The partial Temperley–Lieb algebra and its representations

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Abstract. In this paper, we give a combinatorial description of a new diagram algebra, the partial Temperley–Lieb algebra, arising as the generic centralizer algebra $\mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k})$, where $V = V(0) \oplus V(1)$ is the direct sum of the trivial and natural module for the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{gl}_2)$. It is a proper subalgebra of the Motzkin algebra (the $\mathbf{U}_q(\mathfrak{sl}_2)$-centralizer) of Benkart and Halverson. We prove a version of Schur–Weyl duality for the new algebras, and describe their generic representation theory.

Dedicated to the memory of Georgia Benkart

1. Introduction

The Temperley–Lieb algebra $\mathsf{TL}_k(\delta)$ arose in [33] in connection with the Potts model in mathematical physics. It was rediscovered by Vaughan Jones in his seminal work [15–19] on subfactors, in the guise of a von Neumann algebra, enabling spectacular applications to knot theory and many subsequent developments (see, e.g., [9, 22, 27, 32, 34]). An important feature of these algebras is that when the ground ring is a field and $\delta = \pm(q + q^{-1})$,

$$\mathsf{TL}_k(\delta) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(V(1)^{\otimes k})$$

for almost all values of the parameter q, where $V(1)$ is the 2-dimensional type-**1** simple $\mathbf{U}_q(\mathfrak{sl}_2)$-module (the "natural" module). Kauffman [21] (see also [4]) found the now standard realization of $\mathsf{TL}_k(\delta)$ in terms of planar Brauer diagrams in the Brauer algebra.

The partial Brauer algebra $\mathsf{PB}_k(\delta, \delta')$, the span of all partition k-diagrams with blocksize at most two, was studied in [12, 28]. It comes naturally equipped with two independent parameters δ, δ' of disparate topological significance. In [1] (see also [5]) Benkart and Halverson introduced the Motzkin algebra $\mathsf{M}_k(\delta, \delta')$, the subalgebra of $\mathsf{PB}_k(\delta, \delta')$ spanned by planar partial Brauer diagrams. It is known [7] that

$\mathsf{PB}_k(\delta, \delta') \cong \mathsf{PB}_k(\delta, 1)$ for any $\delta' \neq 0$, so it is natural to restrict one's attention to $\mathsf{PB}_k(\delta, 1)$ and $\mathsf{M}_k(\delta, 1)$. For simplicity, set $\mathsf{M}_k(\delta) := \mathsf{M}_k(\delta, 1)$; this is the context of [1]. When the ground ring is a field and $\delta = 1 \pm (q + q^{-1})$, they obtain an isomorphism

$$\mathsf{M}_k(\delta) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(V^{\otimes k}) \tag{1}$$

for almost all values of q, where $V = V(1) \oplus V(0)$ is the direct sum of the natural module $V(1)$ (as above) and the trivial module $V(0)$. Actually, (1) is proved in [1] only for the case $\delta = 1 - (q + q^{-1})$, but it is easily extended to the above statement. Notice that the right hand side of (1) is independent of the choice of sign, so

$$\mathsf{M}_k(1 + (q + q^{-1})) \cong \mathsf{M}_k(1 - (q + q^{-1})).$$

The isomorphism (1) was unfortunately misstated in [1], where $\mathbf{U}_q(\mathfrak{gl}_2)$ incorrectly appeared in place of $\mathbf{U}_q(\mathfrak{sl}_2)$ on the right hand side. The two centralizers differ; indeed, their dimensions do not agree (see Section 10).

The purpose of this paper is to identify a subalgebra $\mathsf{PTL}_k(\delta)$ of $\mathsf{M}_k(\delta)$, the *partial Temperley–Lieb algebra* of the title, such that when $\delta = 1 \pm (q + q^{-1})$,

$$\mathsf{PTL}_k(\delta) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k}) \tag{2}$$

for almost all values of q. The algebra $\mathsf{PTL}_k(\delta)$ has a basis indexed by the set of *balanced* Motzkin k-diagrams (diagrams with the same number of cups as caps). It is notable that the basis elements are not diagrams; instead, each basis element is an alternating sum of diagrams with a unique maximal balanced diagram as leading term. As the right hand side of (2) is again independent of the choice of sign, we have

$$\mathsf{PTL}_k(1 + (q + q^{-1})) \cong \mathsf{PTL}_k(1 - (q + q^{-1})).$$

The use of the adjective "partial" in describing the algebras $\mathsf{PTL}_k(\delta)$ fits into a more general scheme of "partialization" that goes back at least to [29].

When the ground ring is a field, all irreducible representations of a semisimple cellular algebra are absolutely irreducible. In other words, cellular algebras over a field are semisimple if and only if they are split semisimple, so the adjective "split" is often omitted in describing these algebras in the semisimple case. This applies to the algebras $\mathsf{TL}_k(\delta)$, $\mathsf{M}_k(\delta)$, and $\mathsf{PTL}_k(\delta)$ appearing in this paper.

Our main results are as follows:

(i) In Theorem 4.1, we find two natural bases $\{\overline{d}\}$, $\{\widetilde{d}\}$ of $\mathsf{PTL}_k(\delta)$, each indexed by the set of balanced Motzkin k-diagrams. Theorem 3.5 works out a multiplication rule for each basis.

(ii) We show that $\mathsf{PTL}_k(\delta)$ is an iterated inflation of Temperley–Lieb algebras, in the sense of [11, 24, 25], hence is cellular in the sense of [10]; see Remark 5.3.

More precisely, in Theorem 5.2 we prove the stronger result that $\mathsf{PTL}_k(\delta)$ is Morita equivalent to a direct sum of Temperley–Lieb algebras with parameter $\delta - 1$.

(iii) We construct the cell modules for $\mathsf{PTL}_k(\delta)$, see Theorem 8.2, and prove that when the ground ring is a field and $\mathsf{TL}_n(\delta - 1)$ is semisimple for all $n \leq k$ then the same is true of $\mathsf{PTL}_k(\delta)$, see Theorem 5.4.

(iv) In Theorem 11.2, we slightly extend the aforementioned Schur–Weyl duality result of [1], showing that there are many choices in how to make $\mathsf{M}_k(\delta)$ act on $V^{\otimes k}$, all of which imply that $\mathsf{M}_k(\delta)$ is isomorphic to the $\mathbf{U}_q(\mathfrak{sl}_2)$-centralizer of $V^{\otimes k}$ under suitable hypotheses.

(v) For $\delta = 1 \pm (q + q^{-1})$, under suitable hypotheses, we define a faithful action of $\mathsf{PTL}_k(\delta)$ on $V^{\otimes k}$ commuting with the action of $\mathbf{U}_q(\mathfrak{gl}_2)$, and prove the isomorphism (2) in Theorem 12.4. Thus we obtain a version of Schur–Weyl duality for $V^{\otimes k}$ regarded as a bimodule for $\mathsf{PTL}_k(\delta)$, $\mathbf{U}_q(\mathfrak{gl}_2)$.

Finally, in Appendix A, we reinterpret results of [9] to obtain a precise semisimplicity criterion for $\mathsf{TL}_k(\delta)$, when $\delta = \pm(q + q^{-1})$ and the ground ring is a field.

2. Some diagram algebras

In this section, we define the standard diagram algebras needed for this paper. Unless stated otherwise, we work over an arbitrary unital commutative ring $\Bbbk$.

2.1. Terminology

Let $[k] := \{1, \ldots, k\}$ and $[k]' := \{1', \ldots, k'\}$. The set P_k is the collection of all set partitions (equivalence relations) on $[k] \cup [k]'$. If $d = \{B_1, \ldots, B_l\}$ belongs to P_k where $B_1, \ldots, B_l$ are pairwise disjoint, we call the B_i the *blocks* of d. Typically, d is depicted by a graph on $2k$ vertices arranged in two parallel rows in a rectangle, with vertices in the top (resp., bottom) row indexed by $[k]$ (resp., $[k]'$) in order from left to right. Edges are drawn in the interior of the rectangle in any way such that the resulting connected components coincide with the blocks. Although this graphical depiction of elements of P_k is not in general unique, the lack of uniqueness causes no difficulty. To be precise, we define a *k-diagram* d to be the equivalence class of all graphs depicting its underlying set partition d, where two such depictions are equivalent if and only if they have the same blocks. Henceforth, we identify elements of P_k with their corresponding k-diagrams.

If d_1, d_2 are k-diagrams, their *composite configuration* $\Gamma(d_1, d_2)$ is the graph obtained by placing d_1 above d_2 and identifying the corresponding vertices in the middle row. Let $d_1 \circ d_2$ be the diagram obtained by retaining the edges with endpoints in

the union of the top and bottom rows of vertices in the composite configuration, along with those vertices, and discarding the rest of the configuration. The multiplication ∘ is associative, so $(\mathsf{P}_k, \circ)$ is a monoid, the *partition monoid*.

One often identifies diagrams with morphisms in a suitable category. In this paper, the reader should think of a diagram as depicting a morphism going from its bottom to top row, so that products depict compositions in which morphisms act on the left of arguments.

2.2. The partition algebra

We refer the reader to [13] for basic properties of partition algebras. For any $\delta \in \mathbb{k}$, the *partition algebra* $\mathsf{P}_k(\delta)$ is a twisted semigroup algebra on P_k. As a $\mathbb{k}$-module, $\mathsf{P}_k(\delta) = \mathbb{k}[\mathsf{P}_k]$, the collection of $\mathbb{k}$-linear combinations of elements of P_k. Given k-diagrams d_1, d_2,

$$d_1 d_2 = \delta^{N(d_1, d_2)} d_3 = \delta^{N(d_1, d_2)} (d_1 \circ d_2)$$

where $N(d_1, d_2)$ is the number of interior blocks (connecting vertices in the middle row) in $\Gamma(d_1, d_2)$ that get discarded in forming $d_3 = d_1 \circ d_2$. Extending this multiplication rule to linear combinations bilinearly as usual, the set $\mathsf{P}_k(\delta)$ becomes an associative algebra with unit.
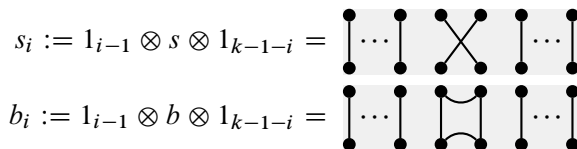
Given diagrams $d_1 \in \mathsf{P}_k$, $d_2 \in \mathsf{P}_l$, let $d_1 \otimes d_2 \in \mathsf{P}_{k+l}$ be the diagram obtained by placing d_1 to the left of d_2. (The notation $\otimes$ in this context is not a tensor product.) The following basic diagrams

$$1 := \Big|\,, \quad p := \begin{matrix} \bullet \\ \bullet \end{matrix}\,, \quad s := \times\,, \quad b := \Big\Vert$$
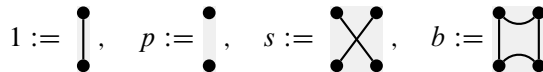
are fundamental building blocks for all partition diagrams. Notice that $1_k = 1 \otimes \cdots \otimes 1$ (with k factors) is the identity element of $\mathsf{P}_k(\delta)$; in the sequel we will often abuse notation by writing 1 in place of 1_k. In the set P_k, define

$$p_j := 1_{j-1} \otimes p \otimes 1_{k-j} = \Big|\cdots\Big| \begin{matrix} \bullet \\ \bullet \end{matrix} \Big|\cdots\Big|$$
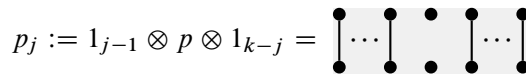
with the isolated vertices in the jth column, for $j = 1, \ldots, k$, and define

$$s_i := 1_{i-1} \otimes s \otimes 1_{k-1-i} = \Big|\cdots\Big| \times \Big|\cdots\Big|$$

$$b_i := 1_{i-1} \otimes b \otimes 1_{k-1-i} = \Big|\cdots\Big| \Vert \Big|\cdots\Big|$$

for $i = 1, \ldots, k-1$. The elements p_j, s_i, b_i form a set of generators of $\mathsf{P}_k(\delta)$; its defining relations are given in [13, Theorem 1.11]. (Note that b_i is denoted by $p_{i+\frac{1}{2}}$

in that reference.) We also need the diagrams

$$e := \quad\text{and}\quad e_i := 1_{i-1} \otimes e \otimes 1_{k-1-i} = $$

for any $i = 1, \ldots, k - 1$. The reader may easily check that the elements e_i satisfy the identity

$$e_i = b_i \, p_i \, p_{i+1} b_i$$

in $\mathsf{P}_k(\delta)$; this identity provides an alternative definition of e_i.

2.3. The partial Brauer algebra

The subalgebra of $\mathsf{P}_k(\delta)$ spanned by the k-diagrams in which each block has cardinality at most 2 is the (one-parameter) *partial Brauer algebra* $\mathsf{PB}_k(\delta, \delta)$ of [12, 28], who studied its more general two-parameter variant $\mathsf{PB}_k(\delta, \delta')$, with multiplication defined by

$$d_1 d_2 = \delta^{N_1(d_1, d_2)} \delta'^{N_2(d_1, d_2)} d_3 = \delta^{N_1(d_1, d_2)} \delta'^{N_2(d_1, d_2)} (d_1 \circ d_2), \tag{3}$$

where $N_1(d_1, d_2)$ (resp., $N_2(d_1, d_2)$) is the number of interior loops (resp., interior paths, including paths consisting of a single vertex) in the middle row of $\Gamma(d_1, d_2)$, and $d_3 = d_1 \circ d_2$ is the product in the partition monoid P_k.

Note that e_i, s_i ($i \in [k-1]$) and p_j ($j \in [k]$) belong to $\mathsf{PB}_k(\delta, \delta')$; in fact, this is a set of generators of that algebra. Both e_i and p_i are pseudo-idempotents, satisfying

$$e_i^2 = \delta e_i, \quad p_i^2 = \delta' p_i.$$

For any $\delta' \neq 0$, we have $\mathsf{PB}_k(\delta, \delta') \cong \mathsf{PB}_k(\delta, 1)$; see (8) in Section 3. Thus it makes sense to focus on $\mathsf{PB}_k(\delta, 1)$.

2.4. The Motzkin algebra

A k-diagram that can be drawn without any intersections is said to *planar*. The *Motzkin algebra* $\mathsf{M}_k(\delta, \delta')$ studied in [1] is the subalgebra of $\mathsf{PB}_k(\delta, \delta')$ spanned by the planar partial Brauer k-diagrams; that is, planar diagrams with blocks of cardinality at most 2.

