J. Comb. Algebra 7 (2023), 347–400
DOI 10.4171/JCA/77

New structure on the quantum alcove model with applications to representation theory and Schubert calculus

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Abstract. The quantum alcove model associated to a dominant weight plays an important role in many branches of mathematics, such as combinatorial representation theory, the theory of Macdonald polynomials, and Schubert calculus. For a dominant weight, it is proved by Lenart–Lubovsky that the quantum alcove model does not depend on the choice of a reduced alcove path, which is a shortest path of alcoves from the fundamental one to its translation by the given dominant weight. This is established through quantum Yang–Baxter moves, which biject the objects of the models associated to two such alcove paths, and can be viewed as a generalization of jeu de taquin slides to arbitrary root systems. The purpose of this paper is to give a generalization of quantum Yang–Baxter moves to the quantum alcove model corresponding to an arbitrary weight, which was used to express a general Chevalley formula for the equivariant K-group of semi-infinite flag manifolds. The generalized quantum Yang–Baxter moves give rise to a "sijection" (bijection between signed sets), and are shown to preserve certain important statistics, including weights and heights. As an application, we prove that the generating function of these statistics does not depend on the choice of a reduced alcove path. Also, we obtain an identity for the graded characters of Demazure submodules of level-zero extremal weight modules over a quantum affine algebra, which can be thought of as a representation-theoretic analogue of the mentioned Chevalley formula.

1. Introduction

The quantum alcove model was introduced in [12]. In [17] it was proved to be a uniform model for tensor products of single-column Kirillov–Reshetikhin crystals of quantum affine algebras, and its relevance to the theory of Macdonald polynomials was also discussed. Crystals are colored directed graphs encoding the structure of quantum algebra representations when the quantum parameter q goes to 0 (see [7]). The quantum alcove model generalizes the alcove model in [19], which has a similar representation-theoretic application [20].

2020 Mathematics Subject Classification. Primary 05E10; Secondary 14N15, 14M15.
Keywords. Quantum Yang–Baxter move, quantum Bruhat graph, quantum alcove model, level-zero Demazure module, semi-infinite flag manifold.

Let λ be a dominant weight for a fixed finite-type root system Δ, which is irreducible or of type $A_1 \times A_1$, and let Γ be a reduced λ-chain (of roots), or equivalently, a shortest path of alcoves from the fundamental one to its translation by λ (see [19]). One associates to Γ (viewed as a sequence) a certain family $\mathcal{A}(\Gamma)$ of subsets of its indices, called admissible subsets. Here we remark that there are two or more reduced λ-chains in general, and therefore the model is not uniquely determined (for the fixed dominant weight λ). However, for any two reduced λ-chains Γ_1 and Γ_2, there exists a bijection between $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$ which preserves the corresponding crystal operators, as well as some important statistics: $\mathrm{wt}(\cdot)$, $\mathrm{height}(\cdot)$, $\mathrm{down}(\cdot)$, and $\mathrm{end}(\cdot)$; the precise definitions of these statistics are given in Section 2.3. The construction of this bijection was given in [13] in terms of the so-called quantum Yang–Baxter moves, which are explicitly described by reduction to the rank 2 root systems. The main idea is the following: given Γ_1 and Γ_2 as above (and, in fact, for arbitrary λ), it is known from [19] that Γ_2 is obtained from Γ_1 by repeated application of a certain procedure called a "Yang–Baxter transformation", see Section 3.1; hence it suffices to construct a bijection (i.e., a quantum Yang–Baxter move) between $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$ when Γ_1 and Γ_2 are related by a Yang–Baxter transformation.

The quantum Yang–Baxter moves generalize the Yang–Baxter moves for the alcove model, which were defined and studied in [11]. It is pointed out in [13] that the quantum Yang–Baxter moves realize the combinatorial R-matrix, namely the (unique) affine crystal isomorphism permuting factors in a tensor product of single-column Kirillov–Reshetikhin crystals. It is also explained that these moves can be viewed as a generalization of jeu de taquin slides (for semi-standard Young tableaux, relevant to type A) to arbitrary root systems.

For an arbitrary (not necessarily dominant) weight λ, we also consider a (not necessarily reduced) λ-chain Γ (of not necessarily positive roots). For an arbitrary element w of the finite Weyl group W, let $\mathcal{A}(w, \Gamma)$ denote the collection of w-admissible subsets. This generalization is introduced in [14,15] to describe the Chevalley formula for the equivariant K-group of semi-infinite flag manifolds, and for the equivariant quantum K-theory of flag manifolds G/B (both of arbitrary type), cf. also [8, 21]. We also define statistics $\mathrm{wt}(A)$, $\mathrm{height}(A)$, $\mathrm{down}(A)$, $\mathrm{end}(A)$ for $A \in \mathcal{A}(w, \Gamma)$ in the same way as for $A \in \mathcal{A}(\Gamma) = \mathcal{A}(e, \Gamma)$ with λ dominant, where $e \in W$ is the identity; in addition, we define $n(A) \in \mathbb{Z}_{\geq 0}$.

Our main result is the existence of a very good map from $\mathcal{A}(w, \Gamma_1)$ to $\mathcal{A}(w, \Gamma_2)$ which preserves the statistics above, where Γ_2 is obtained from Γ_1 by a Yang–Baxter transformation.

Theorem 1 (Theorems 3.2 and 3.4). *Let $\mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}(w, \Gamma_2)$ be quantum alcove models associated to the same weight such that Γ_2 is obtained from Γ_1 by a Yang–*

Baxter transformation. Then, there exist subsets

$$\mathcal{A}_0(w,\Gamma_1) \subset \mathcal{A}(w,\Gamma_1) \quad \textit{and} \quad \mathcal{A}_0(w,\Gamma_2) \subset \mathcal{A}(w,\Gamma_2),$$

which satisfy the following.

(1) *There exists a "sign-preserving" bijection* $Y\colon \mathcal{A}_0(w,\Gamma_1) \to \mathcal{A}_0(w,\Gamma_2)$, *which also preserves the statistics* $\mathrm{wt}(\cdot)$, $\mathrm{height}(\cdot)$, $\mathrm{down}(\cdot)$, *and* $\mathrm{end}(\cdot)$.

(2) *If we set*

$$\mathcal{A}_0^C(w,\Gamma_1) := \mathcal{A}(w,\Gamma_1) \setminus \mathcal{A}_0(w,\Gamma_1),$$
$$\mathcal{A}_0^C(w,\Gamma_2) := \mathcal{A}(w,\Gamma_2) \setminus \mathcal{A}_0(w,\Gamma_2),$$

then there exists a "sign-reversing" involution I_1 *(resp.,* I_2*) on* $\mathcal{A}_0^C(w,\Gamma_1)$ *(resp.,* $\mathcal{A}_0^C(w,\Gamma_2)$*), which preserves the statistics* $\mathrm{wt}(\cdot)$, $\mathrm{height}(\cdot)$, $\mathrm{down}(\cdot)$, *and* $\mathrm{end}(\cdot)$.

Here we should mention that, in contrast to the case of dominant weights, there does not necessarily exist a bijection from the whole of $\mathcal{A}(w,\Gamma_1)$ onto the whole of $\mathcal{A}(w,\Gamma_2)$. Indeed, the cardinalities of the sets $\mathcal{A}(w,\Gamma_1)$ and $\mathcal{A}(w,\Gamma_2)$ are, in general, different; for details, see Example 3.1.

The map Y in Theorem 1 can be regarded as a generalization of the bijection described in terms of quantum Yang–Baxter moves when λ is a dominant weight. Although the map Y is not a bijection from the whole of $\mathcal{A}(w,\Gamma_1)$ onto the whole of $\mathcal{A}(w,\Gamma_2)$, there exist nice involutions I_1, I_2 outside the domain of Y and outside the image of Y. If we regard $\mathcal{A}(w,\Gamma_i)$, $i = 1, 2$, as a signed set equipped with the sign function $A \mapsto (-1)^{n(A)}$, then the collection (I_1, I_2, Y) of maps is a "sijection" (i.e., a signed bijection) $\mathcal{A}(w,\Gamma_1) \Rightarrow \mathcal{A}(w,\Gamma_2)$ which preserves $\mathrm{wt}(\cdot)$, $\mathrm{height}(\cdot)$, $\mathrm{down}(\cdot)$, and $\mathrm{end}(\cdot)$; the notion of a sijection was introduced in [4, Section 2].

Recall that an element of the affine Weyl group W_{af} can be written as $x = wt_\xi$, with w in the finite Weyl group W and ξ in the coroot lattice $Q^\vee$. For $\mathcal{A}(w,\Gamma)$, with Γ a (not necessarily reduced) λ-chain for an arbitrary weight λ, and $x = wt_\xi$ in W_{af}, we define a generating function $\mathbf{G}_\Gamma(x)$ of the statistics $\mathrm{wt}(\cdot)$, $\mathrm{end}(\cdot)$, $\mathrm{height}(\cdot)$, and $\mathrm{down}(\cdot)$ as follows:

$$\mathbf{G}_\Gamma(x) := \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)}\, \mathrm{end}(A) t_{\xi + \mathrm{down}(A)}.$$

We also think of $\mathbf{G}_\Gamma$ as a linear function on the group algebra of W_{af} with the coefficients introduced above. In the case that λ is a dominant weight and $x = e$, this function is a refinement of the specialization at $t = 0$ of the symmetric Macdonald polynomial $P_\lambda(q,t)$, since we know from [17, Theorem 7.9] that

$$P_\lambda(q,0) = \sum_{A \in \mathcal{A}(\Gamma)} q^{\mathrm{height}(A)} e^{\mathrm{wt}(A)}.$$

There is a similar relationship in the case of nonsymmetric Macdonald polynomials [18].

The existence of our generalized quantum Yang–Baxter moves implies the independence of the generating function $\mathbf{G}_\Gamma(x)$, and thus of the quantum alcove model for an arbitrary weight, from the associated chain of roots Γ. Here we need Γ to be "weakly reduced," which means that it does not contain both a simple root and its negative.

Theorem 2 (Theorem 5.6). *Let λ be an arbitrary weight, and $x \in W_{\mathrm{af}}$. Given weakly reduced λ-chains Γ_1 and Γ_2, we have $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$.*

We will now discuss several applications of Theorem 2 and, implicitly, of the generalized quantum Yang–Baxter moves underlying it.

We give a combinatorial realization of the symmetry of the general Chevalley formula in [14, 15] coming from commutativity in equivariant K-group. Indeed, given arbitrary weights μ, ν, we can successively apply the Chevalley formula for the multiplication by the classes of the line bundles corresponding to them, in either order. The fact that the result is the same is expressed by the following identity, where Γ_1 is a μ-chain, Γ_2 is a ν-chain, and $\circ$ indicates composition:

$$\mathbf{G}_{\Gamma_1} \circ \mathbf{G}_{\Gamma_2}(x) = \mathbf{G}_{\Gamma_2} \circ \mathbf{G}_{\Gamma_1}(x). \tag{1.1}$$

It will be shown that (1.1) is realized combinatorially via successive application of the sijection in Theorem 1, assuming that the concatenation of Γ_1 and Γ_2 is weakly reduced.

On another hand, we use Theorem 2 to obtain an identity for the graded characters of Demazure submodules of level-zero extremal weight modules over a quantum affine algebra, which can be viewed as a representation-theoretic analogue of the general Chevalley formula in [14, 15]. For a dominant weight μ and an element x of the affine Weyl group, let $V_x^-(\mu)$ denote the Demazure submodule of the level-zero extremal weight module $V(\mu)$ of extremal weight μ over a quantum affine algebra. For an arbitrary weight λ, let $\overline{\mathrm{Par}}(\lambda)$ denote the set of certain tuples $\boldsymbol{\chi}$ of partitions bounded by λ, to which we assign the quantities $|\boldsymbol{\chi}|$ and $\iota(\boldsymbol{\chi})$; for the definitions of $\overline{\mathrm{Par}}(\lambda)$, $|\boldsymbol{\chi}|$, and $\iota(\boldsymbol{\chi})$, see (5.3) and (5.4) in Section 5.3.

Theorem 3 (Theorem 5.16). *Let μ be a dominant weight, and $x = wt_\xi \in W_{\mathrm{af}}$. Take an arbitrary weight λ such that $\mu + \lambda$ is dominant, and let Γ be a reduced λ-chain. Then we have*

$$\begin{aligned} &\mathrm{gch}\, V_x^-(\mu+\lambda) \\ &= \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle - |\boldsymbol{\chi}|} e^{\mathrm{wt}(A)}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\chi})}}(\mu). \end{aligned}$$

The right-hand side of the above formula is proved to be identical to zero if $\mu + \lambda \notin P^+$, see Appendix B. In the case $\mu = 0$, this proof provides a combinatorial analogue of the vanishing of the 0-th cohomology of the semi-infinite flag manifold for line bundles associated to weights that are not dominant, see [8,21] and the details in Appendix B.

Here we should mention that, in [3], an identity for generalized Weyl modules similar to the one in Theorem 3 is obtained in the case that λ is a fundamental weight ϖ_i, $i \in I$; a generalized Weyl module can be viewed as the $\mathsf{q} = 1$ limit of a certain finite-dimensional quotient of a Demazure submodule of a level-zero extremal weight module over the quantum affine algebra $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ associated to the affine Lie algebra $\mathfrak{g}_{\mathrm{af}}$ (see [22] for an explicit relation between the graded characters of these modules).

The proof of the general Chevalley formula for semi-infinite flag manifolds in [14, 15] can be considerably simplified by using Theorem 2, in a way similar to the proof of Theorem 3; for a sketch of the combinatorial derivation of the general Chevalley formula, see [10, Section 4, Proof of Theorem 10]. Alternatively, the general Chevalley formula can be deduced from Theorem 3 by exactly the same argument as that in [8] and [21]. Conversely, the general Chevalley formula implies Theorem 3 for μ sufficiently dominant, but not for an arbitrary dominant μ; in particular, we cannot set $\mu = 0$. In this sense, Theorem 3 and the corresponding vanishing mentioned above are slightly stronger than the general Chevalley formula.

In conclusion, the generalized quantum Yang–Baxter moves add very useful structure to the quantum alcove model for an arbitrary weight.

This paper is organized as follows. In Section 2, we fix our basic notation, and recall the definitions and some properties of the quantum Bruhat graph and the quantum alcove model. In Section 3, we state our main results precisely; the proofs are given in Section 4. Finally, we prove the equality between the generating functions associated to two reduced λ-chains, and derive the identity above for the graded characters of (level-zero) Demazure submodules in Section 5.

2. Preliminaries

We fix our basic notation in this paper. Also, we recall the definitions and some properties of the quantum Bruhat graph and the quantum alcove model.

2.1. Basic notation

Throughout this paper, let $\mathfrak{g}$ be a complex simple Lie algebra or the complex Lie algebra of type $A_1 \times A_1$, with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We denote by $\langle \cdot, \cdot \rangle$ the canonical pairing of $\mathfrak{h}$ and $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$.

Let Δ denote the root system of $\mathfrak{g}$, with $\Delta^+ \subset \Delta$ the set of all positive roots. Let I be the set of indices of the Dynkin diagram of $\mathfrak{g}$, and let α_i, $i \in I$, be the simple roots of Δ. For $\alpha \in \Delta$, we define $\mathrm{sgn}(\alpha) \in \{1, -1\}$ and $|\alpha| \in \Delta^+$ by

$$\mathrm{sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \Delta^+, \\ -1 & \text{if } \alpha \in -\Delta^+, \end{cases}$$
$$|\alpha| := \mathrm{sgn}(\alpha)\alpha.$$

We set

$$Q := \sum_{i \in I} \mathbb{Z}\alpha_i \quad \text{and} \quad Q^\vee := \sum_{i \in I} \mathbb{Z}\alpha_i^\vee,$$

where $\alpha^\vee$ is the coroot of $\alpha \in \Delta$; also, we set

$$Q^{\vee,+} := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee.$$

Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$, with length function $\ell\colon W \to \mathbb{Z}_{\geq 0}$ and the longest element $w_\circ \in W$; here, for $\alpha \in \Delta^+$, $s_\alpha \in W$ denotes the reflection corresponding to α, and $s_i = s_{\alpha_i}$ is the simple reflection for $i \in I$.

For each $i \in I$, let ϖ_i denote the fundamental weight corresponding to α_i. Let

$$P := \sum_{i \in I} \mathbb{Z}\varpi_i$$

be the weight lattice of $\mathfrak{g}$, with

$$P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i$$

the set of dominant weights; also, we set $\mathfrak{h}^*_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R}$.

2.2. The quantum Bruhat graph

We recall the definition of the *quantum Bruhat graph*, introduced in [1]. We set

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

Definition 2.1 ([1, Definition 6.1]). The *quantum Bruhat graph* $\mathrm{QBG}(W)$ is the Δ^+-labeled directed graph whose vertices are the elements of W and whose edges are of the following form: $x \xrightarrow{\alpha} y$, with $x, y \in W$ and $\alpha \in \Delta^+$, such that $y = xs_\alpha$ and either of the following (B) or (Q) holds:

(B) $\ell(y) = \ell(x) + 1$;

(Q) $\ell(y) = \ell(x) - 2\langle \rho, \alpha^\vee \rangle + 1$.

If (B) (resp., (Q)) holds, then the edge $x \xrightarrow{\alpha} y$ is called a *Bruhat edge* (resp., *quantum edge*).

Let $\mathbf{p}\colon w_0 \xrightarrow{\beta_1} w_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_r} w_r$ be a directed path in $\mathrm{QBG}(W)$. We set

$$\begin{aligned} \ell(\mathbf{p}) &:= r, \\ \mathrm{end}(\mathbf{p}) &:= w_r, \\ \mathrm{wt}(\mathbf{p}) &:= \sum_{\substack{k \in \{1,\dots,r\} \\ w_{k-1} \xrightarrow{\beta_k} w_k \text{ is a quantum edge}}} \beta_k^\vee. \end{aligned}$$

Definition 2.2 ([2, (2.2)]). A total order $\lhd$ on Δ^+ is a *reflection order* if for all $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta^+$, either $\alpha \lhd \alpha + \beta \lhd \beta$ or $\beta \lhd \alpha + \beta \lhd \alpha$ holds.

Let $\lhd$ be a reflection order on Δ^+. A directed path $\mathbf{p}$ in $\mathrm{QBG}(W)$ of the form:

$$\mathbf{p}\colon w_0 \xrightarrow{\beta_1} w_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_r} w_r,$$

with $\beta_1 \lhd \cdots \lhd \beta_r$, is called a *label-increasing* directed path with respect to $\lhd$.

Theorem 2.3 ([1, Theorem 6.4]). *Let $\lhd$ be a reflection order on Δ^+. For all $v, w \in W$, there exists a unique label-increasing directed path from v to w in* $\mathrm{QBG}(W)$ *with respect to $\lhd$. Moreover, the unique label-increasing directed path from v to w has the minimum length.*

The property of $\mathrm{QBG}(W)$ in Theorem 2.3 is called *shellability*.

For all $v, w \in W$, there exists at least one shortest directed path $\mathbf{p}$ from v to w; we set

$$\ell(v \Rightarrow w) := \ell(\mathbf{p}), \quad \mathrm{wt}(v \Rightarrow w) := \mathrm{wt}(\mathbf{p}).$$

Note that by [23, Lemma 1 (2)] or [16, Proposition 8.1], $\mathrm{wt}(v \Rightarrow w)$ is well defined.

We consider a "generalization" of label-increasing directed paths in $\mathrm{QBG}(W)$. Let $\Pi = (\gamma_1, \dots, \gamma_r)$ be a sequence of roots, i.e., $\gamma_1, \dots, \gamma_r \in \Delta$; assume that $\gamma_1, \dots, \gamma_r$ are distinct. Then we say that a directed path $\mathbf{p}$ is Π-*compatible* if $\mathbf{p}$ is of the following form:

$$\mathbf{p} : w_0 \xrightarrow{|\gamma_{j_1}|} w_1 \xrightarrow{|\gamma_{j_2}|} \cdots \xrightarrow{|\gamma_{j_p}|} w_p,$$

with $1 \le j_1 < \cdots < j_p \le r$. For $w \in W$, we denote by $\mathscr{P}(w, \Pi)$ the set of all Π-compatible directed paths in $\mathrm{QBG}(W)$ which start at w.

Remark 2.4. If $\{\gamma_1, \dots, \gamma_r\} \subset \Delta^+$, and if there exists a reflection order $\lhd$ on Δ^+ such that $\gamma_1 \lhd \cdots \lhd \gamma_r$, then a Π-compatible directed path in $\mathrm{QBG}(W)$ is a label-increasing directed path with respect to $\lhd$.

Let $\Pi = (\gamma_1, \dots, \gamma_r)$ be a sequence of roots, with $\gamma_1, \dots, \gamma_r$ not necessarily distinct. For a directed path $\mathbf{p}$ of the form:

$$\mathbf{p}\colon w_0 \xrightarrow{|\gamma_{j_1}|} w_1 \xrightarrow{|\gamma_{j_2}|} \cdots \xrightarrow{|\gamma_{j_p}|} w_p,$$

with $1 \le j_1 < \cdots < j_p \le r$, we define $\mathrm{neg}(\mathbf{p})$ by

$$\mathrm{neg}(\mathbf{p}) := \#\{k \in \{1, \dots, p\} \mid \gamma_{j_k} \in -\Delta^+\}.$$

2.3. The quantum alcove model

We briefly review the quantum alcove model, introduced in [12].

First, we recall from [19] the definition of alcove paths. For $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we set $H_{\alpha,k} := \{\nu \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \nu, \alpha^\vee \rangle = k\}$; $H_{\alpha,k}$ is a hyperplane in $\mathfrak{h}^*_{\mathbb{R}}$. Also, for $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection with respect to $H_{\alpha,k}$. Note that $s_{\alpha,k}(\nu) = \nu - (\langle \nu, \alpha^\vee \rangle - k)\alpha$ for $\nu \in \mathfrak{h}^*_{\mathbb{R}}$. Each connected component of the space

$$\mathfrak{h}^*_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta^+, k \in \mathbb{Z}} H_{\alpha,k}$$

is called an *alcove*. If two alcoves A and B have a common wall, then we say that A and B are *adjacent*. For adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$, $\beta \in \Delta$, if the common wall of A and B is contained in $H_{\beta,k}$ for some $k \in \mathbb{Z}$, and β points in the direction from A to B.

