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The shape of the dust-likeness locus of self-similar sets

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Abstract. We prove that the dust-likeness locus in the deformation space of a contractive holomorphic linear iterated function system is coincident with the quasiconformal deformation space. Also, we determine explicitly the dust-likeness locus restricted to several slices which are different from the Mandelbrot one, and provide specific examples that show diversity of self-similar sets.

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1. Introduction

Let m, n be positive integers and $n \geq 2$. We say that an affine map $S\colon \mathbb{R}^m \to \mathbb{R}^m$ is a *contracting similarity*, or simply a CS, if there is an $r_S \in (0,1)$ such that $|S(x) - S(y)| = r_S|x - y|$ for every $x, y \in \mathbb{R}^m$. We call r_S the *contraction ratio* of S. For a given family $\mathcal{S} = \{S_k\}_{k=1}^n$ of n CSs, $\mathrm{sgp}(\mathcal{S})$ denotes the semigroup generated by $\{S_k\}_{k=1}^n$. The pair $(\mathcal{S}, \mathbb{R}^m)$ of the family $\mathcal{S}$ and the space $\mathbb{R}^m$ is called an *contractive linear IFS* (*iterated function system*) on $\mathbb{R}^m$. The *self-similar set E for $\mathcal{S}$* is the unique non-empty compact set E in $\mathbb{R}^m$ which satisfies

$$E = \bigcup_{k=1}^{n} S_k(E),$$

which is also called the *forward limit set* of $\mathrm{sgp}(\mathcal{S})$ and denoted by $\Lambda^+(\mathcal{S})$.

Definition 1.1. We say that two families $\mathcal{S}_1 = \{S_{k,1}\}_{k=1}^n$ and $\mathcal{S}_2 = \{S_{k,2}\}_{k=1}^n$ of ordered n CSs of $\mathbb{R}^m$ with mutually distinct fixed points are *equivalent* if there is a similarity $T\colon \mathbb{R}^m \to \mathbb{R}^m$ such that $S_{k,2} = T^{-1} \circ S_{k,1} \circ T$ for every k.

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We call the set of all equivalence classes $[\mathcal{S}]$ of such families $\mathcal{S}$ the *deformation space* $D(n, m)$ of IFSs.

The deformation space $D(n, m)$ can be identified with the space of all normalized IFSs $(\mathcal{S}, \mathbb{R}^m)$, where an $\mathcal{S} = \{S_k\}_{k=1}^n$ is *normalized* if the points $\mathbf{0} = (0, \cdots, 0)$ and $\mathbf{e}_1 = (1, 0, \cdots, 0)$ in $\mathbb{R}^m$ are the fixed points of S_1 and S_2, respectively. Hence, $D(n, m)$ can be naturally identified with the product space

$$\mathrm{ND}(n, m) = ((0, 1) \times O(m))^n \times \mathrm{Conf}(n-2, \mathbb{R}^m - \{\mathbf{0}, \mathbf{e}_1\}),$$

where $O(m)$ is the m-dimensional orthogonal group, i.e., the group of all orthogonal transformations of $\mathbb{R}^m$, and $\mathrm{Conf}(n-2, \mathbb{R}^m - \{\mathbf{0}, \mathbf{e}_1\})$ is the configuration space of *ordered* distinct $n-2$ points on $\mathbb{R}^m - \{\mathbf{0}, \mathbf{e}_1\}$. For every $\mathcal{S} = \{S_k\}_{k=1}^n$, we call the vector $\mathbf{r}_{\mathcal{S}} = (r_{S_1}, \cdots, r_{S_n}) \in (0, 1)^n$ of the contraction ratios the *contraction vector* of $\mathcal{S}$.

Definition 1.2. We say that the forward limit set $\Lambda^+(\mathcal{S})$ of $\mathrm{sgp}(\mathcal{S})$ is *dust-like* if $\mathcal{S}$ satisfies SSC (the *strong separation condition*), i.e., for every distinct i and j,

$$S_i(\Lambda^+(\mathcal{S})) \cap S_j(\Lambda^+(\mathcal{S})) = \emptyset.$$

We set

$$\mathrm{DL}(n, m) = \{\mathcal{S} \in \mathrm{ND}(n, m) \mid \Lambda^+(\mathcal{S}) \text{ is dust-like}\},$$

and call $\mathrm{DL}(n, m)$ the *dust-likeness locus* of $\mathrm{ND}(n, m)$.

