J. Comb. Algebra 6 (2022), 185–222
DOI 10.4171/JCA/65

Jucys–Murphy elements and Grothendieck groups for generalized rook monoids

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Abstract. We consider a tower of generalized rook monoid algebras over the field $\mathbb{C}$ of complex numbers and observe that the Bratteli diagram associated to this tower is a simple graph. We construct simple modules and describe Jucys–Murphy elements for generalized rook monoid algebras.

Over an algebraically closed field $\Bbbk$ of positive characteristic p, utilizing Jucys–Murphy elements of rook monoid algebras, for $0 \leq i \leq p-1$ we define the corresponding i-restriction and i-induction functors along with two extra functors. On the direct sum $\mathcal{G}_{\mathbb{C}}$ of the Grothendieck groups of module categories over rook monoid algebras over $\Bbbk$, these functors induce an action of the tensor product of the universal enveloping algebra $U(\widehat{\mathfrak{sl}}_p(\mathbb{C}))$ and the monoid algebra $\mathbb{C}[\mathcal{B}]$ of the bicyclic monoid $\mathcal{B}$. Furthermore, we prove that $\mathcal{G}_{\mathbb{C}}$ is isomorphic to the tensor product of the basic representation of $U(\widehat{\mathfrak{sl}}_p(\mathbb{C}))$ and the unique infinite-dimensional simple module over $\mathbb{C}[\mathcal{B}]$, and also exhibit that $\mathcal{G}_{\mathbb{C}}$ is a bialgebra. Under some natural restrictions on the characteristic of $\Bbbk$, we outline the corresponding result for generalized rook monoids.

1. Introduction

The aim of this paper is to prove several results on the representation theory of certain inverse semigroups called generalized rook monoids and on the structure of their semigroup algebras. These results are motivated by the corresponding results for the wreath products of symmetric and cyclic groups. The latter groups appear as maximal subgroups in generalized rook monoids. Below we explain our motivation and result in more detail.

Let R_n be the set consisting of all $n \times n$ matrices with entries from $\{0, 1\}$ and with the further condition that each row and each column contains at most one nonzero entry. The matrix multiplication defines on R_n the structure of a monoid, called the *rook monoid*, cf. [33]. The monoid R_n is alternatively known as the symmetric inverse semigroup; see [8, 25]. It is very well known, see for example [30], that the

2020 Mathematics Subject Classification. Primary 20M30; Secondary 16G99.

Keywords. Generalized rook monoids, Jucys–Murphy elements, Gelfand–Zeitlin basis, bicyclic monoid, Grothendieck group.

rook monoid algebra $\mathbb{C}[R_n]$ is semisimple, moreover, all simple modules over this algebra are very well understood; see [8, 10, 35].

For a positive integer r, let C_r denote the multiplicative cyclic group of order r. We can consider the wreath product $C_r \wr R_n$, called the *generalized rook monoid* in [34], whose elements are all $n \times n$ matrices with entries from $C_r \cup \{0\}$ and with the condition that each row and each column contains at most one non-zero entry. Many of the results on the representations of the rook monoid obtained in [33] were extended to the case of the generalized rook monoid in [34].

Motivated by the construction of the irreducible representations as seminormal representations in the case of symmetric groups and generalized symmetric groups, in this article, we give a similar construction of the irreducible representations of $C_r \wr R_n$ in Theorem 3.2. The set of elements of $C_r \wr R_n$ whose (n,n)-th entry is equal to 1 is a submonoid of $C_r \wr R_n$ and this submonoid is isomorphic to $C_r \wr R_{n-1}$. Now, viewing $C_r \wr R_{n-1}$ as a submonoid of $C_r \wr R_n$ in this way, we have the following tower of generalized rook monoid algebras:

$$\mathbb{C}[C_r \wr R_0] \subset \mathbb{C}[C_r \wr R_1] \subset \cdots \subset \mathbb{C}[C_r \wr R_n] \subset \cdots . \tag{1}$$

For $r = 1$, the branching rule for the restriction of an irreducible representation for each successive inclusion of algebras in (1) is multiplicity-free by [11, Section 3]. This means that, in this case, the Bratteli diagram of (1) is a simple graph. In this article, we prove a similar result for an arbitrary positive integer r in Corollary 3.4. In particular, this gives a natural basis of each irreducible representation of an algebra in the tower (1) indexed by certain paths in the Bratteli diagram, usually called the *Gelfand–Zeitlin basis*. If we replace, in (1), $\mathbb{C}$ by an algebraically closed field $\Bbbk$ of positive characteristic, then our method gives a modular branching rule as well. We construct a Gelfand model for $\mathbb{C}[C_r \wr R_n]$ in Proposition 3.5 which is a generalization of the case $r = 1$ as considered in [23], see also [27] and [13].

The construction of seminormal representations of the symmetric group S_n is closely connected to the existence of some special elements, called Jucys–Murphy elements, in the group algebra $\mathbb{C}[S_n]$; see the introduction of [32]. Jucys–Murphy elements for $\mathbb{C}[R_n]$ were constructed in [29]. In Section 4, we construct Jucys–Murphy elements for $\mathbb{C}[C_r \wr R_n]$ (these elements are defined over any field in which r is non-zero). Moreover, we observe that the expression for Jucys–Murphy elements of $\mathbb{C}[R_n]$, given in Section 4, is simpler than the one in [29]. We also show that Jucys–Murphy elements satisfy the fundamental properties similar to the ones from the classical setup of symmetric groups. In particular, we have:

(a) Proposition 4.1 shows that these elements commute with each other.

(b) Theorem 4.2 proves that these elements act as scalars on all elements of the Gelfand–Zeitlin basis of every simple $\mathbb{C}[C_r \wr R_n]$-module.

(c) Corollary 4.4 states that the eigenvalues of the action of Jucys–Murphy elements on elements of the Gelfand–Zeitlin basis distinguish non-isomorphic simple modules.

Now let $\Bbbk$ be an algebraically closed field of positive characteristic p. For a finite-dimensional associative $\Bbbk$-algebra A, let A-mod denote the category of finite-dimensional left A-modules. Consider the Grothendieck group $K_0(A\text{-mod})$ of A-mod and the complexified Grothendieck group $G_0(A) = \mathbb{C} \otimes_{\mathbb{Z}} K_0(A\text{-mod})$, where $\mathbb{Z}$ denotes the ring of integers.

Let $\mathbb{N}$ denote the set of all non-negative integers. A classical result, proved in [24], asserts that

$$\bigoplus_{n\in\mathbb{N}} G_0\big(\Bbbk[S_n]\big)$$

has the natural structure of a module over the universal enveloping algebra $U(\widehat{\mathfrak{sl}}_p(\mathbb{C}))$ of the affine Lie algebra $\widehat{\mathfrak{sl}}_p(\mathbb{C})$ of type $A^{(1)}_{p-1}$. Moreover, this module can be identified as the basic representation $V(\Lambda_0)$ of $U(\widehat{\mathfrak{sl}}_p(\mathbb{C}))$. This result was also established in [9] for a more general setting with different techniques. One of the ways to obtain these results is to define the i-restriction and i-induction functors, for $0 \le i \le p-1$, using Jucys–Murphy elements of $\Bbbk[S_n]$. Then one can show that, at the level of Grothendieck group, the functors satisfy the relations for the Chevalley generators of $\widehat{\mathfrak{sl}}_p(\mathbb{C})$.

Motivated by these classical results, we use our Jucys–Murphy elements for rook monoid algebras to define, for $0 \le i \le p-1$, the i-restriction functor res_i and the i-induction functor ind_i in the rook monoid setup; see (19) and (20). We also define two extra functors $\mathbb{A}$ and $\mathbb{B}$ which correspond to the additional edges in the Bratteli diagram for rook monoids; see (20). In Theorem 6.16, we show that, at the level of the direct sum

$$\bigoplus_{n\in\mathbb{N}} G_0\big(\Bbbk[R_n]\big), \tag{2}$$

of the Grothendieck groups, the functors res_i and ind_i, for $0 \le i \le p-1$, satisfy the relations for the Chevalley generators of $\widehat{\mathfrak{sl}}_p(\mathbb{C})$. Additionally, the functors $\mathbb{A}$ and $\mathbb{B}$ satisfy the relation of the generators of the bicyclic monoid $\mathcal{B}$ and commute with all res_i and ind_i. Furthermore, we show that the Grothendieck group (2) is isomorphic to the tensor product of the basic representation $V(\Lambda_0)$ with the unique simple infinite-dimensional $\mathbb{C}[\mathcal{B}]$-module $V_{\mathbb{N}}$, as modules over $U(\widehat{\mathfrak{sl}}_p(\mathbb{C})) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{B}]$.

Assume that r is non-zero in $\Bbbk$. Then, using the result for the generalized symmetric group algebras similar to the ones proved in [36], and also in [37], in Subsection 6.4

we conclude that the Grothendieck group

$$\bigoplus_{n\in\mathbb{N}} G_0\big(\Bbbk[C_r \wr R_n]\big)$$

is isomorphic to $V(\Lambda_0)^{\otimes r} \otimes_{\mathbb{C}} V_{\mathbb{N}}$ as a module over the algebra $U(\widehat{\mathfrak{sl}}_p(\mathbb{C}))^{\otimes r} \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{B}]$.

It is well known that $\bigoplus_{n\in\mathbb{N}} G_0(\Bbbk[S_n])$ is a Hopf algebra, where the multiplication and the comultiplication are obtained by using appropriate induction and restriction functors, respectively, e.g., see [26, Chapter I]. In Theorem 6.20, we prove that (2) is a bialgebra where the multiplication and the comultiplication are again obtained by using certain induction and restriction functors, respectively.

2. Generalized rook monoids

In what follows, $\Bbbk$ is an algebraically closed field.

Recall that $C_r \wr R_n$ denote the generalized rook monoid. For $0 \le i \le n$, let $f_i \in C_r \wr R_n$ be the diagonal matrix whose (k,k)-th entry is 0, when $i+1 \le k \le n$, and the remaining diagonal entries are equal to 1. Note that f_0 and f_n are the zero matrix and the identity matrix in $C_r \wr R_n$, respectively.

Green's left cell $\mathbb{L}_i^n$ of $C_r \wr R_n$ corresponding to the idempotent f_i consists, by definition, of all $\sigma \in C_r \wr R_n$ satisfying

$$(C_r \wr R_n)\sigma = (C_r \wr R_n) f_i.$$

Then $\mathbb{L}_i^n$ consists of all rank i matrices in $C_r \wr R_n$ whose j-th column is zero, for all $i+1 \le j \le n$. The maximal subgroup of $C_r \wr R_n$ corresponding to f_i is the group of units of the submonoid $f_i(C_r \wr R_n) f_i$. The latter consists of all matrices in $C_r \wr R_n$ whose non-zero entries lie inside the first $i \times i$ block. This subgroup is evidently isomorphic to the generalized symmetric group $C_r \wr S_i$, where S_i is the symmetric group on i letters. Unless stated otherwise, we use this identification throughout the manuscript. Note that $C_r \wr S_i$ acts on $\mathbb{L}_i^n$ from the right in the obvious way.

Let $[n] := \{1,2,\dots,n\}$ and $\mathcal{X}_i := \{Z \subseteq [n] \mid |Z| = i\}$. For $Z \in \mathcal{X}_i$, write

$$Z = \{r_1 < r_2 < \cdots < r_i\},$$

and let $h_Z^n \in C_r \wr R_n$ be such that the non-zero entries of h_Z^n are equal to 1 and they are at the coordinates $(r_1,1),\dots,(r_i,i)$. Note that $h_Z^n \in \mathbb{L}_i^n$ and, moreover, these matrices form a cross-section of the orbits of the right action of $C_r \wr S_i$ on $\mathbb{L}_i^n$. In other words, $\Bbbk\mathbb{L}_i^n$ is a free right $\Bbbk[C_r \wr S_i]$-module which admits a $\Bbbk[C_r \wr S_i]$-basis consisting of all matrices of the form h_Z^n, where $Z \in \mathcal{X}_i$ (we use this basis often in what follows). For an element $\sigma \in S_n$, $\sigma(Z)$ denotes the image of Z under the function σ on n letters.

The space $\Bbbk\mathbb{L}_i^n$ is also naturally a left $\Bbbk[C_r \wr R_n]$-module where, for $\tau \in C_r \wr R_n$ and $\sigma \in \mathbb{L}_i^n$, the action is given by

$$\tau\sigma = \begin{cases} \tau\sigma & \text{if } \tau\sigma \in \mathbb{L}_i^n, \\ 0 & \text{otherwise.} \end{cases}$$

These two actions on $\Bbbk\mathbb{L}_i^n$ obviously commute, making $\Bbbk\mathbb{L}_i^n$ a $(\Bbbk[C_r \wr R_n], \Bbbk[C_r \wr S_i])$-bimodule. The associated functor

$$\mathcal{L}_i^n := (\Bbbk\mathbb{L}_i^n \otimes_{\Bbbk[C_r \wr S_i]} -) \colon \Bbbk[C_r \wr S_i]\text{-mod} \to \Bbbk[C_r \wr R_n]\text{-mod}$$

is full, faithful and exact; see [35, Chapter 4].

Lemma 2.1. *The following functor is an equivalence of categories*

$$\bigoplus_{i=0}^{n} \mathcal{L}_i^n \colon \bigoplus_{i=0}^{n} \Bbbk[C_r \wr S_i]\text{-mod} \to \Bbbk[C_r \wr R_n]\text{-mod}.$$

Proof. This follows by combining the standard facts that, for $0 \leq i \leq n$, the right $\Bbbk[C_r \wr S_i]$-module $\Bbbk\mathbb{L}_i^n$ is free and that

$$\bigoplus_{i=0}^{n} \mathrm{End}_{\Bbbk[C_r \wr S_i]}(\Bbbk\mathbb{L}_i^n) \cong \Bbbk[C_r \wr R_n],$$

see [35, Section 10.2]. ∎

Generators. For $1 \leq j \leq n-1$, let s_j denote the simple transposition $(j, j+1)$ in S_n. Fix a primitive r-th root of unity ξ in C_r. Denote by $P \in C_r \wr R_n$ the diagonal matrix whose $(1,1)$-th entry is 0 and the remaining diagonal entries are equal to 1. Denote by $Q \in C_r \wr R_n$ the diagonal matrix whose $(1,1)$-th entry is ξ and the remaining diagonal entries are equal to 1. Then it is easy to check that $C_r \wr R_n$ is generated by P, Q and all s_j, where $1 \leq j \leq n-1$.

3. Seminormal representations

3.1. Bases of irreducible representations

In this section, we construct the irreducible representations of $C_r \wr R_n$ over $\mathbb{C}$, give a basis of an irreducible representation of $C_r \wr R_n$ and describe the actions of generators of $C_r \wr R_n$. We also give the branching rule for the restriction of an irreducible representation of $C_r \wr R_n$ to $C_r \wr R_{n-1}$. To obtain these results we need the following notation and definitions.

