J. Fractal Geom. 1 (2014), 349–390
DOI 10.4171/JFG/11

Journal of Fractal Geometry

Intersections of multiplicative translates of 3-adic Cantor sets

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Abstract. This paper is motivated by questions concerning the discrete dynamical system on the 3-adic integers $\mathbb{Z}_3$ given by multiplication by 2. The exceptional set $\mathcal{E}(\mathbb{Z}_3)$ is defined to be the set of all elements of $\mathbb{Z}_3$ whose forward orbits under this action intersect the 3-adic Cantor set $\Sigma_{3,\bar{2}}$ (of 3-adic integers whose expansions omit the digit 2) infinitely many times. It has been shown that this set has Hausdorff dimension at most $\frac{1}{2}$, and it is conjectured that it has Hausdorff dimension 0. Upper bounds on its Hausdorff dimension can be obtained with sufficient knowledge of Hausdorff dimensions of intersections of multiplicative translates of Cantor sets by powers of 2. This paper studies more generally the structure of finite intersections of general multiplicative translates $S = \Sigma_{3,\bar{2}} \cap \frac{1}{M_1}\Sigma_{3,\bar{2}} \cap \cdots \cap \frac{1}{M_n}\Sigma_{3,\bar{2}}$ by integers $1 < M_1 < M_2 < \cdots < M_n$. These sets are describable as sets of 3-adic integers whose 3-adic expansions have one-sided symbolic dynamics given by a finite automaton. This paper gives a method to determine the automaton for given data $(M_1, \ldots, M_n)$ and to compute the Hausdorff dimension, which is always of the form $\log_3(\beta)$ where β is an algebraic integer. Computational examples indicate that in general the Hausdorff dimension of such sets depends in a very complicated way on the integers $M_1, \ldots, M_n$. Exact answers are obtained for certain infinite families, which show as a corollary that a relaxed notion of generalized exceptional set has a positive Hausdorff dimension.

Mathematics Subject Classification (2010). Primary: 11K55; Secondary: 11S82, 28A80, 28D20, 37B10.

Keywords. p-adic arithmetic, finite automata, graph-directed systems, Hausdorff dimension.

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[1] The first author acknowledges the support of an NSF Graduate Research Fellowship and of the University of Michigan, where this work was carried out.

[2] The second author received support from NSF grants DMS-0801029 and DMS-1101373.

Contents

1. Introduction

We call the subset of the 3-adic integers $\mathbb{Z}_3$ whose 3-adic expansions use only digits 0 and 1 the 3-*adic Cantor set*, and denote it $\Sigma_3 := \Sigma_{3,\bar{2}}$. By a *multiplicative translate* of such a Cantor set we mean a multiplicatively rescaled set $\lambda\Sigma_3 = \{\lambda x \colon x \in \Sigma_3\}$, with $\lambda \in \mathbb{Z}_3$; choosing $\lambda = 2$ recovers the Cantor set using digits 0 and 2. In this paper we mainly restrict attention to cases where $\lambda = \frac{p}{q} \in \mathbb{Q} \cap \mathbb{Z}_3$ is a rational number that is 3-*integral*, meaning that 3 does not divide q; the case of general $\lambda \in \mathbb{Z}_3$ is addressed in Section 2.4. This paper studies sets given as finite intersections of multiplicative translates of the form

$$C(r_1, r_2, \ldots, r_n) := \bigcap_{i=1}^{n} \frac{1}{r_i}\Sigma_3, \tag{1.1}$$

where each fraction $1/r_i \in \mathbb{Q}$ is 3-integral. These sets are fractals and our object is to obtain bounds on their Hausdorff dimensions.

Our results are based on the fact that the Hausdorff dimension of a given $C(r_1, r_2, \ldots, r_n)$ above is effectively computable in a closed form. We show that each such set has the property that the 3-adic expansions of all the members of $C(r_1, r_2, \ldots, r_n)$ are characterizable as the output labels of all infinite paths in a labeled finite automaton which start from a marked initial vertex. General sets of such path labels associated to a finite automaton form symbolic dynamical systems that we call *path sets* and which we have studied in [2]. The sets $C(r_1, r_2, \ldots, r_n)$ are then special cases of a p-adic path set fractal (with $p = 3$), a notion introduced in [3]. Here *p-adic path set fractals* are collections of all p-adic numbers whose p-adic expansions have digits described by the labels along infinite paths according to a digit assignment map taking path labels in the graph to p-adic digits. In [3, Theorem 2.10] we showed that a p-adic path set fractal is any set Y in $\mathbb{Z}_p$ constructed by a p-adic analogue of a real number graph-directed fractal construction, as given in Mauldin and Williams [16]. This geometric object Y is given as the set-valued fixed point of a dilation functional equation using a set of p-adic affine maps, cf. [3, Theorem 2.6]. We showed in [3, Theorem 1.4] that if Y is a p-adic path set fractal then any multiplicative translate rY by a p-integral rational number r is also a p-adic path set fractal. In addition p-adic path set fractals are closed under set intersection, a property they inherit from path sets, see [2, Theorem 1.2]. Since the 3-adic Cantor set is a 3-adic path set fractal, whose underlying symbolic dynamical system is the one-sided shift on two symbols, these closure properties immediately imply that every set $C(r_1, r_2, \dots, r_n)$ is a 3-adic path set fractal.

In [3, Theorem 1.1] we showed that the Hausdorff dimension of a p-adic path set fractal Y is directly computable from the adjacency matrix of a suitable presentation of Y. One has

$$\dim_H(Y) = \log_p \beta,$$

in which β is the spectral radius $\rho(\mathbf{A})$ of the adjacency matrix $\mathbf{A}$ of a finite automaton which gives a suitable presentation of the given path set; see Section 3. This spectral radius coincides with the *Perron eigenvalue* ([13, Definition 4.4.2]) of the nonnegative integer matrix $\mathbf{A} \neq 0$, which is the largest real eigenvalue $\beta \geq 0$ of $\mathbf{A}$. For adjacency matrices of graphs containing at least one directed cycle, which are nonnegative integer matrices, the Perron eigenvalue is necessarily a real algebraic integer, and also has $\beta \geq 1$. In the case at hand we know a priori that $1 \leq \beta \leq 2$, as detailed below.

Our motivation for studying the intersection sets (1.1) originally arose from a problem of Erdős [7] described in Section 1.1 below. It concerns an ergodic-theoretic question on the behavior of the dynamical system that iterates the map $x \to 2x$ acting on the 3-adic integers, called here the exceptional set problem, which was raised in [11]. This problem directly leads to the study of various intersection sets (1.1). Such sets also provide a vehicle to formulate various relaxations of this problem, including the generalized exceptional set problem given in Section 1.2. The study of Hausdorff dimension of intersection sets seems directly of interest in its own right, to shed light on a class of semigroup intersection problems initiated by Furstenberg [8] in 1970, see Section 2.4.

1.1. Erdős ternary expansion problem.

Erdős [7] conjectured that for every $n \geq 9$, the ternary expansion of 2^n contains the ternary digit 2. A weak version of this conjecture asserts that there are only finitely many n such that the ternary expansion of 2^n consists of only 0's and 1's. Both versions of this conjecture appear to be very difficult problems.

In [11] the second author proposed a 3-adic generalization of this problem, as follows. Let $\mathbb{Z}_3$ denote the 3-adic integers, and let a 3-adic integer α have 3-adic expansion

$$(\alpha)_3 := (\dots a_2 a_1 a_0)_3 = a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + \dots, \quad \text{with all } a_i \in \{0, 1, 2\}.$$

Definition 1.1. The 3-*adic exceptional set* $\mathcal{E}(\mathbb{Z}_3)$ is given by

$$\mathcal{E}(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3 \colon \text{for infinitely many } n \geq 0,\ (2^n \lambda)_3 \text{ omits the digit } 2\}.$$

The weak version of Erdős's conjecture above is equivalent to the assertion that $\mathcal{E}(\mathbb{Z}_3)$ does not contain the integer 1.

The exceptional set seems an interesting object in its own right. It is forward invariant under multiplication by 2, and one may expect it to be a very small set in terms of measure or dimension. At present it remains possible that $\mathcal{E}(\mathbb{Z}_3)$ is a countable set, or even that it consists of the single element $\{0\}$. The second author previously put forward a conjecture asserting that the exceptional set is small in the sense of Hausdorff dimension ([11, Conjecture 1.7]), as follows.

Conjecture 1.2. (Exceptional Set Conjecture) *The* 3-*adic exceptional set* $\mathcal{E}(\mathbb{Z}_3)$ *has Hausdorff dimension zero, i.e.*

$$\dim_H(\mathcal{E}(\mathbb{Z}_3)) = 0. \tag{1.2}$$

The definition of p-adic Hausdorff dimension can be found in Abercrombie [1, p. 311–312], and is analogous to the real case. The paper [11] showed that the Hausdorff dimension of $\mathcal{E}(\mathbb{Z}_3)$ is at most $\frac{1}{2}$, as explained below. That paper initiated a strategy to obtain upper bounds for $\dim_H(\mathcal{E}(\mathbb{Z}_3))$ based on the containment relation

$$\mathcal{E}(\mathbb{Z}_3) \subseteq \bigcap_{k=1}^{\infty} \mathcal{E}^{(k)}(\mathbb{Z}_3), \tag{1.3}$$

where

$$\mathcal{E}^{(k)}(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3 \colon \text{at least } k \text{ values of } (2^n \lambda)_3 \text{ omit the digit } 2\}. \tag{1.4}$$

These sets form a nested family

$$\Sigma_{3,\bar{2}} = \mathcal{E}^{(1)}(\mathbb{Z}_3) \supseteq \mathcal{E}^{(2)}(\mathbb{Z}_3) \supseteq \mathcal{E}^{(3)}(\mathbb{Z}_3) \supseteq \dots .$$

The containment relation (1.3) yields inequalities relating the Hausdorff dimension of these sets,

$$\dim_H(\mathcal{E}(\mathbb{Z}_3)) \leq \Gamma, \tag{1.5}$$

where the *nesting constant* Γ is defined by

$$\Gamma := \lim_{k\to\infty} \dim_H(\mathcal{E}^{(k)}(\mathbb{Z}_3)). \tag{1.6}$$

This upper bound leads to the subsidiary problem of obtaining upper bounds for Γ, which in turn requires obtaining bounds for the individual $\dim_H(\mathcal{E}^{(k)}(\mathbb{Z}_3))$. We note the possibility that $\dim_H(\mathcal{E}(\mathbb{Z}_3)) < \Gamma$ may hold.

The analysis of the sets $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ for $k \geq 2$ leads to the study of particular sets of the kind (1.1) considered in this paper. We have

$$\mathcal{E}^{(k)}(\mathbb{Z}_3) = \bigcup_{0\leq m_1<\cdots<m_k} \mathcal{C}(2^{m_1},\dots,2^{m_k}). \tag{1.7}$$

We next give a reduction, showing that for the purposes of computing Hausdorff dimension we may, without loss of generality, restrict this set union to subsets having $m_1 = 0$, so that $2^{m_1} = 1$.

Definition 1.3. The *restricted* 3*-adic exceptional set* $\mathcal{E}_1(\mathbb{Z}_3)$ is given by

$\mathcal{E}_1(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3\colon$ for $n = 0$ and infinitely many other n, $(2^n\lambda)_3$ omits digit $2\}$.

It is easy to see that

$$\mathcal{E}(\mathbb{Z}_3) = \bigcup_{n=0}^{\infty} \frac{1}{2^n}\mathcal{E}_1(\mathbb{Z}_3).$$

Since the right side is a countable union of sets we obtain

$$\dim_H(\mathcal{E}(\mathbb{Z}_3)) = \sup_{n\geq 0}\Big(\dim_H\Big(\frac{1}{2^n}\mathcal{E}_1(\mathbb{Z}_3)\Big)\Big) = \dim_H(\mathcal{E}_1(\mathbb{Z}_3)),$$

and we also have $\mathcal{E}_1(\mathbb{Z}_3) \subset \Sigma_{3,\bar{2}}$. We now define

$$\mathcal{E}_1^{(k)}(\mathbb{Z}_3) := \{\lambda \in \Sigma_{3,\bar{2}}\colon \text{there are } k \text{ values of } n \geq 0, \text{ including } n = 0, \text{ for which } (2^n\lambda)_3 \text{ omits the digit } 2\}.$$

Lemma 1.4. *The nesting constant*

$$\Gamma = \lim_{k\to\infty} \dim_H(\mathcal{E}_1^{(k)}(\mathbb{Z}_3)). \tag{1.8}$$

In addition

$$\dim_H(\mathcal{E}_1^{(k)}(\mathbb{Z}_3)) = \sup_{0<m_1<\cdots<m_{k-1}} (\dim_H(\mathcal{C}(1, 2^{m_1},\dots,2^{m_{k-1}}))). \tag{1.9}$$

Proof. For $0 < m_1 < m_2 < \cdots < m_k$ we have the set identities

$$\mathcal{C}(2^{m_1}, \ldots, 2^{m_k}) = \frac{1}{2^{m_1}} \mathcal{C}(1, 2^{m_2 - m_1}, \ldots, 2^{m_k - m_1}).$$

These identities yield $\mathcal{E}^{(k)}(\mathbb{Z}_3) = \bigcup_{n=0}^{\infty} 2^{-n} \mathcal{E}_1^{(k)}(\mathbb{Z}_3)$. Again, since this is a countable union of sets, we obtain the equality

$$\dim_H(\mathcal{E}^{(k)}(\mathbb{Z}_3)) = \sup_{k \geq 1}(\dim_H(2^{-n}\mathcal{E}_1^{(k)}(\mathbb{Z}_3)) = \dim_H(\mathcal{E}_1^{(k)}(\mathbb{Z}_3)).$$

It also follows that (1.8) holds. We also have

$$\mathcal{E}_1^{(k)}(\mathbb{Z}_3) = \bigcup_{0 < m_1 < \cdots < m_{k-1}} \mathcal{C}(1, 2^{m_1}, \ldots, 2^{m_{k-1}}). \tag{1.10}$$

The right side of this expression is a countable union of sets, so (1.9) follows. □

Upper bounds for the right side of equation (1.9) are obtained by bounding above the Hausdorff dimensions of all the individual sets $\mathcal{C}(1, 2^{m_1}, \ldots, 2^{m_{k-1}})$, of the form (1.1). Lower bounds may be obtained by determining the Hausdorff dimension of specific individual sets $\mathcal{C}(1, 2^{m_1}, \ldots, 2^{m_{k-1}})$. By this means the second author [11, Theorem 1.6 (ii)] obtained the upper bound

$$\Gamma \leq \dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3)) = \dim_H(\mathcal{E}_1^{(2)}(\mathbb{Z}_3)) \leq \frac{1}{2}, \tag{1.11}$$

which by (1.5) yields the upper bound

$$\dim_H(\mathcal{E}(\mathbb{Z}_3)) \leq \frac{1}{2}. \tag{1.12}$$

1.2. Generalized exceptional set problem. One may consider approaches to upper bounding the Hausdorff dimension of the exceptional set $\mathcal{E}(\mathbb{Z}_3)$ that proceed by relaxing its defining conditions. Here we consider the relaxation that allows arbitrary positive integers M in place of powers of 2 in its definition. Since the 3-adic Cantor set $\Sigma_{3,\bar{2}}$ is forward invariant under multiplication by 3, we will restrict to integers $M \not\equiv 0 \pmod 3$. Furthermore Lemma 1.4 indicates that we obtain a relaxation if we consider only the restricted family of sets $\mathcal{C}(1, M_1, \ldots, M_n)$, i.e. taking $M_0 = 1$. Therefore we define a relaxed version of the restricted 3-adic exceptional set, as follows.

