots, \lambda_1)$. For ε real we let $\overline{\lambda}_\varepsilon$ be $(\lambda_n - \varepsilon, \ldots, \lambda_1 - \varepsilon)$. By [7, Lemma 7.1], we have

$$\psi_{\mathcal{H}/C}(B, \overline{\lambda}) = (-1)^{\binom{n}{2}} \cdot \sum_{\pi \in \Pi_n^{\mathrm{ord}}(\lambda)} (-1)^{|\pi|} \cdot (-1)^{g(\pi)},$$

where $g\colon \Pi_n^{\mathrm{ord}} \to \mathfrak{S}_n$ is the function defined at the beginning of [7, Section 6].[2] Finally, using the fact that $\Pi_n^{\mathrm{ord}}(\lambda_\varepsilon) = \mathcal{P}(\lambda)$ for sufficiently small $\varepsilon > 0$, then the sum $S(\lambda)$ of [7, Section 6] is given by the expression:

$$S(\lambda) = (-1)^{\binom{n}{2}} \cdot \psi_{\mathcal{H}/C}(B, \overline{\lambda}_\varepsilon) \tag{4.3}$$

for any sufficiently small $\varepsilon > 0$.

We now find an expression for the sum $T(\lambda)$ of [7, Section 6] in terms of 2-structures. As in [7], we denote by M_n the set of maximal matchings on $\{1, 2, \dots, n\}$. Then we have a bijection $M_n \xrightarrow{\sim} \mathcal{T}(\Phi)$ sending a matching $p = \{p_1, \dots, p_m\}$, where

$$p_1 = \{i_1 < j_1\}, \dots, p_m = \{i_m < j_m\}$$

are the edges of p, to the 2-structure

$$\varphi_p = \{\pm(e_{i_1} - e_{j_1}), \dots, \pm(e_{i_m} - e_{j_m})\}.$$

Moreover, we have $(-1)^p = \epsilon(\varphi_p)$. We can calculate $\psi_{\mathcal{H}_{\varphi_p}/C_{\varphi_p}}(B_{\varphi_p}, \lambda)$ using Lemma 4.1.9 for the decomposition $V = V_0 \times V_1 \times \cdots \times V_m$, where $V_k = \mathbb{R}e_{i_k} + \mathbb{R}e_{j_k}$ for $1 \le k \le m$, $V_0 = \{0\}$ if n is even, and $V_0 = \mathbb{R}e_i$ if n is odd and i is the unique unmatched element of $\{1, \dots, n\}$. By Lemma 4.1.9, we have

$$\psi_{\mathcal{H}_{\varphi_p}/C_{\varphi_p}}(B_{\varphi_p}, \lambda) = \prod_{k=1}^{m} d_2(\lambda_{i_k}, \lambda_{j_k}) \cdot \begin{cases} 1 & \text{if } n \text{ is even,} \\ d_1(\lambda_i) & \text{if } n \text{ is odd,} \end{cases}$$

where:

(a) The function $d_1\colon \mathbb{R} \to \mathbb{R}$ is defined by $d_1(a) = \psi_{\mathcal{H}_1/C_1}(B_1, a)$, where $\mathcal{H}_1$ is the hyperplane arrangement (H_e) on $\mathbb{R}e$ and $B_1 = C_1 = \mathbb{R}_{>0}e$.

(b) The function $d_2\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $d_2(a,b) = \psi_{\mathcal{H}_2/C_2}(B_2, (a,b))$, where $\mathcal{H}_2$ is the hyperplane arrangement $(H_e, H_f, H_{e-f}, H_{e+f})$ on $\mathbb{R}e \oplus \mathbb{R}f$, $C_2 = \{\alpha e + \beta f : \alpha = \beta > 0\}$ and $B_2 = \{\alpha e + \beta f : \alpha > \beta > 0\}$.

In other words, the functions d_1 and d_2 are precisely the function $\psi_{\mathcal{H}/C}(B, \lambda)$ that we are trying to determine in the cases $n = 1$ and $n = 2$. A direct calculation yields:

$$d_1(a) = \begin{cases} -1 & \text{if } a \ge 0, \\ 0 & \text{if } a < 0, \end{cases} \quad \text{and} \quad d_2(a,b) = \begin{cases} -1 & \text{if } a, b \ge 0, \\ -2 & \text{if } b \ge -a > 0, \\ 0 & \text{otherwise.} \end{cases}$$

[2] The map $f\colon \Pi_n^{\mathrm{ord}} \to \mathfrak{S}_n$ takes an ordered partition, orders the elements in each block in increasing order and then maps them to the permutation formed by reading the elements from left to right. The map g is similarly defined, except the elements in each block are reordered in decreasing order.

Comparing this with the formula defining $c(p,\lambda)$ in [7, Section 6], we see that, for all $a,b \in \mathbb{R}$, if $\varepsilon > 0$ is sufficiently small relative to a and b, we have $d_1(a-\varepsilon) = -c_1(a)$ and $d_2(a-\varepsilon, b-\varepsilon) = -c_2(b,a)$, and hence

$$\begin{aligned}\psi_{\mathcal{H}_{\varphi_p}/C_{\varphi_p}}(B_{\varphi_p}, \overline{\lambda}_\varepsilon) &= c(p,\lambda) \cdot \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even,} \\ (-1)^{(n+1)/2} & \text{if } n \text{ is odd} \end{cases} \\ &= (-1)^n \cdot (-1)^{\binom{n}{2}} \cdot c(p,\lambda),\end{aligned}$$

if $\varepsilon > 0$ is sufficiently small relative to the λ_i. Combining all these calculations, we see that if $\varepsilon > 0$ is sufficiently small, then

$$\begin{aligned}\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \overline{\lambda}_\varepsilon) &= (-1)^n \cdot (-1)^{\binom{n}{2}} \cdot \sum_{p \in M_n} (-1)^p \cdot c(p,\lambda) \\ &= (-1)^n \cdot (-1)^{\binom{n}{2}} \cdot T(\lambda).\end{aligned}$$

The identity $S(\lambda) = (-1)^n \cdot T(\lambda)$ in [7, Theorem 6.4] now follows from Theorem 4.2.2, applied to $\overline{\lambda}_\varepsilon$ for $\varepsilon > 0$ sufficiently small.

4.5. Proof of Theorem 4.2.2

We assume for now that $\mathcal{H} = (H_{\alpha_e})_{e \in E}$ is an arbitrary central hyperplane arrangement on V. The following definition will be useful.

Definition 4.5.1. Let $C \in \mathcal{L}$ and $\lambda \in V$. If $D, D' \in \mathcal{L}_{\geq C}$, we define $\psi_{D/C}(D',\lambda)$ by the sum

$$\psi_{D/C}(D',\lambda) = \sum_{\substack{C' \in \mathcal{L}_{\lambda,\geq C} \\ C' \circ D' \leq D}} (-1)^{\dim(C')},$$

where $\mathcal{L}_{\geq \lambda, C} = \mathcal{L}_\lambda \cap \mathcal{L}_{\geq C}$.

Remark 4.5.2. Suppose that D' is a chamber. Then $C' \circ D'$ is a chamber for every $C' \in \mathcal{L}$, so $\psi_{D/C}(D',\lambda) = 0$ unless D is also a chamber. If D is a chamber, we have

$$\psi_{D/C}(D',\lambda) = \sum_{\substack{C' \in \mathcal{L}_{\lambda,\geq C} \\ C' \circ D' = D}} (-1)^{\dim(C')}.$$

The functions $\psi_{D/C}(D',\lambda)$ are related to $\psi_{\mathcal{H}/C}(B,\lambda)$ by the following lemma.

Lemma 4.5.3. *Let $C \in \mathcal{L}$ and $B \in \mathfrak{T} \cap \mathcal{L}_{\geq C}$. Then for every $\lambda \in V$ the following identity holds:*

$$\psi_{\mathcal{H}/C}(B,\lambda) = \sum_{T \in \mathfrak{T} \cap \mathcal{L}_{\geq C}} (-1)^{|S(B,T)|} \cdot \psi_{T/C}(B,\lambda).$$

Proof. Indeed, if $D \in \mathcal{L}_{\geq C}$ then the chamber $D \circ B$ is also in $\mathcal{L}_{\geq C}$. Hence, using equation (4.2) in Definition 4.1.5 and Remark 4.5.2, we obtain:

$$\begin{aligned}\psi_{\mathcal{H}/C}(B,\lambda) &= \sum_{T \in \mathcal{T} \cap \mathcal{L}_{\geq C}} (-1)^{|S(B,T)|} \cdot \sum_{\substack{D \in \mathcal{L}_{\lambda, \geq C} \\ D \circ B = T}} (-1)^{\dim(D)} \\ &= \sum_{T \in \mathcal{T} \cap \mathcal{L}_{\geq C}} (-1)^{|S(B,T)|} \cdot \psi_{T/C}(B,\lambda). \qquad \blacksquare\end{aligned}$$

Before Corollary A.1.11 of Appendix A, we define, for K a closed convex polyhedral cone in V, a function $\psi_K \colon V \times V^\vee \to \mathbb{R}$. For fixed $(x,\ell) \in V \times V^\vee$, the function $K \mapsto \psi_K(x,\ell)$ is a valuation on the set of closed convex polyhedral cones in V; see Definition A.1.3. This function is related to the functions $\psi_{D/C}(D',\lambda)$ in the following way.

Lemma 4.5.4. *Let $C \in \mathcal{L}$, let $D \in \mathcal{L}_{\geq C}$ and let $\lambda \in V$. Denote by $\ell \in V^\vee$ the linear functional $(\cdot,\lambda)$. Then for every $D' \in \mathcal{L}_{\geq C}$ the following identity holds:*

$$\psi_{D/C}(D',\lambda) = \psi_{\bar{D}}(x,\ell),$$

where x is any point in $D'_1 = (-C) \circ D'$.

Proof. As before we write $E(C) = \{e \in E : C \subset H_{\alpha_e}\}$. Note that $s(D'_1)_e = s(D')_e$ for $e \in E(C)$, and $s(D'_1) = -s(C)_e \neq 0$ for $e \in E - E(C)$. Also, by definition of $\mathcal{L}_{\geq C}$, if e is any index of $E - E(C)$ and $C' \in \mathcal{L}_{\geq C}$ then $s(C)_e = s(C')_e \neq 0$.

We claim that for every $C' \in \mathcal{L}$, we have $C' \circ D'_1 \leq D$ if and only if $C' \in \mathcal{L}_{\geq C}$ and $C' \circ D' \leq D$. Suppose first that $C' \in \mathcal{L}_{\geq C}$ and $C' \circ D' \leq D$. Then for every $e \in E(C)$, we have

$$s(C' \circ D'_1)_e = s(C' \circ D')_e \leq s(D)_e.$$

Moreover, if $e \in E - E(C)$ then $s(C')_e = s(C)_e = s(D)_e \neq 0$, so $s(C' \circ D'_1)_e = s(C')_e = s(D)_e$. This shows that $C' \circ D'_1 \leq D$. Conversely, suppose that C' is a face of $\mathcal{L}$ such that $C' \circ D'_1 \leq D$. If $e \in E - E(C)$ then

$$0 \neq s(C)_e = s(D)_e = -s(D'_1)_e,$$

thus $s(C')_e \neq 0$, and so $s(C')_e = s(C' \circ D'_1)_e = s(D)_e$. This implies that $C' \in \mathcal{L}_{\geq C}$. Moreover, if $e \in E(C)$, we have $s(D'_1)_e = s(D')_e$, thus

$$s(C' \circ D')_e = s(C' \circ D'_1)_e \leq s(D)_e.$$

Hence, we conclude that $C' \circ D' \leq D$.

By the claim, we obtain

$$\psi_{D/C}(D',\lambda) = \sum_{\substack{C'\in\mathcal{L}_\lambda \\ C'\circ D'_1\le D}} (-1)^{\dim(C')} = \psi_D(D'_1,\lambda).$$

We wish to show that this is equal to $\psi_{\bar{D}}(x,\ell)$, if $x\in D'_1$. As in Appendix A, we denote by $\mathcal{F}(\bar{D})$ the set of closed faces of the closed convex polyhedral cone $\bar{D}$. We have $\mathcal{F}(\bar{D})=\{\overline{C'}:C'\in\mathcal{L},\ C'\le D\}$, and the set $\{C'\in\mathcal{L}:C'\circ D'_1\le D\}$ is included in the set $\{C'\in\mathcal{L}:C'\le D\}$. To prove the equality above, it suffices to show that the two following statements hold for $C'\in\mathcal{L}$ such that $C'\le D$ (see Lemma A.1.10 for the definition of ψ_x and ψ_ℓ, and the beginning of Section A.1 for the notation $\overline{C'}^{\perp,\bar{D}}$):

(a) The face C' belongs to $\mathcal{L}_\lambda$ if and only if $\psi_\ell(\overline{C'})=1$.

(b) The inequality $C'\circ D'_1\le D$ holds if and only if $\psi_x(\overline{C'}^{\perp,\bar{D}})=1$.

Statement (a) is just a direct translation of the definition of $\mathcal{L}_\lambda$, and statement (b) is proved in Lemma 3.3.1. ■

We are now ready to prove Theorem 4.2.2, so we assume that we are in the situation of that theorem.

Let ℓ be the linear functional $(\cdot,\lambda)$ on V, let $x\in(-C)\circ B$, and consider the valuation ν on the set of closed convex polyhedral cones in V sending such a cone K to $\psi_{K\cap\bar{\mathcal{C}}}(x,\ell)$, where

$$\mathcal{C}=\mathcal{C}_C=\bigcap_{e\in E^{(2)}} H^+_{\alpha_e}.$$

The function $K\mapsto\psi_K(x,\ell)$ is a priori defined only on the set of closed convex polyhedral cones. However, as it is a valuation, we can extend it to the set of all finite intersections of closed and open half-spaces in V; see Remark A.1.13. As x is not on any hyperplane of $\mathcal{H}$, the valuation ψ_x vanishes on any cone contained in a hyperplane of $\mathcal{H}$. It follows from the definition of the function $K\mapsto\psi_K(x,\ell)$ in the discussion before Corollary A.1.11 that we have $\psi_K(x,\ell)=0$ if K is contained in a hyperplane of $\mathcal{H}$. In particular, for D a face of $\mathcal{H}$, we have $\nu(D)=0$ unless D is a chamber, and if D is a chamber then $\nu(D)=\nu(\bar{D})$.

Let $\varphi\subset\Phi$ be a pseudo-root system (we do not assume that φ is a 2-structure), let $\varphi^+=\varphi\cap\Phi^+$ and $\mathcal{H}_\varphi=(H_{\alpha_e})_{e\in\varphi^+\sqcup E^{(2)}}$, and denote by B_φ and C_φ the unique faces of $\mathcal{H}_\varphi$ containing B and C. We have

$$C_\varphi=\bigcap_{e\in\varphi^+} H_{\alpha_e}\cap\bigcap_{e\in E^{(2)}} H^+_{\alpha_e}.$$

We also set $\mathcal{H}^{(1)}_\varphi=(\mathcal{H}_{\alpha_e})_{e\in\varphi^+}$, $\mathcal{L}_\varphi=\mathcal{L}(\mathcal{H}_\varphi)$ and $\mathcal{T}_\varphi=\mathcal{T}(\mathcal{H}_\varphi)$.

As in statement (i) of Lemma 2.1.3, we denote by $\iota \colon \mathcal{L}_{\varphi,\geq C_\varphi} \to \mathcal{L}(\mathcal{H}_\varphi^{(1)})$ the map sending a face $D \geq C_\varphi$ of $\mathcal{H}_\varphi$ to the unique face of $\mathcal{H}_\varphi^{(1)}$ that contains it. By the lemma we just cited, we know that this is an order-preserving bijection, and that its inverse sends a face $D^{(1)}$ of $\mathcal{H}_\varphi^{(1)}$ to $D^{(1)} \cap \mathcal{C}$, where $\mathcal{C} = \bigcap_{e \in E^{(2)}} H_{\alpha_e}^+$ as before. We claim that, if $D^{(1)}$ is a face of $\mathcal{H}_\varphi^{(1)}$, then

$$\overline{D^{(1)} \cap \mathcal{C}} = \overline{D^{(1)}} \cap \overline{\mathcal{C}}.$$

Indeed, let $s \in \{+,-,0\}^{\varphi^+}$ be the sign vector of $D^{(1)}$. Then the sign vector $t \in \{+,-,0\}^E$ of $D^{(1)} \cap \mathcal{C}$ is given by $t_e = s_e$ if $e \in \varphi^+$, and $t_e = +$ if $e \in E^{(2)}$. We set

$$\mathbb{R}_+ = \mathbb{R}_{\geq 0}, \quad \mathbb{R}_- = \mathbb{R}_{\leq 0}, \quad \mathbb{R}_0 = \{0\}.$$

Let $y \in V$. Then $y \in \overline{D^{(1)}}$ if and only if $(\alpha_e, y) \in \mathbb{R}_{s_e}$ for every $e \in \varphi^+$, while $y \in \overline{D^{(1)} \cap \mathcal{C}}$ if and only if $(\alpha_e, y) \in \mathbb{R}_{t_e}$ for every $e \in E$ and $y \in \overline{\mathcal{C}}$ if and only if $(\alpha_e, y) \geq 0$ for every $e \in E^{(2)}$. This immediately implies the claim.

Let $D^{(1)}$ be a face of $\mathcal{H}_\varphi^{(1)}$. By Lemma 4.5.4, we have

$$\psi_{\iota^{-1}(D^{(1)})/C_\varphi}(B_\varphi, \lambda) = \psi_{\overline{D^{(1)} \cap \mathcal{C}}}(x, \ell) = \psi_{\overline{D^{(1)}} \cap \overline{\mathcal{C}}}(x, \ell) = \nu(\overline{D^{(1)}}),$$

because $x \in (-C) \circ B \subset (-C_\varphi) \circ B_\varphi$. Moreover, by Lemma 4.5.3, we have

$$\psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \lambda) = \sum_{T \in \mathcal{T}_\varphi \cap \mathcal{L}_{\varphi,\geq C_\varphi}} (-1)^{|S(B_\varphi, T)|} \cdot \psi_{T/C_\varphi}(B_\varphi, \lambda).$$

So, using statements (i) and (iii) of Lemma 2.1.3, we obtain that

$$\begin{aligned}
\psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \lambda) &= \sum_{T^{(1)} \in \mathcal{T}(\mathcal{H}_\varphi^{(1)})} (-1)^{|S(\iota(B_\varphi), T^{(1)})|} \psi_{\iota^{-1}(T^{(1)})/C_\varphi}(B_\varphi, \lambda) \\
&= \sum_{T^{(1)} \in \mathcal{T}(\mathcal{H}_\varphi^{(1)})} (-1)^{|S(\iota(B_\varphi), T^{(1)})|} \nu(\overline{T^{(1)}}) \\
&= \sum_{C^{(1)} \in \mathcal{T}(\mathcal{H}_\varphi^{(1)})} (-1)^{|S(\iota(B_\varphi), C^{(1)} \circ \iota(B_\varphi))|} \nu(\overline{C^{(1)}}).
\end{aligned}$$

In other words, using the notation of Section 3.2, we obtain

$$\psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \lambda) = \Pi(\mathcal{H}_\varphi, \nu). \tag{4.4}$$

Applying this identity to $\varphi = \Phi$, we have $\psi_{\mathcal{H}/C}(B, \lambda) = \Pi(\mathcal{H}, \nu)$. By Theorem 3.2.1, we deduce that

$$\psi_{\mathcal{H}/C}(B, \lambda) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \Pi(\mathcal{H}_\varphi, \nu).$$

The conclusion of Theorem 4.2.2 follows from this equality and from equation (4.4) for all $\varphi \in \mathcal{T}(\Phi)$.