Now, following is one of the fundamental problems concerning dust-like self-similar sets.

Problem 1.1. Determine when two elements $\mathcal{S}_k \in \mathrm{DL}(n_k, m)$ $(k = 1, 2)$ are *quasi-isometrically equivalent* (bi-Lipschitz equivalent) to each other, i.e., there is a bi-Lipschitz homeomorphism between $\Lambda^+(\mathcal{S}_1)$ and $\Lambda^+(\mathcal{S}_2)$.

Recently, various answers to this problem have been obtained. Some characterizations of mutually quasi-isometrically equivalent pairs are given in [17] and [23], for instance. Note that one of the rather explicit sufficient conditions using the contraction vectors is: for two $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathrm{DL}(n, m)$, if $\mathbf{r}_{\mathcal{S}_1} = \mathbf{r}_{\mathcal{S}_2}$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are quasi-isometrically equivalent. Also recall the famous necessary condition by Falconer and Marsh. See for instance, [8] and [10]. (Also cf. [19].)

On the other hand, it is clear that the answers to Problem 1.1 will become complete only after we know the shape of $\mathrm{DL}(n, m)$ explicitly.

Problem 1.2. Determine the shape of the dust-likeness locus $\mathrm{DL}(n, m)$.

This problem has been considered since early research on fractal geometry of self-similar sets, but chiefly on the "connectedness locus" where the forward limit sets are connected, and for the special slice called the "Mandelbrot set". See the references cited after Theorem 1.2.

Remark 1.1. If $n > 2$, then the dust-likeness locus is not the complement of the connectedness locus (cf. [20]), though it is well-known that it is if $n = 2$. Recently, the dust-likeness locus has been investigated from various viewpoints. For instance, the dust-likeness locus in the deformation space of IFSs without nomalization, together with a natural kind of boundary, were discussed in [12]. See also the references therein.

More specifically, we consider in the sequel, the deformation space $\mathrm{ND}(n)$ of normalized contractive *holomorphic* linear IFSs in $\mathrm{ND}(n, 2)$ which consists of CSs

$$\begin{aligned} S_1(z) &= \lambda_1 z, \\ S_2(z) &= \lambda_2(z-1)+1, \\ S_3(z) &= \lambda_3(z-\alpha_3)+\alpha_3, \\ &\ldots, \\ S_n(z) &= \lambda_n(z-\alpha_n)+\alpha_n \end{aligned}$$

of $\mathbb{C} = \mathbb{R}^2$. $\mathrm{ND}(n)$ can be considered as the complex manifold $(U^*)^n \times \mathrm{Conf}(n-2, \mathbb{C} - \{0, 1\})$, where $U^* = \{0 < |z| < 1\}$.

Let $\mathrm{DL}(n)$ be the dust-likeness locus of $\mathrm{ND}(n)$. Fix a point $\mathcal{S}_0$ in $\mathrm{DL}(n)$. Then, we obtain a canonical semigroup isomorphism

$$\iota_{\mathcal{S}} \colon \mathrm{sgp}(\mathcal{S}_0) \longrightarrow \mathrm{sgp}(\mathcal{S})$$

for every $\mathcal{S} \in \mathrm{DL}(n)$, and hence a canonical holomorphic map

$$\mathrm{FP} \colon \mathrm{DL}(n) \longrightarrow \mathbb{C}^{\mathrm{sgp}(\mathcal{S}_0)} = \mathbb{C}^{\infty},$$

by setting the T-th component of $\mathrm{FP}(\mathcal{S})$ to be the fixed point of $\iota_{\mathcal{S}}(T)$ for every $T \in \mathrm{sgp}(\mathcal{S}_0)$. Also, FP induces a holomorphic motion of the set in $\mathbb{C}$ comprising all components of $\mathrm{FP}(\mathcal{S}_0)$. (See for instance, [1] and [13].)

Remark 1.2. It can be easily seen that the fixed points of elements in $\mathrm{sgp}(\mathcal{S})$ determine $\mathcal{S} = \{S_k\}_{k=1}^n$ for every $\mathcal{S} \in \mathrm{DL}(n)$, or more precisely, that $S_1, \cdots, S_n$ are determined by the $(2n-2)$ fixed points of $S_3, \cdots, S_n, S_2S_1, S_1S_2, \cdots, S_1S_n$, for instance.