Definition 1.5. The 3-*adic generalized exceptional set* is the set

$$\begin{aligned} \mathcal{E}_\star(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3 : \; & \text{there are infinitely many } M \geq 1,\, M \not\equiv 0 \pmod 3, \\ & \text{including } M = 1, \text{ such that} \\ & \text{the 3-adic expansion } (M\lambda)_3 \text{ omits the digit } 2\}. \end{aligned}$$

When considering intersection sets $C(1, M_1, \ldots, M_n)$, we can then further restrict to require all $M_i \equiv 1 \pmod 3$, since any $M \equiv 2 \pmod 3$ has $C(1, M) = \{0\}$. We have $\mathcal{E}_1(\mathbb{Z}_3) \subset \mathcal{E}_\star(\mathbb{Z}_3) \subset \Sigma_{3,\bar{2}}$ and therefore

$$\dim_H(\mathcal{E}(\mathbb{Z}_3)) = \dim_H(\mathcal{E}_1(\mathbb{Z}_3)) \le \dim_H(\mathcal{E}_\star(\mathbb{Z}_3)). \tag{1.13}$$

Thus upper bounds for the Hausdorff dimension of the generalized exceptional set yield upper bounds for that of the exceptional set.

Problem 1.6 (generalized exceptional set problem). *Determine upper and lower bounds for the Hausdorff dimension of the generalized exceptional set* $\mathcal{E}_\star(\mathbb{Z}_3)$. *In particular, determine whether* $\dim_H(\mathcal{E}_\star(\mathbb{Z}_3)) = 0$ *or* $\dim_H(\mathcal{E}_\star(\mathbb{Z}_3)) > 0$ *holds.*

To approximate $\mathcal{E}_\star(\mathbb{Z}_3)$ we next introduce a family of sets in parallel to $\mathcal{E}_1^{(k)}(\mathbb{Z}_3)$ above. We define

$$\begin{aligned}\mathcal{E}_\star^{(k)}(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3 :\ & \text{there exist } 1 = M_0 < M_1 < \cdots < M_k, \\ & \text{for all } M_i \equiv 1 \pmod 3 \text{ such that} \\ & \text{the 3-adic expansion } (M_i\lambda)_3 \text{ omits the digit } 2\}.\end{aligned}$$

Next we define the *generalized nesting constant*

$$\Gamma_\star := \lim_{k\to\infty} \dim_H(\mathcal{E}_\star^{(k)}(\mathbb{Z}_3)) \tag{1.14}$$

and note that Lemma 1.4 yields $\Gamma \le \Gamma_\star$.

In parallel to the case above, we have

$$\mathcal{E}_\star^{(k)}(\mathbb{Z}_3) = \bigcup_{\substack{1<M_1<\cdots<M_{k-1} \\ M_i \equiv 1 \pmod 3}} \mathcal{C}(1, M_1, \ldots, M_{k-1}).$$

In consequence we have the inclusion

$$\mathcal{E}_\star(\mathbb{Z}_3) \subseteq \bigcap_{k=1}^{\infty} \mathcal{E}_\star^{(k)}(\mathbb{Z}_3),$$

which yields the upper bound

$$\dim_H(\mathcal{E}_\star(\mathbb{Z}_3)) \le \Gamma_*. \tag{1.15}$$

A priori it is possible that $\dim_H(\mathcal{E}_\star(\mathbb{Z}_3)) < \Gamma_*$.

The second author [11, Theorem 1.6] previously obtained the upper bound

$$\Gamma_* \le \dim_H(\mathcal{E}_\star^{(2)}(\mathbb{Z}_3)) \le \frac{1}{2}. \tag{1.16}$$

Improved bounds on Γ_* translate to improved bounds on the Hausdorff dimension of the generalized exceptional set. Our motivation in studying this set stems in part from the fact that if it were true that $\dim_H(\mathcal{E}_\star(\mathbb{Z}_3)) = 0$, then the Exceptional Set Conjecture 1.2 would follow.

1.3. Overview. This paper presents theoretical and experimental results concerning intersection sets (1.1). The main results are stated in Section 2. These results are based on an efficient presentation of the underlying path set of $\mathcal{C}(1, M)$ for integers $M \geq 1$, with an algorithm to compute it given in Section 4. This algorithm is slightly simpler than the general construction given in [2, Proposition 4.3] and we extend it to multiple intersections $\mathcal{C}(1, M_1, M_2, \ldots, M_n)$. In Section 5 we describe two infinite families of integers $\{N_k : k \geq 1\}$ whose 3-adic expansions have an especially simple form, for which the associated path set presentation automata for $\mathcal{C}(1, N_k)$ can be completely analyzed.

An interesting feature of this investigation is the nature of the dependence of $\dim_H(C(r_1, r_2, \ldots, r_n))$ on the rational numbers $(r_1, \ldots, r_n)$. In Sections 4 and 5 we present examples showing that the structure of the associated automata depends in a very complicated way on 3-adic arithmetic properties of the integers M_i, as do their Hausdorff dimensions. This complexity of the examples indicates a very complicated behavior of the Hausdorff dimension function under intersection. From the perspective of analogous problems concerning intersections of general additive translates (see Section 2.4), this complexity is perhaps not so surprising. However the exact nature of how the Hausdorff dimension depends on arithmetic properties of the data r_i certainly remains to be better understood.

This investigation led to a resolution of the generalized exceptional set problem. In Section 6 we first establish, using one infinite family, that $\Gamma_* \geq \frac{1}{2}\log_3 2$. We then present a result using the same infinite family, due to A. Bolshakov, establishing the same lower bound for the Hausdorff dimension of the generalized exceptional set, i.e. $\dim_H(\mathcal{E}_\star(\mathbb{Z}_3)) \geq \frac{1}{2}\log_3 2$. This result limits the upper bounds attainable on $\dim_H(\mathcal{E}(\mathbb{Z}_3))$ via the inclusion of $\mathcal{E}(\mathbb{Z}_3)$ in the generalized exceptional set. In Section 6 we present further computational results related to the exceptional set.

The results of this paper concern very special cases of properties of intersections of p-adic path set fractals (as defined in [3]). These more general intersection sets may be studied by similar methods. The algorithmic methods obtained in this paper also apply to p-adic numbers for any prime p and to the g-adic numbers considered by Mahler [14] for any integer $g \geq 2$.

2. Results

The paper presents both algorithmic results and exact results on the Hausdorff dimension of specific infinite families of intersections.

2.1. Algorithmic results. We study the size of intersections of multiplicative translates of the 3-adic Cantor set $\Sigma_3 := \Sigma_{3,\bar{2}}$, as measured by Hausdorff dimension. We study the sets

$$\mathcal{C}(1, M_1, \dots, M_n) := \Sigma_{3,\bar{2}} \cap \frac{1}{M_1}\Sigma_{3,\bar{2}} \cap \dots \cap \frac{1}{M_n}\Sigma_{3,\bar{2}}.$$

where $1 < M_1 < \dots < M_n$ are positive integers. As remarked above, via results in [2] and [3] these sets have a nice description, with their members having p-adic expansions describable by finite automata, which permits effective computation of their Hausdorff dimension. These results are reviewed in Section 3, and the necessary definitions for presentations of a path set used in the following theorem appear there.

Theorem 2.1 (dimension of $C(1, M_1, \dots, M_n)$). (1) *There is a terminating algorithm that takes as input any finite set of integers $1 \le M_1 < \dots < M_n$, and gives as output a labeled directed graph $\mathcal{G} = (G, \mathcal{L})$ with a marked starting vertex v_0, which is a presentation of a path set $X = X(1, M_1, M_2, \dots, M_n)$ describing the 3-adic expansions of the elements of the space*

$$\mathcal{C}(1, M_1, \dots, M_n) := \Sigma_3 \cap \frac{1}{M_1}\Sigma_3 \cap \dots \cap \frac{1}{M_n}\Sigma_3.$$

This presentation is right-resolving and all vertices are reachable from the marked vertex. The graph G has at most $\prod_{i=1}^{n}(1 + \lfloor \frac{1}{2}M_i \rfloor)$ vertices.

(2) *The topological entropy β of the path set X is the Perron eigenvalue of the adjacency matrix A of the directed graph G. It is a real algebraic integer satisfying $1 \le \beta \le 2$. Furthermore the Hausdorff dimension*

$$\dim_H(\mathcal{C}(1, M_1, \dots, M_n)) = \log_3 \beta.$$

This dimension falls in the interval $[0, \log_3 2]$.

This construction is quite explicit in the special case $\mathcal{C}(1, M)$. In that case already the associated graphs G can be very complicated, and there exist examples where the graph has an arbitrarily large number of strongly connected components, cf. [4].

We have computed Hausdorff dimensions of many examples of such intersections. They exhibit a bewildering complexity in general. However in the process we have found several infinite families of integers (whose members satisfy a linear recurrence) where the graph structures are analyzable, see Section 5 and further examples in [4]. From the viewpoint of fractal constructions, the sets so constructed give specific interesting examples of graph-directed fractals, which appear to have structure depending on the integers $(M_1, .., M_n)$ in an intricate way.

2.2. Hausdorff dimension results: two infinite families. There are some simple properties of the 3-adic expansion of M (which coincides with the ternary expansion of M, read backwards) which restrict the Hausdorff dimension of sets $\mathcal{C}(1, M)$. We begin with some simple restrictions on the Hausdorff dimension which can be read off from the 3-adic expansion of M. We write the ternary expansion

$$(M)_3 := (a_k a_{k-1} \dots a_1 a_0)_3 \quad \text{for } M = \sum_{j=0}^{k} a_j 3^j.$$

If the first nonzero 3-adic digit $a_0 = 2$, then $\mathcal{C}(1, M) = \{0\}$, whence its Hausdorff dimension $\dim_H(\mathcal{C}(1, M)) = 0$. On the other hand, if the positive integers $M_1, \dots, M_k$ each have all digits $a_j = 0$ or $a_j = 1$ in their 3-adic expansions, then the Hausdorff dimension $\dim_H(\mathcal{C}(1, M_1, M_2, \dots, M_k))$ must be positive.

We have found several infinite families of integers having ternary expansions of a simple form, whose path set presentations have a regular structure in the family parameter k that permits their Hausdorff dimension to be determined. The simplest family takes $M_1 = 3^k = (10^k)_3$. In this trivial case $\mathcal{C}(1, 3^k) = \Sigma_{3,\bar{2}}$, whence

$$\dim_H(\mathcal{C}(1, M_k)) = \dim_H(\Sigma_{3,\bar{2}}) = \log_3 2 \approx 0.630929. \tag{2.1}$$

In Section 5 we analyze two other infinite families in detail, as follows. The first of these families is $L_k = \frac{1}{2}(3^k - 1) = (1^k)_3$, for $k \geq 1$.

Theorem 2.2 (infinite family $L_k = \frac{1}{2}(3^k - 1)$). (1) *Let* $L_k = \frac{1}{2}(3^k - 1) = (1^k)_3$. *The path set presentation* $(\mathcal{G}, v_0)$ *for the path set* $X(1, L_k)$ *underlying* $\mathcal{C}(1, L_k)$ *has exactly* k *vertices and is strongly connected.*

(2) *For every* $k \geq 1$,

$$\dim_H(\mathcal{C}(1, L_k)) = \dim_H \mathcal{C}(1, (1^k)_3) = \log_3 \beta_k,$$

where β_k *is the unique real root greater than* 1 *of* $\lambda^k - \lambda^{k-1} - 1 = 0$.

(3) *For all* $k \geq 3$ *there holds*

$$\dim_H(\mathcal{C}(1, L_k)) = \frac{\log_3 k}{k} + O\Big(\frac{\log\log(k)}{k}\Big).$$

The Hausdorff dimension of the set $\dim_H(\mathcal{C}(1, L_k))$ is positive but approaches 0 as $k \to \infty$. This result is proved in Section 5.2.

Secondly, we consider the family $N_k = 3^k + 1 = (10^{k-1}1)_3$. Our main results concern this family.

