Minkowski dimension of Brownian motion with drift

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Abstract. We study fractal properties of the image and the graph of Brownian motion in $\mathbb{R}^d$ with an arbitrary càdlàg drift f. We prove that the Minkowski (box) dimension of both the image and the graph of $B + f$ over $A \subseteq [0, 1]$ are a.s. constants. We then show that for all $d \geq 1$ the Minkowski dimension of $(B + f)(A)$ is at least the maximum of the Minkowski dimension of $f(A)$ and that of $B(A)$. We also prove analogous results for the graph. For linear Brownian motion, if the drift f is continuous and $A = [0, 1]$, then the corresponding inequality for the graph is actually an equality.

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1. Introduction

Let (B_t) be a standard Brownian motion in $\mathbb{R}^d$ and $f : [0, 1] \to \mathbb{R}^d$ a càdlàg function. By the Cameron-Martin theorem, the law of $B + f$ is equivalent to the law of B when f is in the Dirichlet space

$$D[0, 1] = \left\{ f \in C[0, 1] : f(t) = \int_0^t g(s)ds \text{ for some function } g \in L^2[0, 1] \right\},$$

and singular to the law of B otherwise.

In [11] it is shown that if f is any continuous function, then the Hausdorff dimension of the image and the graph of $B + f$ are almost surely constants. In the same paper it is also proved that if A is a closed subset of $[0, 1]$, then the Hausdorff dimension of $(B + f)(A)$ is at least $\max\{\dim_H B(A), \dim_H f(A)\}$ and similarly for the dimension of the graph of $B + f$ over A.

In this paper we prove analogous results for the Minkowski (or otherwise called box) dimension of the same sets. We would like to emphasize that the presence of the drift f implies that we cannot use techniques relying on self-similarity of the paths.

Before stating our main results, we recall the definition of Minkowski dimension. For other equivalent definitions and properties see [4], Definition 3.1.

Definition 1.1. Let A be a non-empty bounded subset of $\mathbb{R}^d$. For $\varepsilon > 0$ let $P(A, \varepsilon)$ be the maximum number of disjoint balls of radius ε with centers in A:

$$P(A, \varepsilon) = \max\{k : \exists\, x_1, \ldots, x_k \in A \text{ s.t. } \mathcal{B}(x_i, \varepsilon) \cap \mathcal{B}(x_j, \varepsilon) = \emptyset \text{ if } i \neq j\}.$$

The *upper* and *lower Minkowski dimensions* of A are defined as

$$\overline{\dim}_M(A) = \limsup_{\varepsilon \to 0} \frac{\log P(A, \varepsilon)}{\log \varepsilon^{-1}} \quad \text{and} \quad \underline{\dim}_M(A) = \liminf_{\varepsilon \to 0} \frac{\log P(A, \varepsilon)}{\log \varepsilon^{-1}}$$

respectively. Whenever these two limits are equal, we call the common value the Minkowski dimension of A.

Let $f : [0, 1] \to \mathbb{R}^d$ be a càdlàg function and A a subset of $[0, 1]$. In this paper we first prove that the Minkowski dimension of the image and the graph of $B + f$ over the set A are a.s. constants. The 0–1 law (Theorem 2.1) from [11] used to prove the a.s. constancy of the Hausdorff dimension of $(B + f)(A)$ cannot be used to prove the a.s. constancy in this case, since the Minkowski dimension does not satisfy the countable stability property; this means that the Minkowski dimension of a countable union of sets is not in general the supremum of their dimensions.

Theorem 1.2. *Let (B_t) be a standard Brownian motion in d dimensions. Let $f : [0, 1] \to \mathbb{R}^d$ be a càdlàg function and let A be a subset of $[0, 1]$. Then, there exist constants c_1 and c_2 such that, almost surely,*

$$\underline{\dim}_M(B + f)(A) = c_1 \quad \text{and} \quad \overline{\dim}_M(B + f)(A) = c_2.$$

For a function $h : [0, 1] \to \mathbb{R}^d$ and a set $A \subseteq [0, 1]$ we denote by $G_A(h) = \{(t, h(t)) : t \in A\}$ the graph of h over A.

