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Cohomology ring of the flag variety vs Chow cohomology ring of the Gelfand–Zetlin toric variety

Kiumars Kaveh and Elise Villella

Abstract. We compare the cohomology ring of the flag variety $\mathcal{F}\ell_n$ and the Chow cohomology ring of the Gelfand–Zetlin toric variety X_{GZ}. We show that $H^*(\mathcal{F}\ell_n, \mathbb{Q})$ is the Poincaré duality quotient of the subalgebra of $A^*(X_{\mathrm{GZ}}, \mathbb{Q})$ generated by degree 1 elements. We compute these algebras for $n = 3$ and see that, in general, this subalgebra does not have Poincaré duality.

Introduction

Throughout the paper, the base field is assumed to be $\mathbb{C}$. The complete flag variety $\mathcal{F}\ell_n$ is the variety whose points parametrize complete flags of subspaces in $\mathbb{C}^n$, namely:

$$F = \big(\{0\} \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_n = \mathbb{C}^n\big).$$

The variety $\mathcal{F}\ell_n$ can be identified with the homogeneous space $\mathrm{GL}(n, \mathbb{C})/B$, where B is the subgroup of upper triangular matrices. The geometry of flag variety plays an important role in representation theory of $\mathrm{GL}(n, \mathbb{C})$ and combinatorics related to the permutation group. More generally there is a notion of flag variety for any reductive algebraic group G.

We recall that $\dim(\mathcal{F}\ell_n) = N = n(n-1)/2$. The classes of Schubert varieties form an important $\mathbb{Z}$-basis for $H^*(\mathcal{F}\ell_n, \mathbb{Z})$. Since $\mathcal{F}\ell_n$ has a paving by affine cells (Schubert cells), it has no odd cohomology. Moreover, $H^*(\mathcal{F}\ell_n, \mathbb{Z})$ is generated by degree 2 elements. Also its Chow ring $A^*(\mathcal{F}\ell_n)$ is isomorphic to $H^*(\mathcal{F}\ell_n, \mathbb{Z})$, where the isomorphism doubles the degree. The famous Borel description states that $H^*(\mathcal{F}\ell_n, \mathbb{Z})$ is isomorphic to the polynomial algebra in n variables quotient by the ideal generated by non-constant symmetric polynomials.

We identify the weight lattice $\Lambda = \Lambda_{\mathrm{GL}(n,\mathbb{C})}$ with the additive group $\mathbb{Z}^n$ and the semigroup of dominant weights $\Lambda^+ = \Lambda^+_{\mathrm{GL}(n,\mathbb{C})}$ (respectively, the positive Weyl

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chamber $\Lambda^+_{\mathbb{R}}$) with the collection of all increasing sequences $\lambda = (\lambda_1 \le \cdots \le \lambda_n)$ of integers (respectively, real numbers). If $\lambda_1 < \cdots < \lambda_n$ we call λ a regular dominant weight. We also denote the weight lattice $\Lambda(\mathrm{SL}(n,\mathbb{C}))$ of $\mathrm{SL}(n,\mathbb{C})$ by Λ'. It can be identified with the quotient $\Lambda/\mathbb{Z}(1,\dots,1)$.

In their fundamental work [10], Gelfand and Zetlin[1] construct a certain vector space basis B_λ for an irreducible representation V_λ of $\mathrm{GL}(n,\mathbb{C})$ with highest weight λ, and they explicitly describe the action of $\mathfrak{gl}(n,\mathbb{C}) = \mathrm{Lie}(\mathrm{GL}(n,\mathbb{C}))$ on basis elements in B_λ. The Gelfand–Zetlin basis B_λ has the remarkable property that its elements are indexed by the lattice points in a convex polytope $\Delta_\lambda \subset \mathbb{R}^N$, where $N = n(n-1)/2$, called the *Gelfand–Zetlin polytope* (or GZ polytope) associated to λ. The defining inequalities of Δ_λ can be explicitly written down. If $\lambda = (\lambda_1 \le \cdots \le \lambda_n)$ the polytope Δ_λ is the collection of $(x_{ij} \mid 1 \le i \le n-1, 1 \le j \le n-i) \in \mathbb{R}^N$ satisfying the following array of inequalities:

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_n \\
& x_{11} & & x_{12} & & \cdots & & x_{1(n-1)} & \\
& & x_{21} & & x_{22} & & \cdots & & \\
& & & \ddots & & ⋰ & & & \\
& & & & x_{(n-1)1} & & & &
\end{array} \tag{1}$$

where each small triangle $\begin{smallmatrix} a & & b \\ & c & \end{smallmatrix}$ corresponds to the inequalities $a \le c \le b$. For example, if $\lambda = (-1, 0, 1)$, the Gelfand–Zetlin polytope Δ_λ is given by the inequalities (see Figure 1):

$$-1 \le x \le 0, \quad 0 \le y \le 1, \quad x \le z \le y.$$

Since there is a one-to-one correspondence between the elements of the Gelfand–Zetlin basis B_λ and the lattice points in Δ_λ one immediately sees that

$$\dim(V_\lambda) = \#(\Delta_\lambda \cap \mathbb{Z}^N).$$

It is well known that a weight λ gives rise to a $\mathrm{GL}(n,\mathbb{C})$-linearized line bundle $\mathcal{L}_\lambda$ on the flag variety $\mathcal{F}\ell_n$. When λ is regular dominant the line bundle $\mathcal{L}_\lambda$ is very ample. By the Borel–Weil theorem, $H^0(\mathcal{F}\ell_n, \mathcal{L}_\lambda) \cong V^*_\lambda$ as a $\mathrm{GL}(n,\mathbb{C})$-module. Thus, in particular, we have

$$\dim(H^0(\mathcal{F}\ell_n, \mathcal{L}_\lambda)) = \#(\Delta_\lambda \cap \mathbb{Z}^N).$$

[1]Warning to the reader: several different spellings of Zetlin's name appear in the English literature such as Tsetlin, Cetlin, Zeitlin or Tzetlin. Following Valentina Kiritchenko we use the spelling Zetlin, justified by the fact that while he was Russian his last name seems to have German origins.

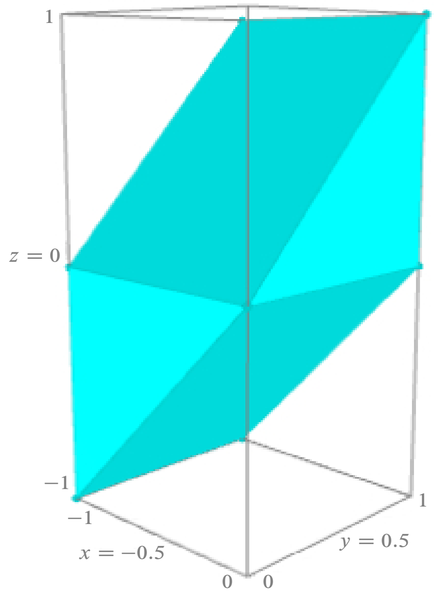

Figure 1. Gelfand–Zetlin for $\lambda = (-1, 0, 1)$.

A general philosophy, suggested in the work of several authors and in particular A. Okounkov [19], is that GZ polytopes play a role for the flag variety similar to that of Newton polytopes for toric varieties. In this direction in [14] the first author obtains a description of $H^*(\mathcal{F}\ell_n, \mathbb{Q})$ in terms of volumes of GZ polytopes. This description is very similar to the Khovanskii–Pukhlikov description of cohomology ring of a smooth projective toric variety in terms of volumes of Newton polytopes. The description in [14] turns out to be equivalent to the Borel description via a theorem of Kostant (see [14, Remark 5.4]). Making the connection between geometry of $\mathcal{F}\ell_n$ and GZ polytopes stronger, in [17] the authors make a correspondence between Schubert varieties and certain unions of faces of GZ polytopes. They use this correspondence to give applications in Schubert calculus.

