Crystals, regularisation and the Mullineux map

Matthew Fayers

Abstract. The Mullineux map is a combinatorial function on partitions which describes the effect of tensoring a simple module for the symmetric group in characteristic p with the one-dimensional sign representation. It can also be interpreted as a signed isomorphism between crystal graphs for $\widehat{\mathfrak{sl}}_p$. We give a new combinatorial description of the Mullineux map by expressing this crystal isomorphism as a composition of isomorphisms between different crystals. These isomorphisms are defined in terms of new generalised regularisation maps introduced by Millan Berdasco.

We then given two applications of our new realisation of the Mullineux map, by providing purely combinatorial proofs of a conjecture of Lyle relating the Mullineux map with regularisation, and a theorem of Paget describing the Mullineux map in RoCK blocks of symmetric groups.

1. Introduction

In the representation theory of Kac–Moody algebras and their quantised enveloping algebras, an important role is played by *crystal bases* for integrable modules, and the associated *crystal graphs*, which facilitate a combinatorial approach to studying these modules. An important theme in combinatorial representation theory is to construct a combinatorial model for the crystal of a given module, in which vertices are labelled by simple combinatorial objects, with a combinatorial rule to determine where the arrows go.

In this paper we concentrate on the simplest case of an affine Kac–Moody algebra, namely the algebra $\widehat{\mathfrak{sl}}_e$ for a fixed integer $e \geqslant 2$. This algebra is implicated in the representation theory of Iwahori–Hecke algebras of type A in quantum characteristic e. In particular, the crystal $B(\Lambda_0)$ of the irreducible highest-weight module with highest weight Λ_0 underlies the Brundan–Kleshchev "modular branching rules" for these Hecke algebras. This application highlights two particular combinatorial models of $B(\Lambda_0)$: the Misra–Miwa model [25], in which the vertices are labelled by

---

e-restricted partitions, and its dual, in which the vertices are labelled by *e*-regular partitions.

In [1], Berg found a third model for B(Λ_0), called the *ladder crystal*, and showed that it is isomorphic to the *e*-regular model, with an isomorphism being given by James's *e*-regularisation function. In [9], the author showed that (provided $e \geq 3$) these three models are members of an infinite family of models based on partitions. The first aim of the present paper is to provide a new proof of the main result of [9], by constructing explicit isomorphisms between the crystals in this family. This is done by generalising Berg's proof using a new generalisation of *e*-regularisation due to Millan Berdasco.

We then apply these results to give a new description of the *Mullineux map*. This is an involution on the set of *e*-regular partitions which arises in the modular representation theory of symmetric groups, where it describes the effect of tensoring an irreducible module with the one-dimensional sign module. The *Mullineux problem* asks for a combinatorial description of this map, and several solutions to this problem are now known. The Mullineux map can also be understood in the crystal setting: via the Brundan–Kleshchev branching rules, the Mullineux map describes the isomorphism between the *e*-restricted and *e*-regular models for B(Λ_0). We construct this isomorphism as a composition of regularisation isomorphisms between the different crystals in our family, thereby giving another solution to the Mullineux problem. We use this new algorithm to give a new (and purely combinatorial) proof of a conjecture of Lyle relating the Mullineux map and regularisation, and a generalisation of a theorem of Paget describing the effect of the Mullineux map in RoCK blocks.

2. Partitions

In this section we review some standard background on partitions, before giving some new results due to Millan Berdasco on regularisation.

2.1. Elementary notation

We begin with some standard notation. We write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of non-negative integers. Given a $\mathcal{L} \subseteq \mathbb{Z}^2$ and an element $(s, d) \in \mathbb{Z}^2$, we will write $\mathcal{L} + (s, d)$ to mean $\{(r + s, c + d) \mid (r, c) \in \mathcal{L}\}$.

Throughout this paper, we fix an integer $e \geq 2$. We write $\mathbb{Z}/e\mathbb{Z}$ for the set of cosets $a + e\mathbb{Z}$ for $a \in \mathbb{Z}$. (We do not employ the usual abuse of notation where $\mathbb{Z}/e\mathbb{Z}$ is identified with the set $\{0, \dots, e - 1\}$.)

2.2. Partitions and Young diagrams

A *partition* means an infinite weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is finite. When writing partitions, we usually group together equal parts and omit trailing zeroes. The partition $(0, 0, \dots)$ is written as $\varnothing$.

We say that a partition λ is *e-regular* if it does not have e equal positive parts, and that λ is *e-restricted* if $\lambda_r - \lambda_{r+1} < e$ for all r.

If λ is a partition, the *Young diagram* of λ is the set

$$\{(r, c) \in \mathbb{N}^2 \mid c \leqslant \lambda_r\}$$

whose elements we call the *nodes* of λ. In general, a *node* means an element of $\mathbb{N}^2$.

We may abuse notation and identify λ with its Young diagram; so for example if λ and μ are partitions we may write $\lambda \subseteq \mu$ to mean that $\lambda_r \leqslant \mu_r$ for all r. We draw a Young diagram as an array of boxes in the plane using the English convention, in which the first coordinate increases down the page and the second increases from left to right. We use words such as "higher" and "lower" with this convention in mind, so that the node $(1, 2)$ is higher than, and to the right of, the node $(2, 1)$.

A node (r, c) of λ is *removable* if it can be removed from the Young diagram of λ to leave the Young diagram of a smaller partition, i.e., if $c = \lambda_r > \lambda_{r+1}$. A node which does not belong to λ is an *addable node* of λ if it can be added to λ to leave the Young diagram of a larger partition. If (r, c) is an addable node of λ, then we may write $\lambda \cup (r, c)$ for the partition (whose Young diagram is) obtained by adding this node; similarly we may write $\lambda \setminus (r, c)$ for the partition obtained by removing a removable node (r, c).

The *residue* of $(r, c) \in \mathbb{Z}^2$ is defined to be $c - r + e\mathbb{Z}$. If $i \in \mathbb{Z}/e\mathbb{Z}$, then we use the term *i-node* to mean a node of residue i. We define the *e-content* of a partition to be the multiset of the residues of its nodes.

2.3. Conjugation and the dominance order

If λ is a partition, the conjugate partition λ' is the partition whose Young diagram is obtained by reflecting the Young diagram of λ in the main diagonal; in other words, λ' is the partition with $\lambda'_r = |\{c \in \mathbb{N} \mid \lambda_c \geqslant r\}|$ for each r.

The *dominance order* is a natural partial order on partitions: we say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if $|\lambda| = |\mu|$ and

$$\lambda_1 + \cdots + \lambda_r \geqslant \mu_1 + \cdots + \mu_r$$

for all r. Another way to describe this is to say that $\lambda \trianglerighteq \mu$ if μ can be obtained from λ by replacing some of the nodes of λ with lower nodes.

We will use the well-known simple result [17, Lemma 1.4.11] that conjugation reverses the dominance order: $\lambda \trianglerighteq \mu$ if and only if $\lambda' \trianglelefteq \mu'$.

2.4. Hooks

Suppose (r, c) is a node of a partition λ. The (r, c)-*hook* of λ is defined to be the set of all nodes of λ directly to the right of or directly below (r, c), including (r, c) itself. The *length* of this hook is the number of nodes it contains, i.e., $\lambda_r - c + \lambda'_c - r + 1$, and we refer to the hook as an *e-hook* it is has length e. The *arm length* of the hook is the number of nodes to the right of (r, c), i.e., $\lambda_r - c$, and the *leg length* is $\lambda'_c - r$. The *hand node* of the hook is the node (r, λ_r), and the *foot node* is (λ'_c, c).

For example, the $(2, 3)$-hook of the partition $(4^3, 3, 1)$ has length 4, arm length 1, leg length 2, hand node $(2, 4)$ and foot node $(4, 3)$, as we see from the following Young diagram.
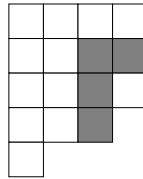
Removing a hook H from a Young diagram means deleting all the nodes in the hook, and then moving all the nodes below and to the right of H diagonally up and to the left to create a new Young diagram. (Equivalently, we can remove the nodes of the corresponding *rim hook*, i.e., all the nodes along the bottom-right edge of the Young diagram from the hand node of H to the foot node.) For example, removing the $(2, 3)$-hook of the partition $(4^3, 3, 1)$ from the last example results in the partition $(4, 3, 2^2, 1)$.

2.5. The abacus

Combinatorics of partitions is often facilitated by using the abacus displays introduced by James. We take an abacus with e vertical runners labelled with the elements of $\mathbb{Z}/e\mathbb{Z}$, with the labels $0 + e\mathbb{Z}, 1 + e\mathbb{Z}, \ldots, -1 + e\mathbb{Z}$ occurring from left to right. We mark positions on these runners labelled with the non-negative integers, reading from left to right along successive rows from the top down. For example, when $e = 5$, the positions are labelled as follows.
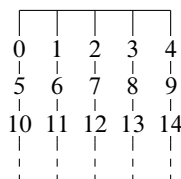
Given two positions $a < b$ on the abacus, we will say that a comes *before*, or *earlier than*, b, and that b comes *after*, or *later than*, a.

Given a partition λ and an integer $n \geqslant \lambda'_1$, we define the *n-bead abacus display* for λ by placing a bead on the abacus at position $\lambda_r + r - 1$ for $r = 1, \ldots, n$. For example, when $e = 5$, the 7-bead abacus display for the partition $(6, 4, 2, 1^2)$ is as follows.
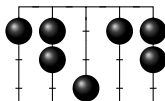

In an abacus display, we say that a position is *occupied* if there is a bead at that position, and *empty* otherwise. When we draw abacus displays, all positions below those drawn should be interpreted as being empty. Every configuration of finitely many beads on the abacus uniquely defines a partition: if the beads are in positions $b_1 > \cdots > b_n$, then the corresponding partition λ is given by

$$\lambda_r = \begin{cases} b_r + r - n & \text{for } r = 1, \ldots, n, \\ 0 & \text{for } r > n. \end{cases}$$

An abacus display for a partition λ is particularly useful for visualising removal of hooks: the hooks of length e in λ correspond to the positions $b \geqslant e$ such that position b is occupied in the abacus display while position $b - e$ is empty. Removing the hook corresponds to sliding the bead up from position b to position $b - e$. The leg length of the hook is the number of occupied positions in the range $[b - e + 1, b - 1]$, and the arm length is the number of empty positions in $[b - e + 1, b - 1]$.

It is well-known that abacus displays behave well with regard to conjugation. Given the n-bead abacus display for a partition λ, we take a large integer m, and let $b_1, \ldots, b_{m-n}$ be the empty positions before position m. Now if we construct a new abacus display in which the occupied positions are $m - b_1, \ldots, m - b_{m-n}$, this will be an abacus display for λ'. In other words, we can construct an abacus display for λ' from an abacus display for λ by truncating at some point after all the beads have appeared, and then replacing each bead with an empty space and each empty space with a bead and rotating through $180°$.

For example, taking the above abacus display for the partition $\lambda = (6, 4, 2, 1^2)$ and choosing $m = 15$, we obtain the following abacus display for $\lambda' = (5, 3, 2^2, 1^2)$.
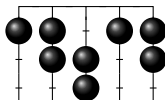
This has the following consequence, which we will need later.

Lemma 2.1. *Suppose λ is a partition, and take the n-bead abacus display for λ with e runners, for sufficiently large n. Let $t_1 < \cdots < t_m$ be the first m empty positions in the abacus display, and construct a new abacus display by moving the bead at position $t_i - e$ to position t_i, for $i = 1, \ldots, m$ in turn. The resulting configuration is the abacus display for the partition obtained by increasing the length of each of the first m columns of λ by e.*

Proof. Let μ be the partition obtained by this procedure. Take an abacus display for λ', and let $b_1 > \cdots > b_m$ be the positions of the last m beads. Then an abacus display for μ' is obtained by moving the bead at position b_i to position $b_i + e$, for $i = 1, \ldots, m$ in turn. The definition of the abacus display then means that μ' is obtained from λ' by adding e to each of its first m parts, which is the same as saying that μ is obtained from λ by adding e to each of its first m columns. ∎

Another important well-known result is the following.

Lemma 2.2. *Suppose λ and μ are partitions with the same e-content, and construct the m-bead abacus displays for λ and μ. Then for each $i \in \mathbb{Z}/e\mathbb{Z}$ the number of beads on runner i of the abacus is the same for λ as it is for μ.*

The proof of this lemma is just a combination of the fact that two partitions with the same e-content have the same e-core (a result which goes back to Littlewood [22]), and the fact that an abacus display for the e-core of a partition is obtained by sliding all the beads up their runners as far as they will go. This is due to James; we refer the reader to [17, Section 2.7] for more information on e-cores and the abacus.

2.6. (e, y)-regularisation

Now we describe some new results due to Millan Berdasco [24], on regularisation of partitions. This generalises e-regularisation introduced by James [15], and Berg's deregularisation operation [1].

Choose an integer $y \in \{1, \ldots, e - 1\}$. Given $r, c \in \mathbb{Z}$, we define the (e, y)-*ladder*

$$\mathcal{L}(r, c) = \{(r + k(y - e), c + ky) \mid k \in \mathbb{Z}\}.$$

The different (e, y)-ladders are disjoint, and comprise a partition of $\mathbb{Z}^2$. If $\mathcal{L}$ is an (e, y)-ladder and $(s, d) \in \mathbb{Z}^2$, then $\mathcal{L} + (s, d)$ is also an (e, y)-ladder.

We say that two partitions λ, μ are (e, y)-*equivalent* if $|\mathcal{L} \cap \lambda| = |\mathcal{L} \cap \mu|$ for every (e, y)-ladder $\mathcal{L}$. This is an equivalence relation on the set of partitions, and we call an equivalence class under this relation an (e, y)-*ladder class*.

Now say that a partition is (e, y)-*singular* if it has a hook with length et and arm length $yt - 1$ for some $t \geqslant 1$, and (e, y)-*regular* otherwise. The next theorem is the main result of [24].

Theorem 2.3. *Each (e, y)-ladder class $\mathcal{C}$ contains a unique (e, y)-regular partition. This partition dominates every partition in $\mathcal{C}$.*

Example. The $(3, 2)$-ladder class containing the partition $(5, 1)$ contains three other partitions. We illustrate these partitions as follows, labelling nodes with letters so that nodes in the same $(3, 2)$-ladder get the same label.

a	b	c	d	e
c				

a	b	c	d
c			
e			

a	b	c
c	d	e

a	b	c
c	d	
e		
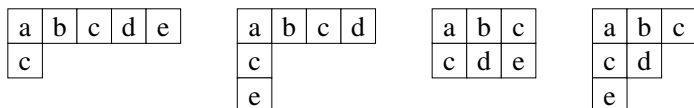
The unique most dominant partition in this class is $(5, 1)$, and this is the only $(3, 2)$-regular partition in the class: each of the other partitions has either a 3-hook with arm length 1 or a 6-hook with arm length 3.