Theorem 1.3. *Let (B_t) be a standard Brownian motion in $d \geq 1$ dimensions, $f : [0, 1] \to \mathbb{R}^d$ a càdlàg function and A a subset of $[0, 1]$. Then, there exist constants c_3 and c_4 such that, almost surely,*

$$\underline{\dim}_M G_A(B + f) = c_3 \quad \text{and} \quad \overline{\dim}_M G_A(B + f) = c_4.$$

We prove Theorems 1.2 and 1.3 in Section 2 by relating the Minkowski dimension to the expected volume of the "sausage" around the graph or the image. In the same section we also give an alternative proof of Theorem 1.3 using Lévy's construction of Brownian motion.

Having established that the Minkowski dimension of the image and the graph of Brownian motion with a càdlàg drift are a.s. constants we show that adding a deterministic drift to the Brownian motion cannot decrease the dimension of the image and the graph.

Theorem 1.4. *Let (B_t) be a standard Brownian motion in $d \geq 1$ dimensions. Let A be a subset of $[0,1]$ and $f : [0,1] \to \mathbb{R}^d$ a càdlàg function. Then almost surely*

$$\overline{\dim}_M (B + f)(A) \geq \max\{\overline{\dim}_M B(A), \overline{\dim}_M f(A)\},$$

$$\underline{\dim}_M (B + f)(A) \geq \max\{\underline{\dim}_M B(A), \underline{\dim}_M f(A)\}.$$

McKean's theorem (see for instance [10], Theorem 4.33) states that if A is a closed subset of $[0, \infty)$, then $\dim_H B(A) = (2 \dim_H A) \wedge d$, where $\dim_H$ stands for the Hausdorff dimension. In the case of Minkowski dimension there cannot be such a formula as the following corollary shows.

Corollary 1.5. *Let (B_t) be a standard Brownian motion in $d \geq 1$ dimensions and $f : [0,1] \to \mathbb{R}^d$ a càdlàg function. Then, for every subset A of $[0,1]$, if $d = 1$, then almost surely,*

$$\underline{\dim}_M (B + f)(A) \geq \frac{2\underline{\dim}_M A}{\underline{\dim}_M A + 1}, \quad and \quad \overline{\dim}_M (B + f)(A) \geq \frac{2\overline{\dim}_M A}{\overline{\dim}_M A + 1}.$$

The lower bounds can be achieved. If $d \geq 2$, then the right hand side in these inequalities is replaced by $2\underline{\dim}_M A$ and $2\overline{\dim}_M A$ respectively.

Remark 1.6. Inequalities analogous to Corollary 1.5 for packing dimension of images $X(A)$ where X is a multi-parameter fractional Brownian motion were established by Talagrand and Xiao in [15].

We prove Theorem 1.4 and Corollary 1.5 in Section 3.

We now state our results concerning the Minkowski dimension of the graph of $B + f$. We prove them in Section 4.

Theorem 1.7. *Let (B_t) be a standard Brownian motion in d dimensions and let $f : [0,1] \to \mathbb{R}^d$ be a càdlàg function. Then, for every subset A of $[0,1]$, we have, almost surely,*

$$\overline{\dim}_M G_A(B + f) \geq \max\{\overline{\dim}_M G_A(B), \overline{\dim}_M G_A(f)\}$$

$$\underline{\dim}_M G_A(B + f) \geq \max\{\underline{\dim}_M G_A(B), \underline{\dim}_M G_A(f)\}.$$

In one dimension, when the drift function f is continuous and $A = [0, 1]$, equality is achieved in the inequalities of Theorem 1.7. This follows from [5], Lemmata 2.2 and 2.3, but we include the proof of the following theorem in Section 4 for the convenience of the reader.