Let $\mathcal{P}$ denote the set of all partitions of all non-negative integers. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of a positive integer, its Young diagram $[\lambda]$ is given as

$$\{(p, q) \mid 1 \leq p \leq r \text{ and } 1 \leq q \leq \lambda_p\}.$$

We use the usual English notation for Young diagrams. The elements of $[\lambda]$ are called *boxes*. By convention, the Young diagram of 0 is denoted $\varnothing$. For $\lambda \in \mathcal{P}$, let $|\lambda|$ denote the number of boxes in $[\lambda]$. For $\lambda \in \mathcal{P}$ with $|\lambda| \leq n$, let $\mathcal{Y}(\lambda, n)$ denote the set of all fillings of boxes of $[\lambda]$ with different elements from $[n]$ such that the entries increase along the rows from left to right and along the columns from top to bottom. Consider the following set of multipartitions of weight n:

$$\Lambda_r(n) := \Big\{\lambda^{(r)} = (\lambda_{(1)}, \ldots, \lambda_{(r)}) \mid \lambda_{(i)} \in \mathcal{P} \text{ for } 1 \leq i \leq r, \text{ and } \sum_{i=1}^{r} |\lambda_{(i)}| = n\Big\}.$$

Let $\lambda^{(r)} \in \Lambda_r(n)$ and m be a non-negative integer such that $n \leq m$. We define $\mathcal{Y}(\lambda^{(r)}, m)$ as the set

$$\big\{(L_1, \ldots, L_r) \mid L_i \in \mathcal{Y}(\lambda_{(i)}, m) \text{ for } 1 \leq i \leq r; \\ L_i \text{ and } L_j \text{ don't have common entries for } 1 \leq i \neq j \leq r\big\}.$$

Let $L = (L_1, \ldots, L_r) \in \mathcal{Y}(\lambda^{(r)}, m)$ and $1 \leq b \leq m$. We write $b \in L$ if b appears in one of L_k, for $1 \leq k \leq r$, and we also say "$b \in L$ at the position k" if b appears in L_k. Define the sign of b in L as

$$\operatorname{sgn}_L(b) := \begin{cases} \xi^{k-1} & \text{if } b \in L \text{ at the position } k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $b \in L$ be at the position k and, further, assume that b is in the box (m_1, m_2) in L_k. Define the content of b as $\operatorname{ct}(L(b)) := (m_2 - m_1)$. If both i and $i + 1$ appear in L at the position k (in particular, $\operatorname{sgn}_L(i) = \operatorname{sgn}_L(i + 1)$), define

$$a_L(i) := \frac{1}{\operatorname{ct}(L(i + 1)) - \operatorname{ct}(L(i))}.$$

Given $L \in \mathcal{Y}(\lambda^{(r)}, n)$, let $s_i L$ be obtained from L by replacing i by $i + 1$ if $i \in L$, and by replacing $i + 1$ by i if $i + 1 \in L$. Note that it may happen that $s_i L$ does not lie in $\mathcal{Y}(\lambda^{(r)}, n)$. For the next statement, we refer, e.g., to [12, p. 169]; see also [3].

Theorem 3.1. (a) *The elements of $\Lambda_r(n)$ index the isomorphism classes of irreducible representations of $\mathbb{C}[C_r \wr S_n]$.*

(b) *For* $\lambda^{(r)} \in \Lambda_r(n)$*, the corresponding irreducible representation* $W^n_{\lambda^{(r)}}$ *of* $C_r \wr S_n$ *has a basis* $\{w_L | L \in \mathcal{Y}(\lambda^{(r)}, n)\}$ *on which the generators* s_j *(for* $1 \le j \le n-1$*) and* Q *act as follows:*

$$s_j w_L = \begin{cases} w_{s_j L} & \text{if } \operatorname{sgn}_L(j) \neq \operatorname{sgn}_L(j+1), \\ a_L(j) w_L + (1 + a_L(j)) w_{s_j L} & \text{if } \operatorname{sgn}_L(j) = \operatorname{sgn}_L(j+1), \end{cases} \tag{3}$$

$$Q w_L = \xi^{k-1} w_L \quad \text{if } 1 \in L \text{ at the position } k. \tag{4}$$

Here, $w_{s_j L} = 0$ *if* $s_j L \notin \mathcal{Y}(\lambda^{(r)}, n)$*.*

The next claim is a generalization of Theorem 3.1 to the case of $C_r \wr R_n$.

Theorem 3.2. (a) *The elements of* $\Lambda_r(\le n) := \bigcup_{i=0}^{n} \Lambda_r(i)$ *index the irreducible representations of* $\mathbb{C}[C_r \wr R_n]$ *in the following way: for* $\lambda^{(r)} \in \Lambda_r(i)$*, the corresponding irreducible representation is*

$$V^n_{\lambda^{(r)}} := \mathbb{C}\mathbb{L}^n_i \otimes_{\mathbb{C}[C_r \wr S_i]} W^i_{\lambda^{(r)}}.$$

(b) $V^n_{\lambda^{(r)}}$ *has a basis* $\{v_L | L \in \mathcal{Y}(\lambda^{(r)}, n)\}$ *on which the generators* P*,* Q*, and* s_j*, for* $1 \le j \le n-1$*, act as follows:*

$$s_j v_L = \begin{cases} v_{s_j L} & \text{if } j \in L,\ j+1 \notin L, \\ v_{s_j L} & \text{if } j \notin L,\ j+1 \in L, \\ v_L & \text{if } j \notin L,\ j+1 \notin L, \\ v_{s_j L} & \text{if } j \in L,\ j+1 \in L \\ & \text{and } \operatorname{sgn}_L(j) \neq \operatorname{sgn}_L(j+1), \\ a_L(j) v_L + (1 + a_L(j)) v_{s_j L} & \text{if } j \in L,\ j+1 \in L \\ & \text{and } \operatorname{sgn}_L(j) = \operatorname{sgn}_L(j+1), \end{cases} \tag{5}$$

$$P v_L = \begin{cases} v_L & \text{if } 1 \notin L, \\ 0 & \text{otherwise,} \end{cases} \qquad Q v_L = \begin{cases} \xi^{k-1} v_L & \text{if } 1 \in L \text{ at the position } k, \\ v_L & \text{otherwise.} \end{cases} \tag{6}$$

Here, $v_{s_j L} = 0$ *if* $s_j L \notin \mathcal{Y}(\lambda^{(r)}, n)$*.*

Proof. The first claim follows directly from the general theory; see [35], so we only prove the second claim. (One can also see it by combining Lemma 2.1 and Theorem 3.2.) Recall that, as a right $\Bbbk[C_r \wr S_i]$-module, $\Bbbk\mathbb{L}^n_i$ has a basis consisting of matrices of the form h^n_Z, where $Z \in \mathcal{X}_i$.

Fix $\lambda^{(r)} \in \Lambda_r(i)$. For $Z = \{r_1 < r_2 < \cdots < r_i\} \subseteq [n]$ and $L' \in \mathcal{Y}(\lambda^{(r)}, i)$, define $L \in \mathcal{Y}(\lambda^{(r)}, n)$ by replacing $l \in L$ by r_l for all $1 \le l \le i$. Conversely, given $L \in \mathcal{Y}(\lambda^{(r)}, n)$, let Z be the set of the entries in L. We can arrange these entries in the increasing order to get $Z = \{r_1 < r_2 < \cdots < r_i\}$. Now, replacing $r_l \in L$ by l, we obtain an element $L' \in \mathcal{Y}(\lambda^{(r)}, i)$. Then

$$\{v_L := h^n_Z \otimes w_{L'} \mid Z \in \mathcal{X}_i \text{ and } L' \in \mathcal{Y}(\lambda^{(r)}, i)\} = \{v_L \mid L \in \mathcal{Y}(\lambda^{(r)}, n)\}$$

is, by construction, a basis of $V^n_{\lambda^{(r)}}$.

Next we compute the action of s_j. For $1 \leq j \leq n-1$, we have

$$s_j v_L = s_j(h^n_Z \otimes w_{L'}) = s_j h^n_Z \otimes w_{L'} = h^n_{s_j(Z)} \otimes (h^n_{s_j(Z)})^{\mathrm{tr}} s_j h^n_Z w_{L'},$$

where $(h^n_{s_j(Z)})^{\mathrm{tr}}$ denotes the transpose of $h^n_{s_j(Z)}$.

Case 1. Suppose that we have $j \notin L$ or $j+1 \notin L$. This means that $j \notin Z$ or $j+1 \notin Z$, respectively. In this case, $(h^n_{s_j(Z)})^{\mathrm{tr}} s_j h^n_Z = f_i \in C_r \wr S_i$, which is the identity of $C_r \wr S_i$. Therefore, we have

$$h^n_{s_j(Z)} \otimes (h^n_{s_j(Z)})^{\mathrm{tr}} s_j h^n_Z w_{L'} = h^n_{s_j(Z)} \otimes w_{L'} = v_{s_j L}.$$

Furthermore, if both $j \notin L$ and $j+1 \notin L$, then $v_{s_j L} = v_L$. This completes the description of the action of s_j for the first three cases in (5).

Case 2. Suppose that $j \in L$ and $j+1 \in L$ or, equivalently, $j \in Z$ and $j+1 \in Z$. Then $(h^n_{s_j(Z)})^{\mathrm{tr}} s_j h^n_Z$ is a $(j, j+1)$ transposition in $C_r \wr S_i$. Then the remaining two cases in (5) follow from (3).

To compute the action of P, we start with $Pv_L = P(h^n_Z \otimes w_{L'}) = Ph^n_Z \otimes w_{L'}$. Note that $Ph^n_Z \in \mathbb{L}^n_i$ if and only if $1 \notin Z$. In particular, we have

$$Ph^n_Z = \begin{cases} h^n_Z & \text{if } 1 \notin Z, \\ 0 & \text{otherwise.} \end{cases}$$

This implies the formula for the action of P in (6).

The action of Q in (6) can be computed similarly using (4). ■

3.2. The restriction functor

As we already mentioned, the set consisting of all matrices in $C_r \wr R_n$ whose (n, n)-th entry is 1 is a submonoid of $C_r \wr R_n$ and it is isomorphic to $C_r \wr R_{n-1}$. This defines an embedding $C_r \wr R_{n-1} \subset C_r \wr R_n$. Similarly, we have $C_r \wr S_{n-1} \subset C_r \wr S_n$.

Denote by $\mathcal{F}$ the functor

$$\mathcal{F}: \bigoplus_{i=0}^{n} \Bbbk[C_r \wr S_i]\text{-mod} \to \bigoplus_{j=0}^{n-1} \Bbbk[C_r \wr S_j]\text{-mod}$$

given by

$$\mathcal{F}|_{K[C_r \wr S_i]\text{-mod}} = \begin{cases} \mathrm{Id}_{\Bbbk[C_r \wr S_0]\text{-mod}} & \text{if } i = 0, \\ \mathrm{Res}^{\Bbbk[C_r \wr S_n]}_{\Bbbk[C_r \wr S_{n-1}]} & \text{if } i = n, \\ \mathrm{Id}_{\Bbbk[C_r \wr S_i]\text{-mod}} \oplus \mathrm{Res}^{\Bbbk[C_r \wr S_i]}_{\Bbbk[C_r \wr S_{i-1}]} & \text{if } 0 < i < n. \end{cases}$$

Theorem 3.3. *The following diagram commutes up to a natural isomorphism of functors:*

$$\begin{tikzcd} \bigoplus_{i=0}^{n} \Bbbk[C_r \wr S_i]\text{-mod} \arrow[rr, "\bigoplus_{i=0}^{n} \mathcal{L}_i^n"] \arrow[d, "\mathcal{F}"] & & \Bbbk[C_r \wr R_n]\text{-mod} \arrow[d, "\mathrm{Res}^{\Bbbk[C_r \wr R_n]}_{\Bbbk[C_r \wr R_{n-1}]}"] \\ \bigoplus_{j=0}^{n-1} \Bbbk[C_r \wr S_j]\text{-mod} \arrow[rr, "\bigoplus_{j=0}^{n-1} \mathcal{L}_j^{n-1}"] & & \Bbbk[C_r \wr R_{n-1}]\text{-mod.} \end{tikzcd}$$

Proof. Recall that, as a right $\Bbbk[C_r \wr S_i]$-module, $\Bbbk\mathbb{L}_i^n$ has a basis consisting of matrices of the form h_Z^n, where $Z \in \mathcal{X}_i$. We need to consider several cases.

Case 1. Assume $0 < i < n$ and let $V \in \Bbbk[C_r \wr S_i]$-mod. Then the linear span of all $h_Z^n \otimes V$, where $Z \in \mathcal{X}_i$ is such that $n \notin Z$, is a subspace of $\mathcal{L}_i^n(V)$ which is stable under the action of $\Bbbk[C_r \wr R_{n-1}]$. It is easy to see that this $\Bbbk[C_r \wr R_{n-1}]$-module is isomorphic to $\mathcal{L}_i^{n-1}(V)$.

Similarly, the linear span of all $h_Z^n \otimes V$, where $Z \in \mathcal{X}_i$ is such that $n \in Z$, is a subspace of $\mathcal{L}_i^n(V)$ which is stable under the action of $\Bbbk[C_r \wr R_{n-1}]$. It is easy to see that this $\Bbbk[C_r \wr R_{n-1}]$-module is isomorphic to $\mathcal{L}_{i-1}^{n-1}(V)$.

Case 2. In the case $i = n$, we have that $\Bbbk\mathbb{L}_n^n$ is the right regular $\Bbbk[C_r \wr S_n]$-module and the necessary claim follows from the construction.

Case 3. In the case $i = 0$, both $\Bbbk\mathbb{L}_0^n$ and the group algebra of $C_r \wr S_0$ are isomorphic to $\Bbbk$ and the claim is trivial. ∎

Now we give some applications of Theorem 3.3.

For $\lambda \in \mathcal{P}$, an *outer corner* of $[\lambda]$ (also known as *removable node*) is a box $(i, j) \in [\lambda]$ such that $[\lambda] \setminus \{(i, j)\}$ is still a Young diagram. For $\lambda^{(r)} \in \Lambda_r(\leq n)$, we define $(\lambda^{(r)})^{-,=}$ as the set consisting of $\lambda^{(r)}$ and all elements in $\Lambda_r(\leq n-1)$, which are obtained from $\lambda^{(r)}$ by removing an outer corner in one of the Young diagrams which constitute $\lambda^{(r)}$. Further, let $(\lambda^r)^- := (\lambda^r)^{-,=} \setminus \{\lambda^{(r)}\}$.

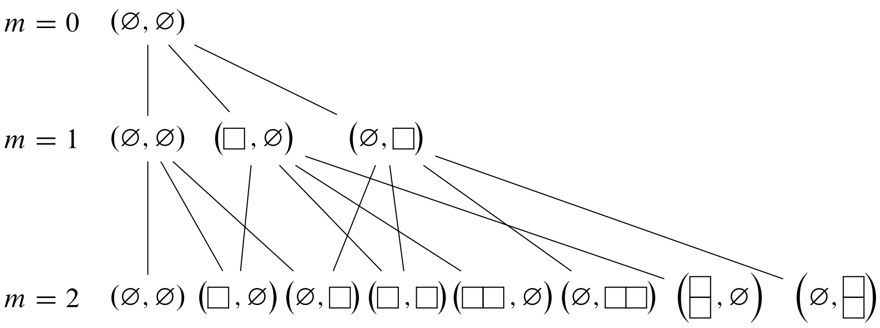

Figure 1. Bratteli diagram for the tower of generalized rook monoid algebras in the case $r = 2$, up to level 2.