Theorem 2.3 (infinite family $N_k = 3^k + 1$). (1) *Let* $N_k = 3^k + 1 = (10^{k-1}1)_3$. *The path set presentation* $(\mathcal{G}, v_0)$ *for the path set* $X(1, N_k)$ *underlying* $\mathcal{C}(1, N_k)$ *has exactly* 2^k *vertices and is strongly connected.*

(2) *For every integer* $k \geq 1$, *there holds*

$$\dim_H(\mathcal{C}(1, N_k)) = \dim_H \mathcal{C}(1, (10^{k-1}1)_3) = \log_3\Big(\frac{1+\sqrt{5}}{2}\Big) \approx 0.438018.$$

Here the Hausdorff dimension is constant as $k \to \infty$. Theorem 2.3 is a direct consequence of results established in Section 5.3 (Theorem 5.5 and Proposition 5.6).

The particular path sets associated to the families L_k and N_k actually have shift-invariant symbolic dynamics, and are one-sided shifts of finite type, as was pointed out by a reviewer (see Remarks 5.4 and 5.8). Consequently, the Hausdorff dimension results in Theorems 2.2 and 2.3 obtained from the constructed path set graphs are obtainable using standard facts in symbolic dynamics, without requiring the extension of the theory to path sets. Nevertheless many $\mathcal{C}(1, M)$ do correspond to path sets that are not shift-invariant, cf. Example 4.6.

In a sequel [4] we will analyze a third infinite family $P_k = (20^{k-1}1)_3 = 2 \cdot 3^k + 1$, whose underlying path set graphs exhibit much more complicated behavior; they have an unbounded number of strongly connected components as $k \to \infty$.

2.3. Hausdorff dimension results: generalized exceptional set. We give several applications to bounding Hausdorff dimensions in Section 6. It is easy to see that for each of the infinite families L_k and N_k above, the Hausdorff dimensions of arbitrarily large intersections are always positive. In Theorem 6.2 we exhibit by explicit construction multiple intersection sets using N_k that establish $\Gamma_\star \geq \frac{1}{2}\log_3 2$.

Following this result, A. Bolshakov noted a simple construction for members of the family N_k, allowing infinite intersections. It yields the following result, which we present in Section 6.2.

Theorem 2.4 (generalized exceptional set lower bound). *The generalized exceptional set $\mathcal{E}_\star$ satisfies*

$$\dim_H(\mathcal{E}_\star) \geq \frac{1}{2}\log_3 2 \approx 0.315464.$$

This result is based on showing that

$$\dim_H(\{\lambda \in \Sigma_{3,\bar{2}} \colon N_{2k+1}\lambda \in \Sigma_{3,\bar{2}} \ \text{ for all } \ k \geq 1\}) \geq \frac{1}{2}\log_3 2, \qquad (2.2)$$

which is done in Theorem 6.3. This bound (2.2) has a short, self-contained proof. Theorem 2.4 establishes that the generalized exceptional set cannot be used to resolve the Exceptional Set Conjecture 1.2 affirmatively.

In Section 6.3 we give numerical improvements on the lower bounds in [11], for small k, for the Hausdorff dimension of the enclosing sets $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ that upper bound that of the exceptional set $\mathcal{E}(\mathbb{Z}_3)$. These improvements come via explicit examples.

2.4. Extensions of results: $\boldsymbol{\lambda \in \mathbb{Z}_3}$. One may consider the problem of understanding the behavior of the Hausdorff dimension of general intersection sets

$$C(\lambda_1, \lambda_2, \ldots, \lambda_n) := \bigcap_{i=1}^{n} \frac{1}{\lambda_i}\Sigma_3,$$

as a function of the parameters $\lambda_i \in \mathbb{Z}_3^\times$. The simplest case of such variation concerns the one-parameter function

$$f(\lambda) := \dim_H(\mathcal{C}(1,\lambda)) = \dim_H\Big(\Sigma_3 \cap \frac{1}{\lambda}\Sigma_3\Big).$$

As a by-product of our Hausdorff dimension calculations for one infinite family, we deduce that $f(\lambda)$ is a discontinuous function of λ with respect to the 3-adic metric on $\mathbb{Z}_3^\times$ at $\lambda = 1$, see Remark 5.9.

Analogous problems concern the behavior of Hausdorff dimension of intersections of additive translates of the classical middle-third Cantor set $X \subset [0, 1]$, whose members have ternary expansions that omit the digit 1. This problem was first raised by Furstenberg [8] in 1970, in a more general context. Set

$$Y_t := X \cap (X + t) \quad \text{for } 0 \leq t \leq 1.$$

The following two contrasting results are known.

(1) Hawkes [9] showed in 1975 that for a set of t of full Lebesgue measure the sets Y_t have constant Hausdorff dimension $\dim_H(Y_t) = \frac{1}{3}\log_3 2$. That is, the Hausdorff dimension of Y_t takes a “generic” value. Later Kenyon and Peres [10] substantially generalized this result.

(2) On the other hand, Davis and Hu [5] showed in 1995 for each $0 \le \alpha \le \log_3 2$ that the set of t for which $\dim_H(Y_t) = \alpha$ is dense in $[-1, 1]$.

These two results together show that the function $g(t) = \dim_H(Y_t)$ must be discontinuous everywhere in $[-1, 1]$ in a very strong sense.

It would be interesting to determine to what extent analogues of these results hold for the function $f(\lambda)$, and for more general multiplicative intersection sets in the p-adic setting. The results for additive translates above do not directly carry over to the multiplicative setting, in part because p-adic addition and multiplication have significant differences compared to the real number operations, as shown in [3]. If an analogue of the “generic” Hausdorff dimension result in (1) above were valid in the 3-adic case, then it would follow that the particular $\lambda = \frac{1}{r} \in \mathbb{Z}_3^\times \cap \mathbb{Q}$ studied in this paper will give “non-generic” values of the Hausdorff dimension.

2.5. Notation. The notation $(m)_3$ refers to either the base 3 radix expansion of the positive integer m, or to the 3-adic expansion of a general 3-adic integer $(m)_3$. In the 3-adic case this expansion is to be read right to left, to be compatible with the ternary expansion. That is, $\alpha = \sum_{j=0}^{\infty} a_j 3^j$ would be written $(\ldots a_2 a_1 a_0)_3$, unless explicitly stated otherwise. However when we write out symbolic dynamics of paths as regular expressions, these are to be read from left to right, e.g. $1100(10)^\infty$ corresponds to the 3-adic integer $\ldots 01010011$.

3. Symbolic dynamics and graph-directed constructions

The constructions of this paper are based on the fact that the points in intersections of multiplicative translates of 3-adic Cantor sets have 3-adic expansions that are nicely describable in terms of symbolic dynamics. This section recalls these basic connections.

3.1. Symbolic dynamics, graphs and finite automata. We consider symbolic dynamics on certain closed subsets of the one-sided shift space $\Sigma = \mathcal{A}^{\mathbb{N}}$ with fixed symbol alphabet $\mathcal{A}$, which for our application will be specialized to $\mathcal{A} = \{0, 1, 2\}$. A basic reference for directed graphs and symbolic dynamics, which we follow, is Lind and Marcus [13].

By a *graph* we mean a finite directed graph, allowing loops and multiple edges. A *labeled graph* is a graph assigning labels to each directed edge; these labels are drawn from a finite symbol alphabet. A labeled directed graph can be interpreted as a *finite automaton* in the sense of automata theory. In our applications to 3-adic digit sets, the labels are drawn from the alphabet $\mathcal{A} = \{0, 1, 2\}$. In a directed graph, a vertex is a *source* if all directed edges touching that vertex are outgoing; it is a *sink* if all directed edges touching that edge are incoming. A vertex is *essential* if it is neither a source nor a sink, and is called *stranded* otherwise. A graph is *essential* if all of its vertices are essential. A graph G is *strongly connected* if for each two vertices i, j there is a directed path from i to j. We let $SC(G)$ denote the set of strongly connected component subgraphs of G.

A (vertex-vertex) *adjacency matrix* $\mathbf{A} = \mathbf{A}_G$ of a directed graph G has entry a_{ij} counting the number of directed edges from vertex i to vertex j. The adjacency matrix is *irreducible* (i.e. for each entry (i, j) some power of the matrix is positive in this entry) if and only if the associated graph is strongly connected; and we term the graph *irreducible* in this case. The adjacency matrix of a directed graph is *primitive* (i.e some power of the matrix has strictly positive entries) if and only if the graph is strongly connected and aperiodic, i.e. the greatest common divisor of its (directed) cycle lengths is 1.

We use basic facts from Perron–Frobenius theory of nonnegative matrices. The *Perron eigenvalue* ([13, Definition 4.4.2]) of a nonnegative real matrix $\mathbf{A} \neq 0$ is the largest real eigenvalue $\beta \geq 0$ of $\mathbf{A}$. A nonnegative matrix is *irreducible* if for each row and column (i, j) some power $\mathbf{A}^m$ has (i, j)-th entry nonzero. A nonnegative matrix $\mathbf{A}$ is *primitive* if some power $\mathbf{A}^k$ for an integer $k \geq 1$ has all entries positive; primitivity implies irreducibility but not vice versa. The *Perron–Frobenius theorem* [13, Theorem 4.2.3] for an irreducible nonnegative matrix $\mathbf{A}$ states that

(1) the Perron eigenvalue β is geometrically and algebraically simple, and has an everywhere positive eigenvector $\mathbf{v}$;
(2) all other eigenvalues μ have $|\mu| \leq \beta$, so that $\beta = \sigma(\mathbf{A})$, the spectral radius of $\mathbf{A}$;
(3) any everywhere positive eigenvector must be a positive multiple of $\mathbf{v}$.

For a general nonnegative real matrix $\mathbf{A} \neq 0$, the Perron eigenvalue need not be simple, but it still equals the spectral radius $\sigma(\mathbf{A})$ and it has at least one everywhere nonnegative eigenvector. For an adjacency matrix of a graph containing at least one directed cycle, its Perron eigenvalue is necessarily a real algebraic integer $\beta \geq 1$ of a special kind called a *Perron number.* See Lind [12] for a characterization of these numbers.

3.2. p-adic path sets and one-sided sofic shifts. Our basic symbolic dynamics objects are special cases of the following notion of path set, which we study in detail in [2]. A *pointed graph* is a pair $(\mathcal{G}, v)$ consisting of a directed labeled graph $\mathcal{G} = (G, \mathcal{E})$ and a marked vertex v of $\mathcal{G}$. Here G is a (directed) graph and $\mathcal{E}$ is an assignment of labels $(e, \ell) = (v_1, v_2, \ell)$ to the edges of G, where every edge gets a unique label, and no two triples are the same (but multiple edges and loops are permitted otherwise).

Definition 3.1. Given a pointed graph $(\mathcal{G}, v)$ its associated *path set* $\mathcal{P} = X_{\mathcal{G}}(v) \subset \mathcal{A}^{\mathbb{N}}$ is the set of all infinite one-sided symbol sequences $(x_0, x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$, giving the successive labels of all one-sided infinite walks in $\mathcal{G}$ issuing from the distinguished vertex v. Some graphs may have finite walks from the given vertex that cannot be further extended; such walks do not belong to $X_{\mathcal{G},v}$.

Path sets are closed sets in the shift topology, but in general they are not invariant under the one-sided shift operator, $\sigma(a_0 a_1 a_2 a_3 \ldots) = a_1 a_2 a_3 \ldots$ in the sense that $\sigma(X) \not\subset X$ may occur. The symbolic dynamics literature typically treats shift-invariant sets ($\sigma(X) \subseteq X$), and this theory requires some minor extensions to cover the case of path sets, for which see [2]. We require the extra generality of path sets here because the multiplicative translation operation can yield non-shift invariant sets even if one starts with shift-invariant inputs, cf. Example 4.6 below.

Many different pointed graphs $(\mathcal{G}, v)$ may give rise to the same path set $\mathcal{P}$, and we call any such $(\mathcal{G}, v)$ a *presentation* of the path set $\mathcal{P}$. An important class of path set presentations have the following extra properties. We say that a directed labeled graph $\mathcal{G} = (G, v)$ is *right-resolving* if for each vertex of $\mathcal{G}$ all directed edges outward have distinct labels. (In automata theory $\mathcal{G}$ is called a *deterministic automaton*.) We say it is *reachable* if every vertex of $\mathcal{G}$ can be reached by a directed path from the initial vertex. Every path set possesses a right-resolving presentation that is reachable ([2, Theorem 3.2]).

Basic properties of path sets include the following.

(1) The collection of path sets $X := X_{\mathcal{G},v_0}$ in a given alphabet is closed under finite union and intersection ([2, Theorem 1.2]).

(2) The *shift-closure* of a path set X is the set of all paths obtainable by applying the one-sided shift repeatedly to paths in X. For a reachable presentation, the shift-closure is the set union of the path sets starting from all vertices of $\mathcal{G}$. The class of shift-invariant path sets $\mathcal{P}$ coincides with the class of *one-sided sofic shifts* in symbolic dynamics, see Theorem 1.4 of [2]. The notion of sofic shift was first introduced in the two-sided case by Weiss [18].

(3) The symbolic dynamics analogue of Hausdorff dimension is topological entropy. The *topological entropy* of a path set $H_{\text{top}}(X)$ is given by

$$H_{\text{top}}(X) := \limsup_{n\to\infty} \frac{1}{n} \log N_n(X),$$

where $N_n(X)$ counts the number of distinct blocks of symbols of length n appearing in elements of X. The topological entropy is easy to compute for a right-resolving presentation of a path set. By [2, Theorem 1.13], it is

$$H_{\text{top}}(X) = \log \beta \tag{3.1}$$

where β is the Perron eigenvalue of the adjacency matrix $\mathbf{A} = \mathbf{A}_G$ of the underlying directed graph G of $\mathcal{G}$, e.g. the spectral radius of $\mathbf{A}$.