5. Properties of the weighted complex

This section is independent of Sections 3 and 4, except for Definition 4.1.1 and Remark 4.1.2. We prove that the weighted complex is shellable for Coxeter arrangements, and more generally, for arrangements satisfying a condition on the dihedral angles between their hyperplanes (Condition (A)). This implies that the weighted complex is a PL ball for arrangements satisfying Condition (A).

5.1. Shellable polytopal complexes

We introduce the following definition. For instance, see [3, Definition 4.7.14].

Definition 5.1.1. A pure n-dimensional polytopal complex Δ is *shellable* if it is 0-dimensional (and hence a collection of a finite number of points), or if there is a linear order of the facets $F_1, F_2, \dots, F_k$ of Δ, called a *shelling order*, such that:

(i) The boundary complex of F_1 is shellable.

(ii) For $1 < j \le k$ the intersection of $\overline{F_j}$ with the union of the closures of the previous facets is nonempty and is the beginning of a shelling of the $(n-1)$-dimensional boundary complex of F_j, that is,

$$\overline{F_j} \cap (\overline{F_1} \cup \overline{F_2} \cup \cdots \cup \overline{F_{j-1}}) = \overline{G_1} \cup \overline{G_2} \cup \cdots \cup \overline{G_r},$$

where $G_1, G_2, \dots, G_t$ is a shelling order of ∂F_j and $r \ge 1$.

We then have the following result.

Theorem 5.1.2 ([3, Theorem 4.3.3]). *Let $\mathcal{H}$ be a central hyperplane arrangement on the vector space V, let $\mathcal{L} = \mathcal{L}(\mathcal{H})$ and $\mathcal{T} = \mathcal{T}(\mathcal{H})$, and let B be a chamber in $\mathcal{T}$. Then any linear extension of the chamber poset with base chamber B is a shelling order on the facets of $\Sigma(\mathcal{L})$.*

Let $\mathcal{H}$ be a central hyperplane arrangement on V. We write $\mathcal{L} = \mathcal{L}(\mathcal{H})$ and $\mathcal{T} = \mathcal{T}(\mathcal{H})$. Let $B \in \mathcal{T}$, and let $B = T_1, T_2, \dots, T_r = -B$ be a linear ordering of $\mathcal{T}$ refining the partial order $\preceq_B$. By Theorem 5.1.2 the linear order $T_1, T_2, \dots, T_r$ is a shelling order of the chambers of $\Sigma(\mathcal{L})$. In particular, the shelling order defines a partition of the faces of $\Sigma(\mathcal{L})$:

$$\mathcal{L} = \coprod_{i=1}^{r} \{C \in \mathcal{L} : C \le T_i \text{ and } C \not\le T_j \text{ for } 1 \le j < i\}.$$

We will give a formula for the blocks of this partition (see Proposition 5.1.5), which implies in particular that the partition is independent of the linear refinement of $\preceq_B$.

Definition 5.1.3. Given a chamber $B \in \mathcal{T}$, we define a function f_B from the face poset $\mathcal{L}$ to the set of chambers $\mathcal{T}$ by $f_B(C) = C \circ B$.

The next proposition gives some basic properties of the function f_B.

Proposition 5.1.4. *The following two statements hold:*

(i) *Fix a face $C \in \mathcal{L}$ and consider the poset isomorphism $\iota_C : \mathcal{L}_{\geq C} \xrightarrow{\sim} \mathcal{L}_{\mathcal{H}(C)}$ of Lemma 2.1.3. If $B \in \mathcal{T} \cap \mathcal{L}_{\geq C}$ then for every $D \in \mathcal{L}_{\geq C}$, we have*

$$\iota_C(f_B(D)) = f_{\iota_C(B)}(\iota_C(D)).$$

(ii) *Suppose that $\mathcal{H}$ is a Coxeter arrangement with a chamber B that is on the positive side of every hyperplane, and let (W, S) be the associated Coxeter system. Identify $\mathcal{L}$ with the Coxeter complex $\Sigma(W)$ as in Section 2.2. If the face $C \in \mathcal{L}$ corresponds to a standard coset $c \subset W$, then the element $w \in W$ corresponding to the chamber $f_B(C)$ is the shortest element of c and also the minimal element of the coset c in the right weak Bruhat order.*

In particular, part (ii) implies that, for the type A Coxeter complex, the function f_B defined here (for B the chamber corresponding to $1 \in W$) is equal to the function f defined at the beginning of [7, Section 4]. Note that the existence of a minimal element in every standard coset is proved in [2, Proposition 2.4.4].

Proof of Proposition 5.1.4. Statement (i) follows immediately from Lemma 2.1.3.

We now prove (ii). By definition of the composition $\circ$, the chamber $f_B(C) = C \circ B$ is the element of $\mathcal{T} \cap \mathcal{L}_{\geq C}$ closest to B in the chamber graph; in other words, it is the minimal element of $\mathcal{T} \cap \mathcal{L}_{\geq C}$ for the order $\preceq_B$; see Section 2.1. As we know that $\preceq_B$ corresponds to the right weak Bruhat order on W (see the discussion after Theorem 2.2.1), and as the elements of W corresponding to the chambers of $\mathcal{T} \cap \mathcal{L}_{\geq C}$ are the elements of the coset c, the result follows. ■

The link between the function f_B and the shellings of Theorem 5.1.2 is established in the following proposition. For the type A Coxeter complex, this result appeared implicitly in the proof of [7, Proposition 4.1].

Proposition 5.1.5. *Let $B \in \mathcal{T}$, and let $B = T_1, T_2, \ldots, T_r = -B$ be a linear ordering of $\mathcal{T}$ refining the partial order $\preceq_B$. Then for every index $1 \leq i \leq r$ the fiber of f_B over T_i is given by*

$$f_B^{-1}(T_i) = \{C \in \mathcal{L} : C \subset \overline{T}_i - (\overline{T}_1 \cup \overline{T}_2 \cup \cdots \cup \overline{T}_{i-1})\} \tag{5.1}$$

$$= \{C \in \mathcal{L} : C \leq T_i \text{ and } C \not\leq T_j \text{ for } 1 \leq j < i\}. \tag{5.2}$$

Proof. The equivalence between equalities (5.1) and (5.2) is an immediate consequence of the definition of the order $\le$ on $\mathcal{L}$. Let us prove equality (5.2).

Let $C \in f_B^{-1}(T_i)$, that is, $C \circ B = T_i$. In particular, we have $C \le C \circ B = T_i$. Suppose that $T \in \mathcal{T}$ is another chamber such that $C \le T$. Then for every $e \in S(B, T_i)$, we have $s(T_i)_e \neq s(B)_e$, but $s(C \circ B)_e = s(T_i)_e$, so $0 \neq s(C)_e = s(T_i)_e$. As $C \le T$, this implies that

$$s(T)_e = s(C)_e = s(T_i)_e \neq s(B)_e,$$

hence that $e \in S(B, T)$. So we have proved that $S(B, T_i) \subseteq S(B, T)$, which means that $T_i \preceq_B T$. In particular, if $1 \le j \le i - 1$, then $C \not\le T_j$.

Conversely, let $C \in \mathcal{L}$ be such that $C \le T_i$ and $C \not\le T_j$ for $1 \le j < i$, and let $T = C \circ B$. If $e \in S(B, T)$, then $0 \neq s(C)_e = s(T)_e$. As $C \le T_i$, this implies that $s(C)_e = s(T_i)_e$, so

$$s(T_i)_e = s(C)_e = s(T)_e \neq s(B)_e,$$

that is, $e \in S(B, T_i)$. So we have proved that $S(B, T) \subseteq S(B, T_i)$, which means that $T \preceq_B T_i$. Hence, there exists an index $1 \le i' \le i$ such that $T = T_{i'}$. As $C \le T$ and $C \not\le T_j$ for $1 \le j < i$, we must have $i' = i$, that is, $f_B(C) = C \circ B = T = T_i$. ∎

5.2. A condition on hyperplane arrangements

We now introduce a geometric condition on the hyperplane arrangement $\mathcal{H}$ that will imply the shellability of the weighted complex.

Condition (A). *Denote by* (A) *the following condition on the family* $(\alpha_e)_{e \in E}$ *(or the corresponding arrangement): For every* $T \in \mathcal{T}$ *and for every* $e \in E$ *such that* $S = \overline{T} \cap H_{\alpha_e}$ *is of dimension* $\dim(V) - 1$*, that is,* S *is a facet of the convex cone* $\overline{T}$*, the following inclusions hold:*

$$T \subseteq \mathring{S} + \mathbb{R}_{>0}\alpha_e \quad \text{if } T \subseteq H_{\alpha_e}^{+},$$
$$T \subseteq \mathring{S} + \mathbb{R}_{<0}\alpha_e \quad \text{if } T \subseteq H_{\alpha_e}^{-},$$

where $\mathring{S}$ *is the relative interior of the cone* S*, that is, the interior of* S *in* $\mathrm{Span}(S)$.

Geometrically, Condition (A) means that if $T \in \mathcal{T}$ then the dihedral angle between any two adjacent facets (facets whose intersection is a face of dimension $\dim(V) - 2$) of the convex polyhedral cone $\overline{T}$ is acute, that is, less than or equal to $\pi/2$.

Proposition 5.2.1. *Suppose that the arrangement* $\mathcal{H}$ *satisfies Condition (A). Let* $T, T' \in \mathcal{T}$ *and* $e \in E$ *such that* $S(T, T') = \{e\}$*, the inner product* (α_e, λ) *is nonnegative and the inclusion* $T' \subset H_{\alpha_e}^{-}$ *holds. Then* $T' \in \mathcal{L}_\lambda$ *implies that* $T \in \mathcal{L}_\lambda$.

Proof. The hypothesis implies that $\overline{T} \cap \overline{T}' = \overline{T} \cap H_{\alpha_e} = \overline{T}' \cap H_{\alpha_e}$. We denote this intersection by S. It is a facet of both $\overline{T}$ and $\overline{T}'$. By Condition (A), we have

$$T \subset \mathring{S} + \mathbb{R}_{>0}\alpha_e \quad \text{and} \quad T' \subset \mathring{S} + \mathbb{R}_{<0}\alpha_e.$$

In particular, if $x \in T$ then there exists $c > 0$ such that $x - c \cdot \alpha_e \in T'$. Then we have

$$(x, \lambda) = (x - c \cdot \alpha_e, \lambda) + c \cdot (\alpha_e, \lambda) \geq 0.$$

This implies that $T \subset \overline{H_\lambda^+}$, that is, $T \in \mathcal{L}_\lambda$. ∎

Corollary 5.2.2. *Suppose the arrangement $\mathcal{H}$ satisfies Condition (A). If $(\lambda, \alpha_e) \geq 0$ for every $e \in E$ and if there exists $B \in \mathcal{T}$ such that $B \subset H_{\alpha_e}^+$ for every $e \in E$, then $\mathcal{T} \cap \mathcal{L}_\lambda$ is a lower order ideal in $\mathcal{T}_B$. More generally, if $C \in \mathcal{L}$ and*

$$E(C) = \{e \in E : C \subset H_{\alpha_e}\},$$

if $(\lambda, \alpha_f) \geq 0$ for every $f \in E(C)$ and if there exists $B \in \mathcal{T} \cap \mathcal{L}_{\geq C}$ such that $B \subset H_{\alpha_f}^+$ for every $f \in E(C)$, then $\mathcal{T} \cap \mathcal{L}_{\lambda, \geq C}$ is a lower order ideal in $\mathcal{T}_B$.

Proof. It suffices to prove the second statement. Let T, T' be such that $S(B, T) \subset S(B, T')$ and $T' \in \mathcal{L}_{\lambda, \geq C}$. We want to show that $T \in \mathcal{L}_{\lambda, \geq C}$. As $\mathcal{T}_B$ is a graded poset and the intersection $\mathcal{T} \cap \mathcal{L}_{\geq C}$ is a lower order ideal in $\mathcal{T}_B$ (see Remark 2.1.4), we know that $T \in \mathcal{T} \cap \mathcal{L}_{\geq C}$, and it suffices to treat the case where $S(T', B) - S(T, B)$ is a singleton. Let f be the single index of $S(B, T') - S(B, T)$. As $B, T' \in \mathcal{T} \cap \mathcal{L}_{\geq C}$, we have $f \in E(C)$ by Lemma 2.1.3 (iii), so $B \subset H_{\alpha_f}^+$. As $f \in S(B, T') - S(B, T)$, we have $T' \subset H_{\alpha_f}^-$ and $T \subset H_{\alpha_f}^+$. Also, as $f \in E(C)$, we have $(\lambda, \alpha_f) \geq 0$. So we may apply Proposition 5.2.1, and we obtain that $T \in \mathcal{L}_\lambda$. ∎

Corollary 5.2.3. *Suppose that the arrangement $\mathcal{H}$ satisfies Condition (A). Then the complex $\Sigma(\mathcal{L}_\lambda)$ is shellable. Moreover, there exists a shelling order on its chambers which is an initial shelling of $\Sigma(\mathcal{L})$. In particular, if $\lambda \neq 0$ then $\Sigma(\mathcal{L}_\lambda)$ is a shellable PL ball of dimension* $\dim(V/V_0) - 1$.

Proof. If $\lambda = 0$ then $\mathcal{L}_\lambda = \mathcal{L}$ and $\Sigma(\mathcal{L}_\lambda) = \Sigma(\mathcal{L})$, and the corollary is just Theorem 5.1.2.

We now assume that $\lambda \neq 0$. By Theorem 5.1.2 and Corollary 5.2.2, it suffices to find a family of signs $(\varepsilon_e) \in \{\pm 1\}^E$ such that:

- for every $e \in E$, we have $(\lambda, \varepsilon_e \alpha_e) \geq 0$;
- there exists a chamber $B \in \mathcal{L}_\lambda$ with $B \subset H_{\varepsilon_e \alpha_e}^+$ for every $e \in E$.

Indeed, Corollary 5.2.2 will then imply that $\mathcal{T} \cap \mathcal{L}_\lambda$ is a lower order ideal in $\mathcal{T}_B$, so it will be an initial segment for at least one linear extension of $\preceq_B$.

Let $F = \{e \in E : (\lambda, \alpha_e) \neq 0\}$. For every $e \in F$, we choose $\varepsilon_e \in \{\pm 1\}$ such that $(\lambda, \varepsilon_e \alpha_e) > 0$. Let x_0 be a point in V not on any hyperplane of $\mathcal{H}$, that is,

$$x_0 \in V - \bigcup_{e \in E} H_{\alpha_e}.$$

Then for every $e \in F$, the inner product $(x_0 + c \cdot \lambda, \varepsilon_e \alpha_e) = (x_0, \varepsilon_e \alpha_e) + c \cdot (\lambda, \varepsilon_e \alpha_e)$ tends to $+\infty$ as c tends to $+\infty$, so it is positive for c large enough. Similarly, the inner product $(x_0 + c \cdot \lambda, \lambda) = (x_0, \lambda) + c \cdot (\lambda, \lambda)$ is positive for c large enough. On the other hand, if $e \in E - F$, then $(x_0 + c \cdot \lambda, \alpha_e) = (x_0, \alpha_e) \neq 0$ for every $c \in \mathbb{R}$. So, if $c \in \mathbb{R}$ is large enough, then

$$x = x_0 + c \cdot \lambda \in V - \bigcup_{e \in E} H_{\alpha_e},$$

and x is in $H^+_{\varepsilon_e \alpha_e}$ for every $e \in F$ and in H^+_λ. In particular, there exists a chamber $B \in \mathcal{T}$ such that $x \in B$, and B is included in $H^+_{\varepsilon_e \alpha_e}$ for every $e \in F$ and in H^+_λ. Now, if $e \in E - F$, we choose $\varepsilon_e \in \{\pm 1\}$ such that $B \subset H^+_{\varepsilon_e \alpha_e}$. As $(\lambda, \alpha_e) = 0$, we clearly have $(\lambda, \varepsilon_e \alpha_e) \geq 0$. ■

5.3. The case of Coxeter arrangements

Lemma 5.3.1. *Every Coxeter arrangement $\mathcal{H}$ satisfies Condition (A).*

Proof. In a Coxeter arrangement, the dihedral angle between any two adjacent facets is π/n, with $n \geq 2$. ■

In particular, Corollaries 5.2.2 and 5.2.3 apply to Coxeter arrangements. But we can actually prove a stronger result in this case.

We fix a Coxeter arrangement $\mathcal{H}$ on an inner product space V, and we use the notation introduced above. We say that a vector $\lambda \in V$ is *dominant* if $(\lambda, \alpha) \geq 0$ for every $\alpha \in \Phi^+$.