We say that a normalized quasiconformal self-map f of $\mathbb{C}$ *preserves the markings* of $\mathcal{S}_1$ and $\mathcal{S}_2$ if f maps every T-th component of $\mathrm{FP}(\mathcal{S}_1)$ to the T-th component of $\mathrm{FP}(\mathcal{S}_2)$ for every $T \in \mathrm{sgp}(\mathcal{S}_0)$. Then we have the following facts.

Theorem 1.1. *The dust-likeness locus* $\mathrm{DL}(n)$ *is a domain contained in*

$$\left\{\mathcal{S} \in \mathrm{ND}(n) \,\middle|\, \|\mathbf{r}_{\mathcal{S}}\| \left(= \sqrt{\sum r_{\mathcal{S}_j}^2} \right) < 1 \right\}.$$

For every two points $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *in* $\mathrm{DL}(n)$*, there is a quasiconformal self-map of* $\mathbb{C}$ *preserving the markings of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$.

The assertions of Theorem 1.1 are rather well-known (cf. [6] and [9]). But for the sake of convenience, we provide a proof in §2.

As shown in Theorem 1.1, $\mathrm{DL}(n)$ is contained in the quasiconformal deformation space of a contractive holomorphic linear IFS $(\mathcal{S}, \mathbb{C})$ of n CSs with the dust-like forward limit set. Hence, we can introduce the Teichmüller distance on $\mathrm{DL}(n)$ in a canonical manner as follows.

Definition 1.3 (Cf. [15] and [18]). The *Teichmüller distance* $d_T(\mathcal{S}_1, \mathcal{S}_2)$ between two points $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathrm{DL}(n)$ is the infimum of $\log K(f)$ among all quasiconformal maps of $\mathbb{C}$ that preserve the markings of $\mathcal{S}_1$ and $\mathcal{S}_2$, where $K(f)$ is the maximal dilatation of f.

Remark 1.3. To conclude that d_T is a distance, we need to show that $\mathcal{S}_1 = \mathcal{S}_2$ if $d_T(\mathcal{S}_1, \mathcal{S}_2) = 0$ for $\mathcal{S}_k \in \mathrm{DL}(n)$. This follows from the fact that a normalized conformal map of $\mathbb{C} - \Lambda^+(\mathcal{S}_1)$ onto $\mathbb{C} - \Lambda^+(\mathcal{S}_2)$ is actually the identical map, which in turn follows, for instance, from the classical fact that $\Lambda^+(\mathcal{S}_1)$ has absolute area 0. See Corollary 2.1 below.

Now, we will prove in §2 the following theorem.

Theorem 1.2. *The Teichmüller distance* d_T *on* $\mathrm{DL}(n)$ *is complete. In particular,* $\mathrm{DL}(n)$ *is coincident with the quasiconformal deformation space of a contractive holomorphic linear IFS in* $\mathrm{DL}(n)$.

On the other hand, it seems that little is known about the explicit shape of $\mathrm{DL}(n)$. Much research on Problem 1.2 has been done for the special slice defined by

$$\mathcal{M} = \{\mathcal{S} \in \mathrm{ND}(2) \mid \lambda_1 = \lambda_2\},$$

which is called the *Mandelbrot set*. Here, another clearly equivalent family consisting of $\widehat{S}_1(z) = \lambda z$ and $\widehat{S}_2(z) = \lambda z + 1$ is usually considered. See for instance [2], [4], [5], and [21].

In this paper, we investigate other slices $\mathrm{Sl}(2;\omega_1,\omega_2)$ of $\mathrm{DL}(2)$ which consist of all $\mathcal{S}(r,s;\omega_1,\omega_2)=\{S_1,S_2\}$ with

$$S_1(z)=r\omega_1 z,\quad S_2(z)=s\omega_2(z-1)+1,$$

where $r>0, s>0$ and $\omega_k^4=1$. Then, we can determine *explicitly* the shape of $\mathrm{Sl}(2;\omega_1,\omega_2)$.