It can be shown that for regular dominant weights λ, all the polytopes Δ_λ have the same normal fan (Proposition 1.1). We call this common normal fan the *Gelfand–Zetlin fan* and denote it by Σ_{GZ}. It is well known that, for each regular dominant λ the pair $(\mathcal{F}\ell_n, \mathcal{L}_\lambda)$ can be degenerated, in a flat family with reduced irreducible fibers, to $(X_{\mathrm{GZ}}, \mathcal{L}_{\Delta_\lambda})$. Here $\mathcal{L}_{\Delta_\lambda}$ is the equivariant line bundle on the toric variety X_{GZ} corresponding to the lattice polytope Δ_λ (see [18]). Such degenerations have been used to study mirror symmetry for the flag variety and partial flag varieties (see [1]). This motivates the problem of comparing the geometry and topology of $\mathcal{F}\ell_n$ with that of X_{GZ}.

The variety X_{GZ} is not smooth and hence its Chow group does not have a ring structure. There is a dual version of the Chow ring, due to Fulton and MacPherson [8], that works for singular varieties as well. It is called the *operational Chow ring* or simply *Chow cohomology ring*. For a variety X we denote its Chow cohomology ring by $A^*(X)$.

Let $\mathbf{k}$ be a field. Given a graded algebra $A = \bigoplus_{i=0}^n A^i$ with $A^0 \cong A^n \cong \mathbf{k}$, one can form the largest quotient A/I of A such that A/I has Poincaré duality (Lemma 4.1). We call this the *Poincaré duality quotient of* A and denote it by $\mathrm{PD}(A)$. The main result of the paper is the following (Theorem 5.1):

Theorem 1. *The cohomology ring $H^*(\mathcal{F}\ell_n, \mathbb{Q})$ is isomorphic to the Poincaré duality quotient of the subalgebra of $A^*(X_{\mathrm{GZ}}, \mathbb{Q})$ generated by degree* 1 *elements.*

One key combinatorial ingredient in the proof is the following statement suggested to us by Valentina Kiritchenko (Proposition 1.3):

Proposition 2. *Let P be a polytope whose normal fan is Σ_{GZ}, then $P = c + \Delta_\lambda$ for some $\lambda \in \Lambda^+$ and $c \in \mathbb{R}^N$.*

Another ingredient in the proof of Theorem 1 is an algebra lemma which states that a Poincaré duality algebra $A = \bigoplus_{i=0}^n A^i$ that is finite dimensional as a vector space and is generated (over A^0) by A^1, is uniquely determined by its top product polynomial $p\colon A^1 \to A^n \cong A^0$, $p(x) = x^n$ (Theorem 4.2).

In [9] it is shown that the Chow cohomology ring of a toric variety X_Σ is naturally isomorphic to the ring of *Minkowski weights* on its fan Σ. A degree k Minkowski weight on a fan Σ is an assignment of integers to k-dimensional cones in Σ which satisfies certain balancing condition. One defines a product of Minkowski weights that makes the collection of all Minkowski weights into a ring (see Section 6, see also [9, 16]). There is also an alternative description of the Chow cohomology ring of a toric variety in terms of piecewise linear functions on its fan (see [20]).

In Section 7, we use the Minkowski weights description of the Chow cohomology ring, to compute $A^*(X_{\mathrm{GZ}}, \mathbb{Q})$ for $n = 3$ and see directly that it coincides with its subalgebra generated by degree 1 elements. We also see $A^*(X_{\mathrm{GZ}}, \mathbb{Q})$ does not have Poincaré duality.

The second author has written a Sage code that verifies that for $n = 4, 5$ the Chow cohomology ring of X_{GZ} is not generated in degree 1, and moreover the subalgebra generated in degree 1 does not have Poincaré duality. See https://github.com/evillella/minkowski. Also see the appendix in the second author's PhD thesis [21].

In geometric terms, the isomorphism between the Picard groups of $\mathcal{F}\ell_n$ and X_{GZ} can be constructed by means of a *toric degeneration*. A toric degeneration of $\mathcal{F}\ell_n$ to X_{GZ} is a flat family $\pi\colon \mathcal{X} \to \mathbb{C}$ with reduced fibers and an action of $\mathbb{C}^*$ lifting the $\mathbb{C}^*$ action on the base $\mathbb{C}$ such that the general fiber $X_t := \pi^{-1}(t)$, $t \neq 0$, is $\mathcal{F}\ell_n$ and its unique special fiber $X_0 := \pi^{-1}(0)$ is X_{GZ}. Then any divisor class $[D]$ in $\mathrm{Pic}(\mathcal{F}\ell_n)$ can be extended to the whole family $\mathcal{X}$ and then specialized to the special fiber X_{GZ} to get a divisor class $[D_0]$ on X_{GZ}. For a general toric degeneration, $[D_0]$ may not be a Cartier divisor class. But one shows that this is the case, for example, for the family constructed in [18] (see [18, Proposition 11]). In fact, under this specialization map the class of a line bundle $\mathcal{L}_\lambda$ on $\mathcal{F}\ell_n$ goes to the class of the line bundle on X_{GZ} determined by the polytope Δ_λ. We do not know if this construction extends to give a homomorphism between the Chow cohomology rings.

1. Some facts about Gelfand–Zetlin polytopes

In this section we prove some basic facts about GZ polytopes. We start with the normal fan to a GZ polytope Δ_λ. Recall that the normal fan Σ_Δ of a polytope Δ is constructed as follows: for each face F let C_F be the face cone of F and let σ_F be the dual cone to C_F. Then $\Sigma_\Delta = \{\sigma_F \mid F \text{ face of } \Delta\}$ (see [3, Section 2.3]).

Proposition 1.1. *For a regular dominant weight λ, the normal fan Σ_λ of Δ_λ is independent of λ.*

Proof. The facets of Δ_λ correspond to single equalities in the array (1), and lower dimensional faces of Δ_λ correspond to multiple equalities in the array. There are two types of equality that can occur: (i) those of the form $x_{1i} = \lambda_j$, and (ii) those of the form $x_{ij} = x_{(i-1)k}$. The second type of equality is clearly independent of λ and the first type depends on λ, so that the faces corresponding to various λ values differ only by translation. It follows that for a face F, which is defined by a combination of these two types of equalities, the corresponding face cone C_F and hence its dual cone σ_F is independent of λ. This proves the claim. ∎

Definition 1.2 (Gelfand–Zetlin fan). We refer to the common normal fan of the Δ_λ, where λ is regular dominant, as the *Gelfand–Zetlin fan* and denote it by Σ_{GZ}.