Lemma 5.3.2. *Suppose that $\lambda \in V$ is dominant. Denote by B the chamber of $\mathcal{H}$ corresponding to $1 \in W$. Let $z, w \in W$ such that $z \leq w$ in the strong Bruhat order of W. Then for every $x \in B$ the following inequality holds:*

$$(z^{-1}(\lambda), x) \geq (w^{-1}(\lambda), x).$$

Proof. We may assume that w covers z, so that there exists $s \in S$ and $u \in W$ such that $w = (usu^{-1})z$. Let α be the unique pseudo-root of Φ^+ such that $u(e_s)$ is a multiple (positive or negative) of α. If s_α is the reflection across H_α, we have $usu^{-1} = s_\alpha$, and so $w = s_\alpha z$ and $s_\alpha w = z$. Since the elements of Φ^+ are unit vectors, s_α is given by the following formula: $s_\alpha(\mu) = \mu - 2 \cdot (\mu, \alpha) \cdot \alpha$ for $\mu \in V$. Hence,

$$(s_\alpha w)^{-1}(\lambda) = (w^{-1} s_\alpha)(\lambda) = w^{-1}(\lambda) - 2 \cdot (\lambda, \alpha) \cdot w^{-1}(\alpha),$$

and so, if $x \in B$,

$$((s_\alpha w)^{-1}\lambda, x) = (w^{-1}(\lambda), x) - 2 \cdot (\lambda, \alpha) \cdot (w^{-1}(\alpha), x).$$

As λ is dominant, we have $(\lambda, \alpha) \geq 0$. By [2, Equation (4.25)], we have the equivalence

$$w^{-1}(\alpha) \in \Phi^+ \Leftrightarrow \ell(w^{-1} s_\alpha) > \ell(w^{-1}),$$

and [2, Proposition 1.4.2(iv)] states that $\ell(v^{-1}) = \ell(v)$ for every $v \in W$. Using these two facts and the condition $\ell(s_\alpha w) < \ell(w)$, we see that $w^{-1}(\alpha) \in \Phi^-$. Thus, we have $(w^{-1}(\alpha), x) < 0$ by definition of B. Hence, the term $-2 \cdot (\lambda, \alpha) \cdot (w^{-1}(\alpha), x)$ is nonnegative, that is,

$$(z^{-1}(\lambda), x) = ((s_\alpha w)^{-1}(\lambda), x) \geq (w^{-1}(\lambda), x). \qquad \blacksquare$$

Proposition 5.3.3. *Let (W, S) be a Coxeter system, and let $\mathcal{H} = (H_\alpha)_{\alpha \in \Phi^+}$ be the associated hyperplane arrangement on the space V of the canonical representation of (W, S). Let $\lambda \in V$ be a dominant vector. Then the set W_λ of $w \in W$ such that the corresponding chamber of $\mathcal{H}$ is in $\mathcal{T} \cap \mathcal{L}_\lambda$ is a lower order ideal with respect to the* strong *Bruhat order on W.*

Proof. We denote by B the chamber of $\mathcal{H}$ corresponding to $1 \in W$. By definition of W_λ, an element w of W is in W_λ if and only if for every $x \in B$, we have $(\lambda, w(x)) = (w^{-1}(\lambda), x) \geq 0$. By Lemma 5.3.2, if $z, w \in W$ and w is greater than z in the strong Bruhat order then for every $x \in B$, we have $(z^{-1}(\lambda), x) \geq (w^{-1}(\lambda), x)$. If moreover $w \in W_\lambda$, this immediately implies that $z \in W_\lambda$. $\blacksquare$

6. Concluding remarks

As mentioned in the introduction, we are not aware whether there is a representation-theoretic interpretation of the identity in Theorem 4.2.2 in general. More precisely, what is the meaning of the constants $\psi_{\mathcal{H}/C}(B, \lambda)$ for different values of λ?

The main results in this paper are in the setting of Coxeter arrangements. However, the sum $\psi_{\mathcal{H}}$ makes sense for general hyperplane arrangements, our original proof of Theorem 4.2.2 used an induction formula that linked the sum $\psi_{\mathcal{H}}$ to similar sums for subarrangements of the restricted arrangements on the hyperplanes of H, and this induction formula is valid for general hyperplane arrangements, and with some adaptions, for oriented matroids. In the paper [9], we use a similar type of induction argument, this time to calculate the alternating sum of another valuation on the chambers of a hyperplane arrangement, that is, the volume of the intersection of the chamber with some set of finite volume. Is there is some analogue of 2-structures for more general hyperplane arrangements?

It is natural to ask whether there is some analogue of Theorem 3.2.1 for Coxeter systems with possible infinite Coxeter groups.

In Section 5, we proved that the weighted complexes of a hyperplane arrangement are shellable under a geometric condition on the arrangement that we call Condition (A). This implies that the weighted complexes are PL balls for arrangements satisfying Condition (A). Are the weighted complexes always PL balls? By Remark 4.1.2, this extends a conjecture of Zaslavsky (see [33, Chapter I, Section 3C, p. 33]) that the bounded complex of a simple hyperplane arrangement is always a PL ball. As a consequence to Corollary 5.2.3, we have the following result.

Corollary 6.1. *Zaslavsky's conjecture holds for affine arrangements obtained by intersecting an affine hyperplane with an arrangement satisfying Condition (A).*

In a paper of Dong (see [6]), the author claims to have proven Zaslavsky's conjecture. However, we do not understand the proof of the crucial Lemma 4.7 in that paper: In the second paragraph of case 2, Dong chooses a linear extension $\leq$ of $\mathcal{T}(\mathcal{L}/g, d_i)$ such that $[d_i, d_j]$ is an initial segment. This linear extension is a shelling order, and Dong deduces that there exists $d_k \in [d_i, d_j)$ such that $d_k \wedge d_j \lessdot d_j$. But the only thing that we can deduce from the fact that we have a shelling order is that $d_i \leq d_k < d_j$, which does not imply that $d_k \in [d_i, d_j)$ for the order on $\mathcal{T}(\mathcal{L}/g, d_i)$. If we do not know that $d_k \in [d_i, d_j)$, the rest of the argument fails.

In Appendix A, we construct a ring structure on the set of valuations on convex closed polyhedral cones with values in a fixed ring. Are there other products of valuations that also yield valuations? For instance, can the ring structure of Corollary A.1.7 be extended to valuations on (not necessarily polyhedral) cones in Euclidean space?

In Appendix B, the proof of Proposition B.2.4 that the group W acts transitively on the set of 2-structures $\mathcal{T}$ consists of verifying the result for all irreducible pseudo-root systems. Is there a general proof that does not use the classification of irreducible pseudo-root systems?

A. Extending the construction of a valuation by Goresky, Kottwitz and MacPherson

We introduce a ring structure on the set of valuations defined on closed convex polyhedral cones in a finite-dimensional real vector space with values in a fixed ring. As a special case we obtain in Corollary A.1.11 a valuation due to Goresky, Kottwitz and MacPherson; see [13, Proposition A.4].

Section A.1 of this appendix contains definitions and statements of results. The proofs are relegated to Section A.2.

A.1. The ring of valuations

Let V be a finite-dimensional real vector space and $V^\vee$ its dual. A *closed convex polyhedral cone* in V is a nonempty subset of the form

$$\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \cdots + \mathbb{R}_{\geq 0}v_k,$$

where $v_1, v_2, \ldots, v_k \in V$ and $k \geq 0$.

For a subset X of the space V, define

$$X^\perp = \{\alpha \in V^\vee : \forall x \in X \ \langle \alpha, x\rangle = 0\} \quad \text{and} \quad X^* = \{\alpha \in V^\vee : \forall x \in X \ \langle \alpha, x\rangle \geq 0\}.$$

Note that $X^\perp$ is a subspace of $V^\vee$ and depends only on the linear span of X, and that X^* is a convex cone in $V^\vee$ and depends only on the closed convex polyhedral cone generated by X.

For F a face[3] of a closed convex polyhedral cone K, define $F^{\perp,K} = F^\perp \cap K^*$. The map $F \mapsto F^{\perp,K}$ is an order-reversing bijection from the set $\mathcal{F}(K)$ of faces of K to the set of faces of K^*. This statement and other basic properties of closed convex polyhedral cones are proved in [11, Section 1.2].

Remark A.1.1. For two closed convex cones X_1 and X_2 such that $X_1 \cup X_2$ is convex then the set $X_1^* \cup X_2^*$ is also convex and we have the two identities

$$(X_1 \cup X_2)^* = X_1^* \cap X_2^* \quad \text{and} \quad (X_1 \cap X_2)^* = X_1^* \cup X_2^*.$$

Definition A.1.2. We denote by $\mathcal{C}(V)$ the set of closed convex polyhedral cones in V. Denote the free abelian group on $\mathcal{C}(V)$ by $\bigoplus_{K\in\mathcal{C}(V)} \mathbb{Z}[K]$ and let $K(V)$ be its quotient by the relations $[K \cup K'] + [K \cap K'] = [K] + [K']$ for all $K, K' \in \mathcal{C}(V)$ such that $K \cup K'$. For $K \in \mathcal{C}(V)$, we still denote its image in $K(V)$ by $[K]$.

[3] In this appendix, we take all faces to be closed faces, unlike in the rest of the article.

For $\lambda \in V^\vee$, we define the hyperplane H_λ and the two open half-spaces H_λ^+ and H_λ^- by

$$H_\lambda = \{x \in V : \langle \lambda, x \rangle = 0\},$$
$$H_\lambda^+ = \{x \in V : \langle \lambda, x \rangle > 0\}, \quad H_\lambda^- = \{x \in V : \langle \lambda, x \rangle < 0\}.$$

The closed half-spaces are given by

$$\overline{H_\lambda^+} = \{x \in V : \langle \lambda, x \rangle \geq 0\} \quad \text{and} \quad \overline{H_\lambda^-} = \{x \in V : \langle \lambda, x \rangle \leq 0\}.$$

Definition A.1.3. A *valuation* on $\mathcal{C}(V)$ with values in an abelian group A is a function $f: \mathcal{C}(V) \to A$ such that $f(\varnothing) = 0$ and that for any $K, K' \in \mathcal{C}(V)$ such that $K \cup K' \in \mathcal{C}(V)$, we have

$$f(K \cup K') + f(K \cap K') = f(K) + f(K'). \tag{A.1}$$

By the definition of $K(V)$, saying that a function $f: \mathcal{C}(V) \to A$ is a valuation is equivalent to saying that there exists a morphism (necessarily unique) $K(V) \to A$ sending $[K]$ to $f(K)$ for every $K \in \mathcal{C}(V)$. We also denote this morphism $K(V) \to A$ by f.

Example A.1.4. By Remark A.1.1, the function $\delta: \mathcal{C}(V) \to K(V^\vee)$ sending $K \in \mathcal{C}(V)$ to $[K^*]$ is a valuation. Thus, it induces a morphism $\delta: K(V) \to K(V^*)$.

We have the following criterion for recognizing valuations on closed convex polyhedral cones. This is known as *Groemer's first extension theorem* and is proved in [15, Theorem 2].

Theorem A.1.5 (Groemer). *Let A be an abelian group and $f: \mathcal{C}(V) \to A$ be a function such that $f(\varnothing) = 0$. Suppose that for every $K \in \mathcal{C}(V)$ and every $\mu \in V^\vee$ the following holds:*

$$f(K) + f(K \cap H_\mu) = f(K \cap \overline{H_\mu^+}) + f(K \cap \overline{H_\mu^-}). \tag{A.2}$$

Then the function f is a valuation.

The main result of this appendix is the following theorem whose proof is in Section A.2. To make the notation more compact in this appendix, we denote the linear span of a subset S of vector space by $\langle S \rangle$.

Theorem A.1.6. (i) *Consider the function $\Delta: \mathcal{C}(V) \to K(V) \otimes_{\mathbb{Z}} K(V)$ defined by*

$$\Delta(K) = \sum_{F \in \mathcal{F}(K)} (-1)^{\dim F} [F] \otimes [\langle F \rangle + K],$$

for every $K \in \mathcal{C}(V)$. Then Δ is a valuation and it induces a morphism

$$\Delta : K(V) \to K(V) \otimes_{\mathbb{Z}} K(V).$$

Moreover, this morphism Δ is coassociative, that is, we have

$$(\Delta \otimes \mathrm{id}_{K(V)}) \circ \Delta = (\mathrm{id}_{K(V)} \otimes \Delta) \circ \Delta.$$

(ii) *Consider the function $\varepsilon : \mathcal{C}(V) \to \mathbb{Z}$ defined by $\varepsilon(K) = (-1)^{\dim K}$ if K is a vector subspace of V and $\varepsilon(K) = 0$ otherwise. Then ε is a valuation and it induces a morphism $\varepsilon : K(V) \to \mathbb{Z}$. This morphism is a counit of Δ, in other words, we have*

$$(\varepsilon \otimes \mathrm{id}_{K(V)}) \circ \Delta = \mathrm{id}_{K(V)} = (\mathrm{id}_{K(V)} \otimes \varepsilon) \circ \Delta.$$

In short, Theorem A.1.6 says that the morphisms $\Delta : K(V) \to K(V) \otimes_{\mathbb{Z}} K(V)$ and $\varepsilon : K(V) \to \mathbb{Z}$ are well-defined and make $K(V)$ into a $\mathbb{Z}$-coalgebra.

Corollary A.1.7. *Let A be a ring. Let $f_1, f_2 : \mathcal{C}(V) \to A$ be two valuations. Then the function $f_1 * f_2 : \mathcal{C}(V) \to A$ defined by*

$$(f_1 * f_2)(K) = \sum_{F \in \mathcal{F}(K)} (-1)^{\dim F} f_1(F) f_2\big(\langle F \rangle + K\big)$$

is also a valuation. This operation $*$ makes the group of valuations $\mathcal{C}(V) \to A$ into a ring. The unit element of this ring is the composition of $\varepsilon : \mathcal{C}(V) \to \mathbb{Z}$ and the canonical ring morphism $\mathbb{Z} \to A$.

Proof. The valuations f_1 and f_2 induce two morphisms $f_1, f_2 : K(V) \to A$, hence a morphism $f_1 \otimes f_2 : K(V) \otimes_{\mathbb{Z}} K(V) \to A$, $x \otimes y \mapsto f_1(x) \otimes f_2(y)$. As we have

$$(f_1 * f_2)(K) = (f_1 \otimes f_2)\big(\Delta\big([K]\big)\big)$$

for every $K \in \mathcal{C}(V)$, this shows that $f_1 * f_2$ descends to a morphism $K(V) \to A$, hence is a valuation.

The operation $*$ is clearly linear in each variable, and it is associative by the coassociativity of Δ. The last statement follows immediately from the fact that ε is a counit of Δ. ■

Corollary A.1.8. *Let A be a ring, and let $f_1 : \mathcal{C}(V) \to A$ and $g_2 : \mathcal{C}(V^\vee) \to A$ be valuations. Then the function $f_1 \star g_2 : \mathcal{C}(V) \to A$ defined by*

$$(f_1 \star g_2)(K) = \sum_{F \in \mathcal{F}(K)} (-1)^{\dim F} f_1(F) g_2(F^{\perp, K})$$

is also a valuation.

Proof. Consider the valuation $\delta: \mathscr{C}(V) \to K(V^\vee)$ of Example A.1.4. Then the map $f_2 = g_2 \circ \delta: \mathscr{C}(V) \to A$ is also a valuation. As $F^{\perp,K} = (\langle F \rangle + K)^*$ for every $K \in \mathscr{C}(V)$ and every face F of K, we have $f_1 \star g_2 = f_1 * f_2$, so the statement follows from Corollary A.1.7. ■

Remark A.1.9. Let $\mathscr{H}$ be a central hyperplane arrangement on V, let $\mathscr{C}_{\mathscr{H}}(V)$ be the set of closed convex polyhedral cones that are intersections of closed half-spaces bounded by hyperplanes of $\mathscr{H}$, and let $K_{\mathscr{H}}(V)$ be the quotient of the free abelian group $\bigoplus_{K \in \mathscr{C}_{\mathscr{H}}(V)} \mathbb{Z}[K]$ on $\mathscr{C}_{\mathscr{H}}(V)$ by the relations

$$[\varnothing] = 0 \quad \text{and} \quad [K] + [K'] = [K \cup K'] + [K \cap K']$$

for all $K, K' \in \mathscr{C}_{\mathscr{H}}(V)$ such that $K \cup K' \in \mathscr{C}_{\mathscr{H}}(V)$. Then the formulas of Theorem A.1.6 also define a coalgebra structure on $K_{\mathscr{H}}(V)$. Indeed, if $K \in \mathscr{C}_{\mathscr{H}}(K)$ and $F \in \mathscr{F}(K)$, then F and $\langle F \rangle + K$ are also in $\mathscr{C}_{\mathscr{H}}(V)$.

In particular, the products in Corollaries A.1.7 and A.1.8 also make sense if the first valuation is only defined on $\mathscr{C}_{\mathscr{H}}(V)$.

We now explain how to use Corollary A.1.8 to recover [13, Proposition A.4].

Lemma A.1.10. *Let X be a subset of V such that the complement $V - X$ is convex. Then the function $\phi_X: \mathscr{C}(V) \to \mathbb{Z}$ defined by*

$$\phi_X(K) = \begin{cases} 1 & \text{if } \varnothing \subsetneq K \subseteq X, \\ 0 & \text{otherwise}, \end{cases}$$

is a valuation. In particular, if $\lambda \in V^\vee$ then the function $\psi_\lambda = \phi_{\overline{H_\lambda^+}}$ is a valuation.

Proof. Let $K \in \mathscr{C}(V)$ be nonempty and let $\mu \in V^\vee$. Let

$$K_0 = K \cap H_\mu, \quad K_+ = K \cap \overline{H_\mu^+}, \quad K_- = K \cap \overline{H_\mu^-}.$$

We must check criterion (A.2) in Theorem A.1.5, that is,

$$\phi_X(K) + \phi_X(K_0) = \phi_X(K_+) + \phi_X(K_-).$$

If $K \subseteq X$ then K_0, K_+ and K_- are also included in X, and the equality above is clear. If $K_+ \subseteq X$ but $K_- \not\subseteq X$, then $K_0 \subseteq X$ and $K \not\subseteq X$, so again the desired equality holds. The case where $K_- \subseteq X$ and $K_+ \not\subseteq X$ is symmetric. Finally, suppose that $K_+, K_- \not\subseteq X$. Then $K \not\subseteq X$, and so we must show that $K_0 \not\subseteq X$. Take $x \in K_+ - X$ and $y \in K_- - X$. Then the segment $[x, y]$ is contained in the convex set $V - X$. As this segment intersects K_0, this shows that $K_0 \not\subseteq X$. ■

Given $x \in V$ and $\lambda \in V^\vee$, we have two valuations

$$\psi_\lambda : \mathcal{C}(V) \to \mathbb{Z} \quad \text{and} \quad \psi_x : \mathcal{C}(V^\vee) \to \mathbb{Z}$$

defined in Lemma A.1.10. Let $K \mapsto \psi_K(x, \lambda)$ be the function defined by

$$\psi_K(x, \lambda) = (\psi_\lambda \star \psi_x)(K)$$

for every $K \in \mathcal{C}(V)$. This function is defined in [13, Appendix A, top of p. 540].

Corollary A.1.11. *For every $x \in V$ and every $\lambda \in V^\vee$, the function $K \mapsto \psi_K(x, \lambda)$ from $\mathcal{C}(V)$ to $\mathbb{R}$ is a valuation.*

Since any valuation satisfies the additivity property, we obtain the next corollary [13, Proposition A.4].

Corollary A.1.12 (Goresky–Kottwitz–MacPherson). *Let K be a closed convex polyhedral cone. Suppose that its relative interior K° is the disjoint union of the relative interiors $K_1^\circ, K_2^\circ, \dots, K_r^\circ$ of r closed convex polyhedral cones $K_1, K_2, \dots, K_r$. Then for every $x \in V$ and every $\lambda \in V^\vee$*

$$\psi_K(x, \lambda) = \sum_{i=1}^{r} (-1)^{\dim(K) - \dim(K_i)} \cdot \psi_{K_i}(x, \lambda).$$

Remark A.1.13. Valuations on $\mathcal{C}(V)$ can be extended to relatively open cones as well. Let G be a collection of sets that is closed under finite intersections. Define $B(G)$ to be the Boolean algebra generated by G, that is, the smallest collection of sets that contains G and is closed under finite unions, finite intersections and complements. Groemer's Integral Theorem states that a valuation on G can be extended to a valuation on the Boolean algebra $B(G)$; see [15] and also [23, Chapter 2]. In the case where $G = \mathcal{C}(V)$, that is, the collection of closed convex polyhedral cones in V, the associated Boolean algebra $B(\mathcal{C}(V))$ contains all cones that are obtained by intersecting closed and open half-spaces.