3.3. p-adic path set fractals, graph directed constructions and Hausdorff dimension. We can view the elements of a path set X on the particular alphabet $\mathcal{A} = \{0, 1, 2, \ldots, p-1\}$ geometrically as describing the digits in the p-adic expansion of a p-adic integer. The associated map $f_p : \mathcal{A}^{\mathbb{N}} \to \mathbb{Z}_p$ having $f_p((a_0, a_1, a_2, \ldots)) = \sum_{j=0}^{\infty} a_j\, p^j$ is a homeomorphism of topological spaces.

Definition 3.2. Given a path set $X = X_{\mathcal{G}}(v) \subset \mathcal{A}^{\mathbb{N}}$ on the alphabet $\mathcal{A} = \{0, 1, 2, \ldots, p-1\}$, the image set $K = f_p(X) \subset \mathbb{Z}_p$ is called a *p-adic path set fractal.*

We study p-adic path set fractals in detail in [3], whose Proposition 2.9 shows that the definition of p-adic path set fractal above is equivalent to the definition used in that paper. The class $\mathcal{C}(\mathbb{Z}_p)$ of all p-adic path set fractals is closed under finite union and intersection, a property such sets inherit from path sets. Furthermore [3] shows that $\mathcal{C}(\mathbb{Z}_p)$ is closed under both p-adic addition and p-adic multiplication by rational numbers $r \in \mathbb{Q}$ that lie in $\mathbb{Z}_p$. The effects of these operations are effectively computable at the level of path set presentations; the p-adic arithmetic operations applied to p-adic path set fractals are treated in [3], and union and intersection of path sets are treated in [2].

The paper [3] shows that p-adic path set fractals are alternatively obtainable via p-adic analogues of graph-directed fractal constructions over real numbers, for which see Mauldin and Williams [16] and [17], Mauldin and Urbanski [15], and Edgar [6, Chapter 6]. This graph-directed fractal interpretation has the consequence that the Hausdorff dimension of a p-adic path set fractal is computable directly from a suitable presentation of the underlying path set $X = X_{\mathcal{G}}(v)$.

Proposition 3.3. *Let p be a prime, and let K be a set of p-adic integers whose allowable p-adic expansions are described by the symbolic dynamics of a p-adic path set X. Let $(\mathcal{G}, v_0)$ be a presentation of this path set that is right-resolving.*

(1) *The map $\phi_p\colon \mathbb{Z}_p \to [0,1]$ taking $\alpha = \sum_{k=0}^{\infty} a_k p^k \in \mathbb{Z}_p$ to the real number with base p expansion $\phi_p(\alpha) := \sum_{k=0}^{\infty} \frac{a_k}{p^{k+1}}$ is a continuous map. The image of K under this map, $K' := \phi_p(K) \subset [0,1]$, is a graph-directed fractal in the sense of Mauldin and Williams.*

(2) *The Hausdorff dimension of the p-adic path set fractal K is*

$$\dim_H(K) = \dim_H(K') = \log_p \beta, \tag{3.2}$$

where β is the spectral radius of the adjacency matrix $\mathbf{A}$ of the underlying graph G of $\mathcal{G}$.

Proof. These results are proved in Theorem 3.1 of [3]. The Hausdorff dimension formula in (2) matches the standard formula for graph-directed fractals of Mauldin and Williams [17] in the real number case. The map in (1) sends a path set to a Mauldin-Williams construction sub-object, see Theorem 2.10 of [3]. □

The 3-adic Cantor set $\Sigma_{3,\bar{2}}$ is a 3-adic path set fractal, and the general closure properties above guarantee that the intersection sets $C(r_1, r_2, \ldots, r_n)$ for rational r_i are also 3-adic path set fractals. In principle path set presentations for $C(r_1, r_2, \ldots, r_n)$ are effectively computable by general algorithms. In the next section we formulate an algorithm which directly computes a path set presentation for an intersection set $C(1, M)$ with M a positive integer satisfying $M \equiv 1 \pmod 3$. This algorithm takes advantage of the special form of the intersection set and of M to reduce the size of the state space of the presentation; it is sufficient for our applications.

4. Structure of intersection sets $\mathcal{C}(1, M_1, M_2, \ldots, M_n)$

We show directly that the sets $\mathcal{C}(1, M_1, \ldots, M_n)$ consist of those 3-adic integers whose 3-adic expansions are describable as path sets $X(1, M_1, \ldots, M_n)$. We also present an algorithm which when given the data $(M_1, \ldots, M_n)$ as input produces as output a presentation $\mathcal{G} = (G, v_0)$ of the path set $X(1, M_1, \ldots, M_n)$.

4.1. Constructing a path set presentation of $X(1, M)$. We describe an algorithmic procedure to obtain a path set presentation of $X(1, M)$ for the 3-adic expansions of elements in $\mathcal{C}(1, M)$. Since $\mathcal{C}(1, 3^j M) = \mathcal{C}(1, M)$, we may reduce to the case $M \not\equiv 0 \pmod 3$ and since $\mathcal{C}(1, M) = \{0\}$ if $M \equiv 2 \pmod 3$ it suffices to consider the case $M \equiv 1 \pmod 3$.

Theorem 4.1. *For $M \geq 1$, with $M \equiv 1 \pmod 3$, the set $\mathcal{C}(1, M) = \Sigma_3 \cap \frac{1}{M}\Sigma_3$ has 3-adic expansions given by a path set $X(1, M)$ which has an algorithmically computable path set presentation $(\mathcal{G}, v_0)$, in which the vertices v_m are labeled with a subset of the integers $0 \leq m \leq \lfloor \frac{1}{2}M \rfloor$, always including $m = 0$, and of cardinality at most $\lfloor \frac{M}{2} \rfloor$. This presentation is right-resolving, connected, and essential.*

Proof. The labeled graph $\mathcal{G} = (G, \mathcal{L})$ will have path labels drawn from $\{0, 1\}$ and the vertices v_N of the underlying directed graph G will be labeled by a subset of the integers N satisfying $0 \leq N \leq \lfloor \frac{M}{2} \rfloor$. The marked vertex v_0 corresponds to $N = 0$ and is the starting vertex of the algorithm.

The idea is simple. Suppose that

$$\alpha := \sum_{j=0}^{\infty} a_j 3^j \in \Sigma_3 \cap \frac{1}{M}\Sigma_3.$$

Here all $a_j \in \{0, 1\}$, and in addition

$$M\alpha = \sum_{j=0}^{\infty} b_j 3^j \in \Sigma_3.$$

Suppose that the first n digits

$$\alpha_n = \sum_{j=0}^{n-1} a_j 3^j,$$

are chosen. Since $M \equiv 1 \pmod 3$ this uniquely specifies the first n digits of

$$M\alpha_n := \sum_{j=0}^{m+n-1} b_j^{(n)} 3^j,$$

namely

$$b_j^{(n)} = b_j, \quad \text{for } 0 \leq j \leq n-1,$$

which have $b_j \in \{0, 1\}$, for $0 \le j \le n-1$. Here the remaining digits $b_{n+k}^{(n)}$ for $1 \le k \le m$ are unrestricted, with

$$m = \lfloor \log_3 M \rfloor + 1.$$

We have followed a path in the graph G corresponding to edges labeled $(a_0, a_1, \ldots, a_{n-1})$. The vertex we arrive at after these steps will be labeled by the value of the "carry-digit" part of β_n, which is

$$N = \sum_{j=n}^{m+n-1} b_j^{(n)} 3^{j-n}.$$

The value of the bottom 3-adic digit $b_n^{(n)}$ of N will determine the allowable exit edges from vertex v_N, and the label of the vertices reached. The requirement is that the next digit a_n satisfy

$$a_n + b_n^{(n)} \equiv 0, 1 (\mathrm{mod}\ 3) \tag{4.1}$$

If such a value is chosen, then we will be able to create a valid α_{n+1}, and $\beta_{n+1} := M\alpha_{n+1}$ will have

$$b_n^{(n+1)} = a_n + b_n^{(n)} \ (\mathrm{mod}\ 3).$$

There always exists at least one exit edge from each reachable vertex v_N, since for $b_n^{(n)} = 0$ the admissible $a_n = 0, 1$; for $b_n^{(n)} = 1$ the only admissible $a_n = 0$, and for $b_n^{(n)} = 2$ the only admissible $a_n = 1$, in order that the next digits a_{n+1}, b_{n+1} both belong to $\{0, 1\}$.

The important point is that the vertex label N fully determines the admissible exit edges possible in the next step, since its bottom digit determines the allowable exit edge values $a \subset \{0, 1\}$ by requiring

$$a + N \equiv 0, 1 \ (\mathrm{mod}\ 3), \tag{4.2}$$

and for an exit edge labeled a one can determine the new vertex label $v_{N'}$ as

$$N' := \left\lfloor \frac{N + Ma}{3} \right\rfloor. \tag{4.3}$$

To the graph G one adds a directed edge for each allowable value $a_n = 0$ or 1 from N to N' labeled by a_n.

Furthermore it is easy to see that the are only finitely many vertices v_N that can be reached from the vertex v_0. One shows by induction on the number of steps n taken that any reachable vertex v_N has vertex label

$$0 \le N \le \left\lfloor \frac{M}{2} \right\rfloor.$$

This property holds for the initial vertex, and for the induction step, we obtain from (4.3) that

$$N' \leq \frac{N + Ma}{3} \leq \frac{M/2 + M}{3} \leq \frac{M}{2}.$$

Thus the process of constructing the graph will halt.

It is easily seen that the presentation $\mathcal{G} = (G, v_0)$ obtained this way has the desired properties.

(1) The graph G is right-resolving because every vertex has exit edges with distinct edge-labels by construction.

(2) The graph G is essential because every vertex has at least one admissible exit edge, as shown above.

(3) The graph G is connected since we include in it only vertices reachable from v_0.

Since G is essential, $\mathcal{G}$ is a presentation of a certain 3-adic path set via the correspondence taking infinite walks beginning at the v_0-state in $\mathcal{G}$ to words in the edges traversed. Denote this path set $X_{\mathcal{G},0}$.

It remains to prove the claim that $X_{\mathcal{G},0}$ is the path set $X(1, M)$ corresponding to $\mathcal{C}(1, M)$. To prove the claim, let $\Phi\colon X_{\mathcal{G},0} \to \mathbb{Z}_3$ be the map

$$\ldots a_2 a_1 a_0 \longmapsto \sum_{k=0}^{\infty} a_k 3^k.$$

Φ is clearely an injection. $\Phi(X_{\mathcal{G},0}) \subset \mathcal{C}(1, M)$: since $\ldots a_2 a_1 a_0 \in X_{\mathcal{G},0}$ is a word in the full shift on $\{0, 1\}$, $\Phi(\ldots a_2 a_1 a_0) = \sum_{k=0}^{\infty} a_k 3^k$ omits the digit 2, so that $\Phi(X_{\mathcal{G},0}) \subset \Sigma_3$. But the algorithm was constructed specifically so that, given a path $\pi = a_l a_{l-1} \ldots a_2 a_1 a_0$ in $\mathcal{G}$ originating at 0, there is an edge labeled $a_{l+1} \in \{0, 1\}$ from the terminal vertex $t(\pi)$ if and only if each digit of the 3-adic expansion of $M \cdot \left(\sum_{k=0}^{l+1} c_k 3^k\right)$ which cannot be altered by any potential $(l + 2)$nd digit is either 0 or 1. This shows both that $\Phi(X_{\mathcal{G},0}) \subset \frac{1}{M}\Sigma_3$ and $\mathcal{C}(1, M) \subset \Phi(X_{\mathcal{G},0})$, so that

$$\Phi|_{\Phi^{-1}(\mathcal{C}(1,M))}\colon X_{\mathcal{G},0} \longrightarrow \mathcal{C}(1, M)$$

is a bijection. Assigning the appropriate metric to $X_{\mathcal{G},0}$ makes Φ an isomorphism in a now obvious way, proving the claim. $\square$

We obtain an algorithm to construct $\mathcal{G} = (G, v_0)$ based on the construction above. In this algorithm M must be a positive integer, $M \equiv 1 \pmod 3$.

Algorithm A (construction of path set presentation of $X(1, M)$)

(1) INITIAL STEP. Start with initial marked vertex v_0, and initial vertex set $I_0 := \{v_0\}$. Add an exit edge with edge label 0 giving a self-loop to v_0, and add another exit edge with edge label 1 going to new vertex v_m with vertex label $m := \lfloor M/3 \rfloor$, Add these two edges and their labels to form (labeled) edge table E_1. Form the new vertex set $I_1 := \{v_m\}$, and go to Recursive Step with $j = 1$.

(2) RECURSIVE STEP. Given value j, a nonempty new vertex set I_j of level j vertices, a current vertex set V_j and current edge set E_j. At step $j+1$ determine all allowable exit edge labels from vertices v_N in I_j, using the criterion (4.2), and compute vertices reachable by these exit edges, with reachable vertex labels computed by update equation (4.3). Add these new edges and their labels to the current edge set to make the updated current edge set E_{j+1}. Collect all vertices reached that are not in the current vertex set V_j into a new vertex set I_{j+1}. Update the current vertex set $V_{j+1} = V_j \cup I_{j+1}$. Go to Test Step.

(3) TEST STEP. If the current vertex set I_{j+1} is empty, halt, with the complete presentation $\mathcal{G} = (G, v_0)$ given by sets V_{j+1}, E_{j+1}. If I_{j+1} is nonempty, reset $j \mapsto j+1$ and go to Recursive Step.

The correctness of the algorithm follows from the discussion above.

4.2. Constructing a path set presentation of $X(1, M_1, \dots, M_n)$. Given integers $1 \le M_1 < \cdots < M_n$, we now have a way to construct graph presentations of the path sets $X(1, M_i)$ for each i. Since

$$X(1, M_1, \dots, M_n) = \bigcap_{i=1}^{n} X(1, M_i),$$

we need to know how to combine these graphs.

Recall the following definition from Lind and Marcus [13].