Corollary 3.4 (Branching rule over $\mathbb{C}$). *For $\lambda^{(r)} \in \Lambda_r(\leq n)$, we have*

$$\operatorname{Res}^{\mathbb{C}[C_r \wr R_n]}_{\mathbb{C}[C_r \wr R_{n-1}]}(V^n_{\lambda^{(r)}}) \cong \begin{cases} \displaystyle\bigoplus_{\mu^{(r)} \in (\lambda^{(r)})^{-,=}} V^{n-1}_{\mu^{(r)}} & \text{if } \lambda^{(r)} \in \Lambda_r(\leq n-1), \\ \displaystyle\bigoplus_{\mu^{(r)} \in (\lambda^{(r)})^{-}} V^{n-1}_{\mu^{(r)}} & \text{if } \lambda^{(r)} \in \Lambda_r(n). \end{cases}$$

Proof. It is a consequence of the branching rule for $\mathbb{C}[C_r \wr S_{n-1}] \subset \mathbb{C}[C_r \wr S_n]$ and Theorem 3.3. ■

The Bratteli diagram of the tower (1) is an undirected graph whose vertices at the level n are given by the elements of $\Lambda_r(\leq n)$. For two vertices $\lambda^{(r)} \in \Lambda_r(\leq n)$ and $\mu^{(r)} \in \Lambda_r(\leq n-1)$, there is an edge between $\mu^{(r)}$ and $\lambda^{(r)}$ if and only if $\mu^{(r)} \in (\lambda^{(r)})^{-,=}$, cf. [31, p. 584]. A path from the vertex $(\varnothing, \varnothing, \dots, \varnothing)$, at the level $m = 0$, to the vertex $\lambda^{(r)}$, at the level $m = n$, in the Bratteli diagram is a list

$$(\nu^{(r)}_0, \nu^{(r)}_1, \dots, \nu^{(r)}_n = \lambda^{(r)})$$

of vertices such that the vertex $\nu^{(r)}_i$ is at the level $m = i$ for $0 \leq i \leq n$, and there is an edge between $\nu^{(r)}_j$ and $\nu^{(r)}_{j+1}$, for $0 \leq j \leq n-1$. By construction, the Bratteli diagram encodes the branching rule in Corollary 3.4. In order to exhibit the Bratteli diagram, often it is more intuitive to consider the Young diagram corresponding to a partition and in the below we follow this.

In Figure 1, we illustrate the branching rule in the case $r = 2$ by the corresponding Bratteli diagram for the tower of generalized rook monoid algebras, up to the second level.

Observe that the branching rule in Corollary 3.4 is multiplicity-free. Therefore, there is a basis of $V^n_{\lambda^{(r)}}$, defined uniquely up to rescaling of its elements, which is indexed by the paths from the vertex at the level $m = 0$ to a vertex, say $\lambda^{(r)}$, at the level $m = n$ in the Bratteli diagram; see, e.g., [31, p. 585], where such a basis is called a *Gelfand–Zeitlin basis*. We note that the set of all such paths is in a bijective correspondence with $\mathcal{Y}(\lambda^{(r)}, n)$. From Theorem 4.2 which will be proved later, it follows that the basis constructed in Theorem 3.2 coincides with a Gelfand–Zeitlin basis of $V^n_{\lambda^{(r)}}$. For $L \in \mathcal{Y}(\lambda^{(r)}, n)$, the vector v_L given in the part of Theorem 3.2 is called a *Gelfand–Zeitlin basis vector*.

Now we outline the modular branching rule for generalized rook monoids as a consequence of Theorem 3.3 and the corresponding rule for generalized symmetric groups. Let p be a prime number and assume that $\Bbbk$ is of characteristic p. Recall that a partition is called *p-regular* if it does not have more than $p-1$ parts that are equal. Let $\Lambda^p(n)$ denote the set of all p-regular partitions of n. It is known that the simple modules of $\Bbbk[S_n]$ are indexed by the elements of $\Lambda^p(n)$; see [14] or [15, Theorem 11.5]. For $\lambda \in \Lambda^p(n)$, let D^λ denote the corresponding simple $\Bbbk[S_n]$-module. In [20,21] one can find a modular branching rule for symmetric groups, that is a description of the socle of $\mathrm{Res}^{\Bbbk[S_n]}_{\Bbbk[S_{n-1}]}(D^\lambda)$. In particular, this asserts that the socle of $\mathrm{Res}^{\Bbbk[S_n]}_{\Bbbk[S_{n-1}]}(D^\lambda)$ is multiplicity-free. A classification of simple modules over $\Bbbk[C_r \wr S_n]$ can be found in [16]. A modular branching rule result for generalized symmetric group, under the assumption that r is non-zero in $\Bbbk$, was obtained in [36]. Combining these results with Theorem 3.3, one obtains a modular branching rule for the generalized rook monoid algebras, in particular, it follows that this branching is multiplicity-free.

3.3. Gelfand model for $\mathbb{C}[C_r \wr R_n]$

A *Gelfand model* of a finite-dimensional semisimple algebra A is a multiplicity-free additive generator of A-mod; see [1, 13, 27] for further details.

For $\sigma \in C_r \wr S_n$, let $\mathrm{Inv}(\sigma)$ be the set consisting of all $(i, j) \in [n] \times [n]$ such that $i < j$ and the non-zero entry in the i-th column of σ appears in a later row than the non-zero entry in the j-th column. Similarly, for $\sigma \in C_r \wr S_n$, let $\mathrm{Pair}(\sigma)$ be the set consisting of all pairs $(i, j) \in [n] \times [n]$ such that $i < j$, the non-zero entry in the i-th column of σ appears in row j and the non-zero entry in the j-th column appears in row i. For a matrix A, let A^{tr} denote its transpose. Let $\mathcal{W}_n$ be the $\mathbb{C}$-span of $\mathcal{J} = \{\sigma \in C_r \wr S_n \mid \sigma = \sigma^{\mathrm{tr}}\}$. For $\omega \in C_r \wr S_n$ and $\sigma \in \mathcal{J}$, set

$$\omega\sigma = (-1)^{|\mathrm{Inv}(\omega) \cap \mathrm{Pair}(\sigma)|} \omega\sigma\omega^{-1}.$$

In [2], it is shown that $\mathcal{W}_n$ is a Gelfand model for $\mathbb{C}[C_r \wr S_n]$, see also [28, Section 2.7].

Let $\mathcal{V}$ be the $\mathbb{C}$-span of $\mathcal{I} = \{M \in C_r \wr R_n \mid M = M^{\mathrm{tr}}\}$. For $M \in C_r \wr R_n$, let $\mathrm{Pair}(M)$ denote the set consisting of all pairs $(i, j) \in [n] \times [n]$ such that $i < j$, the non-zero entry in the i-th column of M appears in row j and the non-zero entry in the j-th column of M appears in row i. We define the action of the generators of $\mathbb{C}[C_r \wr R_n]$ on $\mathcal{V}$ as follows: for $1 \le i \le n-1$, set

$$s_i M = \begin{cases} -s_i M s_i & \text{if } (i, i+1) \in \operatorname{Pair}(M), \\ s_i M s_i & \text{otherwise}, \end{cases} \tag{7}$$

$$QM = QMQ^{-1}, \quad PM = \begin{cases} M & \text{if the first row of } M \text{ is zero}, \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

Proposition 3.5. The module $\mathcal{V}$ is a Gelfand model for $\mathbb{C}[C_r \wr R_n]$.

Proof. For $0 \le m \le n$, let $\mathcal{V}_m$ be the $\mathbb{C}$-span of $\mathcal{I}_m = \{M \in \mathcal{I} \mid \operatorname{rank}(M) = m\}$. From (7) and (8), we see that each $\mathcal{V}_m$ is closed under the action, moreover, it is easy to see that, directly by construction, we have $\mathcal{V}_m \cong \mathcal{L}^n_m(\mathcal{W}_m)$.

Since $\mathcal{V} = \bigoplus_{m=0}^n \mathcal{V}_m$, the claim of the proposition follows by combining the facts that $\mathcal{W}_m$ is a Gelfand model for $\mathbb{C}[C_r \wr S_m]$ and that $\bigoplus_{m=0}^n \mathcal{L}^n_m$ is an equivalence of categories, see Lemma 2.1. ■

4. Jucys–Murphy elements

Given any subset A of $[n]$, let e_A be the diagonal matrix which has 1 at the (i,i)-th entry for $i \in [n] \setminus A$ and zeros elsewhere. Consider the element

$$E_A := \sum_{B \subseteq A} (-1)^{|B|} e_B.$$

Then $E_A \in \Bbbk[R_n]$ is an idempotent and, moreover, any two such idempotents commute with each other (since all idempotents of R_n commute). Assume that r is non-zero in $\Bbbk$. Consider the following elements in $\Bbbk[C_r \wr R_n]$:

$$X_1 = Q - P, \quad X_j = s_{j-1} X_{j-1} s_{j-1} \qquad \text{for } 2 \le j \le n,$$

$$Y_1 = 0, \quad Y_j = \frac{1}{r} \sum_{m=1}^{j-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi^l_m \xi^{-l}_j (m, j) \qquad \text{for } 2 \le j \le n,$$

where $\xi^l_m \xi^{-l}_j \in C_r \wr S_n$ denotes the diagonal matrix with 1's on the diagonal except for ξ^l at the (m,m)-th entry and ξ^{-l} at the (j,j)-th entry. Further, (m, j) is the usual transposition in S_n. It is easy to observe that

$$\xi^l_m \xi^{-l}_j = (1, m) Q^l (1, m)(1, j) Q^{-l} (1, j). \tag{9}$$

We will call the elements $\{X_i, Y_i \mid 1 \le i,\ j \le n\}$ the *Jucys–Murphy elements* for $C_r \wr R_n$.

For $2 \leq j \leq n$, it is easy to observe the following relation between Y_{j-1} and Y_j:

$$Y_j = s_{j-1} Y_{j-1} s_{j-1} + \frac{1}{2} E_{\{j-1,j\}} \sum_{l=0}^{r-1} \xi_{j-1}^{l} \xi_j^{-l} (j-1, j).$$

Proposition 4.1. For $1 \leq i, j \leq n$, we have

$$X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i, \quad \text{and} \quad X_i Y_j = Y_j X_i.$$

Proof. For $1 \leq j \leq n$, let $P_j = e_{\{j\}}$. Note that $P = P_{\{1\}}$. Let $Q_1 = Q$ and, for $2 \leq j \leq n$, let $Q_j = s_{j-1} Q s_{j-1}$. Then Q_j is a diagonal matrix whose (j, j)-th diagonal entry is equal to ξ and the remaining diagonal entries are equal to 1. For $1 \leq i \leq n$, we can write $X_i = Q_i - P_i$, which is a linear combination of diagonal matrices. Thus, $X_i X_j = X_j X_i$.

To prove $Y_i Y_j = Y_j Y_i$, without loss of generality, we may assume that $i < j$. We can write

$$Y_j = \frac{1}{r} \Bigg(\sum_{m=1}^{i-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l} (m, j) + E_{\{i,j\}} \sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l} (i, j) + \sum_{m=i+1}^{j-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l} (m, j) \Bigg). \tag{10}$$

Now, Y_i commutes with $\sum_{m=i+1}^{j-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l} (m, j)$. The product of Y_i with the middle term of (10) can be written as

$$Y_i E_{\{i,j\}} \sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l} (i, j) = \frac{1}{r} \Bigg(\sum_{m=1}^{i-1} E_{\{m,i,j\}} \Bigg(\sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l} \Bigg) \Bigg(\sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l} \Bigg) (m, i)(i, j) \Bigg), \tag{11}$$

$$E_{\{i,j\}} \sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l} (i, j) Y_i = \frac{1}{r} \Bigg(\sum_{m=1}^{i-1} E_{\{m,i,j\}} \Bigg(\sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l} \Bigg) \Bigg(\sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l} \Bigg) (i, j)(m, i) \Bigg). \tag{12}$$

Also, we can write

$$Y_i \sum_{m=1}^{i-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l} (m, j)$$

$$
\begin{aligned}
&= \frac{1}{r}\left(\sum_{p=1}^{i-1} E_{\{p,i\}} \sum_{l=0}^{r-1} \xi_p^l \xi_i^{-l}(p,i)\right)\left(\sum_{m=1}^{i-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l}(m,j)\right) \\
&= \frac{1}{r}\left(\sum_{p=1}^{i-1} E_{\{p,i\}} \sum_{l=0}^{r-1} \xi_p^l \xi_i^{-l}(p,i)\right)\left(\sum_{m=1,m\neq p}^{i-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l}(m,j)\right) \\
&\quad + \frac{1}{r}\left(\sum_{p=1}^{i-1} E_{\{p,i\}} \sum_{l=0}^{r-1} \xi_p^l \xi_i^{-l}(p,i)\right)\left(E_{\{p,j\}} \sum_{l=0}^{r-1} \xi_p^l \xi_j^{-l}(p,j)\right). \qquad (13)
\end{aligned}
$$

In the above, we note that elements in the first term commutes with each other and the second term simplifies to

$$
\sum_{p=1}^{i-1} E_{\{p,i,j\}}\left(\sum_{l=0}^{r-1} \xi_p^l \xi_i^{-l}\right)\left(\sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l}\right)(p,i)(p,j),
$$

which is equal to (12) using, for $1 \leq p \leq i-1$, that $(p,i)(p,j) = (i,j)(p,i)$ and

$$
\left(\sum_{l=0}^{r-1} \xi_p^l \xi_i^{-l}\right)\left(\sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l}\right) = \left(\sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l}\right)\left(\sum_{l=0}^{r-1} \xi_p^l \xi_j^{-l}\right).
$$

Similarly to (13), we can write

$$
\begin{aligned}
&\sum_{m=1}^{i-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l}(m,j) Y_i \\
&\quad = \frac{1}{r}\left(\sum_{m=1}^{i-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l}(m,j)\right)\left(\sum_{p=1,p\neq i}^{i-1} E_{\{p,i\}} \sum_{l=0}^{r-1} \xi_p^l \xi_i^{-l}(p,i)\right) \\
&\qquad + \frac{1}{r}\left(\sum_{m=1}^{i-1} E_{\{m,j\}} \sum_{l=0}^{r-1} \xi_m^l \xi_j^{-l}(m,j)\right)\left(E_{\{m,i\}} \sum_{l=0}^{r-1} \xi_m^l \xi_i^{-l}(m,i)\right),
\end{aligned}
$$

where the elements in the first term commutes with each other and the second term is equal to (11). All of the above, finally, yield that $Y_i Y_j = Y_j Y_i$.