Proposition 1.3. *Let P be a polytope whose normal fan is Σ_{GZ}, then $P = c + \Delta_\lambda$ for some $\lambda \in \Lambda^+$ and $c \in \mathbb{R}^N$. Moreover, if P is a lattice polytope then λ is a dominant weight and $c \in \mathbb{Z}^N$.*

Proof. Since normal fan of P is Σ_{GZ}, the hyperplanes defining P are parallel to the ones defining Δ_λ, for any dominant regular λ (as we have already showed the fan is independent of λ). Let us use y_{ij} (respectively, x_{ij}) for coordinates of a point in P (respectively, a GZ polytope Δ_λ). Recall that there are two types of inequalities defining Δ_λ namely, $\lambda_i \le x_{1i} \le \lambda_{i+1}$ and $x_{(i-1)j} \le x_{ij} \le x_{(i-1)(j+1)}$. Since the facets of P are parallel to those of a GZ polytope we conclude that the inequalities defining P come in two types as well:

$$\begin{aligned} a_j \leq y_{1j} \leq b_j & \quad 1 \leq j \leq n-1, \\ y_{(i-1)j} + a_{ij} \leq y_{ij} \leq y_{(i-1)(j+1)} + b_{ij} & \quad 2 \leq i \leq n-1,\ 1 \leq j \leq n-i. \end{aligned} \tag{2}$$

We wish to find $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n)$ and $c = (c_{ij}) \in \mathbb{R}^N$ such that if $x_{ij} = y_{ij} + c_{ij}$ then the inequalities (2) for the y_{ij} are equivalent to the GZ inequalities (1) for the x_{ij}.

The first type of inequalities $a_j \leq y_{1j} \leq b_j$ tell us what λ to choose. Set $\lambda_1 = a_1$ and $\lambda_2 = b_1$. By induction suppose for $1 \leq j < n-1$ we have picked $\lambda_1, \dots, \lambda_{j+1}$ and $c_{11} = 0, c_{12}, \dots, c_{1j}$ such that

$$\lambda_1 \leq x_{11} = y_{11} \leq \lambda_2 \leq x_{12} \leq \cdots \leq x_{1j} \leq \lambda_{j+1},$$

where $x_{1k} = y_{1k} + c_{1k}$, for all k. Now if we put

$$c_{1(j+1)} = \lambda_{j+1} - a_{j+1} \quad \text{and} \quad \lambda_{j+2} = b_{j+1} + \lambda_{j+1} - a_{j+1},$$

we have $\lambda_{j+1} \leq x_{1(j+1)} \leq \lambda_{j+2}$ as required.

For the remaining rows, we first need to examine the small diamonds $\begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix}$ appearing in the GZ array (1). Since $b \leq d \leq c$, the equalities $b = a$ and $c = a$ imply $d = a$. This gives us linear relations among the ray generators in the fan Σ_{GZ} which in turn translate to relations among the a_{ij}, b_{ij} for the polytope P. Let $1 < i < n-1$ and $1 \leq j \leq n-i$, and by induction suppose we have picked $c_{11}, \dots, c_{i(j-1)}$ so that $x_{11}, \dots, x_{i(j-1)}$ satisfy the GZ triangular array of inequalities. We would like to find c_{ij} so that $x_{ij} = y_{ij} + c_{ij}$ satisfies the diamond

$$\begin{matrix} & x_{(i-2)(j+1)} & \\ x_{(i-1)j} & & x_{(i-1)(j+1)} \\ & x_{ij} & \end{matrix}$$

The second type of inequality in (2) can be written as:

$$x_{(i-1)j} + a'_{ij} \leq y_{ij} \leq x_{(i-1)(j+1)} + b'_{ij}, \tag{3}$$

where $a'_{ij} = a_{ij} + c_{(i-1)j}$ and $b'_{ij} = b_{ij} + c_{(i-1)(j+1)}$. Now when we consider the face of P, where $x_{(i-1)j} = x_{(i-2)(j+1)}$ and $x_{(i-1)(j+1)} = x_{(i-2)(j+1)}$, by what we said above, the inequality (3) becomes two equalities. We thus have

$$x_{(i-2)(j+1)} + a'_{ij} = x_{(i-2)(j+1)} + b'_{ij},$$

which implies $a'_{ij} = b'_{ij}$. Now, if we define $x_{ij} = y_{ij} - a'_{ij}$, i.e., $c_{ij} = -a'_{ij}$, the relation (3) becomes

$$x_{(i-1)j} \leq x_{ij} \leq x_{(i-1)(j+1)},$$

as required. Therefore, $P = c + \Delta_\lambda$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ and $c = (c_{ij})$ as constructed above. Finally, if P is a lattice polytope, then the a_i, b_i, a_{ij}, b_{ij} should be integers (note that none of the inequalities in (2) is redundant and the corresponding equality defines a facet of P). This implies that λ and c are integer vectors as well. ∎

Remark 1.4. Proposition 1.3 was suggested to us by Valentina Kiritchenko. The proof presented above is due to the second author.

Remark 1.5. Observe that there are $n + n(n-1)/2$ parameters present in $c + \Delta_\lambda$, but a GZ polytope is cut out by $n(n-1)$ facets, one for each ray in $\Sigma_{\mathrm{GZ}}(1)$. The dimension of the space of polytopes with normal fan Σ_{GZ} is hence much smaller than the number of rays in the fan due to the fact that Δ_λ is not a simple polytope, or equivalently, the fan Σ_{GZ} is not simplicial.

A third useful property of the GZ polytopes is that they behave well with respect to Minkowski addition. We recall that for polytopes P and Q, the *Minkowski sum* $P + Q$ is the polytope

$$P + Q = \{x + y \mid x \in P, y \in Q\}.$$

Proposition 1.6. *The assignment $\lambda \mapsto \Delta_\lambda$ is additive, that is, for any dominant weights $\lambda, \mu \in \mathbb{Z}^n$, we have*

$$\Delta_{\lambda+\mu} = \Delta_\lambda + \Delta_\mu,$$

where the addition on the right is the Minkowski sum of polytopes.

Proof. The inclusion $\Delta_\lambda + \Delta_\mu \subset \Delta_{\lambda+\mu}$ is clear. We need to show the other direction. Let $x \in \Delta_{\lambda+\mu}$, our goal is to write $x = x' + x''$ with $x' \in \Delta_\lambda$ and $x'' \in \Delta_\mu$. We begin with the first row $x_{1*} = (x_{11}, \dots, x_{1(n-1)})$ satisfying

$$\lambda_1 + \mu_1 \le x_{11} \le \lambda_2 + \mu_2 \le \cdots \le x_{1(n-1)} \le \lambda_n + \mu_n.$$

It is clear that, for each i, the sum of line segments $[\lambda_i, \lambda_{i+1}]$ and $[\mu_i, \mu_{i+1}]$ is

$$[\lambda_i + \mu_i, \lambda_{i+1} + \mu_{i+1}].$$

Thus, we can find $x'_{1*}, x''_{1*} \in \mathbb{R}^{n-1}$ such that $x_{1*} = x'_{1*} + x''_{1*}$, and they satisfy the first row of interlacing inequalities for Δ_λ and Δ_μ, respectively. We can then repeat the same argument for the second row replacing λ, μ with x'_{1*}, x''_{1*} to obtain $x'_{2*}, x''_{2*} \in \mathbb{R}^{n-2}$. Continuing in this way, we find $x' \in \Delta_\lambda$, $x'' \in \Delta_\mu$ with $x = x' + x''$, as required. ∎

Remark 1.7. Proposition 1.6 shows that the collection of Gelfand–Zetlin polytopes is an example of a *linear family of polytopes* (as defined in [15]). In this regard, Proposition 1.1 is related to [15, Proposition 1.3].