Definition 4.2. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be labeled graphs with the same alphabet $\mathcal{A}$, and let their underlying graphs be $G_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $G_2 = (\mathcal{V}_2, \mathcal{E}_2)$. The label product $\mathcal{G}_1 \star \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ has underlying graph G with vertex set $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, edge set $\mathcal{E} = \{(e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \colon e_1$ and e_2 have the same labels$\}$.

In [2, Proposition 4.3], we show that if $(\mathcal{G}_i, v_i)$ is a graph presentation of the path set $\mathcal{P}_i$, then $(\mathcal{G}_1 \star \mathcal{G}_2, (v_1, v_2))$ is a graph presentation for $\mathcal{P}_1 \cap \mathcal{P}_2$. It follows that we can form a presentation of $\mathcal{C}(1, M_1, \ldots, M_n)$ as the label product

$$(\mathcal{G}, v) = (\mathcal{G}_1 \star \mathcal{G}_2 \star \cdots \star \mathcal{G}_n, (v_1, v_2, \ldots, v_n)),$$

where $(\mathcal{G}_i, v_i)$ is the presentation of $\mathcal{C}(1, M_i)$ just constructed.

Theorem 4.3. *For* $1 < M_1 < M_2 < \cdots < M_n$, *with all* $M_i \equiv 1 \pmod 3$, *the set*

$$\mathcal{C}(1, M_1, M_2, \ldots, M_n) = \bigcap_{i=1}^{n} \mathcal{C}(1, M_i) = \Sigma_3 \cap \Big(\bigcap_{i=1}^{n} \frac{1}{M_i} \Sigma_3 \Big)$$

has 3*-adic expansions of its elements given by a path set* $X(1, M_1, M_2, \ldots, M_n)$. *This path set has an algorithmically computable presentation* $(\mathcal{G}, v_{\mathbf{0}})$, *in which the vertices* $v_{\mathbf{N}}$ *are labeled with a subset of integer vectors* $\mathbf{N} = (N_1, N_2, \ldots, N_n)$ *with* $0 \le N_i \le \frac{1}{2} M_i$, *always including the zero vector* $\mathbf{0}$. *The presentation has at most* $\prod_{i=1}^{n} (1 + \lfloor \frac{1}{2} M_i \rfloor)$ *vertices in the underlying graph. This presentation is right-resolving, connected and essential.*

Proof. The presentation is obtained by recursively applying the label product construction to the presentations $\mathcal{C}(1, M_i)$, see Algorithm B below. Each step preserves the properties of the presentation graph being right-resolving, connected and essential. The number of states of the label product construction is at most the product of the number of states in the two presentations being constructed. By Theorem 4.1, the presentation of $\mathcal{C}(1, M_i)$ has at most $1 + \lfloor \frac{1}{2} M \rfloor$ vertices. The bound given follows by induction on the successive label product constructions. ☐

In Algorithm B each M_i must be a positive integer, $M_i \equiv 1 \pmod 3$.

Algorithm B (Construction of path set presentation of $X(1, M_1, \ldots, M_n)$)

(1) Initial step. Construct presentations $\mathcal{G}_i = (G_i, \mathcal{L}_i)$ for $X(1, M_i)$ corresponding to $\mathcal{C}(1, M_i)$ for $1 \le i \le n$, using Algorithm A. Apply the label product construction to form $\mathcal{H}_2 := \mathcal{G}_1 \star \mathcal{G}_2$.

(2) For $2 \le i \le n - 1$, apply the label product construction to form

$$\mathcal{H}_{i+1} = \mathcal{H}_i \star \mathcal{G}_{i+1}.$$

Halt when $\mathcal{H}_n$ is computed.

4.3. Path set characterization of $\mathcal{C}(1, M_1, \ldots, M_n)$. From Theorem 4.3 we easily derive the following result.

Theorem 4.4. *For any integers* $1 \leq M_1 < \cdots < M_n$, *let*

$$\mathcal{C}(1, M_1, \ldots, M_n) := \Sigma_3 \cap \frac{1}{M_1}\Sigma_3 \cap \cdots \cap \frac{1}{M_n}\Sigma_3.$$

This is the set of all 3*-adic integers* $\lambda \in \Sigma_3$ *such that* $M_j\lambda$ *omits the digit* 2 *in its* 3*-adic expansion. Then,*

(1) *the complete set of the* 3*-adic expansions of numbers in* $\mathcal{C}(1, M_1, \ldots, M_n)$ *is a path set in the alphabet* $\mathcal{A} = \{0, 1, 2\}$*;*

(2) *the Hausdorff dimension of* $\mathcal{C}(1, M_1, \ldots, M_n)$ *is* $\log_3 \beta$, *where* $\log \beta$ *is the topological entropy of this path set. Here* β *necessarily satisfies* $1 \leq \beta \leq 2$, *and* β *is a Perron number, i.e. it is a real algebraic integer* $\beta \geq 1$ *such that all of its other algebraic conjugates satisfy* $|\sigma(\beta)| < \beta$.

Proof. Theorem 4.3 gives an explicit construction of a presentation $(\mathcal{G}, v)$ showing that $\mathcal{C}(1, M_1, \ldots, M_n)$ is a p-adic path set.

By Proposition 3.3 the Hausdorff dimension of $\mathcal{C}(1, M_1 \ldots, M_n)$ is $\log_3 \beta$, where β is the spectral radius of the adjacency matrix A of the underlying graph G. Since A is a 0-1 matrix, by Perron–Frobenius theory the spectral radius equals the maximal eigenvalue in absolute value, which is necessarily a positive real number β. It is a solution to a monic polynomial over $\mathbb{Z}$, so that β is necessarily an algebraic integer. By construction, the sum of the entries of any row in A is either 1 or 2, so that we also have $1 \leq \beta \leq 2$. ☐

Remark 4.5. The adjacency matrix A in the sets above can sometimes be *reducible*, i.e. it may have more than one strongly connected component. Example 4.3 below presents a graph $\mathcal{C}(1, 19)$ having a reducible matrix A– the underlying graph $\mathcal{G}$ has two strongly connected components.

Combining the results above yields Theorem 2.1.

Proof of Theorem 2.1. (1) The existence of a terminating algorithm follows from Theorem 4.1 and Theorem 4.3, with the algorithm for constructing a presentation of the path set $X(1, M_1, M_2, \ldots, M_n)$ given by combining Algorithm A and Algorithm B.

(2) This follows from Theorem 4.4. ☐

4.4. Examples. We present several examples of path set presentations.

Example 4.1. The 3-adic Cantor set $\Sigma_3 = \mathcal{C}(1) = \mathcal{C}(1,1)$ has a path set presentation $(\mathcal{G}, v_0)$ pictured in Figure 4.1. It is the full shift on two symbols, and the initial vertex is the vertex labeled 0. The underlying graph G of $\mathcal{G}$ is a double cover of a one vertex graph with two symbols. The advantage of the graph G pictured is that a path for it is completely determined by the set of vertex symbols that it passes through.

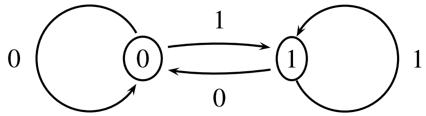

Figure 4.1. Path set presentation of Cantor shift $\Sigma_3 = \mathcal{C}(1)$. The marked vertex is 0.

Example 4.2. A path set presentation of $\mathcal{C}(1,7)$, with $7 = (21)_3$, is shown in Figure 4.2. The vertex labeled 0 is the marked initial state.

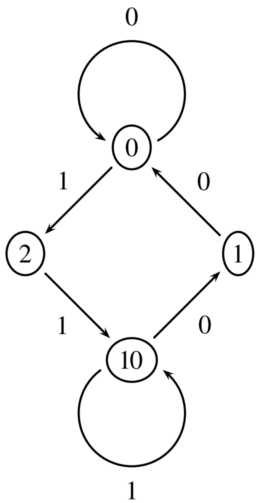

Figure 4.2. Path set presentation of $\mathcal{C}(1,7)$. The marked vertex is 0.

The graph in Figure 4.2 has adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which has Perron–Frobenius eigenvalue $\beta = \frac{1+\sqrt{5}}{2}$, so

$$\dim_H(\mathcal{C}(1,7)) = \log_3\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.438018.$$

Example 4.3. A path set presentation of $\mathcal{C}(1, 19)$, with $19 = (201)_3$, is shown in Figure 4.3. The vertex labeled 0 is the marked initial state.

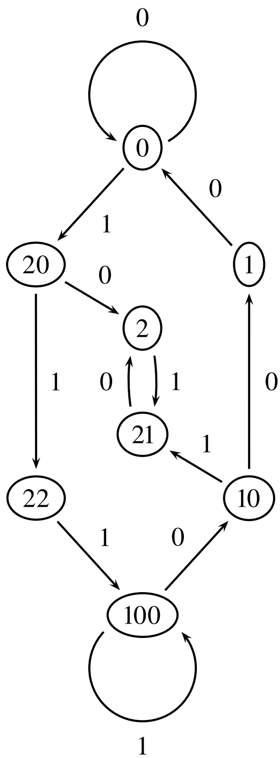

Figure 4.3. Path set presentation of $\mathcal{C}(1, 19)$. The marked vertex is 0.

The graph in Figure 4.3 has adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which has Perron eigenvalue $\beta \approx 1.465571$, so

$$\dim_H(\mathcal{C}(1,19)) = \log_3 \beta \approx 0.347934.$$

Example 4.4. We consider implementation of the algorithm for $\mathcal{C}(1,7,19)$. We start from the presentations of $\mathcal{C}(1,7)$ and $\mathcal{C}(1,19)$ in Example 4.1. Taking the label product gives us a presentation of $\mathcal{C}(1,7,19)$, which is shown in Figure 4.4.

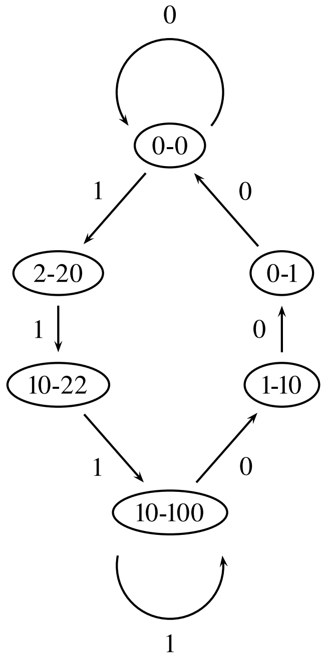

Figure 4.4. Path set presentation of $\mathcal{C}(1,7,19)$. The marked vertex is $0-0$.

This graph G for $\mathcal{C}(1,7,19)$ has adjacency matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Perron eigenvalue $\beta \approx 1.46557$ of this matrix is the largest real root of $\lambda^6 - 2\lambda^5 + \lambda^4 - 1 = 0$: The Hausdorff dimension of $\mathcal{C}(1,7,19)$ is then

$$\dim_H(\mathcal{C}(1,7,19)) = \log_3 \beta \approx 0.347934. \tag{4.4}$$

Example 4.5. The set $\mathcal{C}(1, 43)$, with $N = 43 = (1121)_3$ has $M \equiv 1 \pmod 3$, but nevertheless has Hausdorff dimension 0. A presentation of the path set associated to $\mathcal{C}(1, 43)$ is given in Figure 4.5.

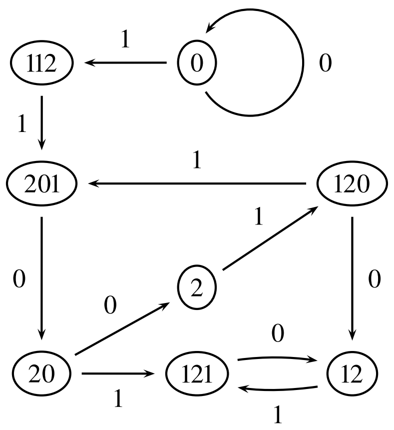

Figure 4.5. Path set presentation of $\mathcal{C}(1, 43)$. The marked vertex is 0.

The graph in Figure 4.5 has four strongly connected components, with vertex sets $\{0\}, \{112\}, \{2, 120, 201, 20\}$, and $\{12, 121\}$ respectively, each of whose topological entropy is 0. The vertex 0 is the marked initial state.

Example 4.6. A path set presentation of $\mathcal{C}(1, 16) = \mathcal{C}(1, (121)_3)$ is shown in Figure 4.6.

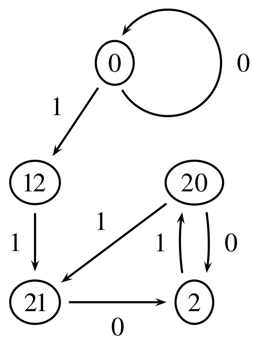

Figure 4.6. Path set presentation of $\Sigma_3 = \mathcal{C}(1)$. The marked vertex is 0.

This path set is not invariant under the shift operator, since the path $0110(01)^\infty$ (read left to right) is an element of the path set $X(1, 16)$, but the infinite path $(01)^\infty$ is not. In particular, this path set is not a one-sided sofic shift in the sense of [2].

5. Infinite families

5.1. Basic properties. We have the following simple result, showing the influence of the digits in the 3-adic expansion of M, M_i on the size of the sets $\mathcal{C}(1, M)$ and $\mathcal{C}(1, M_1, M_2, \ldots, M_n)$.

Theorem 5.1. (1) *If the smallest nonzero* 3*-adic digit in the* 3*-adic expansion of the positive integer* M *is* 2*, then* $\mathcal{C}(1, M) = \{0\}$*, and*

$$\dim_H(\mathcal{C}(1, M)) = 0. \tag{5.1}$$

(2) *If positive integers* $M_1, M_2, \ldots, M_n \in \Sigma_3$ *all have the property that their* 3*-adic expansions* $(M_i)_3$ (*equivalently their ternary expansions*) *contain only digits* 0 *and* 1*, then*

$$\dim_H(\mathcal{C}(1, M_1, M_2, \ldots, M_n)) > 0. \tag{5.2}$$

Remark. For neither (1) nor (2) does the converse hold. The example $M = 43 = (1121)_3$ has $\dim_H(\mathcal{C}(1, M)) = 0$, but its 3-adic expansion has smallest digit 1. The example $M = 64 = (2101)_3$ has $\dim_H(\mathcal{C}(1, M)) > 0$, but its 3-adic expansion has a digit 2.