In view of Theorem 2.3, we can define the (e, y)-*regularisation* of a partition λ to be the unique (e, y)-regular partition in the same ladder class.

In the case $y = 1$, this construction has been known for a long time. The $(e, 1)$-regularisation is the same as the e-regularisation defined by James [15], and is constructed by replacing all the nodes of λ in each ladder with the highest nodes in that ladder. When $y > 1$, the construction is not so straightforward: simply replacing the nodes with the highest nodes in their ladders will not in general result in a Young diagram. Berg [1] addresses the case $y = e - 1$, by introducing a notion of "locked" nodes of λ, and then constructing the (e, y)-regularisation of λ by moving all unlocked nodes to the highest available positions in their ladders. (In fact Berg's convention is the opposite of ours, in that he works with $(e, 1)$-ladders and constructs the least dominant partition in the ladder class of λ; but by conjugation this is equivalent to the case $y = e - 1$ of Theorem 2.3 (cf. Corollary 2.4 below).)

Millan Berdasco [24] gives an algorithm for constructing the (e, y)-regularisation in general using the abacus, generalising the author's algorithm [11] for realising the e-regularisation map on the abacus. This shows in particular that the (e, y)-regularisation is obtained by moving nodes up their (e, y)-ladders. We will see this algorithm in the next section.

We will also need to consider the least dominant partition in each (e, y)-ladder class, and we do this by using conjugate partitions. Define $S' = \{(c, r) \mid (r, c) \in S\}$ for any set $S \subseteq \mathbb{Z}^2$, and observe that if $\mathcal{L}$ is an (e, y)-ladder, then $\mathcal{L}'$ is an $(e, e - y)$-ladder. Hence two partitions λ and μ are (e, y)-equivalent if and only if λ' and μ' are $(e, e - y)$-equivalent. Furthermore, if H is a hook with length et and arm length $yt - 1$, then H' is a hook with length et and arm length $(e - y)t$. Now say that a partition is (e, y)-*restricted* if it has no hooks of length et and arm length yt for any t. This generalises the definition of an e-restricted partition: "e-restricted" is the same as "$(e, e - 1)$-restricted".

The following result follows from Theorem 2.3 and the fact that conjugation reverses the dominance order.

Corollary 2.4. *Each (e, y)-ladder class $\mathcal{C}$ contains a unique (e, y)-restricted partition. This partition is dominated by every partition in $\mathcal{C}$.*

The case $y = 1$ of this corollary is due to Berg. If λ is a partition, then we refer to the least dominant partition in the same (e, y)-ladder class as λ as the (e, y)-*restrictisation* of λ. (We prefer the awkward artificial word "restrictisation" over the more natural "restriction", because the latter word is already widely used.)

Example. Referring back to the last example, we see that the unique least dominant partition in the $(3, 2)$-class containing $(5, 1)$ is $(3, 2, 1)$. This partition is $(3, 2)$-restricted, while each of the other partitions in the class has a 3-hook with arm length 2.

2.7. (e, y)-regularisation on the abacus

Now we give an algorithm due to Millan Berdasco for computing the (e, y)-regularisation of a partition using the abacus. We will use this in Section 4.4 when we consider the Mullineux map on the abacus.

Take an integer $y \in [1, e - 1]$ and a partition ν which is (e, y)-singular. Our task is to construct a more dominant partition which is (e, y)-equivalent to μ; by doing this repeatedly, we will eventually reach the (e, y)-regularisation of ν.

Construct an abacus display for ν. Because ν is (e, y)-singular, there is some $t \in \mathbb{N}$ such that ν has a hook of length et and arm length $yt - 1$. We will assume that the only such t occurring is $t = 1$; if this is not the case, then taking the largest t that occurs, we can replace e, y with et, yt, and apply the same procedure to get a partition $\mu \rhd \nu$ which is (et, yt)-equivalent to ν. Because each (e, y)-ladder is a union of (et, yt)-ladders, μ will also be (e, y)-equivalent to ν, as required.

Assuming $t = 1$ is the only value of t that occurs, there is at least one occupied position b in the abacus such that position $b - e$ is empty and there are exactly y empty positions in the range $[b - e, b]$. We take the largest such b, and let E be the union of the congruence classes modulo e of the empty positions in $[b - e, b]$. Now let $b_1 < \cdots < b_m$ be the occupied positions in E after position $b - e$; in particular, $b_1 = b$. Also, let $t_1 < t_2 < \cdots$ be the empty positions in $\mathbb{N}_0 \setminus E$ after position b. Let $d \in \{1, \ldots, m\}$ be minimal such that either $t_d < b_{d+1}$ or $d = m$. Now construct a new abacus display by moving the bead at position b_i to position $b_i - e$ and moving the bead at position $t_i - e$ to position t_i for $i = 1, \ldots, d$ in turn. This gives the abacus display for a partition κ, and Millan Berdasco [24, Proposition 4.1] proves that $\kappa \rhd \nu$ and κ is (e, y)-equivalent to ν, as required.
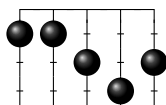
Example. Take $(e, y) = (5, 3)$ and $\lambda = (9, 3^3, 2)$, with the following 5-bead abacus display.
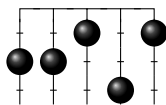
We can see that $b = 5$, with $E = (0 + 5\mathbb{Z}) \cup (1 + 5\mathbb{Z}) \cup (3 + 5\mathbb{Z})$. Hence,

$$(b_1, \dots, b_m) = (5, 6, 13) \quad \text{and} \quad (t_1, t_2, \dots) = (7, 9, 12, 14, \dots).$$

Hence, $d = 2$. So κ is defined by moving the beads at positions 5 and 6 up and the beads at positions 7 and 9 down, giving $\kappa = (9, 6, 5)$.

We can check that κ is $(5, 3)$-equivalent to λ and is $(5, 3)$-regular, so is the $(5, 3)$-regularisation of λ.

3. Crystals

Now we introduce crystals, giving a simplified definition suitable for our purposes. We recall the definition of a family of crystals from [9], and use (e, y)-regularisation to prove that the crystals in this family are isomorphic. Excellent references for crystals are the books by Kashiwara [20], Hong and Kang [13], and Bump and Schilling [6]. Here we provide an abbreviated account, restricting to the special case we need for this paper.

3.1. Introduction to crystals

A $\mathbb{Z}/e\mathbb{Z}$-*crystal* means a set B (not including 0 as an element) together with functions $\mathbf{e}_i, \mathbf{f}_i \colon B \to B \cup \{0\}$ for each $i \in \mathbb{Z}/e\mathbb{Z}$ with the property that if $b, c \in B$ and $i \in \mathbb{Z}/e\mathbb{Z}$ then $\mathbf{e}_i b = c$ if and only if $\mathbf{f}_i c = b$. The associated *crystal graph* is a labelled directed graph with B as its vertex set, and an arrow $b \overset{i}{\to} c$ whenever $\mathbf{f}_i b = c$. Often we abuse notation by not distinguishing between a crystal and the crystal graph.

   Now let $\widehat{\mathfrak{sl}}_e$ denote the affine Kac–Moody algebra of type $A_{e-1}^{(1)}$ (see [13, Chapter 2] for an uncomplicated account of the definition and classification of Kac–Moody algebras). An $\widehat{\mathfrak{sl}}_e$-crystal means a $\mathbb{Z}/e\mathbb{Z}$-crystal B endowed with a weight function wt from B to the weight space for $\widehat{\mathfrak{sl}}_e$, and functions $\varepsilon_i, \phi_i \colon B \to \mathbb{Z} \cup \{-\infty\}$ for each $i \in \mathbb{Z}/e\mathbb{Z}$ which satisfy certain axioms.

Certain types of module for the quantum group $U_q(\widehat{\mathfrak{sl}}_e)$ come equipped with crystals defined in a natural way from a crystal basis for the module. We say that an abstract crystal is *regular* (the term *normal* is used in [6]) if it arises as the crystal of a module in this way. An important theme in the theory of crystals is finding combinatorial models for regular crystals, i.e., finding an abstract crystal isomorphic to the crystal of a given module, with simple combinatorial objects (such as partitions or tableaux) as vertices and a combinatorial rule to determine where the arrows go.

In this paper we are concerned with a family of models for one particular regular crystal: this is the *basic crystal* $B(\Lambda_0)$, which is the crystal of the irreducible integrable highest-weight module $V(\Lambda_0)$ for $U_q(\widehat{\mathfrak{sl}}_e)$. There are two very well known models for this crystal:

- the *Misra–Miwa model* [25], whose vertex set is the set of all e-restricted partitions;

- the dual of the Misra–Miwa model, whose vertex set is the set of all e-regular partitions.

We will now we assume $e \geqslant 3$; this assumption will be in force until the end of Section 3, where we will make some comments on the case $e = 2$. In the case $e \geqslant 3$, Berg [1] found another model for $B(\Lambda_0)$ (the "ladder crystal") with a different set of partitions as its vertex set, and then in [9] the author showed that all three models are members of a continuous family of models for $B(\Lambda_0)$. The proof in [9] is long and technical, using results of Stembridge [29] and Danilov–Karzanov–Koshevoy [7] to show that each of the proposed models is a regular crystal by analysing its local structure, and then appealing to the uniqueness of regular crystals with a unique highest-weight vertex. A geometric proof was given by Sam and Tingley [28], using crystal structure on quiver varieties.

The object in this paper is to give a more direct combinatorial proof that the crystals from [9] are models for $B(\Lambda_0)$, by showing directly that they are all isomorphic to the dual of the Misra–Miwa model. This generalises Berg's approach. A by-product of this is a direct construction of the (unique) isomorphism from the Misra–Miwa model to its dual; this provides a new realisation of the *Mullineux map*, which is the subject of Section 4.

For this paper, it suffices to think of a crystal simply as a directed graph whose arrows are labelled with elements of $\mathbb{Z}/e\mathbb{Z}$. An *isomorphism* between two crystals then just means a bijection α between their vertex sets such that there is an arrow $b \xrightarrow{i} c$ if and only if there is an arrow $\alpha(b) \xrightarrow{i} \alpha(c)$. The additional functions wt, ε_i, ϕ_i and their compatibility with isomorphisms follow naturally in the cases we are concerned with, so we can ignore them in this paper.

3.2. A family of crystals

We now define the family of crystals introduced in [9, Section 2.3]; the Misra–Miwa model and its dual arise as special cases.

Define an *arm sequence* to be a sequence $A = (A_1, A_2, \dots)$ of integers satisfying

- $t - 1 \leqslant A_t \leqslant (e-1)t$ for all $t \geqslant 1$, and

- $A_{t+u} \in \{A_t + A_u, A_t + A_u + 1\}$ for all $t, u \geqslant 1$.

In fact, arm sequences are easily classified. Given any real number $y \in [1, e-1]$, we define two arm sequences A^{y+} and A^{y-} by

$$A_t^{y+} = \lfloor yt \rfloor, \quad A^{y-} = \lceil yt - 1 \rceil$$

for $t \geqslant 1$. (Here, as usual, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x \in \mathbb{R}$, and $\lceil x \rceil$ is the smallest integer greater than or equal to x.) By [9, Lemma 7.4] these sequences are arm sequences, and every arm sequence has one of these forms. Obviously if y is irrational then $A^{y+} = A^{y-}$, but otherwise these arm sequences are distinct.

Now given two arm sequences A and B, write $A \leqslant B$ if $A_t \leqslant B_t$ for all t. It follows from the classification above that $\leqslant$ is a total order on the set of arm sequences. Specifically, if $x, y \in [1, e-1]$ with $x < y$, we get $A^{x\pm} < A^{y\pm}$, while $A^{y-} < A^{y+}$ for every rational $y \in [1, e-1]$.

We will define a crystal $\mathcal{R}_A$ for each arm sequence A. First we define the underlying set of partitions which will be the vertex set of $\mathcal{R}_A$. Say that a partition λ is *A-regular* if it has no hook with length et and arm length A_t for any t. We define $\mathcal{R}_A$ to be the set of A-regular partitions.

Example. Suppose $e = 4$, and let $\lambda = (5, 2, 1^2)$. The hooks of λ of length divisible by 4 are a 4-hook with arm length 1, and an 8-hook with arm length 4. So λ is A-regular for any arm sequence A beginning $(0, \dots)$ or $(2, 5, \dots)$ or $(3, \dots)$.

Now we need to define the crystal operators $\mathbf{e}_i$ and $\mathbf{f}_i$ on $\mathcal{R}_A$, for $i \in \mathbb{Z}/e\mathbb{Z}$. Recall that an *i-node* means a node of residue i. We define a total order (depending on A) on the set of all i-nodes. Given two different i-nodes (r, c) and (s, d), the integer $s - r + c - d$ equals et for some integer t. By interchanging the two nodes if necessary, we assume $t \geqslant 0$. Now we set $(r, c) \prec (s, d)$ if $c - d \leqslant A_t$, and $(r, c) \succ (s, d)$ otherwise. For this purpose, we read A_0 as 0. Then it is easy to check that $\preccurlyeq$ is a total order on the set of i-nodes.

Now take $\lambda \in \mathcal{R}_A$. Let $(r_1, c_1), \dots, (r_m, c_m)$ be the addable and removable i-nodes of λ, ordered so that $(r_1, c_1) \succ \dots \succ (r_m, c_m)$. Define the *i-signature* of λ (with respect to A) to be the sequence of signs $s = s_1 \dots s_m$ defined by $s_k = +$ if (r_k, c_k) is an addable node, and $s_k = -$ if (r_k, c_k) is a removable node. The *reduction* of s is the

sequence red(s) obtained from s by repeatedly deleting adjacent pairs $+-$. If there are any $-$ signs in red(s), then the removable node (r_k, c_k) corresponding to the last one is the *good* i-node of λ, and we define $\mathbf{e}_i \lambda = \lambda \setminus (r_k, c_k)$; otherwise we set $\mathbf{e}_i \lambda = 0$. If there are any $+$ signs in red(s), then the addable node (r_l, c_l) corresponding to the first one is called the *cogood* i-node of λ, and we set $\mathbf{f}_i \lambda = \lambda \cup (r_l, c_l)$; otherwise we set $\mathbf{f}_i \lambda = 0$.

It is an easy consequence of the construction that if $\lambda, \mu \in \mathcal{R}_A$ then $\lambda = \mathbf{e}_i \mu$ if and only if $\mu = \mathbf{f}_i \lambda$. Furthermore, it is shown in [9, Proposition 6.1] that if $\lambda \in \mathcal{R}_A$, then $\mathbf{e}_i \lambda, \mathbf{f}_i \lambda \in \mathcal{R}_A \cup \{0\}$. So the functions $\mathbf{e}_i, \mathbf{f}_i$ for $i \in \mathbb{Z}/e\mathbb{Z}$ endow $\mathcal{R}_A$ with the structure of a crystal. This crystal is connected, and its only source (i.e., the only vertex with no incoming arrows) is $\varnothing$, by [9, Proposition 6.3].