Theorem 1.3. (1) *Suppose that both ω_k are real, i.e., $\omega_k\in\{\pm 1\}$, then*

$$\mathrm{Sl}(2;\omega_1,\omega_2)=\{\mathcal{S}(r,s;\omega_1,\omega_2)\in\mathrm{ND}(2)\mid r+s<1\}.$$

(2) *Suppose that ω_1 is purely imaginary, i.e., $\pm i$, but ω_2 is real. Then*

$$\mathrm{Sl}(2;\omega_1,\omega_2)=\{\mathcal{S}(r,s;\omega_1,\omega_2)\in\mathrm{ND}(2)\mid r^2+s<1\}.$$

(3) *Suppose that both ω_k are purely imaginary. Then*

$$\mathrm{Sl}(2;\omega_1,\omega_2)=\{\mathcal{S}(r,s;\omega_1,\omega_2)\in\mathrm{ND}(2)\mid r^2+sr<1, s^2+sr<1\}.$$

A proof of Theorem 1.3 is given in §3.

Finally in §4, we show by examples that the slices considered in Theorem 1.3 are as rich as the Mandelbrot slice. We provide in §4 explicit examples of the self-similar sets corresponding to IFSs in the slices such as dendrites, closed domains, and porous ones.

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2. Proof of Theorems 1.1 and 1.2

First, we include a proof of Theorem 1.1 for the sake of convenience. Here, recall that SSC implies the *expanded separation condition* (ESC), i.e., there is a non-empty open set O such that $S_j(\overline{O})\subset O$ for every j and that $S_j(\overline{O})\cap S_k(\overline{O})=\emptyset$ for every distinct j and k. Indeed, if $\Lambda^+(\mathcal{S})$ satisfies SSC, then we can take a sufficiently small $\epsilon>0$ so that $\mathcal{S}$ satisfies ESC with the open ϵ-neighborhood $\{z\in\mathbb{C}\mid d(z,\Lambda^+(\mathcal{S}))<\epsilon\}$ of $\Lambda^+(\mathcal{S})$.

Proof of Theorem 1.1. Take an $\mathcal{S}$ in $\mathrm{DL}(n)$. Since it is clear that the same O satisfies ESC for all $\mathcal{S}'$ sufficiently near $\mathcal{S}$, it implies that $\mathrm{DL}(n)$ is open.

Also, the same O satisfies ESC for every such $\mathcal{S}'\in\mathrm{DL}(n)$ that every fixed point α_k' for $\mathcal{S}'$ equals α_k for $\mathcal{S}$ and that $\mathbf{r}_{\mathcal{S}'}$ is not greater than $\mathbf{r}_{\mathcal{S}}$ component-wise. Hence, given two $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathrm{DL}(n)$, we first connect $\mathcal{S}_k$ to an $\mathcal{S}_k'$ as above with an arc in $\mathrm{DL}(n)$ for each k, by shrinking the contraction vectors component-wise. Here, we may assume that $\mathbf{r}_{\mathcal{S}_1'}$ and $\mathbf{r}_{\mathcal{S}_2'}$ have the same sufficiently small components. Then, sliding the fixed points suitably, we can connect $\mathcal{S}_1'$ to $\mathcal{S}_2'$ by an arc in $\mathrm{DL}(n)$. Thus, we conclude that $\mathrm{DL}(n)$ is a domain.

Next, suppose that there is an $\mathcal{S}$ in $\mathrm{DL}(n)$ with $\|\mathbf{r}_{\mathcal{S}}\| \geq 1$. Since $\mathcal{S}$ satisfies the open set condition (OSC), the theorem in [21] implies that the Hausdorff dimension of $\Lambda^+(\mathcal{S})$ should not be less than 2, and hence should be 2. But then $\Lambda^+(\mathcal{S})$ could not be totally disconnected, for instance, by the corollary in [21]. Thus, we conclude the first assertion.

Finally, the so-called λ lemma (cf. for instance [13], [15], and [18]) implies the second assertion. □

The following corollary follows from Theorem 1.1 and the classical nest test. See, for instance, [18].

Corollary 2.1. $\Lambda^+(\mathcal{S})$ *has absolute area* 0 *for every* $\mathcal{S} \in \mathrm{DL}(n)$.