Definition 2.5 ([19, Definition 5.2]). A sequence $(A_0, \dots, A_r)$ of alcoves is called an *alcove path* if A_{i-1} and A_i are adjacent for all $i = 1, \dots, r$. If the length r of an alcove path $(A_0, \dots, A_r)$ is minimal among all alcove paths from A_0 to A_r, we say that $(A_0, \dots, A_r)$ is *reduced*.

The *fundamental alcove* $A_\circ$ is defined by

$$A_\circ := \{\nu \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \nu, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Delta^+\}.$$

Also, for $\lambda \in P$, we define A_λ by

$$A_\lambda := A_\circ + \lambda = \{\nu + \lambda \mid \nu \in A_\circ\}.$$

Definition 2.6 ([19, Definition 5.4]). Let $\lambda \in P$. A sequence $(\beta_1, \dots, \beta_r)$ of roots $\beta_1, \dots, \beta_r \in \Delta$ is called a λ-*chain* if there exists an alcove path $(A_0, \dots, A_r)$, with $A_0 = A_\circ$ and $A_r = A_{-\lambda}$, such that

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_r} A_r = A_{-\lambda}.$$

If such an alcove path $(A_0, \dots, A_r)$ is reduced, then we also say that the corresponding λ-chain $(\beta_1, \dots, \beta_r)$ is *reduced.*

Now, following [14, Section 3.2], we review the quantum alcove model.

Definition 2.7 ([14, Definition 17]). Let $\lambda \in P$, and let $\Gamma = (\beta_1, \dots, \beta_r)$ be a λ-chain. Fix $w \in W$. A subset $A = \{j_1 < \cdots < j_p\} \subset \{1, \dots, r\}$ is said to be w-*admissible* if

$$\mathbf{p}(A) \colon w = w_0 \xrightarrow{|\beta_{j_1}|} w_1 \xrightarrow{|\beta_{j_2}|} \cdots \xrightarrow{|\beta_{j_p}|} w_p$$

is a directed path in $\mathrm{QBG}(W)$. Let $\mathcal{A}(w, \Gamma)$ denote the set of all w-admissible subsets of $\{1, \dots, r\}$.

Remark 2.8. The original definition of admissible subsets in [12] is only for $w = e \in W$. The notion of w-admissible subsets for an arbitrary $w \in W$ is introduced in [14].

Let $\lambda \in P$, and let $\Gamma = (\beta_1, \dots, \beta_r)$ be a λ-chain. By the definition of λ-chains, there exists an alcove path $(A_\circ = A_0, \dots, A_r = A_{-\lambda})$ such that

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_r} A_r = A_{-\lambda}.$$

For $k = 1, \dots, r$, we take $l_k \in \mathbb{Z}$ such that $H_{\beta_k, -l_k}$ contains the common wall of A_{k-1} and A_k, and set $\widetilde{l}_k := \langle \lambda, \beta_k^\vee \rangle - l_k$.

Fix $w \in W$. For $A = \{j_1 < \cdots < j_p\} \in \mathcal{A}(w, \Gamma)$, we set

$$\mathrm{end}(A) := w s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_p}|}, \quad \mathrm{wt}(A) := -w s_{\beta_{j_1}, -l_{j_1}} \cdots s_{\beta_{j_p}, -l_{j_p}}(-\lambda);$$

we call $\mathrm{wt}(A)$ the *weight* of A. Also, we define a subset $A^- \subset A$ by

$$A^- := \left\{ j_k \in A \;\middle|\; w s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_{k-1}}|} \xrightarrow{|\beta_{j_k}|} w s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_k}|} \text{ is a quantum edge} \right\},$$

and set

$$\mathrm{down}(A) := \sum_{j \in A^-} |\beta_j|^\vee, \quad \mathrm{height}(A) := \sum_{j \in A^-} \mathrm{sgn}(\beta_j) \widetilde{l}_j;$$

note that $\mathrm{end}(A) = \mathrm{end}(\mathbf{p}(A))$ and $\mathrm{down}(A) = \mathrm{wt}(\mathbf{p}(A))$. In addition, we define $n(A) \in \mathbb{Z}_{\geq 0}$ by $n(A) := \#\{j \in A \mid \beta_j \in -\Delta^+\}$; note that $n(A) = \mathrm{neg}(\mathbf{p}(A))$.

Remark 2.9. In [18, (31)], an additional statistic, called *coheight*, is introduced. Let us assume that $\lambda \in P^+$ and $w = e$. For $A \in \mathcal{A}(e, \Gamma)$, $\mathrm{coheight}(A) \in \mathbb{Z}_{\geq 0}$ is defined by

$$\mathrm{coheight}(A) := \sum_{j \in A^-} l_j.$$

The coheight is used in [18] to describe the specialization at $t = 0$ of nonsymmetric Macdonald polynomials in terms of the quantum alcove model (see [18, Theorems 29 and 31]).

3. Generalization of quantum Yang–Baxter moves

Quantum Yang–Baxter moves for a dominant weight are introduced in [13].

3.1. Yang–Baxter transformation of λ-chains

Before defining Yang–Baxter transformations, we mention some basic facts about rank 2 root subsystems of Δ. For each $\alpha, \beta \in \Delta$ with $\langle \alpha, \beta^\vee \rangle \leq 0$ and $\alpha \neq -\beta$, the subgroup $\overline{W}$ of W generated by s_α and s_β is a dihedral group with simple reflections $\{s_\alpha, s_\beta\}$. Also, let $\Delta_{\alpha,\beta}$ be the root subsystem of Δ generated by α and β. Then $\Delta_{\alpha,\beta}$ is a root system of rank 2. More precisely, we see that $\Delta_{\alpha,\beta}$ is isomorphic to the root system of type $A_1 \times A_1$, A_2, C_2, or G_2. Let m be the order of $s_\alpha s_\beta \in \overline{W}$. Then $\underbrace{s_\alpha s_\beta s_\alpha s_\beta \cdots}_{m \text{ factors}}$ is the longest element of $\overline{W}$. Hence if m is even (resp., odd), then

$$(\gamma_1, \gamma_2, \ldots, \gamma_q) = (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, \underbrace{s_\alpha s_\beta s_\alpha s_\beta \cdots s_\alpha}_{m-1 \text{ factors}}(\beta)),$$

resp.,

$$(\gamma_1, \gamma_2, \ldots, \gamma_q) = (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, \underbrace{s_\alpha s_\beta s_\alpha s_\beta \cdots s_\beta}_{m-1 \text{ factors}}(\alpha)),$$

forms a sequence of all the (distinct) positive roots of $\Delta_{\alpha,\beta}$ such that $\gamma_q = \beta$.

Let $\lambda \in P$, and let $\Gamma = (\beta_1, \ldots, \beta_r)$ be a λ-chain (of roots). The following procedure (YB) is called the *Yang–Baxter transformation*:

(YB) Take a segment $(\beta_{t+1}, \ldots, \beta_{t+q})$ of Γ of the form

$$(\beta_{t+1}, \ldots, \beta_{t+q}) = (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, s_\beta(\alpha), \beta)$$

for some $\alpha, \beta \in \Delta$ with $\langle \alpha, \beta^\vee \rangle \leq 0$, or equivalently $\langle \beta, \alpha^\vee \rangle \leq 0$, and $\alpha \neq -\beta$, and set

$$\Gamma' := (\beta_1, \beta_2, \ldots, \beta_t, \beta_{t+q}, \beta_{t+q-1}, \ldots, \beta_{t+1}, \beta_{t+q+1}, \beta_{t+q+2}, \ldots, \beta_r),$$

i.e., reverse the segment $(\beta_{t+1}, \ldots, \beta_{t+q})$ of Γ.

Also, we define a procedure (D), called *deletion*, as follows:

(D) Take a segment $(\beta_{t+1}, \beta_{t+2})$ of Γ of the form $(\beta_{t+1}, \beta_{t+2}) = (\beta, -\beta)$ for some $\beta \in \Delta$, and set

$$\Gamma' = (\beta_1, \dots, \beta_t, \beta_{t+3}, \dots, \beta_q),$$

i.e., delete the segment $(\beta_{t+1}, \beta_{t+2})$ of Γ.

Note the procedures (YB) and (D) produce λ-chains; i.e., resulting sequences Γ' above are also λ-chains. In fact, it is known that every λ-chain can be transformed into an arbitrary reduced λ-chain by repeated application of the procedures (YB) and (D) (see [14, Remark 40], or [19, Lemma 9.3]).

3.2. Quantum Yang–Baxter moves

Let $\lambda \in P^+$ be a dominant weight, and let Γ_1, Γ_2 be λ-chains such that Γ_2 is obtained from Γ_1 by the Yang–Baxter transformation (YB). *Quantum Yang–Baxter moves*, introduced in [13, Section 3.1], give a bijection $\mathcal{A}(e, \Gamma_1) \to \mathcal{A}(e, \Gamma_2)$ which preserves weights and heights.

Our main result is the existence of a generalization of quantum Yang–Baxter moves for an arbitrary (not necessarily dominant) weight $\lambda \in P$ and an arbitrary $w \in W$.

Let $\lambda \in P$ be an arbitrary weight, and Γ_1, Γ_2 λ-chains such that Γ_2 is obtained from Γ_1 by the Yang–Baxter transformation (YB). If we write $\Gamma_1 = (\beta_1, \dots, \beta_r)$ and $\Gamma_2 = (\beta'_1, \dots, \beta'_r)$, then there exists $1 \le t \le r$ such that

- $(\beta_{t+1}, \dots, \beta_{t+q}) = (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \dots, s_\beta(\alpha), \beta)$ for some $q \ge 1$ and some $\alpha, \beta \in \Delta$ with $\langle \alpha, \beta^\vee \rangle \le 0$ and $\alpha \ne -\beta$,
- $\Gamma_2 = (\beta'_1, \dots, \beta'_r)$
  $= (\beta_1, \beta_2, \dots, \beta_t, \beta_{t+q}, \beta_{t+q-1}, \dots, \beta_{t+1}, \beta_{t+q+1}, \beta_{t+q+2}, \dots, \beta_r).$

We take the alcove path $(A_\circ = A_0, \dots, A_r = A_{-\lambda})$ corresponding to Γ_1, and take integers $l_k \in \mathbb{Z}$ for $k = 1, \dots, r$ such that for each $k = 1, \dots, r$, the hyperplane $H_{\beta_k, -l_k}$ contains the common wall of A_{k-1} and A_k. Also, we take the alcove path

$$(A_\circ = A'_0, \dots, A'_r = A_{-\lambda})$$

corresponding to Γ_2, and we take integers $l'_k \in \mathbb{Z}$ for $k = 1, \dots, r$ such that for each $k = 1, \dots, r$, the hyperplane $H_{\beta'_k, -l'_k}$ contains the common wall of A'_{k-1} and A'_k. Then it follows that $A'_k = A_k$ and $l'_k = l_k$ for $k = 1, \dots, t, t+q+1, \dots, r$, and that $l'_{t+p} = l_{t+q+1-p}$ for $p = 1, \dots, q$.

Now, we divide Γ_1 into three parts $\Gamma_1^{(1)}$, $\Gamma_1^{(2)}$, and $\Gamma_1^{(3)}$ as follows:

$$\begin{gathered}\Gamma_1^{(1)} := (\beta_1, \dots, \beta_t),\\ \Gamma_1^{(2)} := (\beta_{t+1}, \dots, \beta_{t+q}), \quad \Gamma_1^{(3)} := (\beta_{t+q+1}, \dots, \beta_r).\end{gathered} \tag{3.1}$$

Also, we divide Γ_2 into three parts $\Gamma_2^{(1)}$, $\Gamma_2^{(2)}$, and $\Gamma_2^{(3)}$ as follows:

$$\begin{gathered}\Gamma_2^{(1)} := (\beta'_1, \dots, \beta'_t),\\ \Gamma_2^{(2)} := (\beta'_{t+1}, \dots, \beta'_{t+q}), \quad \Gamma_2^{(3)} := (\beta'_{t+q+1}, \dots, \beta'_r).\end{gathered} \tag{3.2}$$

Note that $\Gamma_1^{(1)} = \Gamma_2^{(1)}$ and $\Gamma_1^{(3)} = \Gamma_2^{(3)}$; in addition, $\beta_{t+1}, \dots, \beta_{t+q}$ are distinct. Next, let $w \in W$. For a w-admissible subset $A \in \mathcal{A}(w, \Gamma_1)$, we define $A^{(1)}$, $A^{(2)}$, and $A^{(3)}$ by

$$\begin{gathered}A^{(1)} := A \cap \{1, \dots, t\},\\ A^{(2)} := A \cap \{t+1, \dots, t+q\}, \quad A^{(3)} := A \cap \{t+q+1, \dots, r\}.\end{gathered} \tag{3.3}$$

Also, for $B \in \mathcal{A}(w, \Gamma_2)$, we define $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ by

$$\begin{gathered}B^{(1)} := B \cap \{1, \dots, t\},\\ B^{(2)} := B \cap \{t+1, \dots, t+q\}, \quad B^{(3)} := B \cap \{t+q+1, \dots, r\}.\end{gathered} \tag{3.4}$$

Unlike the case where λ is dominant, there does not exist a bijection between $\mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}(w, \Gamma_2)$ in general.

Example 3.1. Assume that $\mathfrak{g}$ is of type A_2. We set $\Gamma_1 := (\alpha_2, -\alpha_1, -\theta, -\alpha_1)$ and $\Gamma_2 := (-\theta, -\alpha_1, \alpha_2, -\alpha_1)$, where $\theta = \alpha_1 + \alpha_2$. Then we see that Γ_1 and Γ_2 are $(-2\varpi_1 + \varpi_2)$-chains such that Γ_2 is obtained from Γ_1 by a Yang–Baxter transformation (YB). Let $w = s_2$. By direct calculation, we have

$$\begin{aligned}\mathcal{A}(w, \Gamma_1) = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,4\},\\ &\{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\},\\ \mathcal{A}(w, \Gamma_2) = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\},\\ &\{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.\end{aligned}$$

Hence we have $\#\mathcal{A}(w, \Gamma_1) = 12$, while $\#\mathcal{A}(w, \Gamma_2) = 16$. This shows that there does not exist a bijection $\mathcal{A}(w, \Gamma_1) \to \mathcal{A}(w, \Gamma_2)$.

Thus, towards a generalization of quantum Yang–Baxter moves, we need to give up using bijections and take a new approach. The following theorem is our main result; the proof is given in the next section.

Theorem 3.2. *There exist subsets $\mathcal{A}_0(w,\Gamma_1) \subset \mathcal{A}(w,\Gamma_1)$ and $\mathcal{A}_0(w,\Gamma_2) \subset \mathcal{A}(w,\Gamma_2)$ which satisfy the following:*

(1) *There exists a bijection $Y \colon \mathcal{A}_0(w,\Gamma_1) \to \mathcal{A}_0(w,\Gamma_2)$ such that for all $A \in \mathcal{A}_0(w,\Gamma_1)$, it holds that*
- $(Y(A))^{(1)} = A^{(1)}$, $\mathrm{end}((Y(A))^{(2)}) = \mathrm{end}(A^{(2)})$, $(Y(A))^{(3)} = A^{(3)}$,
- $\mathrm{down}(Y(A)) = \mathrm{down}(A)$, *and*
- $(-1)^{n(Y(A))} = (-1)^{n(A)}$.

(2) *For $k = 1, 2$, we set $\mathcal{A}_0^C(w,\Gamma_k) := \mathcal{A}(w,\Gamma_k) \setminus \mathcal{A}_0(w,\Gamma_k)$. Then, there exists an involution I_k on $\mathcal{A}_0^C(w,\Gamma_k)$ such that for all $A \in \mathcal{A}_0^C(w,\Gamma_k)$, it holds that*
- $(I_k(A))^{(1)} = A^{(1)}$, $\mathrm{end}((I_k(A))^{(2)}) = \mathrm{end}(A^{(2)})$, $(I_k(A))^{(3)} = A^{(3)}$,
- $\mathrm{down}(I_k(A)) = \mathrm{down}(A)$, *and*
- $(-1)^{n(I_k(A))} = -(-1)^{n(A)}$.

Remark 3.3. In order to explain our maps Y, I_1, and I_2 in Theorem 3.2, we have a useful notion, called a *sijection*, introduced in [4]; for the definition of sijections, see [4, Section 2]. For sets S, T equipped with sign functions $S \to \{\pm 1\}$, $T \to \{\pm 1\}$, a sijection from S to T is the collection $(\iota_S, \iota_T, \varphi)$ of a sign-reversing involution ι_S on a subset S_0 of S, a sign-reversing involution ι_T on a subset T_0 of T, and a sign-preserving bijection φ from $S \setminus S_0$ to $T \setminus T_0$ (see [4, p. 9]). In this terminology, our collection (I_1, I_2, Y) in Theorem 3.2 is a sijection from $\mathcal{A}(w,\Gamma_1)$ to $\mathcal{A}(w,\Gamma_2)$. This sijection can be thought of as a generalization of quantum Yang–Baxter moves.

As in the case that λ is dominant, we can prove that the maps Y, I_1, and I_2 preserve weights and heights.

Theorem 3.4. *The following hold:*

(1) *For all $A \in \mathcal{A}_0(w,\Gamma_1)$, it holds that* $\mathrm{wt}(Y(A)) = \mathrm{wt}(A)$ *and* $\mathrm{height}(Y(A)) = \mathrm{height}(A)$.

(2) *Let $k = 1, 2$. For all $A \in \mathcal{A}_0^C(w,\Gamma_k)$, it holds that* $\mathrm{wt}(I_k(A)) = \mathrm{wt}(A)$ *and* $\mathrm{height}(I_k(A)) = \mathrm{height}(A)$.

4. Proofs of the main results

We prove Theorems 3.2 and 3.4 in this section. The proofs are based on a property analogous to shellability of $\mathrm{QBG}(W)$ for the rank 2 root systems (Proposition 4.3). In the proof of this shellability-like property, we take a rank 2 root subsystem of Δ, which is denoted by $\Delta_{\alpha,\beta}$ in Section 4.2, and calculate explicitly certain products of the so-called quantum Bruhat operators for $\Delta_{\alpha,\beta}$ (see Proposition 4.4). In this paper, we omit the explicit calculations in the case that $\Delta_{\alpha,\beta}$ is of type G_2. However, the calculations in type G_2 is needed only in the case that Δ is of type G_2; this is because if the simple Lie algebra $\mathfrak{g}$ is not of type G_2, then the root system Δ of $\mathfrak{g}$ does not contain a root subsystem of type G_2. Hence Propositions 4.3 and 4.4 are sufficient in order to establish Theorems 3.2 and 3.4 for the root system Δ of a complex simple Lie algebra $\mathfrak{g}$ which is not of type G_2.

Although the shellability-like property in type G_2 seems to be slightly different from that in types $A_1 \times A_1$, A_2, and C_2, we can prove it by explicit calculations similar to those in type C_2 given in Section 4.4. Furthermore, based on this property, we can also establish Theorems 3.2 and 3.4 in type G_2. The interested reader can see the precise statement of the shellability-like property and the explicit calculations in type G_2 in our preprint [9] on arXiv.

In the rest of this paper, we assume that Δ is not of type G_2.

4.1. Quantum Bruhat operators

Let K be a field which contains the ring $\mathbb{C}[\![Q^{\vee,+}]\!] := \mathbb{C}[\![Q_i \mid i \in I]\!]$ of formal power series, where Q_i, $i \in I$, are variables, and set

$$Q^{\xi} := \prod_{i \in I} Q_i^{m_i}$$

for $\xi = \sum_{i \in I} m_i \alpha_i^{\vee} \in Q^{\vee,+}$. For $\gamma \in \Delta^+$, following [1], we define the (K-linear) *quantum Bruhat operator* Q_γ on the group algebra $K[W]$ of W by

$$\mathsf{Q}_\gamma v := \begin{cases} v s_\gamma & \text{if } v \xrightarrow{\gamma} v s_\gamma \text{ is a Bruhat edge,} \\ Q^{\gamma^\vee} v s_\gamma & \text{if } v \xrightarrow{\gamma} v s_\gamma \text{ is a quantum edge,} \\ 0 & \text{otherwise} \end{cases}$$

for $v \in W$. We set $\mathsf{Q}_{-\gamma} := -\mathsf{Q}_\gamma$ for $\gamma \in \Delta^+$, and then $\mathsf{R}_\gamma := 1 + \mathsf{Q}_\gamma$ for $\gamma \in \Delta$. The operators $\{\mathsf{R}_\gamma \mid \gamma \in \Delta\}$ satisfy the *Yang–Baxter equation*: for $\alpha, \beta \in \Delta$ (not necessarily positive roots) such that $\langle \alpha, \beta^\vee \rangle \leq 0$ and $\alpha \neq -\beta$, it holds that

$$\mathsf{R}_\alpha \mathsf{R}_{s_\alpha(\beta)} \mathsf{R}_{s_\alpha s_\beta(\alpha)} \cdots \mathsf{R}_{s_\beta(\alpha)} \mathsf{R}_\beta = \mathsf{R}_\beta \mathsf{R}_{s_\beta(\alpha)} \cdots \mathsf{R}_{s_\alpha s_\beta(\alpha)} \mathsf{R}_{s_\alpha(\beta)} \mathsf{R}_\alpha; \tag{4.1}$$

the proof of this equation is the same as that of [14, Proposition 38].

Next, we give some properties of quantum Bruhat operators.