If $i > j$, then $X_i = Q_i - P_i$ commutes with each term in Y_j, which implies that $X_i Y_j = Y_j X_i$. Let us assume that $i \leq j$. Both Q_i and P_i commute with every element in the first and the last terms in (10). Now Q_i commutes with $E_{\{i,j\}}\sum_{l=0}^{r-1}\xi_i^l \xi_j^{-l}(i,j)$ because

$$
\sum_{l=0}^{r-1} \xi_i^{l+1} \xi_j^{-l} = \sum_{l=0}^{r-1} \xi_i^l \xi_j^{-l+1}.
$$

Furthermore, P_i commutes with $E_{\{i,j\}}\sum_{l=0}^{r-1}\xi_i^l \xi_j^{-l}(i,j)$ because

$$
P_i E_{\{i,j\}} = 0 = E_{\{i,j\}} P_j.
$$

This completes the proof. ∎

Theorem 4.2. *For $\lambda^{(r)} \in \Lambda_r(\leq n)$ and $L = (L_1, \dots, L_r) \in \mathcal{Y}(\lambda^{(r)}, n)$, let v_L be the corresponding Gelfand–Zeitlin basis vector of $V^n_{\lambda^{(r)}}$. For $1 \leq i \leq n$, we have*

$$X_i v_L = \begin{cases} \xi^{k-1} v_L & \text{if } i \in L \text{ and } \mathrm{sgn}_L(i) = \xi^{k-1}, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

$$Y_i v_L = \begin{cases} \mathrm{ct}(L(i)) v_L & \text{if } i \in L, \\ 0 & \text{otherwise}. \end{cases} \tag{15}$$

Proof. We begin with proving (14) using induction on i.

From (6), we have

$$X_1 v_L = Q v_L - P v_L = \begin{cases} \xi^{k-1} v_L & \text{if } 1 \in L \text{ and } \mathrm{sgn}_L(1) = \xi^{k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Assume now that (14) is true for $2 \leq i < m \leq n$, and let us prove it for $i = m$. By definition, $X_m = s_{m-1} X_{m-1} s_{m-1}$. We need to consider several cases.

Case 1. $\boldsymbol{m - 1 \in L, m \notin L.}$ Then $s_{m-1} v_L = v_{s_{m-1}L}$ and $s_{m-1}L$ does not contain $m - 1$. From the inductive assumption, we have $X_{m-1} v_{s_{m-1}L} = 0$ and this implies $X_m v_L = 0$.

Case 2. $\boldsymbol{m - 1 \notin L, m \in L.}$ Let us further assume that $\mathrm{sgn}_L(m) = \xi^{k-1}$, for some $1 \leq k \leq r$. Then $s_{m-1} v_L = v_{s_{m-1}L}$ and $\mathrm{sgn}_{s_{m-1}L}(m - 1) = \xi^{k-1}$. Applying the inductive assumption, we get the desired formula.

Case 3. $\boldsymbol{m - 1 \in L, m \in L}$ **and** $\boldsymbol{\mathrm{sgn}_L(m - 1) \neq \mathrm{sgn}_L(m).}$ This case is analogous to Case 2.

Case 4. $\boldsymbol{m - 1 \in L, m \in L}$ **and** $\boldsymbol{\mathrm{sgn}_L(m - 1) = \mathrm{sgn}_L(m) = \xi^{k-1}.}$ In this case, $m - 1 \in L_k$ and $m \in L_k$ for some $1 \leq k \leq r$. Then

$$s_{m-1} v_L = a_L(m - 1) v_L + (1 + a_L(m - 1)) v_{s_{m-1}L}.$$

Now we have to consider two subcases.

Subcase 4.1. $\boldsymbol{s_{m-1}L \notin \mathcal{Y}(\lambda^{(r)}, n).}$ Then $m - 1$ and m appear adjacent to each other either in the same row of L_k or in the same column of L_k. This implies that $a_L(m - 1) = \pm 1$. Also, recall that, by convention, in this case we have $v_{s_{m-1}L} = 0$. Now, applying the inductive assumption, we obtain

$$s_{m-1} X_{m-1} s_{m-1} v_L = a_L^2(m - 1) \xi^{k-1} v_L = \xi^{k-1} v_L.$$

Subcase 4.2. $s_{m-1}L \in \mathcal{Y}(\lambda^{(r)}, n)$. Using the inductive assumption, we have:

$$\begin{aligned} s_{m-1}X_{m-1}s_{m-1}v_L &= \xi^{k-1}\big(a_L(m-1)s_{m-1}v_L + (1+a_L(m-1))s_{m-1}v_{s_{m-1}L}\big) \\ &= \xi^{k-1}a_L(m-1)\big(a_L(m-1)v_L + (1+a_L(m-1))v_{s_{m-1}L}\big) \\ &+ \xi^{k-1}(1+a_L(m-1))\big(-a_L(m-1)v_{s_{m-1}L} + (1-a_L(m-1))v_L\big) \\ &= \xi^{k-1}\big(Av_L + Bv_{s_{m-1}L}\big), \end{aligned}$$

where

$$\begin{aligned} A &= a_L^2(m-1) + (1+a_L(m-1))(1-a_L(m-1)) = 1, \\ B &= a_L(m-1)(1+a_L(m-1)) - (1+a_L(m-1))a_L(m-1) = 0. \end{aligned}$$

This proves (14).

To prove (15), we again use induction on i. For $i = 1$, the claim is obvious. Assume now that (15) is true for $2 \leq i < m \leq n$, and let us prove it for $i = m$.

We have

$$Y_m = s_{m-1}Y_{m-1}s_{m-1} + \frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi_{m-1}^{l}\xi_m^{-l}\right)s_{m-1}.$$

We need to consider several cases. Note that in all the cases below, the action of $\xi_{m-1}^{l}\xi_m^{-l}$ is computed using (5), (6) and (9); in addition, we also use

$$E_{\{m-1,m\}}v_L = \begin{cases} v_L & \text{if } m-1 \in L, m \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Case 1. $m-1 \in L$, $m \notin L$. Then $s_{m-1}v_L = v_{s_{m-1}L}$ and $s_{m-1}L$ does not contain $m-1$. Further, assume that $\mathrm{sgn}_L(m-1) = \xi^{k-1}$, so that $\mathrm{sgn}_{s_{m-1}L}(m) = \xi^{k-1}$. By the inductive assumption, $s_{m-1}Y_{m-1}s_{m-1}v_L = s_{m-1}Y_{m-1}v_{s_{m-1}L} = 0$. Also,

$$\begin{aligned} \frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi_{m-1}^{l}\xi_m^{-l}\right)s_{m-1}v_L &= \frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi_{m-1}^{l}\xi_m^{-l}\right)v_{s_{m-1}L} \\ &= \frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi^{-l(k-1)}\right)v_{s_{m-1}L} \\ &= \frac{1}{r}\left(\sum_{l=0}^{r-1}\xi^{-l(k-1)}\right)E_{\{m-1,m\}}v_{s_{m-1}L} = 0. \end{aligned}$$

Case 2. $m-1 \notin L$, $m \in L$. In this case, $s_{m-1}v_L = v_{s_{m-1}L}$. Further, assume that $\mathrm{sgn}_L(m) = \xi^{k-1}$, so that $\mathrm{sgn}_{s_{m-1}L}(m-1) = \xi^{k-1}$. By the inductive assumption, we have

$$s_{m-1}Y_{m-1}s_{m-1}v_L = \mathrm{ct}((s_{m-1}L)(m-1))v_L = \mathrm{ct}(L(m))v_L.$$

Similarly to Case 1, the term

$$\frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi_{m-1}^{l}\xi_m^{-l}\right)s_{m-1}$$

acts as zero on v_L.

Case 3. $m-1 \in L$, $m \in L$. Assume that $\mathrm{sgn}_L(m-1) = \xi^{k_1-1}$ and $\mathrm{sgn}_L(m) = \xi^{k_2-1}$, where $1 \le k_1 \ne k_2 \le r$. By the inductive assumption, we have

$$s_{m-1}Y_{m-1}s_{m-1}v_L = \mathrm{ct}((s_{m-1}L)(m-1))v_L = \mathrm{ct}(L(m))v_L$$

and

$$\frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi_{m-1}^{l}\xi_m^{-l}\right)s_{m-1}v_L = \frac{1}{r}\left(\sum_{l=0}^{r-1}\xi^{l(k_2-k_1)}\right)v_{s_{m-1}L} = 0.$$

The last equality in the above is a consequence of the following: for an integer s, we have

$$\sum_{l=0}^{r-1}\xi^{ls} = \begin{cases} 0 & \text{if } s \ne 0, \\ r & \text{if } s = 0. \end{cases}$$

Case 4. $m-1 \in L$, $m \in L$ and $\mathrm{sgn}_L(m-1) = \mathrm{sgn}_L(m) = \xi^{k-1}$. This means that $m-1 \in L_k$ and $m \in L_k$ for some $1 \le k \le r$. Then we have

$$s_{m-1}v_L = a_L(m-1)v_L + (1 + a_L(m-1))v_{s_{m-1}L}.$$

We have now to consider two subcases.

Subcase 4.1. $s_{m-1}L \notin \mathcal{Y}(\lambda^{(r)}, n)$. Then $m-1$ and m appear adjacent to each other either in a same row of L_k or in a same column of L_k. This implies that $a_L(m-1) = \pm 1$. Also, recall our convention that $v_{s_{m-1}L} = 0$ in this case. Now, by the inductive assumption, we obtain

$$s_{m-1}Y_{m-1}s_{m-1}v_L = (a_L(m-1))^2\,\mathrm{ct}(L(m-1))v_L = \mathrm{ct}(L(m-1))v_L.$$

Further, we have

$$\frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi_{m-1}^{l}\xi_m^{-l}\right)s_{m-1}v_L = \frac{1}{r}E_{\{m-1,m\}}\left(\sum_{l=0}^{r-1}\xi_{m-1}^{l}\xi_m^{-l}\right)(a_L(m-1)v_L)$$

$$\begin{aligned}
&= \frac{1}{r} E_{\{m-1,m\}} \left(\sum_{l=0}^{r-1} \xi^{l(k-k)} \right) (a_L(m-1) v_L) \\
&= \frac{1}{r} E_{\{m-1,m\}} \left(\sum_{l=0}^{r-1} 1 \right) a_L(m-1) v_L \\
&= a_L(m-1) v_L.
\end{aligned}$$

Since $a_L(m-1) + \mathrm{ct}(L(m-1)) = \mathrm{ct}(L(m))$, we get the desired answer.

Subcase 4.2. $s_{m-1}L \in \mathcal{Y}(\lambda^{(r)}, n)$. Using the inductive assumption, we have

$$\begin{aligned}
&s_{m-1} Y_{m-1} s_{m-1} v_L \\
&\quad = a_L(m-1)\,\mathrm{ct}(L(m-1)) s_{m-1} v_L + (1 + a_L(m-1))\,\mathrm{ct}(L(m)) s_{m-1} v_{s_{m-1}L} \\
&\quad = a_L(m-1)\,\mathrm{ct}(L(m-1)) \big(a_L(m-1) v_L + (1 + a_L(m-1)) v_{s_{m-1}L} \big) \\
&\qquad + (1 + a_L(m-1))\,\mathrm{ct}(L(m)) \big(-a_L(m-1) v_{s_{m-1}L} + (1 - a_L(m-1)) v_L \big).
\end{aligned}$$

Now, we compute the action of second term on v_L:

$$\begin{aligned}
&\frac{1}{r} E_{\{m-1,m\}} \left(\sum_{l=0}^{r-1} \xi_{m-1}^{l} \xi_m^{-l} \right) s_{m-1} v_L \\
&\quad = \frac{1}{r} E_{\{m-1,m\}} \Bigg(\left(\sum_{l=0}^{r-1} \xi_{m-1}^{l} \xi_m^{-l} \right) a_L(m-1) v_L \\
&\qquad\quad + \left(\sum_{l=0}^{r-1} \xi_{m-1}^{l} \xi_m^{-l} \right) (1 + a_L(m-1)) v_{s_{m-1}L} \Bigg) \\
&\quad = \frac{1}{r} E_{\{m-1,m\}} \Bigg(\left(\sum_{l=0}^{r-1} \xi^{l(k-k)} \right) a_L(m-1) v_L \\
&\qquad\quad + \left(\sum_{l=0}^{r-1} \xi^{l(k-k)} \right) (1 + a_L(m-1)) v_{s_{m-1}L} \Bigg) \\
&\quad = \frac{1}{r} E_{\{m-1,m\}} \Bigg(\left(\sum_{l=0}^{r-1} 1 \right) a_L(m-1) v_L + \left(\sum_{l=0}^{r-1} 1 \right) (1 + a_L(m-1)) v_{s_{m-1}L} \Bigg) \\
&\quad = a_L(m-1) v_L + (1 + a_L(m-1)) v_{s_{m-1}L}.
\end{aligned}$$

Combining the coefficients at v_L and $v_{s_{m-1}L}$, we get

$$Y_m v_L = C v_L + D v_{s_{m-1}L},$$

where

$$C = a_L^2(m-1)\,\mathrm{ct}(L(m-1)) + (1 - a_L^2(m-1))\,\mathrm{ct}(L(m)) + a_L(m-1)$$

$$
\begin{aligned}
&= a_L(m-1)\big[a_L(m-1)(\mathrm{ct}(L(m-1)) - \mathrm{ct}(L(m))) + 1\big] + \mathrm{ct}(L(m))\\
&= a_L(m-1)[0] + \mathrm{ct}(L(m))\\
&= \mathrm{ct}(L(m)),\\
D &= a_L(m-1)\,\mathrm{ct}(L(m-1))(1 + a_L(m-1))\\
&\qquad - (1 + a_L(m-1))\,\mathrm{ct}(L(m))a_L(m-1) + (1 + a_L(m-1))\\
&= (1 + a_L(m-1))\big[a_L(m-1)(\mathrm{ct}(L(m-1)) - \mathrm{ct}(L(m))) + 1\big]\\
&= 0.
\end{aligned}
$$

This implies (15) and completes the proof of the theorem. ■

Remark 4.3. For $r = 1$, the corresponding Jucys–Murphy elements were given in [29, equation 2.9]. It is easy to show, using induction, that the elements in [29] and the elements in (15) are the same.

As an immediate consequence of Theorem 4.2, we have the following corollary.

Corollary 4.4. *The eigenvalues of the action of X_i and Y_i, for $1 \le i \le n$, on Gelfand–Zeitlin basis vectors distinguish the latter and, consequently, the isomorphism classes of simple $\mathbb{C}[C_r \wr R_n]$-modules.*

In particular, it follows that the Gelfand–Zeitlin subalgebra of $\mathbb{C}[C_r \wr R_n]$ is generated by Jucys–Murphy elements.

5. Bicyclic monoid

The results of this section should be known to experts. However, we could not trace an explicit reference, so we provide all proofs, for completeness.

Recall, cf. [6], that the bicyclic monoid $\mathcal{B}$ is an infinite monoid generated by two elements a and b and given by the following presentation:

$$\mathcal{B} := \langle a, b \mid ab = 1\rangle.$$

We have $\mathcal{B} = \{b^{n_1}a^{n_2} \mid n_1, n_2 \in \mathbb{N}\}$, where $\mathbb{N} := \{0, 1, 2, \ldots\}$.