Example. Suppose $e = 4$ and A is an arm sequence. Suppose $\lambda = (5, 2, 1^2)$ is A-regular. Then $A_1 \neq 1$ and $A_2 \neq 4$. Take $i = 0 + 4\mathbb{Z}$. The addable and removable i-nodes of λ are $(1, 5)$, $(2, 2)$, and $(5, 1)$. The ordering of these nodes, the i-signature s, the reduced i-signature red(s) depend on A_1 as follows.

A_1	order	s	red(s)	$\mathbf{e}_i \lambda$	$\mathbf{f}_i \lambda$
0	$(1, 5) \succ (2, 2) \succ (5, 1)$	$--+$	$--+$	$(5, 1^3)$	$(5, 2, 1^3)$
2	$(5, 1) \succ (1, 5) \succ (2, 2)$	$+--$	$-$	$(5, 1^3)$	0
3	$(5, 1) \succ (2, 2) \succ (1, 5)$	$+--$	$-$	$(4, 2, 1^2)$	0

We now consider the two extreme special cases of this construction. In the special case where $A = A^{(e-1)+} = (e - 1, 2e - 2, 3e - 3, \dots)$, a partition λ is A-regular if and only if it is e-restricted: the e-restricted condition simply says that λ has no e-hook with arm length $e - 1$, but this automatically implies that λ has no et-hook with arm length $(e - 1)t$ for any t. If λ is e-restricted, then the ordering $(r_1, c_1) \succ \cdots \succ (r_m, c_m)$ on the addable and removable i-nodes is simply the order of these nodes from bottom to top in the Young diagram. This means that $\mathcal{R}_{A^{(e-1)+}}$ is the Misra–Miwa model. Similarly, if $A = A^{1-} = (0, 1, 2, \dots)$, then a partition is A-regular if and only if it is e-regular, in which case the ordering on the addable and removable i-nodes is the order from top to bottom, which means that $\mathcal{R}_{A^{1-}}$ is the dual of the Misra–Miwa model.

3.3. (e, y)-regularisation and crystal isomorphisms

Now we come to our first main result, which uses the regularisation and restrictisation operations from Section 2.6 to give crystal isomorphisms.

Theorem 3.1. *Suppose $y \in [1, e-1] \cap \mathbb{Q}$, and let z be the denominator of y. Then (ez, yz)-regularisation defines an isomorphism of crystals $\mathcal{R}_{A^{y+}} \to \mathcal{R}_{A^{y-}}$. The inverse isomorphism is given by (ez, yz)-restrictisation.*

For the remainder of Section 3.3 we fix $y \in [1, e-1] \cap \mathbb{Q}$, we let z denote the denominator of y, and we write $A = A^{y+}$ and $B = A^{y-}$. We will say "ladder" to mean "(ez, yz)-ladder", and use the terms "ladder equivalent" and "ladder class" similarly.

Proving Theorem 3.1 involves two parts: first, showing that (ez, yz)-regularisation and (ez, yz)-restrictisation give mutually inverse functions between the sets $\mathcal{R}_A$ and $\mathcal{R}_B$, and then showing that these functions commute with the crystal operators $\mathbf{e}_i$ and $\mathbf{f}_i$.

First we show that (ez, yz)-regularisation maps $\mathcal{R}_A$ to $\mathcal{R}_B$. In the case $z = 1$ this is immediate: if y is an integer, then "(e, y)-regular" and "A^{y-}-regular" mean the same thing. But for $z > 1$ more work is needed. Let us assume that $z > 1$, and say that a hook is *y-bad* if it has length te and arm length $\lfloor yt \rfloor$ for some integer t not divisible by z. Let $\mathrm{bad}(\lambda)$ denote the number of y-bad hooks of λ.

Proposition 3.2. *Suppose λ and μ lie in the same ladder class. Then*

$$\mathrm{bad}(\lambda) = \mathrm{bad}(\mu).$$

Proof. Recall that if $r, c \in \mathbb{Z}$ then the ladder $\mathcal{L}(r, c)$ is defined as

$$\mathcal{L}(r, c) = \{(r + k(y - e)z, c + kyz) \mid k \in \mathbb{Z}\}.$$

We need to show that $\mathrm{bad}(\lambda)$ depends only on the number of nodes of λ in each ladder, and we do this by induction on $|\lambda|$. For the inductive step, take a partition ν with an addable node (r, c); then we need to show that $\mathrm{bad}(\nu \cup (r, c))$ depends only on $\mathrm{bad}(\nu)$, the ladder $\mathcal{L}(r, c)$, and the number of nodes of ν in each ladder.

Define the set

$$\mathcal{M} = \bigcup_{t=1}^{z-1} \mathcal{L}(r + A_t - te, c + A_t).$$

Observe that because z is the denominator of y,

$$A_{t+z} = \lfloor y(t + z) \rfloor = \lfloor yt \rfloor + yz = A_t + yz$$

for any t, and so

$$\mathcal{L}(r + A_{t+z} - (t + z)e, c + A_{t+z}) = \mathcal{L}(r + A_t - te, c + A_t).$$

So in fact, $\mathcal{M}$ contains the ladder $\mathcal{L}(r + A_t - te, c + A_t)$ for every $t \in \mathbb{N}$ not divisible by z.

Now we compare the hooks of $\nu \cup (r, c)$ and ν. For every hook of ν with foot node $(r - 1, c)$, there is a hook of $\nu \cup (r, c)$ with the same hand node and with foot node (r, c). Similarly, for every hook of ν with hand node $(r, c - 1)$, there is a hook of $\nu \cup (r, c)$ with the same foot node and with hand node (r, c). Additionally, $\nu \cup (r, c)$ has a 1-hook with hand and foot node both equal to (r, c). Otherwise, the hooks of ν and $\nu \cup (r, c)$ coincide.

Now observe that:

- a hook with foot node (r, c) and hand node (s, d) is y-bad if and only if $(s - 1, d) \in \mathcal{M}$;

- a hook with foot node $(r - 1, c)$ and hand node (s, d) is y-bad if and only if $(s, d) \in \mathcal{M}$;

- a hook with hand node (r, c) and foot node (s, d) is y-bad if and only if $(s, d - 1) \in \mathcal{M}$;

- a hook with hand node $(r, c - 1)$ and foot node (s, d) is y-bad if and only if $(s, d) \in \mathcal{M}$.

So $\operatorname{bad}(\nu \cup (r, c)) - \operatorname{bad}(\nu)$ equals

$$
\begin{aligned}
&|\{(s, d) \in \nu \mid s < r, \ (s, d + 1) \notin \nu \text{ and } (s - 1, d) \in \mathcal{M}\}| \\
&\quad - |\{(s, d) \in \nu \mid s < r, \ (s, d + 1) \notin \nu \text{ and } (s, d) \in \mathcal{M}\}| \\
&\quad + |\{(s, d) \in \nu \mid s > r, \ (s + 1, d) \notin \nu \text{ and } (s, d - 1) \in \mathcal{M}\}| \\
&\quad - |\{(s, d) \in \nu \mid s > r, \ (s + 1, d) \notin \nu \text{ and } (s, d) \in \mathcal{M}\}|.
\end{aligned}
$$

Taking each $(s, d) \in \mathcal{M}$ in turn and examining the possible intersections of ν with the set $\{(s, d), (s + 1, d), (s, d + 1), (s + 1, d + 1)\}$ and its contribution to the above sum, we find that $\operatorname{bad}(\nu \cup (r, c)) - \operatorname{bad}(\nu)$ is the number of configurations

in ν minus the number of configurations
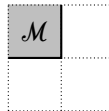
in ν. In these diagrams, the box marked $\mathcal{M}$ indicates an element of $\mathcal{M}$ (either higher or lower than the node (r, c)), the shaded boxes are nodes of ν, and the unshaded boxes are nodes not lying in ν. (We include the case of boxes $(s, d) \in \mathcal{M}$ with s or d equal to 0; for these cases, the shaded boxes can include elements of $\mathbb{Z}^2 \setminus \mathbb{N}^2$.)

Now we obtain

$$\text{bad}(\nu \cup (r, c)) = \text{bad}(\nu) + |\nu \cap \mathcal{M}| - |\nu \cap (\mathcal{M} + (0, 1))|$$
$$- |\nu \cap (\mathcal{M} + (1, 0))| + |\nu \cap (\mathcal{M} + (1, 1))|.$$

Since each of the sets $\mathcal{M}$, $\mathcal{M} + (0, 1)$, $\mathcal{M} + (1, 0)$, $\mathcal{M} + (1, 1)$ is a union of ladders, this is all we need for our inductive step. ∎

Corollary 3.3. (ez, yz)-*regularisation yields a bijection $\mathcal{R}_A \to \mathcal{R}_B$, with inverse given by* (ez, yz)-*restrictisation.*

Proof. Take $\lambda \in \mathcal{R}_A$. Then λ has no hooks of length et and arm length $A_t = \lfloor yt \rfloor$ for any t, and so in particular has no y-bad hooks. Now let μ be the (ez, yz)-regularisation of λ. Then by Theorem 2.3 μ has no hooks of length ezt and arm length $yzt - 1$ for any t, and by Proposition 3.2 μ has no y-bad hooks. Hence, μ has no hooks of length et and arm length B_t for any t, i.e., $\mu \in \mathcal{R}_B$.

So (ez, yz)-regularisation gives a function $\mathcal{R}_A \to \mathcal{R}_B$. By conjugating everything, we can show in the same way that (ez, yz)-restrictisation gives a function from $\mathcal{R}_B$ to $\mathcal{R}_A$. Every partition in $\mathcal{R}_B$ is (ez, yz)-regular, so equals the (ez, yz)-regularisation of its (ez, yz)-restrictisation. Similarly, every partition in $\mathcal{R}_A$ equals the (ez, yz)-restrictisation of its (ez, yz)-regularisation. So the two functions $\mathcal{R}_A \leftrightarrow \mathcal{R}_B$ are inverses of each other. ∎

Now we come to the second part of the proof of Theorem 3.1: showing that the bijections between $\mathcal{R}_A$ and $\mathcal{R}_B$ preserve the crystal operators $\mathbf{e}_i$ and $\mathbf{f}_i$ for each i. For clarity, we will write $\mathbf{e}_i^A$ and $\mathbf{f}_i^A$ for the crystal operators on $\mathcal{R}_A$, and define $\mathbf{e}_i^B$ and $\mathbf{f}_i^B$ similarly.

We continue to use "ladder" to mean "(ez, yz)-ladder". Clearly all the nodes in a given ladder have the same residue, and we say that a ladder in which all nodes have residue i is an i-ladder.

If $(r, c) \in \mathbb{Z}^2$, then we define the *depth* $\text{dep}(r, c) = yr + (e - y)c$. The following lemma comes from [24], and in fact the condition in this lemma is used as the definition of a ladder in [24].

Lemma 3.4 ([24, Lemma 3.2]). *Suppose $(r, c), (s, d) \in \mathbb{Z}^2$. Then (r, c) and (s, d) lie in the same ladder if and only if they have the same depth and the same residue.*

Proof. It is trivial to check that if (r, c) and (s, d) lie in the same ladder then they have the same depth and the same residue. Conversely, suppose (r, c) and (s, d) have the same residue and the same depth. The latter statement says

$$yr + (e - y)c = ys + (e - y)d,$$

which rearranges to

$$yz\frac{r - c - s + d}{e} = z(d - c).$$

The factor $(r - c - s + d)/e$ is an integer because (r, c) and (s, d) have the same residue. By definition z and yz are coprime integers, so yz divides $d - c$; let us say $d - c = kyz$ for $k \in \mathbb{Z}$. But then $(s, d) = (r, c) + k(yz - ez, yz)$, so (r, c) and (s, d) lie in the same ladder. ∎

Lemma 3.4 allows us to impose a total order on the set of i-ladders: we set $\mathcal{K} < \mathcal{L}$ if the depth of the nodes in $\mathcal{K}$ is less than the depth of the nodes in $\mathcal{L}$. In particular, given an i-ladder $\mathcal{L}$, the i-ladder $\mathcal{L} + (1, 1)$ satisfies $\mathcal{L} < \mathcal{L} + (1, 1)$.

Now we study the relationship between ladders and i-signatures. First we need a lemma.

Lemma 3.5. *Suppose λ is a partition with a removable i-node (r, c) and an addable i-node (s, d) with $\mathrm{dep}(s, d) - e < \mathrm{dep}(r, c) < \mathrm{dep}(s, d)$. Then λ has a y-bad hook.*

Proof. Let us assume that $c < d$ (the case $c > d$ is very similar). The fact that

$$\mathrm{dep}(s, d) - e < \mathrm{dep}(r, c)$$

implies that $s < r$, and we claim that the hook with hand node $(s, d - 1)$ and foot node (r, c) is y-bad. The arm length of this hook is $d - c - 1$, and its length is $d - c - s + r$, which is divisible by e because (r, c) and (s, d) are both i-nodes; so we can write

$$d - c - s + r = te,$$

with $t \in \mathbb{Z}$.

The assumption that $\mathrm{dep}(s, d) - e < \mathrm{dep}(r, c) < \mathrm{dep}(s, d)$ says

$$ys + (e - y)d - e < yr + (e - y)c < ys + (e - y)d,$$

which rearranges to

$$d - c - 1 < yt < d - c.$$

This says, in particular, that yt is not an integer, so t is not divisible by z. Furthermore, $d - c - 1 = \lfloor yt \rfloor = A_t$, so the hook is y-bad. ∎

Now for any partition λ and any $u \in \mathbb{Q}$, write $\mathrm{rem}_i^u(\lambda)$ for the number of removable i-nodes of λ of depth u, and $\mathrm{add}_i^u(\lambda)$ for the number of addable i-nodes of λ of depth u. We want to compare these numbers for two partitions which are ladder-equivalent.

Lemma 3.6. *Suppose λ and μ are two ladder-equivalent partitions, and that $i \in \mathbb{Z}/e\mathbb{Z}$ and $u \in \mathbb{Q}$. Then*

$$\mathrm{add}_i^u(\lambda) - \mathrm{rem}_i^{u-e}(\lambda) = \mathrm{add}_i^u(\mu) - \mathrm{rem}_i^{u-e}(\mu).$$

Proof. We prove this lemma by induction on $|\lambda|$. If there are no i-nodes of depth u then the result is trivial, so assume there are such nodes, and let $\mathcal{L}$ be the ladder containing them. Then the ladder containing the i-nodes of depth $u - e$ (if there are any) is $\mathcal{L} - (1, 1)$. To prove our result by induction, it suffices to take a partition ν with an addable node (r, c), and show that the difference

$$D = \mathrm{add}_i^u(\nu \cup (r, c)) - \mathrm{rem}_i^{u-e}(\nu \cup (r, c)) - \mathrm{add}_i^u(\nu) + \mathrm{rem}_i^{u-e}(\nu)$$

depends only on the ladder that contains (r, c). Adding the node (r, c) to ν can only affect the addable nodes in $\mathcal{L}$ if (r, c) lies in $\mathcal{L}$, $\mathcal{L} - (0, 1)$ or $\mathcal{L} - (1, 0)$. Similarly, adding (r, c) can only affect the removable nodes in $\mathcal{L} - (1, 1)$ if (r, c) lies in $\mathcal{L} - (0, 1)$, $\mathcal{L} - (1, 0)$, or $\mathcal{L} - (1, 1)$. So we have four cases to consider.