Lemma 4.1. *Let $\Pi = (\beta_1, \dots, \beta_r)$ be a sequence of roots such that $\beta_1, \dots, \beta_r$ are distinct.*

(1) *For $v \in W$, we have*

$$\mathsf{R}_{\beta_r}\mathsf{R}_{\beta_{r-1}}\cdots\mathsf{R}_{\beta_1}v = \sum_{\mathbf{p}\in\mathcal{P}(v,\Pi)} (-1)^{\mathrm{neg}(\mathbf{p})} Q^{\mathrm{wt}(\mathbf{p})}\,\mathrm{end}(\mathbf{p}).$$

(2) *For $v \in W$, we have*

$$\mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|}\cdots\mathsf{R}_{|\beta_1|}v = \sum_{\mathbf{p}\in\mathcal{P}(v,\Pi)} Q^{\mathrm{wt}(\mathbf{p})}\,\mathrm{end}(\mathbf{p}).$$

Proof. For $J \subset \{1, \dots, r\}$, we set $\mathrm{neg}(J) := \{j \in J \mid \beta_j \in -\Delta^+\}$. We see that

$$\begin{aligned}
\mathsf{R}_{\beta_r}\mathsf{R}_{\beta_{r-1}}\cdots\mathsf{R}_{\beta_1} &= (1+\mathsf{Q}_{\beta_r})(1+\mathsf{Q}_{\beta_{r-1}})\cdots(1+\mathsf{Q}_{\beta_1}) \\
&= \sum_{\{j_1<\cdots<j_s\}\subset\{1,\dots,r\}} \mathsf{Q}_{\beta_{j_s}}\mathsf{Q}_{\beta_{j_{s-1}}}\cdots\mathsf{Q}_{\beta_{j_1}} \\
&= \sum_{\{j_1<\cdots<j_s\}\subset\{1,\dots,r\}} (\mathrm{sgn}(\beta_{j_s})\mathsf{Q}_{|\beta_{j_s}|})(\mathrm{sgn}(\beta_{j_{s-1}})\mathsf{Q}_{|\beta_{j_{s-1}}|})\cdots(\mathrm{sgn}(\beta_{j_1})\mathsf{Q}_{|\beta_{j_1}|}) \\
&= \sum_{J=\{j_1<\cdots<j_s\}\subset\{1,\dots,r\}} (-1)^{\mathrm{neg}(J)}\mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|}. \qquad (4.2)
\end{aligned}$$

Similarly, we see that

$$\mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|}\cdots\mathsf{R}_{|\beta_1|} = \sum_{\{j_1<\cdots<j_s\}\subset\{1,\dots,r\}} \mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|}. \qquad (4.3)$$

For $J = \{j_1, \dots, j_s\} \subset \{1, \dots, r\}$, if we have the edge

$$vs_{|\beta_{j_1}|}\cdots s_{|\beta_{j_{a-1}}|} \xrightarrow{|\beta_{j_a}|} vs_{|\beta_{j_1}|}\cdots s_{|\beta_{j_a}|}$$

in $\mathrm{QBG}(W)$ for all $1 \le a \le s$, then we set $\delta(J) := 1$, and define a directed path $\mathbf{p}(J)$ in $\mathrm{QBG}(W)$ by

$$\mathbf{p}(J)\colon v \xrightarrow{|\beta_{j_1}|} vs_{|\beta_{j_1}|} \xrightarrow{|\beta_{j_2}|} \cdots \xrightarrow{|\beta_{j_s}|} vs_{|\beta_{j_1}|}\cdots s_{|\beta_{j_s}|};$$

otherwise, we set $\delta(J) := 0$. By the definition of quantum Bruhat operators, we have

$$\mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|}v = \begin{cases} Q^{\mathrm{wt}(\mathbf{p}(J))}\,\mathrm{end}(\mathbf{p}(J)) & \text{if } \delta(J)=1, \\ 0 & \text{if } \delta(J)=0. \end{cases}$$

If $\delta(J) = 1$, then we have $\mathrm{neg}(J) = \mathrm{neg}(\mathbf{p}(J))$. Therefore, by (4.2), we deduce that

$$\begin{aligned}
\mathsf{R}_{\beta_r}\mathsf{R}_{\beta_{r-1}}\cdots\mathsf{R}_{\beta_1} v &= \sum_{J=\{j_1<\cdots<j_s\}\subset\{1,\dots,r\}} (-1)^{\mathrm{neg}(J)} \mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|} v \\
&= \sum_{\substack{J\subset\{1,\dots,r\}\\ \delta(J)=1}} (-1)^{\mathrm{neg}(\mathbf{p}(J))} Q^{\mathrm{wt}(\mathbf{p}(J))}\, \mathrm{end}(\mathbf{p}(J)) \\
&= \sum_{\mathbf{p}\in\mathcal{P}(v,\Pi)} (-1)^{\mathrm{neg}(\mathbf{p})} Q^{\mathrm{wt}(\mathbf{p})}\, \mathrm{end}(\mathbf{p}),
\end{aligned}$$

as desired. This proves part (1) of the lemma.

Also, we see from (4.3) that

$$\begin{aligned}
\mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|}\cdots\mathsf{R}_{|\beta_1|} v &= \sum_{\{j_1<\cdots<j_s\}\subset\{1,\dots,r\}} \mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|} v \\
&= \sum_{\substack{J\subset\{1,\dots,r\}\\ \delta(J)=1}} Q^{\mathrm{wt}(\mathbf{p}(J))}\, \mathrm{end}(\mathbf{p}(J)) \\
&= \sum_{\mathbf{p}\in\mathcal{P}(v,\Pi)} Q^{\mathrm{wt}(\mathbf{p})}\, \mathrm{end}(\mathbf{p}).
\end{aligned}$$

This proves part (2) of the lemma. ∎

Remark 4.2. If we set $\mathcal{P}(v,\Pi;w,\xi) := \{\mathbf{p}\in\mathcal{P}(v,\Pi) \mid \mathrm{end}(\mathbf{p}) = w, \mathrm{wt}(\mathbf{p}) = \xi\}$ for $v, w \in W$ and $\xi \in Q^{\vee,+}$, then by Lemma 4.1 (1), we deduce that

$$\mathsf{R}_{\beta_r}\cdots\mathsf{R}_{\beta_1} v = \sum_{w\in W}\sum_{\xi\in Q^{\vee,+}} \left(\sum_{\mathbf{p}\in\mathcal{P}(v,\Pi;w,\xi)} (-1)^{\mathrm{neg}(\mathbf{p})} \right) Q^{\xi} w. \tag{4.4}$$

Also, if we set $c^{v}_{\xi,w} := \#\mathcal{P}(v,\Pi;w,\xi)$ for $v, w \in W$ and $\xi \in Q^{\vee,+}$, then we deduce from Lemma 4.1 (2) that

$$\mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|}\cdots\mathsf{R}_{|\beta_1|} v = \sum_{w\in W}\sum_{\xi\in Q^{\vee,+}} c^{v}_{\xi,w} Q^{\xi} w. \tag{4.5}$$

4.2. Key propositions to a generalization of quantum Yang–Baxter moves

We prove a certain property of $\mathrm{QBG}(W)$, which plays an important role in the proof of Theorem 3.2. Let $\alpha, \beta \in \Delta$ be such that $\langle \alpha, \beta^{\vee}\rangle \le 0$ and $\alpha \ne -\beta$. We define sequences of roots Π, Π' by

$$\begin{aligned}
\Pi &= (\gamma_1, \dots, \gamma_q) := (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \dots, s_\beta(\alpha), \beta), \\
\Pi' &= (\gamma'_1, \dots, \gamma'_q) := (\beta, s_\beta(\alpha), \dots, s_\alpha s_\beta(\alpha), s_\alpha(\beta), \alpha) = (\gamma_q, \dots, \gamma_2, \gamma_1);
\end{aligned}$$

again note that $\gamma_1, \dots, \gamma_q$ are distinct. Also, we recall from the beginning of Section 4 that the root subsystem $\Delta_{\alpha,\beta}$ of Δ generated by α and β is isomorphic to the root system of type $A_1 \times A_1$, A_2, or C_2. Note that $\Delta_{\alpha,\beta}$ is not of type G_2; this is because we are assuming that Δ is not of type G_2. We can prove the following property, which can be thought of as a generalization of shellability of $\mathrm{QBG}(W)$ (Theorem 2.3) for the rank 2 root systems.

Proposition 4.3. *Let $v \in W$, and let $\mathbf{p}$ be a Π-compatible directed path in $\mathrm{QBG}(W)$ which starts at v, i.e., $\mathbf{p} \in \mathcal{P}(v, \Pi)$. Then only one of the following occurs.*

(1) *There exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p})$. This $\mathbf{p}'$ satisfies $(-1)^{\mathrm{neg}(\mathbf{p}')} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p})$.*

(2) *There exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p})$. This $\mathbf{p}'$ satisfies $(-1)^{\mathrm{neg}(\mathbf{p}')} = (-1)^{\mathrm{neg}(\mathbf{p})}$. Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p})$.*

The proof of this proposition can be reduced to the case that Δ is a root system of rank 2; in Appendix A, we explain how to construct the explicit correspondence $\mathbf{p} \mapsto \mathbf{p}'$ through an example. Now we assume that Δ is a root system of type $A_1 \times A_1$, A_2, or C_2. Since it is sufficient to consider the case $\Delta = \Delta_{\alpha,\beta}$, we assume additionally that $\langle \alpha, \beta^\vee \rangle \neq 0$ if Δ is of type C_2; if $\langle \alpha, \beta^\vee \rangle = 0$, then the corresponding root system $\Delta = \Delta_{\alpha,\beta}$ can be thought of as being of type $A_1 \times A_1$. Then we see that there exists some $k = 1, \dots, q$ such that $|\gamma_k|$ and $|\gamma_{k+1}|$ are the simple roots of Δ (for convenience of notation, we set $\gamma_{q+1} := \gamma_1$). If we set

$$(\beta_1, \dots, \beta_q) := (|\gamma_k|, |\gamma_{k-1}|, \dots, |\gamma_1|, |\gamma_q|, \dots, |\gamma_{k+1}|), \tag{4.6}$$

then we have

$$(\beta_1, \dots, \beta_q) = (\beta_1, s_{\beta_1}(\beta_q), s_{\beta_1} s_{\beta_q}(\beta_1), \dots, s_{\beta_q}(\beta_1), \beta_q).$$

Also, if we set

$$\Pi^{\pm} := (\mp\beta_k, \mp\beta_{k-1}, \dots, \mp\beta_1, \pm\beta_q, \pm\beta_{q-1}, \dots, \pm\beta_{k+1}),$$

then $\Pi = \Pi^+$ or $\Pi = \Pi^-$. Note that the total order $\prec$ on $\{\beta_1, \dots, \beta_q\} = \Delta^+$ defined by

$$\beta_1 \prec \beta_2 \prec \cdots \prec \beta_q \tag{4.7}$$

is a reflection order; the total order $\prec'$ defined by

$$\beta_q \prec' \beta_{q-1} \prec' \cdots \prec' \beta_1 \tag{4.8}$$

is also a reflection order. We consider the following operators for $k = 0, 1, \dots, q$:

$$\begin{aligned} \mathsf{T}_k^{\pm} &:= \mathsf{R}_{\pm\beta_{k+1}} \cdots \mathsf{R}_{\pm\beta_q} \mathsf{R}_{\mp\beta_1} \cdots \mathsf{R}_{\mp\beta_k}, \\ \mathsf{S}_k &:= \mathsf{R}_{\beta_{k+1}} \cdots \mathsf{R}_{\beta_q} \mathsf{R}_{\beta_1} \cdots \mathsf{R}_{\beta_k}, \\ \mathsf{S}'_k &:= \mathsf{R}_{\beta_k} \cdots \mathsf{R}_{\beta_1} \mathsf{R}_{\beta_q} \cdots \mathsf{R}_{\beta_{k+1}}; \end{aligned}$$

note that $\mathsf{S}_q = \mathsf{S}_0$ and $\mathsf{S}'_q = \mathsf{S}'_0$ by the definitions.

In the following proposition, the matrices of operators on $K[W]$ are the representation matrices with respect to the basis W of $K[W]$. Note that for a K-linear operator $\mathsf{T}\colon K[W] \to K[W]$, the matrix of T is defined by $(c_{v,w})_{v,w\in W}$ if $\mathsf{T}w = \sum_{v\in W} c_{v,w} v$, $c_{v,w} \in K$.

Proposition 4.4. *The following hold:*

(1) *All the entries of the matrix of S_k, $k = 0, 1, \dots, q$, are of the form $\sum_{j=1}^{r} m_j Q^{\xi_j}$, where all $\xi_j \in Q^{\vee,+}$ are distinct, and $m_j \in \{1, 2\}$.*

(2) *Let $v, w \in W$. Assume that the (v, w)-entry of the matrix of S_k is of the form $\sum_{j=1}^{r} m_j Q^{\xi_j}$ as in* (1)*. Also, assume that the (v, w)-entry of the matrix of $\mathsf{T}_k^{\pm}$ is of the form $\sum_{\xi\in Q^{\vee,+}} n_\xi^{\pm} Q^{\xi}$. For $j = 1, \dots, r$, if $m_j = 2$, then $n_{\xi_j}^{\pm} = 0$, and if $m_j = 1$, then $n_{\xi_j}^{\pm} \in \{1, -1\}$. Moreover, for $\xi \in Q^{\vee,+} \setminus \{\xi_1, \dots, \xi_r\}$, we have $n_\xi^{\pm} = 0$.*

(3) *Let $v, w \in W$. Assume that the (v, w)-entry of the matrix of S_k is of the form $\sum_{j=1}^{r} m_j Q^{\xi_j}$ as in* (1)*. Also, assume that the (v, w)-entry of the matrix of S'_k is of the form $\sum_{\xi\in Q^{\vee,+}} n_\xi Q^{\xi}$. For $j = 1, \dots, r$, if $m_j = 2$, then $n_{\xi_j} = 0$, and if $m_j = 1$, then $n_{\xi_j} = 1$.*

The proof of Proposition 4.4 is based on direct calculations, which we give later.

Proof of Proposition 4.3. First, we show the proposition for the root system Δ of type $A_1 \times A_1$, A_2, or C_2. As in (4.6), we take the sequence

$$(\beta_1, \dots, \beta_q) := (|\gamma_k|, |\gamma_{k-1}|, \dots, |\gamma_1|, |\gamma_q|, \dots, |\gamma_{k+1}|)$$

of roots. Recall from (4.5) that

$$\mathsf{S}_k v = \sum_{w\in W} \sum_{\xi\in Q^{\vee,+}} c_{w,\xi}^{v} Q^{\xi} w,$$

where $c_{w,\xi}^{v} = \#\mathcal{P}(v, \Pi; w, \xi)$. By Proposition 4.4 (1), we have $c_{w,\xi}^{v} \in \{0, 1, 2\}$. Also, again from (4.5), we see that

$$\mathsf{S}'_k v = \sum_{w\in W} \sum_{\xi\in Q^{\vee,+}} (c_{w,\xi}^{v})' Q^{\xi} w,$$

where $(c_{w,\xi}^{v})' = \#\mathcal{P}(v, \Pi'; w, \xi)$.

We write

$$\mathsf{T}_k^{\pm} v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} d_{w,\xi}^{v,\pm} Q^{\xi} w,$$

where $d_{w,\xi}^{v,\pm} \in \mathbb{Z}$. By (4.4), if $\Pi = \Pi^{+}$, then

$$d_{w,\xi}^{v,+} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\mathrm{neg}(\mathbf{q})}, \quad d_{w,\xi}^{v,-} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\ell(\mathbf{q})-\mathrm{neg}(\mathbf{q})}; \tag{4.9}$$

if $\Pi = \Pi^{-}$, then

$$d_{w,\xi}^{v,+} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\ell(\mathbf{q})-\mathrm{neg}(\mathbf{q})}, \quad d_{w,\xi}^{v,-} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\mathrm{neg}(\mathbf{q})}. \tag{4.10}$$

We set $w := \mathrm{end}(\mathbf{p})$ and $\xi := \mathrm{wt}(\mathbf{p})$. Since $\mathbf{p} \in \mathcal{P}(v, \Pi; w, \xi)$, we have $c_{w,\xi}^{v} \neq 0$. First, assume that $c_{w,\xi}^{v} = 2$. Then there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi; w, \xi) \setminus \{\mathbf{p}\}$, i.e., there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p}) = \xi$. By Proposition 4.4 (2), we have $d_{w,\xi}^{v,\pm} = 0$. Hence, by (4.9) and (4.10), we obtain

$$\sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\mathrm{neg}(\mathbf{q})} = (-1)^{\mathrm{neg}(\mathbf{p})} + (-1)^{\mathrm{neg}(\mathbf{p}')} = 0.$$

This shows that $(-1)^{\mathrm{neg}(\mathbf{p}')} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. Here, by Proposition 4.4 (3), we deduce that $(c_{w,\xi}^{v})' = 0$. Hence there does not exist a $\mathbf{q} \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}) = \xi$. This shows the proposition in the case $c_{w,\xi}^{v} = 2$.

Next, assume that $c_{w,\xi}^{v} = 1$. In this case, there does not exist a $\mathbf{q} \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}) = \xi$, since $\mathcal{P}(v, \Pi; w, \xi) = \{\mathbf{p}\}$. We set

$$(\mathsf{T}_k^{\pm})' := \mathsf{R}_{\mp\beta_k} \cdots \mathsf{R}_{\mp\beta_1} \mathsf{R}_{\pm\beta_q} \cdots \mathsf{R}_{\pm\beta_{k+1}}.$$

Then, by the Yang–Baxter equation (4.1), we have $(\mathsf{T}_k^{\pm})' = \mathsf{T}_k^{\pm}$. Hence, if we write

$$(\mathsf{T}_k^{\pm})' v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} (d_{w,\xi}^{v,\pm})' Q^{\xi} w,$$

with $(d_{w,\xi}^{v,\pm})' \in \mathbb{Z}$, then we see that $(d_{w,\xi}^{v,\pm})' = d_{w,\xi}^{v,\pm}$. Here, by Proposition 4.4 (2), we deduce that

$$(d_{w,\xi}^{v,\pm})' = d_{w,\xi}^{v,\pm} \in \{1, -1\}.$$

Again, by Proposition 4.4 (2) (by replacing $(\beta_1, \ldots, \beta_q)$ with $(\beta_q, \ldots, \beta_1)$), we deduce that $(c_{w,\xi}^{v})' = 1$. Hence there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi'; w, \xi)$, i.e., there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p}) = \xi$.

If $\Pi = \Pi^+$, then

$$(-1)^{\mathrm{neg}(\mathbf{p})} = d_{w,\xi}^{v,+} = (d_{w,\xi}^{v,+})' = (-1)^{\mathrm{neg}(\mathbf{p}')};$$

if $\Pi = \Pi^-$, then

$$(-1)^{\mathrm{neg}(\mathbf{p})} = d_{w,\xi}^{v,-} = (d_{w,\xi}^{v,-})' = (-1)^{\mathrm{neg}(\mathbf{p}')}.$$

This shows that $(-1)^{\mathrm{neg}(\mathbf{p}')} = (-1)^{\mathrm{neg}(\mathbf{p})}$, as desired. This completes the proof of the proposition for the root system Δ of type $A_1 \times A_1$, A_2, or C_2.

Now, assume that the root system Δ is of an arbitrary type (except G_2), not necessarily of rank 2. Let $\overline{W}$ be the Weyl group of $\Delta_{\alpha,\beta}$. Note that $\overline{W}$ is a (dihedral) subgroup of W; the quantum Bruhat graph (denoted by $\mathrm{QBG}(\overline{W})$) of $\overline{W}$ is no longer a subgraph of $\mathrm{QBG}(W)$. By [13, Proposition 5.1 and Remarks 5.2 (2)], for each $u \in W$, there exist uniquely $\lfloor u \rfloor \in u\overline{W}$ and $\overline{u} \in \overline{W}$ such that

- $u = \lfloor u \rfloor \overline{u}$, and
- for a positive root γ of $\Delta_{\alpha,\beta}$, we have $\ell(\overline{u}) < \ell(\overline{u}s_\gamma)$ if and only if $\ell(\lfloor u \rfloor \overline{u}) < \ell(\lfloor u \rfloor \overline{u} s_\gamma)$.

We set $w := \mathrm{end}(\mathbf{p})$ and $\xi := \mathrm{wt}(\mathbf{p})$. Suppose, for a contradiction, that there exist two or more directed paths $\mathbf{q} \in \mathcal{P}(v, \Pi; w, \xi) \setminus \{\mathbf{p}\}$. Then we see that

$$\#\mathcal{P}(v, \Pi, w, \xi) \geq 3.$$

By [13, Theorem 5.3], there exists an injection

$$\mathcal{P}(v, \Pi; w, \xi) \hookrightarrow \mathcal{P}(\overline{v}, \Pi; \overline{w}, \gamma),$$

where $\mathcal{P}(\overline{v}, \Pi; \overline{w}, \gamma)$ is the set of all Π-compatible directed paths in $\mathrm{QBG}(\overline{W})$ which starts at $\overline{v}$, ends at $\overline{w}$, and has weight ξ, where Π is considered to be a sequence of roots in the root system $\Delta_{\alpha,\beta}$. Hence, we have $\#\mathcal{P}(\overline{v}, \Pi; \overline{w}, \xi) \geq 3$. This contradicts the proposition for the rank 2 root systems, shown above. Hence we conclude that there exists at most one directed path $\mathbf{q} \in \mathcal{P}(v, \Pi; w, \xi)$ and with $\mathbf{q} \neq \mathbf{p}$. If such a $\mathbf{q}$ exists, then, by the proposition for the rank 2 root systems and [13, Theorem 5.3], we have $(-1)^{\mathrm{neg}(\mathbf{q})} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. Also, a similar argument shows that there exists at most one directed path $\mathbf{r} \in \mathcal{P}(v, \Pi'; w, \xi)$. If such an $\mathbf{r}$ exists, then we have $(-1)^{\mathrm{neg}(\mathbf{r})} = (-1)^{\mathrm{neg}(\mathbf{p})}$.