Next, to prove Theorem 1.2, we recall that the cross-ratio $\chi(z_1, z_2, z_3, z_4)$ of 4 distinct points in $\mathbb{C}$ is defined by

$$\chi(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_1 - z_4}\,\frac{z_3 - z_4}{z_3 - z_2}.$$

For the given ordered 4 distinct elements $\tau = (T_1, \cdots, T_4)$ of $\mathrm{sgp}(\mathcal{S}_0)$, we have a holomorphic function

$$\chi_\tau \colon \mathrm{DL}(n) \longrightarrow \mathbb{C} - \{0, 1\}$$

by setting $\chi_\tau(\mathcal{S}) = \chi(z_{\mathcal{S},T_1}, \cdots, z_{\mathcal{S},T_4})$, where $z_{\mathcal{S},T_k}$ is the fixed point of $\iota_{\mathcal{S}}(T_k)$. We call such holomorphic functions on $\mathrm{DL}(n)$ the *cross ratio functions*. (Cf. [24].) For every vector τ as stated above and every two points $\mathcal{S}_k$ in $\mathrm{DL}(n)$, we know that

$$d_h(\chi_\tau(\mathcal{S}_1), \chi_\tau(\mathcal{S}_2)) \leq d_T(\mathcal{S}_1, \mathcal{S}_2),$$

where d_h is the hyperbolic distance on $\mathbb{C} - \{0, 1\}$. See, for instance, [16].

Proof of Theorem 1.2. Suppose that d_T is not complete. Then, there should be a constant $M > 0$ and a sequence $\{\mathcal{S}_n\}$ such that $\mathcal{S}_n$ tend to "the boundary" of $\mathrm{DL}(n)$ as $n \to \infty$, but satisfies $d_T(\mathcal{S}_0, \mathcal{S}_n) \leq M$ for every n.

First, assume that all components of $\mathbf{r}_{\mathcal{S}_n}$ stay in a compact set in $(0, 1)$ but that the points corresponding to $\mathcal{S}_n$ tend to the boundary of $\mathrm{Conf}(n-2, \mathbb{C} - \{0, 1\})$ in the natural compactification. (For the definition, see for instance [11].) Then, there are two different elements T_1, T_2 of $\mathrm{sgp}(\mathcal{S}_0)$ such that $d_s(z_{\mathcal{S}_n,T_1}, z_{\mathcal{S}_n,T_2}) \to 0$ as $n \to \infty$, where d_s is the spherical distance on the Riemann sphere. Hence, by choosing two other suitable and different T_3, T_4 of $\mathrm{sgp}(\mathcal{S}_0)$, we can construct a cross ratio function χ such that $\chi(\mathcal{S}_n)$ tend to 0 as $n \to \infty$. But this implies that $d_T(\mathcal{S}_0, \mathcal{S}_n)$ tend to $+\infty$, which is a contradiction.

Next, suppose that some of the components of $\mathbf{r}_{\mathcal{S}_n}$ tend to 0 as $n \to \infty$. Then it is easy to construct a cross ratio function such as above, and we have a contradiction in this case also.

Hence by Theorem 1.1, taking a subsequence if necessary, we may assume that $\mathcal{S}_n$ tend to a point $\mathcal{S}_\infty = \{S_{k,\infty}\} \in \mathrm{ND}(n) - \mathrm{DL}(n)$. Since $\mathcal{S}_\infty$ does not satisfy SSC, there is a point contained in two different $S_{k,\infty}(\Lambda^+(\mathcal{S}_\infty))$. Since the set of the components of $\mathrm{FP}(\mathcal{S}_\infty)$ is dense in $\Lambda^+(\mathcal{S}_\infty)$ (cf. for instance, [14]), we can find four different elements $T_1, \cdots, T_4$ of $\mathrm{sgp}(\mathcal{S}_0)$ such that the corresponding cross ratio function χ_τ satisfies

$$\lim_{n\to\infty} d_h(\chi_\tau(\mathcal{S}_0), \chi_\tau(\mathcal{S}_\infty)) > M.$$

Since the fixed points move continuously on $\mathrm{ND}(n)$, we conclude that for every sufficiently large n,

$$d_h(\chi_\tau(\mathcal{S}_0), \chi_\tau(\mathcal{S}_n)) > M,$$

which provides a contradiction. Thus we have finished the proof of Theorem 1.2. □

Remark 2.1. Even if $\mathcal{S}_1 \in \mathrm{DL}(n)$ but $\mathcal{S}_2 \notin \mathrm{DL}(n)$, there can exist a bi-Lipschitz homeomorphism between $\Lambda^+(\mathcal{S}_1)$ and $\Lambda^+(\mathcal{S}_2)$. Cf. [7] and [20].