$(r, c) \in \mathcal{L}$: In this case

$$\mathrm{add}_i^u(\nu \cup (r, c)) = \mathrm{add}_i^u(\nu) - 1,$$

because (r, c) is an addable node of ν but not of $\nu \cup (r, c)$. So, $D = -1$.

$(r, c) \in \mathcal{L} - (1, 0)$: In this case consider the node $(r + 1, c - 1)$.

- If $(r + 1, c - 1) \in \nu$, then

$$\mathrm{add}_i^u(\nu \cup (r, c)) = \mathrm{add}_i^u(\nu) + 1,$$

  because the node $(r + 1, c)$ is an addable node of $\nu \cup (r, c)$ but not of ν; but $\mathrm{rem}_i^{u-e}(\nu \cup (r, c)) = \mathrm{rem}_i^{u-e}(\nu)$.

- If $(r + 1, c - 1) \notin \nu$, then

$$\mathrm{add}_i^u(\nu \cup (r, c)) = \mathrm{add}_i^u(\nu),$$

  but $\mathrm{rem}_i^{u-e}(\nu \cup (r, c)) = \mathrm{rem}_i^{u-e}(\nu) - 1$, because $(r, c - 1)$ is a removable node of ν but not of $\nu \cup (r, c)$.

Either way, $D = 1$ in this case.

$(r, c) \in \mathcal{L} - (0, 1)$: Similarly to the last case, we always have $D = 1$ in this case.

$(r, c) \in \mathcal{L} - (1, 1)$: In this case $D = -1$, similarly to the first case. ∎

Now we can consider i-signatures. Recall from Section 3.2 that the definition of the i-signature of a partition λ depends on the order $\preccurlyeq$ on the set of i-nodes, which

in turn depends on the arm sequence A. Since we are concerned with two different arm sequences $A = A^{y+}$ and $B = A^{y-}$, we will write $\preccurlyeq_A$ and $\preccurlyeq_B$ for the associated orders, and refer to the (A,i)-signature and the (B,i)-signature of a partition, and to A-good and B-good i-nodes. Here is the crucial result.

Proposition 3.7. *Suppose $\lambda \in \mathcal{R}_A$, and let μ be the (ez, yz)-regularisation of λ.*

(1) *The reduced (A,i)-signature of λ equals the reduced (B,i)-signature of μ.*

(2) *If λ has an A-good i-node, then μ has a B-good i-node in the same ladder.*

Proof. Suppose (r,c) and (s,d) are i-nodes. If $\operatorname{dep}(r,c) < \operatorname{dep}(s,d)$, then $(r,c) \prec_A (s,d)$. So when we calculate the (A,i)-signature of λ, the signs $\pm$ corresponding to ladders of greater depth come before those of smaller depth.

If $\operatorname{dep}(r,c) = \operatorname{dep}(s,d)$, then $(r,c) \prec_A (s,d)$ if and only if $r < s$. Now we observe that if λ has an addable i-node (r,c) and a removable i-node (s,d) of the same depth, then $r < s$ (otherwise the hook with foot node (s,d) and hand node $(r-1,c)$ would have length et and arm length A_t for some t, contradicting the assumption that $\lambda \in \mathcal{R}_A$). What this means is that in the (A,i)-signature of λ, the $-$ signs corresponding to removable nodes in a given ladder precede the $+$ signs corresponding to addable nodes in the same ladder.

So we can write down the (A,i)-signature of λ, in terms of the addable and removable nodes in each ladder. Given an integer n, we let $[n]$ denote a string of $+$ signs of length n if $n \geqslant 0$, or a string of $-$ signs of length $-n$ if $n < 0$. Now if we let $u_1 < u_2 < \cdots$ be the possible depths that i-nodes can have, then the first part of the proof shows that the (A,i)-signature of λ equals the concatenation

$$\ldots \left[-\operatorname{rem}_i^{u_3}(\lambda)\right]\left[\operatorname{add}_i^{u_3}(\lambda)\right]\left[-\operatorname{rem}_i^{u_2}(\lambda)\right]\left[\operatorname{add}_i^{u_2}(\lambda)\right]\left[-\operatorname{rem}_i^{u_1}(\lambda)\right]\left[\operatorname{add}_i^{u_1}(\lambda)\right].$$

Because λ has no y-bad hooks, Lemma 3.5 shows that if $v - e < u < v$ then at least one of $\operatorname{rem}_i^u(\lambda)$ and $\operatorname{add}_i^v(\lambda)$ is zero, so the string $[\operatorname{add}_i^v(\lambda)][-\operatorname{rem}_i^u(\lambda)]$ is the same as the string $[-\operatorname{rem}_i^u(\lambda)][\operatorname{add}_i^v(\lambda)]$. Applying this observation repeatedly in the above expression for the (A,i)-signature of λ, we find that this signature equals

$$\ldots \left[\operatorname{add}_i^{u_3}(\lambda)\right]\left[-\operatorname{rem}_i^{u_3-e}(\lambda)\right]\left[\operatorname{add}_i^{u_2}(\lambda)\right]\left[-\operatorname{rem}_i^{u_2-e}(\lambda)\right]\left[\operatorname{add}_i^{u_1}(\lambda)\right]\left[-\operatorname{rem}_i^{u_1-e}(\lambda)\right].$$

(Here we have introduced some empty strings $\operatorname{rem}_i^{u-e}(\lambda)$ for those u such that there are nodes of depth u but no nodes of depth $u - e$, but this is harmless.)

Now observe that if $m, n \in \mathbb{Z}$ with either $m \geqslant 0$ or $n \leqslant 0$, then the reduction of the string $[m][n]$ is $[m+n]$. So the reduced (A,i)-signature of λ is the reduction of the sequence

$$S =$$
$$\ldots \left[\operatorname{add}_i^{u_3}(\lambda) - \operatorname{rem}_i^{u_3-e}(\lambda)\right]\left[\operatorname{add}_i^{u_2}(\lambda) - \operatorname{rem}_i^{u_2-e}(\lambda)\right]\left[\operatorname{add}_i^{u_1}(\lambda) - \operatorname{rem}_i^{u_1-e}(\lambda)\right].$$

The calculation of the (B, i)-signature of μ works in exactly the same way, except that we interchange "above" and "below" when considering nodes of the same depth. So we find that the reduced (B, i)-signature of μ is the reduction of the sequence

$$\ldots \left[\operatorname{add}_i^{u_3}(\mu) - \operatorname{rem}_i^{u_3 - e}(\mu) \right] \left[\operatorname{add}_i^{u_2}(\mu) - \operatorname{rem}_i^{u_2 - e}(\mu) \right] \left[\operatorname{add}_i^{u_1}(\mu) - \operatorname{rem}_i^{u_1 - e}(\mu) \right].$$

But Lemma 3.6 shows that this sequence coincides with S; moreover, the addable or removable nodes of λ and μ corresponding to each sign in this sequence have the same depth. The result follows. ∎

Now we are able to prove our main theorem.

Proof of Theorem 3.1. Corollary 3.3 shows that (ez, yz)-regularisation yields a bijection $\mathcal{R}_A \to \mathcal{R}_B$, with inverse given by (ez, yz)-restrictisation. Now take $\lambda \in \mathcal{R}_A$, and let μ be the (ez, yz)-regularisation of λ. Then for $i \in \mathbb{Z}/e\mathbb{Z}$ Proposition 3.7 shows that $\mathbf{e}_i^A \lambda \neq 0$ if and only if $\mathbf{e}_i^B \mu \neq 0$. Assuming $\mathbf{e}_i^A \lambda \neq 0$, Proposition 3.7 also shows that $\mathbf{e}_i^A \lambda$ and $\mathbf{e}_i^B \mu$ are ladder-equivalent. Since $\mathbf{e}_i^A \lambda \in \mathcal{R}_A$ and $\mathbf{e}_i^B \mu \in \mathcal{R}_B$, this means that $\mathbf{e}_i^B \mu$ is the (ez, yz)-regularisation of $\mathbf{e}_i^A \lambda$ and $\mathbf{e}_i^A \lambda$ is the (ez, yz)-restrictisation of $\mathbf{e}_i^B \mu$. So we have an isomorphism of crystals. ∎

In view of Theorem 3.1, we extend our notation: if $y \in [1, e - 1]$ is rational, then we define (e, y)-*regularisation* to mean (ez, yz)-regularisation, where z is the denominator of y.

3.4. Composing crystal isomorphisms

In this section we extend Theorem 3.1 to show that for any two arm sequences the crystals $\mathcal{R}_A$ and $\mathcal{R}_B$ are isomorphic. We do this by considering finite subcrystals. Given any $n \in \mathbb{N}$, let $\mathcal{R}_A^{(n)}$ denote the induced subgraph of the crystal graph of $\mathcal{R}_A$ comprising only partitions of size at most ne.

Lemma 3.8. *The crystal $\mathcal{R}_A^{(n)}$ depends only on $A_1, A_2, \ldots, A_n$.*

Proof. If λ is a partition of size at most ne, then the hooks of λ have length at most ne, so in particular λ cannot have a hook of length et for $t > n$. So whether λ lies in $\mathcal{R}_A$ depends only on $A_1, \ldots, A_n$. Furthermore, if (r, c) and (s, d) are addable or removable i-nodes of λ, then $|r - s - c + d| \leqslant ne$, so whether $(r, c) \preccurlyeq (s, d)$ depends only on $A_1, \ldots, A_n$. So if $\lambda \in \mathcal{R}_A^{(n)}$ then the i-signature of λ, and hence $\mathbf{e}_i \lambda$, depend only on $A_1, \ldots, A_n$. ∎

Given two arm sequences A and B, our strategy will be to construct an isomorphism $\mathcal{R}_A^{(n)} \to \mathcal{R}_B^{(n)}$ for every n. Such isomorphisms will automatically be compatible, thanks to the following result on uniqueness of isomorphisms.

Proposition 3.9. *Suppose A is an arm sequence. Then the only isomorphism from $\mathcal{R}_A$ to $\mathcal{R}_A$ is the identity. For any $n \in \mathbb{N}$, the only isomorphism from $\mathcal{R}_A^{(n)}$ to $\mathcal{R}_A^{(n)}$ is the identity.*

Proof. Suppose α is an isomorphism from $\mathcal{R}_A$ to $\mathcal{R}_A$ or from $\mathcal{R}_A^{(n)}$ to $\mathcal{R}_A^{(n)}$. We prove that $\alpha(\lambda) = \lambda$ for every λ by induction on $|\lambda|$. As mentioned in Section 3.2, $\varnothing$ is the unique source in $\mathcal{R}_A$. Because $|\mathbf{e}_i \lambda| < |\lambda|$ whenever $\mathbf{e}_i \lambda \neq 0$, this means that $\varnothing$ is also the unique source in $\mathcal{R}_A^{(n)}$. So $\varnothing$ must be preserved by α. If $\lambda \neq \varnothing$, then λ can be written as $\mathbf{f}_{i_1} \dots \mathbf{f}_{i_r} \varnothing$ for some $i_1, \dots, i_r \in \mathbb{Z}/e\mathbb{Z}$, which means that

$$\alpha(\lambda) = \mathbf{f}_{i_1} \dots \mathbf{f}_{i_r} \alpha(\varnothing) = \mathbf{f}_{i_1} \dots \mathbf{f}_{i_r} \varnothing = \lambda. \qquad \blacksquare$$

Corollary 3.10. *Suppose A and B are arm sequences, $m < n$ are natural numbers, and $\alpha_m \colon \mathcal{R}_A^{(m)} \to \mathcal{R}_B^{(m)}$ and $\alpha_n \colon \mathcal{R}_A^{(n)} \to \mathcal{R}_B^{(n)}$ are isomorphisms. Then $\alpha_n(\mathcal{R}_A^{(m)}) = \mathcal{R}_B^{(m)}$, and $\alpha_n|_{\mathcal{R}_A^{(m)}} = \alpha_m$.*

Proof. Obviously $\alpha_m(\varnothing) = \alpha_n(\varnothing) = \varnothing$, since $\varnothing$ is the unique source in any of these crystals. This then implies that $|\alpha_n(\lambda)| = |\lambda|$ for every $\lambda \in \mathcal{R}_A^{(n)}$, since $|\lambda|$ is the length of every directed path from $\varnothing$ to λ in $\mathcal{R}_A$. Hence, $\alpha_n(\mathcal{R}_A^{(m)}) \subseteq \mathcal{R}_B^{(m)}$; then the same argument with α_n^{-1} in place of α shows that $\alpha_n(\mathcal{R}_A^{(m)}) = \mathcal{R}_B^{(m)}$.

So $\alpha_n|_{\mathcal{R}_A^{(m)}}$ gives an isomorphism from $\mathcal{R}_A^{(m)}$ to $\mathcal{R}_B^{(m)}$. By Proposition 3.9 there is only one isomorphism from $\mathcal{R}_A^{(m)}$ to $\mathcal{R}_A^{(m)}$, and so there is at most one isomorphism from $\mathcal{R}_A^{(m)}$ to $\mathcal{R}_B^{(m)}$. So, $\alpha_n|_{\mathcal{R}_A^{(m)}} = \alpha_m$. $\qquad \blacksquare$

So if we can construct an isomorphism $\mathcal{R}_A^{(n)} \to \mathcal{R}_B^{(n)}$ for every n, then we obtain an isomorphism $\mathcal{R}_A \to \mathcal{R}_B$ by gluing these isomorphisms together. The next lemma shows how to realise any given $\mathcal{R}_A^{(n)}$ as $\mathcal{R}_{A^{y+}}^{(n)}$ for some $y \in \mathbb{Q}$.

Lemma 3.11. *Suppose A is an arm sequence and $n \in \mathbb{N}$, and let*

$$y = \max\left\{ \frac{A_t}{t} \mid t \in \{1, \dots, n\} \right\}.$$

Then $A_t = \lfloor yt \rfloor$ for $t = 1, \dots, n$. So, $\mathcal{R}_A^{(n)} = \mathcal{R}_{A^{y+}}^{(n)}$.

Proof. The choice of y means that $A_t \leqslant yt$ for each $t \leqslant n$. So we need to show that $A_t > yt - 1$ for each t. If instead $A_t \leqslant yt - 1$ for some t, then for any u the definition of an arm sequence implies that $A_{tu} \leqslant A_t u + u - 1 < ytu$. But if we choose u so that $A_u = yu$, then the definition of an arm sequence also gives $A_{tu} \geqslant A_u t = ytu$, a contradiction.

