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Annular webs and Levi subalgebras

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Abstract. For any Levi subalgebra of the form $\mathfrak{l} = \mathfrak{gl}_{l_1} \oplus \cdots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n$ we construct a quotient of the category of annular quantum $\mathfrak{gl}_n$ webs that is equivalent to the category of finite-dimensional representations of quantum $\mathfrak{l}$ generated by exterior powers of the vector representation. This can be interpreted as an annular version of skew Howe duality, gives a description of the representation category of $\mathfrak{l}$ by additive idempotent completion, and a web version of the generalized blob algebra.

1. Introduction

Throughout fix $n, l_1, \ldots, l_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^{d} l_i = n$.

1A. Webs, and Schur–Weyl and Howe duality

The so-called *Schur–Weyl duality* has played a key role ever since the early days of representation theory. It relates representations of the symmetric group S_m and the general linear group $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$, and has been generalized in many ways. The representation used to relate these two groups is $(\mathbb{C}^n)^{\otimes m}$.

Let $\mathrm{L} = \mathrm{GL}_{l_1} \times \cdots \times \mathrm{GL}_{l_d} \subseteq \mathrm{GL}_n$, and let us for simplicity stay over $\mathbb{C}$ for now. Two generalizations of Schur–Weyl duality are of crucial importance for this paper. Firstly, the *Schur–Weyl duality of* $(\mathbb{Z}/d\mathbb{Z}) \wr \mathrm{S}_m$ (that is, type $G(d,1,m)$) and L from [5, 19, 33] (see also [26] for a nice and self-contained discussion of this duality). Here the underlying representation is again $(\mathbb{C}^n)^{\otimes m}$. Secondly, *skew (type A) Howe duality*, see [17, 18], relating GL_N and GL_n via their action on the exterior algebra $\bigwedge^{\bullet}(\mathbb{C}^N \otimes \mathbb{C}^n)$.

As explained in [9], a diagrammatic interpretation of skew Howe duality is given by *(exterior* GL_n*) webs*. (The same diagrammatics goes under many names, including birdtracks [11] or spiders [22].) In some sense, in [9] skew Howe duality relating

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$\mathrm{GL}_{N\in\mathbb{Z}_{\geq 0}}$ and GL_n takes the form of an equivalence between the category of webs and the category of GL_n-representations generated by $\{\bigwedge^k\mathbb{C}^n \mid k\in\{1,\dots,n\}\}$, with the web category being obtained by using all GL_N for $N\in\mathbb{Z}_{\geq 0}$. After additive idempotent completion webs even give a diagrammatic interpretation of all finite-dimensional GL_n-representations.

In this paper we show that an explicit quotient of the category of *annular (exterior* GL_n*) webs* is equivalent to the category of L-representations generated by the set $\{\bigwedge^k\mathbb{C}^n \mid k\in\{1,\dots,n\}\}$. As before, additive idempotent completion gives a description of all finite-dimensional L-representations. This, in some sense, is a form of what could be called *annular skew Howe duality* (we avoid the notion affine as its meaning is context depending) or *skew-type* $G(d,1,m\in\mathbb{Z}_{\geq 0})$ *Howe duality*.

1B. The main result and relations to other works

We now give a few details and change to the universal enveloping algebras. We consider a *Levi subalgebra* of the form $\ell=\mathfrak{gl}_{l_1}\oplus\cdots\oplus\mathfrak{gl}_{l_d}\subseteq\mathfrak{gl}_n$ (in this paper we write ℓ instead of the usual notation $\mathfrak{l}$ for readability). Let $\mathbb{K}_q$ denote a field containing an element $q\in\mathbb{K}_q$ that is not a root of unity and additional variables $\mathbb{U}=\{u_1,\dots,u_d\}$ and their inverses, and let further $\mathbb{K}_1$ denote a field of characteristic zero containing $\mathbb{U}$ and their inverses. With these ground fields the category of finite-dimensional $U_q(\ell)$-representations over $\mathbb{K}_q$ respectively of finite-dimensional $U_1(\ell)$-representations over $\mathbb{K}_1$ are semisimple. (We should warn the reader: as explained in the main body of the text there are some nontrivial quantization issues and we carefully need to distinguish the two cases over $\mathbb{K}_q$ and $\mathbb{K}_1$.)

In Section 4 we define a $\mathbb{K}_q$-linear category of annular webs $\mathbf{AWeb}_q\mathfrak{gl}_n$ as well as a quotient $\mathbf{AWeb}_q\ell$ by evaluating essential circles using the variables $\mathbb{U}$ and their inverses. Similarly over $\mathbb{K}_1$, where we write $\mathbf{AWeb}_1\mathfrak{gl}_n$ and $\mathbf{AWeb}_1\ell$. Let $\mathbf{Fund}_q\ell$ respectively $\mathbf{Fund}_1\ell$ denote the categories of $U_q(\ell)$- and $U_1(\ell)$-representations generated by the exterior powers of the vector representation. Our main result is Theorem 6B.3 showing that $\mathbf{AWeb}_q\ell$ is equivalent to $\mathbf{Fund}_q\ell$ and that $\mathbf{AWeb}_1\ell$ is pivotally equivalent to $\mathbf{Fund}_1\ell$. The main ingredients in the proof of Theorem 6B.3 are the usual diagrammatic ideas, the Schur–Weyl-type dualities from [33] as well as the *explosion trick*, which utilizes the semisimplicity.

An almost direct consequence of Theorem 6B.3 is that the endomorphism algebras of annular webs corresponding to tensor products of the vector representation can be described explicitly. As we will see in Section 6D these are given by certain *row quotients of Ariki–Koike algebras* (Ariki–Koike algebras are Hecke algebras of $(\mathbb{Z}/d\mathbb{Z})\wr \mathrm{S}_m$; see, for example, [4,7,10]) as studied in [23]. In the special case of two row quotients, which corresponds to $\ell=\mathfrak{gl}_1\oplus\cdots\oplus\mathfrak{gl}_1$ being the *Cartan subalgebra*, these algebras are Martin–Woodcock's *generalized blob algebras* [25]. We thus obtain a web description of generalized blob algebras; see Section 6E for details. This web description allows us to answer two conjectures of Cautis–Kamnitzer [8, Conjectures 10.2 and 10.3] affirmatively, up to technicalities as detailed in Remark 6F.5.

Let us also mention that our work is inspired by [30] (which gives another, very honest, version of annular skew Howe duality), the aforementioned paper [8], as well as [31] (which proves Theorem 6B.3 for $\mathbb{K}_1 = \mathbb{C}$ and the special case of the Cartan subalgebra). With respect to [30] and [31], which are partially motivated by skein theory, we should also warn the reader that the monoidal structure on $\mathbf{AWeb}_q\ell$ coming from skein theory and the one on $\mathbf{Fund}_q\ell$ coming from the Hopf algebra structure of $U_q(\ell)$ do not seem to be compatible; see Section 6C for a discussion.

2. Notations and conventions

We start by specifying our notations.

Notation 2.1. Recall that we fixed $n, l_1, \dots, l_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^{d} l_i = n$. These will be used via the general linear Lie algebra $\mathfrak{gl}_n$ and a Levi subalgebra ℓ given by

$$\ell = \mathfrak{gl}_{l_1} \oplus \cdots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n.$$

For $l_1 = \cdots = l_d = 1$, so that $\ell = \mathfrak{gl}_1 \oplus \cdots \oplus \mathfrak{gl}_1$ is the Cartan subalgebra, we will write $\mathfrak{h}$ instead of ℓ.

Notation 2.2. We now specify our underlying ground rings.

(a) For essential circles, see Section 4C below, we need extra polynomial elements (these can be ignored otherwise). We denote by $\mathbb{U} = \{u_1, \dots, u_d\}$ variables which will play this role.

(b) Let $\mathbb{Z}_v = \mathbb{Z}[v, v^{-1}, \mathbb{U}, \mathbb{U}^{-1}]$ for some indeterminate v.

(c) We let $\mathbb{K}_q$ denote a field containing $\mathbb{U}$ and $\mathbb{U}^{-1}$ and an element q which is not a root of unity. Let also $\mathbb{K}_1$ be a field of characteristic zero containing $\mathbb{U}$ and $\mathbb{U}^{-1}$. (Note that char($\mathbb{K}_q$) is allowed to be a prime, but we assume that char($\mathbb{K}_1$) = 0.)

(d) We also see $\mathbb{K}_q$ as the specialization and scalar extension $\_ \otimes_{\mathbb{Z}_v} \mathbb{K}_q$ given by $v \mapsto q$, and $\mathbb{K}_1$ as the specialization and scalar extension $\_ \otimes_{\mathbb{Z}_v} \mathbb{K}_1$ given by $v \mapsto 1$. We will apply scalar extension to $\mathbb{Z}_v$-linear categories, and since this will play an important role we will indicate the specialization accordingly. The two important specializations for $\mathbb{K}_q$ and $\mathbb{K}_1$ are distinguished by using q, respectively, 1 as a subscript.

(e) If not specified otherwise, then $\otimes$ denotes the tensor product over the ground ring, which is either $\mathbb{Z}_v$, $\mathbb{K}_q$ or $\mathbb{K}_1$.

Not everything in this paper is defined over $\mathbb{Z}_v$, and statements for $\mathbb{K}_q$ are not strictly related to the ones for $\mathbb{K}_1$. If we use $\mathbb{Z}_v$, then we can specialize without any problem. But if we do not work over $\mathbb{Z}_v$, then, to not double parts of the text, we use the following crucial simplification:

Notation 2.3. Throughout (on the diagrammatic and the representation theoretical sides) we always specify our conventions for $\mathbb{K}_q$ and leave the analog conventions and lemmas for $\mathbb{K}_1$ implicit; the ones for $\mathbb{K}_1$ are always the $q = 1$ versions of the ones for $\mathbb{K}_q$. When lemmas etc. are the same for $\mathbb{K}_q$ and $\mathbb{K}_1$, then we will use simply q (e.g., we write $\mathbb{K}_q$ and not $\mathbb{K}_q$ and $\mathbb{K}_1$) to indicate this. We will stress whenever the statements for $\mathbb{K}_q$ and $\mathbb{K}_1$ are significantly different.

We will use quantum numbers, factorials and binomials viewed as elements of $\mathbb{Z}_v$. That is, for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, we let $[0] = 0$, $[0]! = 1 = \left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]$, $[a] = -[-a]$ for $a < 0$, and otherwise

$$[a] = v^{a-1} + v^{a-3} + \cdots + v^{-a+3} + v^{-a+1}, \quad [b]! = [b][b-1]\ldots[1],$$
$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a][a-1]\ldots[a-b+1]}{[b]!}.$$

The following will be used silently throughout.

Lemma 2.4. *All quantum binomials are invertible in $\mathbb{K}_q$.*

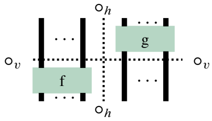
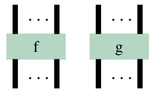
Proof. Easy and omitted. ∎

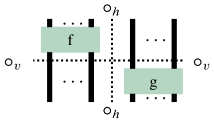
Notation 2.5. We work with strict pivotal (thus, monoidal) categories, were we strictify categories if necessary (by the usual strictification theorems this restriction is for convenience only). We have two directions of composition, vertical $\circ_v$ and horizontal $\circ_h$, as well as a duality * operation. The monoidal unit is denoted by $\mathbb{1}$, and identity morphisms are denoted by id. We will also distinguish objects and morphisms using different fonts, e.g., $\mathtt{K}$ and f.

As we will see, e.g., in Section 6C below, it will turn out to be important to carefully distinguish the monoidal product on the various categories we consider. This will only play a role on the level of morphisms, and we use various symbols for monoidal products between morphisms if necessary.

Notation 2.6. We now summarize the diagrammatic conventions that we will use in this paper.

(a) The following illustration of the interchange law summarizes our reading conventions:

$(\mathrm{id} \circ_h \mathrm{g}) \circ_v (\mathrm{f} \circ_h \mathrm{id}) =$  $=$ 

$=$  $= (\mathrm{f} \circ_h \mathrm{id}) \circ_v (\mathrm{id} \circ_h \mathrm{g}).$

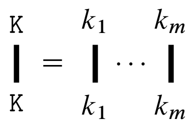
That is, we read diagrams from bottom to top and left to right.

(b) As we will recall below, webs are certain types of labeled (with numbers $a \in \mathbb{Z}_{\geq 0}$) and oriented graphs. Some labels and orientations are determined by others, and we will often omit orientations and labels that can be recovered from the given data to avoid clutter.

(c) If labels or orientations are omitted altogether, then the displayed webs are a shorthand for any web of the same shape and legit labels and orientations.

(d) We use webs with edges labeled by $a \in \mathbb{Z}$, where we use the convention that edges of label 0 are omitted from the illustrations, and edges with label not in $\mathbb{Z}_{\geq 0}$ set the web to zero. (We will use negative labels, but for objects and not for edges in webs.)

(e) We also use strands labeled by objects, i.e., for $\mathtt{K} = (k_1, \dots, k_m)$



to indicate an arbitrary (but finite) number of parallel strings.

3. Web categories in the plane

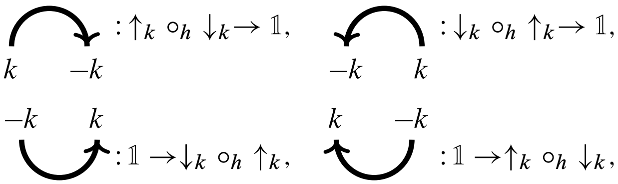
This section serves as a reminder on web categories and their basic properties. Details can be found in many books and papers, e.g., [37] for general diagrammatics, and [9] or, closer to our conventions, [24] and the references therein for web categories. The proofs of the statements below are easy or can be found in loc. cit.

3A. Preliminaries

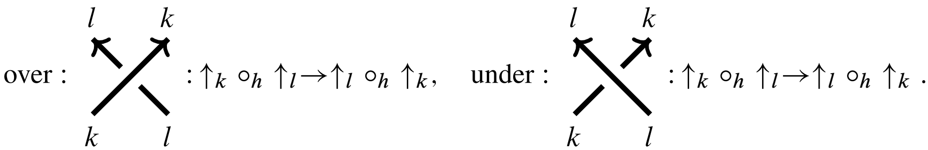
Let R be a commutative ring with unit.

Definition 3A.1. For the purpose of this section, a *diagram category* **Dia** is a pivotal R-linear category with objects $\circ_h$-generated by $\uparrow_k$ with $\downarrow_k = (\uparrow_k)^*$ for $k \in \mathbb{Z}_{\geq 0}$ with $\mathbb{1}$ being the empty word, and a braid group action on upwards objects, meaning morphisms $\widehat{\mathrm{R}}_{k,l}\colon \uparrow_k \circ_h \uparrow_l \to \uparrow_l \circ_h \uparrow_k$ for each simple braid group generator that satisfy the braid relations.