The algebra $\mathsf{PB}_k(\delta, \delta')$ contains elements $r_i := p_i s_i$ and $l_i := s_i p_i$, depicted by

$$r_i = 1_{i-1} \otimes r \otimes 1_{k-1-i} = $$

$$l_i = 1_{i-1} \otimes l \otimes 1_{k-1-i} = $$

for $i \in [k-1]$, where r, l in P_2 are given by

$$r = \quad \diagdown \quad, \qquad l = \quad \diagdown \, .$$

As the r_i, l_i are planar, they belong to the Motzkin algebra $\mathsf{M}_k(\delta, \delta')$. The diagrams p_i are also planar, hence belong to $\mathsf{M}_k(\delta, \delta')$. They satisfy

$$p_i = r_i l_i = l_{i-1} r_{i-1}$$

for all values of the indices for which the equalities are sensible. It is shown in [1] that

$$\mathsf{M}_k(\delta, \delta') \text{ is generated by the } e_i, r_i, l_i \quad (i \in [k-1]).$$

(The element e_i is denoted by t_i in [1].) A set of defining relations for these generators can be found in [31].

2.5. The Temperley–Lieb algebra

The Brauer algebra $\mathsf{B}_k(\delta)$ is the subalgebra of $\mathsf{P}_k(\delta)$ spanned by the k-diagrams in which all blocks have exactly two elements. The Temperley–Lieb algebra $\mathsf{TL}_k(\delta)$ is the subalgebra of $\mathsf{B}_k(\delta)$ spanned by the planar k-diagrams which are also in the Brauer algebra. It is the subalgebra generated by $e_1, \ldots, e_{k-1}$. As such, it is isomorphic to the algebra defined by those generators and satisfying the relations

$$e_i^2 = \delta e_i, \quad e_i e_{i \pm 1} e_i = e_i, \quad e_i e_j = e_j e_i \text{ if } |i - j| > 1. \tag{4}$$

The rank of $\mathsf{TL}_k(\delta)$ over $\Bbbk$ is equal to the kth Catalan number $\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k}$.

It is noteworthy that the algebra morphism defined on generators by $e_i \mapsto -e_i$ for all i defines an isomorphism of algebras

$$\mathsf{TL}_k(\delta) \cong \mathsf{TL}_k(-\delta).$$

So the choice of sign of the parameter is purely a matter of convenience.

If $\Bbbk$ is a field and $0 \neq q$ is an element of $\Bbbk$ such that $q^2 \neq 1$, it is well known that

$$\mathsf{TL}_k(\pm(q + q^{-1})) \cong \mathrm{End}_{\mathbf{U}}(V(1)^{\otimes k}) \tag{5}$$

where $V(1)$ is the 2-dimensional "natural" representation of the quantized enveloping algebra $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_2)$. There is a (unique) copy of the trivial $\mathbf{U}$-module in $V(1) \otimes V(1)$. In the isomorphism (5), we identify

$$e_i = \pm(q + q^{-1})1^{\otimes(i-1)} \otimes \pi \otimes 1^{\otimes(k-i-1)},$$

where π is an orthogonal projection onto that trivial module and 1 denotes the identity map. See Section 10 for details.

Finally, in Appendix A we show that if $\Bbbk$ is a field and q a nonzero element of $\Bbbk$ satisfying the condition $[\![1]\!]_q[\![2]\!]_q \cdots [\![k]\!]_q \neq 0$ (where $[\![k]\!]_q$ is the balanced form of a quantum integer) then $\mathsf{TL}_k(\pm(q + q^{-1}))$ is semisimple over $\Bbbk$. In particular, this semisimplicity statement holds whenever q is not a root of unity.

3. Alternating bases

Vaughan Jones [20] (see also [2, 3, 13]) introduced the *orbit* basis $\{o_d \mid d \in \mathsf{P}_k\}$ of $\mathsf{P}_k(\delta)$, defined as follows. Given k-diagrams d_1, d_2, write

$$d_1 \leq d_2 \iff \text{each block of } d_1 \text{ is contained in some block of } d_2. \qquad (6)$$

The relation $\leq$ is a partial order on the set P_k of k-diagrams. The orbit basis

$$\{o_d \mid d \in \mathsf{P}_k\}$$

is defined by demanding that the unitriangular relation $d = \sum_{d \leq d'} o_{d'}$ hold for every k-diagram d.

A different basis $\{\bar{d} \mid d \in \mathsf{P}_k\}$, also in a unitriangular relation with the diagram basis, is defined by setting

$$\bar{d} = \sum_{d' \leq d} (-1)^{\beta(d) - \beta(d')} d' \qquad (7)$$

for any k-diagram d, where $\beta(d)$ is the number of blocks of d. This basis is the *alternating* basis; it plays a crucial role in this paper.

The blocks of a partial Brauer diagram all have cardinality at most 2, so blocks are either singletons (isolated vertices) or edges (having two vertices as endpoints). For partial Brauer diagrams d, d' the relation $d \leq d'$ defined in (6) holds if and only if all the edges of d are also edges of d'. Equivalently, $d \leq d'$ if and only if d is obtainable from d' by excising zero or more of its edges.

If d is a partial Brauer diagram then any term in the right hand side of the expansion (7) is (up to sign) also a partial Brauer diagram. The same holds for planar partial Brauer diagrams (that is, Motzkin diagrams). Hence, $\bar{d} \in \mathsf{PB}_k(\delta, \delta')$ for any partial Brauer diagram d, and similarly $\bar{d} \in \mathsf{M}_k(\delta, \delta')$ for any planar partial Brauer diagram d.

Lemma 3.1. *For any unital commutative ring $\Bbbk$ and any $\delta, \delta' \in \Bbbk$, the sets*

$$\{\bar{d} \mid d \text{ a partial Brauer diagram}\}, \quad \{\bar{d} \mid d \text{ a Motzkin diagram}\}$$

are bases of $\mathsf{PB}_k(\delta, \delta')$, $\mathsf{M}_k(\delta, \delta')$ respectively.

Proof. By the remarks preceding the lemma, the transition matrix expressing the $\overline{d}$ in terms of the diagram basis in each algebra is unitriangular with respect to any linear order extending $\leq$. ∎

For any invertible $\delta' \in \Bbbk$, by [7, 28] there is an algebra isomorphism

$$\mathsf{PB}_k(\delta, \delta') \cong \mathsf{PB}_k(\delta, 1) \tag{8}$$

defined by replacing the generator p_i by p_i/δ'. Thus, there is no loss of generality in setting $\delta' = 1$, so from now on we work in $\mathsf{PB}_k(\delta, 1)$ and in its subalgebra $\mathsf{M}_k(\delta, 1)$. This is convenient because p_i, $1 - p_i$ become a pair of commuting orthogonal idempotents.

For a given partial Brauer diagram d, consider the subsets $\lfloor d \rfloor$, $\lceil d \rceil$ of $[k]$ (the *bottom*, *top frame*, respectively, of d) defined by

$$\lfloor d \rfloor = \{i \in [k] \mid \text{vertex } i' \text{ is nonisolated in } d\},$$
$$\lceil d \rceil = \{i \in [k] \mid \text{vertex } i \text{ is nonisolated in } d\}.$$

Elements of the set $\lfloor d \rfloor' \cup \lceil d \rceil$ form the *frame* of d and label the endpoints of the edges in d; its complement in $[k]' \cup [k]$ labels the isolated vertices in d.

Proposition 3.2. *Let d be a partial Brauer diagram. The identities:*

(a) $\overline{d} = \prod_{i \in \lceil d \rceil}(1 - p_i)d \prod_{i' \in \lfloor d \rfloor}(1 - p_i)$,

(b) $\overline{d} = \rho_0(d) - \rho_1(d) + \rho_2(d) - \rho_3(d) + \cdots$

hold in $\mathsf{PB}_k(\delta, 1)$, *where* $\rho_i(d)$ *is the sum of all diagrams obtained from d by removing exactly i of its edges. If d is planar (i.e., a Motzkin diagram) then all terms on the right hand side of the identities are also planar, hence belong to* $\mathsf{M}_k(\delta, 1)$.

Proof. (a) It follows from the definition (3) of diagram multiplication that left (resp., right) multiplication by p_i removes the edge with endpoint i (resp., i'), for any $i \in \lceil d \rceil$, $i' \in \lfloor d \rfloor$. Expanding the products on the left and right of identity (a) thus gives

$$\overline{d} = \sum_{d' \leq d} (-1)^{\mathrm{edges}(d) - \mathrm{edges}(d')} d'$$

where $\mathrm{edges}(d)$ is the number of edges in d. Horizontal edges are seemingly removed twice, once for each endpoint, but in fact the second multiplication by the appropriate p_i acts as the identity, thus does not matter. For partial Brauer diagrams, the above expansion coincides with the expression in (7).

(b) This follows from the displayed equation in the proof of (a). ∎

Remark 3.3. The formula in part (b) of the proposition says that, for partial Brauer diagrams d, the element $\bar{d}$ is obtained from d by *inclusion-exclusion edge removal*. As $\rho_0(d) = d$, the leading term in the expansion is d itself.

An edge of a partial Brauer diagram d is a *cup* (resp., *cap*) if both of its endpoints are in $[k]$ (resp., in $[k]'$). Such edges are also called *horizontal*. For a partial Brauer diagram d, we define

$$
\tilde{d} := \prod_{i \in \lceil d \rceil_H} (1 - p_i) d \prod_{i' \in \lfloor d \rfloor_H} (1 - p_i),
$$
$$
\hat{d} := \prod_{i \in \lceil d \rceil_V} (1 - p_i) d \prod_{i' \in \lfloor d \rfloor_V} (1 - p_i),
$$
(9)

where $\lceil d \rceil_H$, $\lfloor d \rfloor_H$ respectively index the endpoints of horizontal edges in the top, bottom rows of d, and similarly $\lceil d \rceil_V$, $\lfloor d \rfloor_V$ respectively index the endpoints of vertical edges in the top, bottom rows of d. Notice that either product in the definition of $\hat{d}$ may be omitted without changing the result. The element $\tilde{d}$ (resp., $\hat{d}$) is obtained from d by *inclusion-exclusion horizontal (resp., vertical) edge removal*. We linearly extend the notations $\bar{d}, \tilde{d}, \hat{d}$ to linear combinations of diagrams.

Proposition 3.4. *Let d be a partial Brauer diagram. Then*

(a) $\bar{d} = \hat{\tilde{d}} = \tilde{\hat{d}}$,

(b) $\tilde{d} = \rho_0^H(d) - \rho_1^H(d) + \rho_2^H(d) - \cdots$,

(c) $\hat{d} = \rho_0^V(d) - \rho_1^V(d) + \rho_2^V(d) - \cdots$,

where $\rho_i^H(d)$ (resp., $\rho_i^V(d)$) is the sum of all diagrams obtained from d by removing i of its horizontal (resp., vertical) edges. Hence the sets

$$
\{\tilde{d} \mid d \in \mathsf{PB}_k\}, \quad \{\tilde{d} \mid d \in \mathsf{M}_k\}
$$

are $\Bbbk$-bases of $\mathsf{PB}_k(\delta, 1)$, $\mathsf{M}_k(\delta, 1)$, respectively.

Proof. (a) follows from the product formula in Proposition 3.2 (a) and the fact that the p_i pairwise commute.

(b), (c) follow by expanding the right hand side in (9).

Applying the operator $d \mapsto \tilde{d}$ to both sides of the identity in part (c) shows that if d is a partial Brauer diagram then $\bar{d}$ is expressible as an alternating sum of the form

$$
\bar{d} = \tilde{d} - \tilde{d}_1 + \tilde{d}_2 - \cdots,
$$

where each $d_i < d$. So the transition between the sets $\{\bar{d} \mid d \in \mathsf{PB}_k\}$ and $\{\tilde{d} \mid d \in \mathsf{PB}_k\}$ is unitriangular. The same holds if we restrict to Motzkin diagrams. The final claim now follows from Lemma 3.1. ∎

We will need to consider various subalgebras of the Motzkin algebra $\mathsf{M}_k(\delta)$. Let R_k be the subalgebra generated by $r_1, \ldots, r_{k-1}$ and L_k the subalgebra generated by $l_1, \ldots, l_{k-1}$. Write RL_k for the subalgebra generated by R_k and L_k. Notice that $\tilde{d} = d$ for any d in RL_k, since such d have no horizontal edges.

Theorem 3.5. *Suppose that d_1, d_2 are partial Brauer k-diagrams. Let $N_1(d_1, d_2)$ be the number of closed loops in the middle of $\Gamma(d_1, d_2)$, as in equation (3), and set $d_3 = d_1 \circ d_2$, the product in the partition monoid P_k. Let*

$$\Omega(d_1, d_2) = \big(([k] \setminus \lfloor d_1 \rfloor) \cap \lceil d_2 \rceil_H\big) \cup \big(([k] \setminus \lceil d_2 \rceil) \cap \lfloor d_1 \rfloor_H\big),$$

the set indexing the vertices in the middle row of $\Gamma(d_1, d_2)$ for which an isolated vertex in one diagram is identified with a horizontal edge endpoint in the other. Then:

(a) $\overline{d}_1 \overline{d}_2 = \begin{cases} (\delta - 1)^{N_1(d_1, d_2)} \overline{d}_3 & \text{if } \lfloor d_1 \rfloor = \lceil d_2 \rceil, \\ 0 & \text{otherwise,} \end{cases}$

(b) $\tilde{d}_1 \tilde{d}_2 = \begin{cases} (\delta - 1)^{N_1(d_1, d_2)} \prod_{i \in S} (1 - p_i) \tilde{d}_3 & \text{if } \Omega(d_1, d_2) \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}$

The set S in formula (b) is the set of indices on the top vertex of any through edge that snakes through some cups and caps in the middle row of $\Gamma(d_1, d_2)$ before emerging to connect to a vertex in the bottom row.

Proof. (a) The proof is based on the formula in Proposition 3.2 (a). First suppose that $\lfloor d_1 \rfloor$ does not match $\lceil d_2 \rceil$. Then there must be at least one isolated vertex that matches up with the endpoint of some edge. We can always insert a copy of p_i corresponding to that vertex, as multiplication by p_i is identity on an isolated vertex in the ith position. This shows that the product $\overline{d}_1 \overline{d}_2 = 0$.