We show that at least one of the directed paths $\mathbf{q}$ and $\mathbf{r}$ exists. We write

$$\mathsf{R}_{\gamma_q} \cdots \mathsf{R}_{\gamma_1} v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} d_{w,\xi}^{v} Q^{\xi} w,$$

$$\mathsf{R}_{\gamma_1} \cdots \mathsf{R}_{\gamma_q} v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} (d_{w,\xi}^{v})' Q^{\xi} w.$$

If there does not exist a directed path $\mathbf{q} \in \mathcal{P}(v, \Pi; w, \xi) \setminus \{\mathbf{p}\}$, then by (4.4), we have $d^v_{w,\xi} = \pm 1$. By the Yang–Baxter equation (4.1), we deduce that $(d^v_{w,\xi})' = d^v_{w,\xi} = \pm 1$. By (4.4), we see that $\mathcal{P}(v, \Pi'; w, \xi) \neq \emptyset$. Therefore, we conclude that there exists a directed path $\mathbf{r} \in \mathcal{P}(v, \Pi'; w, \xi)$ in this case, as desired.

Finally, suppose, for a contradiction, that both $\mathbf{q}$ and $\mathbf{r}$ exist at the same time. Then, by [13, Theorem 5.3], we have

$$\#\mathcal{P}(\overline{v}, \Pi; \overline{w}, \xi) \geq 2 \quad \text{and} \quad \#\mathcal{P}(\overline{v}, \Pi'; \overline{w}, \xi) \geq 1.$$

This contradicts the proposition for the rank 2 root systems, shown above.

This completes the proof of Proposition 4.3. ■

Thus it remains to prove Proposition 4.4. We assume temporarily that Δ is of type $A_1 \times A_1$, A_2, or C_2 with $\langle \alpha, \beta^\vee \rangle \neq 0$. If Δ is of type A_2 (resp., $A_1 \times A_1$, C_2), then we have $q = 3$ (resp., $q = 2, 4$). By shellability of $\mathrm{QBG}(W)$, there exists a unique label-increasing directed path (with respect to $\prec$ or $\prec'$, defined by (4.7) and (4.8)) from v to w in $\mathrm{QBG}(W)$ for all $v, w \in W$. Hence, we have

$$\mathsf{T}^+_0 v = \mathsf{T}^-_q v = \mathsf{S}_0 v = \mathsf{S}'_0 v = \mathsf{S}_q v = \mathsf{S}'_q v = \sum_{w \in W} Q^{\mathrm{wt}(v \Rightarrow w)} w,$$
$$\mathsf{T}^-_0 v = \mathsf{T}^+_q v = \sum_{w \in W} (-1)^{\ell(v \Rightarrow w)} Q^{\mathrm{wt}(v \Rightarrow w)} w$$

for all $v \in W$. Therefore, the proposition is obvious in the case $k = 0, q$. Hence it suffices to prove the proposition in the case $k = 1, q-1$ for all types, and in the case $k = 2$ for type C_2.

4.3. Proof of Proposition 4.4: $k = 1, q - 1$

We prove Proposition 4.4 in the case $k = 1, q-1$; recall that β_1 and β_q are the simple roots of Δ. By (4.4), for all $v \in W$, we have

$$\begin{aligned}
\mathsf{T}^+_1 v &= \mathsf{R}_{\beta_2} \cdots \mathsf{R}_{\beta_q}(\mathsf{R}_{-\beta_1} v) \\
&= \mathsf{R}_{\beta_2} \cdots \mathsf{R}_{\beta_q}((1 - \mathsf{Q}_{\beta_1}) v) \\
&= \mathsf{R}_{\beta_2} \cdots \mathsf{R}_{\beta_q}(v - Q^{\mathrm{wt}(v \to v s_{\beta_1})} v s_{\beta_1}) \\
&= \sum_{\mathbf{q} \in \mathcal{P}(v, (\beta_q, \ldots, \beta_2))} Q^{\mathrm{wt}(\mathbf{q})} \operatorname{end}(\mathbf{q}) - \sum_{\mathbf{q} \in \mathcal{P}(v s_{\beta_1}, (\beta_q, \ldots, \beta_2))} Q^{\mathrm{wt}(v \to v s_{\beta_1}) + \mathrm{wt}(\mathbf{q})} \operatorname{end}(\mathbf{q}).
\end{aligned}$$

Recall that the total order $\prec'$ on $\Delta^+ = \{\beta_1, \ldots, \beta_q\}$, defined by (4.8), is a reflection order. Hence, by shellability of $\mathrm{QBG}(W)$, for all $w \in W$, there exists at most one directed path $\mathbf{q} \in \mathcal{P}(v, (\beta_q, \ldots, \beta_2))$ such that $\operatorname{end}(\mathbf{q}) = w$. For such a $\mathbf{q}$, we have

$$\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(v \Rightarrow w)$$

since $\mathbf{q}$ is a shortest directed path from v to w. The same argument shows that for all $w \in W$, there exists at most one directed path $\mathbf{q} \in \mathcal{P}(vs_{\beta_1}, (\beta_q, \dots, \beta_2))$ such that

$$\mathrm{end}(\mathbf{q}) = w \quad \text{and} \quad \mathrm{wt}(\mathbf{q}) = \mathrm{wt}(vs_{\beta_1} \Rightarrow w).$$

Hence, if we set

$$\delta_{v,w} := \begin{cases} 1 & \text{if there exists } \mathbf{q} \in \mathcal{P}(v, (\beta_q, \dots, \beta_2)) \text{ such that } \mathrm{end}(\mathbf{q}) = w, \\ 0 & \text{otherwise} \end{cases}$$

for $v, w \in W$, then we have

$$\begin{aligned} \mathsf{T}_1^+ v &= \sum_{w \in W} \delta_{v,w}\, Q^{\mathrm{wt}(v \Rightarrow w)} w - \sum_{w \in W} \delta_{vs_{\beta_1},w}\, Q^{\mathrm{wt}(v \to vs_{\beta_1}) + \mathrm{wt}(vs_{\beta_1} \Rightarrow w)} w \\ &= \sum_{w \in W} \bigl(\delta_{v,w}\, Q^{\mathrm{wt}(v \Rightarrow w)} - \delta_{vs_{\beta_1},w}\, Q^{\mathrm{wt}(v \to vs_{\beta_1}) + \mathrm{wt}(vs_{\beta_1} \Rightarrow w)}\bigr) w. \end{aligned} \tag{4.11}$$

Also, by the same argument, we see that

$$\mathsf{T}_1^- v = \sum_{w \in W} (-1)^{\ell(v \Rightarrow w)} \bigl(\delta_{v,w}\, Q^{\mathrm{wt}(v \Rightarrow w)} - \delta_{vs_{\beta_1},w}\, Q^{\mathrm{wt}(v \to vs_{\beta_1}) + \mathrm{wt}(vs_{\beta_1} \Rightarrow w)}\bigr) w; \tag{4.12}$$

note that for a directed path $\mathbf{q}$ from vs_{β_1} to w, it follows that

$$(-1)^{\ell(\mathbf{q})} = (-1)^{\ell(vs_{\beta_1} \Rightarrow w)},$$

and hence

$$(-1)^{\ell(vs_{\beta_1} \Rightarrow w)} = (-1)^{\ell(vs_{\beta_1} \Rightarrow v) + \ell(v \Rightarrow w)} = (-1)^{1 + \ell(v \Rightarrow w)} = -(-1)^{\ell(v \Rightarrow w)}.$$

Let us consider S_1. By the same argument as for T_1^+, we deduce that

$$\begin{aligned} \mathsf{S}_1 v = {} & \sum_{\mathbf{q} \in \mathcal{P}(v, (\beta_q, \dots, \beta_2))} Q^{\mathrm{wt}(\mathbf{q})}\, \mathrm{end}(\mathbf{q}) \\ & + \sum_{\mathbf{q} \in \mathcal{P}(vs_{\beta_1}, (\beta_q, \dots, \beta_2))} Q^{\mathrm{wt}(v \to vs_{\beta_1}) + \mathrm{wt}(\mathbf{q})}\, \mathrm{end}(\mathbf{q}) \\ = {} & \sum_{w \in W} \bigl(\delta_{v,w}\, Q^{\mathrm{wt}(v \Rightarrow w)} + \delta_{vs_{\beta_1},w}\, Q^{\mathrm{wt}(v \to vs_{\beta_1}) + \mathrm{wt}(vs_{\beta_1} \Rightarrow w)}\bigr) w. \end{aligned} \tag{4.13}$$

Hence equations (4.11), (4.12), and (4.13) imply Proposition 4.4 (1), (2) in the case $k = 1$, as desired.

Next, we consider the case $k=q-1$; recall that β_q is a simple root of Δ. By (4.4), we have

$$\begin{aligned}
\mathsf{T}_{q-1}^{+} v &= \mathsf{R}_{\beta_q}(\mathsf{R}_{-\beta_1}\cdots\mathsf{R}_{-\beta_{q-1}} v)\\
&= \mathsf{R}_{\beta_q}\Bigg(\sum_{\mathbf{q}\in\mathcal{P}(v,(-\beta_{q-1},\dots,-\beta_1))} (-1)^{\ell(\mathbf{q})} Q^{\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q})\Bigg)\\
&= \sum_{\mathbf{q}\in\mathcal{P}(v,(-\beta_{q-1},\dots,-\beta_1))} (-1)^{\ell(\mathbf{q})}\\
&\qquad\times\big(Q^{\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q}) + Q^{\mathrm{wt}(\mathbf{q})+\mathrm{wt}(\mathrm{end}(\mathbf{q})\to\mathrm{end}(\mathbf{q})s_{\beta_q})}\,\mathrm{end}(\mathbf{q})s_{\beta_q}\big).
\end{aligned}$$

Hence, if we set

$$\delta'_{v,w} := \begin{cases} 1 & \text{if there exists } \mathbf{q}\in\mathcal{P}(v,(-\beta_{q-1},\dots,-\beta_1)) \text{ such that } \mathrm{end}(\mathbf{q})=w,\\ 0 & \text{otherwise}\end{cases}$$

for $v,w\in W$, then we have

$$\mathsf{T}_{q-1}^{+} v = \sum_{w\in W} (-1)^{\ell(v\Rightarrow w)}\big(\delta'_{v,w}\,Q^{\mathrm{wt}(v\Rightarrow w)} - \delta'_{v,ws_{\beta_q}}\,Q^{\mathrm{wt}(v\Rightarrow ws_{\beta_q})+\mathrm{wt}(ws_{\beta_q}\to w)}\big)w. \tag{4.14}$$

Similarly, we have

$$\mathsf{T}_{q-1}^{-} v = \sum_{w\in W} \big(\delta'_{v,w}\,Q^{\mathrm{wt}(v\Rightarrow w)} - \delta'_{v,ws_{\beta_q}}\,Q^{\mathrm{wt}(v\Rightarrow ws_{\beta_q})+\mathrm{wt}(ws_{\beta_q}\to w)}\big)w. \tag{4.15}$$

Also, we see that

$$\mathsf{S}_{q-1} v = \sum_{w\in W} \big(\delta'_{v,w}\,Q^{\mathrm{wt}(v\Rightarrow w)} + \delta'_{v,ws_{\beta_q}}\,Q^{\mathrm{wt}(v\Rightarrow ws_{\beta_q})+\mathrm{wt}(ws_{\beta_q}\to w)}\big)w. \tag{4.16}$$

Hence equations (4.14), (4.15), and (4.16) imply Proposition 4.4 (1), (2) in the case $k=q-1$.

It remains to prove Proposition 4.4 (3) in the case $k=1,q-1$. It suffices to prove it in the case $k=1$; indeed, if we replace $(\beta_1,\dots,\beta_q)$ with $(\beta_q,\dots,\beta_1)$ and consider the case $k=1$, then we obtain the proposition in the case $k=q-1$. Recall equation (4.13). By the same argument, we see that

$$\mathsf{S}'_1 v = \sum_{w\in W} \big(\varepsilon_{v,w}\,Q^{\mathrm{wt}(v\Rightarrow w)} + \varepsilon_{v,ws_{\beta_1}}\,Q^{\mathrm{wt}(v\Rightarrow ws_{\beta_1})+\mathrm{wt}(ws_{\beta_1}\to w)}\big)w,$$

where

$$\varepsilon_{v,w} := \begin{cases} 1 & \text{if there exists } \mathbf{q}\in\mathcal{P}(v,(\beta_2,\dots,\beta_q)) \text{ such that } \mathrm{end}(\mathbf{q})=w,\\ 0 & \text{otherwise}\end{cases}$$

for $v, w \in W$. Assume that $c^{v}_{w,\xi} = 2$ for some $v, w \in W$ and $\xi \in Q^{\vee,+}$. It suffices to show that $(c^{v}_{w,\xi})' = 0$. In this case, we deduce from (4.13) that

$$\delta_{v,w} = \delta_{vs_{\beta_1},w} = 1, \tag{4.17}$$

$$\mathrm{wt}(v \Rightarrow w) = \mathrm{wt}(v \to vs_{\beta_1}) + \mathrm{wt}(vs_{\beta_1} \Rightarrow w) = \xi. \tag{4.18}$$

By (4.18), the concatenation of the edge $v \to vs_{\beta_1}$ with any shortest directed path from vs_{β_1} to w in $\mathrm{QBG}(W)$ is a shortest directed path from v to w (cf. [1, Lemma 6.7], [23, Lemma 1 (2)], and [16, Proposition 8.1]). Now, take the (unique) label-increasing directed path $\mathbf{r}_0$ from vs_{β_1} to w in $\mathrm{QBG}(W)$ with respect to $\prec$ defined by (4.7), and let $\mathbf{r}$ be the concatenation of the edge $v \to vs_{\beta_1}$ with the path $\mathbf{r}_0$. Note that $\mathbf{r}_0$ is shortest, and hence $\mathbf{r}$ is also shortest. We claim that $\mathbf{r}_0 \in \mathcal{P}(v, (\beta_2, \dots, \beta_q))$; otherwise, the concatenation

$$\mathbf{r}\colon v \xrightarrow{\beta_1} \underbrace{vs_{\beta_1} \xrightarrow{\beta_1} \cdots \to w}_{\mathbf{r}_0}$$

cannot be shortest. Hence $\mathbf{r}$ is the label-increasing directed path from v to w in $\mathrm{QBG}(W)$ such that $\mathbf{r} \notin \mathcal{P}(v, (\beta_2, \dots, \beta_q))$. By the uniqueness of a label-increasing directed path, we conclude that $\varepsilon_{v,w} = 0$.

Since $\delta_{v,w} = 1$ by (4.17), there exists $\mathbf{r}_1 \in \mathcal{P}(v, (\beta_q, \dots, \beta_2))$ such that

$$\mathrm{end}(\mathbf{r}_1) = w.$$

Then the concatenation of the path $\mathbf{r}_1$ with the edge $w \to ws_{\beta_1}$ is label-increasing with respect to $\prec'$, defined by (4.8), and hence this concatenation is shortest. Also, since $\delta_{vs_{\beta_1},w} = 1$ by (4.17), there exists $\mathbf{r}_2 \in \mathcal{P}(vs_{\beta_1}, (\beta_q, \dots, \beta_2))$ such that

$$\mathrm{end}(\mathbf{r}_2) = w.$$

Similarly, the concatenation of the path $\mathbf{r}_2$ with the edge $w \to ws_{\beta_1}$ is label-increasing with respect to $\prec'$, and hence this concatenation is shortest. Since the concatenation of the edge $v \to vs_{\beta_1}$ with any shortest directed path from vs_{β_1} to w is shortest, we obtain:

$$\begin{aligned}
\ell(v \Rightarrow ws_{\beta_1}) &= \underbrace{\ell(v \Rightarrow w)}_{=\ell(\mathbf{r}_1)} + \ell(w \to ws_{\beta_1}) \\
&= \ell(v \to vs_{\beta_1}) + \underbrace{\ell(vs_{\beta_1} \Rightarrow w)}_{=\ell(\mathbf{r}_2)} + \ell(w \to ws_{\beta_1}) \\
&= \ell(v \to vs_{\beta_1}) + \ell(vs_{\beta_1} \Rightarrow ws_{\beta_1}).
\end{aligned}$$

Hence the concatenation of the edge $v \to vs_{\beta_1}$ with any shortest directed path from vs_{β_1} to ws_{β_1} is shortest. Take the (unique) label-increasing directed path $\mathbf{r}_3$ from vs_{β_1}

to ws_{β_1} in QBG(W) with respect to $\prec$. We deduce that $\mathbf{r}_3 \in \mathcal{P}(vs_{\beta_1}, (\beta_2, \dots, \beta_q))$; otherwise, the concatenation

$$v \xrightarrow{\beta_1} \underbrace{vs_{\beta_1} \xrightarrow{\beta_1} \cdots \rightarrow ws_{\beta_1}}_{\mathbf{r}_3}$$

cannot be shortest. Hence we conclude that $\varepsilon_{v,ws_{\beta_1}} = 0$. This completes the proof that $(c^v_{w,\xi})' = 0$.

It remains to show that if $c^v_{w,\xi} = 1$, then $(c^v_{w,\xi})' = 1$. Assume that $c^v_{w,\xi} = 1$. By the above argument (i.e., Proposition 4.4 (2) in the case $k = 1$), we have

$$d^{v,+}_{w,\xi} = \pm 1.$$

By the Yang–Baxter equation (4.1), we see that

$$(d^{v,+}_{w,\xi})' = d^{v,+}_{w,\xi} = \pm 1.$$

Hence we deduce again from the above argument (i.e., Proposition 4.4 (2) in the case $k = 1$, with $(\beta_1, \dots, \beta_q)$ replaced by $(\beta_q, \dots, \beta_1)$) that $(c^v_{w,\xi})' = 1$.

This completes the proof of Proposition 4.4 in the case $k = 1, q - 1$.

4.4. Proof of Proposition 4.4: The case of type C_2

We consider the root system Δ of type C_2. We know that $q = 4$, and

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2)$$

or

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1).$$

Since only the case $k = 2$ is remaining, it suffices to calculate the matrices (with respect to the basis $W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, w_\circ\}$ of $K[W]$) of the following four operators:

(1) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-2\alpha_1-\alpha_2} = \mathsf{R}_{-2\alpha_1-\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$;

(2) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2} = \mathsf{R}_{-\alpha_1-\alpha_2}\mathsf{R}_{-\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$;

(3) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$; and

(4) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$,

where the equalities in (1) and (2) follow from the Yang–Baxter equation (4.1).

The following are the matrices (with respect to the basis W) of operators Q_γ, $\gamma \in \Delta^+ = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$ (cf. [13, Figure 2 (B)]):

$$\mathsf{Q}_{\alpha_1} = \begin{pmatrix} 0 & Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

$$\mathsf{Q}_{2\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & Q_1Q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\mathsf{Q}_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\mathsf{Q}_{\alpha_2} = \begin{pmatrix} 0 & 0 & Q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

By explicit calculations (by using, e.g., SageMath [24]), we obtain

$$\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-2\alpha_1-\alpha_2}$$
$$=\begin{pmatrix}
1 & Q_1Q_2-Q_1 & Q_2 & 0 & -Q_1Q_2 & -Q_1Q_2 & 0 & -Q_1Q_2^2 \\
-1 & 1 & 0 & Q_2 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & -Q_1 & -Q_1Q_2 & 0 & -Q_1Q_2 \\
0 & 1 & 1 & 1 & -Q_1 & -Q_1 & 0 & -Q_1Q_2 \\
0 & -1 & -1 & -Q_2 & 1 & 0 & Q_2 & 0 \\
0 & -1 & -1 & -1 & 1 & 1 & 0 & Q_2 \\
0 & -1 & -1 & -1 & 1 & 0 & 1 & Q_1Q_2-Q_1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{pmatrix},$$

$$\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}$$
$$=\begin{pmatrix}
1 & -Q_1Q_2+Q_1 & -Q_2 & 0 & -Q_1Q_2 & Q_1Q_2 & 0 & -Q_1Q_2^2 \\
1 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & Q_1 & -Q_1Q_2 & 0 & Q_1Q_2 \\
0 & -1 & -1 & 1 & -Q_1 & Q_1 & 0 & -Q_1Q_2 \\
0 & 1 & 1 & -Q_2 & 1 & 0 & -Q_2 & 0 \\
0 & -1 & -1 & 1 & -1 & 1 & 0 & -Q_2 \\
0 & -1 & -1 & 1 & -1 & 0 & 1 & -Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 1
\end{pmatrix}.$$

Also, we obtain

$$\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$$
$$=\begin{pmatrix}
1 & Q_1Q_2+Q_1 & Q_2 & 0 & Q_1Q_2 & Q_1Q_2 & 0 & Q_1Q_2^2 \\
1 & 1 & 0 & Q_2 & 0 & 2Q_1Q_2 & 0 & 0 \\
1 & 2Q_1 & 1 & 0 & Q_1 & Q_1Q_2 & 0 & Q_1Q_2 \\
2 & 2Q_1+1 & 1 & 1 & Q_1 & 2Q_1Q_2+Q_1 & 0 & Q_1Q_2 \\
0 & 1 & 1 & Q_2 & 1 & 0 & Q_2 & 2Q_1Q_2 \\
0 & 1 & 1 & 2Q_2+1 & 1 & 1 & 2Q_2 & 2Q_1Q_2+Q_2 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 2 & 0 & 1 & 1 & 1
\end{pmatrix},$$

$$\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$$
$$=\begin{pmatrix}
1 & Q_1Q_2+Q_1 & 2Q_1Q_2+Q_2 & 2Q_1Q_2 & Q_1Q_2 & Q_1Q_2 & 0 & Q_1Q_2^2 \\
1 & 1 & 2Q_2 & Q_2 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 2Q_1Q_2+Q_1 & Q_1Q_2 & 2Q_1Q_2 & Q_1Q_2 \\
0 & 1 & 1 & 1 & Q_1 & Q_1 & 0 & Q_1Q_2 \\
2 & 1 & 2Q_2+1 & Q_2 & 1 & 0 & Q_2 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & Q_2 \\
0 & 1 & 1 & 1 & 2Q_1+1 & 2Q_1 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 2 & 1 & 1 & 1
\end{pmatrix}.$$

This proves the proposition by direct examination of these matrices.