We also write $\mathtt{K} = (k_1, \dots, k_m) \in \mathbb{Z}^m$ for $m \in \mathbb{Z}_{\geq 0}$ for the objects of **Dia**, where we use the notations $k \leftrightsquigarrow \uparrow_k$ and $-k \leftrightsquigarrow \downarrow_k$ for $k > 0$. We illustrate the (co)evaluation morphisms as



and the braid group action as (k,l)-*crossings* (overcrossings and undercrossings):



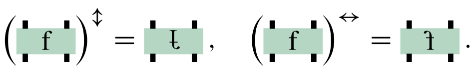
Definition 3A.2. The *diagrammatic antiinvolution* $(\_)^{\updownarrow}$ on a diagram category is defined on objects by

$$(\mathtt{K} \circ_h \mathtt{L} \circ_h \cdots)^{\updownarrow} = \mathtt{K}^* \circ_h \mathtt{L}^* \circ_h \cdots$$

and on morphisms as in equation (3A.3) below. The *diagrammatic involution* $(\_)^{\leftrightarrow}$ on a monoidal diagram category is defined on objects by

$$(\mathtt{K} \circ_h \mathtt{L} \circ_h \cdots)^{\leftrightarrow} = \cdots \circ_h \mathtt{L} \circ_h \mathtt{K}$$

and on morphisms as in equation (3A.3) below:

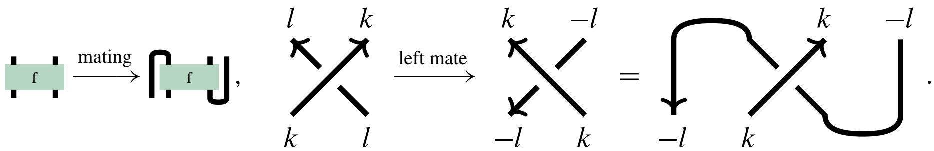


(3A.3)

Lemma 3A.4. *The diagrammatic antiinvolution is an antiinvolution, and the diagrammatic involution is an involution.*

Proof. Easy and omitted. ∎

Definition 3A.5. *Mating* in **Dia** is the process of applying (co)evaluation morphisms. For example,



To be precise, a mate of a morphism f is any morphism obtained from f by applying (co)evaluation morphisms. A left and right mate of f is a mate that only uses left or right (co)evaluation morphisms. Finally, *the* left and right mate of a crossing is as indicated above.

Note that mating produces many morphisms from a given set of morphisms.

Lemma 3A.6. *Suppose that*

$$\mathrm{End}_{\mathbf{Dia}}(\mathbb{1}) \cong \mathrm{End}_{\mathbf{Dia}}(\uparrow_k) \cong R.$$

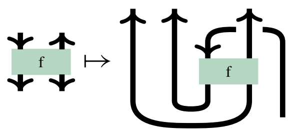
If the left mate of the (k,l)-overcrossing is invertible, then all mates of the (k,l)-overcrossing and its inverse span a pivotal subcategory equivalent to R-linear (labeled) tangles. In general, if the left mate of the (k,l)-overcrossing is invertible, then all mates of the (k,l)-overcrossing and its inverse span a pivotal subcategory equivalent to R-linear (labeled) framed tangles.

Proof. Note that the assumption $\mathrm{End}_{\mathbf{Dia}}(\uparrow_k) \cong R$ implies the Reidemeister I relation up to scalars, which is enough to copy the argument in [21, Theorem X11.2.2]. The second claim follows similarly. ■

We say that $\mathrm{Hom}_{\mathbf{Dia}}(\mathtt{K},\mathtt{L})$ is *determined by* $\mathrm{Hom}_{\mathbf{Dia}}(\mathtt{K}',\mathtt{L}')$ if there exists an isomorphism of R-modules between them.

Lemma 3A.7. *If the left mates of the (k,l)-overcrossings are invertible, then all hom-spaces in* **Dia** *are determined by upwards hom-spaces.*

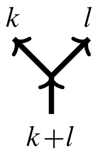
Proof. By Lemma 3A.6, the assumptions imply that

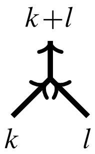


is an isomorphism. This easily generalizes. ■

Definition 3A.8. A left invertible morphism $\uparrow_{k+l} \to \uparrow_k \circ_h \uparrow_l$ is called *explosion*.

Up to rescaling, explosion morphisms and their left inverses can be illustrated by



, respectively,


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Lemma 3A.9. *If the left mates of the (k,l)-overcrossings are invertible and all explosion morphisms exist, then all hom-spaces in* **Dia** *are determined by end-spaces between objects of the form*

$$1^{\circ_h k} = \uparrow_1 \circ_h \cdots \circ_h \uparrow_1$$

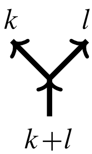
(k factors).

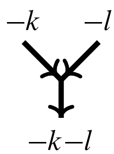
Proof. Turn all strands upwards using Lemma 3A.7, and then explode the strands inductively. ∎

3B. Exterior $\mathfrak{gl}_n$-webs

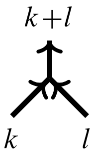
We now recall the category of exterior $\mathfrak{gl}_n$ webs.

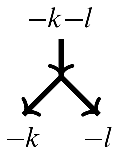
Definition 3B.1. The *(exterior $\mathfrak{gl}_n$) web category* $\mathbf{Web}_v\mathfrak{gl}_n$ is the diagram category for $R = \mathbb{Z}_v$ with $\circ_h$-generating objects of categorical dimension $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ and $\circ_v$-$\circ_h$-generating morphisms



$: \uparrow_{k+l} \to \uparrow_k \circ_h \uparrow_l,$


$: \downarrow_{k+l} \to \downarrow_k \circ_h \downarrow_l,$

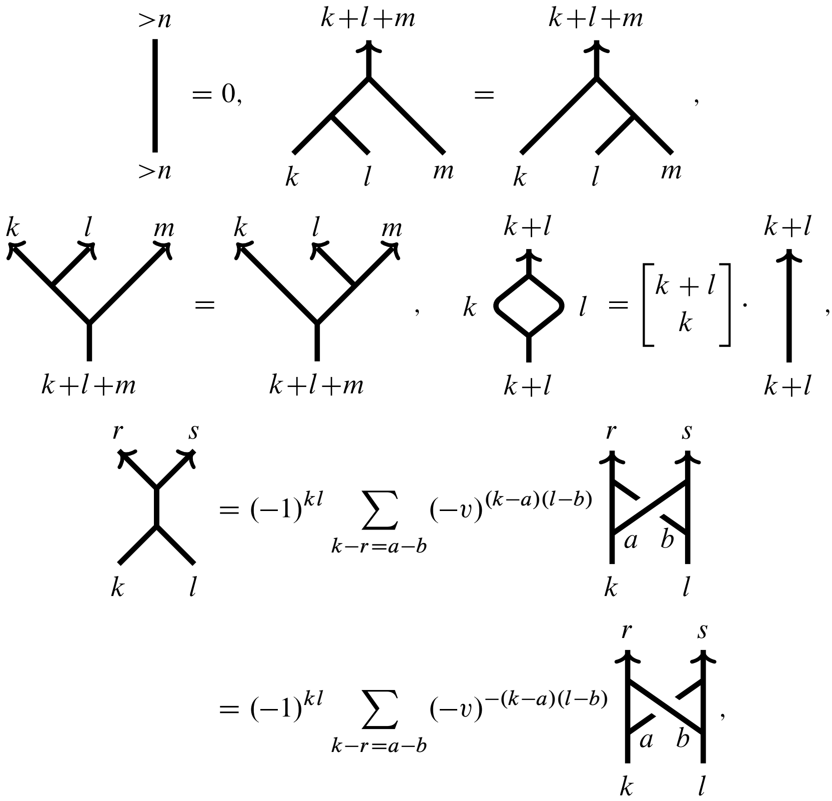


$: \uparrow_k \circ_h \uparrow_l \to \uparrow_{k+l},$


$: \downarrow_k \circ_h \downarrow_l \to \downarrow_{k+l},$

such that the left mates of the (k,l)-overcrossings are invertible. The relations imposed on $\mathbf{Web}_v\mathfrak{gl}_n$ are *isotopies* (not displayed zigzag and trivalent-slide relations; see, e.g., [24, Section 2] for details), the *exterior relation*, *associativity*, *coassociativity*, *digon removal*, and *dumbbell-crossing relation*. That is, we take the quotient by the

$\circ_v$-$\circ_h$-ideal generated by isotopies and



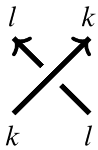
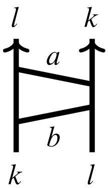
together with their $(\_)^{\updownarrow}$-duals.

We call morphisms in $\mathbf{Web}_v\mathfrak{gl}_n$ *(exterior* $\mathfrak{gl}_n$*) webs*.

Remark 3B.2. The dumbbell-crossing relation is not new and can be deduced from Green's book on the Schur algebra [14] via an interpretation of webs as elements in the Schur algebra. Consequently, this relation is also called the *Schur relation*.

It follows for example from [24, Sections 2 and 5] that there is a well-defined functor from the pivotal category of topological webs (where webs are defined as plane labeled oriented trivalent graphs up to planar isotopy) to $\mathbf{Web}_v\mathfrak{gl}_n$ which we will use to draw webs in a topological fashion.

Lemma 3B.3. *In* $\mathbf{Web}_v\mathfrak{gl}_n$, *we have*


$$= (-1)^{kl} \sum_{b-a=k-l} (-v)^{k-b}$$

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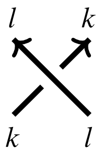
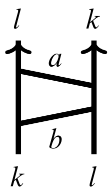
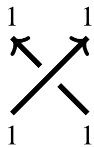
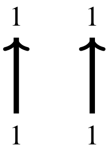

$$= (-1)^{kl} \sum_{b-a=k-l} (-v)^{-k+b}$$

. (3B.4)

Proof. This is explained in [24, Section 5]. ■

For completeness, the $k = l = 1$ case of equation (3B.4) is


$$= v \cdot$$

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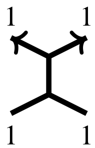
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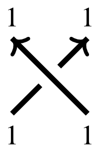
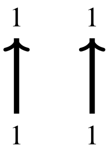

$$= v^{-1} \cdot$$

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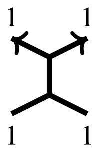
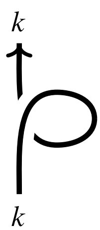
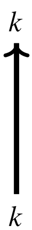
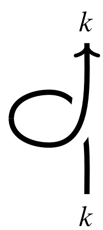

. (3B.5)

Lemma 3B.6. *The following hold in* $\mathbf{Web}_v\mathfrak{gl}_n$.

(a) *The crossings satisfy the* Reidemeister II and III relations, *and the* Reidemeister I relation *holds up to scalars, that is,*


$$= v^{k(-k+n+1)} \cdot$$

$=$

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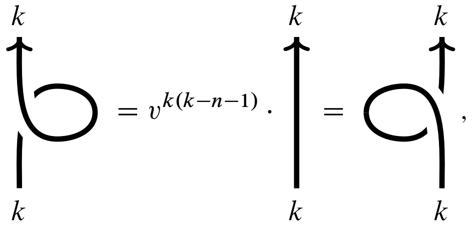
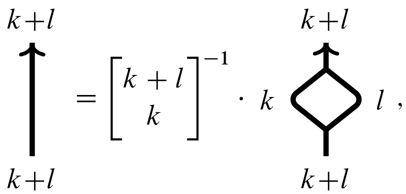


(3B.7)

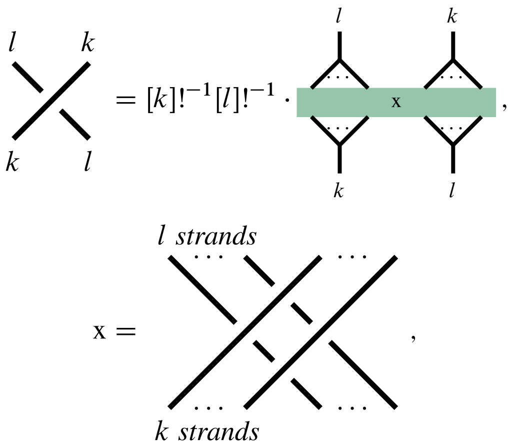
together with their $(\_)^{\updownarrow}$*-duals. Various* naturality relations *hold; see, e.g.,* [24, Section 2].

(b) Square switches*; see, e.g.,* [24, Lemma 5.6]. *Various other relations that we do not explicitly use (see, e.g.,* [24, Section 2]*) also hold.*

(c) Explosion *holds for* $\mathbf{Web}_q\mathfrak{gl}_n$ *(but not in* $\mathbf{Web}_v\mathfrak{gl}_n$*), that is*



together with its $(\_)^{\updownarrow}$*-dual. Moreover, the* (k,l)*-overcrossings satisfy explosion as well, i.e.,*



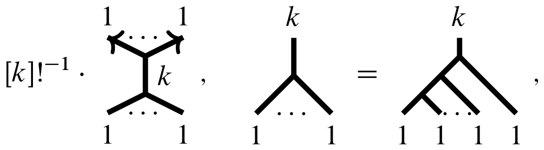
(3B.8)

as well as a similar formula for the (k,l)*-undercrossings.* ■

Note that thus Lemma 3A.6 applies in $\mathbf{Web}_q\mathfrak{gl}_n$. Let us also note the following, partially explaining why explosion works well in practice.

Lemma 3B.9. *The additive idempotent completion of* $\mathbf{Web}_q\mathfrak{gl}_n$ *is semisimple.*

Proof. By Lemma 3A.6 and the existence of certain projectors; see, e.g., [32] or [35, Section 2.3]. To be precise (and to fix notation for the rest of the paper), the projectors are



where the dots indicate an inductive construction as illustrated on the right (the order of how these are constructed is irrelevant due to associativity and coassociativity). ■

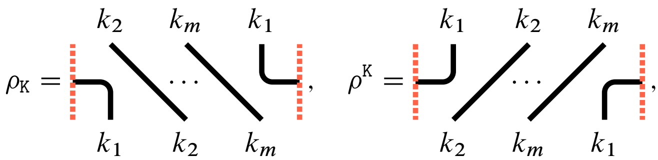
4. Annular webs

This section discusses our main diagram categories of this paper. Similar constructions have appeared in many texts, e.g., [8] or [31].

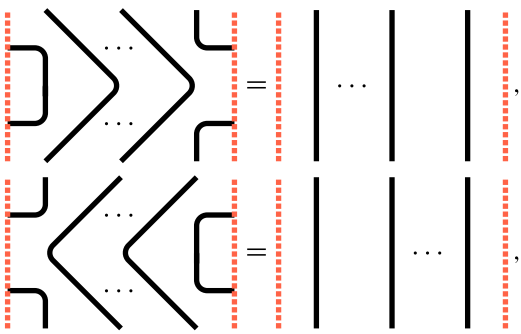
4A. The annular web category

The following definition does not use any $\circ_h$ structure.

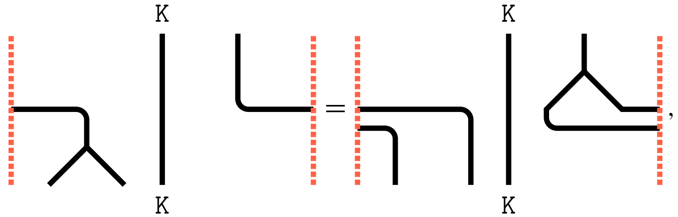
Definition 4A.1. The *(exterior* $\mathfrak{gl}_n$*) annular web category* $\mathbf{AWeb}_v\mathfrak{gl}_n$ is the category obtained from $\mathbf{Web}_v\mathfrak{gl}_n$ by adding extra $\circ_v$-generators



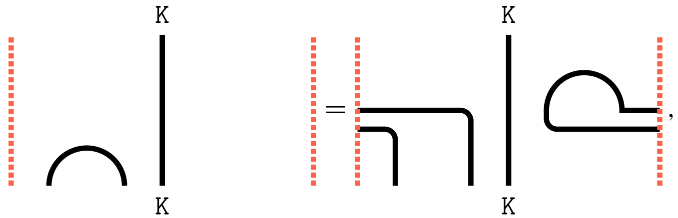
for each $\mathtt{K} = (k_1, \dots, k_m) \in \mathbb{Z}^m$ to $\mathbf{Web}_v\mathfrak{gl}_n$ modulo the $\circ_v$-ideal generated by the relations



(4A.2)



(4A.3)

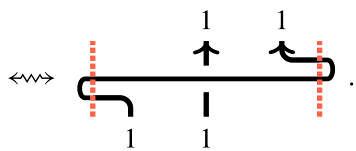


(4A.4)

together with the $(\_)^{\updownarrow}$- and $(\_)^{\leftrightarrow}$-duals of the bottom two relations.