So $(A_1, \dots, A_t) = (A_1^{y+}, \dots, A_t^{y+})$, and so by Lemma 3.8 $\mathcal{R}_A^{(n)} = \mathcal{R}_{A^{y+}}^{(n)}$. $\qquad \blacksquare$

Now we can prove the main result of this section.

Theorem 3.12. *Suppose A and B are arm sequences. Then $\mathcal{R}_A^{(n)}$ and $\mathcal{R}_B^{(n)}$ are isomorphic for every $n \in \mathbb{N}$, and so $\mathcal{R}_A$ is isomorphic to $\mathcal{R}_B$.*

Proof. Recall the total order on arm sequences defined by $A \geqslant B$ if $A_t \geqslant B_t$ for all t. We will assume $A \geqslant B$, and show that $\mathcal{R}_A^{(n)}$ is isomorphic to $\mathcal{R}_B^{(n)}$ by induction on the natural number

$$N = \sum_{t=1}^{n} (A_t - B_t).$$

If $N = 0$, then $(A_1, \ldots, A_n) = (B_1, \ldots, B_n)$, and so the identity map is an isomorphism from $\mathcal{R}_A^{(n)}$ to $\mathcal{R}_B^{(n)}$. If not, then let

$$y = \max\left\{ \frac{A_t}{t} \mid t \in \{1, \ldots, n\} \right\}.$$

Then by Lemma 3.11 $\mathcal{R}_A^{(n)} = \mathcal{R}_{A^{y+}}^{(n)}$.

Since by assumption $A_t > B_t$ for at least one $t \in \{1, \ldots, n\}$, we get $A^{y+} > B$, so that $A^{y-} \geqslant B$. Furthermore, $A_t^{y-} < A_t^{y+}$ for at least one $t \in \{1, \ldots, n\}$ (namely, any value of t for which $A_t = yt$), so

$$\sum_{t=1}^{n} (A_t^{y-} - B_t) < N,$$

and so by induction, $\mathcal{R}_{A^{y-}}^{(n)}$ is isomorphic to $\mathcal{R}_B^{(n)}$. By Theorem 3.1, $\mathcal{R}_{A^{y+}}$ and $\mathcal{R}_{A^{y-}}$ are isomorphic via a regularisation map, and so $\mathcal{R}_A^{(n)} = \mathcal{R}_{A^{y+}}^{(n)}$ is isomorphic to $\mathcal{R}_{A^{y-}}^{(n)}$, and hence to $\mathcal{R}_B^{(n)}$.

So $\mathcal{R}_A^{(n)}$ is isomorphic to $\mathcal{R}_B^{(n)}$ for every n, and so (as explained at the start of this section) $\mathcal{R}_A$ is isomorphic to $\mathcal{R}_B$. ∎

Example. Suppose $e = 4$, and A and B are arm sequences with $(A_1, A_2, A_3, A_4) = (2, 4, 6, 8)$ and $(B_1, B_2, B_3, B_4) = (1, 2, 4, 5)$. We consider the subcrystals $\mathcal{R}_A^{(4)}$ and $\mathcal{R}_B^{(4)}$ obtained by taking partitions of size at most 16. We can show that these are isomorphic via the chain of isomorphisms on the left in Figure 1. In each row of the diagram we give an equality between two crystals $\mathcal{R}_C^{(4)}$ and $\mathcal{R}_D^{(4)}$; this equality holds by Lemma 3.8 because the first four entries of C coincide with the first four entries of D; we give these four entries in each case.

Under the isomorphism $\mathcal{R}_A^{(4)} \to \mathcal{R}_B^{(4)}$, the partition $(4, 3^2, 2, 1^4)$ maps to $(6, 4, 2, 1^4)$, via the sequence shown on the right in Figure 1.

We end this section with some comments on the case $e = 2$. In this case there are only two arm sequences

$$A^{1+} = (1, 2, 3, \ldots), \quad A^{1-} = (0, 1, 2, \ldots),$$

$$\mathcal{R}_A^{(4)} \xmapsto{\ (2,4,6,8)\ } \mathcal{R}_{A^2+}^{(4)} \qquad\qquad (4,3^2,2,1^4)$$

$(4,2)$-regularisation $\qquad\qquad\qquad\qquad (4,2)$-regularisation

$$\mathcal{R}_{A^2-}^{(4)} \xmapsto{\ (1,3,5,7)\ } \mathcal{R}_{A^{7/4}+}^{(4)} \qquad\qquad (5,4,2,1^5)$$

$(16,7)$-regularisation $\qquad\qquad\qquad (16,7)$-regularisation

$$\mathcal{R}_{A^{7/4}-}^{(4)} \xmapsto{\ (1,3,5,6)\ } \mathcal{R}_{A^{5/3}+}^{(4)} \qquad\qquad (5,4,2,1^5)$$

$(12,5)$-regularisation $\qquad\qquad\qquad (12,5)$-regularisation

$$\mathcal{R}_{A^{5/3}-}^{(4)} \xmapsto{\ (1,3,4,6)\ } \mathcal{R}_{A^{3/2}+}^{(4)} \qquad\qquad (6,4,2,1^4)$$

$(8,3)$-regularisation $\qquad\qquad\qquad\qquad (8,3)$-regularisation

$$\mathcal{R}_{A^{3/2}-}^{(4)} \xmapsto{\ (1,2,4,5)\ } \mathcal{R}_B^{(4)} \qquad\qquad (6,4,2,1^4)$$
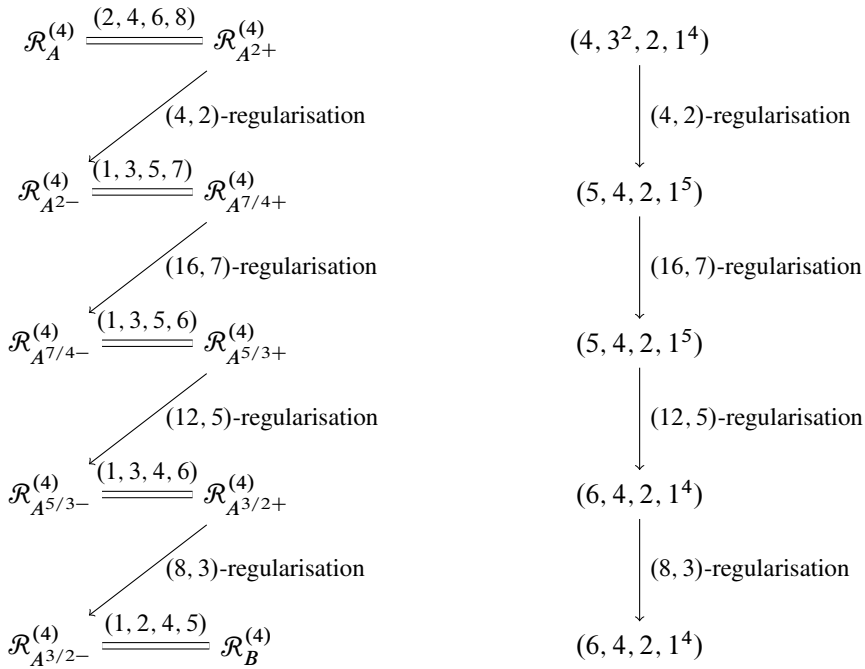
Figure 1. Isomorphism of crystals.

so our family of crystals contains nothing beyond the Misra–Miwa model $\mathcal{R}_{A^1+}$ and its dual $\mathcal{R}_{A^1-}$. Berg's work remains valid in the case $e = 2$, but his model now coincides with the Misra–Miwa crystal. Theorem 3.1 also remains valid when $e = 2$: the only possible value of y now is $y = 1$, and the isomorphisms between $\mathcal{R}_{A^1+}$ and $\mathcal{R}_{A^1-}$ are given by $(2, 1)$-regularisation and -restrictisation. In fact, if λ is a 2-restricted partition then its $(2, 1)$-regularisation is simply λ', and similarly for $(2, 1)$-restrictisation.

So the main results of the present section remain valid for $e = 2$, though they tell us almost nothing. But we will need to allow $e = 2$ in Section 4.4.

4. The Mullineux map

Now we use the results of the previous section to give a new combinatorial algorithm for the Mullineux map.

4.1. Background on the Mullineux map

The Mullineux map first arose in the representation theory of the symmetric group $\mathfrak{S}_n$. When e is a prime number and $\mathbb{F}$ is a field of characteristic e, the irreducible $\mathbb{F}\mathfrak{S}_n$-modules are the *James modules* D^λ indexed by the e-regular partitions λ of n; these were first constructed in [16]. One of these modules is the one-dimensional *sign module* sgn, on which a permutation $\pi \in \mathfrak{S}_n$ acts as $\mathrm{sgn}(\pi)$. Tensoring an irreducible module with a one-dimensional module yields an irreducible module, which means that there is a function m_e (the *Mullineux involution*) on the set of e-regular partitions of n satisfying

$$\mathrm{D}^\lambda \otimes \mathrm{sgn} \cong \mathrm{D}^{\mathrm{m}_e(\lambda)}$$

for each λ. The *Mullineux problem* is to give a combinatorial description of m_e, and there are now several known solutions to this problem. Mullineux [26] gave a combinatorial map M_e, and conjectured that $\mathrm{M}_e = \mathrm{m}_e$. Meanwhile, Kleshchev [21] gave a different combinatorial map K_e, and proved (using his modular branching rules for the symmetric groups) that $\mathrm{K}_e = \mathrm{m}_e$. Shortly afterwards, Ford and Kleshchev [12] proved the purely combinatorial result that $\mathrm{K}_e = \mathrm{M}_e$, thus proving Mullineux's conjecture. Shorter proofs of this statement was given by Bessenrodt and Olsson [2] and by Xu [30], who gave another combinatorial map X_e and showed that $\mathrm{K}_e = \mathrm{X}_e = \mathrm{M}_e$. Later, Brundan and Kujawa [5] gave yet another combinatorial map S_e based on Serganova's work on the general linear supergroup, and showed that $\mathrm{m}_e = \mathrm{S}_e = \mathrm{X}_e$. Very recently, Jacon [14] has given a family of algorithms based on isomorphisms between higher-level crystals; one of these is equivalent to Xu's algorithm, giving a new proof that $\mathrm{X}_e = \mathrm{K}_e$.

   None of the combinatorial parts of this story depend on e being prime, and all the combinatorial maps mentioned in the last paragraph are defined when e is any integer great then 1. In fact there is an algebraic interpretation of this more general situation: the Mullineux involution for arbitrary e can be defined by generalising the construction above from the group algebra of $\mathfrak{S}_n$ in characteristic e to an Iwahori–Hecke algebra of $\mathfrak{S}_n$ in quantum characteristic e; Brundan [4] showed how to generalise Kleshchev's results to this setting, so that this more general Mullineux map is given by all the combinatorial algorithms mentioned in the last paragraph.

4.2. Crystals and a new algorithm for the Mullineux map

The Brundan–Kleshchev approach to the Mullineux problem also provides a link with crystals. Define a *signed isomorphism* between crystals A and B to be a bijection $\alpha: A \to B$ such that $\mathbf{f}_i b = c$ if and only if $\mathbf{f}_{-i}\alpha(b) = \alpha(c)$, for $b, c \in A$ and $i \in \mathbb{Z}/e\mathbb{Z}$. In other words, α is an isomorphism of directed graphs under which each arrow $\xrightarrow{i}$ maps to an arrow $\xrightarrow{-i}$.

Recall that $B(\Lambda_0)$ denotes the crystal of the irreducible highest-weight module for $\widehat{\mathfrak{sl}}_e$ with highest weight Λ_0. There is a signed isomorphism $B(\Lambda_0) \to B(\Lambda_0)$; this arises because of the diagram automorphism of $\widehat{\mathfrak{sl}}_e$ preserving the weight Λ_0. For each of our combinatorial models $\mathcal{R}_A$ for this crystal, this signed isomorphism is realised as an involutory bijection on the set of A-regular partitions. In the particular case of the arm sequence $A = A^{1-} = (0, 1, 2, \dots)$, the crystal $\mathcal{R}_A$ consists of all e-regular partitions, and the Brundan–Kleshchev branching rules show that the signed automorphism in this case (which is unique, in view of Proposition 3.9) is the Mullineux map.

We can use this to realise the Mullineux map in terms of (unsigned) isomorphisms between different crystal models. The following result comes directly from the definitions by conjugating partitions (and using the fact that the residue of the node (c, r) is the negative of the residue of (r, c)).

Proposition 4.1. *Suppose A is an arm sequence, and define the conjugate arm sequence A' by*

$$A'_t = et - A_t - 1.$$

If λ is a partition, then $\lambda \in \mathcal{R}_A$ if and only if $\lambda' \in \mathcal{R}_{A'}$. Furthermore, there is a signed isomorphism $\mathcal{R}_A \to \mathcal{R}_{A'}$ given by $\lambda \mapsto \lambda'$.

If we consider in particular the arm sequence $A = A^{1-}$, then $A' = A^{(e-1)+}$, and the A'-regular partitions are precisely the e-restricted partitions. Proposition 4.1 yields a signed isomorphism $\mathcal{R}_A \to \mathcal{R}_{A'}$. Composing this with the unique isomorphism $\mathcal{R}_{A'} \to \mathcal{R}_A$ gives a signed isomorphism from $\mathcal{R}_A \to \mathcal{R}_A$, which must be the Mullineux map, by uniqueness. So we have proved the following.

Proposition 4.2. *Let α be the unique isomorphism from $\mathcal{R}_{A^{(e-1)+}}$ to $\mathcal{R}_{A^{1-}}$. Then*

$$\mathrm{m}_e(\lambda) = \alpha(\lambda')$$

for any e-regular partition λ.

The results of Section 3.4 allow us to express the isomorphism α as a composition of regularisation maps. We use this description to give a new combinatorial algorithm to compute $\mathrm{m}_e(\lambda)$ for any e-regular partition λ. Recall that if $y \in [1, e-1] \cap \mathbb{Q}$, then we use the term (e, y)-*regularisation* to mean (ez, yz)-regularisation, where z is the denominator of y. Now we can give our algorithm for computing $\mathrm{m}_e(\lambda)$.

(1) Let $\mu = \lambda'$ and $x = e - 1$.

(2) If there exists a rational number $y \in [1, x]$ such that μ is not A^{y-}-regular, then let y be the largest such number. Replace μ with its (e, y)-regularisation and replace x with y.

(3) Repeat Step 2 until μ is A^{y-}-regular for every $y \in [1, x] \cap \mathbb{Q}$.
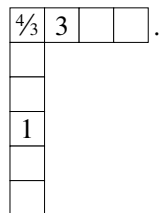
(4) Output μ.