4.5. Proof of Theorem 3.2

Based on Proposition 4.3, we can prove the existence of a generalization of quantum Yang–Baxter moves. In the same way as in (3.3), we divide $\mathbf{p}(A)$ for $A \in \mathcal{A}(w, \Gamma_1)$

into three parts $\mathbf{p}(A)^{(1)}$, $\mathbf{p}(A)^{(2)}$, $\mathbf{p}(A)^{(3)}$. If we write $A = \{a_1, \ldots, a_l\}$, then $\mathbf{p}(A)$ is of the form:

$$\mathbf{p}(A)\colon w = w_0 \xrightarrow{|\beta_{a_1}|} \cdots \xrightarrow{|\beta_{a_l}|} w_l,$$

with $a_1 < \cdots < a_l$; we set $a_0 := 0$. Let $0 \le i_1 \le l$ be maximal such that $a_{i_1} \le t$, and $0 \le i_2 \le l$ maximal such that $a_{i_2} \le t + q$. Then, we set

$$\begin{aligned}
\mathbf{p}(A)^{(1)}&\colon w = w_0 \xrightarrow{|\beta_{a_1}|} \cdots \xrightarrow{|\beta_{a_{i_1}}|} w_{a_{i_1}},\\
\mathbf{p}(A)^{(2)}&\colon w_{a_{i_1}} \xrightarrow{|\beta_{a_{i_1+1}}|} \cdots \xrightarrow{|\beta_{a_{i_2}}|} w_{a_{i_2}},\\
\mathbf{p}(A)^{(3)}&\colon w_{a_{i_2}} \xrightarrow{|\beta_{a_{i_2+1}}|} \cdots \xrightarrow{|\beta_{a_l}|} w_{a_l}.
\end{aligned}$$

Note that the concatenation of $\mathbf{p}(A)^{(1)}$, $\mathbf{p}(A)^{(2)}$, and $\mathbf{p}(A)^{(3)}$ coincides with $\mathbf{p}(A)$.

Also, in the same way as in (3.4), we divide $\mathbf{p}(B)$ for each $B \in \mathcal{A}(w, \Gamma_2)$ into three parts $\mathbf{p}(B)^{(1)}$, $\mathbf{p}(B)^{(2)}$, $\mathbf{p}(B)^{(3)}$. If we write $B = \{b_1, \ldots, b_m\}$, then $\mathbf{p}(B)$ is of the form:

$$\mathbf{p}(B)\colon w = w_0 \xrightarrow{|\beta'_{b_1}|} \cdots \xrightarrow{|\beta'_{b_m}|} w_m,$$

with $b_1 < \cdots < b_m$; we set $b_0 := 0$. Let $0 \le i_1 \le m$ be maximal such that $b_{i_1} \le t$, and $0 \le i_2 \le m$ maximal such that $b_{i_2} \le t + q$. Then, we set

$$\begin{aligned}
\mathbf{p}(B)^{(1)}&\colon w = w_0 \xrightarrow{|\beta'_{b_1}|} \cdots \xrightarrow{|\beta'_{b_{i_1}}|} w_{b_{i_1}},\\
\mathbf{p}(B)^{(2)}&\colon w_{b_{i_1}} \xrightarrow{|\beta'_{b_{i_1+1}}|} \cdots \xrightarrow{|\beta'_{b_{i_2}}|} w_{b_{i_2}},\\
\mathbf{p}(B)^{(3)}&\colon w_{b_{i_2}} \xrightarrow{|\beta'_{b_{i_2+1}}|} \cdots \xrightarrow{|\beta'_{b_m}|} w_{b_m}.
\end{aligned}$$

Note that the concatenation of $\mathbf{p}(B)^{(1)}$, $\mathbf{p}(B)^{(2)}$, and $\mathbf{p}(B)^{(3)}$ coincides with $\mathbf{p}(B)$.

Proof of Theorem 3.2. First we recall that Δ is assumed not to be of type G_2. Let $A \in \mathcal{A}(w, \Gamma_1)$. Then, by Proposition 4.3 with $\Pi = \Gamma_1^{(2)}$ and $\Pi' = \Gamma_2^{(2)}$, we see that only one of the following occurs:

(1) there exists a directed path $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$;

(2) there exists a directed path $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$.

For convenience of explanation, we set

$$\varphi(A) := \begin{cases} 1 & \text{if (1) of the above holds,} \\ 2 & \text{if (2) of the above holds.} \end{cases}$$

We define a set $\mathcal{A}_0(w,\Gamma_1) \subset \mathcal{A}(w,\Gamma_1)$ by

$$\mathcal{A}_0(w,\Gamma_1) := \{A \in \mathcal{A}(w,\Gamma_1) \mid \varphi(A) = 2\}.$$

Then we have

$$\mathcal{A}_0^C(w,\Gamma_1) = \mathcal{A}(w,\Gamma_1) \setminus \mathcal{A}_0(w,\Gamma_1) = \{A \in \mathcal{A}(w,\Gamma_1) \mid \varphi(A) = 1\}.$$

Let us define a map $Y\colon \mathcal{A}_0(w,\Gamma_1) \to \mathcal{A}(w,\Gamma_2)$. Let $A \in \mathcal{A}_0(w,\Gamma_1)$. Then, by applying Proposition 4.3 with $\Pi = \Gamma_1^{(2)}$ and $\Pi' = \Gamma_2^{(2)}$, there exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. We write the $\mathbf{r}_0$ as:

$$\mathbf{r}_0\colon \mathrm{end}(\mathbf{p}(A)^{(1)}) = x_0 \xrightarrow{|\beta'_{j_1}|} \cdots \xrightarrow{|\beta'_{j_p}|} x_p.$$

Since $\mathbf{r}_0$ is $\Gamma_2^{(2)}$-compatible, it follows that $t+1 \le j_1 < \cdots < j_p \le t+q$. Now, we set $B^{(2)} := \{j_1,\dots,j_p\}$, and define $Y(A)$ by $Y(A) := A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)}$; note that $Y(A) \in \mathcal{A}(w,\Gamma_2)$. We define a set $\mathcal{A}_0(w,\Gamma_2)$ by

$$\mathcal{A}_0(w,\Gamma_2) := \{Y(A) \mid A \in \mathcal{A}_0(w,\Gamma_1)\}.$$

We claim that Y defines a bijection $Y\colon \mathcal{A}_0(w,\Gamma_1) \to \mathcal{A}_0(w,\Gamma_2)$. To verify this claim, it suffices to show that Y is injective.

Let $A_1, A_2 \in \mathcal{A}_0(w,\Gamma_1)$, and assume that $Y(A_1) = Y(A_2)$. We show that $A_1 = A_2$. First, we see that

$$A_1^{(1)} = (Y(A_1))^{(1)} = (Y(A_2))^{(1)} = A_2^{(1)},$$

and

$$A_1^{(3)} = (Y(A_1))^{(3)} = (Y(A_2))^{(3)} = A_2^{(3)}.$$

Hence it remains to show that $A_1^{(2)} = A_2^{(2)}$. By the definition of the map Y, we have

$$\mathrm{end}(\mathbf{p}(Y(A_1))^{(2)}) = \mathrm{end}(\mathbf{p}(A_1)^{(2)}) \quad \text{and} \quad \mathrm{wt}(\mathbf{p}(Y(A_1))^{(2)}) = \mathrm{wt}(\mathbf{p}(A_1)^{(2)}).$$

Also, we have

$$\mathrm{end}(\mathbf{p}(Y(A_2))^{(2)}) = \mathrm{end}(\mathbf{p}(A_2)^{(2)}) \quad \text{and} \quad \mathrm{wt}(\mathbf{p}(Y(A_2))^{(2)}) = \mathrm{wt}(\mathbf{p}(A_2)^{(2)}).$$

Since $\mathbf{p}(Y(A_1))^{(2)} = \mathbf{p}(Y(A_2))^{(2)}$, the uniqueness in Proposition 4.3 (2) (with $\Pi = \Gamma_2^{(2)}$ and $\Pi' = \Gamma_1^{(2)}$) implies that $\mathbf{p}(A_1)^{(2)} = \mathbf{p}(A_2)^{(2)}$, from which we obtain $A_1^{(2)} = A_2^{(2)}$, as desired. This shows the injectivity of Y.

To prove that Y satisfies the condition of Theorem 3.2 (1), it remains to show that $\mathrm{end}(Y(A)) = \mathrm{end}(A)$, $\mathrm{down}(Y(A)) = \mathrm{down}(A)$, and $(-1)^{n(Y(A))} = (-1)^{n(A)}$. The first equation is obvious, since

$$\begin{aligned}\mathrm{end}(Y(A)) &= \mathrm{end}(\mathbf{p}(Y(A))) = \mathrm{end}(\mathbf{p}(Y(A))^{(3)}) \\ &= \mathrm{end}(\mathbf{p}(A)^{(3)}) = \mathrm{end}(\mathbf{p}(A)) = \mathrm{end}(A).\end{aligned}$$

The second equation is shown as follows:

$$\begin{aligned}\mathrm{down}(Y(A)) &= \mathrm{wt}(\mathbf{p}(Y(A))) \\ &= \mathrm{wt}(\mathbf{p}(Y(A))^{(1)}) + \mathrm{wt}(\mathbf{p}(Y(A))^{(2)}) + \mathrm{wt}(\mathbf{p}(Y(A))^{(3)}) \\ &= \mathrm{wt}(\mathbf{p}(A)^{(1)}) + \mathrm{wt}(\mathbf{p}(Y(A))^{(2)}) + \mathrm{wt}(\mathbf{p}(A)^{(3)}) \\ &= \mathrm{wt}(\mathbf{p}(A)^{(1)}) + \mathrm{wt}(\mathbf{p}(A)^{(2)}) + \mathrm{wt}(\mathbf{p}(A)^{(3)}) \\ &= \mathrm{wt}(\mathbf{p}(A)) \\ &= \mathrm{down}(A).\end{aligned}$$

Since $(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(2)})} = (-1)^{\mathrm{neg}(\mathbf{p}(A)^{(2)})}$ by Proposition 4.3 (2), the remaining equation is shown as follows:

$$\begin{aligned}(-1)^{n(Y(A))} &= (-1)^{\mathrm{neg}(\mathbf{p}(Y(A)))} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(1)})}(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(2)})}(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(3)})} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(A)^{(1)})}(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(2)})}(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(3)})} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(A)^{(1)})}(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(2)})}(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(3)})} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(A))} \\ &= (-1)^{n(A)}.\end{aligned}$$

Next, we construct an involution I_1 which satisfies the condition of Theorem 3.2 (2). Let $A \in \mathcal{A}_0^C(w, \Gamma_1)$. Then, by applying Proposition 4.3 with $\Pi = \Gamma_1^{(2)}$, we see that there exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that

$$\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)}) \quad \text{and} \quad \mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)}).$$

We write the $\mathbf{r}_0$ as:

$$\mathbf{r}_0 \colon \mathrm{end}(\mathbf{p}(A)^{(1)}) = x_0 \xrightarrow{|\beta_{j_1}|} \cdots \xrightarrow{|\beta_{j_p}|} x_p.$$

Since $\mathbf{r}_0$ is $\Gamma_1^{(2)}$-compatible, it follows that $t+1 \le j_1 < \cdots < j_p \le t+q$. Now, we set $B^{(2)} := \{j_1, \ldots, j_p\}$, and define $I_1(A)$ by

$$I_1(A) := A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)};$$

note also that $I_1(A) \in \mathcal{A}(w, \Gamma_1)$. Since $\mathbf{p}(A)^{(2)} \in \mathscr{P}(\mathrm{end}(\mathbf{p}(I_1(A))^{(1)}), \Gamma_1^{(2)})$ satisfies the condition of Proposition 4.3 (1), with $\mathbf{p} = \mathbf{p}(I_1(A))^{(2)}$, we deduce that $I_1(A) \in \mathcal{A}_0^C(w, \Gamma_1)$, and that

$$I_1(I_1(A)) = I_1(A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)}) = A^{(1)} \sqcup A^{(2)} \sqcup A^{(3)} = A$$

by the definition of I_1. This shows that I_1 is an involution. Hence it remains to show that $\mathrm{end}(I_1(A)) = \mathrm{end}(A)$, $\mathrm{down}(I_1(A)) = \mathrm{down}(A)$, and $(-1)^{n(I_1(A))} = -(-1)^{n(A)}$, which can be shown by the same argument as that for Y. This completes the construction of I_1.

Finally, we show the existence of an involution I_2 on $\mathcal{A}_0^C(w, \Gamma_2)$. To do this, we examine the set $\mathcal{A}_0^C(w, \Gamma_2)$ in detail. Let $B \in \mathcal{A}(w, \Gamma_2)$. Then, in the same way as for $A \in \mathcal{A}(w, \Gamma_1)$, we see by Proposition 4.3, with $\Pi = \Gamma_2^{(2)}$ and $\Pi' = \Gamma_1^{(2)}$, that only one of the following occurs:

(1)′ there exists a directed path $\mathbf{r}_0 \in \mathscr{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_2^{(2)}) \setminus \{\mathbf{p}(B)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$;

(2)′ there exists a directed path $\mathbf{r}_0 \in \mathscr{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_1^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$.

For convenience of explanation, we set

$$\varphi'(B) = \begin{cases} 1 & \text{if (1)}' \text{ of the above holds,} \\ 2 & \text{if (2)}' \text{ of the above holds.} \end{cases}$$

We claim that

$$\mathcal{A}_0(w, \Gamma_2) = \{B \in \mathcal{A}(w, \Gamma_2) \mid \varphi'(B) = 2\}. \tag{4.19}$$

If this equation is shown, then the following holds:

$$\mathcal{A}_0^C(w, \Gamma_2) := \mathcal{A}(w, \Gamma_2) \setminus \mathcal{A}_0(w, \Gamma_2) = \{B \in \mathcal{A}(w, \Gamma_2) \mid \varphi'(B) = 1\}.$$

First, we take $B \in \mathcal{A}_0(w, \Gamma_2)$. Then, by the definition of $\mathcal{A}_0(w, \Gamma_2)$, there exists $A \in \mathcal{A}_0(w, \Gamma_1)$ such that $Y(A) = B$. Since $\mathbf{p}(A)^{(2)} \in \mathscr{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)})$ satisfies the condition of Proposition 4.3 (2), with $\Pi = \Gamma_2^{(2)}$, $\Pi' = \Gamma_1^{(2)}$, and $\mathbf{p} = \mathbf{p}(B)^{(2)}$, we have

$$\varphi'(B) = 2.$$

Next, we take $B \in \mathcal{A}(w, \Gamma_2)$ such that $\varphi'(B) = 2$. Then, by the definition of φ', there exists $\mathbf{r}_0 \in \mathscr{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_1^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$. We write the $\mathbf{r}_0$ as:

$$\mathbf{r}_0 \colon \mathrm{end}(\mathbf{p}(B)^{(1)}) = x_0 \xrightarrow{|\beta_{j_1}|} \cdots \xrightarrow{|\beta_{j_p}|} x_p.$$

Then, we have

$$t+1 \le j_1 < \cdots < j_p \le t+q.$$

Set $A^{(2)} := \{j_1, \dots, j_p\}$, and then $A := B^{(1)} \sqcup A^{(2)} \sqcup B^{(3)}$. We see that $A \in \mathcal{A}_0(w, \Gamma_1)$ and $Y(A) = B$, and hence $B \in \mathcal{A}_0(w, \Gamma_2)$. Thus, equation (4.19) is shown. Hence the existence of the desired involution I_2 on $\mathcal{A}_0^C(w, \Gamma_2)$ can be shown by the same argument as that for the involution I_1 on $\mathcal{A}_0^C(w, \Gamma_1)$. This completes the proof of Theorem 3.2 (for Δ not of type G_2). ■

4.6. Proof of Theorem 3.4

We will prove that the maps Y, I_1 and I_2 preserve weights and heights.

For this purpose, we need additional notation. Let $\Psi = (\psi_1, \dots, \psi_p)$ be a sequence of roots $\psi_1, \dots, \psi_p \in \Delta$, $\mathbf{k} = (k_1, \dots, k_p)$ a sequence of integers $k_1, \dots, k_p \in \mathbb{Z}$, and $w \in W$. For a subset $J = \{j_1 < \cdots < j_a\} \subset \{1, \dots, p\}$ such that

$$w \xrightarrow{|\psi_{j_1}|} ws_{|\psi_{j_1}|} \xrightarrow{|\psi_{j_2}|} \cdots \xrightarrow{|\psi_{j_a}|} ws_{|\psi_{j_1}|} \cdots s_{|\psi_{j_a}|}$$

is a directed path in $\mathrm{QBG}(W)$ (note that if Ψ is a λ-chain for some $\lambda \in P$, then J is a w-admissible subset), we define $\mathrm{height}_{\mathbf{k},\Psi}(w, J)$ by

$$\mathrm{height}_{\mathbf{k},\Psi}(w, J) := \sum_{j \in J^-} \mathrm{sgn}(\psi_j) k_j,$$

where

$$J^- = \left\{ j_i \in J \;\middle|\; ws_{|\psi_{j_1}|} \cdots s_{|\psi_{j_{i-1}}|} \xrightarrow{|\psi_{j_i}|} ws_{|\psi_{j_1}|} \cdots s_{|\psi_{j_i}|} \text{ is a quantum edge} \right\}.$$

Also, we generalize the definition of down:

$$\mathrm{down}_{\Psi}(w, J) := \sum_{j \in J^-} |\psi_j|^{\vee}.$$

Note that if $\Psi = \Gamma_1$, $\mathbf{k} = (\widetilde{l}_1, \dots, \widetilde{l}_r)$, $w \in W$, and $J = A \in \mathcal{A}(w, \Gamma_1)$, then we have

$$\mathrm{down}_{\Psi}(w, A) = \mathrm{down}(A), \quad \mathrm{height}_{\mathbf{k},\Psi}(w, A) = \mathrm{height}(A);$$

if $\Psi = \Gamma_2$, $\mathbf{k} = (\widetilde{l}'_1, \dots, \widetilde{l}'_r) = (\widetilde{l}_1, \dots, \widetilde{l}_t, \widetilde{l_{t+q}}, \dots, \widetilde{l_{t+1}}, \widetilde{l_{t+q+1}}, \dots, \widetilde{l}_r)$, $w \in W$, and $J = B \in \mathcal{A}(w, \Gamma_2)$, then we have

$$\mathrm{down}_{\Psi}(w, B) = \mathrm{down}(B), \quad \mathrm{height}_{\mathbf{k},\Psi}(w, B) = \mathrm{height}(B).$$

In addition, for $A \in \mathcal{A}(w, \Gamma_1)$, it follows that

$$\begin{aligned}\mathrm{down}(A) &= \mathrm{down}_{\Gamma_1^{(1)}}(w, A^{(1)}) \\ &\quad + \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) + \mathrm{down}_{\Gamma_1^{(3)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(3)}),\end{aligned}$$

and that

$$\begin{aligned}\mathrm{height}(A) &= \mathrm{height}_{(\tilde{l}_1,\dots,\tilde{l}_t),\Gamma_1^{(1)}}(w, A^{(1)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+q+1}},\dots,\tilde{l}_r),\Gamma_1^{(3)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(3)});\end{aligned} \tag{4.20}$$

for $B \in \mathcal{A}(w, \Gamma_2)$, it follows that

$$\begin{aligned}\mathrm{down}(B) &= \mathrm{down}_{\Gamma_2^{(1)}}(w, B^{(1)}) \\ &\quad + \mathrm{down}_{\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) + \mathrm{down}_{\Gamma_2^{(3)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(3)}),\end{aligned}$$

and that

$$\begin{aligned}\mathrm{height}(B) &= \mathrm{height}_{(\tilde{l}'_1,\dots,\tilde{l}'_t),\Gamma_2^{(1)}}(w, B^{(1)}) \\ &\quad + \mathrm{height}_{(\widetilde{l'_{t+1}},\dots,\widetilde{l'_{t+q}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\ &\quad + \mathrm{height}_{(\widetilde{l'_{t+q+1}},\dots,\tilde{l}'_r),\Gamma_2^{(3)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(3)}) \\ &= \mathrm{height}_{(\tilde{l}_1,\dots,\tilde{l}_t),\Gamma_2^{(1)}}(w, B^{(1)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+q}},\dots,\widetilde{l_{t+1}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+q+1}},\dots,\tilde{l}_r),\Gamma_2^{(3)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(3)}).\end{aligned} \tag{4.21}$$

Next, we consider weights. For the above J, we set

$$\hat{r}_{\mathbf{k},\Psi}(J) := s_{\psi_{j_1},-k_{j_1}} \cdots s_{\psi_{j_a},-k_{j_a}}.$$

Then, for $A \in \mathcal{A}(w, \Gamma_1)$, we have

$$\begin{aligned}\mathrm{wt}(A) &= -\hat{r}_{(l_1,\dots,l_r),\Gamma_1}(A)(-\lambda) \\ &= -\hat{r}_{(l_1,\dots,l_t),\Gamma_1^{(1)}}(A^{(1)})\hat{r}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(A^{(2)}) \\ &\qquad \times \hat{r}_{(l_{t+q+1},\dots,l_r),\Gamma_1^{(3)}}(A^{(3)})(-\lambda);\end{aligned} \tag{4.22}$$

for $B \in \mathcal{A}(w, \Gamma_2)$, we have

$$\begin{aligned}\mathrm{wt}(B) &= -\widehat{r}_{(l'_1,\dots,l'_r),\Gamma_2}(B)(-\lambda) \\ &= -\widehat{r}_{(l_1,\dots,l_t,l_{t+q},\dots,l_{t+1},l_{t+q+1},\dots,l_r),\Gamma_2}(B)(-\lambda) \\ &= -\widehat{r}_{(l_1,\dots,l_t),\Gamma_2^{(1)}}(B^{(1)})\widehat{r}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(B^{(2)}) \\ &\qquad\qquad \times \widehat{r}_{(l_{t+q+1},\dots,l_r),\Gamma_2^{(3)}}(B^{(3)})(-\lambda). \end{aligned} \tag{4.23}$$

Proof of Theorem 3.4. First, we consider heights. For $A \in \mathcal{A}(w, \Gamma_1)$, we see that

$$\begin{aligned}&\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \\ &= \sum_{j\in(A^{(2)})^-} \mathrm{sgn}(\beta_j)(\langle\lambda, \beta_j^\vee\rangle - l_j) \\ &= \sum_{j\in(A^{(2)})^-} \mathrm{sgn}(\beta_j)\langle\lambda, \beta_j^\vee\rangle - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \\ &= \left\langle\lambda, \sum_{j\in(A^{(2)})^-} \mathrm{sgn}(\beta_j)\beta_j^\vee\right\rangle - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \\ &= \langle\lambda, \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)})\rangle \\ &\qquad - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}). \end{aligned} \tag{4.24}$$

Let us assume that $A \in \mathcal{A}_0(w, \Gamma_1)$, and set $B := Y(A)$. Then we have

$$\begin{aligned}&\mathrm{height}_{(\widetilde{l_{t+q}},\dots,\widetilde{l_{t+1}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\ &= \langle\lambda, \mathrm{down}_{\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)})\rangle \\ &\qquad - \mathrm{height}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}). \end{aligned} \tag{4.25}$$

Here, since $\mathrm{down}(Y(A)) = \mathrm{down}(A)$, it follows that

$$\mathrm{down}_{\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) = \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}).$$

Also, by [13, Lemma 3.5], we know that

$$\bigcap_{k=t+1}^{t+q} H_{\beta_k,-l_k} \neq \emptyset.$$

Therefore, by [14, Lemma 46], we have

$$\begin{aligned}&\mathrm{height}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\ &= \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}),\end{aligned}$$

and hence by (4.24) and (4.25), we obtain

$$\begin{aligned}&\mathrm{height}_{(\widetilde{l_{t+q}},\dots,\widetilde{l_{t+1}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}),B^{(2)})\\&\quad=\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}),A^{(2)}).\end{aligned}$$

Now, by (4.20) and (4.21), we deduce that $\mathrm{height}(B)=\mathrm{height}(A)$, as desired.