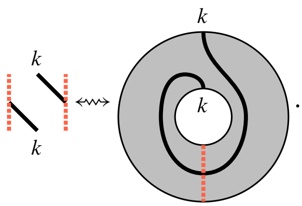
We call morphisms in $\mathbf{AWeb}_v\mathfrak{gl}_n$ *annular (exterior $\mathfrak{gl}_n$) webs*, and $\rho_{\mathtt{K}}$ and $\rho^{\mathtt{K}}$ are called *coils*.

Remark 4A.5. We think of the coils as crossings in front of the annulus, e.g.,

$\rho_{(1,1)} \leftrightsquigarrow$ 

This convention comes because we follow [33] later on for computations. Using the inverse braiding compared to the definitions in [33, Theorem 3.2] translate to coils passing behind the annulus.

Remark 4A.6. The name annular webs comes from the interpretation of the pictures in Definition 4A.1 as embedded in an annulus. For example,



The following elements defined via Lagrange interpolation play a crucial role later on.

Definition 4A.7. For $i \in \{1, \dots, d\}$, define

$$\mathrm{pr}_i^w = \prod_{j \neq i} \frac{\rho_1 - u_j}{u_i - u_j} = \prod_{j\neq i} \frac{\square - u_j}{u_i - u_j} \in \mathrm{End}_{\mathbf{AWeb}_q \mathfrak{gl}_n}(1),$$

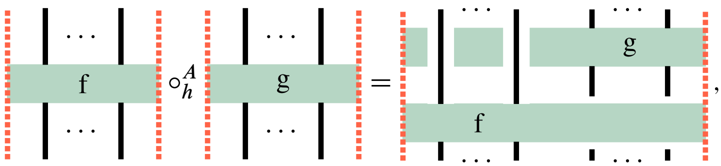
which we call *web block projectors*.

4B. Properties of annular webs

We can endow $\mathbf{AWeb}_v \mathfrak{gl}_n$ with a monoidal structure $\circ_h^A$ as follows. On objects,

$$\mathtt{K} \circ_h \mathtt{K}' = \mathtt{K} \circ_h^A \mathtt{K}'$$

is just the concatenation, i.e., if $\mathtt{K} = (k_1, \dots, k_r)$ and $\mathtt{K}' = (k_1', \dots, k_s')$, then $\mathtt{K} \circ_h \mathtt{K}' = (k_1, \dots, k_r, k_1', \dots, k_s')$. On morphisms $\circ_h^A$, we use



(4B.1)

using the (k, l)-crossings from equation (3B.4) and their mates so that f is in the front and g is in the back.

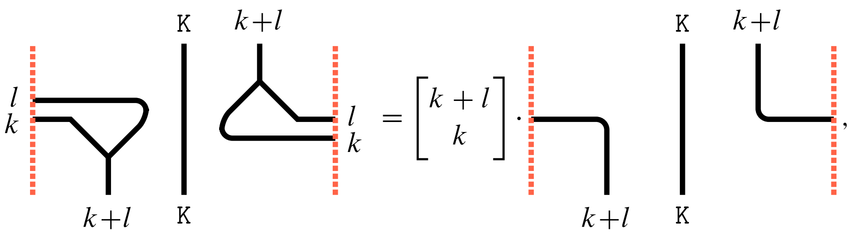
Remark 4B.2. Equation (4B.1) is a standard construction in skein theory; see [29].

Lemma 4B.3. *The monoidal structure $\circ_h^A$ and $\mathbb{1} = \emptyset$ endow $\mathbf{AWeb}_v \mathfrak{gl}_n$ with the structure of a pivotal category with duality given by cups and caps.*

Proof. Easy and omitted. ∎

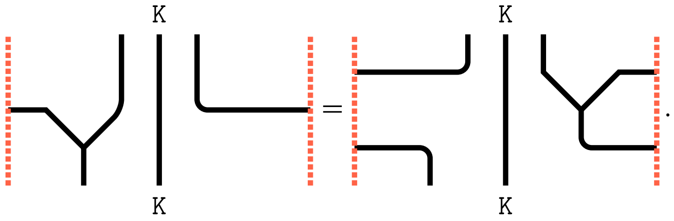
Lemma 4B.4. *The following holds in $\mathbf{AWeb}_v \mathfrak{gl}_n$.*

(a) *We have an* annular digon removal, *that is,*

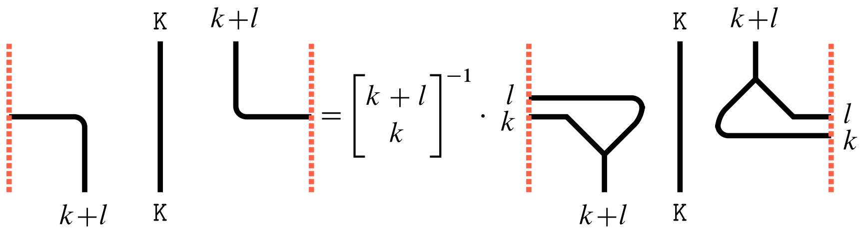


together with its $(\_)^{\updownarrow}$-dual. Various other annular versions of the relations in Lemma 3B.6 hold as well (but are not stated since we do not use them).

(b) *All* half-slides *of merges, splits, cups and caps, e.g.,*

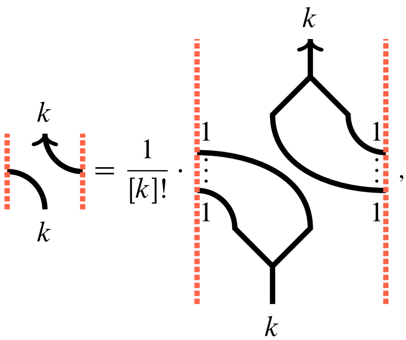


(c) Annular explosion *holds for* $\mathbf{AWeb}_q\mathfrak{gl}_n$ *(but not in* $\mathbf{AWeb}_v\mathfrak{gl}_n$*), that is,*



together with its $(\_)^{\updownarrow}$*-dual.*

Proof. We get



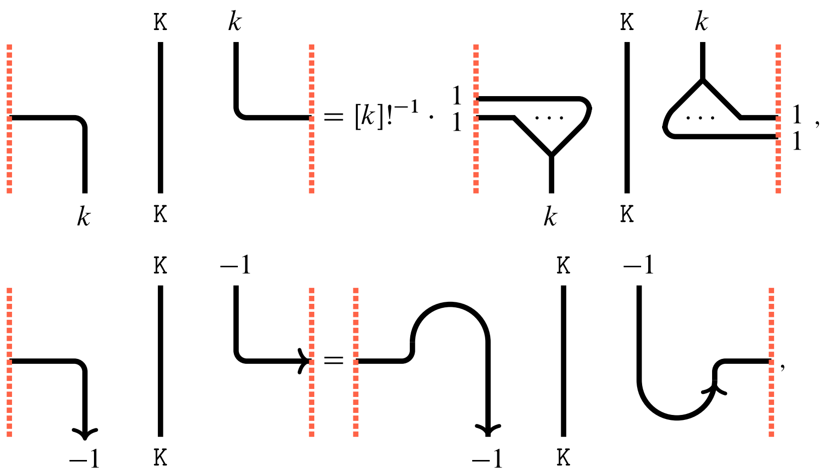
(4B.5)

by (plain) explosion and equation (4A.3). The claimed relations can be proven using this. ■

The thin coils suffice as $\circ_v$-generators:

Lemma 4B.6. *The morphisms* $\rho_{(k,\mathtt{K})}$ *and* $\rho^{(k,\mathtt{K})}$ *in* $\mathbf{AWeb}_q\mathfrak{gl}_n$ *(but not in* $\mathbf{AWeb}_v\mathfrak{gl}_n$*) can be defined inductively from* $\rho_{(\pm 1,\mathtt{K})}$ *and* $\rho^{(\pm 1,\mathtt{K})}$*. Moreover, the morphisms* $\rho_{(-1,\mathtt{K})}$ *and* $\rho^{(-1,\mathtt{K})}$ *in* $\mathbf{AWeb}_v\mathfrak{gl}_n$ *can be defined from* $\rho^{(1,-1,\mathtt{K},-1)}$ *and* $\rho_{(1,-1,\mathtt{K},-1)}$*.*

Proof. The pictures



define the morphisms as claimed. Their inverses are the $(\_)^{\leftrightarrow}$-duals of these pictures. ∎

The following lemma compares $\mathbf{AWeb}_v\mathfrak{gl}_n$ to the construction in [8].

Lemma 4B.7. *The category* $\mathbf{AWeb}_v\mathfrak{gl}_n$ *is equivalent as a diagram category to the affinization* $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ *(in the sense of, e.g.,* [27, Definition 2.1]*) of* $\mathbf{Web}_v\mathfrak{gl}_n$.

Proof. We only sketch the proof: as often in diagrammatic algebra matching a generator-relation presentation with a "all diagrams" definition is lengthy and we omit some details.

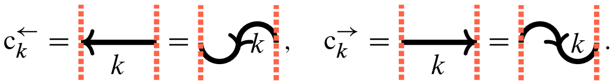
First, there is an essentially surjective functor Γ from $\mathbf{AWeb}_v\mathfrak{gl}_n$ to $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ that puts a plane web into the annulus. Next, $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ is defined by adjoining more morphisms and relations to $\mathbf{AWeb}_v\mathfrak{gl}_n$, namely one coil and its inverse for each $\mathtt{K}$ and relations [27, equation (2.5)]. But using the coils in Definition 4A.1 one can define these more general coils following [27, equation (2.5), left] which satisfy [27, equation (2.5), right], showing that Γ is full. Faithfulness of Γ can then be deduced from Theorem 6B.3 below, by showing that the functor therein factors through $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ via Γ.

Alternatively, one can match the generator-relation presentation of $\mathbf{AWeb}_v\mathfrak{gl}_n$ with the generator-relation presentation from [15], and then the topological presentation of $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ with the topological presentation of [15] and the result follows via [15, Proposition 10]. (With the caveat that [15] only discusses tangles and the description therein needs to be extended to webs. That is straightforward, but lengthy.) ∎

4C. Quotient by essential circles

We now define quotients of $\mathbf{AWeb}_v\mathfrak{gl}_n$.

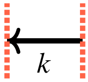
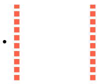
Definition 4C.1. The left and right *essential k-circles* are defined to be

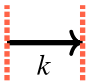
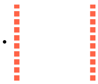


We also say *essential circles* for short.

Note that essential circles are nontrivial endomorphism of $\mathbb{1}$. We want to evaluate them. To this end, let e_k denote the kth elementary symmetric polynomial in n variables, i.e., $e_k = e_k(Z_1, \dots, Z_n)$.

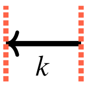
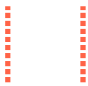
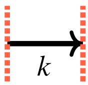
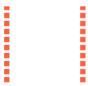
Definition 4C.2. The *Levi evaluation* for ℓ of the essential circles is defined to be

 $- e_k(v^{-1}u_1, v^{-3}u_1, \dots, v^{-2l_1+1}u_1, \dots, v^{-1}u_d, v^{-3}u_d, \dots, v^{-2l_d+1}u_d) \cdot$  ,

 $- e_k(vu_1^{-1}, v^3u_1^{-1}, \dots, v^{2l_1-1}u_1^{-1}, \dots, vu_d^{-1}, v^3u_d^{-1}, \dots, v^{2l_d-1}u_d^{-1}) \cdot$  , (4C.3)

which are elements in $\mathrm{End}_{\mathbf{AWeb}_v\mathfrak{gl}_n}(\mathbb{1})$.

Example 4C.4. As an extreme case, take $\ell = \mathfrak{gl}_n$. The formulas in equation (4C.3) then become

 $- v^{-kn}\begin{bmatrix} n \\ k \end{bmatrix} u_1^k \cdot$  ,  $- v^{kn}\begin{bmatrix} n \\ k \end{bmatrix} u_1^{-k} \cdot$  .

The appearing scalars are multiples of the categorical dimension of $\uparrow_k$, which is the value of the usual circle in the web calculus. Since we want to eventually evaluate essential circles to these scalars, this might be a hint of a connection to annular webs obtained from evaluation representations for the affine Lie algebra; see, e.g., [30, Section 3]. ■

The quotient $\mathbf{AWeb}_q\ell$ of $\mathbf{AWeb}_q\mathfrak{gl}_n$ by an ideal $\mathcal{I}_\ell$ defined later in Section 6B gives a diagrammatic description of quantum ℓ-representations.

For $\mathbb{K}_1$ it will turn out that $\mathcal{I}_\ell$ is the two-sided $\circ_v$-$\circ_h^A$-ideal generated by the Levi evaluations equation (4C.3). Being careful with the underlying monoidal structure (details are given in Theorem 6B.3), the same holds for $\mathbb{K}_q$, hence the name.

5. Representation theory of Levi subalgebras

This section discusses the representation categories of this paper. The below is (partially) well known and we will be brief whenever appropriate. A lot of details and also background can be found in texts such as [20].

5A. The general linear representation category

We start with notations regarding the general linear quantum algebra. Let $U_v(\mathfrak{gl}_n)$ be the *divided power quantum enveloping algebra for* $\mathfrak{gl}_n$, where we use the conventions, excluding the Hopf algebra structure, from [1] in the special case of $\mathfrak{gl}_n$ (using $K_i^{\pm 1} = L_i^{\pm 1}L_{i+1}^{\mp 1}$). The algebra $U_v(\mathfrak{gl}_n)$ specializes to either the $\mathbb{K}_q$-algebra $U_q(\mathfrak{gl}_n)$, for which we now recall the relevant formulas, or the $\mathbb{K}_1$-algebra $U_1(\mathfrak{gl}_n)$.

The algebra $U_q(\mathfrak{gl}_n)$ is generated by $L_i^{\pm 1}$ for $i \in \{1,\dots,n\}$ (these are inverses) and E_i, F_i for $i \in \{1,\dots,n-1\}$ and the Hopf algebra structure used in this paper is

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes L_iL_{i+1}^{-1} + 1 \otimes E_i, \quad \varepsilon(E_i) = 0, \quad S(E_i) = -E_iL_i^{-1}L_{i+1},\\
\Delta(F_i) &= F_i \otimes 1 + L_i^{-1}L_{i+1} \otimes F_i, \quad \varepsilon(F_i) = 0, \quad S(F_i) = -L_iL_{i+1}^{-1}F_i,
\end{aligned}$$

with $L_i^{\pm 1}$ being group like.