Proof. of Theorem 5.1. (1) Suppose the smallest nonzero 3-adic digit in the 3-adic expansion of the positive integer M is 2. Then for any $N \in \Sigma_3$, the smallest nonzero digit of MN is 2, so $MN \notin \Sigma_3$. Thus, $\mathcal{C}(1, M) = \{0\}$, hence $\dim_H(\mathcal{C}(1, M)) = 0$.

(2) Suppose $M_1, \ldots, M_n \in \Sigma_3$ are positive integers such that all of their 3-adic expansions have only the digits 0 and 1. For each M_i, let m_i be the largest nonzero ternary position of M_i (i.e. $M_i = 3^{m_i} +$ *lower order terms*). Then in the graph presentation constructed for $X(1, M_i)$ by Algorithm A, the walk starting at the origin, then moving along an edge labeled 1 (which exists since $(M_i)_3$ omits the digit 2), then moving along m_i consecutive edges labeled 0, is a directed cycle at 0. Since the edge labeled 0 is a loop at 0, if we let $m = \max_{1 \le i \le n} m_i$, then the graph presentation of the path set $X(1, M_1, \ldots, M_n)$ of $\mathcal{C}(1, M_1, \ldots, M_n)$ has a directed cycle at 0 of length $m + 1$ given by first traversing the edge labeled 1, then traversing m consecutive edges labeled 0. This cycle and the loop of length one at 0 are distinct directed cycles at 0. It follows that the associated path set has positive topological entropy, and hence $\mathcal{C}(1, M_1, \ldots, M_n)$ has positive Hausdorff dimension by [3, Theorem 3.1 (iii)]. □

5.2. The family $L_k = (1^k)_3 = \frac{1}{2}(3^k - 1)$. The path set presentations $(\mathcal{G}, v_0)$ of the sets $X(1, L_k)$ are particularly simple to analyze.

Theorem 5.2. (1) *For $k \geq 1$, and $L_k = \frac{1}{2}(3^k - 1)$, there holds*

$$\dim_H(\mathcal{C}(1, L_k)) = \log_3 \beta_k, \tag{5.3}$$

where β_k is the unique real root greater than 1 *of*

$$\lambda^k - \lambda^{k-1} - 1 = 0. \tag{5.4}$$

(2) *For $k \geq 6$, the values β_k satisfy the bounds*

$$1 + \frac{\log k}{k} - \frac{2 \log \log k}{k} \leq \beta_k \leq 1 + \frac{\log k}{k}. \tag{5.5}$$

Then for all $k \geq 3$,

$$\dim_H(\mathcal{C}(1, L_k)) = \frac{\log_3 k}{k} + O\Big(\frac{\log \log k}{\log k}\Big). \tag{5.6}$$

Table 5.1. Hausdorff dimensions of $\mathcal{C}(1, L_k)$ (six decimal places).

Set	Perron eigenvalue	Hausdorff dim
$\mathcal{C}(1, L_1)$	2.000000	0.630929
$\mathcal{C}(1, L_2)$	1.618033	0.438018
$\mathcal{C}(1, L_3)$	1.465571	0.347934
$\mathcal{C}(1, L_4)$	1.380278	0.293358
$\mathcal{C}(1, L_5)$	1.324718	0.255960
$\mathcal{C}(1, L_6)$	1.285199	0.228392
$\mathcal{C}(1, L_7)$	1.255423	0.207052
$\mathcal{C}(1, L_8)$	1.232055	0.189948
$\mathcal{C}(1, L_9)$	1.213150	0.175877

We first analyze the structure of the directed graph $(\mathcal{G}, v_0)$ in this presentation.

Proposition 5.3. *For $L_k = (1^k)_3 = \frac{1}{2}(3^k - 1)$ the path set $X(1, L_k)$ has a presentation $(\mathcal{G}, v_0)$ given by Algorithm A which has exactly k vertices. The vertices v_m have labels $m = 0$ and $m = (1^j)_3$, for $1 \leq j \leq k-1$. The underlying directed graph G is strongly connected and primitive.*

Proof. The presentation $(\mathcal{G}, v_0)$ of $X(1, L_k)$ has an underlying directed graph G having k vertices v_N with $N = 0$ and $N = (1^j)_3$ for $1 \le j \le k-1$. The vertex v_0 has two exit edges labeled 0 and 1, and all other vertices have a unique exit edge labeled 0. The edges form a self-loop at 0 labeled 0, and a directed k-cycle, whose vertex labels are

$$0 \longrightarrow (1^{k-1})_3 \longrightarrow (1^{k-2})_3 \longrightarrow \cdots \longrightarrow (1^2)_3 \longrightarrow (1)_3 \longrightarrow 0.$$

This cycle certifies strong connectivity of the graph G, and in it all edge labels are 0 except the edge $0 \to (1^{k-1})_3$ labeled 1. Primitivity follows because it has a cycle of length 1 at vertex $(0)_3$. □

Proof of Theorem 5.2. (1) By appropriate ordering of the vertices, the adjacency matrix $\mathbf{A}$ of G is the $k \times k$ matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$p_k(\lambda) := \det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{pmatrix} \lambda - 1 & -1 & 0 & \dots & 0 \\ 0 & \lambda & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \lambda & -1 \\ -1 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

Expansion of this determinant by minors on the first column yields

$$p_k(\lambda) = (\lambda - 1)\lambda^{k-1} + (-1)^{k-1}(-1)(-1)^{k-1} = \lambda^k - \lambda^{k-1} - 1. \tag{5.7}$$

The Perron eigenvalue of the nonnegative matrix $\mathbf{A}$ will be a positive real root $\alpha_k \ge 1$ of $p(\lambda)$. By (3.1) the topological entropy of the path set $X(1, L_k)$ associated to $C(1, L_k)$ is $\log \beta_k$, while by Proposition 3.3 the Hausdorff dimension of the 3-adic path set fractal $C(1, L_k)$ itself is $\log_3 \beta_k$.

(2) We estimate the size of β_k. There is at most one real root $\beta_k \geq 1$ since for $\lambda > 1 - 1/k$ one has

$$p_k'(\lambda) = k\lambda^{k-1} - (k-1)\lambda^{k-2} = \lambda^{k-2}(k\lambda - (k-1)) > 0.$$

For the lower bound, we consider $p_k(\lambda)$ for $\lambda > 1$ and define variables $y > 0$, $x > 1$ by $\lambda = 1 + \frac{y}{k}$ with $y > 0$, and $x := \lambda^k > 1$, noting that $x = \lambda^k = (1+\frac{y}{k})^k < e^y$. Now

$$\lambda^{k-1} + 1 = \frac{x}{1+\frac{y}{k}} + 1 \geq x\Big(1 - \frac{y}{k}\Big) + 1 \geq x + \Big(1 - \frac{xy}{k}\Big),$$

which exceeds x whenever $xy \leq k$. Thus we have $p_k(1+\frac{y}{k}) < 0$ whenever $xy < ye^y \leq k$. The choice $y = \log k - 2\log\log k$ gives, for $k \geq 3$,

$$ye^y \leq \log k(e^{\log k - 2\log\log k}) \leq \frac{k}{\log k} \leq k.$$

Thus we have, for $k \geq 3$, $p_k(1 + \frac{\log k}{k} - 2\frac{\log\log k}{k}) < 0$, so

$$\beta_k \geq 1 + \frac{\log(k)}{k} - 2\frac{\log\log k}{k},$$

which is the lower bound in (5.5).

For the upper bound, it suffices to show $p_k(1+\frac{\log k}{k}) > 0$ for $k \geq 6$. We wish to show $(1+\frac{\log k}{k})^{k-1}(\frac{\log k}{k}) > 1$ for $k \geq 6$. This becomes $(1+\frac{\log k}{k})^{k-1} > \frac{k}{\log k}$, and on taking logarithms requires

$$(\log k - 1)\log\Big(1 + \frac{\log k}{k}\Big) > \log k - \log\log k.$$

Using the approximation $\log(1+w) \geq w - \frac{1}{2}w^2$ valid for $0 < w < 1$, we verify this inequality holds for $k \geq 6$, and the upper bound in (5.5) follows. The asymptotic estimate (5.6) for the Hausdorff dimension of $\mathcal{C}(1, L_k)$ immediately follows by taking logarithms to base 3 of the estimates above.

□

The results above imply Theorem 2.2 in the introduction.

Proof of Theorem 2.2. Assertion (1) follows from Proposition 5.3. Assertions (2) and (3) follow from Theorem 5.2. □

Remark 5.4. The path set corresponding to $\mathcal{C}(1, L_k)$ is easily seen to be shift-invariant under the one-sided shift. It is actually a one-sided shift of finite type, i.e. it is characterized as all strings not containing any of a finite set of forbidden blocks. It is easy to see that the 3-adic expansions in $\mathcal{C}(1, L_k)$ are the set of all elements of $\Sigma_{3,\bar{2}}$ that do not contain any of the blocks 10^j1 with $j \in \{0, 1, \ldots, k-1\}$.

5.3. The family $N_k = (10^{k-1}1)_3 = 3^k + 1$. We prove the following result.

Theorem 5.5. *For every integer $k \geq 0$, and $N_k = 3^k + 1 = (10^{k-1}1)_3$,*

$$\dim_H(\mathcal{C}(1, N_k)) = \dim_H \mathcal{C}(1, (10^{k-1}1)_3) = \log_3\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.438018. \quad (5.8)$$

To prove this result we first characterize the presentation $\mathcal{G} = (G, v_0)$ associated to N_k by the construction of Theorem 4.1.

Proposition 5.6. *For $N_k = 3^k + 1$ the path set $X(1, N_k)$ has a presentation*

$$\mathcal{G} = (G, v_0)$$

given by Algorithm A with the following properties.

(1) *The vertices v_m have labels m that comprise those integers*

$$0 \leq m \leq \frac{1}{2}(3^k - 1)$$

whose 3-adic expansion $(m)_3$ omits the digit 2.

(2) *The directed graph G has exactly 2^k vertices.*

(3) *The directed graph G is strongly connected and primitive.*

Proof. (1) Any vertex v_m reachable from v_0 has a 3-adic expansion (equivalently ternary expansion) $(m)_3$ that omits the digit 2, and has at most k 3-adic digits. This is proved by induction on the number of steps n taken. The base case has the vertex $(0)_3$. For the induction step, every vertex in the graph has an exit edge labeled 0, and vertices with labels $m \equiv 0 \pmod 3$ also have an exit edge labeled 1. The exit edges labeled 0 map $m = (b_{k-1}b_{k-2}\ldots b_1b_0)_3$ to $m' = (0b_{k-1}b_{k-2}\ldots b_2b_1)_3$. The exit edges labeled 1 map m to $m' = (1b_{k-1}b_{k-2}\ldots b_2b_1)_3$. For both types of exit edges the new vertex reached at the next step omits the digit 2 from its 3-adic expansion, completing the induction step.

(2) There are exactly 2^k possible such vertex labels m in which $(m)_3$ omits the digit 2. Call such vertex labels *admissible*. The largest such $m = \frac{1}{2}(3^k - 1)$.

(3) To show the graph G_k is strongly connected it suffices to establish that

- (R1) every possible such vertex v_m with admissible label m is reachable by a directed path in G from the initial vertex $0 = (00\ldots0)_3$;
- (R2) all admissible vertices v_m have a directed path in G from v_m to v_0.

Note that (R1) and (R2) together imply that G is strongly connected. To show (R1), write $m = (b_{k-1} \dots b_0)_3$, with all $b_j = 0$ or 1, and let i be the smallest index with $b_i = 1$. Starting from v_0, we may add a directed series of exit edges labeled in order $b_i, b_{i+1}, b_{i+2}, \dots, b_{k-1}$ to arrive at v_m. Such edges exist in G, because all intermediate vertices $v_{m'}$ reached along this path have $m' \equiv 0 \pmod 3$, so that exit edges labeled both 0 and 1 are admissible at that step. Indeed, the j-th step in the path has $(m_j)_3$ having $k - j$ initial 3-adic digits of 0, and $k - 1 - i \leq k - 1$.

To show (R2) we observe that for any vertex v_m, following a path of exit edges all labeled 0 will eventually arrive at the vertex v_0. This is permissible since $(m)_3$ has all digits 0 or 1. Now G_k is strongly connected, and it is primitive since it has a loop at vertex 0. This completes the proof. □

To obtain an adjacency matrix for this graph, we must choose a suitable ordering of the vertex labels. Order the vertices of $\mathcal{G}$ recursively as follows: the $(0^{k-1})_3$-vertex is first, I_1, and the $(10^{k-1})_3$-vertex is second, I_2. Now, suppose that at step j we have ordered the vertices $I_1, \dots, I_m$, in that order, with $m = 2^j$. Then for $1 \leq j < k$, we assert that there will be precisely $2m$ vertices, all distinct from $I_1, \dots, I_m$, to which some I_i has an out edge. We can label these $J_{11}, J_{12}, \dots, J_{m1}, J_{m2}$ so that J_{i1} has an in-edge labeled 0 from I_i, and J_{i2} has an in-edge labeled 1 from I_i. Assuming this assertion, at the j-th step we expand our ordering to $I_1 \dots, I_m, J_{11}, J_{12}, \dots, J_{m1}, J_{m2}$.