From now on, suppose that $\lfloor d_1 \rfloor$ matches $\lceil d_2 \rceil$. There are four cases to consider. First, suppose that two propagating edges meet in the middle row of $\Gamma(d_1, d_2)$ at the ith identified vertex. Then the idempotent $1 - p_i$ can be commuted to both sides of $d_1 d_2$, as $1 - p_j$ on the left and $1 - p_m$ on the right, where the corresponding edge in $d_3 = d_1 d_2$ connects the jth vertex on the top row to the mth on the bottom.
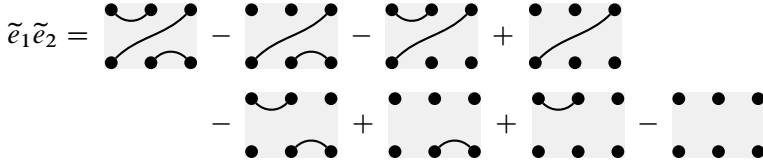
Now suppose that there is an added cup in d_3 that is not present in d_1. This means two propagating edges in d_1 join up with a connected path in the middle of $\Gamma(d_1, d_2)$ to form that additional cup. In this situation, we can commute all the interior idempotents on the path up to the top (to the left of d_3).

The case of an added cap in d_3 that is not present in d_2 is analogous to the previous case.

It remains only to consider loops in the middle row of $\Gamma(d_1, d_2)$. For each such loop, the product of the middle idempotents is equivalent to multiplication by the scalar $\delta - 1$. This completes the proof of (a).

(b) The proof of part (b) is similar to that for part (a), based on the formula in Proposition 3.4 (b). ∎

For example, if $k = 3$ one may verify that $\tilde{e}_1\tilde{e}_2$ is equal to the 8-term linear combination

$$\tilde{e}_1\tilde{e}_2 = \text{[diagram]} - \text{[diagram]} - \text{[diagram]} + \text{[diagram]}$$
$$- \text{[diagram]} + \text{[diagram]} + \text{[diagram]} - \text{[diagram]}$$

illustrating the equality $\tilde{e}_1\tilde{e}_2 = (1 - p_3)\widetilde{e_1e_2}$ in Theorem 3.5 (b).

Corollary 3.6. *If x is a diagram in RL_k and d is a partial Brauer diagram, then*

(a) $\quad x\tilde{d} = \begin{cases} \widetilde{xd} & \text{if } \lceil d \rceil_H \subset \lfloor x \rfloor, \\ 0 & \text{otherwise,} \end{cases}$

(b) $\quad \tilde{d}x = \begin{cases} \widetilde{dx} & \text{if } \lfloor d \rfloor_H \subset \lceil x \rceil, \\ 0 & \text{otherwise.} \end{cases}$

Proof. These formulas follow easily from part (b) of Theorem 3.5. We prove formula (a). Since x belongs to RL_k, it has no horizontal edges, so $\lfloor x \rfloor_H$ is empty. This means that $\Omega(x, d)$ is empty if and only if $\lceil d \rceil_H \subset \lfloor x \rfloor$, and furthermore, there are no interior closed loops in $\Gamma(x, d)$. Formula (a) thus follows. The proof of formula (b) is symmetric. ∎

4. The partial Temperley–Lieb algebra

We are now ready to define the main object of study for this paper. Henceforth, we set $\mathsf{M}_k(\delta) := \mathsf{M}_k(\delta, 1)$.

We say that a Motzkin diagram is *balanced* if it has the same number of cups as caps. Equivalently, a Motzkin diagram is balanced if the number of isolated vertices in each of its rows is the same. Let

$$\mathcal{D}(k) := \{d \mid d \text{ is a balanced Motzkin } k\text{-diagram}\}.$$

We define the *partial Temperley–Lieb algebra* $\mathsf{PTL}_k(\delta)$ to be the linear span of

$$\{\tilde{d} \mid d \in \mathcal{D}(k)\}.$$

That $\mathsf{PTL}_k(\delta)$ is a subalgebra of $\mathsf{M}_k(\delta)$ follows from Theorem 3.5.

Theorem 4.1. *Let $\Bbbk$ be a commutative ring and fix $\delta \in \Bbbk$. Then either of the sets $\{\widetilde{d} \mid d \in \mathcal{D}(k)\}$, $\{\overline{d} \mid d \in \mathcal{D}(k)\}$ is a $\Bbbk$-basis of $\mathsf{PTL}_k(\delta)$.*

Proof. The set $\{\overline{d} \mid d \in \mathcal{D}(k)\}$ is linearly independent by Lemma 3.1. Since it spans the algebra, it is a basis. Furthermore, as in the final paragraph of the proof of Proposition 3.4, the transition matrices between the sets $\{\widetilde{d} \mid d \in \mathcal{D}(k)\}$, $\{\overline{d} \mid d \in \mathcal{D}(k)\}$ are unitriangular, so $\{\widetilde{d} \mid d \in \mathcal{D}(k)\}$ is also a basis. ∎

Let $\mathcal{D}_n(k)$ be the subset of $\mathcal{D}(k)$ consisting of those balanced Motzkin k-diagrams having exactly n edges. Notice that $\mathcal{D}_k(k)$ is the set of Temperley–Lieb diagrams on $2k$ vertices. To any $d \in \mathcal{D}_n(k)$, we associate a triple (A, t, B), where $t \in \mathcal{D}_n(n)$ is the unique Temperley–Lieb diagram on $2n$ vertices obtained by deleting the isolated vertices in d, and A, B are the subsets of $[k]$ respectively indexing the nonisolated vertices in the top, bottom row of d. For example,

$$d = \quad$$
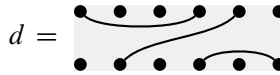
corresponds to the triple (A, t, B), where $A = \{1, 4, 5\}$, $B = \{2, 4, 6\}$, and

$$t = e_1 e_2 = \quad$$
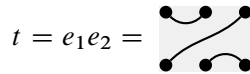
in $\mathcal{D}_3(3)$. Since any diagram d is reconstructible from its triple, the following result is clear.

Lemma 4.2. *The map $d \mapsto (A, t, B)$ defines a bijection between $\mathcal{D}_n(k)$ and the set of triples of the above form.*

From now on, write $d(A, t, B)$ for the k-diagram in $\mathcal{D}_n(k)$ corresponding in the above lemma to a given triple (A, t, B) such that A, B are subsets of $[k]$ of cardinality n and $t \in \mathcal{D}_n(n)$. In other words, the map $(A, t, B) \mapsto d(A, t, B)$ is the inverse of the bijection in Lemma 4.2.

The cardinality of the set TL_n of Temperley–Lieb n-diagrams is equal to the nth Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. As $\mathcal{D}(k) = \bigsqcup_{n=0}^{k} \mathcal{D}_n(k)$ (disjoint union), by combining Lemma 4.2 with Theorem 4.1, we have

$$\mathrm{rank}_{\Bbbk} \, \mathsf{PTL}_k(\delta) = |\mathcal{D}(k)| = \sum_{n=0}^{k} \binom{k}{n}^2 \mathcal{C}_n. \tag{10}$$

If $t \in \mathcal{D}_n(n)$ is a Temperley–Lieb n-diagram, there are two equally natural ways to extend t to a diagram in $\mathcal{D}_n(k)$, for any $k \geq n$. Either of the maps

$$t \mapsto t \otimes \omega_{k-n}, \quad t \mapsto t \otimes 1_{k-n} \tag{11}$$

will do the job, where ω_j is the j-diagram in which all $2j$ vertices are isolated. The following observation is an immediate consequence of Theorem 3.5 (a).

Lemma 4.3. *Write* $t_0 = t \otimes \omega_{k-n}$, $t_1 = t \otimes 1_{k-n}$ *for the image of t in* $\mathcal{D}_n(n)$ *under the map* (11). *For any $n \leq k$, either of the linear maps such that $t \mapsto \overline{t}_0$, $t \mapsto \overline{t}_1$ defines an algebra isomorphism of* $\mathsf{TL}_n(\delta - 1)$ *with a subalgebra of* $\mathsf{M}_k(\delta)$.

Let RP_k (resp., LP_k) be the subalgebra of $\mathsf{M}_k(\delta)$ generated by all r_i, p_i (resp., all l_i, p_i). The following result is a variant of [1, equation (2.12)].

Lemma 4.4. *Let $d = d(A, t, B)$ be a diagram in* $\mathcal{D}_n(k)$*, where t is in* $\mathcal{D}_n(n)$*. In the Motzkin algebra* $\mathsf{M}_k(\delta)$*, we have the factorizations*

$$d = r_A(t \otimes \omega_{k-n})l_B, \quad d = r_A(t \otimes 1_{k-n})l_B$$

where $r_A \in \mathsf{RP}_k$ (resp., $l_B \in \mathsf{LP}_k$) is the unique k-diagram with edges from the first n bottom-row (resp., top-row) vertices connecting to the top-row (resp., top-row) vertices indexed by A (resp., B), in order.

Proof. The second factorization is the RTL factorization in [1, Section 2]. It does not matter in that argument if we replace the identity edges in 1_{k-n} by the isolated vertices in ω_{k-n}, which yields the first factorization. ∎

We illustrate the proof of Lemma 4.4. As an example of the RTL factorization in [1], we have
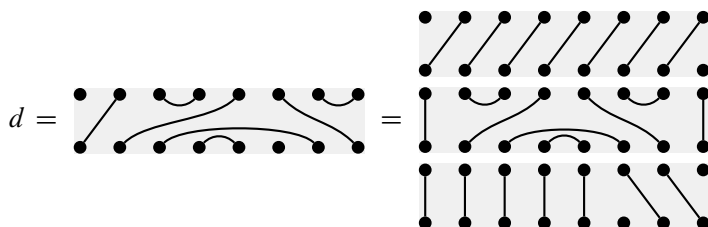
in which the middle diagram is the diagram $t \otimes 1_1$, where $t \in \mathcal{D}_7(7)$ is the diagram

It is clear that replacing $t \otimes 1_1$ in the middle of the above stacked product by $t \otimes \omega_1$ makes no difference in the product.

The next task is to identify a set of generators for $\mathsf{PTL}_k(\delta)$. To that end, we set

$$\varepsilon_i := \widetilde{e}_i = (1 - p_i)e_i(1 - p_i).$$

We note that the equality $\varepsilon_i = (1 - p_i)e_i(1 - p_i)$ remains valid if we replace either or both factors of $(1 - p_i)$ by $(1 - p_{i+1})$. Since the diagrams l_i, r_i, and p_i have no horizontal edges, it is clear that $\widetilde{l}_i = l_i$, $\widetilde{r}_i = r_i$, and $\widetilde{p}_i = p_i$.

Theorem 4.5. $\mathsf{PTL}_k(\delta)$ *is generated by the set* $\{l_i, r_i, \varepsilon_i \mid i \in [k-1]\}$.

Proof. Let P be the subalgebra generated by the given set. By Theorem 4.1, it suffices to show that $\tilde{d}$ belongs to P, for any d in $\mathcal{D}(k)$. Let $d = d(A, t, B)$ as discussed in the paragraph after Lemma 4.2. By Lemma 4.4, we have $d = r_A(t \otimes \omega_{k-n}) \otimes l_B$.

We will need to distinguish generators with differing number of vertices, so we temporarily (in this proof only) write $e_i(n)$ for the diagram e_i in $\mathcal{D}(n)$, for each $n \leq k$. For any $1 \leq n \leq k-1$, let

$$e_{i,n}(k) = e_i(n) \otimes \omega_{k-n} = e_i(k) p_{n+1} \cdots p_k.$$

Then by Corollary 3.6, $\tilde{e}_{i,n}(k) = \tilde{e}_i(k) p_{n+1} \cdots p_k = \varepsilon_i p_{n+1} \cdots p_k$, hence belongs to P. Now let

$$t = e_{i_1}(n) e_{i_2}(n) \cdots e_{i_m}(n)$$

be any word that expresses t in terms of the standard Temperley–Lieb generators of $\mathsf{TL}_n(\delta)$. Then the multiplication rule in Theorem 3.5 (b) together with Corollary 3.6 implies that the equation

$$r_A \tilde{e}_{i_1,n}(k) \tilde{e}_{i_2,n}(k) \cdots \tilde{e}_{i_m,n}(k) l_B = \tilde{d} + \text{lower order terms}$$

holds in $\mathsf{PTL}_k(\delta)$. The left hand side of the above equation belongs to P. The lower order terms are a linear combination of $\tilde{u}$ such that u belongs to $\mathcal{D}_j(k)$ for some $j < n$. By induction, we may assume that each such $\tilde{u}$ belongs to P. Hence, the same conclusion holds for $\tilde{d}$, completing the proof. ∎

5. Semisimplicity of $\mathsf{PTL}_k(\delta)$

In this section, we fix the ground ring $\Bbbk$, $\delta \in \Bbbk$, and k.

Theorem 5.1. *Let $\Bbbk$ be an arbitrary unital commutative ring. Let $X(n)$ be the $\Bbbk$-linear span of $\{\bar{d} \mid d \in \mathcal{D}_n(k)\}$. Then the algebra $\mathsf{PTL}_k(\delta)$ has the direct sum decomposition*

$$\mathsf{PTL}_k(\delta) = \bigoplus_{n=0}^{k} X(n)$$

into pairwise orthogonal two-sided ideals, in the sense that $xy = 0$ if $x \in X(n)$ and $y \in X(n')$, where $n \neq n'$.

Proof. This is an immediate consequence of Theorem 3.5, along with the fact that we are dealing only with balanced diagrams. ∎

Let Q_n be the free $\Bbbk$-module on the set of all cardinality n subsets of $[k]$. Identify $\operatorname{End}_\Bbbk(Q_n)$ with the matrix algebra $\operatorname{Mat}_{\binom{k}{n}}(\Bbbk)$ by means of its basis. All tensor products in this paper are taken over the ground ring $\Bbbk$, so we write $\otimes$ instead of $\otimes_\Bbbk$.

Theorem 5.2. *For any $\delta \in \Bbbk$, where $\Bbbk$ is a commutative unital ring, the ideal $X(n)$ is isomorphic to the matrix algebra*

$$\operatorname{Mat}_{\binom{k}{n}}\big(\mathsf{TL}_n(\delta - 1)\big) \cong \operatorname{Mat}_{\binom{k}{n}}(\Bbbk) \otimes \mathsf{TL}_n(\delta - 1).$$

Furthermore, every $X(n)$-module is isomorphic to one of the form $Q_n \otimes N$, where N is a $\mathsf{TL}_n(\delta - 1)$-module.