The *vector representation* $\mathtt{V}_q = \mathtt{V}_q(\mathfrak{gl}_n) = \mathbb{K}_q\{v_1,\dots,v_n\}$ of $U_q(\mathfrak{gl}_n)$ is the given $\mathbb{K}_q$-vector space with action

$$L_i^{\pm 1} \centerdot v_j = q^{\pm\delta_{i,j}}v_j, \quad E_i \centerdot v_j = \delta_{i,j-1}v_{j-1}, \quad F_i \centerdot v_j = \delta_{i,j}v_{j+1}.$$

Let $T\mathtt{V}_q$ be the tensor algebra. The *kth quantum exterior power* $\bigwedge_q^k \mathtt{V}_q$ is defined as the degree k part (in the usual sense) of the *quantum exterior algebra* given by

$$\begin{aligned}
\textstyle\bigwedge_q^{\bullet}\mathtt{V}_q &= \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \textstyle\bigwedge_q^k \mathtt{V}_q\\
&= T\mathtt{V}_q/\langle v_h \otimes v_h, v_j \otimes v_i + q^{-1}v_i \otimes v_j \mid i < j\rangle_{\text{two-sided }\otimes\text{-ideal}}.
\end{aligned} \tag{5A.1}$$

The exterior powers are $U_q(\mathfrak{gl}_n)$-representations by using the Hopf algebra structure, and so is $\mathbb{1} = \mathbb{K}_q$ itself and all the duals of the above, denoted by using negative powers:

$$\textstyle\bigwedge_q^k \mathtt{V}_q = \big(\bigwedge_q^{-k}\mathtt{V}_q\big)^* \quad \text{for } k \in \mathbb{Z}_{<0}.$$

Lemma 5A.2. *For $k \in \mathbb{Z}_{\geq 0}$, the $\mathbb{K}_q$-vector space $\bigwedge_q^k \mathtt{V}_q$ has a basis given by*

$$\{v_S = v_{i_1} \otimes \cdots \otimes v_{i_k} \mid S = (i_1 < \cdots < i_k) \text{ for } i_j \in \{1,\dots,n\}\}.$$

If $-k \in \mathbb{Z}_{\geq 0}$, then the $\mathbb{K}_q$-vector space $\bigwedge_q^k \mathtt{V}_q$ has a basis given by

$$\{v_S^* = v_{i_{-k}} \otimes \cdots \otimes v_{i_1} \mid S = (i_1 < \cdots < i_{-k}) \text{ for } i_j \in \{1,\dots,n\}\}.$$

Proof. Easy and omitted. ■

Notation 5A.3. We also use the notation v_S from Lemma 5A.2 more generally for any $S = (i_1, \dots, i_k)$ for $i_j \in \{1, \dots, n\}$, and use the usual set operations on them. Recall that such expressions need to potentially be reordered using equation (5A.1) to match the basis of Lemma 5A.2.

We now consider so-called $U_q(\mathfrak{gl}_n)$-representations of type 1 which, as usual, is not a serious restriction; see, e.g., [20, Section 5.2] for details.

Definition 5A.4. Let $\mathbf{Rep}_q\mathfrak{gl}_n$ denote the category of finite-dimensional $U_q(\mathfrak{gl}_n)$-representations of type 1. We view $\mathbf{Rep}_q\mathfrak{gl}_n$ as pivotal using the above Hopf algebra structure on $U_q(\mathfrak{gl}_n)$. Let further $\mathbf{Fund}_q\mathfrak{gl}_n$ denote the full pivotal subcategory with objects of the form

$$\textstyle\bigwedge_q^{\mathtt{K}}\mathtt{V}_q = \bigwedge_q^{k_1}\mathtt{V}_q \otimes \cdots \otimes \bigwedge_q^{k_m}\mathtt{V}_q$$

for $\mathtt{K} = (k_1, \dots, k_m) \in \mathbb{Z}^m$ and $m \in \mathbb{Z}_{\geq 0}$.

We call $\mathbf{Rep}_q\mathfrak{gl}_n$ the *representation category* of $U_q(\mathfrak{gl}_n)$ and $\mathbf{Fund}_q\mathfrak{gl}_n$ its *fundamental category*. (We use the same terminology for ℓ defined below.) The following is crucial, but well known, and will be used throughout. To state it let $X^+_{\mathfrak{gl}_n} \subset \mathbb{Z}^n$ denote the set of dominant integral $\mathfrak{gl}_n$-weight, i.e., tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$.

Lemma 5A.5. *We have the following:*

(a) *The category* $\mathbf{Rep}_q\mathfrak{gl}_n$ *is semisimple, its simple objects can be indexed by* $\lambda \in X^+_{\mathfrak{gl}_n}$ *and their characters are given by Weyl's character formula.*

(b) *The additive idempotent completion of* $\mathbf{Fund}_q\mathfrak{gl}_n$ *is pivotally equivalent to* $\mathbf{Rep}_q\mathfrak{gl}_n$.

We will denote the simple objects in Lemma 5A.5 (a) by $L(\lambda)$ for $\lambda \in X^+_{\mathfrak{gl}_n}$. We do not need them explicitly, but their construction is well known (the $L(\lambda)$ are often called Weyl modules).

Proof. (a) See, e.g., [20, Theorems 5.15 and 5.17] or [1, Section 6].

(b) By (a), classical theory applies; see, e.g., [20, Theorems 5.15 and 5.17] or [1, Section 6] for details. ■

Let us now list generating $U_q(\mathfrak{gl}_n)$-equivariant morphisms that will be the images of the generators of $\mathbf{Web}_q\mathfrak{gl}_n$. The notation is hopefully suggestive.

For tuples S, T as in Notation 5A.3 let $|S < T| = |\{(s,t) \in S \times T | s < t\}|$ and $|S, N| = |S < N| - |N < S|$ for $N = (1, 2, \dots, n)$. Using such tuples and this notation we define (here $k,l\in\mathbb{Z}_{\geq 0}$):

$$\begin{aligned}
&\curlywedge_{k,l}^{k+l}\colon {\textstyle\bigwedge_q^k} \mathtt{V}_q \otimes {\textstyle\bigwedge_q^l}\mathtt{V}_q \to {\textstyle\bigwedge_q^{k+l}}\mathtt{V}_q, && v_S\otimes v_T \mapsto \delta_{S\cap T,\emptyset}(-q)^{-|T<S|}v_{S\cup T},\\
&\curlyvee_{k+l}^{k,l}\colon {\textstyle\bigwedge_q^{k+l}}\mathtt{V}_q \to {\textstyle\bigwedge_q^k} \mathtt{V}_q \otimes {\textstyle\bigwedge_q^l}\mathtt{V}_q, && v_U \mapsto (-1)^{kl}\sum_{S\sqcup T=U,\,|S|=k}(-q)^{|S<T|}v_S\otimes v_T,\\
&\cap_k^{\leftarrow}\colon {\textstyle\bigwedge_q^{-k}}\mathtt{V}_q \otimes {\textstyle\bigwedge_q^k}\mathtt{V}_q \to \mathbb{K}_q, && v_S^*\otimes v_T\mapsto \delta_{S,T},\\
&\cap_k^{\rightarrow}\colon {\textstyle\bigwedge_q^{k}}\mathtt{V}_q \otimes {\textstyle\bigwedge_q^{-k}}\mathtt{V}_q \to \mathbb{K}_q, && v_S\otimes v_T^*\mapsto q^{|S,N|}\delta_{S,T},\\
&\cup_k^{\leftarrow}\colon \mathbb{K}_q \to {\textstyle\bigwedge_q^{k}}\mathtt{V}_q \otimes {\textstyle\bigwedge_q^{-k}}\mathtt{V}_q, && 1\mapsto \sum_{|S|=k} v_S\otimes v_S^*,\\
&\cup_k^{\rightarrow}\colon \mathbb{K}_q \to {\textstyle\bigwedge_q^{-k}}\mathtt{V}_q \otimes {\textstyle\bigwedge_q^{k}}\mathtt{V}_q, && 1\mapsto \sum_{|S|=k} q^{-|S,N|}v_S^*\otimes v_S.
\end{aligned} \tag{5A.6}$$

Lemma 5A.7. *The morphisms in equation (5A.6) are $\circ_v$-$\otimes$-generators of* $\mathbf{Fund}_q\mathfrak{gl}_n$.

Proof. A careful check of the relations shows that these maps are $U_q(\mathfrak{gl}_n)$-equivariant. That they generate follows from Lemma 5A.5 and classical theory. ■

We have an *algebraic version of explosion*.

Lemma 5A.8. Explosion *holds for* $\mathbf{Fund}_q\mathfrak{gl}_n$, *that is*

$$\mathrm{id}_{k+l} = \left[\begin{smallmatrix} k+l \\ k \end{smallmatrix}\right]^{-1} \curlywedge_{k,l}^{k+l}\curlyvee_{k+l}^{k,l}.$$

Proof. A direct computation. ■

We denote by $\curlyvee_k^{1,\dots,1}$ and $\curlywedge_{1,\dots,1}^k$ the successive explosion of k strands.

We also have the following $(1,1)$-*overcrossings* and $(1,1)$-*undercrossings*:

$$\begin{gathered}
\hat{\mathrm{R}}_{1,1} = q\mathrm{id}_{1,1} - \curlyvee_2^{1,1}\curlywedge_{1,1}^2\colon \mathtt{V}_q\otimes\mathtt{V}_q\to\mathtt{V}_q\otimes\mathtt{V}_q,\\
v_i\otimes v_j \mapsto \begin{cases} qv_i\otimes v_i & \text{if } i=j,\\ v_j\otimes v_i & \text{if } i<j,\\ v_j\otimes v_i + (q-q^{-1})v_i\otimes v_j & \text{if } i>j,\end{cases}\\
\hat{\mathrm{R}}_{1,1}^{-1} = q^{-1}\mathrm{id}_{1,1} - \curlyvee_2^{1,1}\curlywedge_{1,1}^2\colon \mathtt{V}_q\otimes\mathtt{V}_q\to\mathtt{V}_q\otimes\mathtt{V}_q,\\
v_i\otimes v_j \mapsto \begin{cases} q^{-1}v_i\otimes v_i & \text{if } i=j,\\ v_j\otimes v_i & \text{if } i>j,\\ v_j\otimes v_i + (q^{-1}-q)v_i\otimes v_j & \text{if } i<j.\end{cases}
\end{gathered}$$

We also get crossings

$$\hat{\mathrm{R}}_{k,l}^{\pm 1}\colon {\textstyle\bigwedge_q^k}\mathtt{V}_q\otimes{\textstyle\bigwedge_q^l}\mathtt{V}_q \to {\textstyle\bigwedge_q^l}\mathtt{V}_q\otimes{\textstyle\bigwedge_q^k}\mathtt{V}_q$$

for all $k,l\in\mathbb{Z}$ by mimicking equation (3B.4) and mating.

Lemma 5A.9. *The crossings satisfy the* Reidemeister II and III relations *and various* naturality relations, *and can be alternatively defined by explosion.*

Proof. Well known and omitted (for the statement about the alternative definition using explosion, see Lemma 3B.6 imported via Theorem 6A.1). ∎

5B. The Levi representation category

Recall that we have fixed $\ell = \mathfrak{gl}_{l_1} \oplus \cdots \oplus \mathfrak{gl}_{l_d}$, which we think of as being

$$\ell = \begin{pmatrix} \mathfrak{gl}_{l_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathfrak{gl}_{l_d} \end{pmatrix} \subset \mathfrak{gl}_n, \quad \text{generators in } \mathfrak{gl}_{l_i}\colon \begin{cases} L_{i,1} & E_{i,1} & & \\ F_{i,1} & \ddots & \ddots & \\ & \ddots & \ddots & E_{i,l_i-1} \\ & & F_{i,l_i-1} & L_{i,l_i} \end{cases},$$

where we reindex the elements $L_i^{\pm}$, E_i and F_i as indicated. (Note that all $L_i^{\pm}$ appear in this reindexing, but not all E_i and F_i.)

Definition 5B.1. Let $U_v(\ell)$ be the $\mathbb{Z}_v$-subalgebra of $U_v(\mathfrak{gl}_n)$ generated by these $L_{i,k}^{\pm}$, $E_{i,k}$ and $F_{i,k}$. We endow $U_v(\ell)$ with the structure of a Hopf algebra by restricting the one for $U_v(\mathfrak{gl}_n)$.

The following lemma gives us a *block decomposition* and will be used without further reference.

Lemma 5B.2. *We have* $U_v(\ell) \cong U_v(\mathfrak{gl}_{l_1}) \otimes \cdots \otimes U_v(\mathfrak{gl}_{l_d})$ *as* $\mathbb{Z}_v$*-algebras.*

Proof. By definition. ∎

The representation theory of $U_q(\ell)$ is easy (knowing the representation theory for $U_q(\mathfrak{gl}_m)$), but nevertheless we state a few lemmas that we will use. For starters, note that all $U_q(\mathfrak{gl}_n)$-representations restrict to $U_q(\ell)$-representations. Note also that the vector representation $\mathtt{V}_q(\mathfrak{gl}_{l_i})$ of $U_q(\mathfrak{gl}_{l_i})$ is a $U_q(\ell)$-representation with action inflated to $U_q(\ell)$. The same holds for the exterior powers.

Lemma 5B.3. *As* $U_q(\ell)$*-representations, we have*

$$\textstyle\bigwedge_q^k \mathtt{V}_q \cong \bigwedge_q^k \bigoplus_{i=1}^{d} \mathtt{V}_q(\mathfrak{gl}_{l_i}) \cong \bigoplus_{k_1+\cdots+k_d=k} \bigwedge_q^{k_1} \mathtt{V}_q(\mathfrak{gl}_{l_1}) \otimes \cdots \otimes \bigwedge_q^{k_d} \mathtt{V}_q(\mathfrak{gl}_{l_d}).$$

We also have

$$\mathtt{V}_q^{\otimes k} \cong \bigoplus_{k_1+\cdots+k_d=k} \mathtt{V}_q(\mathfrak{gl}_{l_1})^{\otimes k_1} \otimes \cdots \otimes \mathtt{V}_q(\mathfrak{gl}_{l_d})^{\otimes k_d}.$$

Note that Lemma 5B.3 implies that $\mathtt{V}_q$ is not simple as a $U_q(\ell)$-representation.

Proof. The case $k = 1$ is clear by, for example, using the usual diagrammatic description of $\mathtt{V}_q$ (the actions of the $L_i^{\pm 1}$ are omitted in the following illustration):

$$v_8 \underset{F_{3,1}}{\overset{E_{3,1}}{\rightleftarrows}} v_7 \underset{F_6}{\overset{E_6}{\rightleftarrows}} v_6 \underset{F_{2,2}}{\overset{E_{2,2}}{\rightleftarrows}} v_5 \underset{F_{2,1}}{\overset{E_{2,1}}{\rightleftarrows}} v_4 \underset{F_3}{\overset{E_3}{\rightleftarrows}} v_3 \underset{F_{1,2}}{\overset{E_{1,2}}{\rightleftarrows}} v_2 \underset{F_{1,1}}{\overset{E_{1,1}}{\rightleftarrows}} v_1 .$$

This illustrates the case $l_1 = 3$, $l_2 = 3$, $l_3 = 2$ and $n = 8$. The case of general k and the second isomorphism are similar and omitted. ■

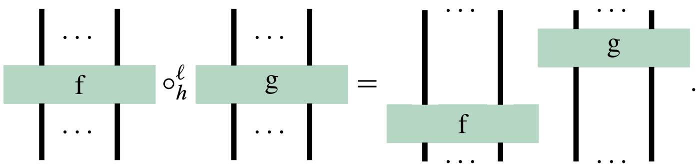
We now reindex the basis of $\mathtt{V}_q$ to $\{v_{1,1}, \dots, v_{1,l_1}, \dots, v_{d,1}, \dots, v_{d,l_d}\}$, which then induces a reindexing of the basis of $\bigwedge_q^k \mathtt{V}_q$ that we will use below.