Proposition 1.8. *Let (B_t) be a standard Brownian motion in one dimension and $f : [0, 1] \to \mathbb{R}$ a continuous function. Then, almost surely,*

$$\underline{\dim}_M G_{[0,1]}(B + f) = \max\{\underline{\dim}_M G_{[0,1]}(B), \underline{\dim}_M G_{[0,1]}(f)\},$$

$$\overline{\dim}_M G_{[0,1]}(B + f) = \max\{\underline{\dim}_M G_{[0,1]}(B), \overline{\dim}_M G_{[0,1]}(f)\}.$$

Remark 1.9. The equalities in Proposition 1.8 can fail if f is not continuous. In Section 5 we describe a càdlàg function f such that $\overline{\dim}_M G_{[0,1]}(f) = 5/3$ and $\underline{\dim}_M G_{[0,1]}(B + f) \geq 7/4$ a.s.

Remark 1.10. Even if f is continuous, the equality in Proposition 1.8 does not hold for Hausdorff dimension. An example is provided in [13], based on an explicit formula for the Hausdorff dimension of the graph of $B + f$.

Related results. Fractal properties of images $X(A)$, where X is a Lévy process or a multi-parameter fractional Brownian motion were investigated in [8], [9], [14], [15], and [17]. Here we restrict attention to Brownian motion; the new feature is the effect of the drift function f. Similar results for the Hausdorff dimension of graphs of sums of fractional Brownian motion and functions above sets can be found in the recent paper [1].

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2. A 0–1 law

In this section we prove Theorems 1.2 and 1.3 by first stating and proving a more general result for any càdlàg adapted process with stationary and independent increments. At the end of the section we give a second proof of Theorem 1.3 using Lévy's construction of Brownian motion.

We introduce some notation that will be used throughout the paper. If

$$g : \mathbb{R}_+ \longrightarrow \mathbb{R}^d$$

is a measurable function and A a subset of $[0, 1]$, then for any $r > 0$ we define

$$V_g(A, r) = \text{vol}\Big(\bigcup_{s \in A} \mathcal{B}(g(s), r) \Big), \tag{2.1}$$

where $\mathcal{B}(x, r)$ stands for the ball centered at x of radius r.

We are now ready to state the main result of this section.

Proposition 2.1. *Let $(\mathcal{F}_t)$ be a right continuous filtration and (X_t) a càdlàg adapted process taking values in $\mathbb{R}^d$, $d \geq 1$, with stationary and independent increments. Let $f : [0, 1] \to \mathbb{R}^d$ be a càdlàg function and A a subset of $[0, 1]$. Then almost surely we have*

$$\overline{\dim}_M(X + f)(A) = \limsup_{\varepsilon \to 0} \frac{\log \mathbb{E}[V_{X+f}(A, \varepsilon)]}{\log \frac{1}{\varepsilon}},$$

$$\underline{\dim}_M(X + f)(A) = \liminf_{\varepsilon \to 0} \frac{\log \mathbb{E}[V_{X+f}(A, \varepsilon)]}{\log \frac{1}{\varepsilon}}.$$

Before proving it we explain how Theorems 1.2 and 1.3 follow.

Proof of Theorem 1.2. Setting $X_t = B_t$, since Brownian motion satisfies the assumptions of Proposition 2.1, the theorem follows. □

Proof of Theorem 1.3. For $t \in \mathbb{R}_+$, let $X_t = (t, B_t)$ and $g(t) = (t, f(t))$. Then X and g clearly satisfy the assumptions of Proposition 2.1, and hence this finishes the proof. □

We now devote the rest of the section to the proof of Proposition 2.1. First we state a standard fact about Minkowski dimensions whose proof can be found e.g. in [4], Proposition 3.2.