Informally, we start with the partition $\mu = \lambda'$, and imagine a variable y decreasing continuously from $e - 1$ to 1; for each rational value of y, we replace μ with its (e, y)-regularisation.

In Step 2 of the algorithm, finding the largest $y \in [1, x]$ such that μ is not A^{y-}-regular is actually straightforward. Suppose μ has a hook with length re and arm length a. The existence of this hook means that μ fails to be A^{y-}-regular whenever $a/r < y \leqslant (a + 1)/r$. So if we define the *slope* of this hook to be $(a + 1)/r$, then the required value of y is the largest $y \in [1, x]$ which occurs as the slope of an re-hook in μ.

Example. Suppose $e = 3$ and $\lambda = (6, 2, 1)$. We set $\mu = \lambda' = (3, 2, 1^4)$ and $x = 2$. We draw the Young diagram of μ; for each hook of length divisible by 3, we fill the corresponding node with the slope of the hook:

	2	
1	
1

.
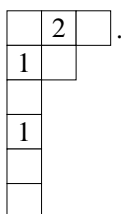
In this case the largest $y \leqslant 2$ for which μ is not A^{y+}-regular is $y = 2$. So we replace μ with its $(3, 2)$-regularisation $(4, 1^5)$, for which we draw a similar diagram:

4/3	3		
1

.

The existence of a 9-hook with arm length 3 means that now $y = \frac{4}{3}$. So we replace μ with its $(3, \frac{4}{3})$-regularisation (that is, its $(9, 4)$-regularisation) $(5, 1^4)$, and draw its diagram:

5/3		3		
1

.
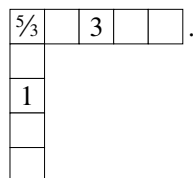
We set $y = 1$, and replace μ with its $(3, 1)$-regularisation $(5, 2^2)$.

| | 2 | 3 | | | .
|---|---|---|---|---|
| 2 | | | | |
| | | | | |

Since y is now 1, we output $\mathrm{m}_3(6, 2, 1) = (5, 2^2)$.

4.3. Regularisation and the Mullineux map

In this section we use our new algorithm to give a new proof of the main result from the author's paper [8] relating the Mullineux map and e-regularisation. Given a partition λ, we write λ^{reg} for the $(e, 1)$-regularisation of λ. Given a hook of λ with arm length a and leg length l, we say that the hook is *steep* if $l \geqslant (e - 1)a$, or *shallow* if $a \geqslant (e - 1)l$. Now we have the following result; this generalises a result of Bessenrodt, Olsson and Xu [3, Theorem 4.8] and confirms a conjecture of Lyle [23].

Theorem 4.3 ([8, Conjecture 1.6]). *Suppose λ is a partition. Then*

$$\mathrm{m}_e(\lambda^{\mathrm{reg}}) \trianglerighteq (\lambda')^{\mathrm{reg}},$$

with equality if and only if every hook of λ with length divisible by e is either steep or shallow.

Remarks.   (1) In fact, the statement given above is stronger than that given in [8], where dominance is not considered; only the statement about when $\mathrm{m}_e(\lambda^{\mathrm{reg}}) = (\lambda')^{\mathrm{reg}}$ is proved. However, the method of proof in [8] can easily be adapted to show the dominance part of the result as well.

(2) In [8], the "if" part of Theorem 4.3 is proved combinatorially, while an algebraic proof (using canonical basis coefficients) is given for the "only if" part. The proof we give here is entirely combinatorial.

First we need a lemma.

Lemma 4.4. *Suppose $e - 1 \geqslant y > 1$. Suppose λ is an (e, y)-singular partition, and let μ be its (e, y)-regularisation. Then $\mu^{\mathrm{reg}} \triangleright \lambda^{\mathrm{reg}}$.*

Proof. Each $(e, 1)$-ladder contains exactly one node in each column. So we can number the $(e, 1)$-ladders $\ldots, \mathcal{L}_1, \mathcal{L}_2, \ldots$ by letting $\mathcal{L}_r$ be the ladder containing $(r, 1)$. We say that ladder $\mathcal{L}_s$ is *later* than ladder $\mathcal{L}_r$ if $s > r$. Then each node in ladder $\mathcal{L}_r$ has a node in $\mathcal{L}_{r+1}$ immediately below it, and as a consequence the number of nodes in $\mathcal{L}_{r+1} \setminus \lambda$ is at least the number of nodes in $\mathcal{L}_r \setminus \lambda$.

Now take $M > 0$ such that $\lambda, \mu \subseteq \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_M$, let $N = |\mathcal{L}_M \cap \mathbb{N}^2|$, and define

$$\widehat{\lambda}_r = N - |\mathcal{L}_r \setminus \lambda|$$

for $r = 1, \ldots, N$, with $\widehat{\lambda}_r = 0$ for $r > N$. Define $\widehat{\mu}$ similarly. The previous paragraph shows that $\widehat{\lambda}$ and $\widehat{\mu}$ are partitions, and obviously $|\widehat{\lambda}| = |\widehat{\mu}|$.

Observe that if (r, c) and (s, d) are nodes in the same (e, y)-ladder with $s < r$, then (s, d) lies in a later $(e, 1)$-ladder than (r, c). Since μ is constructed from λ by moving nodes up their (e, y)-ladders, these nodes are moved to later $(e, 1)$-ladders, which gives $\widehat{\lambda} \rhd \widehat{\mu}$.

Now consider $(\lambda^{\mathrm{reg}})'$. Since λ^{reg} is constructed simply by moving all the nodes of λ up to the highest positions in their $(e, 1)$-ladders, $(\lambda^{\mathrm{reg}})'_d$ equals the number of r for which $|\mathcal{L}_r \setminus \lambda| < d$. Hence, for any $d \geqslant 1$,

$$
\begin{aligned}
(\lambda^{\mathrm{reg}})'_1 + \cdots + (\lambda^{\mathrm{reg}})'_d &= \sum_{i=1}^{d} |\{r \geqslant 1 \mid \widehat{\lambda}_r > N - i\}| \\
&= \widehat{\lambda}'_{N-d+1} + \cdots + \widehat{\lambda}'_N \\
&= |\widehat{\lambda}| - (\widehat{\lambda}'_1 + \cdots + \widehat{\lambda}'_{N-d})
\end{aligned}
$$

and similarly for μ. Now the fact that $\widehat{\lambda} \rhd \widehat{\mu}$ gives $(\lambda^{\mathrm{reg}})' \rhd (\mu^{\mathrm{reg}})'$, and hence $\lambda^{\mathrm{reg}} \lhd \mu^{\mathrm{reg}}$. ∎

Proof of Theorem 4.3. Suppose first that every hook of λ with length divisible by e is either shallow or steep. This means in particular that λ' is $(e, e - 1)$-regular. Since λ and λ^{reg} lie (by definition) in the same $(e, 1)$-ladder class, λ' and $(\lambda^{\mathrm{reg}})'$ lie in the same $(e, e - 1)$-ladder class. So λ' is the $(e, e - 1)$-regularisation of $(\lambda^{\mathrm{reg}})'$.

When we apply our algorithm to compute $\mathrm{m}_e(\lambda^{\mathrm{reg}})$, we first compute $(\lambda^{\mathrm{reg}})'$ and then its $(e, e - 1)$-regularisation, which is λ', and then apply (e, y)-regularisation for every rational $y < e - 1$. But because every hook of λ with length divisible by e is either shallow or steep, λ' is (e, y)-regular whenever $e - 1 > y > 1$. So none of the (e, y)-regularisation maps have any effect until we reach $y = 1$, at which point we replace λ' with its $(e, 1)$-regularisation $(\lambda')^{\mathrm{reg}}$. So $\mathrm{m}_e(\lambda^{\mathrm{reg}}) = (\lambda')^{\mathrm{reg}}$.

Conversely, suppose λ has an er-hook which is neither steep nor shallow. Suppose first that λ' is $(e, e - 1)$-regular. Then the calculation of $\mathrm{m}_e(\lambda^{\mathrm{reg}})$ begins as in the case above, by computing $(\lambda^{\mathrm{reg}})'$ and then $(e, e - 1)$-regularising to get λ'. Then we reach $\mathrm{m}_e(\lambda^{\mathrm{reg}})$ by applying (e, y)-regularisation for some finite list of values $y = y_1, \ldots, y_r$, with $e - 1 > y_1 > \cdots > y_r > 1$, and then finally (if necessary) applying $(e, 1)$-regularisation. So we construct a list of partitions

$$\lambda' = \mu(0) \neq \mu(1) \neq \cdots \neq \mu(r),$$

where $\mu(k)$ is the (e, y_k)-regularisation of $\mu(k - 1)$ for each k, and $\mathrm{m}_e(\lambda^{\mathrm{reg}}) = \mu(r)^{\mathrm{reg}}$.

The fact that λ (and hence λ') has an er-hook which is neither steep nor shallow means that λ' is (e, y)-singular for some y with $e - 1 > y > 1$. (Indeed, suppose λ' has an et-hook with arm length a and leg length l; then λ' is (e, y)-singular for $y = (a - 1)/t$, which lies strictly between 1 and $e - 1$.) Hence, $r \geqslant 1$. Now by Lemma 4.4

$$(\lambda')^{\mathrm{reg}} \lhd \mu(1)^{\mathrm{reg}} \lhd \cdots \lhd \mu(r)^{\mathrm{reg}} = \mathrm{m}_e(\lambda^{\mathrm{reg}}),$$

as required.

The case where λ is $(e, e - 1)$-singular is similar: in this case we begin by computing $(\lambda^{\mathrm{reg}})'$, and then applying (e, y)-regularisation for $y = y_1, \ldots, y_r$ where now $e - 1 = y_1 > \cdots > y_r > 1$. Again this gives a sequence of partitions $\mu(1), \ldots, \mu(r)$ with

$$\mu(1)^{\mathrm{reg}} \lhd \cdots \lhd \mu(r)^{\mathrm{reg}} = \mathrm{m}_e(\lambda^{\mathrm{reg}}).$$

In this case, $\mu(1)$ is the $(e, e - 1)$-regularisation of $(\lambda^{\mathrm{reg}})'$, which is the same as the $(e, e - 1)$-regularisation of λ'. So Lemma 4.4 gives $\mu(1)^{\mathrm{reg}} \rhd (\lambda')^{\mathrm{reg}}$, which is what we need. ∎

## 4.4. The Mullineux map and separated partitions on the abacus

In this section we use our new algorithm for the Mullineux map to prove a reduction theorem which shows that for certain partitions (characterised by a property of their abacus displays) the effect of the Mullineux map can be calculated using Mullineux maps for smaller values of e. Applying this repeatedly yields a new (and purely combinatorial) proof of Paget's theorem [27] describing the effect of the Mullineux map on partitions labelling modules in RoCK blocks of symmetric groups and Iwahori–Hecke algebras.

We fix a set I of integers which is a union of congruence classes modulo e, and set $\bar{I} = \mathbb{Z} \setminus I$. We define c to be the number of congruence classes contained in I, and set $\bar{c} = e - c$. We will assume for now that $2 \leqslant c \leqslant e - 2$, but later we will explain how our results can be extended to the cases where $c = 1$ or $e - 1$. We fix a large integer n divisible by e. Abacus displays with e runners will always be assumed to have n beads.

Given a partition λ, we take the n-bead abacus display with e runners, and construct a new abacus display with c runners by discarding all the positions not in I; this defines a partition which we write as λ_I. Similarly, discarding the positions in I gives a $\bar{c}$-runner abacus display for a partition which we write as $\lambda_{\bar{I}}$.

Example. Take $e = 5$, $I = (0 + 5\mathbb{Z}) \cup (2 + 5\mathbb{Z})$ and $\lambda = (5, 3^2, 2, 1)$. Then $\lambda_I = (2)$ and $\lambda_{\bar{I}} = (2, 1)$. We show the abacus displays for these partitions; in the examples in

this section, when drawing e-runner abacus displays, we will use white beads for the positions in I.

$$\lambda = (5, 3^2, 2, 1) \qquad \lambda_I = (2) \qquad \lambda_{\overline{I}} = (2, 1)$$
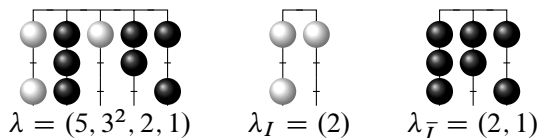
Now say that λ is I-*separated* if in the abacus display for λ the first empty position in I occurs after the last occupied position in $\overline{I}$. (This condition is independent of n, given the assumption that n is divisible by e.) Our aim is to describe the effect of the Mullineux map on I-separated partitions. In fact we will undertake the equivalent task of computing the composite function $\lambda \mapsto m_e(\lambda')$ for e-restricted λ; this composite map preserves the e-content of a partition, so is better suited to the abacus. So we will take an I-separated e-restricted partition λ, and describe $m_e(\lambda')$ in terms of the partitions λ_I and $\lambda_{\overline{I}}$ and the maps m_c and $m_{\overline{c}}$. Very roughly speaking $m_e(\lambda')$ is obtained by replacing λ_I with $m_c(\lambda'_I)$ and $\lambda_{\overline{I}}$ with $m_{\overline{c}}(\lambda'_{\overline{I}})$; however, because $\lambda_{\overline{I}}$ need not be $\overline{c}$-restricted some additional manipulation is needed.

We will use the (e, y)-regularisation maps discussed earlier, so we start with some simple lemmas looking at the (e, y)-regular and (e, y)-restricted conditions for I-separated partitions.

Lemma 4.5. *Suppose λ is an I-separated partition.*

(1) λ *is $(e, \overline{c})$-regular if and only if $\lambda_{\overline{I}}$ is $\overline{c}$-restricted.*

(2) *If λ_I is c-regular, then λ is (e, y)-regular for all $y \in (\overline{c}, \overline{c} + 1) \cap \mathbb{Q}$.*

(3) *If $y \in [\overline{c} + 1, e - 1] \cap \mathbb{Q}$, then λ is (e, y)-regular if and only if λ_I is $(c, y - \overline{c})$-regular.*

(4) λ *is e-restricted if and only if λ_I is c-restricted.*

Proof. We prove only (1) and the "if" part of (3); the proofs for the other parts are similar.

For (1), suppose first that $\lambda_{\overline{I}}$ fails to be $\overline{c}$-restricted. Then in the abacus display for $\lambda_{\overline{I}}$ with $\overline{c}$ runners, there is a bead preceded by $\overline{c}$ empty spaces. Hence in the e-runner abacus display for λ, there is a bead in $\overline{I}$ (at position b, say) such that the preceding $\overline{c}$ positions in $\overline{I}$ are all empty. In particular, position $b - e$ is empty. Now the I-separated condition means that all positions in I before position b are occupied. Hence among the positions $b - e, \ldots, b - 1$, exactly c are occupied (i.e., those lying in I), so λ fails to be $(e, \overline{c})$-regular.