Lemma 5B.4. *As $U_q(\ell)$-representations we have $\mathtt{V}_q^{\otimes k} \cong \bigwedge_q^k \mathtt{V}_q \oplus W$, and no simple constituent of $\bigwedge_q^k \mathtt{V}_q$ appears in W.*

Proof. By looking at highest weight vectors and classical theory, this follows directly from the first and the second decomposition in Lemma 5B.3. ■

Definition 5B.5. Let $\mathbf{Rep}_q \ell$ denote the category of finite-dimensional $U_q(\ell)$-representations of type 1. We view $\mathbf{Rep}_q \ell$ as pivotal using the above Hopf algebra structure on $U_q(\mathfrak{gl}_n)$. Let further $\mathbf{Fund}_q \ell$ denote the full pivotal subcategory with objects of the form $\bigwedge_q^{\mathtt{K}} \mathtt{V}_q$ for $\mathtt{K} = (k_1, \dots, k_m) \in \mathbb{Z}^m$ and $m \in \mathbb{Z}_{\geq 0}$.

We write $\circ_h^\ell$ for the monoidal structure on $\mathbf{Rep}_q \ell$ and $\mathbf{Fund}_q \ell$ on the morphism level, and $\circ_h$ on the object level. With contrast to equation (4B.1), a picture for the monoidal structure on $\mathbf{Fund}_q \ell$ is



(5B.6)

Remark 5B.7. Equation (5B.6) is a standard construction in the theory of Hopf algebras; see, e.g., [37].

Lemma 5B.8. *We have the following:*

(a) *The category $\mathbf{Rep}_q \ell$ is semisimple, and its simple objects are of the form $L(\lambda_1) \otimes \cdots \otimes L(\lambda_d)$ with the factors being simple objects of $\mathbf{Rep}_q \mathfrak{gl}_{l_i}$.*

(b) *$\mathbf{Fund}_q \mathfrak{gl}_n$ is pivotally equivalent to $\mathbf{Rep}_q \mathfrak{gl}_n$ upon additive idempotent completion.*

Proof. Lemma 5A.5 applies componentwise. ■

We define the *Levi* $(1,1)$*-overcrossings* and *Levi* $(1,1)$*-undercrossing* to be

$$\hat{\mathrm{R}}^{\ell}_{1,1}\colon \mathtt{V}_q\otimes\mathtt{V}_q\to\mathtt{V}_q\otimes\mathtt{V}_q,\quad v_{i,j}\otimes v_{k,l}\mapsto\begin{cases}\hat{\mathrm{R}}_{1,1}(v_{i,j}\otimes v_{i,l}) & \text{if } i=k,\\ v_{k,l}\otimes v_{i,j} & \text{else,}\end{cases}$$

$$(\hat{\mathrm{R}}^{\ell}_{1,1})^{-1}\colon \mathtt{V}_q\otimes\mathtt{V}_q\to\mathtt{V}_q\otimes\mathtt{V}_q,\quad v_{i,j}\otimes v_{k,l}\mapsto\begin{cases}(\hat{\mathrm{R}}_{1,1})^{-1}(v_{i,j}\otimes v_{i,l}) & \text{if } i=k,\\ v_{k,l}\otimes v_{i,j} & \text{else.}\end{cases}$$

In other words, $\hat{\mathrm{R}}^{\ell}_{1,1}$ is the respective $\mathfrak{gl}_{l_i}$ braiding within one block, and the swap map otherwise, and similarly for its inverse. Note that these maps are in general not $U_q(\mathfrak{gl}_n)$-equivariant.

Example 5B.9. In the extreme case that $\ell=\mathfrak{h}$ the Levi $(1,1)$-overcrossings equals the Levi $(1,1)$-undercrossings equals the swap map. ■

Definition 5B.10. For $k,l\in\mathbb{Z}_{\geq 0}$ define $\hat{\mathrm{R}}^{\ell}_{k,l}$ as the composition

$$\textstyle\bigwedge_q^k\mathtt{V}_q\otimes\bigwedge_q^l\mathtt{V}_q\xrightarrow{\mathrm{i}}\mathtt{V}_q^{\otimes(k+l)}\xrightarrow{\mathrm{x}}\mathtt{V}_q^{\otimes(k+l)}\xrightarrow{\mathrm{p}}\bigwedge_q^l\mathtt{V}_q\otimes\bigwedge_q^k\mathtt{V}_q,$$

where are i and p are inclusion and projection, respectively, and x is defined as in equation (3B.8) but with Levi crossings. Define $(\hat{\mathrm{R}}^{\ell}_{k,l})^{-1}$ similarly.

We also get various mates of which we think as rotated versions of the ones in Definition 5B.10.

Lemma 5B.11. *The Levi crossings are* $U_q(\ell)$*-equivariant, satisfy the* Reidemeister II and III relations *and various* naturality relations.

Proof. This follows from a calculation, Lemma 5B.2 and Lemma 5A.9. ■

Define now the following *coil maps*:

$$\begin{aligned}\mathrm{r}_{1,\mathrm{K}}\colon \mathtt{V}_q\otimes\textstyle\bigwedge_q^{\mathrm{K}}\mathtt{V}_q\to\bigwedge_q^{\mathrm{K}}\mathtt{V}_q\otimes\mathtt{V}_q,&\quad v_{i,j}\otimes w\mapsto u_i\hat{\mathrm{R}}^{\ell}_{1,\mathrm{K}}(v_{i,j}\otimes w),\\ \mathrm{r}^{1,\mathrm{K}}\colon \textstyle\bigwedge_q^{\mathrm{K}}\mathtt{V}_q\otimes\mathtt{V}_q\to\mathtt{V}_q\otimes\bigwedge_q^{\mathrm{K}}\mathtt{V}_q,&\quad v\otimes w\mapsto u_i^{-1}(\hat{\mathrm{R}}^{\ell}_{1,\mathrm{K}})^{-1}(w\otimes v_{i,j}).\end{aligned}\tag{5B.12}$$

Note that $\mathrm{r}_{1,\mathrm{K}}$ and $\mathrm{r}^{1,\mathrm{K}}$ are inverses. The coil maps are not $U_q(\mathfrak{gl}_n)$-equivariant in general.

Lemma 5B.13. *The morphisms in equation* (5A.6) *and equation* (5B.12) *are* $\circ_v$-$\circ_h^{\ell}$-*generators of* $\mathbf{Fund}_q\ell$.

Proof. The morphisms in equation (5A.6) are $U_q(\mathfrak{gl}_n)$-equivariant, so they are also $U_q(\ell)$-equivariant, and one easily checks that the morphisms in equation (5B.12) are $U_q(\ell)$-equivariant. That these generate follows from Lemmas 5A.7 and 5B.3. ■

For $\mathtt{K} = \emptyset$, we will write $\mathrm{r}_1 = \mathrm{r}_{1,\emptyset}$.

Definition 5B.14. For $i \in \{1, \dots, d\}$, define

$$\mathrm{pr}_i^{\ell} = \prod_{j \neq i} \frac{\mathrm{r}_1 - u_j}{u_i - u_j} \in \mathrm{End}_{\mathbf{Fund}_q \ell}(\mathtt{V}_q),$$

which we call *Levi block projectors*.

Lemma 5B.15. *We have,*

$$\mathrm{pr}_i^{\ell} \mathrm{pr}_j^{\ell} = \delta_{i,j} \mathrm{pr}_i^{\ell} \quad \textit{and} \quad \mathrm{id}_{\mathtt{V}_q} = \sum_{i=1}^{d} \mathrm{pr}_i^{\ell}.$$

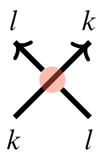
These projectors realize the decomposition

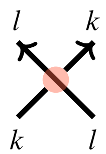
$$\mathtt{V}_q \cong \bigoplus_{i=1}^{d} \mathtt{V}_q(\mathfrak{gl}_{l_i}).$$

Proof. Note that r_1 is given by multiplication by u_i on $v_{i,j}$. Thus, the formula for pr_i^{ℓ} is the usual Lagrange-type interpolation and the claims follow. ∎

5C. Levi crossings

The Levi crossings are not $U_q(\mathfrak{gl}_n)$-equivariant in general, and there is no planar web picture for it. However, it will be helpful to have the following diagrammatic notation. For the Levi (k, l)-overcrossings and the Levi (k, l)-undercrossings, we use

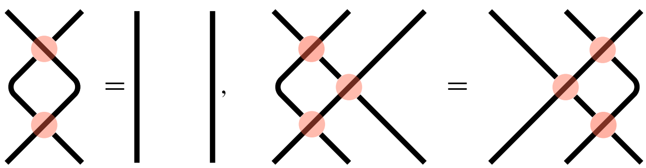
over : $\hat{\mathrm{R}}_{k,l}^{\ell} \leftrightsquigarrow$ 

, under : $(\hat{\mathrm{R}}_{k,l}^{\ell})^{-1} \leftrightsquigarrow$ 

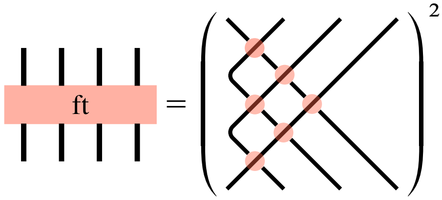
.

We also use rotated pictures for their mates.

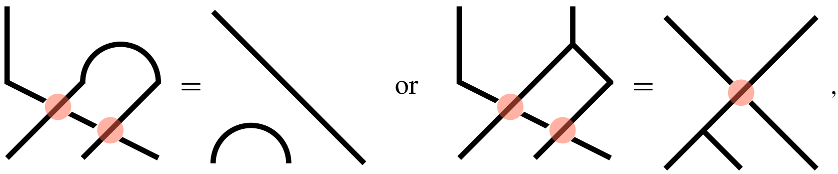
By Lemma 5B.11, we have the *Reidemeister II and III relations*, e.g.,



We can use this to define *Levi braids* associated to any braid word. Of particular importance will be the (positive) *Levi full twist* on k strands (denoted by a box notation). By definition, this map is the square of the positive lift, using Levi over-crossings, of the longest word in the symmetric group on $\{1, \dots, k\}$. For example, for $k = 4$, this full twist is



We also have the usual naturality relations such as



including the various $(\_)^{\updownarrow}$ and $(\_)^{\leftrightarrow}$-duals.

However, we need to be careful with the *Reidemeister I relation* as $\bigwedge_q^k \mathtt{V}_q$ needs not to be simple as a $U_q(\ell)$-representation, see Lemma 5B.3. Nevertheless, we still have the following lemma.

Lemma 5C.1. *The Levi crossings are diagonal matrices in the basis given by the decomposition of* $\bigwedge_q^k \mathtt{V}_q$ *from Lemma* 5B.3.

Proof. Note that the decomposition of $\bigwedge_q^k \mathtt{V}_q$ from Lemma 5B.3 is multiplicity free, and Schur's lemma applies. (We stress that Schur's lemma in this setting does not need the underlying field to be algebraically closed; see, e.g., [1, Corollary 7.4] or [3, Remark 2.29].) ∎

One can check that the diagonal entries mentioned in Lemma 5C.1 are given by products of the Reidemeister I scalars in equation (3B.7).

6. The equivalence

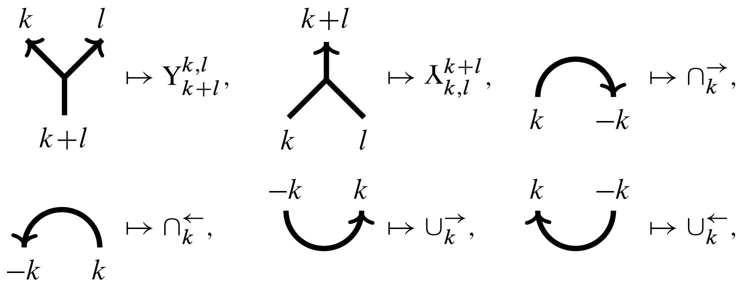
We now state and prove our main result.

6A. A reminder on the $\mathfrak{gl}_n$ story

We first recall the relationship between $\mathfrak{gl}_n$ webs and the representation theory of $U_q(\mathfrak{gl}_n)$. Define a functor

$$\Gamma_q = \Gamma_q^{\text{ext}}(\mathfrak{gl}_n) \colon \mathbf{Web}_q \mathfrak{gl}_n \to \mathbf{Fund}_q \mathfrak{gl}_n$$

sending the object $\mathtt{K}$ to $\bigwedge_q^{\mathtt{K}} \mathtt{V}_q$ and the generating morphisms of $\mathbf{Web}_q\mathfrak{gl}_n$ to the following maps:



and the downwards merges and splits to the respective mates.

Theorem 6A.1. *The functor Γ_q is an equivalence of pivotal categories, and it induces an equivalence of pivotal categories between the additive idempotent completion of* $\mathbf{Web}_q\mathfrak{gl}_n$ *and* $\mathbf{Rep}_q\mathfrak{gl}_n$.

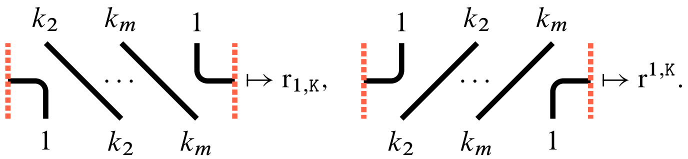
Proof. That Γ_q is fully faithful is [9, Theorem 3.3.1], and the fact that the relevant hom-spaces stay of the same dimension when restricting from $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$. The second claim follows from Lemma 5A.5 (b), or [9, Theorem 3.3.1] and flatness of restriction from $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$. ■

6B. The statement

We now extend the functor Γ_q into a functor

$$\mathbf{A}\Gamma_q = \mathbf{A}\Gamma_q^{\mathrm{ext}}(\ell)\colon \mathbf{AWeb}_q\mathfrak{gl}_n \to \mathbf{Fund}_q\ell.$$

On objects and the generators of $\mathbf{Web}_q\mathfrak{gl}_n$ the functor $\mathbf{A}\Gamma_q$ is defined to be Γ_q. We define $\mathbf{A}\Gamma_q$ on the two coils $\rho_{\mathtt{K}}$ and $\rho^{\mathtt{K}}$ for $k_1 = 1$ by



We call the kernel of the functor $\mathbf{A}\Gamma_q$ the *Levi ideal* and denote it by $\mathcal{I}_\ell$.

Lemma 6B.1. *The Levi ideal $\mathcal{I}_\ell$ is a two-sided $\circ_v$-ideal in* $\mathbf{AWeb}_q\mathfrak{gl}_n$.

Proof. The kernel is a two-sided $\circ_v$-ideal. ■

Thus, we get a well-defined category in the following definition.

Definition 6B.2. Let $\mathbf{AWeb}_q\ell$ denote the quotient of $\mathbf{AWeb}_q\mathfrak{gl}_n$ by the Levi ideal $\mathcal{I}_\ell$.

Theorem 6B.3. *Let* $\mathtt{AdId}(\_)$ *denote additive idempotent completion.*

(a) *We have the commuting diagram of categories*

$$\begin{array}{ccccc}
\mathbf{AWeb}_q\mathfrak{gl}_n & \xrightarrow{\text{equation (4C.3)}} & \mathbf{AWeb}_q\ell & \xrightarrow{\mathtt{AdId}(_)} & \mathtt{AdId}(\mathbf{AWeb}_q\ell) \\
& \searrow^{\mathbf{A\Gamma}_q} & \downarrow{\scriptstyle \mathbf{A\Gamma}_q^\ell}\,\cong & & \cong\,\downarrow{\scriptstyle \mathtt{AdId}(\mathbf{A\Gamma}_q^\ell)} \\
& & \mathbf{Fund}_q\ell & \xrightarrow[\mathtt{AdId}(_)]{} & \mathbf{Rep}_q\ell.
\end{array}$$

The Levi ideal $\mathcal{I}_\ell$ is the two-sided $\circ_v$-$\circ_h^\ell$-ideal generated by the Levi evaluations. Here $\circ_h^\ell$ is the pullback of the monoidal structure from $\mathbf{Fund}_q\ell$ to $\mathbf{AWeb}_q\mathfrak{gl}_n$.