Remark A.1.14. The results of this appendix extend to oriented matroids without much change. Let E be a finite set and consider an oriented matroid $\mathcal{M}$ on E with set of covectors $\mathcal{L} \subseteq \{+, -, 0\}^E$. This set of covectors forms a graded poset with the partial order given by componentwise comparing the entries by $0 < +$ and $0 < -$. We denote its rank function by ρ. For every $F \subseteq E$ and every $s \in \{+, -, 0\}^F$, we write

$$\mathcal{L}_{\le s} = \{x \in \mathcal{L} : x|_F \le s\}.$$

Let $\mathcal{K}$ be the set of lower order ideals of $\mathcal{L}$ of the form $\mathcal{L}_{\le s}$. We order $\mathcal{K}$ by inclusion. If $\mathcal{L}$ is the oriented matroid corresponding to a central hyperplane arrangement $\mathcal{H}$

on V, then $\mathcal{K}$ is the set cones obtained by intersecting closed half-spaces bounded by hyperplanes of $\mathcal{H}$. In general, every element of $\mathcal{K}$ is of the form $\mathcal{L}_{\leq x|_F}$ for some $x \in \mathcal{L}$ and $F \subseteq E$.

We say that an element a of $\mathcal{K}$ is a vector subspace if $a \neq \varnothing$ and a is of the form $\mathcal{L}_{\leq x|_F}$, for some $x \in \mathcal{L}$ and some $F \subseteq E$ such that $x_e = 0$ for every $e \in F$.

A *valuation* on $\mathcal{K}$ with values in an abelian group A is a function $f : \mathcal{K} \to A$ such that $f(\varnothing) = 0$ and, for all $a, b \in \mathcal{K}$ such that $a \cup b \in \mathcal{K}$, we have

$$f(a \cup b) + f(a \cap b) = f(a) + f(b).$$

Giving such a valuation is equivalent to giving a function $w : \mathcal{L} \to A$; the corresponding valuation then sends $a \in \mathcal{K}$ to $\sum_{x \in a} w(x)$.

The analogue of $K(V)$ is the quotient of the free abelian group $\bigoplus_{a \in \mathcal{K}} \mathbb{Z}[a]$ by the relations

$$[\varnothing] = 0 \quad \text{and} \quad [a \cup b] + [a \cap b] = [a] + [b]$$

if $a \cup b \in \mathcal{K}$. We denote this group by $K(\mathcal{L})$. We have an isomorphism

$$K(\mathcal{L}) \xrightarrow{\sim} \bigoplus_{x \in \mathcal{L}} \mathbb{Z}[x]$$

sending $[a]$ to $\sum_{x \in a} [x]$.

Let $F \subseteq E$. We denote the set of covectors of the deletion $\mathcal{M} - (E - F)$ by $\mathcal{L}_F$ and the rank function of $\mathcal{L}_F$ by ρ_F. Let $y \in \mathcal{L}_F$. If $F(y) = \{e \in F : y_e = 0\}$, then we have an order-preserving bijection $\mathcal{L}_{F, \geq y} \xrightarrow{\sim} \mathcal{L}_{F(y)}$ sending any $z \geq y$ in $\mathcal{L}_F$ to $z|_{F(y)}$. This is the analogue of Lemma 2.1.3.

We define the comultiplication $\Delta : K(\mathcal{L}) \to K(\mathcal{L}) \otimes_{\mathbb{Z}} K(\mathcal{L})$ by sending $[\mathcal{L}_{\leq x|_F}]$ to

$$\Delta\big([\mathcal{L}_{\leq x|_F}]\big) = \sum_{y \in \mathcal{L}_{F, \leq x|_F}} (-1)^{\rho_F(y)} [\mathcal{L}_{\leq y}] \otimes [\mathcal{L}_{\leq x|_{F(y)}}].$$

Let $a \in \mathcal{L}$. The counit ε sends $[a]$ to 0 if a is not a vector subspace. If $a = \mathcal{L}_{\leq x|_F}$, with $x \in \mathcal{L}$ and $F \subseteq E$ such that $x_e = 0$ for every $e \in F$, then we set $\varepsilon([a]) = (-1)^{\rho_F(x|_F)}$.

Remark A.1.15. Let $\mathcal{K} = \bigoplus_{n \geq 0} K(\mathbb{R}^n)$. We make $\mathcal{K}$ into a coalgebra using the direct sum of the morphisms Δ and ϵ of Theorem A.1.6. There is a product on $\mathcal{K}$ defined by

$$[K] \cdot [L] = [K \times L]$$

if $K \in \mathcal{C}(\mathbb{R}^n)$ and $L \in \mathcal{C}(\mathbb{R}^m)$, where we identity $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^{n+m}$ in the usual way. This product is associative, and the class of the cone $\{0\} \in \mathcal{C}(\mathbb{R}^0)$ is a unit. It is then straightforward to see that $\mathcal{K}$ is actually a bialgebra. However, it is not a Hopf algebra because if V is a vector subspace of $\mathbb{R}^n$ with $n \geq 1$, then the element $(-1)^{\dim V}[V]$ of $\mathcal{K}$ is group-like but not invertible.

If $K \in \mathcal{C}(\mathbb{R}^n)$ and F is a face of K, then the poset of faces of $\langle F\rangle + K$ is isomorphic to the interval $[F, K]$ in the poset of faces of K. So the bialgebra $\mathcal{K}$ is related to the incidence Hopf algebras defined by Joni and Rota in [22] and further studied by Schmitt in [32], although, unlike those Hopf algebras, it has signs in the definition of its coproduct. Let us make this relation more precise. For every $n \geq 0$, we denote by $K_f(\mathbb{R}^n)$ the free abelian group on the set of closed convex polyhedral cones in $\mathbb{R}^n$; if $K \in \mathcal{C}(\mathbb{R}^n)$, we denote its class in $K_f(\mathbb{R}^n)$ by $[K]_f$. The formulas for Δ and ϵ also define a coalgebra structure on $K_f(\mathbb{R}^n)$ and, if we set

$$\mathcal{K}_f = \bigoplus_{n\geq 0} K_f(\mathbb{R}^n),$$

then the product on $\mathcal{K}_f$ defined by $[K]_f \cdot [L]_f = [K \times L]_f$ makes $\mathcal{K}_f$ into a coalgebra. Let $\mathcal{P}$ be the set of isomorphism classes of finite posets, and let $\mathbb{Z}[\mathcal{P}]$ be the free abelian group on $\mathcal{P}$ equipped with the Hopf algebra structure defined in Sections 3 and 4 of [32]. Then we have bialgebra morphisms $\pi_1 \colon \mathcal{K}_f \to \mathcal{K}$ and $\pi_2 \colon \mathcal{K}_f \to \mathbb{Z}[\mathcal{P}]$ defined as follows: if $K \in \mathcal{C}(\mathbb{R}^n)$ then π_1 sends $[K]_f$ to $[K]$ and π_2 sends $[K]_f$ to $(-1)^d$ times the class of the poset of faces of K, where d is the dimension of the largest vector subspace contained in K.

A.2. Proofs

Before proving Theorem A.1.6, we state and prove the following lemma.

Lemma A.2.1. *Let $K \subseteq V$ be a closed convex polyhedral cone, let F be a closed face of the cone K and let $\mu \in V^\vee$. We write*

$$K_0 = K \cap H_\mu, \quad K_+ = K \cap \overline{H_\mu^+}, \quad K_- = K \cap \overline{H_\mu^-}.$$

(a) *Assume that $F \subseteq \overline{H_\mu^+}$ but $F \not\subseteq H_\mu$, that is, F is a face of K_+ but not of K_0. Then the equality $\langle F\rangle + K = \langle F\rangle + K_+$ holds.*

(b) *Assume that $F \cap H_\mu^+ \neq \varnothing$ and $F \cap H_\mu^- \neq \varnothing$, in other words, the hyperplane H_μ cuts the face F in two. Then the equality $\langle F\rangle + K = \langle F\rangle + K_0$ holds.*

(c) *In the situation of (b), let $F_0 = F \cap H_\mu$. Then the equality $\langle F_0\rangle + K = \langle F\rangle + K$ holds.*

(d) *Let X be a subset of V. Then $X + K = (X + K_+) \cup (X + K_-)$. If moreover $X \subseteq H_\mu$, we also have $X + K_0 = (X + K_+) \cap (X + K_-)$.*

Proof. We first prove (a). The inclusion $\langle F\rangle + K_+ \subseteq \langle F\rangle + K$ clearly holds, so we just need to show the reverse inclusion. Let $x \in \langle F\rangle + K$, and write $x = y + z$, with

$y \in \langle F \rangle$ and $z \in K$. As $F \not\subseteq H_\mu$, there exists $y' \in F$ such that $\langle \mu, y' \rangle > 0$. Then for $a \in \mathbb{R}_{>0}$ large enough we have $\langle \mu, ay' + z \rangle \geq 0$, hence $ay' + z \in K_+$. As

$$x = (y - ay') + (ay' + z),$$

this shows that $x \in \langle F \rangle + K_+$.

We now prove (b). The inclusion $\langle F \rangle + K_0 \subseteq \langle F \rangle + K$ clearly holds, so we just need to verify the reverse inclusion. Let $x \in \langle F \rangle + K$, and write $x = y + z$ with $y \in \langle F \rangle$ and $z \in K$. By the assumption on F, the image of F by μ is not contained in $\mathbb{R}_{\geq 0}$ or in $\mathbb{R}_{\leq 0}$; as this image is a cone in $\mathbb{R}$, we conclude that it is equal to $\mathbb{R}$. In particular, we can find $y' \in F$ such that $\langle \mu, y' \rangle = -\langle \mu, z \rangle$. Then

$$x = (y - y') + (y' + z)$$

with $y - y' \in \langle F \rangle$, $y' + z \in K$ and $\langle \mu, y' + z \rangle = 0$, hence $x \in \langle F \rangle + K_0$.

We prove (c). The inclusion $\langle F_0 \rangle + K \subseteq \langle F \rangle + K$ is clear, so we need to show the reverse inclusion. By the proof of (b), the image of F by μ is equal to $\mathbb{R}$, so we can find $y' \in F$ such that $\langle \mu, y' \rangle = \langle \mu, y \rangle$. Then

$$x = (y - y') + (y' + z)$$

with $y - y' \in \langle F \rangle$, $y' + z \in K$ and $\langle \mu, y - y' \rangle = 0$, hence $x \in \langle F_0 \rangle + K$.

Finally, we prove (d). The inclusion $(X + K_+) \cup (X + K_-) \subseteq X + K$ is clear. Conversely, let $x \in X + K$, and write $x = y + z$ with $y \in X$ and $z \in K$. Then either $z \in K_+$, in which case $x \in X + K_+$, or $z \in K_-$, in which case $x \in X + K_-$. The inclusion $X + K_0 \subseteq (X + K_+) \cap (X + K_-)$ is also clear and holds without any condition on X. Assume that $X \subseteq H_\mu$, and let $x \in (X + K_+) \cap (X + K_-)$. Write

$$x = y_1 + z_1 = y_2 + z_2,$$

with $y_1, y_2 \in X$, $z_1 \in K_+$ and $z_2 \in K_-$. Then $\langle \mu, y_1 \rangle = \langle \mu, y_2 \rangle = 0$, so

$$\langle \mu, z_1 \rangle = \langle \mu, x \rangle - \langle \mu, y_1 \rangle = \langle \mu, x \rangle - \langle \mu, y_2 \rangle = \langle \mu, z_2 \rangle.$$

As $\langle \mu, z_1 \rangle \geq 0$ and $\langle \mu, z_2 \rangle \leq 0$, this implies that $\langle \mu, z_1 \rangle = \langle \mu, z_2 \rangle = 0$, hence that $z_1, z_2 \in K_0$, and so $x \in X + K_0$. ■

Proof of Theorem A.1.6. We first show that $\Delta\colon \mathcal{C}(V) \to K(V) \otimes_{\mathbb{Z}} K(V)$ and ε are valuations. We check the criterion of Theorem A.1.5. Let $K \in \mathcal{C}(V)$ and let $\mu \in V^\vee$. We define as before three closed convex polyhedral cones

$$K_+ = K \cap \overline{H_\mu^+}, \quad K_- = K \cap \overline{H_\mu^-}, \quad K_0 = K \cap H_\mu.$$

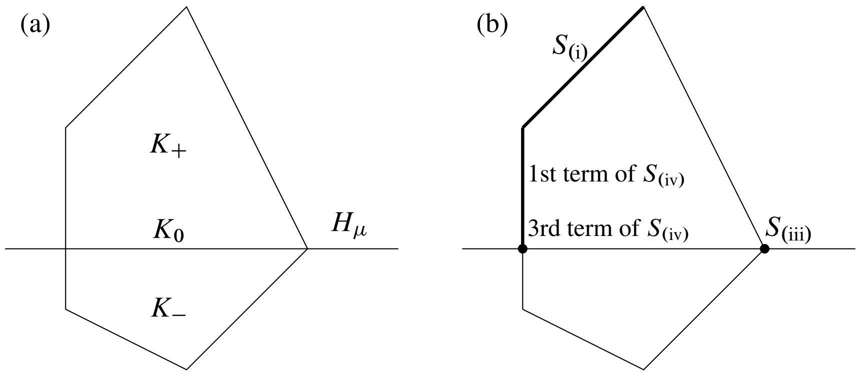

Figure 2. A two-dimensional representation of a five-sided three-dimensional cone K. In (b) the four different contributions to $g(K_+)$ are marked.

We show that

$$\varepsilon(K) + \varepsilon(K_0) = \varepsilon(K_+) + \varepsilon(K_-). \tag{A.3}$$

If $K_+ = K_0$, then $K = K_-$, so equation (A.3) is clear. The case where $K_- = K_0$ is similar. Suppose that $K_+ \neq K_0$ and $K_- \neq K_0$. Then the image of K_+ by μ is $\mathbb{R}_{\geq 0}$, so K_+ cannot be a vector subspace of V, and similarly K_- cannot be a vector subspace of V. This implies that

$$\varepsilon(K_+) = \varepsilon(K_-) = 0,$$

so equation (A.3) holds if and only if $\varepsilon(K) = -\varepsilon(K_0)$. As K_0 is strictly included in K, we have $\dim(K_0) = \dim(K) - 1$, so we need to prove that K is a vector subspace if and only if K_0 is. If K is a vector subspace of V then so is K_0. Suppose that K_0 is a vector subspace of V. We want to prove that K also is a vector subspace of V. Without loss of generality we may assume that $\langle K \rangle = V$. As $K_+ \neq K_0$ and $K_- \neq K_0$, the hyperplane H_μ meets the relative interior of K, and so $\langle K_0 \rangle = H_\mu$, hence

$$H_\mu = K_0 \subseteq K.$$

As K contains points in both open half-spaces cut out by H_μ, this implies that $K = V$.

We now treat the case of Δ. The faces F of the cone K come in four disjoint categories. For each category, we consider the contribution to the sum defining $\Delta(K)$.

(i) F is a face of K_+, but not of K_0, that is, $F^\circ \subseteq H_\mu^+$. Then by Lemma A.2.1 (a), we have $\langle F \rangle + K = \langle F \rangle + K_+$. Hence, the contribution is

$$S_{(\mathrm{i})} = \sum_{\substack{F \in \mathcal{F}(K) \cap \mathcal{F}(K_+) \\ F \notin \mathcal{F}(K_0)}} (-1)^{\dim(F)} \cdot [F] \otimes [\langle F \rangle + K]$$

$$= \sum_{F \in (\mathscr{F}(K_+) \cap \mathscr{F}(K)) - \mathscr{F}(K_0)} (-1)^{\dim(F)} \cdot [F] \otimes [\langle F \rangle + K_+].$$

(ii) F is a face of K_-, but not of K_0, that is, $F^\circ \subseteq H_\mu^-$. As in case (i), we have $\langle F \rangle + K = \langle F \rangle + K_-$, and the contribution is

$$\begin{aligned} S_{\text{(ii)}} &= \sum_{\substack{F \in \mathscr{F}(K) \cap \mathscr{F}(K_-) \\ F \notin \mathscr{F}(K_0)}} (-1)^{\dim(F)} \cdot [F] \otimes [\langle F \rangle + K] \\ &= \sum_{F \in (\mathscr{F}(K_-) \cap \mathscr{F}(K)) - \mathscr{F}(K_0)} (-1)^{\dim(F)} \cdot [F] \otimes [\langle F \rangle + K_-]. \end{aligned}$$

(iii) F is a face of all three cones K_+, K_- and K_0, that is, we have $F \subseteq H_\mu$. Here the contribution is

$$\begin{aligned} S_{\text{(iii)}} &= \sum_{\substack{F \in \mathscr{F}(K) \\ F \subseteq H_\mu}} (-1)^{\dim(F)} \cdot [F] \otimes [\langle F \rangle + K] \\ &= \sum_{F \in \mathscr{F}(K_+) \cap \mathscr{F}(K_-) \cap \mathscr{F}(K_0)} (-1)^{\dim(F)} \cdot [F] \otimes \big([\langle F \rangle + K_+] + [\langle F \rangle + K_-] - [\langle F \rangle + K_0]\big), \end{aligned}$$

since

$$\langle F \rangle + K = \big(\langle F \rangle + K_+\big) \cup \big(\langle F \rangle + K_-\big), \quad \langle F \rangle + K_0 = \big(\langle F \rangle + K_+\big) \cap \big(\langle F \rangle + K_-\big)$$

by Lemma A.2.1 (d).