Consider the $\mathbb{C}$-vector space $V_{\mathbb{N}}$ with $\mathbb{N}$ as a basis. Define the actions of b and a on $V_{\mathbb{N}}$ as follows:

$$bi = i + 1 \quad \text{and} \quad ai = \begin{cases} i-1 & \text{if } i > 0,\\ 0 & \text{otherwise.}\end{cases}$$

In the above, note that a sends basis vectors $0 \in \mathbb{N}$ and $1 \in \mathbb{N}$ to the zero of the vector space $V_{\mathbb{N}}$ and $1 \in \mathbb{N}$, respectively. Then it is easy to check that $V_{\mathbb{N}}$ becomes a simple $\mathbb{C}[\mathcal{B}]$-module.

For a non-zero $\lambda \in \mathbb{C}$, let V_λ be the 1-dimensional $\mathbb{C}$-vector space with basis v_λ. Define the actions of b and a on V_λ as follows:

$$bv_\lambda = \lambda v_\lambda \quad \text{and} \quad av_\lambda = \lambda^{-1} v_\lambda.$$

Then V_λ is a simple $\mathbb{C}[\mathcal{B}]$-module.

Proposition 5.1. Let V be a simple $\mathbb{C}[\mathcal{B}]$-module. Then either $V \cong V_{\mathbb{N}}$ or $V \cong V_\lambda$, for some non-zero $\lambda \in \mathbb{C}$.

Proof. Let L_a and L_b be linear operators on V representing the actions of a and b. Since $ab = 1$, we have $L_a \circ L_b = \mathrm{Id}_V$, in particular, L_a is surjective and L_b is injective. We need to consider two cases.

Case 1. Suppose L_a is injective and hence invertible. Then $L_b = L_a^{-1}$ and in particular $L_a L_b = L_b L_a$, and hence L_a commutes with the action of $\mathbb{C}[\mathcal{B}]$. From Schur–Dixmier lemma, it then follows that $L_a = \lambda \mathrm{Id}_V$, for some $\lambda \in \mathbb{C}$. Obviously, this λ must be non-zero. Consequently, $L_b = \lambda^{-1}\mathrm{Id}_V$. In this case any subspace of V is, clearly, a submodule. Therefore, V must have dimension one by simplicity and hence is isomorphic to V_λ.

Case 2. Suppose L_a is not injective. Then there exists a non-zero $v \in V$ such that $av = 0$. Consider the subspace W spanned by $\{b^n v \mid n \in \mathbb{N}\}$. Since L_b is injective, the set $\{b^n v \mid n \in \mathbb{N}\}$ is linearly independent and, therefore, W is infinite-dimensional. Clearly, W is stable under the action of $\mathbb{C}[\mathcal{B}]$ and is isomorphic to $V_{\mathbb{N}}$. Since V is simple, we must have $V = W$ and, finally, $V \cong V_{\mathbb{N}}$. ∎

6. Grothendieck groups for rook monoid algebras

In this section, we assume that the characteristic of $\Bbbk$ is $p > 0$. We identify the prime subfield $\Bbbk_p$ of $\Bbbk$ with the additive cyclic group of order p. We start by recalling the classical results related to the tower of symmetric groups.

6.1. The case of symmetric groups

We have the following *Jucys–Murphy elements* for the algebra $\Bbbk[S_n]$:

$$\widetilde{Y}_k = \sum_{i=1}^{k-1} (i,k), \quad \text{where } k \in [n]. \tag{16}$$

Here, $(i,k) \in S_n$ is a transposition.

For $V \in \Bbbk[S_n]$-mod, the eigenvalues of the operator $\widetilde{Y}_k$ on V lie in $\Bbbk_p$; see, e.g., [5, Lemma 2.2]. Since the elements in (16) commute with each other, for $\underline{r} = (r_1, r_2, \ldots, r_n) \in \Bbbk_p^n$, the common generalized eigenspace of V with respect to the elements in (16) is

$$V_{\underline{r}} := \{v \in V \mid (\widetilde{Y}_k - r_k)^N v = 0 \text{ for all } k \in [n] \text{ and } N \gg 0\}.$$

We then have the decomposition $V = \bigoplus_{\underline{r} \in \Bbbk_p^n} V_{\underline{r}}$.

For $\underline{r} \in \Bbbk_p^n$ and $l \in \Bbbk_p$, let $\mu_l := |\{k \in [n] \mid r_k = l\}|$. Then $\sum_{l \in \Bbbk_p} \mu_l = n$ and the tuple $\mathrm{wt}(\underline{r}) = (\mu_0, \ldots, \mu_{p-1}) \in \mathbb{N}^p$ is called the *weight* of $\underline{r}$. For

$$\mu \in \mathcal{T}_n := \Big\{(\mu_0, \ldots, \mu_{p-1}) \in \mathbb{N}^p \mid \sum_{r=0}^{p-1} \mu_r = n\Big\},$$

let

$$V[\mu] = \bigoplus_{\substack{\underline{r} \in \Bbbk_p^n \\ \mathrm{wt}(\underline{r}) = \mu}} V_{\underline{r}}. \tag{17}$$

The following lemma is [5, Lemma 2.4] and the key point in their proof is that the center of $\Bbbk[S_n]$ is generated by the elementary symmetric polynomials (see [26, Section 2]) in $\{Y_k \mid k \in [n]\}$, which can be found in [7] and [17].

Lemma 6.1. *For $V \in \Bbbk[S_n]$-mod and $\mu \in \mathcal{T}_n$, the space $V[\mu]$ is a $\Bbbk[S_n]$-submodule of V, moreover, $V \cong \bigoplus_{\mu \in \mathcal{T}_n} V[\mu]$.*

6.1.1. Decompositions of induction and restriction functors. For $i \in \Bbbk_p$ and $\gamma = (\gamma_0, \ldots, \gamma_{p-1}) \in \mathcal{T}_n$, denote by $\gamma + i$ the element in $\mathbb{N}^p$ whose i-th coordinate is $\gamma_i + 1$ and the remaining coordinates are the same as those of γ. Similarly, if $\gamma_i \neq 0$, denote by $\gamma - i$ an element in $\mathbb{N}^p$, whose i-th coordinate is $\gamma_i - 1$ and the remaining coordinates are the same as those of γ.

For $\gamma \in \mathcal{T}_n$ and $V = V[\gamma] \in \Bbbk[S_n]$-mod, define the functors

$$\widetilde{\mathrm{res}}_i \colon \Bbbk[S_n]\text{-mod} \to \Bbbk[S_{n-1}]\text{-mod} \quad \text{and} \quad \widetilde{\mathrm{ind}}_i \colon \Bbbk[S_n]\text{-mod} \to \Bbbk[S_{n+1}]\text{-mod}$$

as follows:

$$\widetilde{\mathrm{res}}_i\big(V[\gamma]\big) := \begin{cases} \big(\mathrm{Res}^{\Bbbk[S_n]}_{\Bbbk[S_{n-1}]} V[\gamma]\big)[\gamma - i] & \text{if } \gamma_i \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\widetilde{\mathrm{ind}}_i\big(V[\gamma]\big) := \big(\mathrm{Ind}^{\Bbbk[S_{n+1}]}_{\Bbbk[S_n]} V[\gamma]\big)[\gamma + i].$$

This definition extends to any object in $\Bbbk[S_n]$-mod and hence completely defines $\widetilde{\mathrm{res}}_i$ and $\widetilde{\mathrm{ind}}_i$ due to Lemma 6.1. The functors $\widetilde{\mathrm{res}}_i$ and $\widetilde{\mathrm{ind}}_i$ are called the *i-restriction* and the i-*induction* functors, respectively. Using these definitions, we get the following decomposition of the restriction and induction functors in terms of the i-restriction and i-induction functors

$$\mathrm{Res}^{\Bbbk[S_n]}_{\Bbbk[S_{n-1}]}(V) \cong \bigoplus_{i\in\Bbbk_p} \widetilde{\mathrm{res}}_i \quad \text{and} \quad \mathrm{Ind}^{\Bbbk[S_{n+1}]}_{\Bbbk[S_n]}(V) \cong \bigoplus_{i\in\Bbbk_p} \widetilde{\mathrm{ind}}_i .$$

Recall that for a finite-dimensional algebra A, $G_0(A)$ denotes the complexified Grothendieck group of A-mod. For $M \in A$-mod, let $[M] \in G_0(\Bbbk[S_n])$ the class of the corresponding of M in $G_0(A)$ If B is another finite-dimensional algebra over $\Bbbk$ and F is a covariant exact functor from A-mod to B-mod, then let $[F]$ denote the induced $\mathbb{C}$-linear map from $G_0(A)$ to $G_0(B)$.

The functors $\widetilde{\mathrm{res}}_i$ and $\widetilde{\mathrm{ind}}_i$ are exact, and hence they induce the following linear maps

$$[\widetilde{\mathrm{res}}_i]\colon G_0\big(\Bbbk[S_n]\big) \to G_0\big(\Bbbk[S_{n-1}]\big) \quad \text{and} \quad [\widetilde{\mathrm{ind}}_i]\colon G_0\big(\Bbbk[S_n]\big) \to G_0\big(\Bbbk[S_{n+1}]\big).$$

For $\lambda \in \Lambda^p(n)$, we denote by D^λ the corresponding simple module of $\Bbbk[S_n]$-mod. Then the set $\{[D^\lambda] \mid \lambda \in \Lambda^p(n)\}$ is a basis of $G_0(\Bbbk[S_n])$. In order to describe actions of $[\widetilde{\mathrm{res}}_i]$ and $[\widetilde{\mathrm{ind}}_i]$ in this basis, we need to recall some terminology.

Definition 6.2. Let λ be a partition and $i \in \Bbbk_p$.

(1) For a box $b \in [\lambda]$, the *residue of* b, denoted by $\mathrm{res}(b)$, is defined as the content of b modulo p.

(2) A box $b \in [\lambda]$ is called i-*removable* if $\mathrm{res}(b) = i$ and $[\lambda] \setminus \{b\}$ is a Young diagram.

(3) A box b adjacent to $[\lambda]$ is called i-*addable* for λ if $\mathrm{res}(b) = i$ and $[\lambda]\bigcup\{b\}$ is a Young diagram.

(4) Let us label all i-addable boxes for λ by $+$ and all i-removable boxes in $[\lambda]$ by $-$. Then, going along the rim of $[\lambda]$ from bottom left to top right, we can read off a sequence of $+$ and $-$. This sequence is called the i-*signature* of λ. We refer to [5, Subsection 2.4] for more details and an example.

(5) The *reduced* i-*signature of* λ is obtained from the i-signature of λ by removing, recursively, all adjacent pairs of the form $-+$. Note that the reduced i-signature is a sequence of $+$'s followed by a sequence of $-$'s. The boxes labeled by $-$ (respectively, $+$) in the reduced i-signature of λ are called i-*normal* (respectively, i-*conormal*). We again refer to [5, Subsection 2.4] for more details and an example.

Finally, we have the following result describing actions of the linear operators $[\widetilde{\mathrm{res}}_i]$ and $[\widetilde{\mathrm{ind}}_i]$ on the classes of simple modules, see [4, Theorems E and E'].

Theorem 6.3. *For $\lambda \in \Lambda^p(n)$, we have*

$$[\widetilde{\mathrm{res}}_i]\big([D^\lambda]\big) = \sum_{\mu \in \Lambda^p(n-1)} \alpha_{\lambda\mu}[D^\mu] \quad \textit{and} \quad [\widetilde{\mathrm{ind}}_i]\big([D^\lambda]\big) = \sum_{\nu \in \Lambda^p(n+1)} \beta_{\lambda\nu}[D^\nu],$$

where

(a) $\alpha_{\lambda\mu} = 0$ *unless there exists an i-normal box $b \in [\lambda]$ such that $[\lambda] \setminus \{b\} = [\mu]$, in which case $\alpha_{\lambda\mu}$ is the number of i-normal boxes in the i-reduced signature of $[\lambda]$ that are to the right to b (including b);*

(b) $\beta_{\lambda\nu} = 0$ *unless there exists an i-conormal box b adjacent to $[\lambda]$ such that $[\lambda] \bigcup \{b\} = [\nu]$, in which case $\beta_{\lambda\nu}$ is the number of i-conormal boxes in the i-reduced signature of $[\lambda]$ that are to the left to b (including b).*

For every $i \in \Bbbk_p$, we can view both $[\widetilde{\mathrm{res}}_i]$ and $[\widetilde{\mathrm{ind}}_i]$ as $\mathbb{C}$-linear endomorphisms of the space

$$\widetilde{\mathscr{G}}_{\mathbb{C}} = \bigoplus_{n \in \mathbb{N}} G_0\big(\Bbbk[S_n]\big).$$

Let $\widehat{\mathfrak{sl}}_p(\mathbb{C})$ denote the affine Lie algebra of type $A^{(1)}_{p-1}$ with the Chevalley generators e_i, f_i for $i \in \Bbbk_p$. Also, let $V(\Lambda_0)$ denote the basic representation of $\widehat{\mathfrak{sl}}_p(\mathbb{C})$ with dominant integral weight Λ_0; see [24, Section 4], [22, Chapter 11], and also [18, 19]. The following theorem, proved in [24] and [9], relates $\widetilde{\mathscr{G}}_{\mathbb{C}}$ to $V(\Lambda_0)$ (see also [5, Theorem 2.7]).

Theorem 6.4. (a) *For $i \in \Bbbk_p$, the endomorphisms $[\widetilde{\mathrm{res}}_i]$ and $[\widetilde{\mathrm{ind}}_i]$ of $\widetilde{\mathscr{G}}_{\mathbb{C}}$ satisfy the defining relations for the Chevalley generators of $\widehat{\mathfrak{sl}}_p(\mathbb{C})$.*

(b) *For $i \in \Bbbk_p$, let e_i and f_i act on $\widetilde{\mathscr{G}}_{\mathbb{C}}$ via $[\widetilde{\mathrm{res}}_i]$ and $[\widetilde{\mathrm{ind}}_i]$, respectively. Then the obtained $\widehat{\mathfrak{sl}}_p(\mathbb{C})$-module is isomorphic to the basic representation $V(\Lambda_0)$.*

6.2. The case of rook monoids

The following elements are the Jucys–Murphy elements of $\mathbb{Z}[R_n]$ which are obtained by taking $r = 1$ for the elements defined in Section 4:

$$X_j := E_{\{k\}} \quad \text{and} \quad Y_j := \sum_{i=1}^{j-1} E_{\{i,j\}}(i,j), \qquad \text{where } j \in [n].$$

Lemma 6.5. *The idempotent $E_{\{n-1,n\}}$ commutes with Y_{n-1}, Y_n and s_{n-1}.*

Proof. This is a straightforward computation. ∎

From Theorem 4.2, we have that the eigenvalues of the actions of X_i and Y_i on all elements of the Gelfand–Zeitlin basis of simple $\mathbb{C}[R_n]$-modules are integers. It follows that the eigenvalues of the actions of X_i and Y_i on all $\mathbb{C}[R_n]$-modules are integers. The following lemma describes an analogue of the latter statement over $\Bbbk$.