Proposition 1.8. *Suppose for two dominant weights λ, $\lambda' \in \Lambda$ and $c \in \mathbb{Z}^N$ we have $c + \Delta_\lambda = \Delta_{\lambda'}$. Then $\lambda - \lambda'$ is a multiple of $(1, \dots, 1)$, that is, λ, λ' represent the same weight in Λ'.*

Proof. Let the (x_{ij}), (x'_{ij}) denote the coordinates of points in Δ_λ, $\Delta_{\lambda'}$, respectively. Also let $c = (c_{ij})$. The assumption that $c + \Delta_\lambda = \Delta_{\lambda'}$ implies that for all $1 \le i \le n-1$, $\lambda_i \le x_{1i} \le \lambda_{i+1}$ if and only if $\lambda'_i \le x_{1i} + c_{1i} \le \lambda'_{i+1}$. It follows that $\lambda'_i = \lambda_i + c_{1i}$ and $\lambda'_{i+1} = \lambda_{i+1} + c_{1i}$, which in turn implies that $c_{1i} = c_{1(i+1)}$. This finishes the proof. ∎

Recall that a virtual polytope is a formal difference of two polytopes. The set of virtual polytopes in $\mathbb{R}^N$ forms an infinite dimensional $\mathbb{R}$-vector space. For a fan Σ in $\mathbb{R}^N$ let $\mathcal{P}(\Sigma)$ denote the subgroup of virtual lattice polytopes in $\mathbb{R}^N$ generated by polytopes whose normal fan is Σ. The group $\mathcal{P}(\Sigma)$ contains a copy of the additive group $\mathbb{Z}^N$ as the virtual lattice polytopes whose support function is linear on the whole $\mathbb{R}^N$.

Corollary 1.9. (1) *The map $\lambda \mapsto \Delta_\lambda$ gives a homomorphism*

$$\phi\colon \Lambda = \Lambda(\mathrm{GL}(n, \mathbb{C})) \to \mathcal{P}(\Sigma_{\mathrm{GZ}}).$$

(2) *The homomorphism ϕ induces an isomorphism*

$$\bar{\phi}\colon \Lambda' = \Lambda(\mathrm{SL}(n, \mathbb{C})) = \Lambda/\mathbb{Z}(1, \dots, 1) \to \mathcal{P}(\Sigma_{\mathrm{GZ}})/\mathbb{Z}^N.$$

(3) *The quotient group Λ' is isomorphic to the Picard group of the toric variety X_{GZ} associated to the fan Σ_{GZ}.*

Proof. The assertion (1) is an immediate corollary of Proposition 1.6. To prove (2), note that surjectivity of $\bar{\phi}$ follows from Proposition 1.3 and the injectivity of $\bar{\phi}$ is the content of Proposition 1.8. Finally, (3) follows from the well-known fact that for a fan Σ, the group $\mathrm{Pic}(X_\Sigma)$ is isomorphic to the group $\mathrm{PL}(\Sigma, \mathbb{Z}^N)$ of integer piecewise linear functions on Σ modulo integer linear functions. This in turn can be identified with the quotient group $\mathcal{P}(\Sigma)/\mathbb{Z}^N$ (see [3, Theorem 4.2.12]). ∎

2. Review of degrees of line bundles on toric and flag varieties

We recall that, for a projective variety X of dimension d embedded into a projective space $\mathbb{P}^s$, the *degree* of X is defined to be

$$\deg(X) = \#(X \cap H_1 \cap \ldots \cap H_d),$$

where the H_i are generic hyperplanes in $\mathbb{P}^s$. Alternatively, let $[H]$ be the class of a hyperplane in $\mathrm{Pic}(\mathbb{P}^s) \cong \mathbb{Z}$ and let $[H']$ be the pullback of $[H]$ to X via the embedding $X \hookrightarrow \mathbb{P}^s$, then $\deg(X) = [H']^d$, the self-intersection number of the divisor class $[H']$.

If the embedding $X \hookrightarrow \mathbb{P}^s$ is given by the sections of a very ample line bundle $\mathcal{L}$, that is, $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$, we will write $\deg(X, \mathcal{L})$ for $\deg(X)$. The asymptotic Riemann–Roch theorem, implies that

$$\deg(X, \mathcal{L}) = d! \lim_{m\to\infty} \frac{\dim H^0(X, \mathcal{L}^{\otimes m})}{m^d}.$$

If $\mathcal{L}$ is not very ample, we still define $\deg(X, \mathcal{L})$ as the self-intersection number of the divisor class of $\mathcal{L}$.

In the case $X = X_\Sigma$ is the toric variety of a fan Σ, we recall that all divisors are linearly equivalent to T-invariant divisors which in turn are generated by codimension 1 orbit closures $D_\rho = \overline{O}_\rho$, $\rho \in \Sigma(1)$. Thus, an arbitrary T-invariant divisor on X_Σ can be written in the form $D = \sum_\rho a_\rho D_\rho$. The associated line bundle will be $\mathcal{L} = \mathcal{O}(D)$, and the dimension of $H^0(X, \mathcal{L})$ is equal to the number of lattice points in the polytope

$$P_D = \{m \mid \langle m, v_\rho\rangle \geq -a_\rho,\ \forall \rho \in \Sigma(1)\},$$

where v_ρ is the primitive vector along the ray ρ. One can also start with a lattice polytope P normal to the fan of X_Σ. The *support numbers* $\{a_\rho\}_{\rho\in\Sigma(1)}$ of the polytope enable us to define a T-invariant divisor $D_P = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ on X_Σ, and $P_{D_P} = P$. One shows that D is ample that is, kD defines an embedding into projective space for sufficiently large $k \in \mathbb{N}$. We have the following proposition (which is a version of the well-known Bernstein–Kushnirenko–Khovanskii theorem).

Proposition 2.1. *Let $\mathcal{L}_P$ be the line bundle associated to the divisor D_P. Then*

$$\deg(X_\Sigma, \mathcal{L}_P) = d!\,\mathrm{Vol}_d(P).$$

Proof. By the asymptotic Riemann–Roch we have

$$\begin{aligned}\deg(X_\Sigma, \mathcal{L}_P) &= d! \lim_{m\to\infty} \frac{\dim H^0(X_\Sigma, \mathcal{L}_P^{\otimes m})}{m^d}\\ &= d! \lim_{m\to\infty} \frac{\#(mP \cap \mathbb{Z}^d)}{m^d} = d!\,\mathrm{Vol}_d(P).\end{aligned}$$ ■

As we are interested in comparing X_{GZ} with the flag variety $\mathcal{F}\ell_n$, we also recall some facts about degrees of embeddings for $\mathcal{F}\ell_n$. Recall that to a weight λ one associates a line bundle $\mathcal{L}_\lambda$ on $\mathcal{F}\ell_n$. This line bundle satisfies the property

$$\mathcal{L}_\lambda^{\otimes m} = \mathcal{L}_{m\lambda}.$$

Similarly to the proof of Proposition 2.1, we can show the following (see, for example, [14, Remark 2.4]).

Proposition 2.2. *For any dominant weight λ, we have*

$$\deg(\mathcal{F}\ell_n, \mathcal{L}_\lambda) = N! \operatorname{Vol}_N(\Delta_\lambda),$$

where $N = n(n-1)/2 = \dim(\mathcal{F}\ell_n)$.