(iv) The face F gets cut into three faces: $F_+ = F \cap K_+$ in K_+, $F_- = F \cap K_-$ in K_- and $F_0 = F \cap K_0$ in K_0. Then we have

$$\langle F \rangle = \langle F_+ \rangle = \langle F_- \rangle.$$

By Lemma A.2.1 (b), we have $\langle F \rangle + K = \langle F \rangle + K_0$, and so

$$\langle F \rangle + K = \langle F_+ \rangle + K_+ = \langle F_- \rangle + K_- = \langle F \rangle + K_0.$$

By Lemma A.2.1 (c), we also have $\langle F \rangle + K = \langle F_0 \rangle + K$, and by Lemma A.2.1 (d) we have

$$[\langle F_0 \rangle + K] = [\langle F_0 \rangle + K_+] + [\langle F_0 \rangle + K_-] - [\langle F_0 \rangle + K_0].$$

So the contribution is

$$S_{\text{(iv)}} = \sum_{\substack{F \in \mathscr{F}(K) \\ F \text{ being cut}}} (-1)^{\dim(F)} \cdot [F] \otimes [\langle F \rangle + K]$$

$$
\begin{aligned}
&= \sum_{\substack{F\in\mathcal{F}(K)\\ F \text{ being cut}}} (-1)^{\dim(F)} \cdot \big([F_+]+[F_-]-[F_0]\big) \otimes [\langle F\rangle + K] \\
&= \sum_{\substack{F\in\mathcal{F}(K)\\ F \text{ being cut}}} (-1)^{\dim(F)} \cdot \big([F_+]\otimes[\langle F_+\rangle + K_+] + [F_-]\otimes[\langle F_-\rangle + K_-] \\
&\qquad - [F_0]\otimes[\langle F_0\rangle + K]\big) \\
&= \sum_{\substack{F\in\mathcal{F}(K)\\ F \text{ being cut}}} (-1)^{\dim(F)} \cdot \big([F_+]\otimes[\langle F_+\rangle + K_+] + [F_-]\otimes[\langle F_-\rangle + K_-] \\
&\qquad - [F_0]\otimes[\langle F_0\rangle + K_+] - [F_0]\otimes[\langle F_0\rangle + K_-] + [F_0]\otimes[\langle F_0\rangle + K_0]\big).
\end{aligned}
$$

Now expand $\Delta(K)$ as $S_{(i)} + S_{(ii)} + S_{(iii)} + S_{(iv)}$. We use the fact that

$$(-1)^{\dim(F)} = -(-1)^{\dim(F_0)}$$

in the third, fourth and fifth terms of $S_{(iv)}$. The contributions to $\Delta(K_+)$, respectively, $\Delta(K_-)$, are given by the sum $S_{(i)}$, respectively, $S_{(ii)}$, the first term in the sum $S_{(iii)}$, respectively, the second term, and the first and third terms in the sum $S_{(iv)}$, respectively, the second and fourth terms; see Figure 2 (b). Finally, the third term of the sum $S_{(iii)}$ and the fifth term of the sum $S_{(iv)}$ yield the sum for $-\Delta(K_0)$, which proves that

$$\Delta(K) = \Delta(K_+) + \Delta(K_-) - \Delta(K_0).$$

We now prove that Δ is coassociative. Let $K \in \mathcal{C}(V)$. Then we have

$$
\begin{aligned}
(\Delta\otimes \mathrm{id}_{K(V)})(\Delta(K)) &= (\Delta\otimes \mathrm{id}_{K(V)})\Bigg(\sum_{F\in\mathcal{F}(K)} (-1)^{\dim F}[F]\otimes[\langle F\rangle + K]\Bigg) \\
&= \sum_{F\in\mathcal{F}(K)} \sum_{G\in\mathcal{F}(F)} (-1)^{\dim F + \dim G}[G]\otimes[\langle G\rangle + F]\otimes[\langle F\rangle + K].
\end{aligned}
$$

We want to compare this expression with $(\mathrm{id}_{K(V)}\otimes\Delta)(\Delta([K]))$. To calculate this last expression, we need a description of the faces of the cone $\langle G\rangle + K$, where G is a face of K. Let $\mathcal{H}$ be the collection of hyperplanes containing a facet of K. Then $\mathcal{H}$ is a finite central hyperplane arrangement in V and, as in Section 2.1, we write $\mathcal{L} = \mathcal{L}(\mathcal{H})$ and $\mathcal{T} = \mathcal{T}(\mathcal{H})$. Let C and T be the relative interiors of G and K respectively. We have $C \in \mathcal{L}$ and $T \in \mathcal{T} \cap \mathcal{L}_{\geq C}$, and there is a bijection

$$\{D \in \mathcal{L}_{\geq C} : D \leq T\} \xrightarrow{\sim} \{F \in \mathcal{F}(K) : G \subseteq F\}$$

sending D to $\overline{D}$. Let $\mathcal{H}(C)$ be the subarrangement of $\mathcal{H}$ whose hyperplanes are the ones containing C (or equivalently, G). By Lemma 2.1.3, the cone $\langle G\rangle + K$ is the closure of the unique chamber of $\mathcal{H}(C)$ containing T, and there is a bijection from the set $\{D \in \mathcal{L}_{\geq C} : D \leq T\}$ to the set of faces of $\langle G \rangle + K$ sending D to $\langle G \rangle + \overline{D}$. We deduce that there is a bijection from the set $\{F \in \mathcal{F}(K) : G \subseteq F\}$ to $\mathcal{F}(\langle G \rangle + K)$ sending F to $\langle G \rangle + F$. Moreover, Lemma 2.1.3 (iv) states that this bijection preserves dimensions. Thus, we obtain

$$\begin{aligned}
(\mathrm{id}_{K(V)} \otimes \Delta)(\Delta(K)) &= (\mathrm{id}_{K(V)} \otimes \Delta)\Big(\sum_{G \in \mathcal{F}(K)} (-1)^{\dim G} [G] \otimes [\langle G \rangle + K] \Big) \\
&= \sum_{G \in \mathcal{F}(K)} \sum_{F' \in \mathcal{F}(\langle G \rangle + K)} (-1)^{\dim F' + \dim G} [G] \otimes [F'] \otimes [\langle F' \rangle + K] \\
&= \sum_{G \in \mathcal{F}(K)} \sum_{F \in \mathcal{F}(K) : G \subseteq F} (-1)^{\dim F + \dim G} [G] \otimes [\langle G \rangle + F] \otimes [\langle F \rangle + K] \\
&= (\Delta \otimes \mathrm{id}_{K(V)})(\Delta(K)).
\end{aligned}$$

This completes the proof of the coassociativity of Δ.

We finally prove that ε is a counit of Δ. Let $K \in \mathcal{C}(V)$. Suppose first that K is not a vector subspace of V. Then the only face of K that is a vector subspace is $\{0\}$, and the only face F such that $\langle F \rangle + K$ is a vector subspace is K. Hence,

$$\begin{aligned}
(\mathrm{id}_{K(V)} \otimes \varepsilon)(\Delta(K)) &= \sum_{F \in \mathcal{F}(K)} (-1)^{\dim F} [F] \otimes \varepsilon\big([\langle F \rangle + K]\big) \\
&= (-1)^{\dim K} [K] \otimes \varepsilon\big([\langle K \rangle]\big) \\
&= (-1)^{\dim K} (-1)^{\dim \langle K \rangle} [K] \otimes 1 = [K], \\
(\varepsilon \otimes \mathrm{id}_{K(V)})(\Delta(K)) &= \sum_{F \in \mathcal{F}(K)} (-1)^{\dim F} \varepsilon\big([F]\big) \otimes [\langle F \rangle + K] \\
&= (-1)^0 \varepsilon\big([\{0\}]\big) \otimes [K] = [K].
\end{aligned}$$

If K is a vector subspace of V then the only face of K is K itself, so

$$\Delta(K) = (-1)^{\dim K} [K] \otimes [K],$$

and we clearly have

$$(\mathrm{id}_{K(V)} \otimes \varepsilon)(\Delta(K)) = (\mathrm{id}_{K(V)} \otimes \varepsilon)(\Delta(K)) = [K]. \qquad \blacksquare$$

B. Review of 2-structures

The concept of 2-structure for a root system was introduced by Herb to calculate discrete series characters on real reductive groups; see, for example, Section 5 of [20] or Section 4 of the review article [19]. In this section we review Herb's constructions and adapt them so that they work for an arbitrary Coxeter system having finite Coxeter group. We also adapt some of her results to this setting and give detailed elementary proofs of these results. Although this is not strictly necessary, we think that it might be valuable, as the proofs of these results in the literature can be very hard to follow for people not already immersed in the representation theory of real groups.

We fix a finite-dimensional $\mathbb{R}$-vector space V and an inner product $(\cdot,\cdot)$ on V. For every $v \in V - \{0\}$, we denote by s_v the (orthogonal) reflection across the hyperplane $v^\perp$.

Whenever we need to describe the irreducible root systems, we use the description given in the tables at the end of [5], except that we write $(e_1, \dots, e_n)$ for the canonical basis of $\mathbb{R}^n$. When we need a system of positive roots in these root systems, we also use the ones given in these tables.

This appendix is organized as follows. Sections B.1 and B.2 contain the definitions and results respectively. Sections B.3 and B.4 contain the technical proofs. The verification that the Coxeter group W acts transitively on the set of 2-structures takes place in the fourth subsection.

B.1. Pseudo-root systems

Definition B.1.1. A finite subset Φ of $V - \{0\}$ is called a *pseudo-root system* if it satisfies the following conditions:

(a) for every $\alpha \in \Phi$, we have $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$;

(b) for every $\alpha, \beta \in \Phi$, the reflection s_α sends β to a vector of the form $c\gamma$, with $c \in \mathbb{R}_{>0}$ and $\gamma \in \Phi$.

If all the elements of Φ are unit vectors, we call Φ a *normalized pseudo-root system*. In that case, condition (b) becomes "$s_\alpha(\beta) \in \Phi$".

Remark B.1.2. We use this definition because it is convenient in the context of Coxeter systems. A root system (in the usual sense) is a pseudo-root system, which is not normalized in general. The converse is not true, even if we allow ourselves to replace the elements of Φ by scalar multiples, because of the existence of noncrystallographic Coxeter systems; see Proposition B.1.6.

Pseudo-root systems are called "root systems" in [21, Section 1.2] and [2, Section 4.4]. We avoid this terminology because it is not compatible with the established definition of root systems in representation theory.

Remark B.1.3. If Φ is normalized or an actual root system then the group W preserves Φ, so the action of W on V restricts to an action of W on Φ. In general, we can still make W act on Φ by declaring that if $w \in W$ and $\alpha \in \Phi$ then $w \cdot \alpha$ is the unique element β of Φ such that $w(\alpha) \in \mathbb{R}_{>0}\beta$. This reduces to the previous action if Φ is normalized or an actual root system. Whenever we write an element of W acting on an element of Φ, this is the action that we mean.

Definition B.1.4. Let $\Phi \subset V$ be a pseudo-root system. A subset Δ of Φ is called a *system of simple pseudo-roots* if

(a) The set Δ is a vector space basis for the linear span of Φ.

(b) For every $\alpha \in \Phi$, we can write $\alpha = \sum_{\beta \in \Delta} n_\beta \beta$, where the coefficients n_β are in $\mathbb{R}$ and they are either all nonnegative or all nonpositive.

The corresponding *system of positive pseudo-roots* is then

$$\Phi^+ = \Phi \cap \left\{ \sum_{\beta \in \Delta} n_\beta \beta : n_\beta \in \mathbb{R}_{\geq 0} \ \forall \beta \in \Delta \right\}.$$

We also write $\Phi^- = -\Phi^+$.

Definition B.1.5. Let $\Phi \subset V$ be a pseudo-root system. We say that Φ is *irreducible* if there is no partition $\Phi = \Phi_1 \sqcup \Phi_2$, with Φ_1 and Φ_2 nonempty pseudo-root systems such that $(\alpha_1, \alpha_2) = 0$ for every $\alpha_1 \in \Phi_1$ and every $\alpha_2 \in \Phi_2$.

Proposition B.1.6. *The following two statements hold:*

(i) *See* [21, Section 1.9] *and* [21, Section 1.4]. *Let $\Phi \subset V$ be a pseudo-root system and $\Delta \subset \Phi$ be a system of simple pseudo-roots. Let $W = W(\Phi)$ be the subgroup of $\mathbf{GL}(V)$ generated by the reflections s_α for $\alpha \in \Phi$, and let $S = \{s_\alpha : \alpha \in \Delta\}$. Then (W, S) is a Coxeter system where W is finite, and the Coxeter graph of (W, S) is connected if and only if Φ is irreducible.*

Moreover, W acts transitively on the set of systems of positive pseudo-roots if we use the action of Remark B.1.3.

(ii) *See* [21, Section 5.4]. *Conversely, let (W, S) be a Coxeter system with W finite, and let $\rho \colon W \to \mathbf{GL}(V)$ be its canonical representation on $V = \bigoplus_{s \in S} \mathbb{R} e_s$; see the beginning of Section* 2.2. *Then $\Phi = \{\rho(w)(e_s) : w \in W,\ s \in S\}$ is a normalized pseudo-root system and $\Delta = \{e_s : s \in S\}$ is a system of simple pseudo-roots in Φ.*

Definition B.1.7. Let $\Phi \subset V$ be an irreducible pseudo-root system. We say that Φ is of type A_n, respectively, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, $I_2(m)$ with $m \geq 3$, if the corresponding Coxeter system is of that type. Here we use the classification of simple finite Coxeter systems proved in [16, Chapter 5]; see Table 1 in [2, Appendix A].

Remark B.1.8. The Coxeter group of type $I_2(m)$ is the dihedral group of order $2m$. Note that types $I_2(3)$ and A_2 are isomorphic, types $I_2(4)$ and B_2 are isomorphic, and types $I_2(6)$ and G_2 are isomorphic. We did not include $I_2(2)$ in the list of irreducible types, because the corresponding Coxeter system is not irreducible, as it is isomorphic to $A_1 \times A_1$.

We will use the following lemma when introducing the sign associated to a 2-structure in Proposition B.2.7. Recall that, if $r \geq 1$, then the *lexicographic order* on $\mathbb{R}^r$ is defined by $(x_1, \ldots, x_r) < (y_1, \ldots, y_r)$ if there exists $1 \leq i \leq r$ such that $x_i < y_i$ and that $x_j = y_j$ for $1 \leq j \leq i-1$. It is a total order. Furthermore, we say that a vector x is *positive* if $x > (0, 0, \ldots, 0)$.

Lemma B.1.9. *Let $\Phi \subset V$ be a pseudo-root system. Let $v_1, v_2, \ldots, v_r$ be linearly independent elements of V such that no element of Φ is orthogonal to every v_i. Define Φ^+ to be the set of $\alpha \in \Phi$ such that the element $((\alpha, v_1), (\alpha, v_2), \ldots, (\alpha, v_r))$ of $\mathbb{R}^r$ is positive with respect to the lexicographic order on $\mathbb{R}^r$. Then Φ^+ is a system of positive pseudo-roots.*

Proof. We complete $(v_1, \ldots, v_r)$ to a basis $(v_1, \ldots, v_n)$ of V, where n is the dimension of V. For $v, w \in V$, we say that $v < w$ if

$$((v, v_1), \ldots, (v, v_n)) < ((w, v_1), \ldots, (w, v_n))$$

in the lexicographic order on $\mathbb{R}^n$. This defines a total order on V in the sense of [21, Section 1.3], and Φ^+ is the corresponding positive system in Φ. By the theorem in [21, Section 1.3], Φ^+ is a system of positive pseudo-roots in the sense of Definition B.1.4. ∎

Definition B.1.10. If $\theta = (v_1, \ldots, v_r)$ is a sequence of linearly independent elements of V such that $\theta^\perp \cap \Phi = \varnothing$, we denote the system of positive pseudo-roots of Lemma B.1.9 by Φ_θ^+.

B.2. 2-structures

We define 2-structures, generalizing a notion introduced by Herb for root systems; see, for example, the beginning of [19, Section 4]. We also generalize some of the results of [20, Section 5] to Coxeter systems with finite Coxeter groups.

We fix a pseudo-root system Φ in V and a system of positive pseudo-roots $\Phi^+ \subset \Phi$. We denote by (W, S) the corresponding Coxeter system; see Proposition B.1.6.

Definition B.2.1. A 2-*structure* for Φ is a subset φ of Φ, that is, a pseudo-root system in V satisfying the following properties:

(a) The subset φ is a disjoint union $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$, where the φ_i are pairwise orthogonal subsets of φ and each of them is an irreducible pseudo-root system in V of type A_1, B_2 or $I_2(2^n)$ for $n \geq 3$.

(b) Let $\varphi^+ = \varphi \cap \Phi^+$. If $w \in W$ is such that $w(\varphi^+) = \varphi^+$, then $\det(w) = 1$.

Remark B.2.2. Although condition (b) involves the set of positive pseudo-roots φ^+ in φ, it does not actually depend on the choice of φ^+, because the Coxeter group of φ acts transitively on sets of positive pseudo-roots in φ.

Remark B.2.3. If $\varphi \subseteq \Phi$ is a 2-structure then there is no $\alpha \in \Phi$ that is orthogonal to every element of φ. Indeed, if such an α existed then the associated reflection s_α would fix every element of φ, and in particular send φ^+ to itself, which would contradict condition (b) of Definition B.2.1.

Let $\mathcal{T}(\Phi) \subseteq 2^\Phi$ be the set of all 2-structures for the pseudo-root system Φ. The following proposition is proved in Section B.4, where we also show that each irreducible pseudo-root system contains a 2-structure and give the type of this 2-structure. This introduces no circularity in the arguments: the only results in this appendix that depend on Proposition B.2.4 are Lemmas B.2.11 and B.2.12, and these lemmas are not used in Sections B.3 and B.4.

Proposition B.2.4. *The group W acts transitively on the collection of 2-structures $\mathcal{T}(\Phi)$.*

Let $\varphi \in \mathcal{T}(\Phi)$. We write $\varphi^+ = \varphi \cap \Phi^+$ and $\varphi^- = \varphi \cap \Phi^-$, and we define

$$W(\varphi, \Phi^+) = \{w \in W : w(\varphi^+) \subset \Phi^+\},$$
$$W_1(\varphi, \Phi^+) = \{w \in W : w(\varphi^+) \subset \varphi^+\} = \{w \in W : w(\varphi^+) = \varphi^+\}.$$

Note that $W_1(\varphi, \Phi^+)$ is a subgroup of W, and that the subset $W(\varphi, \Phi^+)$ of W is stable by right translations by elements of $W_1(\varphi, \Phi^+)$.

Corollary B.2.5. *Let $\varphi \in \mathcal{T}(\Phi)$. Then the map $W \to \mathcal{T}(\Phi)$, $w \mapsto w(\varphi)$ induces a bijection*

$$W(\varphi, \Phi^+)/W_1(\varphi, \Phi^+) \xrightarrow{\sim} \mathcal{T}(\Phi).$$

Proof. We denote by $f : W \to \mathcal{T}(\Phi)$ the map defined by $f(w) = w(\varphi)$.

If $u \in W_1(\varphi, \Phi^+)$, then $u(\varphi) = \varphi$, so $f(wu) = f(w)$ for every $w \in W$. So the map f does induce a map from $W(\varphi, \Phi^+)/W_1(\varphi, \Phi^+)$ to $\mathcal{T}(\Phi)$, that we denote by $\overline{f}$.