Lemma 6.6. *Let $M \in \Bbbk[R_n]$-mod. Then, for $1 \le i \le n$, the eigenvalues of the linear operators $X_i, Y_i \in \mathrm{End}_{\Bbbk}(M)$ lie in $\Bbbk_p$.*

Proof. Since X_i is an idempotent, the eigenvalues of X_i on a finite-dimensional $\Bbbk[R_n]$-module are 0 and 1.

As we mentioned before, from Theorem 4.2 we know that the eigenvalues of the actions Y_i on all $\mathbb{C}[R_n]$-modules are integers. In particular, this applies to the regular $\mathbb{C}[R_n]$-module. This means that the characteristic polynomial $f(t) \in \mathbb{C}[t]$ of Y_i decomposes as $f(t) = (t - a_1) \cdots (t - a_m)$, where m denotes the cardinality of R_n and $a_1, \ldots, a_m \in \mathbb{Z}$.

Since the Jucys–Murphy element Y_i belongs to $\mathbb{Z}[R_n]$, over the field $\Bbbk$, the polynomial

$$\overline{f}(t) = (t - \overline{a}_1) \cdots (t - \overline{a}_m)$$

is the characteristic polynomial of Y_i on the regular module $\Bbbk[R_n]$, where the scalars $\overline{a}_1, \ldots, \overline{a}_m \in \Bbbk_p$ are the mod p reductions of the integers $a_1, \ldots, a_m$, respectively. So the eigenvalues of Y_i on the regular module $\Bbbk[R_n]$ lie in $\Bbbk_p$. A finite-dimensional $\Bbbk[R_n]$-module M is a quotient of the direct sum of some copies of $\Bbbk[R_n]$. Therefore, any eigenvalue of Y_i on M must also be an eigenvalue of Y_i on the regular module $\Bbbk[R_n]$. The claim follows. ∎

An alternative inductive proof of the above lemma can also be given along the lines of the proof of [4, Lemma 2.2].

As an immediate consequence of Lemma 6.6, we have the following corollary.

Corollary 6.7. *For $M \in \Bbbk[R_n]$-mod, we have*

$$M \cong \bigoplus_{\substack{\underline{i} \in \Bbbk_p^n \\ \underline{j} \in \{0,1\}^n}} M_{\underline{i},\underline{j}},$$

where

$$M_{\underline{i},\underline{j}} = \{m \in M \mid (X_k - j_k)^N m = 0,\ (Y_k - i_k)^N m = 0 \text{ for } k \in [n] \text{ and } N \gg 0\}.$$

For $\underline{i} \in \Bbbk_p^n$, $\underline{j} \in \{0,1\}^n$ and $r \in \Bbbk_p$, define $\gamma_r := |\{k \in [n] \mid i_k = r, j_k = 1\}|$. Then

$$\left(\sum_{r=0}^{p-1} \gamma_r\right) \le n$$

and $\mathrm{wt}(\underline{i},\underline{j}) := (\gamma_0,\ldots,\gamma_{p-1}) \in \mathbb{N}^p$ is called the *weight* of $(\underline{i},\underline{j})$. For $\gamma \in \mathbb{N}^p$, define

$$M[\gamma] := \bigoplus_{\substack{\underline{i}\in\Bbbk_p^n,\ \underline{j}\in\{0,1\}^n\\ \mathrm{wt}(\underline{i},\underline{j})=\gamma}} M_{\underline{i},\underline{j}}.$$

The next lemma is motivated by the classical result that the center of $\Bbbk[S_n]$ is generated by the elementary symmetric polynomials in Jucys–Murphy elements; see, e.g., [7] and [17].

Lemma 6.8. *The center of $\Bbbk[R_n]$ is generated by the elementary symmetric polynomials in $\{X_k \mid k \in [n]\}$ and the elementary symmetric polynomials in $\{Y_k \mid k \in [n]\}$.*

Before we prove Lemma 6.8, we need to introduce some notation that we will use in the proof. For $\sigma \in R_n$, let $\mathcal{C}(\sigma)$ and $\mathcal{R}(\sigma)$ be the sets of indexes for all non-zero columns and rows of σ, respectively. When $\mathcal{C}(\sigma) = \mathcal{R}(\sigma)$ and $|\mathcal{R}(\sigma)| = r$, there is a unique order preserving bijection $\mathbf{r} \to \mathcal{C}(\sigma)$ and σ can be thought of as an element σ' in S_r. We define the *cycle type* of $\sigma \in R_n$ as the cycle type of $\sigma' \in S_r$; see [8] for details.

Example 6.9. The element

$$\sigma = \begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{bmatrix} \in R_3$$

can be realized, by the above, as the transposition $(1,2) \in S_2$. So, the cycle type of this σ is (2), i.e., it has one cycle of length two.

Proof. Let us denote by Z the subalgebra of $\Bbbk[R_n]$ spanned by all elementary symmetric polynomials in the X_i and all elementary symmetric polynomials in the Y_i. It is a straightforward exercise to check that Z is contained in the center of $\Bbbk[R_n]$ (in fact, below we explicitly compute the elementary symmetric polynomials in the X_i and the fact that these are central follow e.g., from [34]). Below we show the converse inclusion, i.e., the center of $\Bbbk[R_n]$ is contained in Z.

For $0 \le r \le n$ and $\lambda \vdash r$, let

$$\mathcal{M}_\lambda = \{\sigma \in R_n \mid \mathcal{C}(\sigma) = \mathcal{R}(\sigma) \text{ and the cycle type of } \sigma = \lambda\},$$
$$c_\lambda = \sum_{\sigma\in\mathcal{M}_\lambda} E_{\mathcal{R}(\sigma)}\sigma.$$

Note that $c_\varnothing = e_{[n]}$ is the zero element of R_n.

Our first claim is that $\{c_\lambda \mid \lambda \vdash r, 0 \le r \le n\}$ is a basis of the center of $\Bbbk[R_n]$. This follows easily from the construction by combining the following three well-known facts:

• The center of $\Bbbk[S_n]$ has the obvious basis indexed by the cycle types for S_n, in which the basis element corresponding to a fixed cycle type is just the sum of all elements in S_n which have this cycle type.

• For any m, the center of the algebra of $m \times m$ matrices over $\Bbbk[S_n]$ has the obvious basis indexed by cycle types for S_n, in which the basis element corresponding to a fixed cycle type is just the identity matrix times the corresponding basis element for the center of $\Bbbk[S_n]$.

• Since R_n is an inverse monoid, the monoid algebra $\Bbbk[R_n]$ is isomorphic to a direct sum of matrix algebras corresponding to the equivalence classes (with respect to Green's $\mathcal{D}$-relation) of the maximal subgroups in R_n. The latter subgroups are of the form S_k, for $0 \leq k \leq n$. For each such S_k, the rows and columns in the corresponding matrix algebra are naturally indexes by all k-element subsets of $[n]$ and the idempotents cutting out the $\Bbbk[S_k]$-parts are exactly the elements $E_{\mathcal{R}(\sigma)}$; see, e.g., [34, 35].

Consider the element

$$d_r := \sum_{A \subseteq [n],\, |A|=r} E_A.$$

Note that $X_1 = Q - P = e_\varnothing - e_{\{1\}} = E_{\{1\}}$, and hence $X_k = E_{\{k\}}$, for all $1 \leq k \leq n$. In particular, $d_1 = X_1 + X_2 + \cdots + X_n$. Further, for $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, we have

$$X_{i_1} X_{i_2} \cdots X_{i_r} = E_{\{i_1\}} E_{\{i_2\}} \cdots E_{\{i_r\}} = E_{\{i_1, i_2, \ldots, i_r\}}$$

implying

$$d_r = \sum_{\substack{i_1, \ldots, i_r \in [n] \\ i_1 < \cdots < i_r}} X_{i_1} X_{i_2} \cdots X_{i_r},$$

which is an elementary symmetric polynomial in the elements $\{X_k \mid k \in [n]\}$.

Next, by induction on r, one shows that the element

$$g_r := \sum_{\substack{i_1, \ldots, i_r \in [n] \\ i_1 < \cdots < i_r}} e_{\{i_1\}} e_{\{i_2\}} \cdots e_{\{i_r\}}$$

is a linear combination of $d_0, d_1, \ldots, d_r$, and hence belongs to Z.

Further, note that

$$E_A e_i = 0 \quad \text{if } i \in A. \tag{18}$$

For $\lambda \vdash r$, let $\widetilde{\lambda}$ be the partition of n obtained by adding $1^{(n-r)}$ to λ at the end. Using (18), we can write

$$c_\lambda = d_r \Bigg(\sum_{\substack{\tau \in S_n, \\ \text{cycle type of } \tau = \widetilde{\lambda}}} \tau \Bigg) g_{n-r}.$$

By the classical results for symmetric groups, there exists a symmetric polynomial f_λ in n variables such that

$$f_\lambda(\widetilde{Y}_1, \ldots, \widetilde{Y}_n) = \sum_{\substack{\tau \in S_n, \\ \text{cycle type of } \tau = \widetilde{\lambda}}} \tau.$$

One again, using (18), we conclude that $c_\lambda = d_r f_\lambda(Y_1, \ldots, Y_n) g_{n-r} \in Z$. ■

Note that, over $\mathbb{C}$, an analogue of Lemma 6.8 is also true for the generalized rook monoid algebras.

Proposition 6.10. The center of $\mathbb{C}[C_r \wr R_n]$ is generated by the symmetric polynomials in $\{X_i \mid i \in [n]\}$ and the symmetric polynomials in $\{Y_i \mid i \in [n]\}$.

Proof. Indeed, from Theorem 4.2, it is easy to see that all symmetric polynomials in the X_i and all symmetric polynomials in the Y_i act as scalars on all simple modules and hence are central. At the same time, they separate the isomorphism classes of simple modules and hence generate the center. ■

Over an arbitrary field, we only have the following:

Proposition 6.11. The symmetric polynomials in $\{X_i \mid i \in [n]\}$ and the symmetric polynomials in $\{Y_i \mid i \in [n]\}$ belong to the center of $\Bbbk[C_r \wr R_n]$.

Proof. To prove the result, it is enough to show that the elementary symmetric polynomials in $\{X_i \mid i \in [n]\}$ as well as the elementary symmetric polynomials in $\{Y_i \mid i \in [n]\}$ belong to the center of $\Bbbk[C_r \wr R_n]$. Note that, both P and Q commute with each X_i and Y_i. Furthermore, for $j \in [n-1]$, we have

- $s_i X_i s_i = X_{i+1}$ and $s_j X_i s_j = X_i$ for $j \notin \{i, i+1\}$,
- $s_i Y_i s_i = Y_{i+1} - E_{\{i,i+1\}} \sum_{l=0}^{r-1} \xi_i^l \xi_{i+1}^{-l} s_i$ and $s_j Y_i s_j = Y_i$ for $j \notin \{i, i+1\}$.

From this it follows that the symmetric polynomials in question commute with the generators of $\Bbbk[C_r \wr R_n]$ (see the end of Section 2) and hence are central. ■

Now in order to avoid excessive notation, the remaining part of this subsection is for $r = 1$ and for the arbitrary r we have stated the corresponding results in Section 6.4.

Like in the case of symmetric groups, symmetric polynomials in X_i and the symmetric polynomials in Y_i being central (see Lemma 6.8) give the following statement.

Lemma 6.12. *Let $\mathcal{T}_{\leq n} := \bigcup_{i=0}^n \mathcal{T}_i$. Then, any $M \in \Bbbk[R_n]$-*mod *admits a decomposition*

$$M \cong \bigoplus_{\gamma \in \mathcal{T}_{\leq n}} M[\gamma],$$

as $\Bbbk[R_n]$-modules.

We want to use Lemma 6.12 to define, for $i \in \Bbbk_p$, the following functors:

$$\begin{aligned} \mathrm{res}_i \colon \Bbbk[R_n]\text{-mod} &\to \Bbbk[R_{n-1}]\text{-mod}, & \mathbb{A}\colon \Bbbk[R_n]\text{-mod} &\to \Bbbk[R_{n-1}]\text{-mod},\\ \mathrm{ind}_i \colon \Bbbk[R_n]\text{-mod} &\to \Bbbk[R_{n+1}]\text{-mod}, & \mathbb{B}\colon \Bbbk[R_n]\text{-mod} &\to \Bbbk[R_{n+1}]\text{-mod}. \end{aligned} \tag{19}$$

For $\gamma \in \bigcup_{i=0}^{n} \mathcal{T}_i$ and $M = M[\gamma] \in \Bbbk[R_n]$-mod, let

$$\mathrm{res}_i\big(M[\gamma]\big) := \begin{cases} \big(\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{n-1}]} M[\gamma]\big)[\gamma - i] & \text{if } \gamma_i \neq 0,\\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

$$\mathbb{A}\big(M[\gamma]\big) := \begin{cases} \big(\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{n-1}]}\big(M[\gamma]\big)\big)[\gamma] & \text{if } \gamma \in \mathcal{T}_{\leq n-1},\\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{ind}_i\big(M[\gamma]\big) := \big(\mathrm{Ind}^{\Bbbk[R_{n+1}]}_{\Bbbk[R_n]} M[\gamma]\big)[\gamma + i], \quad \mathbb{B}\big(M[\gamma]\big) := \big(\mathrm{Ind}^{\Bbbk[R_{n+1}]}_{\Bbbk[R_n]}\big(M[\gamma]\big)\big)[\gamma].$$

These definitions extend to any object in $\Bbbk[R_n]$-mod by additivity using Lemma 6.12. The functors res_i and ind_i are called the *i-restriction* and *i-induction*, respectively. Using these definitions, we get the following decompositions

$$\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{n-1}]}(V) \cong \Big(\bigoplus_{i \in \Bbbk_p} \mathrm{res}_i\Big) \oplus \mathbb{A} \quad \text{and} \quad \mathrm{Ind}^{\Bbbk[R_{n+1}]}_{\Bbbk[R_n]}(V) \cong \Big(\bigoplus_{i \in \Bbbk_p} \mathrm{ind}_i\Big) \oplus \mathbb{B}.$$

Since res_i, $\mathbb{A}$, ind_i and $\mathbb{B}$ are all exact functors, we get the induced $\mathbb{C}$-linear maps $[\mathrm{res}_i]$, $[\mathbb{A}]$, $[\mathrm{ind}_i]$, and $[\mathbb{B}]$ on the complexified Grothendieck groups.From Lemma 2.1, for $r = 1$, it follows that

$$\mathcal{Q}_n := \Big\{[\mathcal{L}^n_j(D^\lambda)] \mid \lambda \in \bigcup_{j=0}^{n} \Lambda^p(j)\Big\}$$

is a basis of $G_0(\Bbbk[R_n])$. In order to describe $[\mathrm{res}_i]$, $[\mathbb{A}]$, $[\mathrm{ind}_i]$, and $[\mathbb{B}]$ in this basis, we use the following lemma, which says that via the functor in Lemma 2.1 the decomposition in Lemma 6.1 goes to the decomposition in Lemma 6.12.