(b) *We have the commuting diagram of pivotal categories*

$$\begin{array}{ccccc}
\mathbf{AWeb}_1\mathfrak{gl}_n & \xrightarrow{\text{equation (4C.3)}} & \mathbf{AWeb}_1\ell & \xrightarrow{\mathtt{AdId}(_)} & \mathtt{AdId}(\mathbf{AWeb}_1\ell) \\
& \searrow^{\mathbf{A\Gamma}_1} & \downarrow{\scriptstyle \mathbf{A\Gamma}_1^\ell}\,\cong & & \cong\,\downarrow{\scriptstyle \mathtt{AdId}(\mathbf{A\Gamma}_1^\ell)} \\
& & \mathbf{Fund}_1\ell & \xrightarrow[\mathtt{AdId}(_)]{} & \mathbf{Rep}_1\ell.
\end{array}$$

The Levi ideal $\mathcal{I}_\ell$ is the two-sided $\circ_v$-$\circ_h^A$-ideal generated by the Levi evaluations.

The proof of Theorem 6B.3 is postponed to Section 7, since we want to focus on applications of this theorem first. For the rest of the section we assume that Theorem 6B.3 holds with the exception of the next subsection where we only assume that $\mathbf{A\Gamma}_q$ is well defined.

6C. Monoidal behavior of the main functor

The categories $\mathbf{AWeb}_q\mathfrak{gl}_n$ and $\mathbf{Rep}_q\ell$ are endowed with monoidal structures that are natural from two different perspectives, as explained in Remark 4B.2. However, as we will elaborate now, these need not to be the same under the equivalence Theorem 6B.3.

Lemma 6C.1. *Assume that the functor $\mathbf{A\Gamma}_q$ is well defined.*

(a) *The functor $\mathbf{A\Gamma}_1$ is pivotal.*

(b) *The functor $\mathbf{A\Gamma}_q$ is not monoidal (and thus not pivotal).*

Proof. (a) Let us consider $q = 1$. In this case the braiding on $\mathbf{Rep}_1\ell$ is given by permutation. Comparing equations (4B.1) and (5B.6), and observing that coils in $\mathbf{Rep}_1\ell$

are permutations, up to diagonal entries of the form u_i on blocks, shows that the functor $\mathbf{A}\Gamma_1$ is monoidal. Pivotality is then clear.

(b) For $q \neq 1$, one can check that the images under $\mathbf{A}\Gamma_q$ of equation (4B.1) is not the same as equation (5B.6). Explicitly, taking $\mathtt{K} = \mathtt{L} = (1)$, $\mathrm{f} = \rho_{\mathtt{K}}$ and $\mathrm{g} = \mathrm{id}_{\mathtt{L}}$ verifies that $\mathbf{A}\Gamma_q$ is not monoidal. ∎

Remark 6C.2. We note that Lemma 6C.1 shows that the choice which side goes over or under in equation (4B.1) matters and gives different results on the representation theoretical side. This indicates that one might need to use the notion of module categories rather than monoidal categories to describe the representation theory associated to $\mathbf{AWeb}_q\mathfrak{gl}_n$. This is similar to, for example, [34] or [12] (via [26, Remark 12]), so coideal subalgebras might play a role.

The quantization issue that we are facing in Lemma 6C.1 is also potentially related to the classification of K-matrices as, for example, in [28] where all nondegenerate solutions to the reflection equation in the quantum case have satisfy a minimal polynomial of order ≤ 2. Diagrammatically K-matrices correspond to coils, but for $\ell = \mathfrak{gl}_{l_1} \oplus \cdots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n$ these coils need to have a minimal polynomial of order d.

Remark 6C.3. One could use the equivalence in Theorem 6B.3 to pullback the monoidal structure of $\mathbf{Rep}_q\ell$, resulting in a monoidal structure on $\mathbf{AWeb}_q\ell$ that is distinct from the one we give in Lemma 4B.3 above. This pullback monoidal structure would not satisfy the conditions in [27, (2.11)], which are necessary for it to be unique, see [27, Proposition 2.5]. Conversely, one could push the monoidal structure from $\mathbf{AWeb}_q\ell$ to $\mathbf{Rep}_q\ell$, resulting in a monoidal structure on $\mathbf{Rep}_q\ell$ satisfying [27, (2.11)].

6D. Ariki–Koike algebras and annular webs

The *Ariki–Koike algebra* $\mathcal{H}_q^{m,d}$ from [4,7,10], using different conventions, is defined as follows.

Definition 6D.1. Fix $m \in \mathbb{Z}_{\geq 0}$, the number of strands, and let $\mathcal{H}_v^{m,d}$ denote the $\mathbb{Z}_v$-algebra with algebra generators $\mathrm{T}_0, \mathrm{T}_1, \dots, \mathrm{T}_{m-1}$ modulo the two-sided ideal generated by

$$\prod_{k=1}^{d}(\mathrm{T}_0 - u_k), \quad \mathrm{T}_0\mathrm{T}_1\mathrm{T}_0\mathrm{T}_1 - \mathrm{T}_1\mathrm{T}_0\mathrm{T}_1\mathrm{T}_0,$$

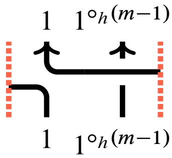
$$\begin{aligned} &(\mathrm{T}_i - q)(\mathrm{T}_i + q^{-1}) && \text{if } i > 0,\\ &\mathrm{T}_i\mathrm{T}_j\mathrm{T}_i - \mathrm{T}_j\mathrm{T}_i\mathrm{T}_j && \text{if } |i-j| = 1,\\ &\mathrm{T}_i\mathrm{T}_j - \mathrm{T}_j\mathrm{T}_i && \text{if } |i-j| > 1, \end{aligned}$$

where $i, j \in \{1, \dots, m-1\}$.

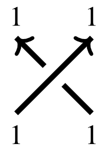
The Ariki–Koike algebra acts on $1^{\circ_h m}$.

Proposition 6D.2. *We have a surjective $\mathbb{K}_q$-algebra homomorphism*

$$\pi_{m,d}\colon \mathcal{H}_q^{m,d} \twoheadrightarrow \mathrm{End}_{\mathbf{AWeb}_q\ell}(1^{\circ_h m}), \quad \mathrm{T}_0 \mapsto$$



$$, \quad \mathrm{T}_i \mapsto$$



.

Here the bottom left strand of the image of T_i *is the* i*th strand from the left.*

Proof. By Theorem 6B.3, this follows from [33, Theorem 4.2] after adjustment of conventions. ∎

We define the usual *Jucys–Murphy elements* as follows.

Definition 6D.3. Define elements of $\mathcal{H}_v^{m,d}$ by $\mathrm{X}_1 = \mathrm{T}_0$ and for $i \in \mathbb{Z}_{\geq 1}$ recursively, $\mathrm{X}_i = \mathrm{T}_i \mathrm{X}_{i-1} \mathrm{T}_i$.

Let $P(m,d)$ denote the set of *d-partitions of m* (which we identify with d-tuples of Young diagrams in the English convention), and for $\boldsymbol{\lambda} \in P(m,d)$ let $\mathrm{Std}(\boldsymbol{\lambda})$ denote the set of all *standard d-tableaux of shape $\boldsymbol{\lambda}$*.

Lemma 6D.4. *For all $\boldsymbol{\lambda} \in P(m,d)$ there exists a simple $\mathcal{H}_q^{m,d}$-representation $V_{\boldsymbol{\lambda}}$, and these form a complete and nonredundant set of simple $\mathcal{H}_q^{m,d}$-representations. Moreover, we have an $\mathbb{K}_q$-algebra isomorphism*

$$\mathcal{H}_q^{m,d} \cong \bigoplus_{\boldsymbol{\lambda} \in P(m,d)} \mathrm{End}_{\mathbb{K}_q}(V_{\boldsymbol{\lambda}}).$$

Finally, $V_{\boldsymbol{\lambda}}$ can be given a $\mathbb{K}_q$-basis $\{v_T \mid T \in \mathrm{Std}(\boldsymbol{\lambda})\}$ such that X_i acts by

$$\mathrm{X}_i \centerdot v_T = u_a q^{2b-2c} v_T, \tag{6D.5}$$

where a is the component of the entry i in T, b is the column number where i appears and c is the row number of i.

Proof. Our assumptions on the involved parameters imply that $\mathcal{H}_q^{m,d}$ is semisimple and the lemma follows from [4, Theorem 3.7] and results in the same section, e.g., [4, Proposition 3.16]. ∎

Let $\mathcal{I}_{>l_1,\dots,>l_d} \subset \mathcal{H}_q^{m,d}$ denote the two-sided ideal generated by the idempotents, realizing the Artin–Wedderburn decomposition in Lemma 6D.4, for d-partitions of m with strictly more than l_i rows in the ith entry.

Example 6D.6. An important special case is $\mathcal{I}_{>1,\dots,>1} \subset \mathcal{H}_q^{m,d}$. In this case the d-partitions indexing the idempotents not in $\mathcal{I}_{>1,\dots,>1}$ are of the form

$$(\underbrace{\square\cdots\square}_{k_1},\dots,\underbrace{\square\cdots\square}_{k_d})$$

for $(k_1,\dots,k_d) \in \mathbb{Z}_{\geq 0}^d$. These are so-called one row d-partitions. ■

Remark 6D.7. The image of $\mathcal{I}_{>l_1,\dots,>l_d} \subset \mathcal{H}_q^{m,d}$ under $\pi_{m,d}$ from Proposition 6D.2 is forcing a condition on the minimal polynomial of coils.

The following definition appears in [23, Section 2C].

Definition 6D.8. Let $\mathcal{B}_q^{\ell}$ be the algebra quotient of $\mathcal{H}_q^{m,d}$ by $\mathcal{I}_{>l_1,\dots,>l_d}$.

Proposition 6D.9. *The map $\pi_{m,d}$ from Proposition 6D.2 induces a $\mathbb{K}_q$-algebra isomorphism*

$$\overline{\pi}_{\ell}\colon \mathcal{B}_q^{\ell} \xrightarrow{\cong} \mathrm{End}_{\mathbf{AWeb}_q\ell}(1^{\circ_h m}).$$

Thus, the kernel of $\pi_{m,d}$ is $\ker(\pi_{m,d}) = \mathcal{I}_{>l_1,\dots,>l_d}$.

Proof. As in the proof of Proposition 6D.2. ■

6E. Cartan subalgebras and generalized blob algebras

We now consider the case of the Cartan subalgebra in detail.

Proposition 6E.1. *The functor $\mathbf{A}\Gamma_q$ descends to an equivalence of categories*

$$\mathbf{A}\Gamma_q^{\mathfrak{h}}\colon \mathbf{AWeb}_q\mathfrak{h} \to \mathbf{Fund}_q\mathfrak{h},$$

and it induces an equivalence of categories between the additive idempotent completion of $\mathbf{AWeb}_q\mathfrak{h}$ and $\mathbf{Rep}_q\mathfrak{h}$.

Proof. Directly from Theorem 6B.3 and Lemma 6C.1. ■

Remark 6E.2. Proposition 6E.1 should be compared with [31, Corollary 43], and can be seen as a quantum version of that corollary.

Now recall the so-called *generalized blob algebra* $\mathcal{B}_q^{m,d}$ in the sense of [25] (which is a special case of Definition 6D.8).

Definition 6E.3. Let $\mathcal{B}_q^{m,d}$ be the algebra quotient of $\mathcal{H}_q^{m,d}$ by $\mathcal{I}_{>1,\dots,>1}$.

Proposition 6E.4. *The map $\pi_{m,d}$ from Proposition 6D.2 induces a $\mathbb{K}_q$-algebra isomorphism*

$$\overline{\pi}_{m,d}\colon \mathcal{B}_q^{m,d} \xrightarrow{\cong} \mathrm{End}_{\mathbf{AWeb}_q\mathfrak{h}}(1^{\circ_h m}).$$

Thus, the kernel of $\pi_{m,d}$ is $\ker(\pi_{m,d}) = \mathcal{I}_{>1,\dots,>1}$.

Proof. By Proposition 6E.1, this follows from [5, Theorem 3.1]. ∎

6F. On two conjectures about end-spaces in annular webs

Recall from Section 4C that e_k denotes the kth elementary symmetric polynomial in d variables. In the special case of the Cartan subalgebra we have $d = n$, and we let $e_k = e_k(u_1, \dots, u_d)$.

Definition 6F.1. Define $e_k^{(1)} = e_k$ and for $i \in \mathbb{Z}_{\geq 1}$ recursively,

$$e_k^{(i)} = e_k^{(i-1)} + (q^2-1)\big(\mathrm{X}_{i-1}e_{k-1}^{(i-1)} - \mathrm{X}_{i-1}^2 e_{k-2}^{(i-1)} + \dots \pm \mathrm{X}_{i-1}^k\big),$$

and let $\mathcal{J}_{>1} \subset \mathcal{H}_q^{m,d}$ denote the two-sided ideal generated by

$$\mathrm{R}_i = \mathrm{X}_i^d - e_1^{(i)}\mathrm{X}_i^{d-1} + \dots \pm e_d^{(i)}.$$

Let $\mathcal{J}_2 \subset \mathcal{H}_q^{m,d}$ denote the two-sided ideal generated by R_2.

Proposition 6F.2. *The kernel of $\pi_{m,d}$ is alternatively given by*

$$\ker(\pi_{m,d}) = \mathcal{J}_2 = \mathcal{J}_{>1}.$$

The same holds for $q = 1$.

Proof. We start by making two claims.

Claim 1. R_2 acts on $V_{\boldsymbol{\lambda}}$ as zero if and only if the d-partition $\boldsymbol{\lambda}$ has at most one row per component.

Proof of Claim 1. The case where $\boldsymbol{\lambda}$ has one node is easy, so assume that $\boldsymbol{\lambda}$ has at least two nodes. We use that $V_{\boldsymbol{\lambda}}$ has a $\mathbb{K}_q$-basis given by v_T for T a standard d-tableaux of shape $\boldsymbol{\lambda}$ on which X_i acts by equation (6D.5). Using equation (6D.5), we can calculate the action of R_2 on the $\mathbb{K}_q$-basis given by the v_T. There are three cases depending on the positions of 1 and 2 in T one needs to check:

(i) $\big(\dots, \boxed{1}, \dots, \boxed{2}, \dots\big)$ or vice versa,

(ii) $\big(\dots, \boxed{1\,|\,2}, \dots\big)$,

(iii) $\left(\dots,\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\dots\right)$.

All of these are annoying but straightforward calculations, and details are omitted. ■

Claim 2. $\mathcal{J}_2 = \mathcal{J}_{>1}$.

Proof of Claim 2. To see this, we note that on $v_T \in V_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda}$ a one row d-partition and T a standard d-tableau of shape $\boldsymbol{\lambda}$, we have

$$e_k^{(i)} \centerdot v_T = e_k(q^{2\alpha_1}u_1, \dots, q^{2\alpha_d}u_d)v_T,$$

where $\alpha_r = |\{s < i \mid s \text{ is in the } r\text{th component of } T\}|$. Using this formula and equation (6D.5) one can recursively check that the above claim holds for R_i for $i \geq 2$, which implies that $\mathcal{J}_{>1} \subset \mathcal{J}_2$, and the proof of the claim is complete. ■

The first claim implies that $\mathcal{J}_2 = \mathcal{I}_2$, and this together with the second claim and Proposition 6E.4 proves the lemma. ■

Recall $\mathbf{Aff}(\mathbf{Web}_q\mathfrak{gl}_n)$ from Lemma 4B.7. Mimicking the construction of $\mathbf{AWeb}_q\ell$ from $\mathbf{AWeb}_q\mathfrak{gl}_n$ as in Definition 6B.2, we denote by $\mathbf{Aff}(\mathbf{Web}_q\mathfrak{gl}_n)/\mathrm{LI}$ (LI stands for Levi ideal) the quotient of $\mathbf{Aff}(\mathbf{Web}_q\mathfrak{gl}_n)$ by essential circles. The following will be compared with [8, Conjectures 10.2 and 10.3] in Proposition 6F.4 below.