We show that $\overline{f}$ is surjective. Let $\varphi' \in \mathcal{T}(\Phi)$. By Proposition B.2.4, there exists $w \in W$ such that $w(\varphi) = \varphi'$. By the theorem in [21, Section 1.3], the set $w^{-1}(\Phi^+) \cap \varphi$ is a system of positive pseudo-roots in φ, and so, by Proposition B.1.6, there exists $v \in W(\varphi)$, where $W(\varphi)$ is the Coxeter group of φ, such that $v(\varphi^+) = w^{-1}(\Phi^+) \cap \varphi$. Then

$$wv(\varphi^+) = \Phi^+ \cap w(\varphi) \subset \Phi^+,$$

so $wv \in W(\varphi, \Phi^+)$, and $wv(\varphi) = w(\varphi) = \varphi'$, that is, $f(wv) = \varphi'$.

We show that $\overline{f}$ is injective. Let $w, w' \in W(\varphi, \Phi^+)$ such that $w(\varphi) = w'(\varphi)$. Then we have $w^{-1}w'(\varphi) = \varphi$, and again by the theorem in [21, Section 1.3], the set

$w^{-1}w'(\varphi^+)$ is a system of positive pseudo-roots in φ, so there exists $v \in W(\varphi)$ such that $v^{-1}w^{-1}w'(\varphi^+) = \varphi^+$. This means that we have $w' = wvu$ with $u \in W_1(\varphi, \Phi^+)$. So we will be done if we show that $v = 1$. Note that

$$wv(\varphi^+) = wvu(\varphi^+) = w'(\varphi^+) \subset \Phi^+.$$

Suppose that $v \neq 1$; then there exists $\alpha \in \varphi^+$ such that $v(\alpha) \in \varphi^-$, and then $wv(\alpha) = -w(-v(\alpha)) \in \Phi^-$ (because $w \in W(\varphi, \Phi^+)$), contradicting the fact that $wv(\varphi^+) \subset \Phi^+$. So $v = 1$. ∎

The following proposition, which follows immediately from Lemma 5.6 of [20] for root systems, can be proved via a direct calculation for the remaining irreducible types. We will not need this result, so we do not go into details.

Proposition B.2.6. *Let $\mathcal{T}_{(a)}(\Phi)$ be the set of $\varphi \subseteq \Phi$ that satisfy condition* (a) *of Definition B.2.1. Then $\mathcal{T}(\Phi)$ is exactly the set of elements of $\mathcal{T}_{(a)}(\Phi)$ that are maximal with respect to inclusion.*

Proposition B.2.7. *Let $\varphi \subseteq \Phi$ be a 2-structure. Define an ordered subset θ of φ as follows. Select a linear order of the irreducible components $\varphi_1, \varphi_2, \ldots, \varphi_r$ of φ. If φ_i is a pseudo-root system of type A_1, let θ_i be the singleton $\varphi_i \cap \varphi^+$. If φ_i is a pseudo-root system of type B_2 or $I_2(2^k)$ for $k \geq 3$, pick two orthogonal elements α and α' from $\varphi_i \cap \varphi^+$ such that $\varphi_i \cap \varphi^+ = \varphi^+_{i,(\alpha,\alpha')}$, that is, such that an element β of φ_i is in φ^+ if and only if either $(\beta, \alpha) > 0$, or $(\beta, \alpha) = 0$ and $(\beta, \alpha') > 0$. Let θ_i be the sequence (α, α'). Finally, let θ be the concatenation of the sequences $\theta_1, \theta_2, \ldots, \theta_r$.*

Let Φ^+_θ be the system of positive pseudo-roots defined by the sequence θ as in Lemma B.1.9, and let w_θ be the unique element of W such that $w_\theta \cdot \Phi^+ = \Phi^+_\theta$. Then the sign $\det(w_\theta)$ *depends only on φ and not on the choices made to form θ.*

Note that there are several choices when producing the ordered set θ. First we have to select an order $\varphi_1, \varphi_2, \ldots, \varphi_r$. There are $r!$ ways to do this. Second, if φ_i is of type B_2 or of type $I_2(2^k)$, there are two possible choices for the pseudo-roots α and α'; see Figure 3. These selections do not influence the sign of w_θ, although they do of course affect the set Φ^+_θ.

Proof of Proposition B.2.7. Let θ and θ' be the results of two possible sequences of choices. For an element w in W, recall that its length $\ell(w)$ [4] is also given by the cardinality of the intersection $w \cdot \Phi^+ \cap \Phi^-$; see [2, Proposition 4.4.4]. Note that

$$\Phi^+_\theta \cap \Phi^-_{\theta'} = w_\theta \cdot \Phi^+ \cap w_{\theta'} \cdot \Phi^- = w_{\theta'} \cdot (w_{\theta'}^{-1} w_\theta \cdot \Phi^+ \cap \Phi^-),$$

[4] By definition, this is the minimal number of factors in an expression of w as a product of reflections corresponding to simple pseudo-roots.

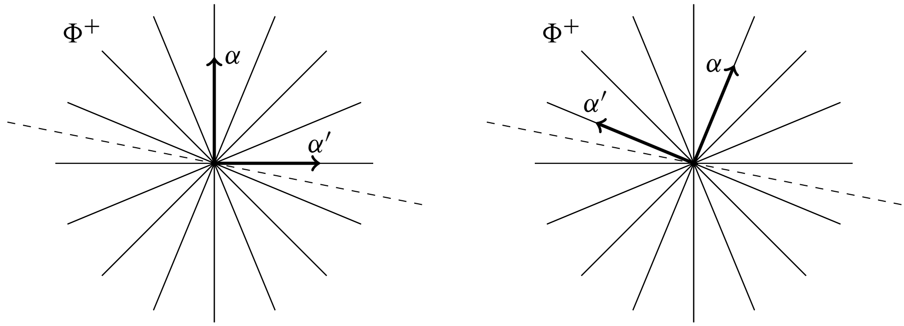

Figure 3. The dihedral pseudo-root system $I_2(8)$ with the two choices of $\theta = \big(\alpha, \alpha'\big)$.

which has cardinality $\ell(w_{\theta'}^{-1} w_\theta)$. Hence, to prove that the signs agree, that is, that $\det(w_\theta) = \det(w_{\theta'})$, it suffices to show that the set $\Phi_\theta^+ \cap \Phi_{\theta'}^-$ has an even number of elements.

We can reduce to the following two cases:

(a) there exists $1 \le i \le r$ such that θ and θ' differ only by the choice of the two pseudo-roots in the factor φ_i;

(b) there exists $1 \le i \le r-1$ such that $\theta_i = \theta'_{i+1}$, $\theta_{i+1} = \theta'_i$ and $\theta_j = \theta'_j$ if $j \ne i, i+1$.

We begin by treating case (a). We write

$$\theta_i = (\alpha, \alpha') \quad \text{and} \quad \theta'_i = (\beta, \beta').$$

Let $\gamma \in \Phi_\theta^+ \cap \Phi_{\theta'}^-$. Then γ is orthogonal to $\varphi_1, \dots, \varphi_{i-1}$, and it is not orthogonal to φ_i. Also, as the sets of positive pseudo-roots in φ_i defined by θ_i and θ'_i are equal by assumption, we cannot have $\gamma \in \varphi_i$. Write $\gamma = c\alpha + c'\alpha' + \lambda$, with $\lambda \in \varphi_1^\perp \cap \cdots \cap \varphi_i^\perp$. By the previous sentence, we have $\lambda \ne 0$. The vector

$$\iota(\gamma) = -(s_\alpha s_{\alpha'})(\gamma) = c\alpha + c'\alpha' - \lambda$$

is also in Φ. It is not equal to γ because $\lambda \ne 0$, and it is in $\Phi_\theta^+ \cap \Phi_{\theta'}^-$ because γ and $\iota(\gamma)$ have the same inner product with any element of the set $\{\alpha, \alpha', \beta, \beta'\}$. Note that we clearly have $\iota(\iota(\gamma)) = \gamma$. We have constructed a fixed-point free involution ι on the set $\Phi_\theta^+ \cap \Phi_{\theta'}^-$, which proves that this set has even cardinality.

We treat case (b). Suppose first that φ_i and φ_{i+1} are both of type A_1, so we can write

$$\theta_i = (\alpha_i) \quad \text{and} \quad \theta_{i+1} = (\alpha_{i+1}).$$

Let Φ' be the pseudo-root system $\Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1})$. If Φ' is of type $I_2(m)$ with $m \ge 3$, then m must be even because Φ' contains two orthogonal pseudo-roots. But then Φ' contains a multiple β of $\alpha_i - \alpha_{i+1}$, and the reflection s_β sends φ^+ to φ^+ because it fixes every element of φ_j for $j = i, i+1$ and exchanges α_i and α_{i+1}, contradicting the definition of a 2-structure. Hence, Φ' is of type $A_1 \times A_1$, and then the fact that $|\Phi^+_\theta \cap \Phi^-_{\theta'}|$ is even follows from Lemma B.3.2.

Suppose that φ_i is of type A_1 and φ_{i+1} is of type $I_2(2^m)$ with $m \geq 2$. Then we can write

$$\theta_i = (\alpha_i) \quad \text{and} \quad \theta_{i+1} = (\alpha_{i+1}, \alpha'_{i+1}).$$

Let Φ', respectively, Φ'', be the pseudo-root system $\Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1})$, respectively, $\Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha'_{i+1})$, and let θ'' be the sequence that we obtain from θ by switching α_i and α_{i+1}. As φ_{i+1} is of type $I_2(2^m)$, it (and hence Φ) contains a pseudo-root β proportional to $\alpha_{i+1} - \alpha'_{i+1}$, and then $s_\beta(\Phi') = \Phi''$, so Φ' and Φ'' are of the same type. By Lemma B.3.2, the cardinalities of the sets $\Phi^+_\theta \cap \Phi^-_{\theta''}$ and $\Phi^+_{\theta''} \cap \Phi^-_{\theta'}$ have the same parity, and so $|\Phi^+_\theta \cap \Phi^-_{\theta'}|$ is even. The case where φ_i is of rank 2 and φ_{i+1} of rank 1 follows from the previous case by switching the roles of φ_i and φ_{i+1}.

Finally, suppose that both φ_i and φ_{i+1} are of rank 2. Then we can write

$$\theta_i = (\alpha_i, \alpha'_i) \quad \text{and} \quad \theta_{i+1} = (\alpha_{i+1}, \alpha'_{i+1}).$$

We move from θ to θ' by the following sequence of operations:

(1) We switch α'_i and α_{i+1}. By Lemma B.3.2 and Remark B.3.3, this changes the sign of w_θ by $(-1)^{m_1/2-1}$, where the pseudo-root system $\Phi_1 = \Phi \cap (\mathbb{R}\alpha'_i + \mathbb{R}\alpha_{i+1})$ is of type $I_2(m_1)$.

(2) We switch α'_i and α'_{i+1}. By the same lemma and remark, this changes the sign by $(-1)^{m_2/2-1}$, where the pseudo-root system $\Phi_2 = \Phi \cap (\mathbb{R}\alpha'_i + \mathbb{R}\alpha'_{i+1})$ is of type $I_2(m_2)$.

(3) We switch α_i and α_{i+1}. By the same lemma and remark, this changes the sign by $(-1)^{m_3/2-1}$, where the pseudo-root system $\Phi_3 = \Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1})$ is of type $I_2(m_3)$.

(4) We switch α_i and α'_{i+1}. By the same lemma and remark, this changes the sign by $(-1)^{m_4/2-1}$, where the pseudo-root system $\Phi_4 = \Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha'_{i+1})$ is of type $I_2(m_4)$.

The reflections $s_i = s_{\alpha_i - \alpha'_i}$ and $s_{i+1} = s_{\alpha_{i+1} - \alpha'_{i+1}}$ are both in W because φ_i contains a multiple of $\alpha_i - \alpha'_i$ and φ_{i+1} contains a multiple of $\alpha_{i+1} - \alpha'_{i+1}$. Observe now that

$$\begin{aligned} s_i(\Phi_1) &= \Phi_3, & s_i(\Phi_2) &= \Phi_4, \\ s_{i+1}(\Phi_1) &= \Phi_2, & s_{i+1}(\Phi_3) &= \Phi_4. \end{aligned}$$

Thus, the four pseudo-root systems Φ_1, Φ_2, Φ_3, and Φ_4 are isomorphic, and hence

$$m_1 = m_2 = m_3 = m_4.$$

Hence, performing operations (1) to (4) changes the sign by $((-1)^{m_1/2-1})^4 = 1$, that is, $\det(w_\theta) = \det(w_{\theta'})$. ∎

Definition B.2.8. Let $\varphi \subseteq \Phi$ be a 2-structure, and let w_θ be as in Proposition B.2.7. Then the sign $(-1)^{r+r'} \det(w_\theta)$, where r is the number of irreducible factors of Φ of type A_{2n} with n odd and r' is the number of irreducible factors of Φ of type $I_2(2n'+1)$ with $n' \geq 3$ odd, is called the *sign* of φ and denoted by $\epsilon(\varphi, \Phi^+)$, or by $\epsilon(\varphi)$ if the system of positive pseudo-roots Φ^+ is understood.

Remark B.2.9. For a root system, this coincides with the definition of the sign of φ from Herb's paper [18], and it differs from the definition in Section 5 of Herb's paper [20]; see of [20, Remark 5.1] and Corollary 3.1.3.

Lemma B.2.10. *Let $\varphi \subseteq \Phi$ be a 2-structure, that is, $\varphi \in \mathcal{T}(\Phi)$.*

(i) *For every $w \in W$, the identity $\epsilon(w(\varphi), w(\Phi^+)) = \epsilon(\varphi, \Phi^+)$ holds.*

(ii) *Let $w \in W$ be such that $w(\varphi^+) \subseteq \Phi^+$. Then the identity $\epsilon(w(\varphi), \Phi^+) = \det(w) \cdot \epsilon(\varphi, \Phi^+)$ holds.*

Proof. Both identities follow easily from the definition of $\epsilon(\varphi, \Phi^+)$. Indeed, let θ be a subset of φ chosen as in Proposition B.2.7. For every $w \in W$, $w(\varphi)$ is a 2-structure for Φ and its subset $w(\theta)$ satisfies the same conditions for the system of positive pseudo-roots $w(\Phi^+)$, and also for the system of positive pseudo-roots Φ^+ if $w(\varphi^+) \subset \Phi^+$. Also, we have $\Phi^+_{w\cdot\theta} = w \cdot \Phi^+_\theta$. This immediately yields (i) and (ii). ∎

Lemma B.2.11. *Let $\alpha_0 \in \Phi$ be a simple pseudo-root, let s_0 be the simple reflection defined by α_0, let $\Phi_0 = \alpha_0^\perp \cap \Phi$ and $\Phi_0^+ = \Phi_0 \cap \Phi^+$. Let $\mathcal{T}''$ be the set of $\varphi \in \mathcal{T}(\Phi)$ such that $s_0(\varphi) = \varphi$; we also consider the subsets $\mathcal{T}_1'' = \{\varphi \in \mathcal{T}'' : \varphi \cap \Phi_0 \in \mathcal{T}(\Phi_0)\}$ and $\mathcal{T}_2'' = \mathcal{T}'' - \mathcal{T}_1''$. Then the following statements hold:*

(0) *Let φ be a 2-structure for Φ. Then $s_0(\varphi) = \varphi$, that is, the 2-structure φ is in $\mathcal{T}''$, if and only if $\alpha_0 \in \varphi$.*

(1) *The map $\mathcal{T}_1'' \to \mathcal{T}(\Phi_0)$, $\varphi \mapsto \varphi \cap \Phi_0$ is bijective.*

(2) *For every $\varphi \in \mathcal{T}_1''$, we have $\epsilon(\varphi, \Phi^+) = \epsilon(\varphi \cap \Phi_0, \Phi_0^+)$.*

(3) *There exists an involution ι of $\mathcal{T}_2''$ such that, for every $\varphi \in \mathcal{T}_2''$, we have $\varphi \cap \Phi_0 = \iota(\varphi) \cap \Phi_0$ and $\epsilon(\iota(\varphi), \Phi^+) = -\epsilon(\varphi, \Phi^+)$.*

Proof. We prove (0). If $\alpha_0 \in \varphi$, then s_0 is in the Coxeter group of φ, so $s_0(\varphi) = \varphi$. Conversely, we have $s_{\alpha_0}(\Phi^+ - \{\alpha_0\}) \subset \Phi^+$ by [2, Lemma 4.4.3], so, if $\varphi \in \mathcal{T}''$ and $\alpha_0 \notin \varphi$, then $s_0(\varphi^+) \subseteq \Phi^+ \cap \varphi = \varphi^+$, contradicting condition (b) in the definition of a 2-structure. Note also that the subset $\varphi \cap \Phi_0$ of Φ_0 always satisfies condition (a) in the definition of a 2-structure, but it does not always satisfy condition (b).

We prove (1). We may assume that Φ is irreducible, and we will freely use the explicit description of 2-structures given in Section B.4. If 2-structures for Φ are all of type A_1^s for some s, which happens in types A_n, D_n, E_6, E_7, E_8, H_3, H_4 and $I_2(m)$ for m odd, then $\varphi \cap \Phi_0 \in \mathcal{T}(\Phi_0)$ for every $\varphi \in \mathcal{T}(\Phi)$, that is, $\mathcal{T}_1'' = \mathcal{T}''$, and we see in the explicit description of 2-structures that the map of statement (1) is a bijection. It is easy to check that the same statement holds in type $I_2(m)$ for m even.

We now suppose that Φ is of type B_n or F_4. (Recall that from the point of view of Coxeter systems types B_n and C_n are isomorphic.) For convenience, in this case, we take Φ to be the actual root system, with possibly non-normalized roots; this does not affect any of the definitions that we made before. To study the map of (1), we may assume that $\alpha_0 = e_n$ or $\alpha_0 = e_1 - e_2$. Suppose first that $\alpha_0 = e_1 - e_2$. Then Φ_0 is reducible. Furthermore, it is of type $A_1 \times B_{n-2}$ if Φ is of type B_n, and of type $A_1 \times B_2$ if Φ is of type F_4, where the A_1 factor is $\{\pm(e_1 + e_2)\}$. In both cases, it is easy to see that $\mathcal{T}_1'' = \mathcal{T}''$ and that (1) holds. Suppose that $\alpha_0 = e_n$. Then Φ_0 is irreducible. Furthermore, it is of type B_{n-1} if Φ is of type B_n, and of type B_3 if Φ is of type F_4. If Φ is of type F_4 or B_n with n even then again it is easy to see that $\mathcal{T}_1'' = \mathcal{T}''$ and that (1) holds.

Finally, suppose that Φ is of type B_n with n odd and that $\alpha_0 = e_n$. If $\varphi \in \mathcal{T}''$ then we have $\varphi \in \mathcal{T}_1''$ if and only if $\{\pm e_n\}$ is an irreducible component of φ. The map sending $\varphi_0 \in \mathcal{T}(\Phi_0)$ to $\varphi_0 \sqcup \{\pm e_n\}$ is thus an inverse to the map of (1), so statement (1) holds.