Lemma 6.13. *For $0 \leq j \leq n$, $\gamma \in \mathcal{T}_j$ and $V = V[\gamma] \in \Bbbk[S_j]$-mod, we have*

$$\mathcal{L}^n_j\big(V[\gamma]\big) = \mathcal{L}^n_j\big(V[\gamma]\big)[\gamma].$$

Proof. Clearly, $\mathcal{L}^n_j(V[\gamma])[\gamma] \subseteq \mathcal{L}^n_j(V[\gamma])$ and we prove the reverse inclusion. By definition,

$$\mathcal{L}^n_j\big(V[\gamma]\big) = \Bbbk\mathbb{L}^n_j \otimes_{\Bbbk[S_j]} V[\gamma].$$

For $\underline{r} = (r_1, r_2, \ldots, r_j)$ with $\mathrm{wt}(\underline{r}) = \gamma$, let $v \in V_{\underline{r}}$. Let $Z = \{\beta_1 < \cdots < \beta_j\} \subseteq [n]$. Then using the decomposition (17), $h^n_Z \otimes v \in \mathcal{L}^n_j(V[\gamma])$ and a general element is a linear combination of these elements. Next, we show that $h^n_Z \otimes v \in \mathcal{L}^n_j(V[\gamma])[\gamma]$.

Define $\underline{l} = (l_1, \ldots, l_n)$, where $l_{\beta_1} = l_{\beta_2} = \cdots = l_{\beta_j} = 1$ and the remaining coordinates are equal to 0. Also, let $\underline{m} = (m_1, \ldots, m_n)$, where $m_{\beta_1} = r_1, \ldots, m_{\beta_j} = r_j$ and the remaining coordinates are equal to 0. Then $\mathrm{wt}((\underline{l}, \underline{m})) = \gamma$.

Let $k \in [n]$. Then

$$X_k(h^n_Z \otimes v) = \begin{cases} h^n_Z \otimes v & \text{if } k \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $(X_k - l_k)h^n_Z \otimes v = 0$. For $q_1, q_2 \in [n]$, observe that

$$E_{\{q_1,q_2\}} h^n_Z = \begin{cases} 0 & \text{if } q_1 \notin Z \text{ or } q_2 \notin Z, \\ h^n_Z & \text{otherwise.} \end{cases} \tag{21}$$

If $k \in Z$, then there exists $s \in [j]$ such that $\beta_s = k$. For $i \in [s-1]$, we observe that $(\beta_i, \beta_s)h^n_Z = h^n_Z(i, s)$. Now this together with (21) imply that $Y_k h^n_Z = h^n_Z \widetilde{Y}_s$. If $k \notin Z$, then again from (21), we get $Y_k h^n_Z = 0$.

Since $v \in V_{\underline{r}}$, by the definition of $V_{\underline{r}}$, we have $(\widetilde{Y}_q - r_q)^N v = 0$ for all $q \in [j]$ and for $N \geq 0$. From the above, we get $(Y_k - m_k)^N (h^n_Z \otimes v) = 0$ for all $k \in [n]$. Thus,

$$h^n_Z \otimes v \in \mathcal{L}^n_i\big(V[\gamma]\big)_{(\underline{l},\underline{m})} \subseteq \mathcal{L}^n_i\big(V[\gamma]\big)[\gamma]. \qquad \blacksquare$$

The following corollary is a consequence of Theorem 3.3 and Lemma 6.13.

Corollary 6.14. *For $i \in \Bbbk_p$ and $0 \leq j \leq n$, we have*

$$\mathrm{res}_i \circ \mathcal{L}^n_j \cong \begin{cases} \mathcal{L}^{n-1}_{j-1} \circ \widetilde{\mathrm{res}}_i & \text{if } j \neq 0, \\ 0 & \text{otherwise}, \end{cases} \qquad \mathbb{A} \circ \mathcal{L}^n_j \cong \begin{cases} \mathcal{L}^{n-1}_j & \text{if } j \neq n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathrm{ind}_i \circ \mathcal{L}^n_j \cong \mathcal{L}^{n+1}_{j+1} \circ \widetilde{\mathrm{ind}}_i, \qquad \mathbb{B} \circ \mathcal{L}^n_j \cong \mathcal{L}^{n+1}_j.$$

As an application of Lemma 2.1 and Corollary 6.14, we obtain the following corollary.

Corollary 6.15. *For $i \in \Bbbk_p$, we have*

$$\mathrm{res}_i \circ \mathbb{A} \cong \mathbb{A} \circ \mathrm{res}_i, \quad \mathrm{ind}_i \circ \mathbb{A} \cong \mathbb{A} \circ \mathrm{ind}_i,$$
$$\mathrm{res}_i \circ \mathbb{B} \cong \mathbb{B} \circ \mathrm{res}_i, \quad \mathrm{ind}_i \circ \mathbb{B} \cong \mathbb{B} \circ \mathrm{ind}_i,$$

and also $\mathbb{A} \circ \mathbb{B} \cong \mathrm{Id}$.

Let $\lambda \in \bigcup_{j=0}^n \Lambda^p(j)$. For $i \in \Bbbk_p$, below in (22) and (23), the first equality is due to Corollary 6.14 and the second equality is due to Theorem 6.3

$$[\mathrm{res}_i]\big([\mathcal{L}^n_j(D^\lambda)]\big) = [\mathcal{L}^n_j]\big([\widetilde{\mathrm{res}}_i][D^\lambda]\big) = \sum_{\mu \in \Lambda^p(n-1)} \alpha_{\lambda\mu}[\mathcal{L}^{n-1}_{j-1}(D^\mu)], \tag{22}$$

$$[\mathrm{ind}_i]\big([\mathcal{L}^n_j(D^\lambda)]\big) = [\mathcal{L}^n_j]\big([\widetilde{\mathrm{ind}}_i][D^\lambda]\big) = \sum_{\nu \in \Lambda^p(n+1)} \beta_{\lambda\nu}[\mathcal{L}^{n+1}_{j+1}(D^\nu)], \tag{23}$$

where $\alpha_{\lambda\mu}$ and $\beta_{\lambda\nu}$ are as given in Theorem 6.3. Once again from Corollary 6.14, we obtain

$$[\mathbb{A}]\big([\mathcal{L}_j^n(D^\lambda)]\big) = \begin{cases} [\mathcal{L}_j^{n-1}(D^\lambda)] & \text{if } j \neq n, \\ 0 & \text{otherwise}, \end{cases} \tag{24}$$

$$[\mathbb{B}]\big([\mathcal{L}_j^n(D^\lambda)]\big) = [\mathcal{L}_j^{n+1}(D^\lambda)]. \tag{25}$$

Define $\mathcal{G}_\mathbb{C} := \bigoplus_{n\in\mathbb{N}} G_0(\Bbbk[R_n])$. Then we can view $[\mathrm{res}_i]$, $[\mathbb{A}]$, $[\mathrm{ind}_i]$, and $[\mathbb{B}]$ as endomorphisms on $\mathcal{G}_\mathbb{C}$.

Theorem 6.16. (a) *For $i \in \Bbbk_p$, the endomorphisms $[\mathrm{res}_i]$ and $[\mathrm{ind}_i]$ on $\mathcal{G}_\mathbb{C}$ satisfy the defining relations of the Chevalley generators of $\widehat{\mathfrak{sl}}_p(\mathbb{C})$.*

(b) *For $i \in \Bbbk_p$, we have the relations*

$$[\mathrm{res}_i][\mathbb{A}] = [\mathbb{A}][\mathrm{res}_i], \quad [\mathrm{res}_i][\mathbb{B}] = [\mathbb{B}][\mathrm{res}_i],$$
$$[\mathrm{ind}_i][\mathbb{A}] = [\mathrm{ind}_i][\mathbb{A}], \quad [\mathrm{ind}_i][\mathbb{B}] = [\mathrm{ind}_i][\mathbb{B}],$$

and also $[\mathbb{A}][\mathbb{B}] = \mathrm{Id}_{\mathcal{G}_\mathbb{C}}$.

(c) *The vector space $\mathcal{G}_\mathbb{C}$ is a module over $U(\widehat{\mathfrak{sl}}_p(\mathbb{C})) \otimes_\mathbb{C} \mathbb{C}[\mathcal{B}]$, and moreover*

$$\mathcal{G}_\mathbb{C} \cong V(\Lambda_0) \otimes_\mathbb{C} V_\mathbb{N}, \quad \text{as } \big(U(\widehat{\mathfrak{sl}}_p(\mathbb{C})) \otimes_\mathbb{C} \mathbb{C}[\mathcal{B}]\big)\text{-modules}.$$

Proof. Claim (a) follows from Theorem 6.3, Theorem 6.4 (a), and formulae (22) and (23). Claim (b) follows from Corollary 6.15.

Let us now prove Claim (6.16). Note that $U(\widehat{\mathfrak{sl}}_p(\mathbb{C})) \otimes \mathbb{C}[\mathcal{B}]$ is generated by $e_i \otimes 1$, $f_i \otimes 1$ for all $i \in \Bbbk_p$, and by $1 \otimes a$, $1 \otimes b$. Let $e_i \otimes 1$ and $f_i \otimes 1$ act on $\mathcal{G}_\mathbb{C}$ by $[\mathrm{res}_i]$ and $[\mathrm{ind}_i]$, respectively, for all $i \in \Bbbk_p$. Likewise, let $1 \otimes a$ and $1 \otimes b$ act on $\mathcal{G}_\mathbb{C}$ by $[\mathbb{A}]$ and $[\mathbb{B}]$, respectively. Then using claims (a) and (b), we see that $\mathcal{G}_\mathbb{C}$ is a module over $U(\widehat{\mathfrak{sl}}_p(\mathbb{C})) \otimes_\mathbb{C} \mathbb{C}[\mathcal{B}]$.

Under the isomorphism in Theorem 6.4 (b), we may consider

$$\Big\{[D^\lambda] \mid \lambda \in \bigcup_{j\in\mathbb{N}} \Lambda^p(j)\Big\}$$

as a basis of $V(\Lambda_0)$. Define the map

$$\Phi\colon \mathcal{G}_\mathbb{C} \to V(\Lambda_0) \otimes V_\mathbb{N}, \tag{26}$$

by

$$\Phi\big([\mathcal{L}_j^n(D^\lambda)]\big) = [D^\lambda] \otimes (n-j).$$

It follows from the above discussion and the constructions that this map is an isomorphism of $U(\widehat{\mathfrak{sl}}_p(\mathbb{C})) \otimes_\mathbb{C} \mathbb{C}[\mathcal{B}]$-modules. ∎

6.3. Bialgebra structure on $\mathscr{G}_{\mathbb{C}}$

6.3.1. Preliminaries. It is well known that $\widetilde{\mathscr{G}}_{\mathbb{C}}$ has the natural structure of a Hopf algebra; see [26, Chapter I]. In this section, we prove that $\mathscr{G}_{\mathbb{C}}$ has the natural structure of a bialgebra.

For $j,k \in \mathbb{N}$, denote $\Bbbk[S_{(j,k)}] := \Bbbk[S_j] \otimes_{\Bbbk} \Bbbk[S_k]$. Further, for $a,b \in \mathbb{N}$, denote $\mathscr{L}^{(a,b)}_{(j,k)} := (\Bbbk\mathbb{L}^a_j \otimes_{\Bbbk} \Bbbk\mathbb{L}^b_k) \otimes_{\Bbbk[S_{(j,k)}]} -$.

Lemma 6.17. *For $n, n_1, n_2 \in \mathbb{N}$ with $n = n_1 + n_2$, the following diagram commutes, up to isomorphism of functors:*

$$\begin{tikzcd}
\bigoplus\limits_{i=0}^{n} \Bbbk[S_i]\text{-mod} \arrow[rr, "\bigoplus\limits_{i=0}^{n} \mathscr{L}^n_i"] \arrow[d, "\mathscr{F}"] & & \Bbbk[R_n]\text{-mod} \arrow[d, "\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{(n_1,n_2)}]}(-)"] \\
\bigoplus\limits_{j=0}^{n_1}\bigoplus\limits_{k=0}^{n_2} \Bbbk[S_{(j,k)}]\text{-mod} \arrow[rr, "\bigoplus\limits_{j=0}^{n_1}\bigoplus\limits_{k=0}^{n_2} \mathscr{L}^{(n_1,n_2)}_{(j,k)}"] & & \Bbbk[R_{(n_1,n_2)}]\text{-mod},
\end{tikzcd}$$

where the functor $\mathscr{F}$ is given by

$$\mathscr{F}|_{\Bbbk[S_i]\text{-mod}} = \bigoplus_{\substack{j \in \{0,\dots,n_1\} \\ k \in \{0,\dots,n_2\} \\ j+k=i}} \mathrm{Res}^{\Bbbk[S_i]}_{\Bbbk[S_{(j,k)}]}(-).$$

Proof. Recall that, as a right $\Bbbk[C_r \wr S_i]$-module, $\Bbbk\mathbb{L}^n_i$ has a basis consisting of matrices of the form h^n_Z, where $Z \in \mathcal{X}_i$.

For Z as above, let

$$Z' = Z \cap \{1,2,\dots,n_1\} \quad \text{and} \quad Z'' = Z \cap \{n_1+1, n_1+2, \dots, n_1+n_2 = n\}.$$

Then $Z = Z' \sqcup Z''$. If $|Z'| = j$ and $|Z''| = k$, then we have $0 \le j \le n_1$ and $0 \le k \le n_2$ such that $j + k = i$.

Then the map

$$\begin{aligned}
&\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{(n_1,n_2)}]}(\mathscr{L}^n_i(V)) \\
&\quad = \Bbbk\mathbb{L}^n_i \otimes_{\Bbbk[S_i]} V \to \bigoplus_{\substack{j \in \{0,\dots,n_1\} \\ k \in \{0,\dots,n_2\} \\ j+k=i}} \big(\Bbbk\mathbb{L}^{n_1}_j \otimes_{\Bbbk} \Bbbk\mathbb{L}^{n_2}_k\big) \otimes_{\Bbbk[S_{(j,k)}]} \mathrm{Res}^{\Bbbk[S_i]}_{\Bbbk[S_{(j,k)}]}(V), \\
&\qquad\qquad h^n_Z \otimes v \mapsto (h^{n_1}_{Z'} \otimes h^{n_2}_{Z''}) \otimes v
\end{aligned}$$

is an isomorphism of $\Bbbk[R_{(n_1,n_2)}]$-modules which is functorial in V, and the claim follows. ∎

The following statement follows from Lemma 6.17 using Frobenius reciprocity.