Assume that $A\in\mathcal{A}_0^C(w,\Gamma_1)$, and set $B:=I_1(A)$. As in (4.24), we have

$$\begin{aligned}&\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}),B^{(2)})\\&\quad=\langle\lambda,\mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}),B^{(2)})\rangle\\&\qquad-\mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}),B^{(2)}).\end{aligned}$$

Again, by the definition of I_1 and [14, Lemma 46], we have

$$\begin{aligned}&\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}),B^{(2)})\\&=\langle\lambda,\mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}),B^{(2)})\rangle-\mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}),B^{(2)})\\&=\langle\lambda,\mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}),A^{(2)})\rangle-\mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}),A^{(2)})\\&=\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}),A^{(2)}),\end{aligned}$$

and hence obtain $\mathrm{height}(B)=\mathrm{height}(A)$, as desired. Now, the assertion for I_2 is shown by the same argument as for I_1.

It remains to consider weights. Again, by [13, Lemma 3.5], we can take $\mu\in\mathfrak{h}_{\mathbb{R}}^*$ such that

$$\mu\in\bigcap_{k=t+1}^{t+q}H_{\beta_k,-l_k}\neq\emptyset.$$

Note that $\langle\mu,\beta_k^\vee\rangle=-l_k$ for $k=t+1,\dots,t+q$.

Let $A\in\mathcal{A}_0(w,\Gamma_1)$, and set $B:=Y(A)$. Recall that $\mathrm{end}(\mathbf{p}(B)^{(2)})=\mathrm{end}(\mathbf{p}(A)^{(2)})$. For simplicity of notation, we set

$$v:=\mathrm{end}(\mathbf{p}(A)^{(1)})=\mathrm{end}(\mathbf{p}(B)^{(1)}).$$

For $\nu\in\mathfrak{h}_{\mathbb{R}}^*$, we denote by t_ν the translation by ν, i.e., we define $\mathrm{t}_\nu\colon\mathfrak{h}_{\mathbb{R}}^*\to\mathfrak{h}_{\mathbb{R}}^*$ by $\mathrm{t}_\nu(\xi):=\xi+\nu$. We see that for $\nu\in\mathfrak{h}_{\mathbb{R}}^*$ and $\gamma\in\Delta^+$,

$$s_\gamma\mathrm{t}_\nu=\mathrm{t}_{s_\gamma(\nu)}s_\gamma,$$

and for $\nu_1,\nu_2\in\mathfrak{h}_{\mathbb{R}}^*$,

$$\mathrm{t}_{\nu_1}\mathrm{t}_{\nu_2}=\mathrm{t}_{\nu_1+\nu_2}.$$

Also, for $\gamma \in \Delta$ and $k \in \mathbb{Z}$, we have $\mathsf{t}_{k\gamma} s_{|\gamma|} = s_{\gamma,k}$. If we write $A^{(2)} = \{j_1, \dots, j_a\}$, then we have

$$\begin{aligned}
\mathsf{t}_\mu v^{-1} \operatorname{end}(\mathbf{p}(A^{(2)}))\mathsf{t}_{-\mu} &= \mathsf{t}_\mu v^{-1}(v s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_a}|})\mathsf{t}_{-\mu} \\
&= (\mathsf{t}_\mu s_{|\beta_{j_1}|}\mathsf{t}_{-\mu}) \cdots (\mathsf{t}_\mu s_{|\beta_{j_a}|}\mathsf{t}_{-\mu}) \\
&= (\mathsf{t}_\mu \mathsf{t}_{s_{|\beta_{j_1}|}(-\mu)} s_{|\beta_{j_1}|}) \cdots (\mathsf{t}_\mu \mathsf{t}_{s_{|\beta_{j_a}|}(-\mu)} s_{|\beta_{j_a}|}) \\
&= (\mathsf{t}_{\langle \mu, |\beta_{j_1}|^\vee \rangle |\beta_{j_1}|} s_{|\beta_{j_1}|}) \cdots (\mathsf{t}_{\langle \mu, |\beta_{j_a}|^\vee \rangle |\beta_{j_a}|} s_{|\beta_{j_a}|}) \\
&= (\mathsf{t}_{\langle \mu, \beta_{j_1}^\vee \rangle \beta_{j_1}} s_{|\beta_{j_1}|}) \cdots (\mathsf{t}_{\langle \mu, \beta_{j_a}^\vee \rangle \beta_{j_a}} s_{|\beta_{j_a}|}) \\
&= (\mathsf{t}_{-l_{j_1} \beta_{j_1}} s_{|\beta_{j_1}|}) \cdots (\mathsf{t}_{-l_{j_a} \beta_{j_a}} s_{|\beta_{j_a}|}) \\
&= s_{\beta_{j_1}, -l_{j_1}} \cdots s_{\beta_{j_a}, -l_{j_a}} \\
&= \hat{r}_{(l_{t+1}, \dots, l_{t+q}), \Gamma_1^{(2)}}(A^{(2)}).
\end{aligned} \tag{4.26}$$

By the same calculation, we have

$$\mathsf{t}_\mu v^{-1} \operatorname{end}(\mathbf{p}(B^{(2)}))\mathsf{t}_{-\mu} = \hat{r}_{(l_{t+q}, \dots, l_{t+1}), \Gamma_2^{(2)}}(B^{(2)}).$$

Since $\operatorname{end}(\mathbf{p}(B)^{(2)}) = \operatorname{end}(\mathbf{p}(A)^{(2)})$, it follows that

$$\hat{r}_{(l_{t+q}, \dots, l_{t+1}), \Gamma_2^{(2)}}(B^{(2)}) = \hat{r}_{(l_{t+1}, \dots, l_{t+q}), \Gamma_1^{(2)}}(A^{(2)}).$$

Hence, by (4.22) and (4.23), we obtain $\mathrm{wt}(B) = \mathrm{wt}(A)$, as desired.

Next, assume that $A \in \mathcal{A}_0^C(w, \Gamma_1)$, and set $B := I_1(A)$. By the same calculation as for (4.26), we have

$$\mathsf{t}_\mu v^{-1} \operatorname{end}(\mathbf{p}(B^{(2)}))\mathsf{t}_{-\mu} = \hat{r}_{(l_{t+1}, \dots, l_{t+q}), \Gamma_1^{(2)}}(B^{(2)}).$$

Hence, from the equality $\operatorname{end}(\mathbf{p}(B)^{(2)}) = \operatorname{end}(\mathbf{p}(A)^{(2)})$, we deduce that

$$\hat{r}_{(l_{t+1}, \dots, l_{t+q}), \Gamma_1^{(2)}}(B^{(2)}) = \hat{r}_{(l_{t+1}, \dots, l_{t+q}), \Gamma_1^{(2)}}(A^{(2)}).$$

Therefore, we conclude by (4.22) and (4.23) that $\mathrm{wt}(B) = \mathrm{wt}(A)$, as desired.

The assertion for I_2 is shown by the same argument as for I_1. This completes the proof of the theorem. ∎

5. Generating functions of certain statistics

We consider a generating function of the statistics associated to the quantum alcove model. We describe the relationship between the generating functions associated to two alcove paths which are related by the procedures (YB) and (D). We also investigate the composition of generating functions. As an application, we derive an identity of "Chevalley type" for the graded characters of Demazure submodules of (level-zero) extremal weight modules over a quantum affine algebra.

5.1. Generating functions

Take an indeterminate q, and consider the ring $R := \mathbb{Z}[q, q^{-1}]$ of Laurent polynomials in q. Recall that an element of the affine Weyl group W_{af} can be written as $x = wt_{\xi}$, with w in the finite Weyl group W and ξ in the coroot lattice $Q^{\vee}$.

Definition 5.1. For each λ-chain Γ and $x = wt_{\xi} \in W_{\mathrm{af}}$, we define a *generating function* $\mathbf{G}_{\Gamma}(x) \in R[P][W_{\mathrm{af}}]$ associated to the set $\mathcal{A}(w, \Gamma)$ of w-admissible subsets by

$$\mathbf{G}_{\Gamma}(x) := \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi + \operatorname{down}(A)}. \tag{5.1}$$

We also think of $\mathbf{G}_{\Gamma}$ as a linear function on $R[P][W_{\mathrm{af}}]$ by $R[P]$-linearly extending the above assignment $x \mapsto \mathbf{G}_{\Gamma}(x)$.

Let $\lambda \in P$, and take λ-chains Γ_1, Γ_2. We consider the relation of the two generating functions $\mathbf{G}_{\Gamma_1}(x)$ and $\mathbf{G}_{\Gamma_2}(x)$ for $x = wt_{\xi} \in W_{\mathrm{af}}$.

First, we consider the case in which Γ_2 is obtained from Γ_1 by the procedure (YB). As a corollary of Theorems 3.2 and 3.4, we obtain the equality between the two generating functions.

Proposition 5.2. *Assume that Γ_2 is obtained from Γ_1 by (YB). Then we have*

$$\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x).$$

Proof. As in Theorem 3.2, we take subsets $\mathcal{A}_0(w, \Gamma_1)$, $\mathcal{A}_0^C(w, \Gamma_1)$ of $\mathcal{A}(w, \Gamma_1)$. Also, we take subsets $\mathcal{A}_0(w, \Gamma_2)$, $\mathcal{A}_0^C(w, \Gamma_2)$ of $\mathcal{A}(w, \Gamma_2)$. Then we have the maps Y, I_1, I_2 as in Theorem 3.2. Note that by Theorem 3.4, Y, I_1, and I_2 preserve weights and heights.

Since I_1 is a sign-reversing involution which preserves weights, heights, and down statistics, we have

$$\sum_{A \in \mathcal{A}_0^C(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi + \operatorname{down}(A)} = 0,$$

and hence

$$\mathbf{G}_{\Gamma_1}(x) = \sum_{A \in \mathcal{A}_0(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi + \operatorname{down}(A)}.$$

We derive the similar result for $\mathbf{G}_{\Gamma_2}(x)$ via the sign-reversing involution I_2. Using the map Y given by a generalization of quantum Yang–Baxter moves, we deduce that

$$\begin{aligned}\mathbf{G}_{\Gamma_1}(x) &= \sum_{A \in \mathcal{A}_0(w,\Gamma_1)} (-1)^{n(Y(A))} q^{-\operatorname{height}(Y(A))-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(Y(A))} \operatorname{end}(Y(A)) t_{\xi+\operatorname{down}(Y(A))} \\ &= \mathbf{G}_{\Gamma_2}(x).\end{aligned}$$

This concludes the proof. ∎

Next, we consider the case in which Γ_2 is obtained from Γ_1 by the procedure (D).

Proposition 5.3. *Assume that Γ_2 is obtained from Γ_1 by the procedure (D), which deletes the segment $(\pm\beta, \mp\beta)$ of Γ_1, where β is not a simple root. Then we have*

$$\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x).$$

Proof. We write $\Gamma_1 = (\beta_1, \dots, \beta_r)$. By the assumption, there exists $u \in \{1, \dots, r-2\}$ such that

- $\beta_{u+2} = -\beta_{u+1}$,
- β_{u+1} and β_{u+2} are not simple roots, and
- $\Gamma_2 = (\beta_1, \dots, \beta_u, \beta_{u+3}, \dots, \beta_r)$.

Set $\beta := |\beta_{u+1}| = |\beta_{u+2}|$. Since β is not a simple root, there does not exist any path of the form

$$v \xrightarrow{|\beta_{u+1}|=\beta} v' \xrightarrow{|\beta_{u+2}|=\beta} v'' = v.$$

Hence, for $A \in \mathcal{A}(w, \Gamma_1)$, we have $A \cap \{u+1, u+2\} \neq \{u+1, u+2\}$. We define a subset $\mathcal{A}_\emptyset(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ by

$$\mathcal{A}_\emptyset(w, \Gamma_1) := \{A \in \mathcal{A}(w, \Gamma_1) \mid A \cap \{u+1, u+2\} = \emptyset\}.$$

Also, we define a subset $\mathcal{A}_\emptyset^C(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ by

$$\begin{aligned}\mathcal{A}_\emptyset^C(w, \Gamma_1) &:= \mathcal{A}(w, \Gamma_1) \setminus \mathcal{A}_\emptyset(w, \Gamma_1) \\ &= \{A \in \mathcal{A}(w, \Gamma_1) \mid A \cap \{u+1, u+2\} = \{u+1\}, \{u+2\}\}.\end{aligned}$$

We define a map $I_D \colon \mathcal{A}_\emptyset^C(w, \Gamma_1) \to \mathcal{A}_\emptyset^C(w, \Gamma_1)$ as follows. If $A \in \mathcal{A}_\emptyset^C(w, \Gamma_1)$ satisfies $A \cap \{u+1, u+2\} = \{u+1\}$, then we set

$$I_D(A) := (A \cap \{1, \dots, u\}) \sqcup \{u+2\} \sqcup (A \cap \{u+3, \dots, r\});$$

if $A \in \mathcal{A}_\emptyset^C(w, \Gamma_1)$ satisfies $A \cap \{u+1, u+2\} = \{u+2\}$, then we set

$$I_D(A) := (A \cap \{1, \dots, u\}) \sqcup \{u+1\} \sqcup (A \cap \{u+3, \dots, r\}).$$

We see that for all $A \in \mathcal{A}_{\emptyset}^{C}(w, \Gamma_1)$, we have $I_D(A) \in \mathcal{A}_{\emptyset}^{C}(w, \Gamma_1)$. Also, we see that $I_D(I_D(A)) = A$. Hence I_D defines an involution on $\mathcal{A}_{\emptyset}^{C}(w, \Gamma_1)$, and it is easy to see that it preserves the statistics down$(\cdot)$, end$(\cdot)$, height$(\cdot)$, and wt$(\cdot)$. Also, it is easy to verify that I_D negates the sign $(-1)^{n(\cdot)}$. Therefore, we obtain

$$\sum_{A \in \mathcal{A}_{\emptyset}^{C}(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi+\operatorname{down}(A)} = 0.$$

Now, we define a bijection $D : \mathcal{A}_{\emptyset}(w, \Gamma_1) \to \mathcal{A}(w, \Gamma_2)$ by

$$D(A) := (A \cap \{1, \dots, u\}) \sqcup \{j - 2 \mid j \in A \cap \{u + 3, \dots, r\}\}.$$

It is again easy to see that this bijection preserves all the statistics. Therefore, we deduce that

$$\begin{aligned}
\mathbf{G}_{\Gamma_1}(x) &= \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi+\operatorname{down}(A)} \\
&= \sum_{A \in \mathcal{A}_{\emptyset}(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi+\operatorname{down}(A)} \\
&= \sum_{A \in \mathcal{A}_{\emptyset}(w,\Gamma_1)} (-1)^{n(D(A))} q^{-\operatorname{height}(D(A))-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(D(A))} \operatorname{end}(D(A)) t_{\xi+\operatorname{down}(D(A))} \\
&= \mathbf{G}_{\Gamma_2}(x),
\end{aligned}$$

as desired. This concludes the proof. ∎

Remark 5.4. In the setting of Proposition 5.3, assume that β is a simple root such that $\pm\beta$ appears in positions $u + 1$ and $u + 2$ in Γ_1. Then the equality $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$ does not hold. This is because there exists a directed path of the form

$$v \xrightarrow{\beta} v' \xrightarrow{\beta} v$$

for all $v \in W$, in contrast to the case that β is not a simple root. In fact, in $\mathcal{A}(w, \Gamma_1)$, we can pair each A for which $A \cap \{u + 1, u + 2\} = \emptyset$ with $A' := A \sqcup \{u + 1, u + 2\}$. Let $h \in \mathbb{Z}$ be the contribution of (one of the) positions $u + 1$, $u + 2$ to height(A'); note that this is independent of A. By using the above pairing, as well as the map D and the cancellations given by the involution I_D in the proof of Proposition 5.3, we derive

$$\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)(1 - q^{-h} t_{\beta^\vee}).$$

We need the following weaker version of the notion of a reduced λ-chain.

Definition 5.5. A λ-chain is *weakly reduced* if it does not contain both a simple root and its negative.

Now let us consider arbitrary weakly reduced λ-chains Γ_1 and Γ_2. Then there exists a sequence $\Gamma_1 = \Psi_0, \Psi_1, \ldots, \Psi_p = \Gamma_1^*$ of λ-chains such that Γ_1^* is reduced, and each Ψ_k is obtained from Ψ_{k-1} by one of the procedure (YB) or (D). In a similar way, we relate Γ_2 to a reduced λ-chain Γ_2^*. Finally, we relate Γ_1^* to Γ_2^* by successively applying the procedure (YB). The weakly reduced property of Γ_1 and Γ_2 implies that, in the above process, the procedure (D) never deletes a segment $(\pm\beta, \mp\beta)$, where β is a simple root. By Propositions 5.2 and 5.3, we derive the following theorem.

Theorem 5.6. *For arbitrary weakly reduced λ-chains Γ_1 and Γ_2, we have*

$$\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x).$$

5.2. Combinatorial realization of commutativity

In this section we realize combinatorially the symmetry of the general Chevalley formula in [14,15] coming from commutativity of line bundle multiplication in equivariant K-theory. As explained in the Introduction, this realization involves commutativity of the composition of two functions $\mathbf{G}_{\Gamma_1}$ and $\mathbf{G}_{\Gamma_2}$, and is based on the generalized quantum Yang–Baxter moves. The main result here will also play an important role in the proof of the character identity in Section 5.3.

We start by developing the notion of a weakly reduced chain of roots in Definition 5.5. Consider an arbitrary weight λ and an arbitrary decomposition of it $\lambda = \lambda_1 + \cdots + \lambda_p$. Let $\lambda_j = \sum_{i \in I} m_{ij} \varpi_i$.

Definition 5.7. The weight decomposition $\lambda = \lambda_1 + \cdots + \lambda_p$ is *cancellation-free* if, for any $i \in I$, all the nonzero coefficients among $m_{i1}, \ldots, m_{ip}$ have the same sign.

Given the above weight decomposition, consider λ_j-chains of roots Γ_j, for $j = 1, \ldots, p$. Their concatenation, defined in the obvious way and denoted by

$$\Gamma = \Gamma_1 * \cdots * \Gamma_p,$$

is clearly a λ-chain. Note that the alcove path corresponding to Γ is obtained by considering the shift of the alcove path for Γ_j by $\lambda_1 + \cdots + \lambda_{j-1}$, for $j = 1, \ldots, p$, and by concatenating them in this order.

Proposition 5.8. *The λ-chain Γ is a weakly reduced if and only if the weight decomposition $\lambda = \lambda_1 + \cdots + \lambda_p$ is cancellation-free and each λ_j-chain Γ_j is weakly reduced.*

Proof. The result is easily derived from the following general fact about a (not necessarily reduced) λ-chain $\Gamma = (\beta_1, \ldots, \beta_r)$, for an arbitrary weight λ and a positive root α (see, e.g., [19, Lemma 5.3]):

$$\langle \lambda, \alpha^\vee \rangle = \#\{j \mid \beta_j = \alpha\} - \#\{j \mid \beta_j = -\alpha\}.$$

This fact is applied to a simple root $\alpha = \alpha_i$, by noting that $\langle \lambda, \alpha_i^\vee \rangle$ is the coefficient of ϖ_i in the expansion of λ. ∎

Let us now consider a cancellation-free weight decomposition $\lambda = \mu + \nu$, and weakly reduced chains of roots Γ_1 and Γ_2 corresponding to μ and ν, respectively. Then, by Proposition 5.8, we have the weakly reduced λ-chain $\Gamma := \Gamma_1 * \Gamma_2$. Observe that there exists a natural bijection

$$\{(A, B) \mid A \in \mathcal{A}(w, \Gamma_1),\ B \in \mathcal{A}(\mathrm{end}(A), \Gamma_2)\} \to \mathcal{A}(w, \Gamma); \tag{5.2}$$

let $A * B \in \mathcal{A}(w, \Gamma)$ denote the image of (A, B) under this bijection. The following lemma relates the statistics of interest under the bijection; its proof is based on arguments completely similar to those in the proof of [14, Theorem 48].