We now prove (3). We have seen in the proof of (1) that $\mathcal{T}_2'' = \varnothing$ unless Φ is of type B_n with n odd and α_0 is the short simple root. Assume that we are in this case, which means that $\alpha_0 = e_n$. Let $\varphi \in \mathcal{T}_2''$. Then there exists $2 \leq i \leq n$ such that

$$\varphi_1 = \{\pm e_n, \pm e_i, \pm e_n \pm e_i\}$$

is an irreducible component of φ. Write $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$, where the φ_k are irreducible and $\varphi_2 = \{\pm e_j\}$ is the unique rank 1 component of φ. Set

$$\iota(\varphi) = \{\pm e_n, \pm e_j, \pm e_n \pm e_j\} \sqcup \{\pm e_i\} \sqcup \varphi_3 \sqcup \cdots \sqcup \varphi_r.$$

This map switches the roles of e_i and e_j. Then $\iota(\varphi)$ is also in $\mathcal{T}_2''$, it is not equal to φ, we have $\iota(\varphi) \cap \Phi_0 = \varphi \cap \Phi_0$ and $\iota(\iota(\varphi)) = \varphi$. To finish the proof of (3), it suffices to show that $\epsilon(\iota(\varphi), \Phi^+) = -\epsilon(\varphi, \Phi^+)$ for every $\varphi \in \mathcal{T}_2''$. But this follows immediately from the definition of $\iota(\varphi)$ and from Lemma B.3.2.

We finally prove (2). Let $\varphi \in \mathcal{T}_1''$. Choose a family $\theta = (\alpha_1, \ldots, \alpha_r)$ of elements of φ as in Proposition B.2.7.We may assume that $\alpha_0 \in \theta$. If α_0 is in an irreducible component of φ of type A_1, we may assume that $\alpha_0 = \alpha_1$. If α_0 is in an irreducible component of φ of rank 2, then, as it is a simple pseudo-root, it cannot be the first element of θ coming from this rank 2 factor of φ (see Figure 3 for an illustration in the case of $I_2(8)$, the general case is similar), so we may assume that $\alpha_0 = \alpha_r$.

Suppose first that α_0 is in an irreducible component of φ of rank 2 and that $\alpha_0 = \alpha_r$. By the description of 2-structures in Section B.4, this can only happen if Φ is of type B_n, F_4 or $I_2(m)$ with m even. The family $(\alpha_1, \dots, \alpha_{r-1})$ is a family of elements of φ_0 satisfying the conditions of Proposition B.2.7, and $\Phi^+_{0,\theta_0} = \Phi^+_\theta \cap \Phi_0$. So the statement of (2) will follow if we can show that

$$X = (\Phi^+_\theta - \Phi^+_{0,\theta_0}) \cap \Phi^-$$

has even cardinality. Let $s = s_{\alpha_r}$. We claim that $s(X) = X$ and that s has no fixed points in X, which implies that X has even cardinality because $s^2 = 1$. The fact that s has no fixed point in X follows from the facts that the fixed points of s are the elements of $\alpha_r^\perp$, that $\Phi \cap \alpha_r^\perp = \Phi_0$ and that $X \cap \Phi_0 = \varnothing$. As $\alpha_r \notin \Phi^-$ and $-\alpha_r \notin \Phi^+_\theta$, we have

$$X = \big(\Phi^- - \{-\alpha_r\}\big) \cap \big(\Phi^+_\theta - (\Phi^+_{0,\theta_0} \cup \{\alpha_r\})\big).$$

As α_r is a simple pseudo-root, we have

$$s(\Phi^- - \{-\alpha_r\}) \subset \Phi^- - \{-\alpha_r\}$$

by [2, Lemma 4.4.3]. So it suffices to prove that s preserves $\Phi^+_\theta - (\Phi^+_{0,\theta_0} \cup \{\alpha_r\})$. If $\beta \in \Phi^+_\theta - \Phi^+_{0,\theta_0}$ is such that $\beta \neq \alpha_r$, then we cannot have $(\beta, \alpha_i) = 0$ for every $i \in \{1, \dots, r-1\}$; indeed, as Φ is of type B_n, F_4 or $I_2(m)$ with m even, the family $(\alpha_1, \dots, \alpha_r)$ is an orthonormal basis of V, so the only element of Φ^+_θ that is orthogonal to $\alpha_1, \dots, \alpha_{r-1}$ is α_r. So $\Phi^+_\theta - (\Phi^+_{0,\theta_0} \cup \{\alpha_r\})$ is the set pseudo-roots $\beta \in \Phi$ such that

$$((\beta, \alpha_1), \dots, (\beta, \alpha_{r-1})) > 0$$

(for the lexicographic order on $\mathbb{R}^{r-1}$) and that $(\beta, \alpha_r) \neq 0$. This set is stable by s, because, for every $\beta \in V$, we have

$$(s(\beta), \alpha_i) = (\beta, s(\alpha_i)) = (\beta, \alpha_i)$$

if $1 \leq i \leq r-1$ and $(s(\beta), \alpha_r) = (\beta, s(\alpha_r)) = -(\beta, \alpha_r)$.

Now we suppose that α_0 is in an irreducible component of φ of rank 1 and that $\alpha_0 = \alpha_1$. Then $\theta_0 = (\alpha_2, \dots, \alpha_r)$ is a family of elements of φ_0 satisfying the conditions of Proposition B.2.7, and $\Phi^+_{0,\theta_0} = \Phi_0 \cap \Phi^+_\theta$, so $\Phi^+_\theta - \Phi^+_{0,\theta_0} = \{\beta \in \Phi : (\beta, \alpha_1) > 0\}$. Statement (2) will follow if we can show that

$$X = (\Phi^+_\theta - \Phi^+_{0,\theta_0}) \cap \Phi^- = \{\beta \in \Phi^- : (\beta, \alpha_1) > 0\}$$

has even cardinality if Φ is not of type A_{2n} or $I_2(2n'+1)$ with n' odd, and odd cardinality otherwise. As φ has an irreducible component of rank 1, we cannot be in type F_4. We can check that X has even cardinality by a computer calculation in the exceptional types E, G and H.

We now go through the remaining types one by one (in cases A, B and D, we use the description of the roots from the tables at the end of [5], and not the normalized pseudo-root systems):

Type $I_2(m)$. If m is even, then φ is a rank 2 pseudo-root system; so m must be odd, and then φ_0 is empty and $\epsilon(\varphi_0) = 1$. There are exactly m pseudo-roots β such that $(\beta, \alpha_1) > 0$, and $(m-1)/2$ of these are in Φ^-. So

$$\epsilon(\varphi) = (-1)^{(m-1)/2}(-1)^{(m-1)/2} = 1,$$

which is what we wanted.

Type A_n. We write $\alpha_0 = e_i - e_{i+1}$, with $1 \leq i \leq n$. Then

$$X = \{e_j - e_k : 1 \leq k < j = i \text{ or } i+1 = k < j \leq n+1\}$$

has cardinality $n-1$, that is, even if and only if n is odd.

Type B_n. As φ has an irreducible component of rank 1, the integer n must be odd and α_0 is the short simple root, that is, $\alpha_0 = e_n$. Then

$$X = \{-e_i + e_n : 1 \leq i \leq n\}$$

has cardinality $n-1$, which is even.

Type D_n. If $\alpha_0 = e_i - e_{i+1}$ with $1 \leq i \leq n-1$, then

$$\begin{aligned} X = \{&e_j - e_k : 1 \leq k < j = i \text{ or } i+1 = k < j \leq n\} \\ &\cup \{-(e_j + e_k) : i+1 = j < k \leq n \text{ or } i \neq j < k = i+1\} \end{aligned}$$

has cardinality $2n-4$. If $\alpha_0 = e_{n-1} + e_n$, then

$$X = \{e_j - e_k : n = j > k \neq n-1 \text{ or } j = n-1 > k\}$$

also has cardinality $2n-4$. ■

Lemma B.2.12. *Let $\varphi \subseteq \Phi$ be a 2-structure. Then $|\Phi^+ - \varphi^+|$ is an even integer. More precisely, if Φ is irreducible, we have*

$$|\Phi^+ - \varphi^+| = \begin{cases} 2n \bmod 4 & \text{if } \Phi \text{ is of type } A_{2n}, \\ 0 \bmod 4 & \text{if } \Phi \text{ is of type } A_{2n+1}, B, D, E, F_4, G_2, \text{ or } H, \\ 2^r(m-1) & \text{if } \Phi \text{ is of type } I_2(2^r m) \text{ with } m \text{ odd.} \end{cases}$$

Proof. This follows from the explicit description of 2-structures for the irreducible types in Section B.4. ■

B.3. Orthogonal sets of pseudo-roots and 2-structures

For Φ a root system (not just a pseudo-root system), let $\mathcal{O}(\Phi)$ be the set of all finite sequences $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ of elements of Φ which are pairwise orthogonal and such that their entries all have the same length, that is, the following two conditions hold:

(a) $(\alpha_i, \alpha_j) = 0$ for all $1 \leq i < j \leq r$;

(b) $\|\alpha_1\| = \|\alpha_2\| = \cdots = \|\alpha_r\|$.

Lemma B.3.1. *Suppose that Φ is a root system. Let $\theta = (\alpha_1, \ldots, \alpha_r)$ and $\theta' = (\beta_1, \ldots, \beta_s)$ be elements of $\mathcal{O}(\Phi)$, and suppose that $\theta^\perp \cap \Phi = (\theta')^\perp \cap \Phi = \varnothing$ and that the elements of θ and θ' have the same length. Then there exists $w \in W$ such that $\{\alpha_1, \ldots, \alpha_r\} = \{w(\beta_1), \ldots, w(\beta_s)\}$. In particular, $r = s$ holds.*

Proof. Let Φ^+_θ, respectively, $\Phi^+_{\theta'}$, be the system of positive roots defined by θ, respectively, θ', as in Definition B.1.10. As W acts transitively on the set of systems of positive roots, there exists $w \in W$ such that $w(\Phi^+_{\theta'}) = \Phi^+_\theta$. As $w(\Phi^+_{\theta'}) = \Phi^+_{w(\beta_1),\ldots,w(\beta_s)}$, we may assume that $\Phi^+_\theta = \Phi^+_{\theta'}$. We then wish to prove that θ and θ' are equal up to reordering their entries. We proceed by induction on the length of θ. If θ is empty then Φ is also empty because of the condition $\Phi \cap \theta^\perp = \varnothing$, so θ' is empty and we are done. Suppose that $r \geq 1$. Let j_0 be the smallest index j such that $(\alpha_1, \beta_j) \neq 0$. Since $\Phi \cap (\theta')^\perp = \varnothing$, this minimum exists. As $\beta_{j_0} \in \Phi^+_{\theta'} = \Phi^+_\theta$, we cannot have $(\alpha_1, \beta_{j_0}) < 0$, so $(\alpha_1, \beta_{j_0}) > 0$. As Φ is a root system and not just a pseudo-root system, the corollary after [5, Chapitre VI, § 1, № 3, Théorème 1] implies that the difference $\gamma = \alpha_1 - \beta_{j_0}$ is an element of $\Phi \cup \{0\}$. Suppose that $\gamma \in \Phi^+_{\theta'}$. As $(\gamma, \beta_j) = 0$ for $1 \leq j < j_0$, we must then have

$$0 \leq (\gamma, \beta_{j_0}) = (\alpha_1, \beta_{j_0}) - (\beta_{j_0}, \beta_{j_0}).$$

The hypothesis states that $\|\alpha_1\| = \|\beta_{j_0}\|$, and hence we deduce that

$$(\alpha_1, \beta_{j_0}) \geq \|\beta_{j_0}\|^2 = \|\alpha_1\| \cdot \|\beta_{j_0}\|.$$

This inequality implies that $\alpha_1 = \beta_{j_0}$, contradicting the fact that γ is nonzero. Suppose that $\gamma \in \Phi^-_\theta$. Then

$$0 \leq (\alpha_1, -\gamma) = (\alpha_1, \beta_{j_0}) - (\alpha_1, \alpha_1),$$

so $(\alpha_1, \beta_{j_0}) \geq \|\alpha_1\|^2$, and again this implies that $\alpha_1 = \beta_{j_0}$ and contradicts the assumption. Hence, we conclude that $\gamma = 0$, that is, $\alpha_1 = \beta_{j_0}$. Let $\Phi_0 = \alpha_1^\perp \cap \Phi = \beta_{j_0}^\perp \cap \Phi$,

$$\theta_0 = (\alpha_2, \ldots, \alpha_r) \quad \text{and} \quad \theta'_0 = (\beta_1, \ldots, \widehat{\beta_{j_0}}, \ldots, \beta_s).$$

Then Φ_0 is a root system, θ_0 and θ'_0 are in $\mathcal{O}(\Phi_0)$, $\theta_0^\perp \cap \Phi_0 = (\theta'_0)^\perp \cap \Phi_0 = \varnothing$, and

$$\Phi^+_{0,\theta_0} = \Phi^+_\theta \cap \Phi_0 = \Phi^+_{\theta'} \cap \Phi_0 = \Phi^+_{0,\theta'_0}.$$

k	E_6	E_7	E_8	F_4 short	F_4 long	H_3	H_4
1	72	126	240	24	24	30	120
2	1080	3780	15120	72	72	60	1800
3	4320	32760	302400	96	96	40	2400
4	2160	75600	1965600	48	48		1200
5		90720	3628800				
6		60480	3628800				
7		17280	2073600				
8			518400				

Table 1. The number of orthogonal sets of roots or pseudo-roots of size k where the elements all have the same length in the exceptional/sporadic reflection arrangements. Note the double occurrence of 10! in the E_8 column. The equality of the columns in type F_4 comes from the fact that there is an automorphism of the underlying vector space that preserves angles, sends short roots to long roots, and sends long roots to doubles of short roots (for instance, the automorphism given by $e_1 \mapsto e_1 + e_2$, $e_2 \mapsto e_1 - e_2$, $e_3 \mapsto e_3 + e_4$ and $e_4 \mapsto e_3 - e_4$).

We can apply the induction hypothesis to conclude that

$$\{\alpha_2, \dots, \alpha_r\} = \{\beta_1, \dots, \widehat{\beta_{j_0}}, \dots, \beta_s\},$$

and this immediately implies that $\{\alpha_1, \dots, \alpha_r\} = \{\beta_1, \dots, \beta_s\}$. ∎

Lemma B.3.2. *Let Φ be a normalized pseudo-root system, let $\theta = (\alpha_1, \dots, \alpha_r)$ be a sequence of pairwise orthogonal elements of Φ such that $\theta^\perp \cap \Phi = \varnothing$, and let θ' be the sequence obtained from θ by exchanging α_i and α_{i+1}. Consider the subroot system*

$$\Phi' = \Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1}).$$

Then Φ' is of type $A_1 \times A_1$ or $I_2(m)$ with $m \geq 4$ even, and the parity of the cardinality of $\Phi_\theta^+ \cap \Phi_{\theta'}^-$ is given by

$$\begin{aligned} |\Phi_\theta^+ \cap \Phi_{\theta'}^-| &\equiv 0 \bmod 2 && \text{if } \Phi' = A_1 \times A_1, \\ |\Phi_\theta^+ \cap \Phi_{\theta'}^-| &\equiv m/2 - 1 \bmod 2 && \text{if } \Phi' = I_2(m). \end{aligned}$$

Proof. As Φ' is a pseudo-root system of rank 2 (because it is contained in a 2-dimensional vector space and contains the two linearly independent pseudo-roots α_i and α_{i+1}), it is of type $A_1 \times A_1$ or $I_2(m)$ with $m \geq 3$. Moreover, Φ' contains two orthogonal pseudo-roots, so it cannot be of type $I_2(m)$ with m odd.

We now set $C = \Phi_\theta^+ \cap \Phi_{\theta'}^-$ and calculate the parity of $|C|$. Let $\gamma \in C$. Then γ is orthogonal to $\alpha_1, \dots, \alpha_{i-1}$, so we can write

$$\gamma = c\alpha_i + d\alpha_{i+1} + \lambda$$

with $\lambda \in \mathrm{Span}(\alpha_1, \dots, \alpha_{i+1})^\perp$ and $c\alpha_i + d\alpha_{i+1} \neq 0$. Set

$$\iota(\gamma) = -s_{\alpha_i} s_{\alpha_{i+1}}(\gamma).$$

Then $\iota(\gamma) \in \Phi$ and $\iota(\gamma) = c\alpha_i + d\alpha_{i+1} - \lambda$, so $\iota(\gamma) \in C$. Also, we clearly have $\iota(\iota(\gamma)) = \gamma$, and $\iota(\gamma)$ is equal to γ if and only if $\lambda = 0$, that is, if and only if $\gamma \in \Phi'$. We have defined an involution ι of C, and we conclude that

$$|C| \equiv |C_0| \bmod 2,$$

where $C_0 = \Phi' \cap C$ is the set of fixed points of ι in C. If Φ' is of type $A_1 \times A_1$ then we easily see that C_0 is empty, so we are done. Suppose that Φ' is of type $I_2(m)$ with m even. Let $\gamma = c\alpha_i + d\alpha_{i+1} \in \Phi'$, with $c, d \in \mathbb{R}$. Then $\gamma \in C$ if and only if $c > 0$ and $d < 0$. The set C_0 contains exactly one quarter of the elements of $\Phi' - \{\pm\alpha_i, \pm\alpha_{i+1}\}$, that is,

$$|C_0| = \frac{2m-4}{4} = \frac{m}{2} - 1. \qquad \blacksquare$$

Remark B.3.3. If we view the root system $A_1 \times A_1$ as the dihedral pseudo-root system $I_2(2)$ then the conclusion of Lemma B.3.2 is that $|\Phi_\theta^+ \cap \Phi_{\theta'}^-| \equiv m/2 - 1 \bmod 2$ if $\Phi' = I_2(m)$ with m even and $m \geq 2$.

Lemma B.3.4. *Suppose that Φ is an irreducible root system (not just a pseudo-root system) and not of type G_2. Let Φ^+ be a system of positive roots of Φ and let $\varphi \subseteq \Phi$ be a 2-structure. Define a subset θ of φ as in Proposition B.2.7. Then there is a choice of the sequences θ_i for which θ is an element of $\mathcal{O}(\Phi)$. Moreover, if Φ is of type B_n or F_4 we can choose θ to consist of short roots. Similarly, if Φ is of type C_n or F_4 we can choose θ to consist of long roots.*

Proof. By Remark B.2.3 we have $\theta^\perp \cap \Phi = \varnothing$. We use the notation of Proposition B.2.7. If all the roots of Φ have the same length (which is the case for A_n, D_n, E_6, E_7 and E_8), then there is nothing to prove. Note also that if φ_i is an actual root system of type B_2 (that is, with the correct root lengths), then the two possible choices for θ_i are the set of short positive roots and the set of long positive roots.