Claim 2.2. *Let A be a bounded subset of $\mathbb{R}^d$. Then*

$$\overline{\dim}_M A = \limsup_{\varepsilon \to 0} \frac{\log \mathrm{vol}(A + \mathcal{B}(0, \varepsilon))}{\log \frac{1}{\varepsilon}} + d,$$

$$\underline{\dim}_M A = \liminf_{\varepsilon \to 0} \frac{\log \mathrm{vol}(A + \mathcal{B}(0, \varepsilon))}{\log \frac{1}{\varepsilon}} + d.$$

The main ingredient of the proof of Proposition 2.1 is the following lemma on the concentration of the volume of the sausage around its mean.

Lemma 2.3. *Let $(\mathcal{F}_t)$ be a right continuous filtration and (X_t) a càdlàg adapted process taking values in $\mathbb{R}^d$, $d \geq 1$, with stationary and independent increments. Let $f : [0, 1] \to \mathbb{R}^d$ be a càdlàg function and A a subset of $[0, 1]$. Then almost surely we have*

$$\limsup_{\varepsilon \to 0} \frac{\log V_{X+f}(A, \varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{\log \mathbb{E}[V_{X+f}(A, \varepsilon)]}{\log \frac{1}{\varepsilon}},$$

$$\liminf_{\varepsilon \to 0} \frac{\log V_{X+f}(A, \varepsilon)}{\log \frac{1}{\varepsilon}} = \liminf_{\varepsilon \to 0} \frac{\log \mathbb{E}[V_{X+f}(A, \varepsilon)]}{\log \frac{1}{\varepsilon}}.$$

Claim 2.4. *Let $(\mathcal{F}_t)$ be a right continuous filtration and (X_t) a càdlàg adapted process taking values in $\mathbb{R}^d$, $d \geq 1$. Let D be an open set in $\mathbb{R}^d$ and F a subset of $[0, 1]$. Then*

$$\tau = \inf\{t \in F : X_t \in D\}$$

is a stopping time.

Proof. Let F_∞ be a countable dense subset of F. Then for all $t \in [0, 1]$ we deduce

$$\{\tau < t\} = \bigcup_{q \in F_\infty, q < t} \{X_q \in D\},$$

since X is càdlàg and D is an open set. Hence $\{\tau < t\} \in \mathcal{F}_t$. Writing

$$\{\tau \leq t\} = \bigcap_n \{\tau < t + 1/n\},$$

we get that $\{\tau \leq t\} \in \mathcal{F}_{t+} = \mathcal{F}_t$. $\square$

Proof of Lemma 2.3. First notice that by the monotonicity of the volume we have

$$\limsup_{\varepsilon \to 0} \frac{\log V_{X+f}(A, \varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{k \to \infty} \frac{\log V_{X+f}(A, 2^{-k})}{\log 2^k}$$

and

$$\liminf_{\varepsilon \to 0} \frac{\log V_{X+f}(A, \varepsilon)}{\log \frac{1}{\varepsilon}} = \liminf_{k \to \infty} \frac{\log V_{X+f}(A, 2^{-k})}{\log 2^k}.$$

Hence it suffices to show that a.s. for all large enough k we have

$$\frac{1}{2k}\mathbb{E}[V_{X+f}(A, 2^{-k})] \leq V_{X+f}(A, 2^{-k}) \leq k^2\mathbb{E}[V_{X+f}(A, 2^{-k})].$$

The upper bound follows easily from Markov's inequality and the Borel-Cantelli lemma. We now show the lower bound. In fact, note that by Borel-Cantelli, it suffices to show that for all k we have

$$\mathbb{P}\left(V_{X+f}(A, 2^{-k}) \geq \frac{1}{2k}\mathbb{E}[V_{X+f}(A, 2^{-k})]\right) \geq 1 - \left(\frac{7}{8}\right)^k. \qquad (2.2)$$

We first show that for any measurable $F \subseteq \mathbb{R}_+$ and any $\delta > 0$ we have

$$\mathbb{E}[(V_{X+f}(F, \delta))^2] \leq 2(\mathbb{E}[V_{X+f}(F, \delta)])^2. \qquad (2.3)$$