Proof. By the construction of the Gelfand–Zetlin polytope [10], for every dominant λ, we have

$$\#(\Delta_\lambda \cap \mathbb{Z}^N) = \dim(V_\lambda) = \dim(V_\lambda^*).$$

On the other hand, by the Borel–Weil theorem, one knows that $H^0(\mathcal{F}\ell_n, \mathcal{L}_\lambda) \cong V_\lambda^*$. We note that for any $m > 0$, we have

$$\mathcal{L}_\lambda^{\otimes m} = \mathcal{L}_{m\lambda} \quad \text{and} \quad m\Delta_\lambda = \Delta_{m\lambda}.$$

Then the asymptotic Riemann–Roch theorem gives us

$$\begin{aligned}\deg(\mathcal{F}\ell_n, \mathcal{L}_\lambda) &= N! \lim_{m\to\infty} \frac{\dim H^0(\mathcal{F}\ell_n, \mathcal{L}_\lambda^{\otimes m})}{m^N} \\ &= N! \lim_{m\to\infty} \frac{\#(m\Delta_\lambda \cap \mathbb{Z}^N)}{m^N} = N! \operatorname{Vol}_N(\Delta_\lambda),\end{aligned}$$

as required. ∎

Proposition 2.2 and Proposition 2.1 show that the map $\operatorname{Pic}(\mathcal{F}\ell_n) \to \operatorname{Pic}(X_{\mathrm{GZ}})$, given by $\mathcal{L}_\lambda \mapsto \mathcal{L}_{\Delta_\lambda}$, preserves degree of line bundles. This observation is important in the proof of our main theorem (Theorem 5.1).

3. Review of intersection theory on toric and flag varieties

In this section we recall some basic facts about Chow rings and Chow cohomology rings of toric and flag varieties.

For an algebraic variety X and $1 \le k \le n = \dim(X)$, the k-th Chow group $A_k(X)$ is the group generated by algebraic k-cycles on X, that is, formal sums of irreducible k-dimensional subvarieties in X, modulo rational equivalence. Two k-cycles are equivalent if their difference is the divisor of a rational function on a $(k+1)$-dimensional subvariety, and the rational equivalence is the equivalence relation generated by this. The total Chow group of X is

$$A_*(X) = \bigoplus_{k=0}^{n} A_k(X).$$

When X is smooth we let $A^k(X) = A_{n-k}(X)$ and

$$A^*(X) = \bigoplus_{k=0}^{n} A^k(X).$$

In this case, the transverse intersection of subvarieties gives a well-defined multiplication on $A^*(X)$ making it into a graded algebra called the Chow ring of X ([6, Proposition 8.3]). More generally, for a commutative ring R, one can define the Chow groups $A_k(X, R)$ and the Chow ring $A^*(X, R)$ whenever X is smooth.

In general, for a smooth variety X, the cohomology ring $H^*(X)$ and the Chow ring $A^*(X)$ are different. Nevertheless, for some nice varieties X these algebras are naturally isomorphic ([6, Example 19.1.11]).

Theorem 3.1. *Suppose X is smooth and has a paving by affine cells, then $H^*(X)$ and $A^*(X)$ are naturally isomorphic.*

The above theorem in particular applies to complete smooth toric varieties and the flag variety $\mathcal{F}\ell_n$.

When $X = X_\Sigma$ is a smooth complete toric variety, there is a nice description of the Chow ring $A^*(X_\Sigma)$. In this case, for each k, the Chow group $A^k(X_\Sigma) = A_{n-k}(X_\Sigma)$ is generated by the orbit closures of codimension k. Although not needed in this paper, we state the following well-known result on description of the Chow ring of a smooth complete toric variety (see [7, Section 5.2]).

Theorem 3.2. *Let X_Σ be a smooth complete toric variety. Let $D_1, \ldots, D_r$ be the codimension 1 orbit closures corresponding to rays $\rho_1, \ldots, \rho_r \in \Sigma(1)$. Then*

$$A^*(X_\Sigma) \cong H^*(X_\Sigma) \cong \mathbb{Z}[D_1, \ldots, D_r]/I,$$

where I is the ideal generated by the following relations:

(1) $D_{i_1} \cdots D_{i_k}$ *for all $\rho_{i_1}, \ldots, \rho_{i_k}$ not contained in any cone of Σ, and*

(2) $\sum_{i=1}^{r} \langle u, v_{\rho_i} \rangle D_i$ *for all $u \in M$.*

There is also a nice description of the ring $A^*(\mathcal{F}\ell_n) \cong H^*(\mathcal{F}\ell_n)$ due to Borel. For each weight λ let $c_1(\mathcal{L}_\lambda)$ be the divisor class (Chern class) of the line bundle $\mathcal{L}_\lambda$ on $\mathcal{F}\ell_n$ (see [2], in particular, Remark 1.4.2 in there).

Theorem 3.3. *We have the following:*

(1) *The map $\lambda \mapsto c_1(\mathcal{L}_\lambda)$ gives an isomorphism of $A^1(\mathcal{F}\ell_n) = \mathrm{Pic}(\mathcal{F}\ell_n)$ with the weight lattice $\Lambda' = \Lambda(\mathrm{SL}(n, \mathbb{C})) = \Lambda/\mathbb{Z}(1, \ldots, 1)$.*

(2) *$A^*(\mathcal{F}\ell_n)$ is generated, as an algebra, by $c_1(\mathcal{L}_\lambda)$, $\lambda \in \Lambda$.*

(3) *$A^*(\mathcal{F}\ell_n) \cong \mathrm{Sym}(\Lambda')/I_W$ where I_W is the ideal generated by non-constant W-invariants.*

In the proof of our main theorem (Theorem 5.1) we will need parts (1) and (2) in Theorem 3.3.

Remark 3.4. Alternatively, $H^*(\mathcal{F}\ell_n, \mathbb{Q})$ can be viewed as the polytope algebra of the Gelfand–Zetlin family (see [14, Corollary 5.3]). There it is shown that

$$H^*(\mathcal{F}\ell_n, \mathbb{Q}) \cong \mathrm{Sym}(\Lambda_{\mathbb{Q}})/I,$$

where I is the ideal of polynomials which, when viewed as differential operators, annihilate the volume polynomial of the Gelfand–Zetlin polytopes. This description of the Chow ring of the flag variety is it is closely related to the proof of Theorem 5.1 but is not directly used there.

We note that the toric variety X_{GZ} is not smooth except when $n = 1, 2$ and hence we need a more general notion of the Chow ring that applies to non-smooth varieties as well. For a (not necessarily smooth) variety X in [8], Fulton and MacPherson construct a variant of the Chow ring called the *operational Chow ring* or *Chow cohomology ring*

$$A^*(X) = \bigoplus_{k=0}^{n} A^k(X).$$

When X is smooth it coincides with the usual Chow ring. When $X = X_\Sigma$ is a complete toric variety one has

$$A^k(X_\Sigma) = \mathrm{Hom}(A_k(X_\Sigma), \mathbb{Z}).$$

Moreover, the ring $A^*(X_\Sigma)$ can be described purely in terms of combinatorial data of Minkowski weights, which are certain integer valued functions on the fan Σ. In Section 7 we will use this combinatorial description for some computations in the Chow cohomology of the Gelfand–Zetlin toric variety for $n = 3$. Section 6 reviews the Minkowski weights description of the Chow cohomology ring.

4. Some algebra lemmas

Let $A = \bigoplus_{i=0}^{n} A^i$ be a graded ring over a field $\mathbf{k}$ which is finite dimensional as a $\mathbf{k}$-vector space and $A^0 \cong A^n \cong \mathbf{k}$. Following [11], we call the graded subalgebra of A generated by A^1, the *Lefschetz subalgebra* of A. We recall that A has *Poincaré duality* if the multiplication maps

$$A^i \times A^{n-i} \to A^n \cong \mathbf{k}$$

are non-degenerate for all i. Our goal is to compare $A^*(\mathcal{F}\ell_n) \cong H^*(\mathcal{F}\ell_n)$, which has Poincaré duality, with the algebra $A^*(X_{\mathrm{GZ}})$, which in general does not. We start by observing how to get a Poincaré duality algebra from a general graded algebra.