3. Proof of Theorem 1.3

The case (1). This case is classical and well known. Indeed,

- if both of $\omega_j = 1$, then set $I = [0, 1]$;
- if $\omega_1 = -1,\ \omega_2 = 1$, then set $I = [-r, 1]$;
- if $\omega_1 = 1,\ \omega_2 = -1$, then set $I = [0, 1+s]$;
- if both of $\omega_j = -1$, then set

$$I = \left[\frac{-r(s+1)}{1-rs}, \frac{s+1}{1-rs}\right].$$

Then, it is clear that $I = \Lambda^+(\mathcal{S})$ if $r+s \geq 1$, and $\Lambda^+(\mathcal{S})$ is dust-like if $r+s < 1$. □

In the sequel, we use the family $\mathcal{S}_t(r, s; \omega_1, \omega_2) = \{S_1^t, S_2^t\}$ with

$$S_1^t(z) = r\omega_1 z, \quad S_2^t(z) = s\omega_2(z - 1 - it) + 1 + it,$$

which is equivalent to $\mathcal{S}(r, s; \omega_1, \omega_2)$. Also, let $[\![\alpha, \beta]\!]$ be the rectangle

$$[\![\alpha, \beta]\!] = \{z = x + iy \in \mathbb{C} \mid a \leq x \leq c, b \leq y \leq d\},$$

for all complex numbers $\alpha = a + ib,\ \beta = c + id$ with $a < c,\ b < d$.

The case (2). First, assume that $(\omega_1, \omega_2) = (i, 1)$. Set $t = r$ and

$$L = [\![-r^2 - r^3 i, 1 + ri]\!].$$

Then, by simple computation, we see that $S_j^t(L) \subset L$ for each j.

Lemma 3.1. *$\Lambda^+(\mathcal{S}_t)$ is dust-like if and only if $r^2 + s < 1$.*

Proof. If $r^2 + s < 1$, then $S_1^t(L) \cap S_2^t(L) = \emptyset$, and hence $\Lambda^+(\mathcal{S}_t)$ is dust-like. If $r^2 + s \geq 1$, then we can see that the diagonal of L having positive inclination is contained in $\Lambda^+(\mathcal{S}_t)$ as illustrated on the left in Figure 1, and hence $\Lambda^+(\mathcal{S}_t)$ cannot be dust-like, and is connected actually. $\square$

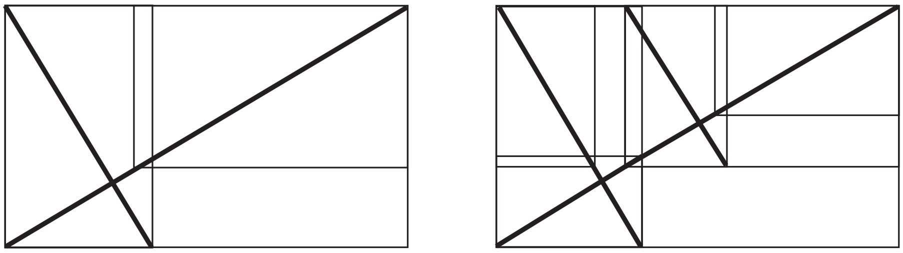

Figure 1. The diagonal in the attractor for the case $(i, 1)$.

Next, assume that $(\omega_1, \omega_2) = (i, -1)$. Then set $t = r$ and

$$L = \left[\!\left[\frac{-r^2(1+s) - r^3(1+s)i}{1 - r^2 s}, \frac{1 + s + r(1+s)i}{1 - r^2 s} \right]\!\right],$$

and we have the same conclusion as in the above lemma.

In the other cases, since the family $\mathcal{S}_t$ is conjugate to one of the above ones by the complex conjugation, we can conclude the assertion of Theorem 1.3 (2). $\square$

The case (3). First, assume that $(\omega_1, \omega_2) = (i, -i)$. Here, we also assume that $s \leq r$.

Set

$$t = \frac{r - s}{1 + rs}$$

and

$$L = \left[\!\left[\frac{-r^2(1+s^2) - r^3(1+s^2)i}{1 - r^2 s^2}, \frac{1 + s^2 + r(1+s^2)i}{1 - r^2 s^2} \right]\!\right].$$

Then, by simple computation, we see that $S_j^t(L) \subset L$ for each j. Also similarly as before, we conclude that $\Lambda^+(\mathcal{S}_t)$ is dust-like if and only if $r^2 + sr < 1$. See Figure 2.