Lemma 5.9 ([14]). *For $A \in \mathcal{A}(w, \Gamma_1)$ and $B \in \mathcal{A}(\mathrm{end}(A), \Gamma_2)$, the following hold:*

(1) $n(A * B) = n(A) + n(B)$;

(2) $\mathrm{end}(A * B) = \mathrm{end}(B)$;

(3) $\mathrm{down}(A * B) = \mathrm{down}(A) + \mathrm{down}(B)$;

(4) $\mathrm{height}(A * B) = \mathrm{height}(A) + \mathrm{height}(B) + \langle \nu, \mathrm{down}(A) \rangle$;

(5) $\mathrm{wt}(A * B) = \mathrm{wt}(A) + \mathrm{wt}(B)$.

We are now ready to prove the main result of this section.

Theorem 5.10. *Given the above setup and any $x = wt_\xi \in W_{\mathrm{af}}$, we have*

$$\mathbf{G}_{\Gamma_1} \circ \mathbf{G}_{\Gamma_2}(x) = \mathbf{G}_{\Gamma_2} \circ \mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma}(x).$$

These identities are realized combinatorially via the bijection (5.2) and the generalized quantum Yang–Baxter moves.

Proof. It suffices to prove the second equality. Indeed, this would imply that

$$\mathbf{G}_{\Gamma_1} \circ \mathbf{G}_{\Gamma_2}(x) = \mathbf{G}_{\Gamma'}(x),$$

where $\Gamma' := \Gamma_2 * \Gamma_1$. The proof is then concluded by using Theorem 5.6 to show that $\mathbf{G}_{\Gamma}(x) = \mathbf{G}_{\Gamma'}(x)$. Recall that the mentioned theorem is proved by applying the generalized quantum Yang–Baxter moves.

By iterating the definition (5.1), we obtain

$$\begin{aligned}
&\mathbf{G}_{\Gamma_2} \circ \mathbf{G}_{\Gamma_1}(x) \\
&= \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A)-\langle \mu,\xi\rangle} e^{\operatorname{wt}(A)} \mathbf{G}_{\Gamma_2}(\operatorname{end}(A) t_{\xi+\operatorname{down}(A)}) \\
&= \sum_{A \in \mathcal{A}(w,\Gamma_1)} \sum_{B \in \mathcal{A}(\operatorname{end}(A),\Gamma_2)} (-1)^{n(A)+n(B)} q^{-\operatorname{height}(A)-\langle \mu,\xi\rangle-\operatorname{height}(B)-\langle \nu,\xi+\operatorname{down}(A)\rangle} \\
&\qquad\qquad \times e^{\operatorname{wt}(A)+\operatorname{wt}(B)} \operatorname{end}(B) t_{\xi+\operatorname{down}(A)+\operatorname{down}(B)} \\
&= \sum_{A \in \mathcal{A}(w,\Gamma_1)} \sum_{B \in \mathcal{A}(\operatorname{end}(A),\Gamma_2)} (-1)^{n(A*B)} q^{-\operatorname{height}(A*B)-\langle \lambda,\xi\rangle} \\
&\qquad\qquad \times e^{\operatorname{wt}(A*B)} \operatorname{end}(A*B) t_{\xi+\operatorname{down}(A*B)} \\
&= \mathbf{G}_{\Gamma}(x).
\end{aligned}$$

The last two equalities are based on the bijection (5.2) and Lemma 5.9. ■

Theorem 5.10 immediately implies the following corollary involving a composition of more than two functions $\mathbf{G}_{\Gamma_i}$. Here we use the corresponding setup that was defined above. Namely, we consider the cancellation-free weight decomposition $\lambda = \lambda_1 + \cdots + \lambda_p$, the weakly reduced λ_j-chains of roots Γ_j, for $j = 1, \ldots, p$, and their concatenation $\Gamma = \Gamma_1 * \cdots * \Gamma_p$.

Corollary 5.11. *In the above setup, the composite of generating functions* $\mathbf{G}_{\Gamma_1} \circ \cdots \circ \mathbf{G}_{\Gamma_p}(x)$ *is invariant under permuting the maps* $\mathbf{G}_{\Gamma_\cdot}$*, and coincides with* $\mathbf{G}_{\Gamma}(x)$.

We now generalize the function $\mathbf{G}_\Gamma$ on $R[P][W_{\mathrm{af}}]$ by defining the function $\widehat{\mathbf{G}}_\Gamma$, which expresses the general K-theory Chevalley formula for semi-infinite flag manifolds in [14, 15]. In order to do this, we need some notation for partitions. Let $\lambda \in P$ and write it as $\lambda = \sum_{i \in I} m_i \varpi_i$. We define the set $\overline{\operatorname{Par}}(\lambda)$ by

$$\overline{\operatorname{Par}}(\lambda) := \left\{ \boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \;\middle|\; \begin{array}{l} \chi^{(i)} \text{ is a partition whose length is} \\ \text{less than or equal to } \max\{m_i, 0\} \end{array} \right\}. \tag{5.3}$$

For $\boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \in \overline{\operatorname{Par}}(\lambda)$, we write it as $\chi^{(i)} = (\chi_1^{(i)} \geq \chi_2^{(i)} \geq \cdots \geq \chi_{l_i}^{(i)} > 0)$, where $0 \leq l_i \leq \max\{m_i, 0\}$ and $\chi_1^{(i)}, \ldots, \chi_{l_i}^{(i)} \in \mathbb{Z}$, and set

$$|\boldsymbol{\chi}| := \sum_{i \in I} \sum_{k=1}^{l_i} \chi_k^{(i)}, \quad \iota(\boldsymbol{\chi}) := \sum_{i \in I} \chi_1^{(i)} \alpha_i^\vee; \tag{5.4}$$

if $\chi^{(i)} = \emptyset$, then we understand that $l_i = 0$ and $\chi_1^{(i)} = 0$.

Definition 5.12. For each λ-chain Γ and $x \in W_{\mathrm{af}}$, we define

$$\widehat{\mathbf{G}}_\Gamma(x) := \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} q^{-|\boldsymbol{\chi}|} \mathbf{G}_\Gamma(x) t_{\iota(\boldsymbol{\chi})}. \tag{5.5}$$

Like above, we now consider a cancellation-free weight decomposition $\lambda = \mu + \nu$, weakly reduced chains of roots Γ_1 and Γ_2 corresponding to μ and ν, respectively, and the weakly reduced λ-chain $\Gamma := \Gamma_1 * \Gamma_2$. Let $\mu = \sum_{i \in I} m_{i1} \varpi_i$ and $\nu = \sum_{i \in I} m_{i2} \varpi_i$, so $\lambda = \sum_{i \in I} m_i \varpi_i$ with $m_i = m_{i1} + m_{i2}$. We will show that there exists a natural bijection

$$\overline{\mathrm{Par}}(\mu) \times \overline{\mathrm{Par}}(\nu) \to \overline{\mathrm{Par}}(\lambda), \quad (\boldsymbol{\psi}, \boldsymbol{\omega}) \mapsto \boldsymbol{\chi} := \boldsymbol{\psi} * \boldsymbol{\omega}, \tag{5.6}$$

which is compatible with the corresponding statistics. The above map is constructed by defining the partition $\chi^{(i)}$ in terms of the partitions $\psi^{(i)}$ and $\omega^{(i)}$, for each $i \in I$; we will identify a partition with its Young diagram. We may assume that $m_{i1}, m_{i2} \geq 0$, and at least one is positive; indeed, otherwise $m_{i1}, m_{i2} \leq 0$, so $\psi^{(i)} = \omega^{(i)} = \emptyset$, and we let $\chi^{(i)} := \emptyset$. In the nontrivial case, we consider a rectangular partition with m_{i2} rows of size $\psi_1^{(i)}$; then $\chi^{(i)}$ is defined as the result of attaching $\omega^{(i)}$ at the right of the rectangle (top justified) and $\psi^{(i)}$ at the bottom of the rectangle (left justified). It is easy to verify that the result is indeed a partition of length at most m_i, as needed, as well as the fact that this map is invertible.

Lemma 5.13. *For $\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)$ and $\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)$, the following hold:*

(1) $\iota(\boldsymbol{\psi} * \boldsymbol{\omega}) = \iota(\boldsymbol{\psi}) + \iota(\boldsymbol{\omega})$;

(2) $|\boldsymbol{\psi} * \boldsymbol{\omega}| = |\boldsymbol{\psi}| + |\boldsymbol{\omega}| + \langle \nu, \iota(\boldsymbol{\psi}) \rangle$.

Proof. We use the above notation, in particular $\boldsymbol{\chi} := \boldsymbol{\psi} * \boldsymbol{\omega}$, as well as the fact that the weight decomposition $\lambda = \mu + \nu$ is cancellation-free. The first relation is clear by construction. The second one follows from the fact that

$$\langle \nu, \iota(\boldsymbol{\psi}) \rangle = \sum_{i \in I} m_{i2} \psi_1^{(i)} = \sum_{i \in I} \max\{m_{i2}, 0\} \psi_1^{(i)};$$

here we note that $m_{i2} \psi_1^{(i)}$ is the size of the rectangle used to construct $\chi^{(i)}$ in the nontrivial case. ∎

Theorem 5.14. *Given the above setup and any $x = wt_\xi \in W_{\mathrm{af}}$, we have*

$$\widehat{\mathbf{G}}_{\Gamma_1} \circ \widehat{\mathbf{G}}_{\Gamma_2}(x) = \widehat{\mathbf{G}}_{\Gamma_2} \circ \widehat{\mathbf{G}}_{\Gamma_1}(x) = \widehat{\mathbf{G}}_\Gamma(x).$$

These identities are realized combinatorially via the bijections (5.2), (5.6), and the generalized quantum Yang–Baxter moves.

Proof. As in the proof of Theorem 5.10, it suffices to prove the second equality. By iterating the definition (5.5) and by also using (5.1), we obtain

$$\begin{aligned}
&\widehat{\mathbf{G}}_{\Gamma_2} \circ \widehat{\mathbf{G}}_{\Gamma_1}(x) \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\langle \mu, \xi \rangle - \mathrm{height}(A) - |\boldsymbol{\psi}|} \\
&\qquad\qquad \times e^{\mathrm{wt}(A)} \widehat{\mathbf{G}}_{\Gamma_2}(\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\psi})}) \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\langle \mu, \xi \rangle - \mathrm{height}(A) - |\boldsymbol{\psi}| - |\boldsymbol{\omega}|} \\
&\qquad\qquad \times e^{\mathrm{wt}(A)} \mathbf{G}_{\Gamma_2}(\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\psi})}) t_{\iota(\boldsymbol{\omega})} \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} q^{-|\boldsymbol{\psi}| - |\boldsymbol{\omega}| - \langle \nu, \iota(\boldsymbol{\psi}) \rangle} \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\langle \mu, \xi \rangle - \mathrm{height}(A)} \\
&\qquad\qquad \times e^{\mathrm{wt}(A)} \mathbf{G}_{\Gamma_2}(\mathrm{end}(A) t_{\xi + \mathrm{down}(A)}) t_{\iota(\boldsymbol{\psi}) + \iota(\boldsymbol{\omega})} \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} q^{-|\boldsymbol{\psi}| - |\boldsymbol{\omega}| - \langle \nu, \iota(\boldsymbol{\psi}) \rangle} \mathbf{G}_{\Gamma_2} \circ \mathbf{G}_{\Gamma_1}(w t_\xi) t_{\iota(\boldsymbol{\psi}) + \iota(\boldsymbol{\omega})} \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} q^{-|\boldsymbol{\psi} * \boldsymbol{\omega}|} \mathbf{G}_{\Gamma}(w t_\xi) t_{\iota(\boldsymbol{\psi} * \boldsymbol{\omega})} \\
&= \widehat{\mathbf{G}}_{\Gamma}(x).
\end{aligned}$$

The last two equalities are based on the bijection (5.6), Lemma 5.13, and Theorem 5.10. ∎

Remarks 5.15. A few words are in order:

(1) Theorem 5.14 exhibits a combinatorial realization of the symmetry of the general Chevalley formula [14, Theorem 33] coming from commutativity in equivariant K-theory.

(2) Corollary 5.11 can be extended to the setup of Theorem 5.14.

5.3. Identity of Chevalley type for graded characters

As an application of the results in Sections 5.1 and 5.2, we obtain an identity of "Chevalley type" for the graded characters of Demazure submodules of (level-zero) extremal weight modules over a quantum affine algebra.

Let $\mathfrak{g}_{\mathrm{af}}$ be the untwisted affine Lie algebra whose underlying finite-dimensional simple Lie algebra is $\mathfrak{g}$, and let $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ denote the quantum affine algebra associated to $\mathfrak{g}_{\mathrm{af}}$ with Chevalley generator $E_i, F_i \in U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$, $i \in I_{\mathrm{af}} = I \sqcup \{0\}$, where q is an indeterminate. We denote by

$$U_{\mathsf{q}}^{-}(\mathfrak{g}_{\mathrm{af}}) := \langle F_i \rangle_{i \in I_{\mathrm{af}}} \subset U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$$

the subalgebra of $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ generated by $\{F_i \mid i \in I_{\mathrm{af}}\}$. Also, let

$$W_{\mathrm{af}} = W \ltimes \{t_\xi \mid \xi \in Q^\vee\} \simeq W \ltimes Q^\vee$$

be the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}$, where t_ξ, $\xi \in Q^\vee$, denotes the translation by ξ (see [6, Proposition 6.5]).

For each $\lambda \in P^+$ (regarded as a level-zero affine weight), denote by $V(\lambda)$ the *level-zero extremal weight module* of extremal weight λ over $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$, which is equipped with a family $\{v_x\}_{x \in W_{\mathrm{af}}} \subset V(\lambda)$ of extremal weight vectors, where $v_x \in V(\lambda)$, $x \in W_{\mathrm{af}}$, is an extremal weight vector of weight $x\lambda$ (see [7, Proposition 8.2.2]). For $x \in W_{\mathrm{af}}$ and $\lambda \in P^+$, the *Demazure submodule* $V_x^-(\lambda)$ is defined by

$$V_x^-(\lambda) := U_{\mathsf{q}}^-(\mathfrak{g}_{\mathrm{af}}) v_x.$$

We denote by $\operatorname{gch} V_x^-(\lambda)$ the *graded character* of $V_x^-(\lambda)$ (see [8, Section 2.4]). If $x = wt_\xi$ with $w \in W$ and $\xi \in Q^\vee$, then we know that $\operatorname{gch} V_x^-(\lambda) \in \mathbb{Z}[P][\![q^{-1}]\!]q^{-\langle \lambda, \xi \rangle}$; in fact, we know that $\operatorname{gch} V_w^-(\lambda) \in \mathbb{Z}[\![q^{-1}]\!][P]$ for $w \in W$.

We will prove the following identity for the graded characters of Demazure submodules, which is a representation-theoretic analogue of the general Chevalley formula for the equivariant K-group of semi-infinite flag manifolds ([14, Theorem 33]).

Theorem 5.16. *Let $\mu \in P^+$ and $x \in W_{\mathrm{af}}$. We write x as $x = wt_\xi$, with $w \in W$ and $\xi \in Q^\vee$. Take $\lambda \in P$ such that $\mu + \lambda \in P^+$, and let Γ be an arbitrary reduced λ-chain. Then we have*

$$\operatorname{gch} V_x^-(\mu + \lambda) = \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle - |\boldsymbol{\chi}|} e^{\mathrm{wt}(A)} \times \operatorname{gch} V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\chi})}}(\mu). \tag{5.7}$$

Remark 5.17. The right-hand side of (5.7) is identical to zero if $\mu + \lambda \notin P^+$; the proof is given in Appendix B.

Although Theorem 5.16 can be proved in a parallel way to [14, Theorem 33], we show that it follows immediately from the results in Sections 5.1 and 5.2.

Now we recall two special cases of Theorem 5.16, i.e., the cases that λ is dominant or anti-dominant. The following theorem gives the identity for dominant weights; this is a restatement of [21, Corollary C.1] in terms of the quantum alcove model, which is given by exactly the same argument as for [14, Theorem 29]. Here, for a dominant weight $\lambda \in P^+$, the *lex λ-chain* is a λ-chain constructed in [20, Proposition 4.2].

Theorem 5.18 (cf. [14, Theorem 29] and [21, Corollary C.1]; see also [8, Proposition D.1]). *Let $\mu, \lambda \in P^+$, and $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$. Let Γ be the lex λ-chain. Then we have*

$$\operatorname{gch} V_x^-(\mu+\lambda) = \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\chi \in \overline{\mathrm{Par}}(\lambda)} q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle-|\chi|} e^{\operatorname{wt}(A)} \times \operatorname{gch} V^-_{\operatorname{end}(A)t_{\xi+\operatorname{down}(A)+\iota(\chi)}}(\mu).$$

Also, the following theorem gives the identity for anti-dominant weights; this is a restatement of [21, Corollary 3.15] in terms of the quantum alcove model, which is given by exactly the same argument as for [14, Theorem 32]. Here, following [14, Section 4.2], the *lex λ-chain* for an anti-dominant weight $\lambda \in -P^+$ is defined to be the reverse of the lex $(-\lambda)$-chain with all roots negated in the lex $(-\lambda)$-chain; namely, for the given lex $(-\lambda)$-chain $(\beta_1, \dots, \beta_r)$, the lex λ-chain is defined to be $(-\beta_r, \dots, -\beta_1)$.

Theorem 5.19 (cf. [14, Theorem 32] and [21, Corollary 3.15]; see also [8, Proposition D.1]). *Let $\mu \in P^+$, and $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$. Take $\lambda \in -P^+$ such that $\mu+\lambda \in P^+$, and let Γ be the lex λ-chain. Then we have*

$$\operatorname{gch} V_x^-(\mu+\lambda) = \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{|A|} q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(A)} \operatorname{gch} V^-_{\operatorname{end}(A)t_{\xi+\operatorname{down}(A)}}(\mu).$$

Proof of Theorem 5.16. Let μ, λ, x and Γ be as in the statement of Theorem 5.16. Write $\lambda = \lambda^+ + \lambda^-$, with $\lambda^+ \in P^+$ and $\lambda^- \in -P^+$ given by

$$\lambda^+ := \sum_{i \in I} \max\{\langle\lambda, \alpha_i^\vee\rangle, 0\}\varpi_i, \quad \lambda^- := \sum_{i \in I} \min\{\langle\lambda, \alpha_i^\vee\rangle, 0\}\varpi_i.$$

Note that the weight decomposition $\lambda = \lambda^+ + \lambda^-$ is cancellation-free. Take lex λ^+-chain (resp., lex λ^--chain) Γ^+ (resp., Γ^-). Note that the two chains of roots are reduced, and Γ^+ consists of positive roots, while Γ^- consists of negative roots. Define a λ-chain Γ_0 as the concatenation $\Gamma^+ * \Gamma^-$, which is weakly reduced by Proposition 5.8.

By Theorems 5.14 and 5.6, for $x \in W_{\mathrm{af}}$ we have

$$\widehat{\mathbf{G}}_{\Gamma^-} \circ \widehat{\mathbf{G}}_{\Gamma^+}(x) = \widehat{\mathbf{G}}_{\Gamma_0}(x) = \widehat{\mathbf{G}}_{\Gamma}(x). \tag{5.8}$$

Now consider the correspondence $x \mapsto \operatorname{gch} V_x^-(\mu)$ for $x \in W_{\mathrm{af}}$, which defines an $R[P]$-module homomorphism $R[P][W_{\mathrm{af}}] \to \mathbb{Z}((q^{-1}))[P]$. Under this homomorphism, $\widehat{\mathbf{G}}_\Gamma(x)$ is mapped to the right-hand side of (5.7). By (5.8), we obtain the same result by applying the homomorphism to $\widehat{\mathbf{G}}_{\Gamma^-} \circ \widehat{\mathbf{G}}_{\Gamma^+}(x)$. We observe that doing this parallels the process of expanding

$$\operatorname{gch} V_x^-(\mu+\lambda) = \operatorname{gch} V_x^-((\mu+\lambda^-)+\lambda^+)$$

in terms of gch $V_y^-(\mu+\lambda^-)$ by Theorem 5.18, followed by expanding the result in terms of gch $V_y^-(\mu)$ by Theorem 5.19; here we use the fact that $\mu+\lambda \in P^+$ implies $\mu+\lambda^- \in P^+$. The mentioned observation proves that by applying the above homomorphism to $\widehat{\mathbf{G}}_{\Gamma^-} \circ \widehat{\mathbf{G}}_{\Gamma^+}(x)$, we obtain gch $V_x^-(\mu+\lambda)$. We conclude that the right-hand side of (5.7) coincides with gch $V_x^-(\mu+\lambda)$. ■

Remark 5.20. Theorem 5.16 can also be proved by using the λ-chain $\Gamma_0^* := \Gamma^- * \Gamma^+$ instead of Γ_0.

5.4. Towards a signed crystal structure on the quantum alcove model for an arbitrary weight

Crystals are colored directed graphs encoding the structure of representations of the quantum algebra $U_{\mathsf{q}}(\mathfrak{g})$ in the limit $\mathsf{q} \to 0$, where $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra. The vertices B of the crystal correspond to the elements of the crystal basis for the representation, and the edges correspond to the action of the Chevalley generators e_i, f_i in the above limit. Formally, we define the *crystal operators*

$$\widetilde{e}_i, \widetilde{f}_i : B \to B \sqcup \{\mathbf{0}\},$$

where the value $\mathbf{0}$ corresponds to the operators being undefined, and $\widetilde{e}_i$ is a partial inverse to $\widetilde{f}_i$. These operators are subject to several conditions; see, e.g., [5] for all the background information on crystals.

We define a *signed crystal* simply as a crystal together with a sign function on the vertex set B; note that we do not require the crystal operators to preserve signs. An isomorphism of signed crystals B and B' is a sijection between B and B' which commutes with the crystal operators.