Lemma 4.1. *Let $A = \bigoplus_{i=0}^{n} A^i$ with $A^0 \cong A^n \cong \mathbf{k}$. There exists a homogeneous ideal $I \subset A$ such that A/I has Poincaré duality and is the smallest homogeneous ideal (with respect to inclusion) with this property.*

Proof. Consider the ideal I generated by all the homogeneous elements $x \in A$ such that

$$x \cdot A^{n-\deg(x)} = 0.$$

It is straightforward to check that I has the required properties. ∎

We call the algebra A/I in Lemma 4.1, the *Poincaré duality quotient* $\mathrm{PD}(A)$ of A. We next recall a useful algebra fact (see [14, Theorem 1.1] and [4, Exercise 21.7]), which we will need later. It states that a Poincaré duality algebra is determined by its top power polynomial.

Theorem 4.2. *Let $A = \bigoplus_{i=0}^{n} A^i$ be a finite dimensional graded algebra over a field $\mathbf{k}$, which is generated by A^1, satisfies $A^0 \cong \mathbf{k} \cong A^n$, and has Poincaré duality. Fix a basis $\{a_1, \dots, a_r\}$ for A^1, and consider the polynomial $P : \mathbf{k}^r \to \mathbf{k}$ defined by*

$$P(x_1, \dots, x_r) = (x_1 a_1 + \cdots + x_r a_r)^n \in A^n \cong \mathbf{k}.$$

Then we have an isomorphism of graded algebras

$$A \cong \mathbf{k}[\partial_1, \dots, \partial_r]/I,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and I is the ideal of polynomials in the operators $\partial_1, \dots, \partial_r$, which annihilate P. The isomorphism sends each a_i to the image of ∂_i in $\mathbf{k}[\partial_1, \dots, \partial_r]/I$.

A generalization of Theorem 8.1 for commutative algebras A with Poincaré duality that are not necessarily generated by $A^0 \cong \mathbf{k}$ and A^1 can be found in [5].

We now use Theorem 4.2 to prove the following key lemma required in the proof of our main result (Theorem 5.1).

Lemma 4.3. *Suppose $A = \bigoplus_{i=0}^{n} A^i$ and $B = \bigoplus_{i=0}^{n} B^i$ are $\mathbf{k}$-algebras which are finite dimensional $\mathbf{k}$-vector spaces and have the following properties:*

(1) $A^0 \cong A^n \cong B^0 \cong B^n \cong \mathbf{k}$.

(2) *A and B are generated in degree one.*

(3) *A has Poincaré duality.*

(4) *There exists a linear isomorphism $\varphi : A^1 \to B^1$ such that for all $a_1, \dots, a_n \in A^1$, we have*

$$a_1 \cdots a_n = \varphi(a_1) \cdots \varphi(a_n)$$

using fixed isomorphisms $A^n \cong \mathbf{k} \cong B^n$.

Then φ extends to give a $\mathbf{k}$-algebra isomorphism $\widetilde{\varphi}$ between A and the Poincaré duality quotient of B.

Proof. We apply Theorem 4.2 to A and to the Poincaré duality quotient $\mathrm{PD}(B)$. Since A already satisfies the conditions of Theorem 4.2 we know that

$$A \cong \mathbf{k}[\partial_1, \ldots, \partial_r]/I,$$

where $r = \dim_{\mathbf{k}}(A^1)$ and I is the annihilator of the top power polynomial P described in Theorem 4.2. We need to show that $\mathrm{PD}(B)$ also satisfies these conditions. First note that $B^0 \cong \mathbf{k} \cong B^n$, so the multiplication $B^0 \times B^n \to B^n \cong \mathbf{k}$ is already nondegenerate and thus the ideal I in Lemma 4.1 contains neither B^0 nor B^n. This gives us

$$\mathrm{PD}(B)^0 \cong \mathbf{k} \cong \mathrm{PD}(B)^n.$$

Also, by construction $\mathrm{PD}(B)$ has Poincaré duality. Finally, $\mathrm{PD}(B)$ is generated in degree one since B is generated in degree 1. Now consider the map on degree one pieces

$$A^1 \xrightarrow{\varphi} B^1 \xrightarrow{q} \mathrm{PD}(B)^1,$$

where q is the quotient map. It suffices to show $\widetilde{\varphi} := q \circ \varphi \colon A^1 \to \mathrm{PD}(B)^1$ is an isomorphism. Since φ is an isomorphism and q is surjective, $\widetilde{\varphi}$ is surjective and we only need to verify injectivity. Suppose for contradiction that some nonzero $a \in A^1$ has image $\widetilde{\varphi}(a) = q(\varphi(a)) = 0$. Then $b = \varphi(a)$ is in the ideal in Lemma 4.1, so it is a linear combination of the x_i satisfying $x_i \cdot B^{n-\deg(x_i)} = 0$. Since $b \in B^1$, the x_i must be in degree 0 or 1. One knows that $B^0 \cap I = \{0\}$, so we can only have $x_i \in B^1$. It follows that $b \cdot B^{n-1} = 0$. But the assumption (4) then implies that $a \cdot A^{n-1} = 0$, which contradicts that A has Poincaré duality. Thus, $\mathrm{PD}(B)$ satisfies the conditions required for Theorem 4.2, and hence $\mathrm{PD}(B) \cong \mathbf{k}[\partial_1, \ldots, \partial_r]/I$. We have already seen that A is isomorphic to this quotient algebra, and thus $A \cong \mathrm{PD}(B)$. ∎

5. Main theorem

We now state and prove our main theorem relating the cohomology ring of the flag variety $\mathcal{F}\ell_n$ and the Chow cohomology ring of the toric variety X_{GZ} associated to the GZ fan $\Sigma = \Sigma_{\mathrm{GZ}}$.

Theorem 5.1. *The cohomology ring $H^*(\mathcal{F}\ell_n, \mathbb{Q}) \cong A^*(\mathcal{F}\ell_n, \mathbb{Q})$ is isomorphic to the Poincaré duality quotient of the Lefschetz subalgebra of $A^*(X_{\mathrm{GZ}}, \mathbb{Q})$. For each dominant weight λ, the isomorphism sends the divisor class of the line bundle $\mathcal{L}_\lambda$ on $\mathcal{F}\ell_n$ to the image of the cohomology class in X_{GZ} associated to the GZ polytope Δ_λ.*

Proof. We claim that there is an isomorphism of groups $A^1(\mathcal{F}\ell_n) \cong A^1(X_{\mathrm{GZ}})$. One knows that

$$A^1(\mathcal{F}\ell_n) = A_{N-1}(\mathcal{F}\ell_n) = \mathrm{Pic}(\mathcal{F}\ell_n) \\ \cong \Lambda(\mathrm{SL}(n,\mathbb{C})) = \Lambda(\mathrm{GL}(n,\mathbb{C}))/\mathbb{Z}(1,\dots,1).$$

Also for a complete toric variety X_Σ, where Σ is a complete fan in $\mathbb{R}^N$, the Chow cohomology group $A^1(X_\Sigma)$ is naturally isomorphic to $\mathrm{Pic}(X_\Sigma)$ (see [9, Corollary 3.4]). Now the claim follows from Corollary 1.9.