Suppose that Φ is of type B_n. If φ has no irreducible component of type A_1, then we choose the two short positive roots in each φ_i. Suppose that φ has a factor of type A_1. We show that this factor cannot contain long roots. Suppose on the contrary that this occurs. Without loss of generality, we may assume that $\varphi_1 = \{\pm(e_1 + e_2)\}$. The rank 2 factors of φ cannot contain $e_1 - e_2$, so they are all in $e_1^\perp \cap e_2^\perp$. All the

Type of root system Φ	2-structures are isomorphic to ...	Type of 2-structure
A_n	$\{\pm(e_1 - e_2), \ \pm(e_3 - e_4), \ \dots, \ \pm(e_{2m-1} - e_{2m})\}$	A_1^m
D_n	$\{\pm e_1 \pm e_2, \ \pm e_3 \pm e_4, \ \dots, \ \pm e_{2m-1} \pm e_{2m}\}$	A_1^{2m}
E_6	$\{\pm e_1 \pm e_2, \ \pm e_3 \pm e_4\}$	A_1^4
E_7	$\{\pm e_1 \pm e_2, \ \pm e_3 \pm e_4, \ \pm e_5 \pm e_6, \ \pm(e_7 - e_8)\}$	A_1^7
E_8	$\{\pm e_1 \pm e_2, \ \pm e_3 \pm e_4, \ \pm e_5 \pm e_6, \ \pm e_7 \pm e_8\}$	A_1^8

Table 2. The 2-structures in types A, D and E, where $m = \lfloor (n+1)/2 \rfloor$ in type A and $m = \lfloor n/2 \rfloor$ in type D.

rank 1 factors that do not contain $e_1 - e_2$ must also be in $e_1^\perp \cap e_2^\perp$. If $e_1 - e_2$ were not in φ then the reflection $s_{e_1 - e_2}$ would act as the identity on all the elements on φ, which contradicts the definition of a 2-structure. Hence, $\{\pm(e_1 - e_2)\}$ is another rank 1 factor of φ. But then the reflection s_{e_1} preserves φ^+, which is impossible. Hence, all the A_1 factors of φ contain only short roots, and we choose the θ_i in the B_2 factors to contain the two short positive roots.

The case for C_n is similar, with the roles of short and long roots uniformly exchanged.

Finally, suppose that $\Phi = F_4$. In this case we can similarly show that the 2-structure φ has type B_2^2, allowing us to pick either short or long roots in each factor. ∎

B.4. 2-structures in the irreducible types

In this subsection we prove Proposition B.2.4, that is, the fact that the group W acts transitively on the collection of 2-structures $\mathcal{T}(\Phi)$. It is enough to prove this result for irreducible pseudo-root systems. We proceed by a case by case analysis.

Types A_n, D_n, E_6, E_7 and E_8. Suppose that Φ is a root system of type A_n, D_n or E_m with $m \in \{6, 7, 8\}$. As all the roots of Φ have the same length and as Φ contains no B_2 root system, the 2-structures for Φ are exactly the maximal sets $\varphi = \{\pm\alpha_1, \dots, \pm\alpha_r\}$ such that $(\alpha_1, \dots, \alpha_r) \in \mathcal{O}(\Phi)$. By Lemma B.3.1, for any $(\alpha_1, \dots, \alpha_r)$ and $(\beta_1, \dots, \beta_s)$ on $\mathcal{O}(\Phi)$, there exists $w \in W$ such that

$$\{\alpha_1, \dots, \alpha_r\} = \{w(\beta_1), \dots, w(\beta_s)\}.$$

Hence, the group W acts transitively on $\mathcal{T}(\Phi)$. In particular, all the 2-structures for Φ are isomorphic, so we can determine their type; see Table 2.

Types B_n and C_n. Suppose that Φ is a root system of type B_n. This will also give the type C_n case, since B_n and C_n correspond to the same Coxeter system. We claim that W acts transitively on $\mathcal{T}(\Phi)$. In particular, all the 2-structures for Φ are isomorphic to

$$\varphi_0 = \begin{cases} \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\} \sqcup \cdots \\ \qquad \cdots \sqcup \{\pm e_{2m-1}, \pm e_{2m}, \pm e_{2m-1} \pm e_{2m}\} & \text{if } n = 2m, \\ \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\} \sqcup \cdots \\ \qquad \cdots \sqcup \{\pm e_{2m-1}, \pm e_{2m}, \pm e_{2m-1} \pm e_{2m}\} \sqcup \{\pm e_{2m+1}\} & \text{if } n = 2m+1, \end{cases}$$

so they are of type B_2^m if $n = 2m$ is even, and of type $B_2^m \times A_1$ if $n = 2m+1$ is odd.

We prove the claim by induction on n. The case $n = 1$ is clear. Suppose that $n \geq 2$. Let $\varphi, \varphi' \in \Phi$. By Lemma B.3.4, we can choose sequences θ of φ and θ' of φ' as in Proposition B.2.7 such that $\theta, \theta' \in \mathcal{O}(\Phi)$ and that these subsets contain only short roots. By Lemma B.3.1, we may assume that θ and θ' coincide up to the order of their elements. Denote by

$$\varphi = \varphi_1 \sqcup \cdots \sqcup \varphi_s \quad \text{and} \quad \varphi' = \varphi'_1 \sqcup \cdots \sqcup \varphi'_t$$

the decomposition into irreducible systems that gave rise to θ and θ'. We can always change the order on the φ_i and the φ'_j.

Suppose that φ_1 is of rank 1, so that $\varphi_1 = \{\pm\alpha_1\}$. We may assume that $\alpha_1 \in \varphi'_1$. If φ'_1 is of rank 1 then $\varphi'_1 = \varphi_1$. As $\Phi \cap \varphi_1^\perp$ is an irreducible root system of type B_{n-1}, the conclusion follows by the induction hypothesis.

If φ'_1 is of rank 2 then φ'_1 is a B_2 root system whose short positive roots are α_1 and some α_2, and we may assume that $\alpha_2 \in \varphi_2$. In particular, the vector $\beta = \alpha_1 - \alpha_2$ is in Φ. If $\varphi_2 = \{\pm\alpha_2\}$ then the reflection s_β preserves φ^+, which is not possible. So φ_2 is of rank 2 (in particular, $n \geq 3$), which means that it is a B_2 root system whose short roots are α_2 and some α_3. We may assume that $\alpha_3 \in \varphi'_2$. In particular, $\alpha_2 - \alpha_3 \in \Phi$, so $\gamma = s_\beta(\alpha_2 - \alpha_3) = \alpha_1 - \alpha_3$ is also a root. The irreducible components of $s_\gamma(\varphi)$ are $\varphi'_1, \{\pm\alpha_3\}, \varphi_3, \ldots, \varphi_s$. As $\Phi \cap (\varphi'_1)^\perp$ is a root system of type B_{n-2}, the induction hypothesis implies that there is a $w \in W$ such that $w(\varphi') = s_\gamma(\varphi)$, which finishes the proof in this case.

Suppose that φ_1 is of rank 2, and call its other short positive root α_2. We may assume that $\alpha_1 \in \varphi'_1$. If φ'_1 is of rank 1 then $\varphi'_1 = \{\pm\alpha_1\}$, and we can repeat the reasoning of the previous paragraph with the roles of φ and φ' exchanged. If $\varphi'_1 = \varphi_1$ then the conclusion follows from the induction hypothesis applied to the B_{n-2} root system $\varphi_1^\perp \cap \Phi$. Finally, suppose that φ'_1 is of rank 2 and $\varphi'_1 \neq \varphi_1$. Let α_3 be the other short positive root of φ'_1. As α_2 and α_3 are both short roots, $\beta = \alpha_2 - \alpha_3 \in \Phi$. Note that the irreducible components of $s_\beta(\varphi)$ are $\varphi'_1, s_\beta(\varphi_2), \ldots, s_\beta(\varphi_s)$, so again the induction hypothesis implies that there exists $w \in W$ such that $s_\beta(\varphi) = w(\varphi')$, and we are done.

Type F_4. Suppose that Φ is a root system of type F_4. Then we can show that W acts transitively on $\mathcal{T}(\Phi)$ exactly as in type B_n. In particular, any 2-structure is isomorphic to $\varphi_0 = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\} \sqcup \{\pm e_3, \pm e_4, \pm e_3 \pm e_4\}$, so it is of type B_2^2.

Dihedral types. Suppose that Φ is a pseudo-root system of type $I_2(m)$ with $m \geq 5$ (this includes the type G_2 root system). It is straightforward to see that W acts transitively on $\mathcal{T}(\Phi)$. If m is odd then all the 2-structures for Φ are isomorphic to $\varphi_0 = \{\pm e_1\}$, and in particular of type A_1. If m is even then all the 2-structures for Φ are of type $I_2(2^r)$, where 2^r is the largest power of 2 dividing m.

Types H_3 and H_4. Suppose that Φ is of type H_3 or H_4. We use the description of the pseudo-root systems H_3 and H_4 given in [16, Table 5.2] where they are called I_3 and I_4. In particular, we choose Φ to be normalized. We claim that W acts transitively on $\mathcal{T}(\Phi)$, and so every 2-structure for Φ is isomorphic to

$$\varphi_0 = \begin{cases} \{\pm e_1, \pm e_2, \pm e_3\} & \text{if } \Phi = H_3, \\ \{\pm e_1, \pm e_2, \pm e_3, \pm e_4\} & \text{if } \Phi = H_4, \end{cases} \tag{B.1}$$

and in particular it is of type A_1^3 if $\Phi = H_3$ and of type A_1^4 if $\Phi = H_4$.

It is clear by the chosen description of Φ that all of the inner products of elements of Φ are in $\mathbb{Q}[\sqrt{5}]$, and in particular $1/\sqrt{2}$ never appears. So there are no pseudo-roots in Φ with an angle of $\pi/4$ between them, which implies that Φ does not contain any pseudo-root system of type $I_2(m)$ with m a multiple of 4, and so 2-structures for Φ (if they exist) can only have irreducible components of type A_1.

We check easily that the set φ_0 given in equation (B.1) is a 2-structure, so it remains to show that all the maximal sets of pairwise orthogonal pseudo-roots are conjugate under W to ζ_0, where $\zeta_0 = \{e_1, e_2, e_3\}$ if $\Phi = H_3$ and $\zeta_0 = \{e_1, e_2, e_3, e_4\}$ if $\Phi = H_4$. Any element of the stabilizer W_0 of ζ_0 in W must act on $\mathrm{Span}(\Phi)$ by a permutation of the coordinates, and it must be an even permutation to be in W. This implies that the cardinality of W_0 is 3 for $\Phi = H_3$ and 12 for $\Phi = H_4$. Using a computer, it is not hard to count all the maximal sets of pairwise orthogonal pseudo-roots in H_3 and H_4; see Table 1. We find that there are 40 such sets for H_3 and 1200 such sets for H_4. In both cases, this number is equal to $|W|/|W_0|$, so W does act transitively on the set of maximal sets of pairwise orthogonal pseudo-roots, and hence also on $\mathcal{T}(\Phi)$.

C. Relationship with locally symmetric spaces

In this appendix, aimed at specialists of Shimura varieties, we give more details about the connection between some of the objects introduced in this article and the calculation of the weighted cohomology of locally symmetric spaces.

This is a continuation of the discussion in the first part of the introduction, and we return to the notation of this discussion. We do not suppose yet that the group $G(\mathbb{R})$ has a discrete series. In the introduction, we only considered cohomology of X_K with constant coefficients, but now we need to introduce a coefficient system. Let F be an irreducible algebraic representation of G. Then, via the $G(\mathbb{Q})$-covering

$$(G(\mathbb{R}) \times G(\mathbb{A}^\infty))/(K_\infty \times K) \to X_K$$

and the action of $G(\mathbb{Q})$ on F, we get a locally constant sheaf L_F on X_K, and we will write $H^*(X_K, F)$ instead of $H^*(X, L_F)$ for every reasonable cohomology theory H^*.[5] If $T \subset B$ are a maximal torus and a Borel subgroup of $G_{\mathbb{C}}$, respectively, then the representation F has a highest weight λ_B in the Lie algebra of T that is dominant with respect to B.[6] The space V of the article will typically be this Lie algebra with the inner product coming from the Killing form of the Lie algebra of G in the usual way. The pseudo-root system Φ that defines the hyperplane arrangement will be the root system of T in the Lie algebra of G, with the positive system determined by B; sometimes T will be defined over $\mathbb{R}$, and Φ will be the real root system of T.

The first cohomology theory that we consider is *weighted cohomology*, from which the weighted complex and the weighted sum get their names. Weighted cohomology was introduced by Goresky–Harder–MacPherson in the paper [12]. It depends on an auxiliary parameter called a "weight profile" and is the cohomology of a sheaf of truncated differential forms on the reductive Borel–Serre compactification of X_K, where the truncation depends on the weight profile. The Hecke algebra acts on the weighted cohomology groups, and they are explicit enough to make the calculation of the traces of Hecke operators possible; see the paper [14] of Goresky and MacPherson. Also, there are two "middle" weight profiles for any group G and, if X_K is a Shimura variety, then the two middle weighted cohomology groups are both isomorphic to the intersection cohomology of the Baily–Borel compactification of X_K. What we call the "weighted sum" in this article appears in the calculation of the trace of a Hecke operator on the weighted cohomology groups, hence the name. This calculation is carried out in [14] and summarized in Section 7 of [13]. Very roughly, the trace of a Hecke operator on a weighted cohomology group is a sum over conjugacy classes of rational Levi subgroups M of G and certain conjugacy classes of $\gamma \in M(\mathbb{Q})$ of the product of:

- a normalizing factor;

[5] We need a different construction of L_F if X_K is the set of complex points of a Shimura variety and H^* is étale (intersection) cohomology, but this is not the point of this appendix.

[6] The representation F might not stay irreducible when seen as a representation of $G_{\mathbb{C}}$, but we ignore this technical complication.

- an orbital integral on the conjugacy class of γ in $M(\mathbb{A}^\infty)$ that depends on the Hecke operator but not on the weight profile or on the coefficient system;
- a "term at infinity" $L_M(\gamma)$ that depends on the weight profile and the coefficient system but not on the Hecke operator.

See [13, formula (7.14.7)]. Goresky, Kottwitz and MacPherson then introduce a stable virtual character Θ on $G(\mathbb{R})$ (this notion is defined in [13, p. 495]) that depends on the coefficient system via the highest weight of F and on the weight profile. We can recover the function L_M as the restriction of Θ to M up to chasing some denominators depending on M; this last statement is Theorem 5.1 of [13], and it works for any weight profile. While the expression for the function L_M involves "relative" weighted sums $\psi_{\mathcal{H}/C}$ where we are in the situation of Example 4.2.1 (see [13, pp. 504–505]), the virtual character Θ only involves the simpler weighted sums $\psi_{\mathcal{H}}$, where we are in the situation of Section 4.3. In both cases, the space V is the real Lie algebra of a maximal torus T of G, the pseudo-root system is the set of real roots of T in G, and the element λ of V is, up to a shift depending on the weight profile, of the form $w(\lambda_B + \rho_B) - \rho_B$, where $B \supset T$ is a Borel subgroup (defined over $\mathbb{C}$), λ_B is the highest weight of F corresponding to B, ρ_B is half the sum of the positive roots and w is an element of the Weyl group.

We now assume that the weight profile is one of the middle profiles and that X_K is a Shimura variety. Then, as explained in the introduction, we know that weighted cohomology is isomorphic to L^2 cohomology, for which we have a spectral description known as Matsushima's formula (even though it was proved by Borel and Casselman in this generality). This implies in particular that the virtual character Θ is equal to the stable discrete series character corresponding to the dual of the representation F, hence that the weighted sum $\psi_{\mathcal{H}}$ is equal to what are known as *stable discrete series constants*; see, for example, [13, pp. 493, 498–500] for a quick review of these constants. The first statement of the previous sentence is proved directly in Theorem 5.2 of [13], and the second statement is proved directly in Theorem 3.1 of the same paper. The stable discrete series constants can be expressed in terms of 2-structures by the work of Herb (see, for example, [19, Theorem 4.2]), and this is the expression on the right-hand side of the identity of Corollary 4.3.1.

We can go further and relate the traces of Hecke operators on L^2 cohomology to the Arthur–Selberg trace formula for a particular test function. This is done in Arthur's paper [1]. The resulting trace formula can then be stabilized. Although this is a very complicated process in the general case, it is slightly less involved for our test function, by the work of Kottwitz (unpublished) and Zhifeng Peng [29]. Thus, we get character formulas relating the virtual character Θ and stable discrete series characters on endoscopic groups of G. However, when we express everything in terms of 2-structures, the distinction between G and its endoscopic groups disappears. Indeed, endoscopic groups of G have root systems that are subsystems of the root system of G, and 2-structures, being very small root systems, can be shared between G and its endoscopic groups. (We are summarily ignoring many complications, due in particular to the appearance of transfer factors in the character identities.)

We finally come to the case where X_K is a Shimura variety defined over some number field E and we are interested, not just in the action of the Hecke algebra on the intersection cohomology $IH^*(\overline{X}_K, F)$ of its Baily–Borel compactification $\overline{X}_K$, but also in the action of the absolute Galois group of E. There is a calculation of the trace of a Hecke operator times a power of the Frobenius morphism (at an unramified place p) that parallels the calculation of [14]: see [26] for the algebraic version of weighted cohomology, the papers [27] and [28] for the trace calculation in the cases of unitary and symplectic groups (over $\mathbb{Q}$), and [34] for the trace calculation in the case of orthogonal groups. We obtain an expression for this trace that is reminiscent of formula (7.14.7) of [13], that we quickly described above, except that the orbital integral at p is twisted and that the terms $L_M(\gamma)$ are slightly different. Nevertheless, by using techniques similar to those of the proof of Theorem 5.1 of [13], in particular the Weyl character formula and Kostant's theorem, we can still relate $L_M(\gamma)$ to the relative weighted sum $\psi_{\mathcal{H}/C}$ in the situation of Example 4.2.1. For symplectic groups over $\mathbb{Q}$, this calculation is done in the proof of Proposition 3.3.1 of [28]. The difference with the situation of [13] is that L^2 cohomology does not have an action of the absolute Galois of E, so we do not have a nice spectral expression for our trace, and in particular we do not know if there is a stable virtual character "interpolating" the function L_M as in Theorem 5.1 of [13]. Fortunately, we are still able to relate our trace expression directly to a sum of stable trace formulas for well-chosen test functions on endoscopic groups of G, and this is where Theorem 4.2.2 comes into play: We must express the function L_M in terms of stable discrete series constants for endoscopic groups of G. Via Herb's formula, this reduces to giving a formula for L_M involving 2-structures for the root systems of these endoscopic groups, but, as we explained above, these 2-structures can also be seen as 2-structures for the root system of G. Again, we are sweeping many technical complications under the rug, and the story is by no means finished once we have Theorem 4.2.2.

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Finally, we used SageMath and Maple for innumerable root system computations.

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