Proof. Let (A_i, d_i', B_i) be the triples in Lemma 4.2 corresponding to diagrams $d_i \in \mathcal{D}_n(k)$, for $i = 1, 2$. Theorem 3.5 implies that

$$\bar{d}_1 \bar{d}_2 = \begin{cases} (\delta - 1)^N \bar{d}_3 & \text{if } B_1 = A_2, \\ 0 & \text{otherwise,} \end{cases}$$

where, in the nonzero case, $d_3' = d_1' \circ d_2'$ in the partition monoid P_n and the triple corresponding to $d_3 \in \mathcal{D}_n(k)$ is (A_1, d_3', B_2). Notice that $d_1' d_2' = (\delta - 1)^N d_3'$ in the Temperley–Lieb algebra $\mathsf{TL}_n(\delta - 1)$. This proves the first claim. The second claim follows from the well-known isomorphism

$$\operatorname{Mat}_m(\mathcal{A}) \cong \operatorname{Mat}_m(\Bbbk) \otimes \mathcal{A}$$

as $\Bbbk$-algebras, for any $\Bbbk$-algebra $\mathcal{A}$ and any m. The isomorphism is given for any $a_{ij} \in \mathcal{A}$ by $\sum a_{ij} e_{ij} \mapsto \sum e_{ij} \otimes a_{ij}$, where e_{ij} is the (i, j)th matrix unit. The final claim is a standard feature of Morita theory. The action of

$$\operatorname{Mat}_{\binom{k}{n}}(\Bbbk) \otimes \mathsf{TL}_n(\delta - 1) \cong \operatorname{End}_\Bbbk(Q_n) \otimes \mathsf{TL}_n(\delta - 1)$$

on $Q_n \otimes N$ is the obvious one, in which $(f \otimes t)(A \otimes v) = f(A) \otimes tv$, for any $f \in \operatorname{End}_\Bbbk(Q_n)$, $t \in \mathsf{TL}_n(\delta - 1)$, $A \in Q_n$, $v \in N$. $\blacksquare$

Remark 5.3. (i) Theorem 5.2 implies that $X(n)$ is Morita equivalent to $\mathsf{TL}_n(\delta)$; that is, there is a category equivalence between their (left, or, equivalently, right) modules. In particular, this implies (rather trivially) that $\mathsf{PTL}_k(\delta)$ is an iterated inflation of Temperley–Lieb algebras, in the sense of [11, 24, 25].

(ii) The bijection in Lemma 4.2 between $\mathcal{D}_n(k)$ and triples induces an isomorphism

$$X(n) \cong Q_n \otimes \mathsf{TL}_n(\delta - 1) \otimes Q_n^*,$$

where $Q_n^* := \operatorname{Hom}_\Bbbk(Q_n, \Bbbk)$ is the linear dual of Q_n. This is an isomorphism of $\Bbbk$-modules, and also an isomorphism of $\Bbbk$-algebras, where $Q_n^* \otimes Q_n$ is identified with $\operatorname{End}_\Bbbk(Q_n)$ by the usual isomorphism.

Theorem 5.4. *Suppose that $\Bbbk$ is a field. Then:*

(a) *$X(n)$ is semisimple if and only if the same is true of $\mathsf{TL}_n(\delta - 1)$.*

(b) *If $X(n)$ is semisimple, its simple modules are of the form $Q_n \otimes \mathsf{TL}^\lambda$, where TL^λ is a simple $\mathsf{TL}_n(\delta - 1)$-module.*

(c) *$\mathsf{PTL}_k(\delta)$ is semisimple for any $\delta \in \Bbbk$ such that $\mathsf{TL}_n(\delta - 1)$ is semisimple for all $n = 0, 1, \ldots, k$.*

Proof. (a) follows from Theorem 5.2; see, e.g., [30, Lemma 4.5]. Part (b) is clear from the isomorphism in Theorem 5.2, and part (c) follows from part (a). ∎

Remark 5.5. By Theorem A.1, if q is a nonzero element of the field $\Bbbk$ such that $[\![n]\!]_q^! \neq 0$, then $\mathsf{PTL}_n(1 \pm (q + q^{-1}))$ is semisimple over $\Bbbk$. In particular, this holds whenever q is not a root of unity.

6. Representations of $\mathsf{TL}_k(\delta)$

The purpose of this section is to recall [10, Example 1.4] the standard combinatorial construction of the cell modules for a Temperley–Lieb algebra $\mathsf{TL}_k(\delta)$. A *planar involution* with λ fixed points is a sequence of k points arranged in a line with $k - \lambda$ of them joined in pairs and λ of them with an "end" attached. The pairs correspond to interchanges and ends to fixed points. The planar condition requires that the involution can be drawn in a half-plane if the ends are extended to infinity, without intersections. For example, the two diagrams

depict planar involutions on $k = 7$ points with 3 fixed points. One can think of planar involutions as products of disjoint cycles of length ≤ 2, or as "half" Temperley–Lieb diagrams. Given two planar involutions with the same number of fixed points, there is a unique way to join the ends to create a Temperley–Lieb diagram. For instance, the above involutions join to make the Temperley–Lieb diagram

with the leftmost involution at the top and rightmost involution (inverted) at the bottom.

By labeling the fixed points and left endpoint of interchanges by 1 and labeling right endpoints of interchanges by -1, a planar involution of length k produces a sequence $a = (a_1, \ldots, a_k)$ such that:

(i) $a_i = \pm 1$ for all $1 \leq i \leq k$,

(ii) each partial sum $a_1 + \cdots + a_n \geq 0$ for all $1 \leq n \leq k$.

We call any sequence satisfying these conditions a *Temperley–Lieb path*. Given any Temperley–Lieb path, we can construct a unique planar involution that produces the sequence. To see this, pair each i such that some $j > i$ exists with $a_i + a_{i+1} + \cdots + a_j = 0$ with the unique minimal such j. Such pairings define the interchanges in the involution, and all unpaired vertices are fixed points. Thus, we have a bijection between the sets of planar involutions and Temperley–Lieb paths of the same length, so we may as well identify these sets.

A Temperley–Lieb diagram t acts on a planar involution a by the usual diagram multiplication, producing a multiple of another planar involution b having at most as many fixed points as a. So the free $\Bbbk$-module W_0 on the set of planar involutions of length k is a $\mathsf{TL}_k(\delta)$-module. Let $W_0^{\leq \lambda}$ (resp., $W_0^{<\lambda}$) denote the span of the planar involutions with at most (resp., fewer than) λ fixed points. Then

$$\mathsf{TL}^\lambda := W_0^{\leq \lambda} / W_0^{<\lambda}$$

is a $\mathsf{TL}_k(\delta)$-module, for any λ such that $k - \lambda$ is an even number.

Theorem 6.1 ([10]). *Let $\Bbbk$ be a commutative unital ring. The collection*

$$\{\mathsf{TL}^\lambda \mid \lambda \equiv k \,(\mathrm{mod}\ 2)\}$$

is a complete set of cell modules for $\mathsf{TL}_k(\delta)$. If $\Bbbk$ is a field and $\delta \in \Bbbk$ is chosen such that $\mathsf{TL}_k(\delta)$ is semisimple, then the same set is a complete set of isomorphism classes of simple $\mathsf{TL}_k(\delta)$-modules.

Remark 6.2. There is another way to index the cell modules for $\mathsf{TL}_k(\delta)$, by the set Λ_k of partitions of k with at most two parts. The map $(\lambda_1, \lambda_2) \mapsto \lambda_1 - \lambda_2$ defines a bijection of Λ_k with the set of integers in $\{0, 1, \dots, k\}$ which are congruent to k mod 2, where the value $\lambda_2 = 0$ is allowed. In this indexing scheme, for $\lambda = (\lambda_1, \lambda_2)$ in Λ_k, the notation $\mathsf{TL}^\lambda = W_0^{\trianglelefteq \lambda} / W_0^{\triangleleft \lambda}$ replaces the previous notation $\mathsf{TL}^m = W_0^{\leq m} / W_0^{<m}$, where $m = \lambda_1 - \lambda_2$. Here $\trianglerighteq$ is the usual dominance order on partitions. We will switch to the partition notation starting in Section 8.

7. Motzkin paths and representations

We now recall the combinatorial construction of the cell modules for the Motzkin algebra, which generalizes the previous section. This will be applied in the next section to construct the cell modules for $\mathsf{PTL}_k(\delta)$. The remainder of this section closely follows [1, Section 4], to which the reader should refer for additional details.

A *Motzkin path* (in the sense of [1]) of length k is a sequence $a = (a_1, \ldots, a_k)$ such that $a_i \in \{-1, 0, 1\}$ and each partial sum $a_1 + \cdots + a_n \geq 0$ for all $1 \leq n \leq k$. The *rank* of a Motzkin path $a = (a_i)$ is defined to be

$$\mathrm{rank}(a) := a_1 + \cdots + a_k.$$

For each index i with $a_i = 1$, let j be the smallest index (if any) such that $i < j \leq k$ and $a_i + a_{i+1} + \cdots + a_j = 0$. Whenever this happens, $(a_i, a_j) = (1, -1)$ are said to be *paired*; otherwise, $a_i = 1$ is *unpaired*. By omitting the zeros in a Motzkin path we obtain a Temperley–Lieb path in the sense of the previous section. Connecting paired indices by an edge and extending unpaired indices by a line to infinity, we recover the planar involution of that path. By including the discarded zeros as isolated vertices, we obtain a graph on $[k]$ called a 1-*factor*. Since paired indices cancel one another in the sum,

$$\mathrm{rank}(a) = \text{the number of fixed points (lines to infinity)}$$

in the corresponding 1-factor. For example, the graph

$$\alpha = \quad \text{}$$

is the 1-factor produced by the Motzkin path $(1, 1, 1, -1, 0, -1, 1, 1, 0, -1)$.

In general, a 1-factor on k vertices is a graph that gives a planar involution once its isolated points are removed. This means that fixed points of the involution may not appear between paired vertices. Labeling paired vertices in a 1-factor α by $1, -1$ respectively, labeling fixed points by 1, and labeling all other vertices by 0 produces a Motzkin path a given by the sequence of labels whose corresponding 1-factor is α. So there is a bijection between Motzkin paths and 1-factors of the same length. From now on, we identify Motzkin paths with 1-factors by means of this bijection.

Remark 7.1. In [1], the fixed points of a 1-factor are depicted by white-colored vertices instead of lines to infinity.

For any given pair (α, β), where α, β are 1-factors on k vertices of the same rank λ, there is a unique Motzkin k-diagram $C_{\alpha, \beta}^{\lambda}$ such that the fixed points in α are connected to those in β. For example, if α is the 1-factor displayed above and β the 1-factor (of rank 2) below

$$\beta = \quad \text{},$$

then $C_{\alpha, \beta}^{\lambda}$ is obtained by reflecting β across its horizontal axis and then drawing edges connecting the fixed points in α, β in order, which gives

$$C_{\alpha, \beta}^{2} = \quad \text{}.$$

Notice that the zeros in the Motzkin paths corresponding to α, β label the isolated vertices in the diagram $C_{\alpha,\beta}^{\lambda}$. For any k, the disjoint union

$$\bigsqcup_{\lambda=0}^{k} \{C_{\alpha,\beta}^{\lambda} \mid \text{rank}(\alpha) = \text{rank}(\beta) = \lambda\}$$

is a cellular basis of $\mathsf{M}_k(\delta)$. This is the basis of Motzkin k-diagrams.

Now we describe the cell modules for the Motzkin algebra. For a Motzkin diagram d and a Motzkin path a (viewed as a 1-factor) of the same length, the Motzkin path da is given by

$$da = \delta^{N(d,a)} b,$$

where the multiplication is carried out by the usual graphical stacking procedure. The integer $N(d, a)$ is the number of loops in the bottom row of $\Gamma(d, a)$, and b is the resulting 1-factor in the top row of $\Gamma(d, a)$, after erasing the second row of vertices.

For example, if d is the Motzkin diagram and a the 1-factor given by the pictures

$$d = \quad\text{,}$$
$$a = \quad\text{,}$$
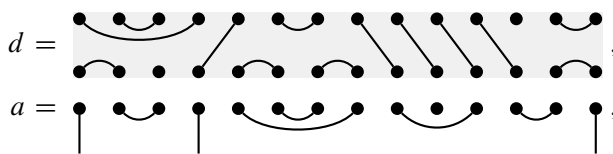
then the resulting $\Gamma(d, a)$ is the configuration

$$\Gamma(d, a) = \quad\text{,}$$

and thus $da = \delta b$, where b is the 1-factor

$$b = \quad\text{.}$$

There is a factor of δ in this example because there is a single loop in $\Gamma(d, a)$.

Let W be the free $\Bbbk$-module on the Motzkin paths of length k. The above action extends linearly to make W into an $\mathsf{M}_k(\delta)$-module. Since

$$\text{rank}(da) \leq \min(\text{rank}(d), \text{rank}(a)),$$

the $\Bbbk$-submodule $W^{\leq\lambda}$ of W spanned by the Motzkin paths of rank at most λ is an $\mathsf{M}_k(\delta)$-submodule, for any $\lambda = 0, 1, \ldots, k$. Thus we have a filtration

$$(0) \subseteq W^{\leq 0} \subseteq W^{\leq 1} \subseteq \cdots \subseteq W^{\leq k} = W$$

of $\mathsf{M}_k(\delta)$-submodules. For each λ, the quotient module

$$\mathsf{M}^\lambda := W^{\leq\lambda}/W^{<\lambda}$$

is an $\mathsf{M}_k(\delta)$-module, where we set $W^{<\lambda} := W^{\leq\lambda-1}$ (and $W^{<0} = (0)$).

Theorem 7.2 ([1, Theorems 4.7, 4.16]). *Let $\Bbbk$ be a commutative unital ring. The cell modules for $\mathsf{M}_k(\delta)$ are given by*

$$\{\mathsf{M}^\lambda \mid 0 \leq \lambda \leq k\}.$$

If $\Bbbk$ is a field then M^λ is indecomposable, and if $\delta \in \Bbbk$ is chosen so that $\mathsf{M}_k(\delta)$ is semisimple then the above set is a complete set of pairwise nonisomorphic simple $\mathsf{M}_k(\delta)$-modules.

By restricting our attention to the Motzkin paths with no zeros, we obtain copies of the cell modules for $\mathsf{TL}_k(\delta)$. The following was observed in [1, Remark 4.9].