For any $x \in \mathbb{R}^d$ we write

$$\tau_x = \inf\{t \in F : X(t) + f(t) \in \mathcal{B}(x, \delta)\}$$

with the convention that τ_x is infinite, if $X + f$ does not hit the ball $\mathcal{B}(x, \delta)$. We have

$$
\begin{aligned}
\mathbb{E}[(V_{X+f}(F, \delta))^2] &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{P}(\tau_x < \infty, \tau_y < \infty) dx \, dy \\
&= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{P}(\tau_x \leq \tau_y < \infty) dx \, dy \\
&= 2 \int_{\mathbb{R}^d} \mathbb{P}(\tau_x < \infty) \int_{\mathbb{R}^d} \mathbb{P}(\tau_x \leq \tau_y < \infty \mid \tau_x < \infty) \, dy \, dx \\
&= 2 \int_{\mathbb{R}^d} \mathbb{P}(\tau_x < \infty) \, \mathbb{E}[V_{X+f}(F \cap [\tau_x, \infty), \delta) \mid \tau_x < \infty] \, dx.
\end{aligned}
$$
$$(2.4)$$

Since X and f are càdlàg and the filtration is right continuous, it follows from Claim 2.4 that τ_x is a stopping time. By the stationarity, the independence of increments and the càdlàg property of X, we get that X satisfies the strong Markov property (see [2], Proposition I.6). Thus the conditional law of the process $\{X(\tau_x + s) - X(\tau_x)\}_{s \geq 0}$ given that $\{\tau_x < \infty\}$, is identical to the law of $\{X(s)\}_{s \geq 0}$. Let X' be a process independent of X but with the same law as X. The Markov property of X' and its independence from τ_x implies that the conditional law of the process $\{X'(\tau_x + s) - X'(\tau_x)\}_{s \geq 0}$ given that $\{\tau_x < \infty\}$, is also identical to the law of $\{X(s)\}_{s \geq 0}$. Therefore given $\tau_x < \infty$, the two random paths $\{X(t) - X(\tau_x)\}_{t \geq \tau_x}$ and $\{X'(t) - X'(\tau_x)\}_{t \geq \tau_x}$ have the same law. Since the volume is unaffected by translation,

$$
\begin{aligned}
\mathbb{E}\big[V_{X+f}(F \cap [\tau_x, \infty), \delta) \mid \tau_x < \infty\big] &= \mathbb{E}[V_{X'+f}(F \cap [\tau_x, \infty), \delta) \mid \tau_x < \infty] \\
&\leq \mathbb{E}[V_{X'+f}(F, \delta)[= \mathbb{E}[V_{X+f}(F, \delta)],
\end{aligned}
$$

and hence this together with (2.4) concludes the proof of (2.3).

Therefore, from (2.3), applying the second moment method to the random variable $V_{X+f}(F, \delta)$ we get that for any set F and any $\delta > 0$

$$\mathbb{P}\left(V_{X+f}(F, \delta) \geq \frac{1}{2}\mathbb{E}[V_{X+f}(F, \delta)]\right) \geq \frac{1}{8}. \tag{2.5}$$

We set $t_0 = 0$. It is easy to see that $\mathbb{E}[V_{X+f}(A \cap [0, t], 2^{-k})]$ is continuous as a function of t. Hence for $j = 1, \ldots, k$ we can define

$$t_j = \inf \left\{t \geq 0 : \mathbb{E}[V_{X+f}(A \cap [0, t], 2^{-k})] = \frac{j}{k}\mathbb{E}[V_{X+f}(A, 2^{-k})]\right\}$$

and we write

$$I_j = [t_{j-1}, t_j].$$

By the subadditivity property of the volume, we get that for all j

$$\mathbb{E}[V_{X+f}(A \cap I_j, 2^{-k})]$$

$$\geq \mathbb{E}[V_{X+f}(A \cap [0, t_j], 2^{-k})] - \mathbb{E}[V_{X+f}(A \cap [0, t_{j-1}], 2^{-k})]$$