Corollary 6.18. *For $n, n_1, n_2 \in \mathbb{N}$ with $n = n_1 + n_2$, the following diagram commutes, up to isomorphism of functors:*

$$\begin{tikzcd}
\bigoplus_{i=0}^{n} \Bbbk[S_i]\text{-mod} \arrow[rr, "\bigoplus_{i=0}^{n} \mathcal{L}_i^n"] & & \Bbbk[R_n]\text{-mod} \\
\bigoplus_{j=0}^{n_1}\bigoplus_{k=0}^{n_2} \Bbbk[S_{(j,k)}]\text{-mod} \arrow[u, "\mathcal{F}'"] \arrow[rr, "\bigoplus_{j=0}^{n_1}\bigoplus_{k=0}^{n_2} \mathcal{L}_{(j,k)}^{(n_1,n_2)}"] & & \Bbbk[R_{(n_1,n_2)}]\text{-mod}, \arrow[u, "\mathrm{Ind}_{\Bbbk[R_{(n_1,n_2)}]}^{\Bbbk[R_n]}(-)"']
\end{tikzcd}$$

where the functor $\mathcal{F}'$ is given by

$$\mathcal{F}'|_{\Bbbk[S_{(j,k)}]\text{-mod}} = \bigoplus_{\substack{i\in\{0,\ldots,n\}\\ j+k=i}} \mathrm{Ind}_{\Bbbk[S_{(j,k)}]}^{\Bbbk[S_i]}(-).$$

An immediate consequence of Corollary 6.18 is the following statement.

Corollary 6.19. *For $n, n_1, n_2 \in \mathbb{N}$ with $n = n_1 + n_2$, the functor $\mathrm{Ind}_{\Bbbk[R_{(n_1,n_2)}]}^{\Bbbk[R_n]}(-)$ is exact.*

For $n_1, n_2, \ldots, n_s \in \mathbb{N}$, let $R_{(n_1,n_2,\ldots,n_s)} := R_{n_1} \times R_{n_2} \times \cdots \times R_{n_s}$. Then, as usual, we have the following decomposition involving the corresponding monoid algebras:

$$\Bbbk[R_{(n_1,n_2,\ldots,n_s)}] = \Bbbk[R_{n_1}] \otimes_{\Bbbk} \Bbbk[R_{n_2}] \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \Bbbk[R_{n_s}].$$

Now we are ready to discuss the bialgebra structure on $\mathcal{G}_{\mathbb{C}}$.

6.3.2. Multiplication. Since we have to deal with modules over $\Bbbk[R_n]$ for all $n \in \mathbb{N}$ in the same course of a proof or a statement, for the sake of clarity we decorate a module over $\Bbbk[R_n]$ by putting superscript n on it. This notational convention applies for modules over symmetric group algebras as well. For $V^n \in \Bbbk[R_n]$-mod and $W^m \in \Bbbk[R_m]$-mod, we have that $V^n \otimes_{\Bbbk} W^m \in \Bbbk[R_{(n,m)}]$-mod. Define:

$$[V^n][W^m] = \big[\mathrm{Ind}_{\Bbbk[R_{(n,m)}]}^{\Bbbk[R_{n+m}]}(V^n \otimes_{\Bbbk} W^m)\big]. \tag{27}$$

Since the functor $\mathrm{Ind}_{\Bbbk[R_{(n,m)}]}^{\Bbbk[R_{n+m}]}(-)$ is exact, (27) gives rise to a well-defined multiplication on $\mathcal{G}_{\mathbb{C}}$. Associativity of tensor products and also of the induction functor imply the associativity of (27). Let $\Bbbk^0$ denote the trivial $\Bbbk[R_0]$-module. Then $[\Bbbk^0] \in G_0(\Bbbk[R_0])$ is the unit with respect to this multiplication. Thus, $\mathcal{G}_{\mathbb{C}}$ becomes a unital algebra with respect to the multiplication given by (27).

6.3.3. Comultiplication. Define

$$\Delta\big([V^n]\big) = \sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}} \big[\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{(n_1,n_2)}]} V^n\big]. \tag{28}$$

Using the identification

$$\bigoplus_{n_1,n_2\in\mathbb{N}} G_0\big(\Bbbk[R_{(n_1,n_2)}]\big) \cong \mathcal{G}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{G}_{\mathbb{C}},$$

we have that $\Delta([V^n]) \in \mathcal{G}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{G}_{\mathbb{C}}$. Then, both sides of the coassociativity condition for (28) reduce, essentially, to computation of every $\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{(n_1,n_2,n_3)}]}(V^n)$ for $n = n_1 + n_2 + n_3$. Consider the map $\varepsilon\colon \mathcal{G}_{\mathbb{C}} \to \Bbbk$ which sends the basis element $[\Bbbk^0] \in G_0(\Bbbk[R_0])$ to $1 \in \Bbbk$ and is zero on all other basis elements. It is straightforward to see that the map ε is a counit of $\mathcal{G}_{\mathbb{C}}$, and so $\mathcal{G}_{\mathbb{C}}$ becomes a coalgebra with respect to Δ and ε.

6.3.4. Compatibility. The vector space $V_{\mathbb{N}}$ is isomorphic to the monoid algebra $\mathbb{C}[\mathbb{N}]$ of the monoid $(\mathbb{N}, +)$ of natural numbers. Therefore, $V_{\mathbb{N}}$ inherits from $\mathbb{C}[\mathbb{N}]$ the structure of a bialgebra, where the multiplication is given by the monoid operation (addition) and the value of the comultiplication on $i \in \mathbb{N}$ is

$$\sum_{\substack{i_1,i_2\in\mathbb{N}\\ i_1+i_2=i}} i_1 \otimes i_2.$$

It is well known that $V(\Lambda_0)$ is a Hopf algebra, where the multiplication and comultiplication are given by replacing R_n by S_n in (27) and in (28), respectively. As a consequence, we obtain that $V(\Lambda_0) \otimes_{\mathbb{C}} V_{\mathbb{N}}$ is, naturally, a bialgebra.

Next we prove that the respective multiplication and comultiplication maps are preserved under the isomorphism (26). In particular, this implies that Δ is compatible with multiplication and thereby $\mathcal{G}_{\mathbb{C}}$ possess the structure of a bialgebra.

Theorem 6.20. *The isomorphism (26) Φ preserves multiplication and comultiplication. In particular, $\mathcal{G}_{\mathbb{C}}$ is a bialgebra.*

Proof. For the comultiplication maps $\Delta_{V(\Lambda_0)}$ and $\Delta_{V_{\mathbb{N}}}$ of $V(\Lambda_0)$ and $V_{\mathbb{N}}$, respectively, the comultiplication on $V(\Lambda_0) \otimes_{\mathbb{C}} V_{\mathbb{N}}$ is given by

$$\Delta_{V(\Lambda_0)\otimes_{\mathbb{C}} V_{\mathbb{N}}} := (\mathrm{id}_{V(\Lambda_0)} \otimes \tau \otimes \mathrm{id}_{V_{\mathbb{N}}}) \circ (\Delta_{V(\Lambda_0)} \otimes \Delta_{V_{\mathbb{N}}}),$$

where $\tau\colon V(\Lambda_0) \otimes_{\mathbb{C}} V_{\mathbb{N}} \to V_{\mathbb{N}} \otimes_{\mathbb{C}} V(\Lambda_0)$ is the swap of the tensor factors.

For $i \le n$, let $M^i \in \Bbbk[S_i]$-mod. We want to show that

$$\big((\Phi \otimes \Phi) \circ \Delta\big)\big([\mathcal{L}_i^n(M^i)]\big) = (\Delta_{V(\Lambda_0)\otimes_{\mathbb{C}} V_{\mathbb{N}}} \circ \Phi)\big([\mathcal{L}_i^n(M^i)]\big). \tag{29}$$

Under the identification

$$\bigoplus_{n_1,n_2\in\mathbb{N}} G_0\big(\Bbbk[S_{(n_1,n_2)}]\big) \cong \widetilde{\mathscr{G}}_{\mathbb{C}} \otimes_{\mathbb{C}} \widetilde{\mathscr{G}}_{\mathbb{C}},$$

we fix a decomposition of $[\mathrm{Res}^{\Bbbk[S_i]}_{\Bbbk[S_{(j,k)}]}(M^i)]$ of the form

$$\sum_{a,b\in\mathbb{N}} [V_a^j]\otimes[V_b^k],$$

where all but finitely many summands are zero. Now, using Lemma 6.17, the left-hand side of (29) can be computed as follows:

$$\begin{aligned}
&\big((\Phi\otimes\Phi)\circ\Delta\big)\big([\mathscr{L}_i^n(M^i)]\big)\\
&\quad=\Phi\otimes\Phi\Bigg(\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\Big[\mathrm{Res}^{\Bbbk[R_n]}_{\Bbbk[R_{(n_1,n_2)}]}\big(\mathscr{L}_i^n(M^i)\big)\Big]\Bigg)\\
&\quad=\Phi\otimes\Phi\Bigg(\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\Bigg[\bigoplus_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}}\mathscr{L}^{(n_1,n_2)}_{(j,k)}\circ\mathscr{F}(M^i)\Bigg]\Bigg)\\
&\quad=\Phi\otimes\Phi\Bigg(\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\Bigg[\bigoplus_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}}\mathscr{L}^{(n_1,n_2)}_{(j,k)}\,\mathrm{Res}^{\Bbbk[S_i]}_{\Bbbk[S_{(j,k)}]}(M^i)\Bigg]\Bigg)\\
&\quad=\Phi\otimes\Phi\Bigg(\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\ \sum_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}}\big[\mathscr{L}^{(n_1,n_2)}_{(j,k)}\,\mathrm{Res}^{\Bbbk[S_i]}_{\Bbbk[S_{(j,k)}]}(M^i)\big]\Bigg)\\
&\quad=\Phi\otimes\Phi\Bigg(\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\ \sum_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}}\big[\mathscr{L}^{(n_1,n_2)}_{(j,k)}\big]\big[\mathrm{Res}^{\Bbbk[S_i]}_{\Bbbk[S_{(j,k)}]}(M^i)\big]\Bigg)\\
&\quad=\Phi\otimes\Phi\Bigg(\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\ \sum_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}}\big[\mathscr{L}^{(n_1,n_2)}_{(j,k)}\big]\sum_{a,b\in\mathbb{N}}[V_a^j]\otimes[V_b^k]\Bigg)\\
&\quad=\Phi\otimes\Phi\Bigg(\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\ \sum_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}}\ \sum_{a,b\in\mathbb{N}}\big[\mathscr{L}_j^{n_1}(V_a^j)\big]\otimes\big[\mathscr{L}_k^{n_2}(V_b^k)\big]\Bigg)\\
&\quad=\sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}}\ \sum_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}}\ \sum_{a,b\in\mathbb{N}}\Phi\big[\big(\mathscr{L}_j^{n_1}(V_a^j)\big)\big]\otimes\Phi\big[\big(\mathscr{L}_k^{n_2}(V_b^k)\big)\big]
\end{aligned}$$

$$= \sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1+n_2=n}} \sum_{\substack{j\in\{0,\dots,n_1\}\\ k\in\{0,\dots,n_2\}\\ j+k=i}} \sum_{a,b\in\mathbb{N}} \big([V_a^j]\otimes(n_1-j)\big)\otimes\big([V_b^k]\otimes(n_2-k)\big).$$

On the other hand, the right-hand side of (29) can be computed as follows:

$$\begin{aligned}
&(\Delta_{V(\Lambda_0)\otimes_{\mathbb{C}}V_{\mathbb{N}}}\circ\Phi)\big([\mathcal{L}_i^n(M^i)]\big)\\
&\quad= \Delta_{V(\Lambda_0)\otimes_{\mathbb{C}}V_{\mathbb{N}}}\big([M^i]\otimes(n-i)\big)\\
&\quad= (\mathrm{id}_{V(\Lambda_0)}\otimes\tau\otimes\mathrm{id}_{V_{\mathbb{N}}})\big(\Delta_{V(\Lambda_0)}[M^i]\otimes\Delta_{V_{\mathbb{N}}}(n-i)\big)\\
&\quad= (\mathrm{id}_{V(\Lambda_0)}\otimes\tau\otimes\mathrm{id}_{V_{\mathbb{N}}})\Bigg(\bigg(\sum_{\substack{p,q\in\mathbb{N}\\ p+q=i}}[\mathrm{Res}^{\Bbbk[S_i]}_{\Bbbk[S_{(p,q)}]}(M^i)]\bigg)\otimes\bigg(\sum_{\substack{r,s\in\mathbb{N}\\ r+s=n-i}} r\otimes s\bigg)\Bigg)\\
&\quad= (\mathrm{id}_{V(\Lambda_0)}\otimes\tau\otimes\mathrm{id}_{V_{\mathbb{N}}})\Bigg(\bigg(\sum_{\substack{p,q\in\mathbb{N}\\ p+q=i}}\sum_{c,d\in\mathbb{N}}[V_c^p]\otimes[V_d^q]\bigg)\otimes\bigg(\sum_{\substack{r,s\in\mathbb{N}\\ r+s=n-i}} r\otimes s\bigg)\Bigg)\\
&\quad= \sum_{c,d\in\mathbb{N}}\sum_{\substack{p,q\in\mathbb{N}\\ p+q=i}}\bigg(\sum_{\substack{r,s\in\mathbb{N}\\ r+s=n-i}}\big([V_c^p]\otimes r\big)\otimes\big([V_d^q]\otimes s\big)\bigg).
\end{aligned}$$

This implies (29) and we are done. ■

6.4. The case of generalized rook monoids

Suppose p does not divide r. For $r\in[n]$, we have $X_i^r = X_i$ and hence the eigenvalues of each X_i, considered as an operator on a $\Bbbk[C_r\wr R_n]$-finite-dimensional module, are either r-th roots of unity or 0. Similarly, the eigenvalues of Jucys–Murphy elements Y_i as an operator on a finite dimensional module over $\Bbbk[C_r\wr R_n]$ lie in $\Bbbk_p$. Using this and Proposition 6.11, we get a decomposition of every object in $\Bbbk[C_r\wr R_n]$-mod as in Lemma 6.12. This allows us to define the i-induction and i-restriction functors as well as the functors corresponding to the two generators of the bicyclic monoid $\mathcal{B}$. Using the results for the generalized symmetric groups $C_r\wr S_n$ from [36] (see also [37] for the more general setting), one shows that the direct sum, over all n, of $G_0(\Bbbk[C_r\wr R_n])$ is a $U(\widehat{\mathfrak{sl}}_p(\mathbb{C}))^{\otimes r}\otimes\mathbb{C}[\mathcal{B}]$-module isomorphic to $V(\Lambda_0)^{\otimes r}\otimes V_{\mathbb{N}}$.

Remark 6.21. The results of Section 6 have the obvious analogues in characteristic zero (with the same proofs), where the field $\Bbbk_p$ is replaced by the ring $\mathbb{Z}$ of integers and, consequently, the Lie algebra $\widehat{\mathfrak{sl}}_p(\mathbb{C})$ is replaced by $\mathfrak{sl}_\infty(\mathbb{C})$. Also, the basic representation $V(\Lambda_0)$ now becomes the Fock space representation of $\mathfrak{sl}_\infty(\mathbb{C})$.

Acknowledgments. We thank Weiqiang Wang for bringing the reference [37] to our attention. The second author also thanks Arun Ram for fruitful discussions. We thank the referee for helpful comments.

Funding. The first author is partially supported by the Swedish Research Council and Göran Gustafsson Stiftelse.

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Received 14 May 2021; revised 2 May 2022.

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