Proposition 5.7. *The ordering of the vertices above is valid, and the adjacency matrix* $\mathbf{A}$ *of the underlying graph* G *of* $\mathcal{G}$ *is the* $2^k \times 2^k$ *matrix* $\mathbf{A} = (a_{ij})$,

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq i \leq 2^{k-1} \text{ and } j \in \{2i-1, 2i\}, \\ 1 & \text{if } 2^{k-1} < 1 \text{ and } j = 2(i - 2^{k-1}) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

This description is consistent and exhaustive, characterizing $\mathbf{A}$.

To illustrate this proposition, we have for $k = 3$,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. First, we address the ordering of the vertices of $\mathcal{G}$. According to the prescription of the proposition, $I_1 = (0)_3$, $I_2 = (10^{k-1})_3$. In the next step, there is an out-edge labeled 1 from vertex $(10^{k-1})_3$ to $(110^{k-2})_3$, and an out-edge labeled 0 from vertex $(10^{k-1})_3$ to vertex $(10^{k-2})_3$. This gives $I_3 = (110^{k-2})_3$, $I_4 = (10^{k-2})_3$. In general, for $k_1 + \cdots + k_r < k$ all nonnegative, if we have a vertex

$$(1^{k_1}0^{k_2}1^{k_3}\ldots 1^{k_r}0^{k-\Sigma k_i})_3,$$

then it has an out-edge labeled 1 to a vertex

$$(1^{k_1+1}0^{k_2}1^{k_3}\ldots 1^{k_r}0^{k-1-\Sigma k_i})_3$$

and an out-edged labeled 0 to a vertex

$$(1^{k_1}0^{k_2}1^{k_3}\ldots 1^{k_r}0^{k-1-\Sigma k_i})_3.$$

On the other hand, a vertex labeled

$$(1^{k_1}0^{k_2}1^{k_3}\ldots 1^{k_r})_3$$

ending in 1 has a single out-edge labeled 0 to the vertex

$$(1^{k_1}0^{k_2}1^{k_3}\ldots 1^{k_r-1})_3.$$

Thus, if an edge-walk originating at the 0-vertex has label $(e_r e_{r-1}\ldots e_1)_3$, the terminal vertex of this edge walk is the vertex $(e_r e_{r-1}\ldots e_1 0^{k-r})_3$. Now, for any vertex ending in 0, edges labeled 0 and 1 are both admissible, which means that an edge walk labeled $e_1 e_2 \ldots e_k$ is admissible for all values $e_j = 0$ or $e_j = 1$ for all $1 \le j \le k$. But this, then, says that all possible vertex labels from $\{0, 1\}^k$ are achieved. Moreover, we showed above that a vertex with label from $\{0, 1\}^k$ has out-edges only to other vertices labeled from $\{0, 1\}^k$, so this is precisely the set of vertices of $\mathcal{G}$. The r^{th} step of the vertex ordering procedure adds precisely those vertices which end in 0^{k-r}, of which there are $2^{r-1} = 2\cdot 2^{r-2}$. The procedure ends at the kth step with those vertices which end in 1. In all, there are 2^k vertices, one for each label from $\{0, 1\}^k$.

Now we can understand the definition of the coefficients a_{ij} of the adjacency matrix $\mathbf{A}$ of the underlying graph G of $\mathcal{G}$. Vertex $(0)_3$ maps into itself and vertex $(10^k)_3$, which are ordered first and second with respect to the ordering. Thus $a_{11} = a_{12} = 1, a_{1j} = 0$ for $j > 2$. Now suppose a vertex is ordered i^{th} (I_i) at the r^{th} stage, and $r \le k-1$, so that not all vertices have yet been ordered. There are 2^r vertices ordered so far (so $1 \le i \le 2^r$), and the $(r+1)^{st}$ stage of the construction orders the next 2^r vertices precisely so that the out-edges from vertex I_i go to vertices I_{2i-1} and I_{2i}. This gives the prescription for a_{ij} for $1 \le i \le 2^{k-1}$.

Observe that the vertices $I_{2^{k-1}+1}, I_{2^{k-1}+2}, \ldots, I_{2^k}$ have labels ending in 1. Such a vertex labeled m has a single out-edge to the vertex labeled $(m-1)/3$. But if m is the label of $I_{2^{k-1}+r}$, then $(m-1)/3$ is the label of I_{2r-1}. But $(2^{k-1}+r, 2r-1)$ can be rewritten $(i, 2(i-2^{k-1})-1)$. This gives the result. □

We are now ready to prove Theorem 5.5.

Proof of Theorem 5.5. Let $\mathbf{A}_k$ be the adjacency matrix of the presentation of the set $X(1, N_k)$ constructed via our algorithm. We directly find a strictly positive eigenvector $\mathbf{v}_k$ of $\mathbf{A}_k$ having $\mathbf{A}_k\mathbf{v}_k^T = (\frac{1+\sqrt{5}}{2})\mathbf{v}_k^T$. Here $\mathbf{v}_k$ is a $2^k \times 1$ row vector, with $\mathbf{v}_k^T$ its transpose, and let $\mathbf{v}_k^{(j)}$ denote its j-th entry. The Perron–Frobenius Theorem [13, Theorem 4.2.3] then implies that $\alpha = \frac{1+\sqrt{5}}{2}$ is the Perron eigenvalue of $\mathbf{A}_k$. Theorem 2.1 will then give us that

$$\dim_H(\mathcal{C}(1, N_k)) = \log_3\Big(\frac{1+\sqrt{5}}{2}\Big).$$

Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. We define the vector $\mathbf{v}_k$ recursively as follows:

(1) $\mathbf{v}_1 = (\phi, 1) = (\phi^1, \phi^0)$;

(2) If $\mathbf{v}_{j-1} = (\phi^{k_1}, \phi^{k_2}, \ldots, \phi^{k_{2^{j-1}}})$, then

$$\mathbf{v}_j = (\phi^{k_1+1}, \phi^{k_2+1}, \ldots, \phi^{k_{2^{j-1}}+1}, \phi^{k_1}, \phi^{k_2}, \ldots, \phi^{k_{2^{j-1}}}).$$

Note that $\mathbf{v}_j$ is obtained from $\mathbf{v}_{j-1}$ by adjoining $\phi\mathbf{v}_{j-1}$ to the front of $\mathbf{v}_{j-1}$.

We need now to check that $\mathbf{A}\mathbf{v}_k^T = \phi\mathbf{v}_k^T$. We will argue by induction on k. The base case is easy. Now observe that if we write

$$\mathbf{A_k} = \begin{pmatrix} T_k \\ B_k \end{pmatrix}$$

for T_k and B_k each $2^{k-1} \times 2^k$ blocks, then we have

$$B_{k+1} = \begin{pmatrix} B_k & 0 \\ 0 & B_k \end{pmatrix},$$

and

$$T_{k+1} = \begin{pmatrix} T_k & 0 \\ 0 & T_k \end{pmatrix}.$$

It follows easily from this and the definition of the vectors $\mathbf{v}_k$ that if $\mathbf{A}_k\mathbf{v}_k^T = \phi\mathbf{v}_k^T$, then $\mathbf{A}_{k+1}\mathbf{v}_{k+1}^T = \phi\mathbf{v}_{k+1}^T$. This proves the theorem. □

Proof of Theorem 2.3. Here (1) follows from Proposition 5.6, and (2) follows from Theorem 5.5. ☐

Remark 5.8. The path set corresponding to $\mathcal{C}(1, N_k)$ is also one-sided shift-invariant, and is a one-sided shift of finite type. It is easy to see that the 3-adic expansions of elements in $\mathcal{C}(1, N_k)$ are the set of all $\sum_{j=0}^{\infty} a_j 3^j \in \Sigma_3$ such that for every $j \geq 0$, $a_j a_{j+k} \neq 11$

Remark 5.9. Using the infinite family $N_k = 3^k + 1$ it is easy to see that $f(\lambda) := \dim_H(\mathcal{C}(1, \lambda))$ for $\lambda \in \mathbb{Z}_3$ is discontinuous in the 3-adic topology. In this topology we have $\lim_{k\to\infty} N_k = 1$. Theorem 5.5 now gives

$$\lim_{k\to\infty} \dim_H \left(\mathcal{C}\left(1, \frac{1}{N_k}\right)\right) = \log_3\left(\frac{1+\sqrt{5}}{2}\right) < \dim_H(\mathcal{C}(1,1)) = \log_3(2),$$

exhibiting a discontinuity at $\lambda = 1$.

6. Applications

6.1. Hausdorff dimension bounds for $\mathcal{C}(1, M_1, \ldots, M_n)$ with M_i in families. The path set structures of each of the three infinite families are compatible with each other, as a function of k, so that the associated $\mathcal{C}(1, M_1, \ldots, M_n)$ all have positive Hausdorff dimension. We treat them separately.

Theorem 6.1. *For the family $L_k = \frac{1}{2}(3^k - 1) = (1^k)_3$, for $1 \leq k_1 < k_2 < \cdots < k_n$, the pointed presentation $\mathcal{G}(0, \ldots, 0)$ of the path set $X(1, L_{k_1}, \ldots L_{k_m})$ associated to $\mathcal{C}(1, L_{k_1}, \ldots, L_{k_n})$ is isomorphic to the pointed graph $(\mathcal{G}_{k_n}, 0)$ presenting $X(1, L_{k_n})$. In particular*

$$\dim_H(\mathcal{C}(1, L_{k_1}, L_{k_2}, \ldots, L_{k_n})) = \dim_H(\mathcal{C}(1, L_{k_n})). \tag{6.1}$$

Proof. The presentation $(\mathcal{G}_k, 0)$ of $\mathcal{C}(1, L_k)$ constructed with Algorithm A consists of a self-loop at the 0-vertex and a cycle of length k at the 0-vertex. Taking in Algorithm B the label product $\mathcal{G}_{k_1} \star \cdots \star \mathcal{G}_{k_n}$ gives a graph $\mathcal{G}$ with a self-loop at the $(0, \ldots, 0)$-vertex plus a cycle of length k_n sending $(0, \ldots, 0)$ to $(1^{k_1-1}, \ldots, 1^{k_n-1})$ to $(1^{k_1-2}, \ldots, 1^{k_n-2})$ which eventually reaches $(0, \ldots, 0, 1)$ and $(0, 0, \ldots, 0)$. The graph $\mathcal{G}$ is isomorphic to $\mathcal{G}_{k_n}$ by an isomorphism sending $(0, \ldots, 0)$ to 0. ☐

We next treat multiple intersections drawn from the second family N_k.

Theorem 6.2. *For the family* $N_k = 3^k + 1 = (10^{k-1}1)_3$ *the following hold.*

(1) *For* $1 \le k_1 < k_2 < \cdots < k_n$, *one has*

$$\dim_H(\mathcal{C}(1, N_{k_1}, N_{k_2}, \ldots, N_{k_n})) \ge \dim_H(\mathcal{C}(1, L_{k_n+1})) \tag{6.2}$$

Equality holds when $k_j = j$ *for* $1 \le j \le n$.

(2) *For fixed* $n \ge 1$, *choosing* N_{k-1+j} $(1 \le j \le n)$ *one has*

$$\liminf_{k\to\infty} \dim_H(\mathcal{C}(1, N_k, N_{k+1}, \ldots, N_{k+n-1})) \ge \frac{1}{2}(\log_3 2) \approx 0.315464. \tag{6.3}$$

In particular, $\Gamma_\star \ge \frac{1}{2}(\log_3 2)$.

Proof. (1) It is easy to see that the set $\mathcal{C}(1, N_{k_1}, N_{k_2}, \ldots, N_{k_n})$ contains the set

$$Y_{k_n} := \Big\{\lambda = \sum_{j=1}^{\infty} 3^{\ell_1+\cdots+\ell_j} \in \mathbb{Z}_3 \colon \text{all } \ell_j \ge k_n + 1\Big\},$$

where here we allow finite sums, corresponding to some $\ell_j = +\infty$. This fact holds by observing that if $\lambda \in Y_{k_n}$ then $N_{k_j}\lambda \in \Sigma_{3,\bar{2}}$ for $1 \le j \le n$, because

$$N_{k_j}\lambda = \Big(\sum_{j=1}^{\infty} 3^{\ell_1+\cdots+\ell_j}\Big) + \Big(\sum_{j=1}^{\infty} 3^{\ell_1+\cdots+\ell_j+k_j}\Big),$$

and the 3-adic addition has no carry operations since all exponents are distinct. The set Y_{k_n} is a 3-adic path set fractal and it is easily checked to be identical with $\mathcal{C}(1, L_{n_k+1})$, using the structure of its associated graph. This proves (6.2). To show equality holds, one must show that allowable sequences for each of $N_1, N_2, \ldots, N_n$ require gaps of size at least $n + 1$ between each successive nonzero 3-adic digit in an element of $\mathcal{C}(1, N_1, N_2, \ldots, N_n)$. This can be done by induction on the current non-zero 3-adic digit; we omit details.

(2) We study the symbolic dynamics of the elements of the underlying path sets in $\mathcal{C}(1, N_{k+j-1})$, for $1 \le j \le n$, given in Theorem 5.5, and use this to lower bound the Hausdorff dimension.

Claim. *The 3-adic path set underlying* $\mathcal{C}(1, N_k, \ldots, N_{k+n})$ *contains all symbol sequences which, when subdivided into successive blocks of length* $2k + n$, *have every such block of the form*

$$(00\ldots00a_k a_{k-1}\ldots a_3 a_2 1)_3 \quad \textit{with each } a_i \in \{0, 1\}.$$

Proof of claim. It suffices to show that all sequences split into blocks of length $2k + n$ of the form $(00\ldots 00a_k a_{k-1}\ldots a_3 a_2 1)_3$ occur in $\mathcal{C}(1, N_j)$ for each $k \leq j \leq k + n$, since this will imply the statement for the label product. Consider the presentation $(\mathcal{G}_j, 0)$ of $X(1, N_j)$ given by our algorithm. Beginning at the 0-vertex, an edge labeled 1 takes us to the state $(10^{j-1})_3$. From a vertex whose label ends in 0, one may traverse an edge with label 1 or 0. But if we are at a vertex labeled $a0$, an edge labeled 0 takes us to a vertex labeled a, and an edge labeled 1 takes us to a vertex labeled $1a$ (this is specific to the case of N_j). In other words, we apply the truncated shift map to our vertex label and either concatenate with 1 on the left or not. It follows that from the vertex $(10^{j-1})_3$ the next $j - 1$ edges traversed may be labeled either 0 or 1.