Theorem 7.3. *Let $\mathcal{T}^\lambda$ be the $\Bbbk$-span of the Motzkin paths a in M^λ with no zeros. The span of the Motzkin diagrams in $\mathsf{M}_k(\delta)$ having no isolated vertices is a subalgebra isomorphic to $\mathsf{TL}_k(\delta)$. When restricted to that subalgebra, the action of $\mathsf{M}_k(\delta)$ on M^λ makes $\mathcal{T}^\lambda$ into a $\mathsf{TL}_k(\delta)$-module which is isomorphic to the cell module TL^λ, for any λ such that $k - \lambda$ is even.*

8. Representations of $\mathsf{PTL}_k(\delta)$

We continue to fix k and $\delta \in \Bbbk$, where $\Bbbk$ is a given unital commutative ring. Recall from Theorem 5.1 that $\mathsf{PTL}_k(\delta) = \bigoplus_{n=0}^{k} X(n)$. From now on, we will index representations by partitions of not more than two parts, instead of by integers as in the previous two sections; see Remark 6.2.

The *type* of a Motzkin path $a = (a_1, \ldots, a_k)$ is the pair $\lambda = (\lambda_1, \lambda_2)$ such that λ_1 (resp., λ_2) be the number of i such that $a_i = 1$ (resp., $a_i = -1$). Then $\lambda_1 \geq \lambda_2 \geq 0$ and $\mathrm{rank}(a) = \lambda_1 - \lambda_2$. We set

$$|\lambda| := \lambda_1 + \lambda_2.$$

The 1-factor corresponding to a has λ_2 cups and a has $k - |\lambda|$ zeros. Write

$$\Lambda := \{(\lambda_1, \lambda_2) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq k\}$$

for the set of types that occur as the type of some Motzkin path. We identify elements of Λ with partitions of at most two parts. Thus Λ is the disjoint union of the Λ_n (see Remark 6.2) as n runs from 0 to k.

For Motzkin paths $a = (a_1, \ldots, a_k)$, $a' = (a'_1, \ldots, a'_k)$ we write $a' \leq a$ if the following two conditions are satisfied:

(i) $a_i = 0$ implies that $a'_i = 0$,

(ii) $a'_i = 0$ if and only if $a'_j = 0$ whenever a_i, a_j are paired in a.

In other words, $a' \leq a$ if and only if the 1-factor corresponding to a' is obtainable from the 1-factor corresponding to a by erasing zero or more edges or lines to infinity. Let $\chi(a, a')$ be the number of such changes. Define $\bar{a}$ to be the linear combination

$$\bar{a} := \sum_{a' \leq a} (-1)^{\chi(a,a')} a'.$$

For example, if $a = (1, -1, 1)$ then

$$\bar{a} = (1, -1, 1) - (0, 0, 1) - (1, -1, 0) + (0, 0, 0).$$

If i is any index such that $a_i = \pm 1$ then $p_i a$ is the unique Motzkin path obtained from a by changing a_i to 0 (and also changing a_j to 0 if a_i, a_j are paired).

We define $\lceil a \rceil$ to be the set of all i such that $a_i = \pm 1$. In the corresponding 1-factor, this set indexes the vertices that are endpoints of edges or lines to infinity. Notice that we have

$$\bar{a} = \prod_{i \in \lceil a \rceil} (1 - p_i) a \tag{12}$$

in the Motzkin algebra. Notice that $\{\bar{a} \mid a$ is a Motzkin path of length $k\}$ is a $\Bbbk$-basis for the free $\Bbbk$-module W on the set of Motzkin paths of length k. The next result explains how $\mathsf{PTL}_k(\delta)$ acts on W.

Theorem 8.1. *Let $\Bbbk$ be a commutative unital ring. Suppose that a is a Motzkin path in W of type λ in Λ. Let b be the unique Motzkin path such that $da = \delta^N b$, for some N, where $d \in \mathcal{D}(k)$ is balanced.*

$$\bar{d}\bar{a} = \begin{cases} (\delta - 1)^{N(d,a)} \bar{b} & \text{if } \lfloor d \rfloor = \lceil a \rceil, \\ 0 & \text{otherwise.} \end{cases}$$

If $\lfloor d \rfloor = \lceil a \rceil$ then the type of b is some μ in Λ such that $|\mu| = |\lambda|$, and $\mathrm{rank}(a) - \mathrm{rank}(b)$ is an even number.

Proof. Thanks to the identity (12), the action of $\bar{d}$ by $\bar{a}$ is given by precisely the same rule as the multiplication rule in Theorem 3.5 (a), giving the first claim. Suppose that $\lfloor d \rfloor = \lceil a \rceil$. Then the zeros in a appear at the same places at the isolated vertices in the bottom row of d. Since d is balanced, it has the same number of isolated vertices in its top row, so $|\mu| = |\lambda|$. The rank of b cannot be larger but may be strictly less than that of a. (For instance, if $k = 3$ and $a = (1, 1, 1)$, then $e_1 a = b = (1, -1, 1)$, so $\bar{e}_1 \bar{a} = \bar{b}$

where rank$(a) = 3$, rank$(b) = 1$.) The rank decreases only if one or more pairs of unpaired indices of 1 in a are replaced by pairings in b, so the rank can decrease only in steps of 2. ∎

Recall that the usual *dominance order* on partitions is defined by declaring that $\lambda \trianglerighteq \mu$ if and only if $\lambda - \mu$ can be written as a sum of positive roots (in the root system of $\mathfrak{gl}_n$). In our situation, if $|\lambda| = |\mu|$ and $\lambda, \mu \in \Lambda$, this is equivalent to

$$\lambda \trianglerighteq \mu \iff \lambda - \mu = m(1, -1) \text{ for some integer } m \geq 0.$$

This is equivalent to $(\lambda_1 - \lambda_2) - (\mu_1 - \mu_2)$ being a nonnegative even integer. Write $\lambda \triangleright \mu$ whenever $\lambda \trianglerighteq \mu$ but $\lambda \neq \mu$.

For any λ in Λ, let $\bar{W}^{\trianglelefteq \lambda}$ and $\bar{W}^{\triangleleft \lambda}$ be the $\Bbbk$-span of the sets

$$\{\bar{a} \mid a \text{ has type } \mu \text{ and } \lambda \trianglerighteq \mu\} \quad \text{and} \quad \{\bar{a} \mid a \text{ has type } \mu \text{ and } \lambda \triangleright \mu\},$$

respectively. By Theorem 8.1, both $\bar{W}^{\trianglelefteq \lambda}$, $\bar{W}^{\triangleleft \lambda}$ are $\mathsf{PTL}_k(\delta)$-submodules of W. We define

$$\mathsf{PTL}^{\lambda} := \bar{W}^{\trianglelefteq \lambda} / \bar{W}^{\triangleleft \lambda}$$

to be the corresponding quotient module. So the collection of $\bar{a} + \bar{W}^{\triangleleft \lambda}$ such that a has type λ is a basis of PTL^{λ}.

Theorem 8.1 above and Theorems 5.2, 5.4 imply the following result.

Theorem 8.2. *Let $\Bbbk$ be a commutative unital ring. The PTL^{λ} for λ in Λ are the cell modules for $\mathsf{PTL}_k(\delta)$. For any $\lambda \in \Lambda$, with $n = |\lambda|$, we have an isomorphism*

$$\mathsf{PTL}^{\lambda} \cong Q_n \otimes \mathsf{TL}^{\lambda}$$

as $X(n)$-modules, where TL^{λ} is the cell module for $\mathsf{TL}_n(\delta - 1)$ indexed by λ in accordance with Remark 6.2. Hence, if $\Bbbk$ is a field and $\delta \in \Bbbk$ is chosen so that $\mathsf{PTL}_k(\delta)$ is semisimple, then

$$\{\mathsf{PTL}^{\lambda} \mid \lambda \in \Lambda\}$$

is a complete set of pairwise nonisomorphic simple $\mathsf{PTL}_k(\delta)$-modules.

Proof. Suppose that a is a Motzkin path of type λ, for λ in Λ. Let $A = \lceil a \rceil$. If $n = |A|$ then A is an element of Q_n. Let b be the Motzkin path obtained from a by removing all its zero entries; that is, b is the part of a supported by A. Then b may be regarded as a Temperley–Lieb half diagram (a planar involution in the terminology of Graham and Lehrer; see Section 6). Suppose that $|A| = n$. The linear map sending

$$\bar{a} \mapsto A \otimes b$$

defines an isomorphism

$$\overline{W}^{\trianglelefteq \lambda} \to Q_n \otimes W_0^{\trianglelefteq \lambda}$$

that restricts to an isomorphism $\overline{W}^{\triangleleft \lambda} \to Q_n \otimes W_0^{\triangleleft \lambda}$. Passing to quotients induces the desired isomorphism $\mathsf{PTL}^\lambda \cong Q_n \otimes \mathsf{TL}^\lambda$. This shows that the PTL^λ are inflations of the various cell modules for $\mathsf{TL}_n(\delta - 1)$, which proves the first claim. The last claim follows from the first two. ∎

As a consequence of the last result, if λ is in Λ and satisfies $\lambda_1 - \lambda_2 = n$, we have

$$\operatorname{rank}_{\Bbbk} \mathsf{PTL}^\lambda = \binom{k}{n} \operatorname{rank}_{\Bbbk} \mathsf{TL}^\lambda.$$

Remark 8.3. The subalgebra of $\mathsf{PTL}_k(\delta)$ spanned by $\{\overline{d} \mid d \in \mathcal{D}_k(k)\}$ is isomorphic to $\mathsf{TL}_k(\delta - 1)$. For $\lambda \in \Lambda$ with $|\lambda| = k$, the action of $\mathsf{PTL}_k(\delta)$ on PTL^λ, when restricted to that subalgebra, is isomorphic to TL^λ as a $\mathsf{TL}_k(\delta - 1)$-module.

9. $\mathbf{U}_q(\mathfrak{gl}_2)$ and $\mathbf{U}_q(\mathfrak{sl}_2)$

We assume henceforth that $\Bbbk$ is a field and that $0 \neq q \in \Bbbk$ is not a root of unity. Let $\mathbf{U} = \mathbf{U}_q(\mathfrak{gl}_2)$ be the quantized enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_2$. By definition, $\mathbf{U}$ is the associative algebra with 1 generated by E, F, $K_i^{\pm 1}$ (for $i = 1, 2$) subject to the defining relations

$$
\begin{aligned}
K_1 K_2 &= K_2 K_1, & K_i K_i^{-1} &= K_i^{-1} K_i = 1 \quad (i = 1, 2), \\
K_1 E K_1^{-1} &= q E, & K_2 E K_2^{-1} &= q^{-1} E, \\
K_1 F K_1^{-1} &= q^{-1} F, & K_2 F K_2^{-1} &= q F, \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}, & \text{where } K &:= K_1 K_2^{-1}.
\end{aligned}
$$

Furthermore, $\mathbf{U}$ is a Hopf algebra with coproduct $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ and counit $\epsilon : \mathbf{U} \to \Bbbk$ given on generators by

$$
\begin{aligned}
\Delta(E) &= E \otimes K + 1 \otimes E, \\
\Delta(F) &= F \otimes 1 + K^{-1} \otimes F, \\
\Delta(K_i) &= K_i \otimes K_i \quad (i = 1, 2), \\
\epsilon(E) = \epsilon(F) &= 0, \quad \epsilon(K_i) = 1 \quad (i = 1, 2).
\end{aligned}
\tag{13}
$$

The subalgebra of $\mathbf{U} = \mathbf{U}_q(\mathfrak{gl}_2)$ generated by E, F, $K^{\pm 1}$ is the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{sl}_2)$.

Remark 9.1. The coproduct Δ defined in (13) is the one used in [1]. It differs from the usual one in [14, 26]. One could use either convention in this paper, but for the sake of consistency, we stick with the choice made in [1].

We refer to [14, Chapter 2] for basic facts on the representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$. For each integer $n \geq 0$, by [14, Theorem 2.6] there exist simple $\mathbf{U}_q(\mathfrak{sl}_2)$-modules $L(n, +)$, $L(n, -)$ of dimension $n + 1$. (In characteristic 2, $L(n, +) \cong L(n, -)$.) Any simple $\mathbf{U}_q(\mathfrak{sl}_2)$-module of dimension $n + 1$ is isomorphic to either $L(n, +)$ or $L(n, -)$, and $L(n, +)$ is of type **1**.

Lemma 9.2. *Suppose that $\Bbbk$ is a field and that $0 \neq q \in \Bbbk$ is not a root of unity. For any $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{Z} \times \mathbb{Z}$ with $\lambda_1 - \lambda_2 \geq 0$, there is a unique $\mathbf{U}_q(\mathfrak{gl}_2)$-module $V(\lambda)$ such that $V(\lambda) \cong L(\lambda_1 - \lambda_2, +)$ as $\mathbf{U}_q(\mathfrak{sl}_2)$-modules, and*

$$K_i v_+ = q^{\lambda_i} v_+ \quad \text{for } i = 1, 2,$$

where v_+ is a highest weight vector of weight $\lambda_1 - \lambda_2$ for the $\mathbf{U}_q(\mathfrak{sl}_2)$-module structure.

Proof. This follows from the results in [14, Chapter 2], using the fact that any $\mathbf{U}_q(\mathfrak{gl}_2)$-module is also (by restriction) a $\mathbf{U}_q(\mathfrak{sl}_2)$-module. (See also [23, Section 7.3].) ∎

Under the same hypotheses, every finite-dimensional $\mathbf{U}_q(\mathfrak{gl}_2)$-module M is semisimple. Furthermore, if M is of type **1**, it is isomorphic to a direct sum of modules of the form $V(\lambda)$ as in the above lemma.

The "polynomial" $\mathbf{U}_q(\mathfrak{gl}_2)$-modules are direct sums of $V(\lambda)$ such that $\lambda = (\lambda_1, \lambda_2)$ belongs to $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and $\lambda_1 \geq \lambda_2$. If M is a simple polynomial $\mathbf{U}_q(\mathfrak{gl}_2)$-module, we may (and do) identify its highest weight λ with a partition of at most two parts. In particular, if $\lambda = (n)$ is a partition of one part, then we write $V(n)$ for $V(\lambda)$. Thus, $V(n) \cong L(n, +)$ as $\mathbf{U}_q(\mathfrak{sl}_2)$-modules.

Consider the simple $\mathbf{U}_q(\mathfrak{gl}_2)$-modules $V(0)$, $V(1)$. Then $V(0) \cong \Bbbk$ is the trivial module, with $\mathbf{U}_q(\mathfrak{gl}_2)$ acting via the counit ϵ. This means that if v_0 is a chosen basis of $V(0)$; then on v_0 the operators E, F act as zero and each K_i acts as 1. Fix a choice of v_0. Fix also a basis $\{v_1, v_{-1}\}$ of weight vectors of $V(1)$, where v_1 has weight $(1, 0)$ and v_{-1} has weight $(0, 1)$ as $\mathbf{U}_q(\mathfrak{gl}_2)$-modules, such that the action of E, F is given by

$$E v_1 = 0, \quad E v_{-1} = v_1, \quad F v_1 = v_{-1}, \quad F v_{-1} = 0.$$

In other words, with respect to the basis $\{v_1, v_{-1}\}$, the matrices representing the action of E, F, K_i are

$$E \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K_1 \to \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}, \quad K_2 \to \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}.$$

Following [1], we set $V := V(0) \oplus V(1)$, with basis $\{v_1, v_0, v_{-1}\}$.