One knows that for an N-dimensional toric variety X_Σ, under the isomorphism $A^1(X_\Sigma) \cong \mathrm{Pic}(X_\Sigma)$ the top product of an element in $A^1(X_\Sigma) \cong \mathrm{Pic}(X_\Sigma)$ coincides with the self-intersection number of the corresponding divisor in $\mathrm{Pic}(X_\Sigma)$. Applying this to the Gelfand–Zetlin toric variety X_{GZ}, from Propositions 2.1 and 2.2, we now conclude that the isomorphism $\mathrm{Pic}(\mathcal{F}\ell_n) = \mathrm{Pic}(X_{\mathrm{GZ}})$ respects the multiplication, i.e., it satisfies the assumption (4) in Lemma 4.3 (alternatively this can be deduced from [12, Theorem 4.3 and Corollary 4.5]). Applying Lemma 4.3 to $A = A^*(\mathcal{F}\ell_n)$ and $B =$ the Lefschetz subalgebra of $A^*(X_{\mathrm{GZ}})$ finishes the proof. ■

6. Minkowski weights

In this section we recall the description of the Chow cohomology ring of a toric variety in terms of Minkowski weights (see [9], see also [16]). We will use it in Section 7 to compute the Gelfand–Zetlin Chow cohomology ring for $n = 3$. Let Σ be a complete fan in N. Recall that $\Sigma(k)$ is the set of cones of dimension k in Σ.

Definition 6.1. A function $c\colon \Sigma(n-k) \to \mathbb{Z}$ is a *Minkowski weight* of codimension k on Σ if it satisfies the *balancing condition* for all $\tau \in \Sigma(n-k-1)$:

$$\sum_{\sigma\in\Sigma(n-k),\sigma\supset\tau} \langle u, n_{\sigma,\tau}\rangle c(\sigma) = 0 \quad \forall u \in M(\tau) := M \cap \tau^\perp. \tag{4}$$

Here $n_{\sigma,\tau}$ is a lattice point in σ which generates the rank 1 lattice N_σ/N_τ, the quotient of the lattices spanned by $\sigma \cap N$ and $\tau \cap N$, respectively.

Let MW^k denote the set of all Minkowski weights of codimension k. For two Minkowski weights $c \in MW^p$ and $\tilde{c} \in MW^q$, the product $c \cup \tilde{c} \in MW^{p+q}$ is defined by:

$$(c \cup \tilde{c})(\gamma) = \sum_{(\sigma,\tau)\in\Sigma(n-p)\times\Sigma(n-q)} m^\gamma_{\sigma,\tau} c(\sigma)\tilde{c}(\tau) \quad \forall \gamma \in \Sigma(n-p-q),$$

where $m^\gamma_{\sigma,\tau} = [N : N_\sigma + N_\tau]$, and the sum is over all pairs of cones (σ, τ), which both contain γ and σ meets $\tau + v$ for fixed generic vector v (see [9, Theorem 4.2]).

In [9] an isomorphism between the ring of Minkowski weights and the operational Chow ring of a complete toric variety X_Σ is given. In fact, it is shown that $MW^k \cong A^k(X_\Sigma)$ (see [9, Theorem 3.1]). In particular,

$$\mathrm{Pic}(X_\Sigma) \cong A^1(X_\Sigma). \tag{5}$$

Example 6.2 (Hypersimplex). The following is an example of a fan where the ring MW^* is not generated by MW^1 (see [9, Example 3.5] or [13, Example 4.2]). Consider the fan Σ_H over the cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. The rays in the fan Σ_H are:

$$\begin{aligned}
\rho_1 &= \langle 1,1,1\rangle, & \rho_5 &= -\rho_1,\\
\rho_2 &= \langle 1,1,-1\rangle, & \rho_6 &= -\rho_2,\\
\rho_3 &= \langle 1,-1,1\rangle, & \rho_7 &= -\rho_3,\\
\rho_4 &= \langle -1,1,1\rangle, & \rho_8 &= -\rho_4.
\end{aligned}$$

One computes that

$$MW^1 \cong \mathbb{Z} \quad \text{and} \quad MW^2 \cong \mathbb{Z}^5.$$

Thus, MW^* is not generated by MW^1.

7. Gelfand–Zetlin example, $n = 3$

In this section we compute the Chow cohomology ring of X_{GZ} for $n = 3$ using the Minkowski weights and show that while it is generated in degree 1, it does not have Poincaré duality. We consider the GZ polytope of the weight $\lambda = (-1, 0, 1)$ for ease of computation. The polytope Δ_λ is defined by the following array of inequalities

$$\begin{array}{ccccc}
-1 & & 0 & & 1\\
 & x & & y & \\
 & & z & &
\end{array}$$

and has normal fan Σ_{GZ} as in Figure 2. We enumerate the rays as follows:

$$\begin{aligned}
\rho_1 &= (1,0,0), & \rho_3 &= (0,1,0), & \rho_5 &= (1,0,-1),\\
\rho_2 &= (-1,0,0), & \rho_4 &= (0,-1,0), & \rho_6 &= (0,-1,1).
\end{aligned}$$

Likewise, we let σ_{ij} denote the 2-dimensional cone spanned by rays ρ_i and ρ_j:

$$\begin{array}{cccc}
\sigma_{13} & \sigma_{23} & \sigma_{24} & \\
\sigma_{15} & \sigma_{25} & \sigma_{35} & \sigma_{45}\\
\sigma_{16} & \sigma_{26} & \sigma_{36} & \sigma_{46}
\end{array}$$

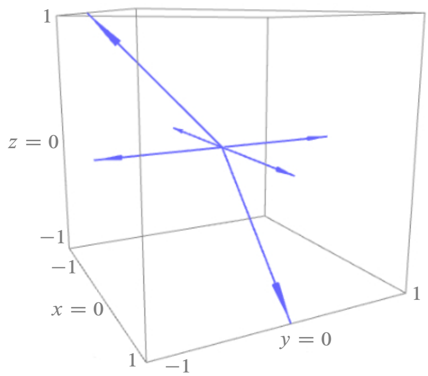

Figure 2. Rays of Σ_{GZ} for $n = 3$.

Similarly, the collection of 3-dimensional cones are:

$$\begin{array}{cccc} \gamma_{135} & \gamma_{235} & \gamma_{245} & \gamma_{1456} \\ \gamma_{136} & \gamma_{236} & \gamma_{246} & \end{array}$$

We now compute MW^k, $k = 0, \ldots, 3$. A Minkowski weight in MW^3 is any map $\{0\} \to \mathbb{Z}$ and hence $MW^3 \cong \mathbb{Z}$. A Minkowski weight $c \in MW^2$ is a function on rays ρ_i. Let $c(\rho_i) = c_i$, then the single relation coming from the cone $\tau = 0$ is given by $\sum_{i=1}^{6} c_i v_{\rho_i} = 0$. From this we get the three relations:

$$\begin{aligned} c_1 - c_2 + c_5 &= 0, \\ c_3 - c_4 - c_6 &= 0, \\ -c_5 + c_6 &= 0. \end{aligned}$$