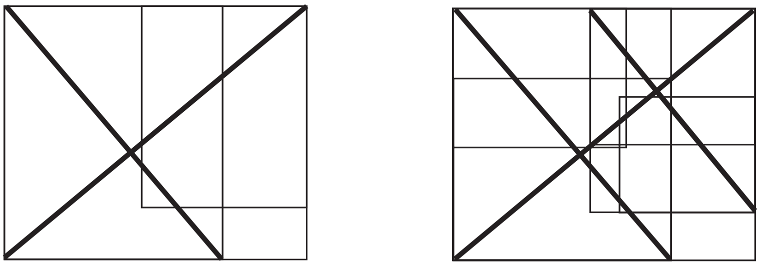

Figure 2. The diagonal in the attractor for the case $(i, -i)$.

Next, assume that $(\omega_1, \omega_2) = (i, i)$. Again, we also assume that $s \le r$. Set

$$t = \frac{r(1 - s^2 r^2)}{1 - s(r^3 + r - s)}$$

and

$$L = \left[\!\left[\frac{-r^2(1+s^2) - r^3(1+s^2)i}{1 - s(r^3 + r - s)}, \frac{1 + s^2 + r(1+s^2)i}{1 - s(r^3 + r - s)} \right]\!\right].$$

Then similarly as before we conclude the same assertion, where the segment in the attractor is not a diagonal but a segment with positive inclination as in Figure 3.

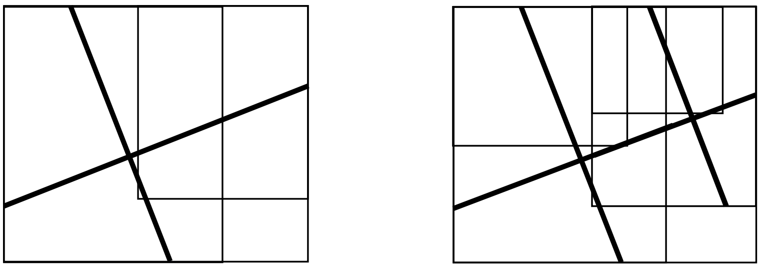

Figure 3. The segment in the attractor for the case (i, i).

In the other cases, since the family $\mathcal{S}_t$ is equivalent to one of the above ones and the complex conjugation of them, we can conclude the assertion of Theorem 1.3 (3). $\square$

4. Examples

We provide examples of $\mathcal{S} \in \mathrm{ND}(2)$ such that

(1) $\Lambda^+(\mathcal{S})$ is a closed domain.

(2) $\Lambda^+(\mathcal{S})$ is a *dendrite* in a sense that $\Lambda^+(\mathcal{S})$ is a connected and locally connected set with the connected complement and has no interior points.

(3) $\Lambda^+(\mathcal{S})$ is *porous*, or equivalently, $\Lambda^+(\mathcal{S})$ is connected but not so for its complement.

First, simple computation show the following fact.

Example 4.1. In the case of $\mathrm{Sl}(2; i, -i)$, if $1 \leq 2rs$, then $\Lambda^+(\mathcal{S})$ is $S_1^t(L) \cup S_2^t(L)$, and hence a closed domain.

Next, recall that, in the case of ND(2), $\Lambda^+(\mathcal{S})$ is either dust-like or connected. Hence, boundary points of the dust-likeness locus in the slice as shown in Theorem 1.3 give dendrites.

Lemma 4.1. *For the boundary point $\mathcal{S}$ of the slice as in case* (3) *of Theorem* 1.3*, $\Lambda^+(\mathcal{S})$ is a dendrite except for* $(r, s) = (1/\sqrt{2}, 1/\sqrt{2})$.

Proof. First, from the proof of Theorem 1.3, $\Lambda^+(\mathcal{S}_t)$ is connected, and we can see from the construction that so is the complement of $\Lambda^+(\mathcal{S})$, when $\mathcal{S}_t$ is on the boundary of $\mathrm{DL}(n)$. Also, recall that the connected self-similar sets are locally connected, as is easily seen from the definition.