We will need the following general fact about tensor powers of bialgebra representations, for which we were unable to find a suitable reference. If $\mathbf{U}$ is a bialgebra and V a $\mathbf{U}$-module, then $V^{\otimes k}$ is a $\mathbf{U}$-module for any $k > 1$, with $u \in \mathbf{U}$ acting on $V^{\otimes k}$ by

$$\Delta^{(k)} \colon \mathbf{U} \to \mathbf{U}^{\otimes k},$$

where Δ is the coproduct on $\mathbf{U}$, lifted to $\Delta^{(k)}$ inductively by (for example) defining $\Delta^{(2)} = \Delta$ and $\Delta^{(k+1)} = (\Delta \otimes 1^{\otimes(k-1)})\Delta^{(k)}$ for $k \geq 2$.

Lemma 9.3. *Let V be a $\mathbf{U}$-module, where $\mathbf{U}$ is a bialgebra with coproduct Δ. Suppose that ψ is in $\mathrm{End}_{\mathbf{U}}(V \otimes V)$; that is,*

$$\psi\Delta(u) = \Delta(u)\psi \quad \text{for any } u \in \mathbf{U}.$$

Then $1^{\otimes(i-1)} \otimes \psi \otimes 1^{k-1-i}$ commutes with the action of $\Delta^{(k)}(u)$, for any $u \in \mathbf{U}$.

Proof. This is of course well known; we sketch a proof for completeness. It follows by induction from the coassociativity axiom

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$

that

$$\Delta^{(k)} = (1^{\otimes a} \otimes \Delta \otimes 1^{\otimes b})\Delta^{(k-1)}$$

for any a, b such that $a + b = k - 2$. Taking $a = i - 1$, $b = k - 1 - i$ proves the result. ∎

10. Structure of $V \otimes V$ for $V = V(0) \oplus V(1)$

We continue to assume in this section that $\Bbbk$ is a field and $0 \neq q \in \Bbbk$ is not a root of unity. We wish to analyze the structure of $V \otimes V$, as module for both $\mathbf{U}_q(\mathfrak{gl}_2)$ and $\mathbf{U}_q(\mathfrak{sl}_2)$. We have

$$V \otimes V = (V(0) \otimes V(0)) \oplus (V(0) \otimes V(1)) \oplus (V(1) \otimes V(0)) \oplus (V(1) \otimes V(1))$$

and since the first three direct summands on the right hand side are respectively isomorphic to $V(0)$, $V(1)$, and $V(1)$, understanding the structure of $V \otimes V$ reduces to understanding the structure of $V(1) \otimes V(1)$.

To simplify notation, set $v_{i,j} := v_i \otimes v_j$. Then $\{v_{i,j} \mid i, j \in \{1, 0, -1\}\}$ is a basis of $V \otimes V$, and $\{v_{i,j} \mid i, j \in \{1, -1\}\}$ is a basis of $V(1) \otimes V(1)$. A simple direct computation shows that

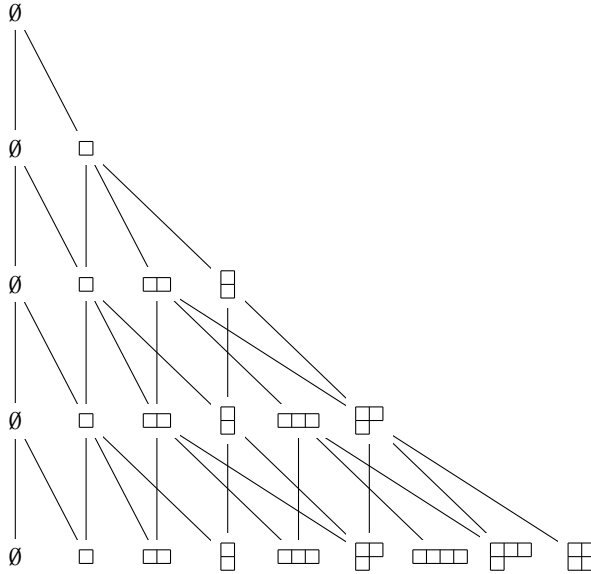$$\{v_{1,1}, q^{-1}v_{1,-1} + v_{-1,1}, v_{-1,-1}\}$$

Figure 1. Bratteli diagram for $\mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k})$, $k \leq 4$.

is a basis of weight vectors for a submodule of $V(1) \otimes V(1)$ isomorphic to $V(2)$. In order to pick a complement to this submodule, we observe that the vector

$$Z_0 := -qv_{1,-1} + v_{-1,1} \tag{14}$$

is orthogonal to the submodule, with respect to the standard bilinear form on $V(1)$ extended to $V(1) \otimes V(1)$, and this property uniquely determines Z_0 up to a scalar multiple. The line $\Bbbk Z_0$ in $V(1) \otimes V(1)$ is isomorphic to the trivial module $V(0)$ as $\mathbf{U}_q(\mathfrak{sl}_2)$-modules. Since K_i for $i = 1, 2$ both act as q on Z_0, it follows that $\Bbbk Z_0 \cong V(1, 1)$ as $\mathbf{U}_q(\mathfrak{gl}_2)$-modules. Hence,

$$V(1) \otimes V(1) \cong \begin{cases} V(2) \oplus V(1,1) & \text{as } \mathbf{U}_q(\mathfrak{gl}_2)\text{-modules}, \\ V(2) \oplus V(0) & \text{as } \mathbf{U}_q(\mathfrak{sl}_2)\text{-modules}. \end{cases} \tag{15}$$

It follows from (15) that

$$V \otimes V \cong \begin{cases} V(0) \oplus 2V(1) \oplus V(2) \oplus V(1,1) & \text{as } \mathbf{U}_q(\mathfrak{gl}_2)\text{-modules}, \\ 2V(0) \oplus 2V(1) \oplus V(2) & \text{as } \mathbf{U}_q(\mathfrak{sl}_2)\text{-modules}. \end{cases}$$

From this, it is immediate that

$$\dim \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V \otimes V) = 1^2 + 2^2 + 1^2 + 1^2 = 7,$$
$$\dim \mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(V \otimes V) = 2^2 + 2^2 + 1^2 = 9.$$

k	$\varnothing$	(1)	(2)	(1^2)	(3)	$(2,1)$	(4)	$(3,1)$	$(2,2)$	$\dim \mathcal{Z}_k(q)$
0	1									1
1	1	1								2
2	1	2	1	1						7
3	1	3	3	3	1	2				33
4	1	4	6	6	4	8	1	3	2	183

Table 1. Dimension of $\mathcal{Z}_k(q)$ for $k \leq 4$.

This dichotomy illustrates the error in [1], where it is implicitly assumed that the centralizers are the same in both cases.

The first decomposition given in equation (15) is a special case of the decomposition:

$$V(1) \otimes V(\lambda) \cong \bigoplus_{\mu} V(\mu), \tag{16}$$

where λ, μ are partitions of at most two parts and μ varies over the set of such partitions obtainable from λ by adding one box. The rule (16) is itself a special case of the standard Pieri rule. By repeated application of (16), we now construct the Bratteli diagram (see Figure 1) for the centralizer algebra

$$\mathcal{Z}_k(q) := \operatorname{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k}).$$

Since q is not a root of unity, the dimension of the simple $\mathcal{Z}_k(q)$-module indexed by a partition λ of at most two parts is the number of paths from the top vertex to its label in the Bratteli diagram. The sum of the squares of those dimensions is the dimension of $\mathcal{Z}_k(q)$, as shown in Table 1; and they differ from the dimensions of $\operatorname{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(V^{\otimes k})$ given in [1, Figure 1]. Notice that the dimension of $\mathcal{Z}_k(q)$ agrees with the dimension in (10) of the algebra $\mathsf{PTL}_k(\delta)$, at least up to degree 4. We will prove in Theorem 12.4 that they agree in general, when $\delta = 1 \pm (q + q^{-1})$.

11. Schur–Weyl duality for the Motzkin algebras

We continue to assume that $\Bbbk$ is a field, and $0 \neq q \in \Bbbk$ is not a root of unity. We endow $V = V(0) \oplus V(1)$ with the standard nondegenerate bilinear form such that $\langle v_i, v_j \rangle = \delta_{i,j}$ for all $i, j = 1, 0, -1$ and extend the form to $V \otimes V$ in the natural way. Let π be the orthogonal projection of $V(1) \otimes V(1)$ onto the line $\Bbbk Z_0 \cong V(0)$, where $Z_0 = -q v_{1,-1} + v_{-1,1}$ is the invariant in equation (14). With respect to the ordered

basis $v_{1,1}, v_{1,-1}, v_{-1,1}, v_{-1,-1}$ the matrix of π is

$$A(\pi) = \frac{1}{q + q^{-1}} \begin{bmatrix} 0 & & & \\ & q & -1 & \\ & -1 & q^{-1} & \\ & & & 0 \end{bmatrix}$$

in which omitted entries should be interpreted as zero, as usual. Recall from Section 2.5 that the action of the generator e_i in $\mathsf{TL}_k(\pm(q + q^{-1}))$ on $V(1)^{\otimes k}$ is given by the operator

$$\pm(q + q^{-1})1^{\otimes(i-1)} \otimes \pi \otimes 1^{\otimes(k-i-1)}.$$

This defines a faithful action of $\mathsf{TL}_k(\pm(q + q^{-1}))$, and it is well known that

$$\mathsf{TL}_k(\pm(q + q^{-1})) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(V(1)^{\otimes k}) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V(1)^{\otimes k}). \qquad (17)$$

In order to extend the above action to a faithful action of $\mathsf{M}_k(1 \pm (q + q^{-1}))$ on $V^{\otimes k}$, we first consider how to do this for the case $k = 2$.

The algebra $\mathsf{M}_2(1 \pm (q + q^{-1}))$ is generated by the elements r, l, and e defined in Section 2. It has basis consisting of the nine Motzkin diagrams:

In order from left to right, the above elements are expressible in terms of the generators r, l, e as follows:

$$1, \quad r, \quad l, \quad lr, \quad rl, \quad e, \quad re = le, \quad er = el, \quad r^2 = l^2 = rer = lel.$$

Since (in the partition algebra) the elements r, l satisfy the identities

$$r = p_1 s = s p_2, \quad l = s p_1 = p_2 s$$

where s is the swap operator defined in Section 2 and p_1, p_2 are projections onto $V(0) \otimes V$, $V \otimes V(0)$, respectively, it is natural to define the action of r, l on basis elements by

$$r v_{i,j} = \delta_{j,0} v_{0,i}, \quad l v_{i,j} = \delta_{i,0} v_{j,0}$$

for all $i, j \in \{1, 0, -1\}$, as in [1, Section 3.4].

The lines $\Bbbk v_{0,0}$, $\Bbbk Z_0$ are isomorphic copies of the trivial $\mathbf{U}_q(\mathfrak{sl}_2)$-module in $V \otimes V$, so it is natural to let e act as a projection onto some linear combination of the form

$$Y_0 := v_{0,0} + \alpha Z_0 = v_{0,0} + \alpha(-q v_{1,-1} + v_{-1,1}) \quad (\alpha \neq 0).$$

We demand that on restriction to $V(1) \otimes V(1)$ (resp., $V(0) \otimes V(0)$) the action restricts to the Temperley–Lieb action defined above (resp., the trivial action). This forces $\alpha \neq 0$ and implies that each $v_{i,j}$ is sent to a multiple $\beta_{i,j} Y_0$ of Y_0. The multipliers $\beta_{i,j}$ are forced by our demands to be

$$\beta_{1,-1} = \mp\alpha^{-1}, \quad \beta_{0,0} = 1, \quad \beta_{-1,1} = \pm\alpha^{-1}q^{-1}$$

with all other $\beta_{i,j} = 0$. In other words, the matrix of the action of e on the 0-weight space $(V \otimes V)_0$ with respect to the ordered basis $v_{1,-1}, v_{0,0}, v_{-1,1}$ is given by

$$B(\alpha, \pm) := \begin{bmatrix} \pm q & -\alpha q & \mp 1 \\ \mp\alpha^{-1} & 1 & \pm\alpha^{-1}q^{-1} \\ \mp 1 & \alpha & \pm q^{-1} \end{bmatrix}.$$

The matrix $B(\alpha, \pm)$ satisfies the relation

$$B(\alpha, \pm)^2 = (1 \pm (q + q^{-1}))B(\alpha, \pm)$$

so it is a scalar multiple of a projection in the above sense.

We now define two nondegenerate bilinear forms $\langle -, - \rangle_t$, $\langle -, - \rangle_b$ on V by the rules:

$$\begin{aligned}
\langle v_1, v_{-1} \rangle_t &= -\alpha q, & \langle v_0, v_{-0} \rangle_t &= 1, & \langle v_{-1}, v_1 \rangle_t &= \alpha, \\
\langle v_1, v_{-1} \rangle_b &= \mp\alpha^{-1}, & \langle v_0, v_{-0} \rangle_b &= 1, & \langle v_{-1}, v_1 \rangle_b &= \pm\alpha^{-1}q^{-1},
\end{aligned} \tag{18}$$

with all other $\langle v_i, v_j \rangle_t = 0$, $\langle v_i, v_j \rangle_b = 0$. It is worth noticing that the nonzero values of $\langle v_i, v_j \rangle_t$ (resp., $\langle v_i, v_j \rangle_b$) are encoded in the middle column (resp., middle row) of the matrix $B(\alpha, \pm)$.

Just as in [1, Section 3.4], the forms may be applied to give explicit formulas for the action of $\mathsf{M}_k(1 \pm (q + q^{-1}))$ on $V^{\otimes k}$, as follows. Given a Motzkin k-diagram d and i_α, j_α in $\{-1, 0, 1\}$ for $\alpha = 1, \ldots, k$, label the top row vertices of d from left to right with basis elements $v_{j_1}, \ldots, v_{j_k}$ and similarly label the bottom row vertices with $v_{i_1}, \ldots, v_{i_k}$. The blocks of d are either isolated vertices or edges. Then

$$d(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1,\ldots,j_k} (d)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k} \, v_{j_1} \otimes \cdots \otimes v_{j_k}$$

defines the action of d, where the scalar $(d)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}$ is the product of the weights taken over the various blocks of d. Table 2 shows the weight $(\beta)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}$ of a labeled block β of d. Here $\delta_{i,j}$ is the usual Kronecker delta function.