Proposition 6F.3. *We have $\mathbb{K}_q$-algebra isomorphisms*

$$\mathcal{H}_q^{m,d}/\mathcal{J}_2 = \mathcal{H}_q^{m,d}/\mathcal{J}_{>1} \xrightarrow{\cong} \mathrm{End}_{\mathbf{AWeb}_q\ell}(1^{\circ_h m}) \xrightarrow{\cong} \mathrm{End}_{\mathbf{Aff}(\mathbf{Web}_q\mathfrak{gl}_n)/LI}(1^{\circ_h m}),$$

with the first map being induced by Proposition 6D.2 and the second being induced by Lemma 4B.7.

Proof. Combine Lemma 4B.7 and Proposition 6F.2. ■

Proposition 6F.4. *Proposition 6F.3 answers* [8, Conjectures 10.2 and 10.3] *affirmatively (up to different ground rings; we address that in Remark 6F.5 below).*

Proof. The common object of interest when comparing [8] to this paper is the annular web category and its various flavors:

(i) First, we have $\mathbf{AWeb}_q\mathfrak{gl}_n$ and $\mathbf{AWeb}_q\ell$ and these are the main objects of study in this paper.

(ii) We also have $\mathbf{Aff}(\mathbf{Web}_q\mathfrak{gl}_n)$ and $\mathbf{Aff}(\mathbf{Web}_q\mathfrak{gl}_n)/\mathrm{LI}$, the versions defined by affinization. These are studied in [8] for $\mathfrak{sl}_n$ instead of $\mathfrak{gl}_n$.

We first note that the difference between $\mathfrak{sl}_n$ instead of $\mathfrak{gl}_n$ plays no key role in the sense that the relevant hom-spaces are of the same dimension and all maps are defined verbatim. On the representation theoretical side this is well known, for the webs see, e.g., [35, Remark 1.1].

In [8, Section 10] Cautis–Kamnitzer define a map from the affine Hecke algebra $\mathcal{H}_q^{\mathrm{aff}}$ to the category $\mathbf{Aff(Web}_q\mathfrak{gl}_n)$. In [8, Conjecture 10.3] they conjecture what the kernel of this map is. In Proposition 6F.3 we identify the kernel of the respective map from the Ariki–Koike algebra $\mathcal{H}_q^{m,d}$ to $\mathbf{AWeb}_q\ell$. Thus, we get the following comparison diagram:

$$\begin{array}{ccccc}
\mathcal{H}_q^{\mathrm{aff}} & \xrightarrow{\text{map in [8, Section 10]}} & \mathbf{Aff(Web}_v\mathfrak{gl}_n) & \xleftrightarrow[\text{Proposition 6F.3}]{\cong} & \mathbf{AWeb}_q\mathfrak{gl}_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{H}_q^{m,d} & \xrightarrow[\text{via the right equivalence}]{\text{map in Proposition 6D.2}} & \mathbf{Aff(Web}_q\mathfrak{gl}_n)/\mathrm{LI} & \xleftrightarrow[\cong]{\text{Proposition 6F.3}} & \mathbf{AWeb}_q\ell.
\end{array}$$

Comparison of definitions shows that this diagram commutes. Similarly for [8, Conjecture 10.2] which follows from the $q = 1$ version of the above. ∎

Remark 6F.5. Recall from Notation 2.2 that we work over a field containing variables $\mathbb{U} = \{u_1, \dots, u_d\}$ as well as $\mathbb{U}^{-1}$. We use this crucially in equation (4C.3) where we evaluate essential circles to elementary symmetric polynomials e_k in these variables and their inverses.

On the other hand, the ground ring used in [8, Section 10] is

$$E = \mathbb{C}(q)[\tilde{e}_1, \dots, \tilde{e}_n],$$

where $\tilde{e}_k$ is the kth elementary symmetric function (not the polynomial), and these elementary symmetric function compare to our variables $\mathbb{U}$.

Note that [8, Section 10] does not have $\mathbb{U}^{-1}$ and in this sense Proposition 6F.3 and [8, Conjectures 10.2 and 10.3] are strictly speaking not comparable.

6G. Working integrally

Note that our main result Theorem 6B.3 is not stated or proven over $\mathbb{Z}_v$, and we work over $\mathbb{K}_q$ in which case $\mathbf{AWeb}_q\ell$ and $\mathbf{Rep}_q\ell$ are semisimple. Working *integrally*, that is, over $\mathbb{Z}_v$ or even $\mathbb{Z}[v, v^{-1}, \mathbb{U}]$ needs some nontrivial extra steps:

(i) Theorem 6A.1 works over $\mathbb{Z}_v$, see [13, Theorem 2.58], which uses the light ladder strategy from that paper and [2, Theorem 3.1]. Passing to an appropriate field (e.g., $\overline{\mathbb{F}}_p$ for a prime p) one gets an equivalence of pivotal categories between the additive idempotent completion of $\mathbf{Web}_q\mathfrak{gl}_n$ and the category of tilting modules $\mathbf{Tilt}_v\mathfrak{gl}_n$.

(ii) The relation from diagram categories to tilting modules is a folk observation in the field; see, e.g., [13, Theorem 2.58], [2, Section 5A], [36, Proposition 2.28] or [6, Theorem 1.1] for some examples.

(iii) Thus, it is tempting to conjecture that integral versions of Theorem 6B.3 and its consequences involve $\mathbf{Tilt}_v\ell$, e.g., under appropriate assumptions on the underlying field the additive idempotent completion of $\mathbf{AWeb}_v\ell$ should be equivalent to $\mathbf{Tilt}_v\ell$. However, there is a nontrivial catch: the quantization does not behave very well; see, e.g., Section 6C or [8, Section 10]. As sketched in Remark 6C.2, this might indicate that quantum groups are not the correct objects to use in this setting.

(iv) Note that the blob algebra is not defined integrally, and it is also not clear from the definition how to work integrally. Let us however point out that the description of $\mathcal{B}_q^{m,d}$ from [23, Theorem 2.15] works integrally and might play a role in the integral story.

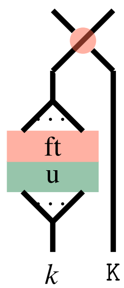
We decided not to pursue these points further in this work.

7. Proof of the main theorem

7A. Well-definedness

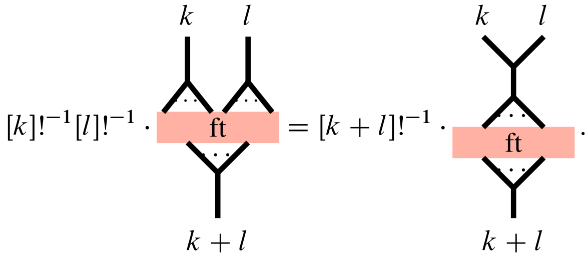
Recall that the images of the coils are defined by explosion, that is, we mimic equation (4B.5) on the side of the representation theory. We define $\mathrm{u}\colon \mathtt{V}_q^{\otimes k} \to \mathtt{V}_q^{\otimes k}$ as the map sending $v_{i_1,j_1} \otimes \cdots \otimes v_{i_k,j_k}$ to $u_{i_1} \cdots u_{i_k} v_{i_1,j_1} \otimes \cdots \otimes v_{i_k,j_k}$. We use a box to denote this map in illustrations.

Lemma 7A.1. *We have*

$\mathbf{A}\Gamma_q(\rho_{(k,\mathtt{K})}) = [k]!^{-1} \cdot$  .

Proof. Using the Levi crossings introduced in Section 5C, we easily check that $\mathbf{A}\Gamma_q$ sends the coil $\rho_{(k,\mathtt{K})}$ to the claimed picture. ∎

Lemma 7A.2. *In* $\mathbf{Fund}_q\ell$*, we have*



Proof. We first suppose that the strands are oriented upward. The image of $\mathrm{Y}_{k+l}^{1,\dots,1}$ is an $U_q(\ell)$-subrepresentation of $\mathtt{V}_q^{\otimes k+l}$. The Levi full twist ft is $U_q(\ell)$-invariant, so this subrepresentation remains invariant. But this subrepresentation is isomorphic to $\bigwedge_q^{k+l}\mathtt{V}_q$ and since the weight spaces of $\bigwedge_q^{k+l}\mathtt{V}_q$ are of dimension one, we deduce that the image of a vector v_S by $\mathrm{ft}\circ\mathrm{Y}_{k+l}^{1,\dots,1}$ is a multiple of $\mathrm{Y}_{k+l}^{1,\dots,1}(v_S)$, say $s_S\mathrm{Y}_{k+l}^{1,\dots,1}(v_S)$. Hence, $\mathrm{Y}_{k+l}^{k,l}\circ\lambda_{1,\dots,1}^{k+l}\circ\mathrm{ft}\circ\mathrm{Y}_{k+l}^{1,\dots,1}$ sends v_S to

$$s_S\mathrm{Y}_{k+l}^{k,l}\circ\lambda_{1,\dots,1}^{k+l}\circ\mathrm{Y}_{k+l}^{1,\dots,1}(v_S) = s_S[k+l]!\mathrm{Y}_{k+l}^{k,l}(v_S)$$

and $(\lambda_{1,\dots,1}^{k}\otimes\lambda_{1,\dots,1}^{l})\circ\mathrm{ft}\circ\mathrm{Y}_{k+l}^{1,\dots,1}$ sends v_S to

$$s_S(\lambda_{1,\dots,1}^{k}\otimes\lambda_{1,\dots,1}^{l})\circ\mathrm{Y}_{k+l}^{1,\dots,1}(v_S) = s_S[k]![l]!\mathrm{Y}_{k+l}^{k,l}(v_S).$$

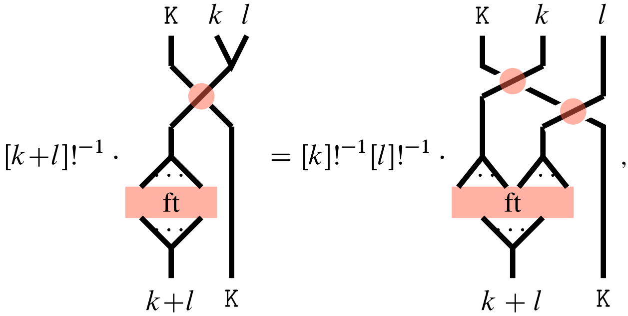
The same arguments shows that the equality also holds with strands oriented downward. ■

Lemma 7A.3. *The relation* (4A.2) *is satisfied after applying the functor* $\mathbf{A}\Gamma_q$.

Proof. Clear. ■

Lemma 7A.4. *The relation* (4A.3) *is satisfied after applying the functor* $\mathbf{A}\Gamma_q$.

Proof. Using the associativity of splits and merges and the fact that Levi crossings satisfy the Reidemeister III relation, it remains to prove

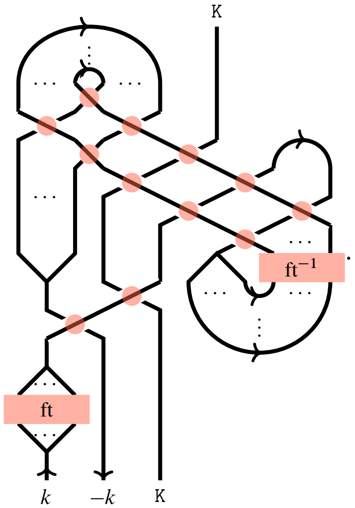


which follows immediately from Lemma 7A.2. ■

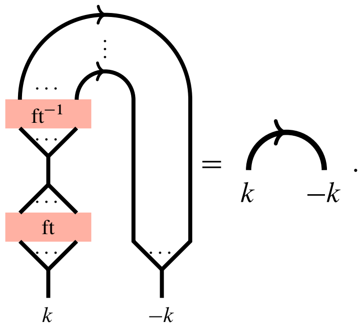
We can show similarly that merges slide through coils.

Lemma 7A.5. *The relation* (4A.4) *is satisfied after applying the functor* $\mathbf{A}\Gamma_q$.

Proof. The image of the right-hand side of equation (4A.4) is given, up to the multiplication by elements of $\mathbb{U}$ that cancel out, by

$[k]!^{-2}\cdot$ 

Using Reidemeister II relations and the fact that mates of merges are splits and vice versa, it remains to prove the following equality:

$[k]!^{-2}\cdot$ 

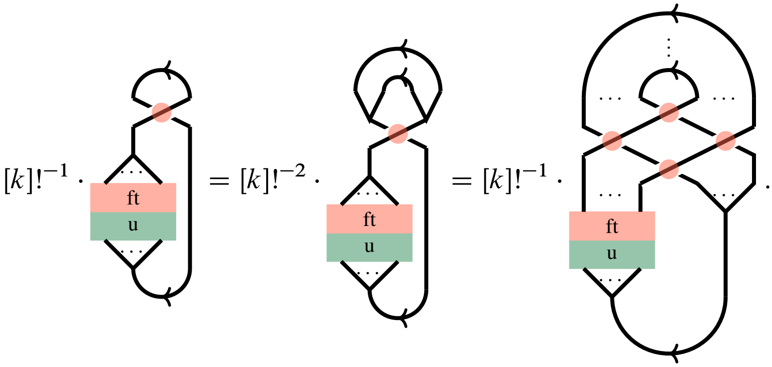
In order to get rid of the explosions between the Levi full twists, we apply repeatedly Lemma 7A.2 with $l=1$ and use the fact that the Levi full twist on k strands can be obtain from the Levi full twist on $k-1$ strands and some extra crossings. We can conclude using explosion as in Lemma 3B.6.

The argument for the leftward oriented cap is similar and omitted. ■

We can show similarly that cups slide through coils.

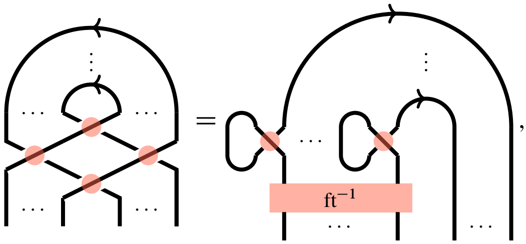
Lemma 7A.6. *The relations in equation* (4C.3) *are in the kernel of* $\mathbf{A}\Gamma_q$.

Proof. The image of the leftward oriented essential circle with a strand of thickness k through $\mathbf{A}\Gamma_q$ is given by

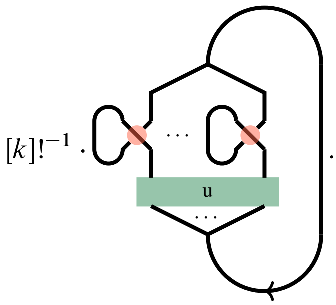


In this calculation the first equality is obtained from explosion of strands, and the last equality is obtained from Lemma 7A.2.