We see from this that any weight $c \in MW^2$ is determined by its values on three rays $c(\rho_2) = a$, $c(\rho_4) = b$, and $c(\rho_6) = c$. Thus, $MW^2 \cong \mathbb{Z}^3$. Next take $c \in MW^1$. It is a function on codimension 1 cones σ_{ij}. Let $c(\sigma_{ij}) = c_{ij}$. The relations among the c_{ij} come from the rays. The relation for $\tau = \rho_1$ involves the cones σ_{13}, σ_{15}, and σ_{16}. Let $n_{\sigma\tau}$ be the lattice point in σ which generates the one-dimensional lattice N_σ / N_τ. We compute:

$$n_{13} = (0, 1, 0), \quad n_{15} = (0, 0, -1), \quad n_{16} = (0, -1, 1),$$

where all vectors are considered modulo $\rho_1 = (1, 0, 0)$. The balancing condition then becomes

$$c_{13}(0, 1, 0) + c_{15}(0, 0, -1) + c_{16}(0, -1, 1) = (0, 0, 0),$$

which implies $c_{13} = c_{15} = c_{16}$. Similar computations for the other rays yield the following results:

$$c_{13} = c_{15} = c_{16} = c_{25} = c_{26},$$
$$c_{24} = c_{35} = c_{36} = c_{45} = c_{46},$$
$$c_{23} = c_{13} + c_{24}.$$

For later computations, we let

$$a = c_{13} = c_{15} = c_{16} = c_{25} = c_{26},$$
$$b = c_{24} = c_{35} = c_{36} = c_{45} = c_{46},$$
$$c_{23} = a + b.$$

Finally, a weight $c \in MW^0$ is a function on top-dimensional cones subject to relations coming from 2-dimensional cones. Each 2-dimensional cone σ_{ij} separates two top-dimensional cones, and the corresponding relation gives equality between the values of c on each pair of top-dimensional cones. Hence, $MW^0 \cong \mathbb{Z}$ as the value of c on each 3-dimensional cone must be the same. In summary, we have the following:

$$MW^0 \cong \mathbb{Z},$$
$$MW^1 \cong \mathbb{Z}^2,$$
$$MW^2 \cong \mathbb{Z}^3,$$
$$MW^3 \cong \mathbb{Z}.$$

Before understanding the product structure on MW^*, it is already clear that this ring does not have Poincaré duality as the rank of MW^2 is greater than that of MW^1.

Recall from Section 6 that for weights $c \in MW^p$, $\widetilde{c} \in MW^q$, their product is a function on cones of codimension $p + q$, and its value on a cone $\gamma \in \Sigma(3 - p - q)$ is given by

$$(c \cup \widetilde{c})(\gamma) = \sum_{(\sigma,\tau)} m^{\gamma}_{\sigma\tau} c(\sigma)\widetilde{c}(\tau), \tag{6}$$

where the sum is over certain pairs $(\sigma, \tau) \in \Sigma(3 - p) \times \Sigma(3 - q)$ and

$$m^{\gamma}_{\sigma\tau} = [N : N_\sigma + N_\tau].$$

We compute products of Minkowski weights in our example to determine whether $MW^*(X_{\mathrm{GZ}})$ is generated in degree 1. Let $c, \widetilde{c} \in MW^1(X_{\mathrm{GZ}})$ with

$$c\colon \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{25}, \sigma_{26}\} \mapsto a,$$
$$c\colon \{\sigma_{24}, \sigma_{35}, \sigma_{36}, \sigma_{45}, \sigma_{46}\} \mapsto b,$$
$$c\colon \{\sigma_{23}\} \mapsto a + b,$$

$$\widetilde{c}\colon \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{25}, \sigma_{26}\} \mapsto \widetilde{a},$$
$$\widetilde{c}\colon \{\sigma_{24}, \sigma_{35}, \sigma_{36}, \sigma_{45}, \sigma_{46}\} \mapsto \widetilde{b},$$
$$\widetilde{c}\colon \{\sigma_{23}\} \mapsto \widetilde{a} + \widetilde{b}.$$

The Minkowski weight $c \cup \widetilde{c} \in MW^2$ is evaluated on rays and from the arguments above it is enough to determine the value of this weight on the rays ρ_2, ρ_4 and ρ_5. Moreover, in equation (6) for $(c \cup \widetilde{c})(\rho_2)$ the sum is over all pairs $(\sigma, \tau) \in \Sigma(2) \times \Sigma(2)$, where σ and τ both contain ρ_2 and σ meets $\tau + v$ for a generic fixed $v \in N$. The cones in $\Sigma(2)$ which contain ρ_2 are $\{\sigma_{23}, \sigma_{24}, \sigma_{25}, \sigma_{26}\}$, so σ, τ will come from this collection. Since all these cones involve $\rho_2 = (-1, 0, 0)$, we can sketch the relevant cones in the yz-plane where for example σ_{23} can be viewed as $\rho_3 = (1, 0)$.

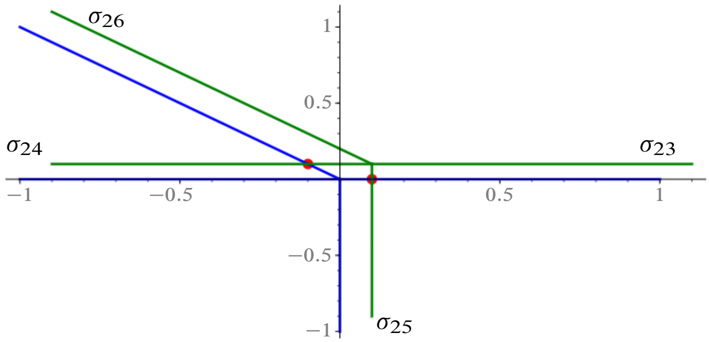

Figure 3. Intersection of σ and $\tau + v$.

In Figure 3, we see the cones for c in blue, and for $\widetilde{c}$ in green using a shift of $v = (0.1, 0.1, 0.1)$. Then there are two pairs (σ, τ) which meet for this vector v, either $(\sigma, \tau) = (\sigma_{23}, \sigma_{25})$ or $(\sigma, \tau) = (\sigma_{26}, \sigma_{24})$. The last ingredient required to compute this product are the coefficients $m^{\rho_2}_{\sigma\tau}$ for the sum. In both cases, $N_\sigma + N_\tau = N$, so $m^{\rho_2}_{\sigma\tau} = 1$. Thus, we have

$$\begin{aligned}(c \cup \widetilde{c})(\rho_2) &= c(\sigma_{23})\widetilde{c}(\sigma_{25}) + c(\sigma_{26})\widetilde{c}(\sigma_{24}) \\ &= (a + b)\widetilde{a} + a(\widetilde{b}) \\ &= a\widetilde{a} + b\widetilde{a} + a\widetilde{b}.\end{aligned}$$

Similar computations for $(c \cup \widetilde{c})(\rho_4)$ and $(c \cup \widetilde{c})(\rho_5)$ yield

$$\begin{aligned}(c \cup \widetilde{c})(\rho_4) &= b\widetilde{b}, \\ (c \cup \widetilde{c})(\rho_5) &= b\widetilde{a} + a\widetilde{b}.\end{aligned}$$

Thus, we see that the products $c \cup \tilde{c}$ in fact generate the entire 3-dimensional space MW^2, and hence MW^* for Σ_{GZ} is generated in degree 1 for the case $n = 3$.

Finally, the second author has written a Sage code which shows that for $n = 4, 5$, the ring MW^* of Σ_{GZ} is not generated in degree 1, and moreover its Lefschetz subalgebra does not have Poincaré duality. It can be found at https://github.com/evillella/minkowski. See also the appendix in [21].

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Kiumars Kaveh
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA;
kaveh@pitt.edu

Elise Villella
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA;
emv23@pitt.edu