Finally, if $s < r$ and $r^2 + rs = 1$, then $\Lambda^+(\mathcal{S}_t)$ has area 0, since such an $\mathcal{S}$ satisfies OSC and $r^2 + s^2 < r^2 + rs = 1$. Hence, there are no interior points by the corollary in [21]. $\square$

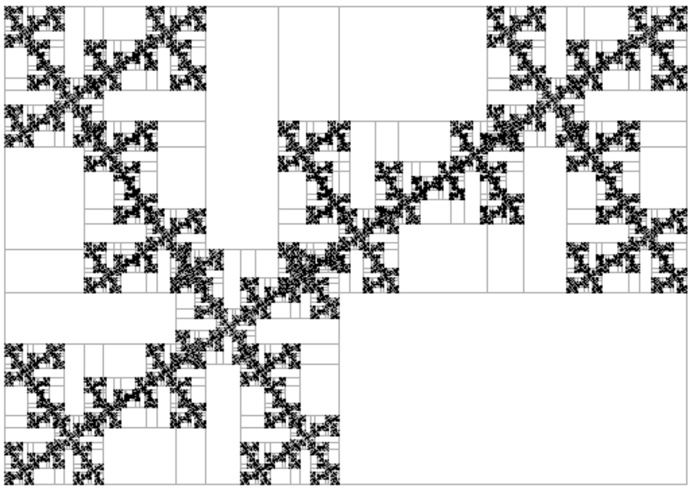

Figure 4. A dendrite on the boundary of $\mathrm{ND}(n)$.

Finally, we give an explicit example of $\mathcal{S}$ with the porous attractor. Presently, the locus of IFSs corresponding to the porous attractors is not clear.

Example 4.2. Set

$$S_1^t(z) = \frac{7i}{10}z, \quad S_2^t(z) = -\frac{100}{149}\left(z - 1 - \frac{7i}{10}\right) + 1 + \frac{7i}{10}.$$

Then $\Lambda^+(\mathcal{S}_t)$ is porous.

Indeed, we can see that there appears to be a hole in the union of 5-th crones of L. On the other hand, the union of 5-th crones of the diagonal in the attractor surrounds the hole, which shows that $\Lambda^+(\mathcal{S}_t)$ has a hole and hence, has the porous attractor. See Figure 5.

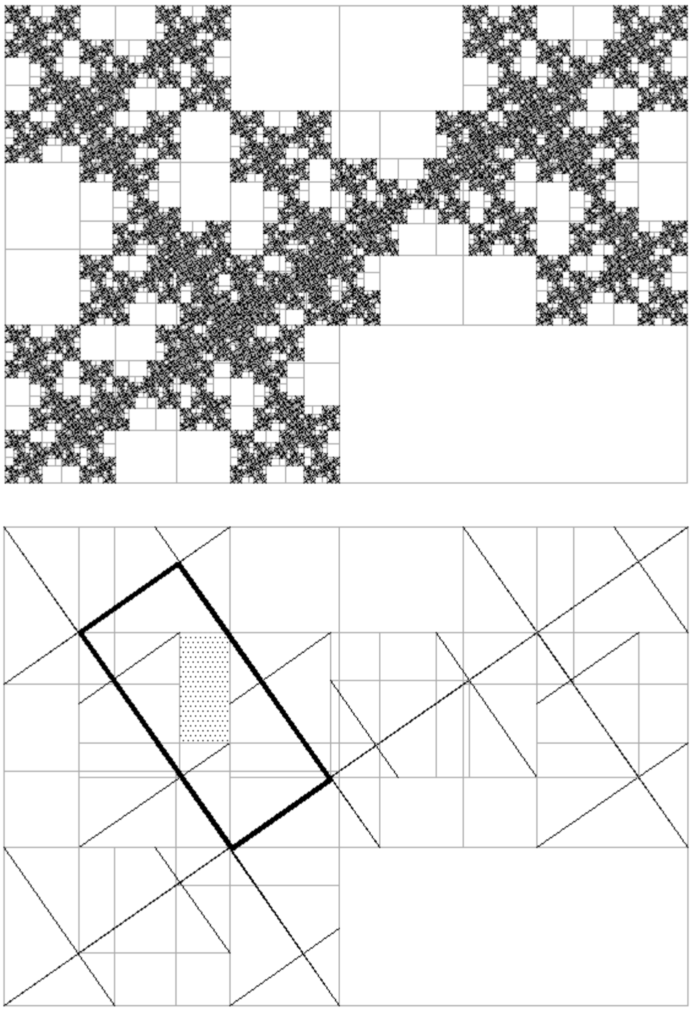

Figure 5. A hole and a surrounding closed curve in the attractor.

Similarly, we can construct examples of IFSs with porous attractors, also in other subcases of case (2) in Theorem 1.3.

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