Conversely, if λ is not $(e, \overline{c})$-regular, then in the e-runner abacus display for λ there is an occupied position b such that for some $t \geqslant 1$ position $b - et$ is empty and exactly ct of the positions $b - et, \ldots, b - 1$ are occupied. Now we claim that b lies in $\overline{I}$.

If not, then the I-separated condition means that all positions in $[b - et, b - 1] \cap \overline{I}$ are empty. Because position $b - et$ is also empty, this means that there are at most $et - 1$ occupied positions among $b - et, \ldots, b - 1$, a contradiction.

So $b \in \overline{I}$, as claimed. Now the I-separated condition again means that all the positions in $[b - et, b - 1] \cap I$ are occupied. This gives ct occupied positions, so all the positions in $[b - et, b - 1] \cap \overline{I}$ are empty. Hence, in the $\overline{c}$-runner abacus display for $\lambda_{\overline{I}}$ there are $\overline{c}t$ consecutive empty spaces followed by a bead, so $\lambda_{\overline{I}}$ is not $\overline{c}$-restricted.

Now we prove the "if" part of (3). Suppose λ is (e, y)-singular. Then λ has a hook with length et and arm length $yt - 1$ for some $t \in \mathbb{N}$ for which $yt \in \mathbb{N}$. So in the abacus display for λ there is an occupied position $b \geqslant et$ such that position $b - et$ is empty, and there are exactly yt empty positions in the range $[b - et, b]$. At most $\overline{c}t$ of these positions lie in $\overline{I}$, so (because $y > \overline{c}$) at least one must lie in I. Now the assumption that λ is I-separated means that $b \in I$. Hence the empty position $b - et$ lies in I, so the I-separated assumption again means that all the positions in $[b - et, b] \cap \overline{I}$ are empty. So the c-runner abacus display for λ_I has an occupied position a (corresponding to position b in the e-runner abacus) with position $a - ct$ empty, and exactly $yt - \overline{c}t$ empty positions in the range $[a - ct, a]$. So λ_I is $(c, y - \overline{c})$-singular. ∎

Lemma 4.5 (4) allows us to formulate the main theorem of this section, but for this we need some more notation. With our large integer n fixed as above, we fix an integer $0 \leqslant u \leqslant n$. We will say that a partition λ is a (u, I)-partition if the n-bead abacus display for λ has exactly u beads in positions in I (and therefore $n - u$ beads in positions in $\overline{I}$). A (u, I)-partition λ is determined by the two partitions λ_I and $\lambda_{\overline{I}}$. Conversely, for any two partitions β, γ there is a unique (u, I)-partition λ with $\lambda_I = \beta$ and $\lambda_{\overline{I}} = \gamma$.

Given two partitions α, β, we write $\alpha \,|\overline{c}|\, \beta$ for the partition

$$(\overline{c}\alpha_1 + \beta_1, \overline{c}\alpha_2 + \beta_2, \ldots)$$

and $\alpha \,\underline{\overline{c}}\, \beta$ for the partition obtained by taking the parts of β together with c copies of each of the parts of α, and arranging these parts in decreasing order.

Now we give the set-up for our main theorem. We keep I, u fixed as above. We take three partitions α, β, γ, with β being c-restricted and γ being $\overline{c}$-restricted. We define two (u, I)-partitions λ and μ by

$$\lambda_I = \beta, \quad \lambda_{\overline{I}} = \alpha \,|\overline{c}|\, \gamma, \quad \mu_I = \alpha' \,\underline{\overline{c}}\, \mathrm{m}_c(\beta'), \quad \mu_{\overline{I}} = \mathrm{m}_{\overline{c}}(\gamma').$$

Now we can state the main theorem of this section.

Theorem 4.6. *Suppose λ and μ defined as above. If λ and μ are both I-separated, then $m_e(\lambda') = \mu$.*

Examples.   (1) Take $e = 5$ and $I = (1 + 5\mathbb{Z}) \cup (4 + 5\mathbb{Z})$, so that $(c, \bar{c}) = (2, 3)$. Taking $n = 15$, we choose $u = 10$, and

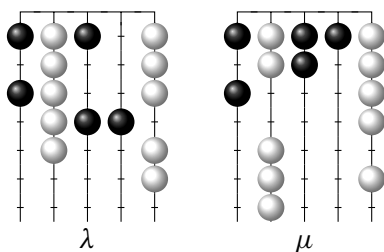$$\alpha = (2^2, 1), \quad \beta = (2, 1^2), \quad \gamma = (1^3).$$

We know that $m_2(3, 1) = (3, 1)$ and we can calculate $m_3(3) = (2, 1)$, so we get

$$\lambda_I = (2, 1^2), \quad \lambda_{\bar{I}} = (7^2, 4), \quad \mu_I = (3^3, 2^2, 1), \quad \mu_{\bar{I}} = (2, 1),$$

and hence

$$\lambda = (15, 11, 9, 7^3, 6, 4^3, 2, 1), \quad \mu = (17, 16, 14, 10, 9, 5, 2^2, 1^2).$$

We see from the following abacus displays that λ and μ are both I-separated, and we can check that $m_5(\lambda') = \mu$.
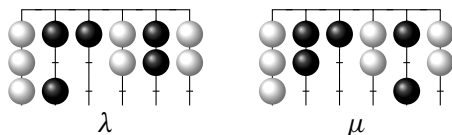

(2) We give an example to show that it is necessary to assume that λ and μ are both I-separated in Theorem 4.6. Take $e = 6$, $I = (0 + 6\mathbb{Z}) \cup (3 + 6\mathbb{Z}) \cup (5 + 6\mathbb{Z})$, $n = 12$ and $u = 7$. If we take

$$\alpha = \beta = \varnothing, \quad \gamma = (2^2),$$

then we get

$$\lambda = (2^5), \quad \mu = (5, 2^2, 1).$$

From the abacus displays we see that λ is I-separated but μ is not, and we can check that $m_6(\lambda') \neq \mu$.

We will prove Theorem 4.6 using Proposition 4.2, which can be rephrased as saying that the map $\lambda \mapsto \mathrm{m}_e(\lambda')$ is the unique isomorphism between the crystals $\mathcal{R}_{A^{(e-1)+}}$ and $\mathcal{R}_{A^{1-}}$. This map is given by applying (e, y)-regularisation for all rational numbers in $[1, e-1]$ in decreasing order. In fact we factorise this isomorphism as a product of three isomorphisms

$$\mathcal{R}_{A^{(e-1)+}} \to \mathcal{R}_{A^{\bar{c}+}} \to \mathcal{R}_{A^{\bar{c}-}} \to \mathcal{R}_{A^{1-}}$$

and deal with each of these separately. For this we will need to define two intermediate partitions. Keeping the notation from above, we define π and ρ to be the (u, I)-partitions given by

$$\pi_I = \mathrm{m}_f(\beta'), \quad \pi_{\bar{I}} = \alpha \, |\bar{c}| \, \gamma, \quad \rho_I = \alpha' \, \underline{\bar{c}} \, \mathrm{m}_f(\beta'), \quad \rho_{\bar{I}} = \gamma.$$

We fix this notation for the rest of this section, and assume from now on that λ and μ are I-separated.

The results of Section 3.4 show that the isomorphism $\mathcal{R}_{A^{(e-1)+}} \to \mathcal{R}_{A^{\bar{c}+}}$ is defined by applying (e, y)-regularisation for all rational $y \in (\bar{c}, e-1]$ in decreasing order. The next proposition looks at what happens when we apply one of these regularisation maps.

Proposition 4.7. *Suppose $y \in [\bar{c}+1, e-1] \cap \mathbb{Q}$ and ν is an I-separated partition. Let ξ be the (e, y)-regularisation of ν. Then ξ is I-separated, $\xi_{\bar{I}} = \nu_{\bar{I}}$, and ξ_I is the $(c, y - \bar{c})$-regularisation of ν_I.*

Proof. By Lemma 4.5 (3) ν is (e, y)-regular if and only if ν_I is $(c, y - \bar{c})$-regular, and in this case the result is immediate. So assume ν is (e, y)-singular. We claim that we can find a partition $\kappa \rhd \nu$ such that κ and ν are (e, y)-equivalent, κ_I and ν_I are $(c, y - \bar{c})$-equivalent and $\kappa_{\bar{I}} = \nu_{\bar{I}}$.

The assumption that ν is (e, y)-singular means that for some $t \in \mathbb{N}$ there is an occupied position b in the abacus display for ν such that position $b - et$ is empty, and there are exactly yt empty positions in the range $[b - et, b]$. We take the largest such b, and we will assume that $t = 1$. If this is not the case, then we can just replace $e, c, \bar{c}, y$ with $et, ct, \bar{c}t, yt$ (keeping the set I unchanged). Assuming without loss of generality that n is divisible by et, all the hypotheses then still apply with these new parameters, and the partition κ obtained will have the desired properties.

So we assume $t = 1$, which means in particular that y is an integer. Now we construct the partition κ as explained in Section 2.7. Recall that for this we define E to be the union of the congruence classes containing the empty positions in the range $[b - e, b]$, and let $b_1 < \cdots < b_m$ be the occupied positions in $\{c \in E \mid c \geqslant b\}$. We also let $t_1 < t_2 < \cdots$ be the empty positions in $\{c \in \mathbb{N}_0 \setminus E \mid c \geqslant b\}$; then we move the beads at positions b_i and $t_i - e$ to positions $b_i - e$ and t_i for $i = 1, \ldots, d$, where d is minimal such that $t_d < b_{d+1}$. This then gives a partition $\kappa \rhd \nu$ which is (e, y)-equivalent to ν.

The fact that ν is I-separated means (as in the proof of Lemma 4.5) that $b \in I$, so that all positions in $[b - e, b] \cap \overline{I}$ are empty, and hence $\overline{I} \subseteq E$. Moreover, $b_1, \ldots, b_m \in I \cap E$. Hence, when we construct κ from ν we make no changes to any positions in $\overline{I}$, so $\kappa_{\overline{I}} = \nu_{\overline{I}}$, as required. Furthermore, the changes we make in the positions in I correspond exactly to applying the same algorithm (with e, y replaced by c, $y - \overline{c}$) to ν_I. Hence, κ_I is $(c, y - \overline{c})$-equivalent to ν_I.

So our claim is proved. The fact that $\kappa_I \rhd \nu_I$ while $\kappa_{\overline{I}} = \nu_{\overline{I}}$ means that κ is also I-separated: the last occupied position in the abacus in $\overline{I}$ is the same for κ as for ν, while the fact that $\kappa \rhd \nu$ implies that $\kappa_1' \leqslant \nu_1'$, so the first empty position in I in the abacus display for κ is the same as, or later than, the first empty position in I in the abacus display for ν.

So the hypotheses of the proposition apply with ν replaced by κ. By induction on the dominance order the proposition holds for κ, and so it holds for ν. ∎

Applying Proposition 4.7 repeatedly allows us to complete the first step in the proof of Theorem 4.6. We recall the partitions λ, π from above.

Corollary 4.8. *Let ϕ be the unique isomorphism from $\mathcal{R}_{A^{(e-1)+}}$ to $\mathcal{R}_{A^{(\overline{c}+1)-}}$, and ψ the unique isomorphism from $\mathcal{R}_{A^{(e-1)+}}$ to $\mathcal{R}_{A^{\overline{c}+}}$. Then $\phi(\lambda) = \psi(\lambda) = \pi$.*

Proof. Using the results in Section 3.4, the isomorphism ϕ is given by applying (e, y)-regularisation for all numbers $y \in [\overline{c} + 1, e - 1] \cap \mathbb{Q}$ in decreasing order. Applying Proposition 4.7 at every step, we find that $\phi(\lambda)_{\overline{I}} = \lambda_{\overline{I}}$, while $\phi(\lambda)_I$ is obtained from λ_I by applying (c, y)-regularisation for all $y \in [1, c - 1] \cap \mathbb{Q}$ in decreasing order. So (from Section 4.2) $\phi(\lambda)_I = \mathrm{m}_c(\lambda_I')$. Finally, λ and $\phi(\lambda)$ have the same e-content, so by Lemma 2.2 $\phi(\lambda)$ is a (u, I)-partition. So, $\phi(\lambda) = \pi$.

The isomorphism $\psi \circ \phi^{-1}$ from $\mathcal{R}_{A^{(\overline{c}+1)-}}$ to $\mathcal{R}_{A^{\overline{c}+}}$ is given by applying (e, y)-regularisation for all numbers $y \in (\overline{c}, \overline{c} + 1) \cap \mathbb{Q}$ in decreasing order. But π_I is c-regular, so π is (e, y)-regular for all $y \in (\overline{c}, \overline{c} + 1) \cap \mathbb{Q}$, by Lemma 4.5 (2). So the isomorphism $\psi \circ \phi^{-1}$ fixes π, and therefore $\psi(\lambda) = \pi$. ∎

Reversing the roles of λ and μ' and applying conjugation throughout, we can complete the third step in the factorisation as well.

Corollary 4.9. *Let χ be the unique isomorphism from $\mathcal{R}_{A^{\overline{c}-}}$ to $\mathcal{R}_{A^{1-}}$. Then $\chi(\rho) = \mu$.*

It remains to show that the isomorphism $\mathcal{R}_{A^{\overline{c}+}} \to \mathcal{R}_{A^{\overline{c}-}}$ sends π to ρ. This isomorphism is given by $(e, \overline{c})$-regularisation. For this we use the following proposition.

Proposition 4.10. *Suppose ν is an I-separated (u, I)-partition. Suppose $\nu_{\overline{I}}$ is not $\overline{c}$-restricted, and let $s \geqslant 1$ be minimal such that*

$$(\nu_{\overline{I}})_s - (\nu_{\overline{I}})_{s+1} \geqslant \overline{c}.$$

Write $v_{\bar{I}} = (1^s) |\bar{c}| \tau$, and define a (u, I)-partition ξ by

$$\xi_I = (s) \,\underline{\bar{c}}\, v_I, \quad \xi_{\bar{I}} = \tau.$$

Then v and ξ are $(e, \bar{c})$-equivalent.

Proof. By Lemma 4.5 (1), the fact that $v_{\bar{I}}$ is not e-restricted means that v is not $(e, \bar{c})$-regular. We apply a single step of the algorithm in Section 2.7 to construct the $(e, \bar{c})$-regularisation of v on the abacus; we will show that this step produces the partition ξ.

We can find a position b in the abacus display for v and $t \in \mathbb{N}$ such that position $b - te$ is empty and there are $\bar{c}t$ empty positions in the range $[b - et, b]$. We take the maximal such b. If $b \in I$, then the I-separated condition means that all the positions in $[b - et, b] \cap \bar{I}$ are empty, so that there are at least $\bar{c}t + 1$ empty positions in $[b - et, b]$, a contradiction. So $b \in \bar{I}$. This means that all the positions in $[b - et, b] \cap I$ are occupied, and therefore all the positions in $[b - et, b - 1] \cap \bar{I}$ are empty. In particular, we can assume $t = 1$, and the set E defined in Section 2.7 now coincides with $\bar{I}$.