Proposition 11.1. *For any $\alpha \neq 0$, the above rules define a left action of $\mathsf{M}_k(\delta)$ on $V^{\otimes k}$, where $\delta = 1 \pm (q + q^{-1})$.*

$\delta_{i,0}$	if β is an isolated vertex labeled by v_i
$\delta_{i,j}$	if β is a vertical edge with endpoints labeled by v_i, v_j.
$\langle v_i, v_j \rangle_t$	if β is a horizontal top edge with left and right endpoints labeled by v_i and v_j, resp.
$\langle v_i, v_j \rangle_b$	if β is a horizontal bottom edge with left and right endpoints labeled by v_i and v_j, resp.

Table 2. Weight of a labeled block β.

Proof. This is proved by the same argument as the proof of [1, Proposition 3.29]. In that argument, the notations $v_1^* = v_{-1}$, $v_0^* = v_0$, $v_1^* = v_1$ were defined. The argument depends on the following properties of the forms:

$$\langle v_a^*, v_a \rangle_b \langle v_a, v_a^* \rangle_t = 1, \quad \langle v_a, v_a^* \rangle_b \langle v_a^*, v_a \rangle_t = 1,$$

$$\langle v_a, v_a^* \rangle_t \langle v_a, v_a^* \rangle_b = \begin{cases} -q & \text{if } a = 1, \\ 1 & \text{if } a = 0, \\ -q^{-1} & \text{if } a = -1, \end{cases}$$

and only on those properties. One checks that for any $\alpha \neq 0$, all of these properties still hold when we use equation (18) to define the forms. ∎

The following is a slight generalization of results proved in [1, Section 3.4].

Theorem 11.2. *Suppose that $\Bbbk$ is a field, $0 \neq q \in \Bbbk$ is not a root of unity, and $0 \neq \alpha \in \Bbbk$. For any $i = 1, \ldots, k - 1$, let e_i, r_i, l_i act on $V^{\otimes k}$ in tensor positions i, $i + 1$ as the operator $1^{\otimes(i-1)} \otimes g \otimes 1^{\otimes(k-i-1)}$, where $g = e, r, l$ (as operators), respectively. This extends to an action of $\mathsf{M}_k(1 \pm (q + q^{-1}))$ that commutes with the action of $\mathbf{U}_q(\mathfrak{sl}_2)$. The corresponding representation*

$$\rho \colon \mathsf{M}_k(1 \pm (q + q^{-1})) \to \mathrm{End}_{\Bbbk}(V^{\otimes k})$$

is faithful, thus induces an algebra isomorphism

$$\mathsf{M}_k(1 \pm (q + q^{-1})) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(V^{\otimes k}).$$

Proof. The defining relations for $\mathsf{M}_k(\delta)$ are given in [31]. It is a tedious yet elementary calculation to verify that our operators e_i, r_i, l_i satisfy precisely the same relations (for any $\alpha \neq 0$). It suffices to do the calculation in $M_3(\delta)$. We used a computer algebra system to create explicit matrices for the generating operators and verified the defining

relations accordingly. It follows that when $\delta = 1 \pm (q + q^{-1})$, the action determines a representation

$$\rho \colon \mathsf{M}_k(1 \pm (q + q^{-1})) \to \mathrm{End}_{\Bbbk}(V^{\otimes k}).$$

Another way to see this is to repeat the proof of [1, Proposition 3.29] with the appropriate substitutions. One can check by an elementary direct computation that e, r, and l (as operators on $V \otimes V$) commute with the action of $\mathbf{U}_q(\mathfrak{sl}_2)$; here again a computer algebra system is useful. It then follows from Lemma 9.3 that the action of any of the generators e_i, r_i, l_i commutes with the action of $\mathbf{U}_q(\mathfrak{sl}_2)$. Finally, the proof of [1, Theorem 3.31] also applies to our situation to show that ρ is faithful, and the result follows. ∎

Remark 11.3. (i) Working with $\mathsf{M}_k(1 - q - q^{-1})$, Benkart and Halverson [1] choose to define the action of e (which they denote by t) on $(V \otimes V)_0$ in terms of the matrix

$$B(q^{-1/2}, -) = \begin{bmatrix} -q & -q^{1/2} & 1 \\ q^{1/2} & 1 & -q^{-1/2} \\ 1 & q^{-1/2} & -q^{-1} \end{bmatrix},$$

and define their bilinear forms $\langle -, - \rangle_t$, $\langle -, - \rangle_b$ accordingly. In other words, they are setting $\alpha = q^{-1/2}$ and making a particular choice of sign. Thus they implicitly assume that $q^{1/2}$ exists in $\Bbbk$. Our analysis shows that this assumption is avoidable by simply taking $\alpha = 1$ (or any other convenient nonzero value).

(ii) If we work instead in $\mathsf{M}_k(1 + q + q^{-1})$, where $e^2 = (1 + q + q^{-1})e$, it is also possible to define the action of e as a multiple of the *orthogonal* projection onto Y_0 (with respect to the standard bilinear form). The matrix giving the action of e on $(V \otimes V)_0$ with respect to the same ordered basis as above is

$$B(\pm q^{-1/2}, +) = \begin{bmatrix} q & \mp q^{1/2} & -1 \\ \mp q^{1/2} & 1 & \pm q^{-1/2} \\ -1 & \pm q^{-1/2} & q^{-1} \end{bmatrix}.$$

With this choice, the corresponding bilinear forms $\langle -, - \rangle_t$, $\langle -, - \rangle_b$ become identical.

12. Schur–Weyl duality for $\mathsf{PTL}_k(\delta)$

We now turn to $\mathsf{PTL}_k(\delta)$, with $\delta = 1 \pm (q + q^{-1})$, continuing to assume that $0 \neq q \in \Bbbk$ is not a root of unity and $\Bbbk$ is a field. This algebra acts faithfully on $V^{\otimes k}$ by restriction of the action of $\mathsf{M}_k(\delta)$. Recall from (9) that

$$\varepsilon_i = \tilde{e}_i = (1 - p_i)e_i(1 - p_i).$$

We have shown in Theorem 4.5 that $\mathsf{PTL}_k(\delta)$ is generated by the ε_i, r_i, l_i ($i \in [k-1]$). By Lemma 9.3, it suffices to understand the action in the case $k = 2$. Let ε be the operator on $V \otimes V$ given by the action of ε_1 in $\mathsf{PTL}_2(\delta)$. In terms of Motzkin diagrams we have the identity

An explicit calculation with the formulas in the preceding section reveals that ε acts as

$$v_{1,-1} \mapsto \pm(qv_{1,-1} - v_{-1,1}), \quad v_{-1,1} \mapsto \pm(-v_{1,-1} + q^{-1}v_{-1,1})$$

with all other $v_{i,j} \mapsto 0$. This is independent of the choice of $\alpha \neq 0$. We note that $\varepsilon^2 = \pm(q + q^{-1})\varepsilon$. In other words, the restriction of ε to $V(1) \otimes V(1)$ coincides with the standard action of the generator e in $\mathsf{TL}_2(\pm(q + q^{-1}))$, as described at the beginning of Section 11.

Thus ε_i acts on $V^{\otimes k}$ as the operator $1^{\otimes(i-1)} \otimes \varepsilon \otimes 1^{(k-i-1)}$, for any i in $[k-1]$.

Proposition 12.1. *The action of $\mathsf{PTL}_k(1 \pm (q + q^{-1}))$ on $V^{\otimes k}$ commutes with the action of $\mathbf{U}_q(\mathfrak{gl}_2)$.*

Proof. By Lemma 9.3 it suffices to check that the action of ε on $V \otimes V$ commutes with that of $\mathbf{U}_q(\mathfrak{gl}_2)$. We already know that it commutes with the action of E, F, so we only need to check commutation with the action of K_1, K_2. When restricted to $\Bbbk v_{1,-1} \oplus \Bbbk v_{-1,1}$, each K_i acts as q, so K_i acts as q times the identity operator on that subspace. The result follows. ∎

With $\delta = 1 \pm (q + q^{-1})$, let φ be the restriction of the representation

$$\rho \colon \mathsf{M}_k(\delta) \to \operatorname{End}_\Bbbk(V^{\otimes k})$$

to $\mathsf{PTL}_k(\delta)$. Thus

$$\varphi \colon \mathsf{PTL}_k(\delta) \to \operatorname{End}_\Bbbk(V^{\otimes k})$$

is an injective algebra morphism.

Proposition 12.2. *Suppose that $d = d(A, t, B)$ belongs to $\mathcal{D}(k)$, so that $d = r_A t_0 l_B$ as in Lemma 4.4, where $t_0 = t \otimes \omega_{k-n}$. Then $\bar{d} = r_A \bar{t}_0 l_B$.*

Proof. Observe that by the definition of the bar elements,

$$\bar{t}_0 = \prod_{i=1}^n (1 - p_i) t_0 \prod_{i=1}^n (1 - p_i).$$

Then considerations similar to those in the proof of Theorem 3.5 show that

$$r_A \prod_{i=1}^{n}(1-p_i)t_0 \prod_{i=1}^{n}(1-p_i)l_B = \prod_{i\in A}(1-p_i)r_A t_0 l_B \prod_{i\in B}(1-p_i)$$

and the result follows. ∎

To proceed, write $1 = \dot{p} + (1-\dot{p})$, where $1 = 1_V$ is the identity map on V and $\dot{p}$ is the projection map $V \to V(0)$. This decomposition (a sum of orthogonal idempotents) induces the defining decomposition

$$V = \dot{p}V \oplus (1-\dot{p})V = V(0) \oplus V(1).$$

As $V(0)$, $V(1)$ are $\mathbf{U}_q(\mathfrak{gl}_2)$-submodules of V, the operators $\dot{p}$, $1-\dot{p}$ commute with the action of $\mathbf{U}_q(\mathfrak{gl}_2)$. Since p_i acts as $\dot{p}$ on the ith tensor factor and as identity in the remaining factors, we can apply the same reasoning to the ith tensor position in $V^{\otimes k}$ to obtain the decomposition

$$V^{\otimes k} = p_i V^{\otimes k} \oplus (1-p_i)V^{\otimes k} = V^{\otimes i-1} \otimes \big(V(0) \oplus V(1)\big) \otimes V^{\otimes k-i}.$$

Expanding the operator $1^{\otimes k} = (\dot{p} + (1-\dot{p}))^{\otimes k}$ binomially produces an identity

$$1^{\otimes k} = \sum_{A\subseteq[k]} \prod_{i\in A}(1-p_i) \prod_{j\in[k]\setminus A} p_j.$$

Applying the above expansion to the space $V^{\otimes k}$ produces the decomposition

$$V^{\otimes k} = \bigoplus_{A\subseteq[k]} V[A] \tag{19}$$

where, for a given subset A of $[k]$, $V[A] := V_1 \otimes V_2 \otimes \cdots \otimes V_k$, where $V_i = V(1)$ if $i \in A$ and $V_i = V(0)$ otherwise.

Proposition 12.3. *Let $d \in \mathcal{D}(k)$ and let (A, d', B) be the corresponding triple under the bijection in Lemma 4.2. If $n = |A| = |B|$ then the representation $\bar{d} \mapsto \varphi(\bar{d})$ maps $V[B]$ into $V[A]$ and maps all other $V[B']$ with $B' \neq B$ to zero.*

Proof. Let $d = rtl$ be the factorization in Proposition 12.2, so that $\bar{d} = r\bar{t}l$. Then

$$\varphi(\bar{d}) = \varphi(r)\varphi(\bar{t})\varphi(l).$$

Furthermore, $\varphi(r)$ induces an isomorphism $V[\{1,\ldots,n\}] \to V[A]$ and $\varphi(r) = 0$ on all $V[Y]$ such that $Y \neq \{1,\ldots,n\}$. (The inverse map is obtained by flipping the diagram r upside down.) Similarly, $\varphi(l)$ induces an isomorphism $V[B] \to V[\{1,\ldots,n\}]$ and $\varphi(l) = 0$ on all $V[Y]$ such that $Y \neq B$. As $\varphi(\bar{t})$ induces a map from $V[\{1,\ldots,n\}] \cong V(1)^{\otimes n}$ into itself, and is zero on all other components, the result follows. ∎

We are now ready to prove the following.

Theorem 12.4. *Suppose that $\Bbbk$ is a field and $0 \neq q \in \Bbbk$ is not a root of unity. Set $\delta = 1 \pm (q + q^{-1})$. Then*

$$\mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k}) \cong \mathsf{PTL}_k(\delta).$$

Hence, $V^{\otimes k}$ satisfies Schur–Weyl duality with respect to the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_2)$, $\mathsf{PTL}_k(\delta)$.

Proof. Since the actions commute, $\varphi(\mathsf{PTL}_k(\delta))$ is contained in the commuting algebra $\mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k})$. The action of $\mathsf{PTL}_k(\delta)$ is faithful, so the desired isomorphism will follow once we show the inclusion is an equality. We do this by comparing dimensions. By the $\mathbf{U}_q(\mathfrak{gl}_2)$-module decomposition in equation (19), we have

$$\dim_{\Bbbk} \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k}) = \sum_{A,B} \dim_{\Bbbk} \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V[A], V[B]),$$

where the sum is over all pairs (A, B) of subsets of $[k]$. By classical Schur–Weyl duality, the simple $\mathbf{U}_q(\mathfrak{gl}_2)$-modules appearing as constituents of $V[A] \cong V(1)^{\otimes n}$, where $|A| = n$, are all indexed by partitions of n with not more than two parts. Thus,

$$\mathrm{Hom}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V[B], V[A]) = (0)$$

unless $|A| = |B|$. Furthermore, if $|A| = |B| = n$ for subsets A, B of $[k]$, we have

$$\mathrm{Hom}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V[B], V[A]) \cong \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V(1)^{\otimes n}) \cong \mathsf{TL}_n(\delta - 1)$$

by Schur–Weyl duality for Temperley–Lieb algebras (17), so

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V[B], V[A]) = \dim_{\Bbbk} \mathsf{TL}_n(\delta - 1) = \mathcal{C}_n$$

where $\mathcal{C}_n$ is the nth Catalan number. Putting these facts together yields the equality

$$\dim_{\Bbbk} \mathrm{End}_{\mathbf{U}_q(\mathfrak{gl}_2)}(V^{\otimes k}) = \sum_{n=0}^{k} \binom{k}{n}^2 \mathcal{C}_n,$$

which by equation (10) agrees with the dimension of $\mathsf{PTL}_k(\delta)$. This proves the first statement in the theorem. The remaining claims then follow by standard facts in the theory of semisimple algebras. ∎

Corollary 12.5. *Under the same hypotheses, we have the decomposition*

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda} V(\lambda) \otimes \mathsf{PTL}^{\lambda}$$

as $(\mathbf{U}_q(\mathfrak{gl}_2), \mathsf{PTL}_k(\delta))$-bimodules, where the indexing set Λ is the set of partitions of n of not more than two parts, for $0 \leq n \leq k$, as in Section 8.