Now, since



it remains to compute the following scalar:



We easily check that the twist sends $v_{i,j}$ to $q^{-n+2(l_1+\dots+l_d)}v_{i,j}$, and therefore the previous scalar is equal to

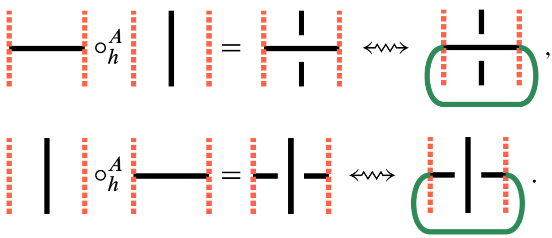
$$\sum_{k_1+\dots+k_d=k}\ \sum_{\substack{1\leq j_{1,1}<\dots<j_{1,k_1}\leq l_1,\\ \dots\\ 1\leq j_{d,1}<\dots<j_{d,k_d}\leq l_d}} u_1^{k_1}\cdots u_d^{k_d}$$
$$\times\prod_{i=1}^{d} q^{-n+2(l_1+\dots+l_d)k_i+\sum_{r=1}^{k_i}(n+1-2(l_1+\dots+l_{i-1}+j_{i,r}))}$$
$$=\sum_{k_1+\dots+k_d=k}\ \sum_{\substack{1\leq j_{1,1}<\dots<j_{1,k_1}\leq l_1,\\ \dots\\ 1\leq j_{d,1}<\dots<j_{d,k_d}\leq l_d}} u_1^{k_1}\cdots u_d^{k_d}\prod_{i=1}^{d} q^{k_i-\sum_{r=1}^{k_i}2j_{i,r}}$$
$$=e_k(q^{-1}u_1,q^{-3}u_1,\dots,q^{-2l_1+1}u_1,\dots,q^{-1}u_d,q^{-3}u_d,\dots,q^{-2l_d+1}u_d),$$

which is what we needed to show. ■

Lemma 7A.7. *The functor* $\mathbf{A}\Gamma_q$ *is well defined and descends to the functor* $\mathbf{A}\Gamma_q^{\ell}$. *Moreover, for* $q\neq 1$ *the Levi ideal* $\mathcal{I}_\ell$ *contains the two-sided* $\circ_v$*-ideal and right* $\circ_h^A$*-ideal generated by the Levi evaluations, while for* $q=1$ *the Levi ideal* $\mathcal{I}_\ell$ *contains the two-sided* $\circ_v$-$\circ_h^A$*-ideal generated by the Levi evaluations.*

Proof. We need to check that the relations in Definition 4A.1 are satisfied and that the Levi evaluations from equation (4C.3) are in the kernel of $\mathbf{A}\Gamma_q$. This follows as a combination of Lemma 7A.5 and Lemma 7A.6. By Lemma 6C.1, the statement about the Levi ideal for $q=1$ follows from that. To verify the claim for $q\neq 1$, we recall from Remark 4A.5 that the coils pass in front of the annulus. Now we observe that, for example, (the ⇝ refers to Remark 4A.5)

right $\circ_h^A$-ideal:
left $\circ_h^A$-ideal:



(7A.8)

In the top picture the essential circle is not pierced by the identity morphism, while in the bottom it is. More generally, multiplying essential circles from the right (but not from the left) by any morphism f produces a picture with f not interfering with the essential circles and the essential circles are on the outside. Thus, by our convention from Remark 4A.5, the essentially circle evaluations will not change by right multiplication. ■

For the remainder of the paper, we assume Lemma 7A.7.

7B. Proof for $q = 1$

Recall that a colored permutation on $\{1,\dots,d\}^k$ is a permutation on $\{1,\dots,k\}$ that preserves the labels. In terms of classical permutation diagrams these are permutations with colored strands such that crossings preserve the colors, e.g.,

, where $i, j \in \{1,\dots,d\}$.

Recall also the web block projectors pr_i^w from Definition 4A.7, respectively, the Levi block projectors pr_i^ℓ from Definition 5B.14. We use these to define a certain basis in the following definition, where S_k is the symmetric group on $\{1,\dots,k\}$.

Definition 7B.1. Fix $k \in \mathbb{Z}_{\geq 0}$. Let $s = (s_1,\dots,s_k), t \in \{1,\dots,d\}^k$ and let σ be a colored permutation from s to t such that the longest word of $\mathrm{S}_{l_i+1} \subset \mathrm{S}_k$ does not appear in the permutation of color i. (This condition is vacuous for $l_i \geq k$ or if the color i does not appear strictly more than l_i times.) Define

$$\nu^1_{s,t,\sigma} = \sigma \circ_v (\mathrm{pr}^w_{s_1} \circ^A_h \dots \circ^A_h \mathrm{pr}^w_{s_d}), \quad \mathrm{n}^1_{s,t,\sigma} = \sigma \circ_v (\mathrm{pr}^\ell_{s_1} \circ^\ell_h \dots \circ^\ell_h \mathrm{pr}^\ell_{s_d}),$$

where we view σ as an element of $\mathbf{AWeb}_1\ell$ and $\mathbf{Fund}_1\ell$, respectively. The respective sets (collecting these elements for all $k \in \mathbb{Z}_{\geq 0}$) of these are denoted by $B(\mathbf{AWeb}_1\ell)$ and $B(\mathbf{Fund}_1\ell)$.

Lemma 7B.2. *We have* $\mathrm{A}\Gamma^\ell_1(B(\mathbf{AWeb}_1\ell)) = B(\mathbf{Fund}_1\ell)$ *and this set is a* $\mathbb{K}_1$*-linear independent set in* $\coprod_k \mathrm{End}_{\mathbf{Fund}_1\ell}(\mathrm{V}_1^{\otimes k})$.

Proof. We get $\mathrm{A}\Gamma^\ell_1(B(\mathbf{AWeb}_1\ell)) = B(\mathbf{Fund}_1\ell)$ directly from Lemma 6C.1 and the definition.

To prove faithfulness, let $\mathrm{pr}^\ell_s = \mathrm{pr}^w_{s_1} \circ^A_h \dots \circ^A_h \mathrm{pr}^w_{s_d}$. By Lemma 5B.15 and construction, we have

$$\mathrm{n}^1_{s,t,\sigma} \circ_v \mathrm{pr}^\ell_s = \mathrm{pr}^\ell_t \circ_v \mathrm{n}^1_{s,t,\sigma} = \mathrm{n}^1_{s,t,\sigma} \quad \text{and} \quad \mathrm{n}^1_{s,t,\sigma} \circ_v \mathrm{pr}^\ell_q = \mathrm{pr}^\ell_r \circ_v \mathrm{n}^1_{s,t,\sigma} = 0$$

for $q \neq s$ and $r \neq t$. Thus, it suffices to show that $(\mathrm{n}^1_{s,t,\sigma})_\sigma$ is $\mathbb{K}_1$-linear independent (that is, we can fix s and t). After sorting the colors of s and t, it remains to verify faithfulness within a block, i.e., for $\mathfrak{gl}_{l_i}$. Thus, classical theory applies: for $\mathfrak{gl}_{l_i}$ and S_k it is known that Schur–Weyl duality gives the generalized Temperley–Lieb algebra as the endomorphism algebra. This algebra admits a description in terms of a quotient of S_k by the longest word of S_{l_i+1}; see, e.g., [16, Section 3], and the elements of $B(\mathbf{Fund}_1\ell)$ describe the associated standard-type basis within each block. ■

Lemma 7B.3. *The set $B(\mathbf{AWeb}_1\ell)$ $\mathbb{K}_1$-linearly spans $\coprod_k \mathrm{End}_{\mathbf{AWeb}_1\ell}(1^{\circ_h k})$. Moreover, the set $B(\mathbf{Fund}_1\ell)$ is a $\mathbb{K}_1$-linear spanning set of $\coprod_k \mathrm{End}_{\mathbf{Fund}_1\ell}(\mathtt{V}_1^{\otimes k})$.*

Proof. We first recall that the full antisymmetrizer is zero in $\mathbf{Fund}_1\ell$; see, e.g., [26, Lemma 11]. Thus, the same holds in $\mathbf{AWeb}_1\ell$ by Lemma 7B.2. That is, in formulas we have

$$\Big(\sum_{\sigma\in S_{l_i+1}}(-1)^{\mathrm{l}(\sigma)}\sigma(\mathrm{pr}_i^{\ell})^{\circ_h^{\ell}(l_i+1)}=0\Big)\Rightarrow\Big(\sum_{\sigma\in S_{l_i+1}}(-1)^{\mathrm{l}(\sigma)}\sigma(\mathrm{pr}_i^{w})^{\circ_h^{A}(l_i+1)}=0\Big),$$

and both hold.

To address the first statement of the lemma, observe that $\coprod_k \mathrm{End}_{\mathbf{AWeb}_1\ell}(1^{\circ_h k})$ is generated as a $\mathbb{K}_1$-algebra by crossings and coils since we can remove essential circles by Lemma 7A.7. Moreover, the sliding relations (4A.3) and (4A.4) imply that we have the usual $\mathbb{K}_1$-linear spanning set given by first coils and then crossings. Observe next that the web block projectors $\mathbb{K}_1$-linear span the subalgebra generated by coils, so it remains to see that the symmetric group part is $\mathbb{K}_1$-linear spanned by σ such that the longest word of S_{l_i+1} does not appear in the permutation of color i. This however is a consequence of the vanishing of the antisymmetrizer.

For the second statement of the lemma we use Lemmas 5B.3, 5B.15 and Schur's lemma. (As explained in the proof of Lemma 5C.1, Schur's lemma still holds in this setting although $\mathbb{K}_1$ is not necessary algebraically closed.) ■

Lemma 7B.4. *The set $B(\mathbf{AWeb}_1\ell)$ is a $\mathbb{K}_1$-basis of $\coprod_k \mathrm{End}_{\mathbf{AWeb}_1\ell}(1^{\circ_h k})$. Furthermore, the set $B(\mathbf{Fund}_1\ell)$ is a $\mathbb{K}_1$-basis of $\coprod_k \mathrm{End}_{\mathbf{Fund}_1\ell}(\mathtt{V}_1^{\otimes k})$.*

Proof. Combine Lemma 7B.2 and Lemma 7B.3. (Note that Lemma 7B.2 also proves that $B(\mathbf{AWeb}_1\ell)$ is a $\mathbb{K}_1$-linear independent set.) ■

Proposition 7B.5. *The functor $\mathbf{A}\Gamma_1^{\ell}$ is fully faithful and the Levi ideal $\mathcal{I}_\ell$ is the two-sided $\circ_v$-$\circ_h^A$-ideal generated by the Levi evaluations.*

Proof. Lemma 3A.9 implies that we only need to show that $\coprod_k \mathrm{End}_{\mathbf{AWeb}_1\ell}(1^{\circ_h k})$ and $\coprod_k \mathrm{End}_{\mathbf{Fund}_1\ell}(\mathtt{V}_1^{\otimes k})$ are isomorphic $\mathbb{K}_1$-vector spaces with $\mathbf{A}\Gamma_1^{\ell}$ inducing an isomorphism, and this follows from Lemma 7B.2 and Lemma 7B.4. The proof is complete. ■

Proof of Theorem 6B.3 (b). Lemma 7A.7 shows that the functors $\mathbf{A}\Gamma_1$ and $\mathbf{A}\Gamma_1^{\ell}$ are well defined and Lemma 6C.1 shows the statements involving the pivotal structure. Fully faithfulness follows from Proposition 7B.5, and Lemma 5B.8 ensures that we have that $\mathtt{AdId}(\mathbf{Fund}_1\ell)$ is equivalent to $\mathbf{Rep}_1\ell$. These statements taken together complete the proof using the usual properties of the additive idempotent completion. ■

7C. Proof for $q \neq 1$

We start with the following lemma.

Lemma 7C.1. *The functor* $\mathrm{A}\Gamma_q$ *is full.*

Proof. It is clear that the image of the crossings and the coils span. (Note that there is no issue with essential circles in $\mathbf{Fund}_q\ell$.) ■

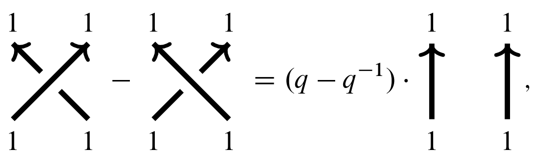
For $q \neq 1$, we note that we can mimic Definition 7B.1 on the Levi side (the only difference is that we use a positive lift, in Levi crossings, of σ instead of σ itself) to define $\mathrm{n}^q_{s,t,\sigma}$ as well as $B(\mathbf{Fund}_q\ell)$. Now we use that and Lemma 7C.1 to define $\nu^q_{s,t,\sigma}$ as well as $B(\mathbf{AWeb}_q\ell)$ by pulling back the elements from $\mathrm{n}^q_{s,t,\sigma}$ by choosing a preimage.

Lemma 7C.2. *The set* $B(\mathbf{AWeb}_q\ell) \subset \coprod_k \mathrm{End}_{\mathbf{AWeb}_q\ell}(1^{\circ_h k})$ *is* $\mathbb{K}_q$*-linearly independent. Moreover, the set* $B(\mathbf{Fund}_q\ell)$ *is* $\mathbb{K}_q$*-linearly independent in* $\coprod_k \mathrm{End}_{\mathbf{Fund}_q\ell}(\mathtt{V}_q^{\otimes k})$.

Proof. The claim on the Levi side can be proven verbatim as in Lemma 7B.2, so our focus is on the web side. However, by construction, the set $B(\mathbf{AWeb}_q\ell)$ is then sent to $B(\mathbf{Fund}_q\ell)$, so $B(\mathbf{AWeb}_q\ell)$ is $\mathbb{K}_q$-linearly independent because $B(\mathbf{Fund}_q\ell)$ is. ■

Lemma 7C.3. *The set* $B(\mathbf{AWeb}_q\ell)$ $\mathbb{K}_q$*-linearly spans* $\coprod_k \mathrm{End}_{\mathbf{AWeb}_q\ell}(1^{\circ_h k})$*. Moreover, the set* $B(\mathbf{Fund}_q\ell)$ *is a* $\mathbb{K}_q$*-linear spanning set of* $\coprod_k \mathrm{End}_{\mathbf{Fund}_q\ell}(\mathtt{V}_q^{\otimes k})$.

Proof. We can remove essential circles in front of the cylinder by definition of the monoidal product, see also equation (7A.8), and the Levi ideal. The essential circles in the back of the cylinder labeled by 1 can then be inductively removed by using



which is the classical skein relation (that holds in our setting by using equation (3B.5)). Using explosion, the rest of the argument is the same as in the proof of Lemma 7B.3 by using that the $q \neq 1$ basis agrees with $B(\mathbf{AWeb}_1\ell)$ on the associated graded by filtration by number of crossings (using the skein relations). ■

Lemma 7C.4. *The set $B(\mathbf{AWeb}_q\ell)$ is a $\mathbb{K}_q$-basis of $\coprod_k \mathrm{End}_{\mathbf{AWeb}_q\ell}(1^{\circ_h k})$. Furthermore, the set $B(\mathbf{Fund}_q\ell)$ is a $\mathbb{K}_q$-basis of $\coprod_k \mathrm{End}_{\mathbf{Fund}_q\ell}(\mathbb{V}_q^{\otimes k})$.*

Proof. By Lemma 7C.2 and Lemma 7C.3. ∎

Remark 7C.5. We do not have or need any explicit description of the elements of $B(\mathbf{AWeb}_q\ell)$ in terms of webs.

Proposition 7C.6. *The functor $\mathrm{A}\Gamma_q^\ell$ is fully faithful and the Levi ideal $\mathcal{I}_\ell$ is the two-sided $\circ_v$-$\circ_h^\ell$-ideal generated by the Levi evaluations. Here $\circ_h^\ell$ is the pullback of the monoidal structure from $\mathbf{Fund}_q\ell$ to $\mathbf{AWeb}_q\mathfrak{gl}_n$.*

Proof. As in the proof of Proposition 7B.5. ∎

Proof of Theorem 6B.3 (a). Using the above statements, this can be proven verbatim as Theorem 6B.3 (b). ∎

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