Consider the $\bar{c}$-runner abacus display D obtained by discarding all the positions in I, and let a be the position in D corresponding to position b in the e-runner abacus display for v. From the way abacus displays are constructed,

$$a = (v_{\bar{I}})_r + n - u - r$$

for some r. All the positions in $[a - \bar{c}, a - 1]$ are empty in D, which means that

$$(v_{\bar{I}})_r - (v_{\bar{I}})_{r+1} \geqslant \bar{c}.$$

The choice of b means that r is minimal with this property, so $r = s$. So there are s occupied positions at or after position a in D, corresponding to the first s parts of $v_{\bar{I}}$. If we let $b_1 < \cdots < b_s$ be the corresponding positions in the abacus display for v, then $b_1, \ldots, b_s$ are precisely the occupied positions in E at or after position b. Now let $t_1 < t_2 < \cdots$ be the empty positions in I after position b. The I-separated condition means that $b_s < t_1$, so (according to the recipe in Section 2.7) we define a new partition κ by moving a bead from position b_i to position $b_i - e$ and moving a bead from position $t_i - e$ to position t_i, for $i = 1, \ldots, s$ in turn. The effect of moving the bead from b_i to $b_i - e$ for each i is to reduce each of the first s parts of v_I by $\bar{c}$; so $\kappa_{\bar{I}} = \tau$. By Lemma 2.1 the effect of moving a bead from $t_i - e$ to t_i for each i is to increase the first s columns of v_I by c. So $\kappa_I = (s) \,\underline{\bar{c}}\, v_I = \xi_I$, and therefore $\kappa = \xi$. From Section 2.7 κ is $(e, \bar{c})$-equivalent to v. ∎

Corollary 4.11. *The partitions π and ρ are $(e, \bar{c})$-equivalent, and hence ρ is the $(e, \bar{c})$-regularisation of π.*

Proof. Let $a = \alpha_1$, and for each $r = 0, \ldots, a$ define $\hat{\alpha}(r)$ to be the partition obtained by removing the last r non-zero columns from α, and define $\check{\alpha}(r)$ to be the partition obtained by removing the first $a - r$ columns from α. Now define (u, I)-partitions $\nu(0), \ldots, \nu(a)$ by

$$\nu(r)_I = \check{\alpha}(r)' \; \overline{c} \; \pi_I, \quad \nu(r)_{\overline{I}} = \hat{\alpha}(r) \, |\overline{c}| \, \gamma.$$

Then $\nu(0) = \pi$ and $\nu(a) = \rho$, and by Proposition 4.10 $\nu(r-1)$ is $(e, \overline{c})$-equivalent to $\nu(r)$ for $r = 1, \ldots, a$. So π and ρ are $(e, \overline{c})$-equivalent. By Lemma 4.5 (1) ρ is $(e, \overline{c})$-regular, and so ρ is the $(e, \overline{c})$-regularisation of π. ∎

We have now done enough to prove Theorem 4.6: by Corollary 4.8 the isomorphism from $\mathcal{R}_{A(e-1)+}$ to $\mathcal{R}_{A\overline{c}+}$ sends λ to π; by Corollary 4.11 the isomorphism from $\mathcal{R}_{A\overline{c}+}$ to $\mathcal{R}_{A\overline{c}-}$ sends π to ρ, and by Corollary 4.9 the isomorphism from $\mathcal{R}_{A\overline{c}-}$ to $\mathcal{R}_{A1-}$ sends ρ to μ. Hence, the isomorphism from $\mathcal{R}_{A(e-1)+}$ to $\mathcal{R}_{A1-}$ sends λ to μ, which is the same as saying that $\mathrm{m}_e(\lambda') = \mu$.

Now we consider the case where c or $\overline{c}$ equals 1. In this case the construction of the partitions in Theorem 4.6 still makes sense, but we need to clarify our definitions. The only 1-regular partition is the empty partition $\varnothing$ (and this is also the only 1-restricted partition), and we can define the Mullineux map m_1 by $\mathrm{m}_1(\varnothing) = \varnothing$. With this convention, the construction of the partitions λ and μ makes sense, and in fact Theorem 4.6 still holds. The proof just needs to be simplified slightly: if $c = 1$, then $\lambda = \pi$ and the second equality in Corollary 4.8 holds trivially; similarly if $\overline{c} = 1$, then $\rho = \mu$ and Corollary 4.9 is trivial. Corollary 4.11 needs no modification at all, and the steps still combine to give the main result.

We end this section by explaining briefly how Theorem 4.6 can be use to give a new proof of Paget's theorem [27, Theorem 2.1] describing the effect of the Mullineux map for a certain class of partitions; these partitions label modules in *RoCK blocks* of symmetric groups and Iwahori–Hecke algebras. (RoCK blocks – also called *Rouquier blocks* – are blocks with especially nice properties, which have played an important role in the resolution of several conjectures for symmetric groups. These blocks are discussed in detail in [18].) In fact the version we give here of Paget's theorem will be slightly more general, but it is not hard to show that it is implied by Paget's version.

We take a partition λ and a large integer n divisible by e, and construct the n-bead abacus display for λ with e runners. For each $i \in \mathbb{Z}/e\mathbb{Z}$ we regard runner i of the abacus as a 1-runner abacus display, and let $\lambda^{(i)}$ be the corresponding partition. The e-tuple $(\lambda^{(i)} \mid i \in \mathbb{Z}/e\mathbb{Z})$ is called the *e-quotient* of λ. We say that λ is *e-quotient separated* if there is a bijection $\sigma: \{1, \ldots, e\} \to \mathbb{Z}/e\mathbb{Z}$ such that for every $1 \leq k < l \leq e$, the last occupied position on runner $\sigma(k)$ in the abacus display for λ is earlier than the first empty position on runner $\sigma(l)$. The definition of e-quotient separated

partitions was introduced in the case $e = 2$ by James and Mathas [19], and the general version has been considered several times; for example in [10, Section 5.4].

If λ is e-quotient separated, then it is easy to check (and in fact is a special case of Lemma 4.5 (4)) that λ is e-restricted if and only if $\lambda^{(\sigma(e))} = \varnothing$. Now we can state a version of Paget's theorem as follows.

Theorem 4.12. *Suppose λ is an e-restricted e-quotient separated partition. Let μ be the partition with the same e-core as λ and with e-quotient $(\mu^{(i)} \mid i \in \mathbb{Z}/e\mathbb{Z})$ defined by*

$$\mu^{(\sigma(1))} = \varnothing, \qquad \mu^{(\sigma(k))} = (\lambda^{(\sigma(k-1))})' \quad \text{for } k = 2, \dots, e.$$

If μ is also e-quotient separated, then $m_e(\lambda') = \mu$.

Example. Take $e = 4$, and $\lambda = (11, 10, 9, 8, 7, 5^2, 4, 3, 2, 1^5)$.
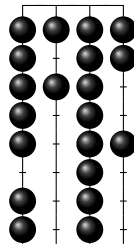
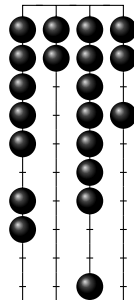From the abacus display we see that λ is 4-quotient separated, with

$$(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) = (1 + 4\mathbb{Z}, 3 + 4\mathbb{Z}, 0 + 4\mathbb{Z}, 2 + 4\mathbb{Z}),$$
$$(\lambda^{(\sigma(1))}, \lambda^{(\sigma(2))}, \lambda^{(\sigma(3))}, \lambda^{(\sigma(4))}) = ((1), (2), (1^2), \varnothing).$$

So we define μ by

$$(\mu^{(\sigma(1))}, \mu^{(\sigma(2))}, \mu^{(\sigma(3))}, \mu^{(\sigma(4))}) = (\varnothing, (1), (1^2), (2)),$$

giving $\mu = (19, 10, 9, 8, 7, 4, 3^3, 2, 1)$.

The abacus display shows that μ is also 4-quotient separated, and we can check that $m_4(\lambda') = \mu$.

Remark. We point out that for an e-quotient separated partition λ, the bijection σ need not be unique; however, if the partition μ is also e-quotient separated for some choice of σ, then in fact μ is independent of the choice of σ.

We can deduce Theorem 4.12 from Theorem 4.6 by induction on e. The case $e = 2$ is straightforward because m_2 is the identity map. Assuming $e \geqslant 3$, we apply Theorem 4.6 with $I = \sigma(e)$ (so that $c = 1$). We take $\beta = \varnothing$ and $\alpha = \lambda^{(\sigma(e-1))}$, and we let γ be the partition obtained from λ by moving all the beads on runner $\sigma(e-1)$ up to the highest positions on the runner and then deleting runner $\sigma(e)$. Then (with $u = u_{\sigma(e)}$) λ is the partition defined from α, β and γ in Theorem 4.6. Furthermore, γ is $(e-1)$-quotient separated, so we can compute $\mathrm{m}_{e-1}(\gamma')$ by induction: we find that $\mathrm{m}_{e-1}(\gamma') = \mu_{\bar{I}}$. Moreover, $\mu_I = \mu^{(e)} = \alpha'$. Hence, μ is the partition defined from α, β and γ in Theorem 4.6, and so the result follows.

References

[1] C. Berg, The ladder crystal. *Electron. J. Combin.* **17** (2010), no. 1, Research Paper 97 MR 2661400

[2] C. Bessenrodt and J. B. Olsson, On residue symbols and the Mullineux conjecture. *J. Algebraic Combin.* **7** (1998), no. 3, 227–251   Zbl 0906.05077   MR 1616083

[3] C. Bessenrodt, J. B. Olsson, and M. Xu, On properties of the Mullineux map with an application to Schur modules. *Math. Proc. Cambridge Philos. Soc.* **126** (1999), no. 3, 443–459   Zbl 0931.05088   MR 1684242

[4] J. Brundan, Modular branching rules and the Mullineux map for Hecke algebras of type A. *Proc. London Math. Soc. (3)* **77** (1998), no. 3, 551–581   Zbl 0904.20007   MR 1643413

[5] J. Brundan and J. Kujawa, A new proof of the Mullineux conjecture. *J. Algebraic Combin.* **18** (2003), no. 1, 13–39   Zbl 1043.20006   MR 2002217

[6] D. Bump and A. Schilling, *Crystal bases*. World Scientific, Hackensack, NJ, 2017 Zbl 1440.17001   MR 3642318

[7] V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy, Combinatorics of regular A_2-crystals. *J. Algebra* **310** (2007), no. 1, 218–234   Zbl 1158.17004   MR 2307791

[8] M. Fayers, Regularisation and the Mullineux map. *Electron. J. Combin.* **15** (2008), no. 1, Research Paper 142   Zbl 1183.20009   MR 2465766

[9] M. Fayers, Partition models for the crystal of the basic $U_q(\widehat{\mathfrak{sl}}_n)$-module. *J. Algebraic Combin.* **32** (2010), no. 3, 339–370   Zbl 1211.17020   MR 2721055

[10] M. Fayers, The irreducible representations of the alternating group which remain irreducible in characteristic p. *Trans. Amer. Math. Soc.* **368** (2016), no. 8, 5807–5855 Zbl 1344.20018   MR 3458400

[11] M. Fayers, Regularising a partition on the abacus. https://webspace.maths.qmul.ac.uk/m.fayers/papers/abreg.pdf

[12] B. Ford and A. S. Kleshchev, A proof of the Mullineux conjecture. *Math. Z.* **226** (1997), no. 2, 267–308   Zbl 0958.20018   MR 1477629

[13] J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*. Grad. Stud. Math. 42, Amer. Math. Soc., Providence, RI, 2002   Zbl 1134.17007   MR 1881971

[14] N. Jacon, Mullineux involution and crystal isomorphisms. 2021, arXiv:2107.02641

[15] G. James, On the decomposition matrices of the symmetric groups. II. *J. Algebra* **43** (1976), no. 1, 45–54   Zbl 0347.20006   MR 430050

[16] G. James, The irreducible representations of the symmetric groups. *Bull. London Math. Soc.* **8** (1976), no. 3, 229–232   Zbl 0358.20019   MR 417272

[17] G. James and A. Kerber, *The representation theory of the symmetric group*. Encyclopedia Math. Appl. 16, Addison-Wesley, Reading, MA, 1981   Zbl 0491.20010   MR 644144

[18] G. James, S. Lyle, and A. Mathas, Rouquier blocks. *Math. Z.* **252** (2006), no. 3, 511–531   Zbl 1096.20007   MR 2207757

[19] G. James and A. Mathas, Hecke algebras of type **A** with $q = -1$. *J. Algebra* **184** (1996), no. 1, 102–158   Zbl 0865.20011   MR 1402572

[20] M. Kashiwara, *Bases cristallines des groupes quantiques*. Cours Spéc. 9, Soc. Math. France, Paris, 2002   Zbl 1066.17007   MR 1997677

[21] A. S. Kleshchev, Branching rules for modular representations of symmetric groups. III. Some corollaries and a problem of Mullineux. *J. London Math. Soc. (2)* **54** (1996), no. 1, 25–38   Zbl 0854.20014   MR 1395065

[22] D. E. Littlewood, Modular representations of symmetric groups. *Proc. Roy. Soc. London Ser. A* **209** (1951), 333–353   Zbl 0044.25702   MR 49896

[23] S. Lyle, Some topics in the representation theory of the symmetric and general linear groups. PhD thesis, University of London, 2003

[24] D. Millan Berdasco, Generalised regularisation maps on partitions. 2021, arXiv:2105.05550

[25] K. Misra and T. Miwa, Crystal base for the basic representation of $U_q(\mathfrak{sl}(n))$. *Comm. Math. Phys.* **134** (1990), no. 1, 79–88   Zbl 0724.17010   MR 1079801

[26] G. Mullineux, Bijections of p-regular partitions and p-modular irreducibles of the symmetric groups. *J. London Math. Soc. (2)* **20** (1979), no. 1, 60–66   Zbl 0401.05011   MR 545202

[27] R. Paget, The Mullineux map for RoCK blocks. *Comm. Algebra* **34** (2006), no. 9, 3245–3253   Zbl 1142.20001   MR 2252669

[28] S. V. Sam and P. Tingley, Combinatorial realizations of crystals via torus actions on quiver varieties. *J. Algebraic Combin.* **39** (2014), no. 2, 271–300   Zbl 1292.05266   MR 3159253

[29] J. R. Stembridge, A local characterization of simply-laced crystals. *Trans. Amer. Math. Soc.* **355** (2003), no. 12, 4807–4823   Zbl 1047.17007   MR 1997585

[30] M. Xu, On p-series and the Mullineux conjecture. *Comm. Algebra* **27** (1999), no. 11, 5255–5265   Zbl 0947.20012   MR 1713034

Matthew Fayers
School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, UK; m.fayers@qmul.ac.uk