Proof. This is a standard fact in semisimple representation theory.   ∎

Remark 12.6. (i) It makes sense to set $q = 1$ in $\mathsf{PTL}_k(1 \pm (q + q^{-1}))$, thus obtaining $\mathsf{PTL}_k(1 \pm 2)$. If the field $\Bbbk$ has characteristic zero, the analogue of Theorem 12.4 holds. In particular,

$$\mathrm{End}_{U(\mathfrak{gl}_2)}(V^{\otimes k}) \cong \mathsf{PTL}_k(1 \pm 2),$$

where $U(\mathfrak{gl}_2)$ is the ordinary universal enveloping algebra of $\mathfrak{gl}_2$. There is of course also a version of Corollary 12.5 for this situation.

(ii) If q is a root of unity then Theorem 12.4 (but not Corollary 12.5) still holds, provided that $\mathbf{U}_q(\mathfrak{gl}_2)$ is replaced by an appropriate $\Bbbk$-form, but the proof is very different. One needs to work with the Lusztig "integral" form of the quantized enveloping algebra and to appeal to the paper [8], which established a version of Jimbo's Schur–Weyl duality at roots of unity.

(iii) The image of $\mathbf{U}_q(\mathfrak{gl}_2)$ in $\mathrm{End}_{\Bbbk}(V^{\otimes k})$ is isomorphic to a generalized q-Schur algebra in type A, in the sense of [6], defined by the set Λ.

We now derive explicit formulas for the action of $\bar{d}, \tilde{d}$, where $d \in \mathcal{D}(k)$. Recall from Section 3 that

$$\bar{d} = \prod_{i \in \lceil d \rceil} (1 - p_i) d \prod_{i' \in \lfloor d \rfloor} (1 - p_i), \quad \tilde{d} = \prod_{i \in \lceil d \rceil_H} (1 - p_i) d \prod_{i' \in \lfloor d \rfloor_H} (1 - p_i).$$

In the representation on tensor space, the element $1 - p_i$ is the operator

$$1^{\otimes(i-1)} \otimes (1 - \dot{p}) \otimes 1^{\otimes(k-i)},$$

where $\dot{p}$ is projection onto $V(0)$ and hence $1 - \dot{p}$ projects onto $V(1)$. This observation gives the following result.

Proposition 12.7. *Suppose that d is in $\mathcal{D}(k)$. Given i_α, j_α in the set $\{-1, 0, 1\}$ for $\alpha = 1, \ldots, k$, label the top row vertices of d from left to right with $v_{j_1}, \ldots, v_{j_k}$ and similarly label the bottom row vertices with $v_{i_1}, \ldots, v_{i_k}$. Then the action $\bar{d}$ on $V^{\otimes k}$ is given by*

$$\bar{d}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1, \ldots, j_k} (\bar{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k} \, v_{j_1} \otimes \cdots \otimes v_{j_k},$$

and similarly the action $\tilde{d}$ on $V^{\otimes k}$ is given by

$$\tilde{d}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1, \ldots, j_k} (\tilde{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k} \, v_{j_1} \otimes \cdots \otimes v_{j_k},$$

where the scalars $(\bar{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k}$ and $(\tilde{d})^{j_1, \ldots, j_k}_{i_1, \ldots, i_k}$ are the product over the modified weights of the labeled blocks of d. The modified weight of a block β in d is the same is its

weight minus a correction term of $\delta_{i,0}\,\delta_{j,0}$ applied to all (resp., all horizontal) edges of d in computing the action of $\overline{d}$ (resp., $\widetilde{d}$).

Proof. This follows from the observation preceding the proposition and the table of weights preceding Proposition 11.1, by considering the cases separately. ∎

A. Semisimplicity criterion for $\mathsf{TL}_k(\pm(q+q^{-1}))$

The purpose of this appendix is to highlight a precise semisimplicity criterion for Temperley–Lieb algebras that arose in the work of Vaughan Jones, following the exposition of [9]. Assume that $\Bbbk$ is a field. For $q \in \Bbbk$, let

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$$

in $\mathbb{Z}[q]$ be the usual quantum integer (regarded as an element of $\Bbbk$) and define $[k]_q^! = [1]_q[2]_q \cdots [k]_q$. If $q \neq 0$, then the balanced form $[\![n]\!]_q$ of the quantum integer is

$$[\![n]\!]_q := q^{-(n-1)}[n]_{q^2} \in \mathbb{Z}[q, q^{-1}].$$

We also define $[\![k]\!]_q^! = [\![1]\!]_q[\![2]\!]_q \cdots [\![k]\!]_q$. Here is the criterion.

Theorem A.1 ([9]). *If $\Bbbk$ is a field and $0 \neq q \in \Bbbk$ satisfies $[\![k]\!]_q^! \neq 0$, then*

$$\mathsf{TL}_k(\pm(q+q^{-1}))$$

is semisimple over $\Bbbk$.

Proof. For $0 \neq \beta \in \Bbbk$, the Jones algebra $A_k(\beta)$ is the algebra with 1 on generators $u_1, \ldots, u_{k-1}$ subject to the defining relations

$$u_i^2 = u_i, \quad \beta u_i u_{i\pm 1} u_i = u_i, \quad u_i u_j = u_j u_i \quad \text{if } |i - j| > 1.$$

By [9, Proposition 2.8.5 (a)], $A_k(\beta)$ is (split) semisimple over $\Bbbk$ if

$$P_1(\beta^{-1}) P_2(\beta^{-1}) \cdots P_{k-1}(\beta^{-1}) \neq 0,$$

where $P_n(x)$ are polynomials in $\mathbb{Z}[x]$ satisfying the recursion $P_0(x) = 1$, $P_1(x) = 1$, and $P_{n+1}(x) = P_n(x) - x P_{n-1}(x)$ for all $n \geq 1$. Choose q in $\Bbbk$ such that $q \neq 0$, $q \neq -1$, and $\beta = q + q^{-1} + 2$. (Replace $\Bbbk$ by a suitable quadratic extension if necessary.) By [9, Proposition 2.8.3 (iv)],

$$P_n(\beta^{-1}) = \frac{1 + q + q^2 + \cdots + q^n}{(1+q)^n} = \frac{[n+1]_q}{(1+q)^n}.$$

Hence, $A_k(\beta) = A_k(q + q^{-1} + 2)$ is semisimple over $\Bbbk$ if $[k]_q^! \neq 0$. Now, by setting $e_i = \delta u_i$ for all i we recover the defining relations (4) if and only if $\beta = \delta^2$, so $A_k(\delta^2) \cong \mathsf{TL}_k(\delta)$. We conclude that if $[k]_q^! \neq 0$, then $\mathsf{TL}_k(\pm(q^{1/2} + q^{-1/2}))$ is semisimple over $\Bbbk$. To obtain the final conclusion, we replace $q^{1/2}$ by q. This has the effect of replacing $[k]_q^!$ by $[\![k]\!]_q^!$, up to a power of q. ∎

Remark A.2. It makes sense to specialize q to 1 in Theorem A.1. Then $[\![k]\!]_1^! = k!$ is the ordinary factorial of k, and the semisimplicity criterion coincides with the one appearing in Maschke's theorem for finite symmetric groups. This is no accident, as $\mathsf{TL}_k(\pm(q + q^{-1}))$ is a quotient of an appropriate Iwahori–Hecke algebra of type A.

References

[1] G. Benkart and T. Halverson, Motzkin algebras. *European J. Combin.* **36** (2014), 473–502 Zbl 1284.05333 MR 3131911

[2] G. Benkart and T. Halverson, Partition algebras and the invariant theory of the symmetric group. In *Recent trends in algebraic combinatorics*, pp. 1–41, Assoc. Women Math. Ser. 16, Springer, Cham, 2019 Zbl 1421.05094 MR 3969570

[3] G. Benkart and T. Halverson, Partition algebras $\mathsf{P}_k(n)$ with $2k > n$ and the fundamental theorems of invariant theory for the symmetric group S_n. *J. Lond. Math. Soc. (2)* **99** (2019), no. 1, 194–224 Zbl 1411.05274 MR 3909254

[4] J. S. Birman and H. Wenzl, Braids, link polynomials and a new algebra. *Trans. Amer. Math. Soc.* **313** (1989), no. 1, 249–273 Zbl 0684.57004 MR 992598

[5] I. Dolinka, J. East, and R. D. Gray, Motzkin monoids and partial Brauer monoids. *J. Algebra* **471** (2017), 251–298 Zbl 1406.20065 MR 3569186

[6] S. Doty, Presenting generalized q-Schur algebras. *Represent. Theory* **7** (2003), 196–213 Zbl 1030.17011 MR 1990659

[7] S. Doty and A. Giaquinto, Schur–Weyl duality for twin groups. *Trans. Groups* (2022), online first, DOI 10.1007/s00031-022-09708-w

[8] J. Du, B. Parshall, and L. Scott, Quantum Weyl reciprocity and tilting modules. *Comm. Math. Phys.* **195** (1998), no. 2, 321–352 Zbl 0936.16008 MR 1637785

[9] F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter graphs and towers of algebras*. Math. Sci. Res. Inst. Publ. 14, Springer, New York, 1989 Zbl 0698.46050 MR 999799

[10] J. J. Graham and G. I. Lehrer, Cellular algebras. *Invent. Math.* **123** (1996), no. 1, 1–34 Zbl 0853.20029 MR 1376244

[11] R. Green and R. Paget, Iterated inflations of cellular algebras. *J. Algebra* **493** (2018), 341–345 Zbl 1420.16010 MR 3715216

[12] T. Halverson and E. delMas, Representations of the Rook–Brauer algebra. *Comm. Algebra* **42** (2014), no. 1, 423–443 Zbl 1291.05215 MR 3169580

[13] T. Halverson and A. Ram, Partition algebras. *European J. Combin.* **26** (2005), no. 6, 869–921 Zbl 1112.20010 MR 2143201

[14] J. C. Jantzen, *Lectures on quantum groups*. Grad. Stud. Math. 6, American Mathematical Society, Providence, RI, 1996 Zbl 0842.17012 MR 1359532

[15] V. F. R. Jones, Index for subfactors. *Invent. Math.* **72** (1983), no. 1, 1–25 Zbl 0508.46040 MR 696688

[16] V. F. R. Jones, A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)* **12** (1985), no. 1, 103–111 Zbl 0564.57006 MR 766964

[17] V. F. R. Jones, Braid groups, Hecke algebras and type II$_1$ factors. In *Geometric methods in operator algebras (Kyoto, 1983)*, pp. 242–273, Pitman Res. Notes Math. Ser. 123, Longman Scientific and Technical, Harlow, 1986 Zbl 0659.46054 MR 866500

[18] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials. *Ann. of Math. (2)* **126** (1987), no. 2, 335–388 Zbl 0631.57005 MR 908150

[19] V. F. R. Jones, Subfactors of type II$_1$ factors and related topics. In *Proceedings of the International Congress of Mathematicians, Volumes 1, 2 (Berkeley, CA, 1986)*, pp. 939–947, American Mathematical Society, Providence, RI, 1987 Zbl 0674.46034 MR 934296

[20] V. F. R. Jones, The Potts model and the symmetric group. In *Subfactors (Kyuzeso, 1993)*, pp. 259–267, World Scientific, River Edge, NJ, 1994 Zbl 0938.20505 MR 1317365

[21] L. H. Kauffman, State models and the Jones polynomial. *Topology* **26** (1987), no. 3, 395–407 Zbl 0622.57004 MR 899057

[22] L. H. Kauffman, *Knots and physics*. Fourth edn., Ser. Knots Everything 53, World Scientific, Hackensack, NJ, 2013 Zbl 1266.57001 MR 3013186

[23] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*. Texts Monogr. Phys., Springer, Berlin, 1997 Zbl 0891.17010 MR 1492989

[24] S. König and C. Xi, Cellular algebras: Inflations and Morita equivalences. *J. London Math. Soc. (2)* **60** (1999), no. 3, 700–722 Zbl 0952.16014 MR 1753809

[25] S. König and C. Xi, A characteristic free approach to Brauer algebras. *Trans. Amer. Math. Soc.* **353** (2001), no. 4, 1489–1505 Zbl 0996.16009 MR 1806731

[26] G. Lusztig, *Introduction to quantum groups*. Progr. Math. 110, Birkhäuser, Boston, MA, 1993 Zbl 0788.17010 MR 1227098

[27] P. Martin, *Potts models and related problems in statistical mechanics*. Ser. Adv. Stat. Mech. 5, World Scientific, Teaneck, NJ, 1991 Zbl 0734.17012 MR 1103994

[28] P. Martin and V. Mazorchuk, On the representation theory of partial Brauer algebras. *Q. J. Math.* **65** (2014), no. 1, 225–247 Zbl 1354.16016 MR 3179659

[29] V. Mazorchuk, Endomorphisms of $\mathfrak{B}_n$, $\mathcal{P}\mathfrak{B}_n$, and $\mathfrak{C}_n$. *Comm. Algebra* **30** (2002), no. 7, 3489–3513 Zbl 1008.20056 MR 1915009

[30] W. D. Munn, On semigroup algebras. *Proc. Cambridge Philos. Soc.* **51** (1955), 1–15 Zbl 0064.02001 MR 66355

[31] E. Posner, K. Hatch, and M. Ly, Presentation of the Motzkin monoid. 2013, arXiv:1301.4518

[32] D. Ridout and Y. Saint-Aubin, Standard modules, induction and the structure of the Temperley–Lieb algebra. *Adv. Theor. Math. Phys.* **18** (2014), no. 5, 957–1041 Zbl 1308.82015 MR 3281274

[33] H. N. V. Temperley and E. H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. *Proc. Roy. Soc. London Ser. A* **322** (1971), no. 1549, 251–280 Zbl 0211.56703 MR 498284

[34] B. W. Westbury, The representation theory of the Temperley–Lieb algebras. *Math. Z.* **219** (1995), no. 4, 539–565 Zbl 0840.16008 MR 1343661

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