At this point the initial 1 from $(10^{j-1})_3$ has moved to the far right of our vertex label. Therefore, our choice is restricted: we must traverse an edge labeled 0. Since our vertex label, whatever it is, consists of only 0's and 1's, we can in any case traverse j or more consecutive edges labeled 0 to get back to the 0-vertex. Thus, first traversing an edge labeled 1, then traversing edges labeled 0 or 1 freely for the next $(k-1)$-steps, then traversing $k+n$ edges labeled 0 and returning to the 0-vertex, is possible in the graph $\mathcal{G}_j$ for each $k \leq j \leq k+n$. It follows that all sequences of the desired form are in each $\mathcal{C}(1, N_j)$, and hence in $\mathcal{C}(1, N_k \ldots, N_{k+n})$, proving the claim. ☐

With this claim in hand, we see that each block of size $(2k + n)$ contains at least 2^{k-2} admissible $(2k + n)$-blocks in $\mathcal{C}(1, N_k, \ldots, N_{k+n})$. We conclude that the maximum eigenvalue $\beta_{n,k}$ of the adjacency matrix of the graph $\mathcal{G}_{n,k}$ of $\mathcal{C}(1, N_k, N_{k+1}, \ldots, N_{k+n-1})$ must satisfy $(\beta_{n,k})^{2n+k} \geq 2^{k-2}$. This yields

$$\beta_{n,k} \geq 2^{\frac{k-2}{k+2n}}.$$

and hence $\liminf_{k\to\infty} \beta_{n,k} \geq \sqrt{2}$. The Hausdorff dimension formula in Proposition 3.3 then yields

$$\limsup_{k\to\infty} \dim_H(\mathcal{C}(1, N_k, \ldots, N_{k+n})) \geq \limsup_{k\to\infty} \log_3 \beta_{n,k} \geq \frac{1}{2}\log_3 2, \tag{6.4}$$

as asserted. The lower bound $\Gamma_\star \geq \frac{1}{2}\log_3 2$ follows immediately, see (1.14). ☐

6.2. Hausdorff dimension of the generalized exceptional set $\mathcal{E}_\star(\mathbb{Z}_3)$. Theorem 6.2 (2) shows that there are arbitrarily large families $\mathcal{C}(1, N_{k_1}, \ldots, N_{k_n})$ having Hausdorff dimension uniformly bounded below. We now show that if one properly restricts the choice of the N_{k_j} then one can directly choose an infinite set of N_{k_j} with this property; this observation is due to A. Bolshakov, whom we thank for allowing its inclusion here.

Theorem 6.3. (1) *Consider the subset Y of the 3-adic Cantor set $\Sigma_{3,\bar{2}}$ given by*

$$Y := \left\{\lambda := \sum_{j=0}^{\infty} a_j 3^j : \text{all } a_{2k} \in \{0,1\} \text{ and all } a_{2k+1} = 0\right\} \subset \mathbb{Z}_3.$$

It is a 3-adic path set fractal having $\dim_H(Y) = \frac{1}{2}\log_3 2 \approx 0.315464$. *This set satisfies*

$$Y \subset \mathcal{C}(1, N_{2k+1}), \quad \text{for all } k \geq 0,$$

where $N_k = 3^k + 1$, and in consequence

$$Y \subseteq \bigcap_{k=1}^{\infty} \mathcal{C}(1, N_{2k+1}).$$

(2) *Set $Z := \{\lambda \in \Sigma_{3,\bar{2}} \colon N_{2k+1}\lambda \in \Sigma_{3,\bar{2}}$ for all $k \geq 0\}$. Then*

$$\dim_H(Z) \geq \dim_H(Y) = \frac{1}{2}\log_3 2.$$

Proof. (1) The 3-adic path set fractal property of $Y \subset \Sigma_{3,\bar{2}}$ has the underlying graph of its symbolic dynamics pictured in Figure 6.1, certifying that it is a 3-adic path set fractal. The Perron eigenvalue of its adjacency matrix is $\sqrt{2}$, and its Hausdorff dimension is $\frac{1}{2}\log_3 2$ by Proposition 3.3.

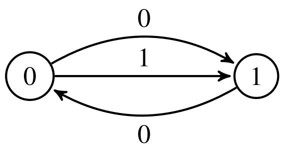

Figure 6.1. Presentation of Y.

The elements of Y can be rewritten in the form $\lambda = \sum_{j=0}^{\infty} b_{2j} 3^{2j}$, with all $b_{2j} \in \{0,1\}$. We then have

$$N_{2k+1}\lambda = \sum_{j=0}^{\infty} b_{2j} 3^{2j} + \sum_{j=0}^{\infty} b_{2j} 3^{2j+2k+1} \in \Sigma_{3,\bar{2}},$$

and the inclusion in the Cantor set $\Sigma_{3,\bar{2}}$ follows because the sets of 3-adic exponents in the two sums on the right side are disjoint, so there are no carry operations in combining them under 3-adic addition. This establishes that $Y \subset \mathcal{C}(1, N_{2k+1})$.

(2) All elements $\lambda \in Y$ have $N_{2k+1}\lambda \in \Sigma_{3,\bar{2}}$ for all $k \geq 1$. Thus

$$Y \subset Z := \{\lambda \in \Sigma_{3,\bar{2}} \colon N_{2k+1}\lambda \in \Sigma_{3,\bar{2}} \text{ for all } k \geq 1\}.$$

Now $\dim_H(Z) \geq \dim_H(Y) \geq \frac{1}{2}\log_3 2$, by (1). □

Proof of Theorem 2.4. The generalized exceptional set $\mathcal{E}_*(\mathbb{Z}_3)$ has Hausdorff dimension bounded below by

$$\dim_H(\mathcal{E}_*(\mathbb{Z}_3)) \geq \dim_H(Z) \geq \frac{1}{2}\log_3 2 \approx 0.315464,$$

the last inequality being Theorem 6.3 (2). □

6.3. Bounds for approximations to the exceptional set $\mathcal{E}(\mathbb{Z}_3)$. We present numerical results concerning Hausdorff dimensions of the upper approximation sets $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ to the exceptional set $\mathcal{E}(\mathbb{Z}_3)$. Recall that the only powers of 2 that are known to have ternary expansions that omit the digit 2 are $2^0 = 1 = (1)_3$, $2^2 = 4 = (11)_3$, and $2^8 = 256 = (10111)_3$.

Theorem 6.4. *The following bounds hold for sets $\mathcal{E}^{(k)}(\mathbb{Z}_3)$:*

$$\dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3)) \geq \log_3\Big(\frac{1+\sqrt{5}}{2}\Big) \approx 0.438018$$

and

$$\dim_H(\mathcal{E}^{(3)}(\mathbb{Z}_3)) \geq \log_3 \beta_1 \approx 0.228392,$$

where $\beta_1 \approx 1.28520$ is a root of $\lambda^6 - \lambda^5 - 1 = 0$.

Proof. We have

$$\dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3)) \geq \dim_H(\mathcal{C}(2^0, 2^2)) = \log_3\Big(\frac{1+\sqrt{5}}{2}\Big)$$

and

$$\dim_H(\mathcal{E}^{(3)}(\mathbb{Z}_3)) \geq \dim_H(\mathcal{C}(2^0, 2^2, 2^8)) = \log_3 \beta_1 \approx 0.228392,$$

where $\beta_1 \approx 1.28520\ldots$ is a root of $\lambda^6 - \lambda^5 - 1 = 0$. □

We conclude with results about the sets $\mathcal{C}(1, 2^{m_1}, \ldots, 2^{m_n})$ obtained via Algorithm A and Algorithm B. Since we are interested in cases of positive Hausdorff dimension we may suppose all $m_i \equiv 0 \pmod 2$ so that $2^{m_i} \equiv 1 \pmod 3$. The data in Table 6.1 reveals that the Hausdorff dimension of $\mathcal{C}(1, 2^{2n})$ oscillates as n increases.

Table 6.1. Hausdorff dim. of $\mathcal{C}(1, 2^{m_1}, \ldots, 2^{m_k})$ (six decimal places).

Set	Hausdorff dim.	Set	Hausdorff dim.
$\mathcal{C}(1, 2^2)$	0.438018	$\mathcal{C}(1, 2^2, 2^4)$	0
$\mathcal{C}(1, 2^4)$	0.255960	$\mathcal{C}(1, 2^2, 2^6)$	0
$\mathcal{C}(1, 2^6)$	0.278002	$\mathcal{C}(1, 2^2, 2^8)$	0.228392
$\mathcal{C}(1, 2^8)$	0.287416	$\mathcal{C}(1, 2^2, 2^{10})$	0
$\mathcal{C}(1, 2^{10})$	0.215201	$\mathcal{C}(1, 2^4, 2^6)$	0
$\mathcal{C}(1, 2^{12})$	0.244002	$\mathcal{C}(1, 2^4, 2^8)$	0
$\mathcal{C}(1, 2^{14})$	0.267112	$\mathcal{C}(1, 2^4, 2^{10})$	0

In addition to the data presented in Table 6.1, we computed the Hausdorff dimensions of $\mathcal{C}(1, 2^6, 2^{2k})$ for $8 \leq 2k \leq 14$ and of $\mathcal{C}(1, 2^2, 2^8, 2^{2k})$ for $10 \leq 2k \leq 16$ and found these dimensions to be 0 in all cases. It is unclear whether $\dim_H(\mathcal{E}^{(k)}(\mathbb{Z}_3))$ is positive for any $k \geq 4$.

Acknowledgments. We thank the reviewer for helpful comments, noting that the two infinite families L_k and N_k in Section 5 have symbolic dynamics given by shifts of finite type. We thank A. Bolshakov for permitting us to include his result Theorem 6.3.

References

[1] A. G. Abercrombie, The Hausdorff dimension of some exceptional sets of p-adic integer matrices. *J. Number Theory* **53** (1995), 311–341. MR 1348767 Zbl 0837.11042

[2] W. C. Abram and J. C. Lagarias, Path sets in one-sided symbolic dynamics. *Adv. in Appl. Math.* **56** (2014), 109–134. MR 3194209 Zbl 06335236

[3] W. C. Abram and J. C. Lagarias, p-adic path set fractals and arithmetic. *J. Fractal Geom.* **1** (2014), 45–81. MR 3166206 Zbl 1291.11107

[4] W. C. Abram, A. Bolshakov, and J. C. Lagarias, Intersections of multiplicative translates of 3-adic Cantor sets, II. In preparation.

[5] G. J. Davis and T.-Y. Hu, On the structure of the intersection of two middle thirds Cantor sets. *Publ. Mat.* **39** (1995), 43–60. MR 1336355 Zbl 0841.28001

[6] G. Edgar, *Measure, topology and fractal geometry.* 2nd ed. Undergraduate Texts in Mathematics. Springer, New York, 2008. MR 2356043 Zbl 1152.28008

[7] P. Erdős, Some unconventional problems in number theory. *Math. Mag.* **52** (1979), 67–70. MR 0527408 Zbl 0407.10001

[8] H. Furstenberg, Intersections of Cantor sets and transversality of semigroups. In R. C. Gunning (ed.), *Problems in analysis*. A symposium in honor of Salomon Bochner, Princeton University, Princeton, N.J., 1–3 April 1969. Princeton University Press, Princeton, N.J., 1970, 41–59. MR 0354562 MR 0337450 (collection) Zbl 0208.32203 Zbl 0208.00301 (collection)

[9] J. Hawkes, Some algebraic properties of small sets. *Quart. J. Math. Oxford Ser.* (2) **26** (1975), 195–201. MR 0374391 Zbl 0315.28002

[10] R. Kenyon and Y. Peres, Intersecting random translates of invariant Cantor sets. *Invent. Math.* **104** (1991), 601–629. MR 1106751 Zbl 0745.28012

[11] J. C. Lagarias, Ternary expansions of powers of 2. *J. Lond. Math. Soc.* (2) **79** (2009), 562–588. MR 2506687 Zbl 1193.11008

[12] D. Lind, The entropies of topological Markov shifts and a related class of algebraic integers. *Ergodic Theory Dynam. Systems* **4** (1984), 283–300. MR 0766106 Zbl 0546.58035

[13] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995. MR 1369092 Zbl 1106.37301

[14] K. Mahler, *Lectures on diophantine approximations*. Part I. g-adic numbers and Roth's theorem. Prepared from the notes by R. P. Bambah of the lectures given at the University of Notre Dame in the Fall of 1957. University of Notre Dame Press, Notre Dame, IN, 1961. MR 0142509 Zbl 0158.29903

[15] R. D. Mauldin and M. Urbański, *Graph directed Markov systems*. Geometry and dynamics of limit sets. Cambridge Tracts in Mathematics, 148. Cambridge University Press, Cambridge, 2003. MR 2003772 Zbl 1033.37025

[16] R. D. Mauldin and S. C. Williams, Random recursive constructions: asymptotic geometric and topological properties. *Trans. Amer. Math. Soc.* **295** (1986), 325–346. MR 0831202 Zbl 0625.54047

[17] R. D. Mauldin and S. C. Williams, Hausdorff dimension in graph directed constructions. *Trans. Amer. Math. Soc.* **309** (1988), 811–829. MR 0961615 Zbl 0706.28007

[18] B. Weiss, Subshifts of finite type and sofic systems. *Monatsh. Math.* **77** (1973), 462–474. MR 0340556 Zbl 0285.28021

Received August 15, 2013

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