Given a dominant weight λ and a reduced λ-chain Γ, an affine crystal structure was constructed on $\mathcal{A}(e,\Gamma)$ in [12], which was then shown in [17] to uniformly describe tensor products of single-column Kirillov–Reshetikhin crystals of quantum affine algebras; note that for all $w \in W$, the set $\mathcal{A}(w,\Gamma)$ can be equipped with an affine crystal structure through the bijection with the affine crystal $\mathcal{A}(e,\Gamma)$, which is afforded by [14, Proposition 28] and quantum Yang–Baxter moves. Also, for an anti-dominant weight λ' and a reduced λ'-chain Γ', an argument similar to that in the proof of [19, Theorem 8.6] yields a signed crystal structure on the set

$$\mathcal{A}(\Gamma', w) := \{A \in \mathcal{A}(\Gamma') \mid \mathrm{end}(A) = w\} \quad \text{with } \mathcal{A}(\Gamma') := \bigsqcup_{w \in W} \mathcal{A}(w, \Gamma'),$$

which is in bijective correspondence with $\mathcal{A}(ww_\circ, (\Gamma')^*)$, where $(\Gamma')^*$ is a reduced $w_\circ(\lambda')$-chain of roots dual to Γ'. In both the dominant and anti-dominant cases, as well as in general (below), we use the same sign function as throughout the paper; in particular, in the dominant case the sign function is identically 1.

We now propose the construction of a (partial) signed crystal structure on

$$\mathcal{A}(\Gamma) := \bigsqcup_{w \in W} \mathcal{A}(w, \Gamma),$$

where Γ is a reduced chain of roots corresponding to an arbitrary weight λ. We use the same objects and facts as in Sections 5.2 and 5.3, namely $\lambda = \lambda^+ + \lambda^-$, the lex λ^+-chain (resp., lex λ^--chain) Γ^+ (resp., Γ^-), their concatenation Γ_0^* (not Γ_0), and the bijection (5.2). Based on these facts, we can define a signed crystal structure on $\mathcal{A}(\Gamma_0^*)$ by decomposing it as

$$\mathcal{A}(\Gamma_0^*) = \bigsqcup_{w \in W} (\mathcal{A}(\Gamma^-, w) * \mathcal{A}(w, \Gamma^+)),$$

where

$$\mathcal{A}(\Gamma^-, w) := \{A \in \mathcal{A}(\Gamma^-) \mid \mathrm{end}(A) = w\}$$

for $w \in W$; note that the concatenation $\mathcal{A}(\Gamma^-, w) * \mathcal{A}(w, \Gamma^+)$ is a well-defined crystal for all $w \in W$. On another hand, we know that Γ_0^* can be related to the reduced λ-chain Γ by the procedures (YB) and (D). By propagating the signed crystal structures through the corresponding generalized quantum Yang–Baxter moves, we end up with a (partial) signed crystal structure on $\mathcal{A}(\Gamma)$, that is, a signed crystal structure for which crystal operators are defined only on a subset of $\mathcal{A}(\Gamma)$.

We state the following conjecture.

Conjecture 5.21. *For each reduced chain of roots Γ corresponding to an arbitrary weight λ, there exists a signed crystal structure on the whole of $\mathcal{A}(\Gamma)$, which extends the (partial) crystal structure defined above. Moreover, the sijection (I_1, I_2, Y) defining a generalized quantum Yang–Baxter move in Theorem 3.2 commutes with the crystal operators defined on $\mathcal{A}(\Gamma)$.*

A. An example of quantum Yang–Baxter moves in type C_2

Based on Proposition 4.3, we explain how to construct quantum Yang–Baxter moves explicitly in a specific case. We assume that $\mathfrak{g}$ is of type C_2. Let Π, Π' be the sequences of roots introduced in Section 4.2. We consider the case that $v = s_2$ and $\Pi = (-2\alpha_1 - \alpha_2, -\alpha_1, \alpha_2, \alpha_1 + \alpha_2)$. Note that $\Pi' = (\alpha_1 + \alpha_2, \alpha_2, -\alpha_1, -2\alpha_1 - \alpha_2)$. Let us construct an explicit matching between a certain subset of $\mathcal{P}(v, \Pi)$ and that of $\mathcal{P}(v, \Pi')$, and also sign-reversing involutions outside of those subsets.

$\mathbf{p} \in \mathcal{P}(v, \Pi)$	$\mathrm{end}(\mathbf{p})$	$\mathrm{down}(\mathbf{p})$
$\mathbf{p}_1$	e	$\alpha_2^\vee$
$\mathbf{p}_2$	s_2	0
$\mathbf{p}_3$	$s_1 s_2$	0
$\mathbf{p}_4$	$s_2 s_1$	0
$\mathbf{p}_5$	$s_1 s_2 s_1$	0
$\mathbf{p}_6$	$s_2 s_1 s_2$	0

Table 1. Statistics of elements $\mathbf{p} \in \mathcal{P}(v, \Pi)$.

$\mathbf{q} \in \mathcal{P}(v, \Pi')$	$\mathrm{end}(\mathbf{q})$	$\mathrm{down}(\mathbf{q})$
$\mathbf{q}_1$	e	$\alpha_1^\vee + \alpha_2^\vee$
$\mathbf{q}_2$	e	$\alpha_1^\vee + \alpha_2^\vee$
$\mathbf{q}_3$	e	$\alpha_2^\vee$
$\mathbf{q}_4$	s_1	$\alpha_2^\vee$
$\mathbf{q}_5$	s_1	$\alpha_2^\vee$
$\mathbf{q}_6$	s_2	0
$\mathbf{q}_7$	$s_1 s_2$	0
$\mathbf{q}_8$	$s_2 s_1$	$\alpha_2^\vee$
$\mathbf{q}_9$	$s_2 s_1$	$\alpha_2^\vee$
$\mathbf{q}_{10}$	$s_2 s_1$	0
$\mathbf{q}_{11}$	$s_1 s_2 s_1$	0
$\mathbf{q}_{12}$	$s_2 s_1 s_2$	0

Table 2. Statistics of elements $\mathbf{q} \in \mathcal{P}(v, \Pi')$.

Recall the matrices of the operators $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$,$\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$ calculated in Section 4.4. In particular, the v-column of the matrix of the operator $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$ (resp., $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$) is ${}^t(Q_2, 0, 1, 1, 1, 1, 1, 0)$ (resp., ${}^t(2Q_1Q_2 + Q_2, 2Q_2, 1, 1, 2Q_2 + 1, 1, 1, 0)$). For example, the (e, v)-entry of the matrix of the operator $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$ is $2Q_1Q_2 + Q_2$. Therefore, we deduce from equation (4.3) that there exist exactly three Π'-compatible directed paths $\mathbf{r}^{(1)}, \mathbf{r}^{(2)}, \mathbf{r}^{(3)}$ such that

- $\mathbf{r}^{(j)}$ starts at $v = s_2$ for $j = 1, 2, 3$,
- $\mathrm{end}(\mathbf{r}^{(j)}) = e$, $j = 1, 2, 3$,
- $\mathrm{down}(\mathbf{r}^{(j)}) = \alpha_1^\vee + \alpha_2^\vee$, $j = 1, 2$, and
- $\mathrm{down}(\mathbf{r}^{(3)}) = \alpha_2^\vee$;

remark that $Q^{\alpha_1^\vee + \alpha_2^\vee} = Q_1Q_2$ and $Q^{\alpha_2^\vee} = Q_2$. Similarly, we see that there exist six Π-compatible directed paths $\mathbf{p}_1, \dots, \mathbf{p}_6$ such that

$$\mathcal{P}(v, \Pi) = \{\mathbf{p}_1, \dots, \mathbf{p}_6\}.$$

Also, there exist twelve Π'-compatible directed paths $\mathbf{q}_1, \dots, \mathbf{q}_{12}$ such that

$$\mathcal{P}(v, \Pi') = \{\mathbf{q}_1, \dots, \mathbf{q}_{12}\}.$$

For $\mathbf{p} \in \mathcal{P}(v, \Pi)$ (resp., $\mathbf{q} \in \mathcal{P}(v, \Pi')$), the statistics $\mathrm{end}(\mathbf{p})$, $\mathrm{down}(\mathbf{p})$ (resp., $\mathrm{end}(\mathbf{q})$, $\mathrm{down}(\mathbf{q})$) are given in Tables 1, 2.

Note that $\mathbf{p} \in \mathscr{P}(v, \Pi)$ and $\mathbf{q} \in \mathscr{P}(v, \Pi')$ are explicitly written as follows:

$\mathbf{p}_1 \colon s_2 \xrightarrow{\alpha_2} e$; $\quad$ $\mathbf{p}_2 \colon s_2$ (the trivial directed path);

$\mathbf{p}_3 \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2$; $\quad$ $\mathbf{p}_4 \colon s_2 \xrightarrow{\alpha_1} s_2 s_1$;

$\mathbf{p}_5 \colon s_2 \xrightarrow{\alpha_1} s_2 s_1 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2 s_1$; $\quad$ $\mathbf{p}_6 \colon s_2 \xrightarrow{\alpha_1} s_2 s_1 \xrightarrow{\alpha_2} s_2 s_1 s_2$;

$\mathbf{q}_1 \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2 \xrightarrow{\alpha_2} s_1 \xrightarrow{\alpha_1} e$; $\quad$ $\mathbf{q}_2 \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2 \xrightarrow{\alpha_1} s_1 s_2 s_1 \xrightarrow{2\alpha_1+\alpha_2} e$;

$\mathbf{q}_3 \colon s_2 \xrightarrow{\alpha_2} e$; $\quad$ $\mathbf{q}_4 \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2 \xrightarrow{\alpha_2} s_1$;

$\mathbf{q}_5 \colon s_2 \xrightarrow{\alpha_2} e \xrightarrow{\alpha_1} s_1$; $\quad$ $\mathbf{q}_6 \colon s_2$ (the trivial directed path);

$\mathbf{q}_7 \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2$; $\quad$ $\mathbf{q}_8 \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2 \xrightarrow{\alpha_2} s_1 \xrightarrow{2\alpha_1+\alpha_2} s_2 s_1$;

$\mathbf{q}_9 \colon s_2 \xrightarrow{\alpha_2} e \xrightarrow{\alpha_1} s_1 \xrightarrow{2\alpha_1+\alpha_2} s_2 s_1$; $\quad$ $\mathbf{q}_{10} \colon s_2 \xrightarrow{\alpha_1} s_2 s_1$;

$\mathbf{q}_{11} \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2 \xrightarrow{\alpha_1} s_1 s_2 s_1$; $\quad$ $\mathbf{q}_{12} \colon s_2 \xrightarrow{\alpha_1+\alpha_2} s_1 s_2 \xrightarrow{2\alpha_1+\alpha_2} s_2 s_1 s_2$.

Thus, if we set

$$\begin{aligned}
\mathscr{P}_0(v, \Pi) &:= \mathscr{P}(v, \Pi), \\
\mathscr{P}_0(v, \Pi') &:= \{\mathbf{q}_3, \mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_{10}, \mathbf{q}_{11}, \mathbf{q}_{12}\} \subset \mathscr{P}(v, \Pi'), \\
\mathscr{P}_0^C(v, \Pi') &:= \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_8, \mathbf{q}_9\} = \mathscr{P}(v, \Pi') \setminus \mathscr{P}_0(v, \Pi'),
\end{aligned}$$

then we obtain the following bijection $Y^{v,\Pi} \colon \mathscr{P}_0(v, \Pi) \to \mathscr{P}_0(v, \Pi')$ and involution $I_2^{v,\Pi'}$ on $\mathscr{P}_0^C(v, \Pi')$ which preserve $\mathrm{end}(\cdot)$ and $\mathrm{down}(\cdot)$:

$$\begin{aligned}
Y^{v,\Pi} &\colon \mathbf{p}_1 \mapsto \mathbf{q}_3, \quad \mathbf{p}_2 \mapsto \mathbf{q}_6, \quad \mathbf{p}_3 \mapsto \mathbf{q}_7, \quad \mathbf{p}_4 \mapsto \mathbf{q}_{10}, \quad \mathbf{p}_5 \mapsto \mathbf{q}_{11}, \quad \mathbf{p}_6 \mapsto \mathbf{q}_{12}; \\
I_2^{v,\Pi'} &\colon \mathbf{q}_1 \mapsto \mathbf{q}_2, \quad \mathbf{q}_2 \mapsto \mathbf{q}_1, \quad \mathbf{q}_4 \mapsto \mathbf{q}_5, \quad \mathbf{q}_5 \mapsto \mathbf{q}_4, \quad \mathbf{q}_8 \mapsto \mathbf{q}_9, \quad \mathbf{q}_9 \mapsto \mathbf{q}_8.
\end{aligned}$$

These maps give the correspondence $\mathbf{p} \mapsto \mathbf{p}'$ in Proposition 4.3.

Now, let us give an example of generalized quantum Yang–Baxter moves. Let $\lambda \in P$. Take λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by the Yang–Baxter transformation (YB). Let $w \in W$. As in equations (3.1) and (3.2), we take $\Gamma_1^{(k)}$, $\Gamma_2^{(k)}$, $k = 1, 2, 3$. Also, as in equations (3.3) and (3.4), we take $A^{(k)}$ (resp., $B^{(k)}$), $k = 1, 2, 3$, for $A \in \mathcal{A}(w, \Gamma_1)$ (resp., $B \in \mathcal{A}(w, \Gamma_2)$). In this example, we consider the case that

$$\Gamma_1^{(2)} = \Pi \quad \text{and} \quad \Gamma_2^{(2)} = \Pi'.$$

By the consideration above, we can give an explicit description of quantum Yang–Baxter moves for $A \in \mathcal{A}(w, \Gamma_1)$ (resp., $B \in \mathcal{A}(w, \Gamma_2)$) such that $\mathrm{end}(A^{(1)}) = s_2$ (resp., $\mathrm{end}(B^{(1)}) = s_2$), as given in Tables 3, 4.

$A^{(2)}$	$\mathbf{p}(A^{(2)})$	$\mathbf{p}(Y(A)^{(2)}) = Y^{v,\Pi}(\mathbf{p}(A^{(2)}))$	$Y(A)^{(2)}$
$\emptyset$	$\mathbf{p}_2$	$\mathbf{q}_6$	$\emptyset$
$\{t+2\}$	$\mathbf{p}_4$	$\mathbf{q}_{10}$	$\{t+3\}$
$\{t+3\}$	$\mathbf{p}_1$	$\mathbf{q}_3$	$\{t+2\}$
$\{t+4\}$	$\mathbf{p}_3$	$\mathbf{q}_7$	$\{t+1\}$
$\{t+2, t+3\}$	$\mathbf{p}_6$	$\mathbf{q}_{12}$	$\{t+1, t+4\}$
$\{t+2, t+4\}$	$\mathbf{p}_5$	$\mathbf{q}_{11}$	$\{t+1, t+3\}$

Table 3. List of $Y(A)$ for $A \in \mathcal{A}_0(w, \Gamma_1)$ such that $\mathrm{end}(A^{(1)}) = s_2$.

$B^{(2)}$	$\mathbf{p}(B^{(2)})$	$\mathbf{p}(I_2(B)^{(2)}) = I_2^{v,\Pi'}(\mathbf{p}(B^{(2)}))$	$I_2(B)^{(2)}$
$\{t+1, t+2\}$	$\mathbf{q}_4$	$\mathbf{q}_5$	$\{t+2, t+3\}$
$\{t+2, t+3\}$	$\mathbf{q}_5$	$\mathbf{q}_4$	$\{t+1, t+2\}$
$\{t+1, t+2, t+3\}$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\{t+1, t+3, t+4\}$
$\{t+1, t+2, t+4\}$	$\mathbf{q}_8$	$\mathbf{q}_9$	$\{t+2, t+3, t+4\}$
$\{t+1, t+3, t+4\}$	$\mathbf{q}_2$	$\mathbf{q}_1$	$\{t+1, t+2, t+3\}$
$\{t+2, t+3, t+4\}$	$\mathbf{q}_9$	$\mathbf{q}_8$	$\{t+1, t+2, t+4\}$

Table 4. List of $I_2(B)$ for $B \in \mathcal{A}_0^C(w, \Gamma_2)$ such that $\mathrm{end}(B^{(1)}) = s_2$.

B. The right-hand side of the identity of Chevalley type for graded characters

We show that the right-hand side of (5.7) is identical to zero if $\mu + \lambda \notin P^+$.

Proposition B.1. *Let $\mu \in P^+$, and $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$. Take $\lambda \in P$ such that $\mu + \lambda \notin P^+$, and let Γ be an arbitrary reduced λ-chain. Then we have*

$$\sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\chi \in \overline{\mathrm{Par}}(\lambda)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle - |\chi|} e^{\mathrm{wt}(A)} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\chi)}}(\mu) = 0.$$

In the proof of Proposition B.1, we make use of the following equalities for graded characters.

Proposition B.2 ([8, Proposition D.1]). *For each $x \in W_{\mathrm{af}}$, $\xi \in Q^\vee$, and $\lambda \in P^+$, we have*

$$\operatorname{gch} V^-_{xt_\xi}(\lambda) = q^{-\langle \lambda, \xi \rangle} \operatorname{gch} V^-_x(\lambda).$$

Proposition B.3 (cf. [21, Appendix B]). *Let $\mu \in P^+$ and $x \in W$. Take $\lambda \in -P^+$ such that $\mu + \lambda \notin P^+$, and let Γ be the lex λ-chain. Then we have*

$$\sum_{A \in \mathcal{A}(x,\Gamma)} (-1)^{|A|} q^{-\operatorname{height}(A)} e^{\operatorname{wt}(A)} \operatorname{gch} V^{-}_{\operatorname{end}(A) t_{\operatorname{down}(A)}}(\mu) = 0.$$

Remark B.4. In [21], Proposition B.3 is stated and proved in terms of semi-infinite Lakshmibai–Seshadri paths.

Proof of Proposition B.1. By considering $\lambda^{\pm}$, $\Gamma^{\pm}$, Γ_0, and by using Theorems 5.14 and 5.6 as in the proof of Theorem 5.16 (cf. (5.8)), we obtain:

$$\begin{aligned}
&\sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\chi \in \overline{\operatorname{Par}}(\lambda)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle - |\chi|} e^{\operatorname{wt}(A)} \operatorname{gch} V^{-}_{\operatorname{end}(A) t_{\xi + \operatorname{down}(A) + \iota(\chi)}}(\mu) \\
&= \sum_{A \in \mathcal{A}(w,\Gamma_0)} \sum_{\chi \in \overline{\operatorname{Par}}(\lambda)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle - |\chi|} e^{\operatorname{wt}(A)} \operatorname{gch} V^{-}_{\operatorname{end}(A) t_{\xi + \operatorname{down}(A) + \iota(\chi)}}(\mu) \\
&= \sum_{A \in \mathcal{A}(w,\Gamma+)} \sum_{B \in \mathcal{A}(\operatorname{end}(A),\Gamma-)} \sum_{\chi \in \overline{\operatorname{Par}}(\lambda+)} (-1)^{|B|} \\
&\qquad \times q^{-\operatorname{height}(A) - \operatorname{height}(B) - \langle \lambda^-, \operatorname{down}(A) + \iota(\chi) \rangle - \langle \lambda, \xi \rangle - |\chi|} \\
&\qquad \times e^{\operatorname{wt}(A) + \operatorname{wt}(B)} \operatorname{gch} V^{-}_{\operatorname{end}(B) t_{\xi + \operatorname{down}(A) + \operatorname{down}(B) + \iota(\chi)}}(\mu) \\
&= \sum_{A \in \mathcal{A}(w,\Gamma+)} \sum_{B \in \mathcal{A}(\operatorname{end}(A),\Gamma-)} \sum_{\chi \in \overline{\operatorname{Par}}(\lambda+)} (-1)^{|B|} \\
&\qquad \times q^{-\operatorname{height}(A) - \operatorname{height}(B) - \langle \lambda^-, \operatorname{down}(A) + \iota(\chi) \rangle - \langle \lambda, \xi \rangle - |\chi|} \\
&\qquad \times q^{-\langle \mu, \xi + \operatorname{down}(A) + \iota(\chi) \rangle} e^{\operatorname{wt}(A) + \operatorname{wt}(B)} \operatorname{gch} V^{-}_{\operatorname{end}(B) t_{\operatorname{down}(B)}}(\mu) \\
&= q^{-\langle \mu + \lambda, \xi \rangle} \sum_{A \in \mathcal{A}(w,\Gamma+)} \sum_{\chi \in \overline{\operatorname{Par}}(\lambda+)} q^{-\operatorname{height}(A) - \langle \lambda^- + \mu, \operatorname{down}(A) + \iota(\chi) \rangle - |\chi|} e^{\operatorname{wt}(A)} \\
&\qquad \times \sum_{B \in \mathcal{A}(\operatorname{end}(A),\Gamma-)} (-1)^{|B|} q^{-\operatorname{height}(B)} e^{\operatorname{wt}(B)} \operatorname{gch} V^{-}_{\operatorname{end}(B) t_{\operatorname{down}(B)}}(\mu);
\end{aligned} \tag{B.1}$$

here the third equality follows by Proposition B.2. Since $\mu + \lambda \notin P^+$, it follows that $\mu + \lambda^- \notin P^+$. Therefore, we deduce by Proposition B.3 that

$$\sum_{B \in \mathcal{A}(\operatorname{end}(A),\Gamma-)} (-1)^{|B|} q^{-\operatorname{height}(B)} e^{\operatorname{wt}(B)} \operatorname{gch} V^{-}_{\operatorname{end}(B) t_{\operatorname{down}(B)}}(\mu) = 0$$

for each $A \in \mathcal{A}(w, \Gamma^+)$, and hence that (B.1) is identical to zero, as needed. ∎

Acknowledgments. The authors would like to thank Daisuke Sagaki for helpful discussions. The authors are also grateful to Shunsuke Tsuchioka for useful advices. An extended abstract of this work appeared in the Proceedings of the 34th International Conference on Formal Power Series and Algebraic Combinatorics [10].

Funding. T. Kouno was partly supported by JSPS Grant-in-Aid for Scientific Research 22J00874, 22KJ2908, and 20J12058. C. Lenart was partly supported by the NSF grant DMS-1855592. S. Naito was partly supported by JSPS Grant-in-Aid for Scientific Research (B) 16H03920 and (C) 21K03198.

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Received 23 August 2022; revised 19 April 2023.

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