$$= \frac{1}{k} \mathbb{E}[V_{X+f}(A, 2^{-k})].$$

Therefore we get

$$\mathbb{P}\Big(V_{X+f}(A, 2^{-k}) \geq \frac{1}{2k} \mathbb{E}[V_{X+f}(A, 2^{-k})]\Big)$$

$$\geq \mathbb{P}\Big(\exists j : V_{X+f}(A \cap I_j, 2^{-k}) \geq \frac{1}{2} \mathbb{E}[V_{X+f}(A \cap I_j, 2^{-k})]\Big)$$

$$= 1 - \prod_{j=1}^{k} \mathbb{P}\Big(V_{X+f}(A \cap I_j, 2^{-k}) < \frac{1}{2} \mathbb{E}[V_{X+f}(A \cap I_j, 2^{-k})]\Big)$$

$$\geq 1 - \Big(\frac{7}{8}\Big)^k,$$

where the equality follows by the independence of the increments of X and the last inequality follows from (2.5). This finishes the proof of (2.2), and hence concludes the proof of the lemma. $\quad\square$

Remark 2.5. We note that if X and f are càdlàg and A is a subset of $[0, 1]$, then $V_{X+f}(A, \varepsilon)$ is a random variable. Indeed, let A_∞ be a countable dense subset of A, then $V_{X+f}(A, \varepsilon) = V_{X+f}(A_\infty, \varepsilon)$. Now $V_{X+f}(A_\infty, \varepsilon) = \lim_{n\to\infty} V_{X+f}(A_n, \varepsilon)$, where A_n are finite sets. By the continuity of the volume (see [12], Lemma 4.1) $V_{X+f}(A_n, \varepsilon)$ is a random variable for each n.

Proof of Proposition 2.1. The statement of the proposition follows directly from Lemma 2.3 and Claim 2.2. $\quad\square$

2.1. Another proof of Theorem 1.3. In this section we give an alternative proof of Theorem 1.3 that relies on Lévy's construction of Brownian motion. The only properties of Minkowski dimension that are used in this proof are stability under finite unions and under adding linear functions.

Proposition 2.6. *Let* $f : [0, 1] \to \mathbb{R}^d$ *be a bounded measurable function, and* $\mu \in \mathbb{R}^d$. *Define* $g : [0, 1] \to \mathbb{R}^d$ *by*

$$g(t) = f(t) + \mu t.$$

Then, for every subset A of $[0, 1]$, *we have*

$$\underline{\dim}_M G_A(f) = \underline{\dim}_M G_A(g) \quad and \quad \overline{\dim}_M G_A(f) = \overline{\dim}_M G_A(g).$$

Proof. For $\varepsilon \in (0, \infty)$ and $k \in \mathbb{N}$ define

$$\mathcal{C}_\varepsilon(k) = [(k-1)\varepsilon, k\varepsilon] \times \{\text{some cube of edge length } \varepsilon \text{ in } \mathbb{R}^d\}$$

and

$$\mathcal{C}_\varepsilon = \bigcup_{k \in \mathbb{N}} \mathcal{C}_\varepsilon(k).$$

Write $N = \lceil \|\mu\|_\infty \rceil$, and consider a covering of $G_A(f)$ by cubes of $\mathcal{C}_\varepsilon$. Consider the cubes of the covering that are in $C_\varepsilon(k)$, and thus form a covering of $G_{A \cap [(k-1)\varepsilon, k\varepsilon]}(f)$. Clearly, shifting them by the vector $(0, \mu_1(k-1)\varepsilon, \dots, \mu_d(k-1)\varepsilon)$ produces a covering of

$$G_{A \cap [(k-1)\varepsilon, k\varepsilon]}(f + \mu(k-1)\varepsilon).$$

But within a time interval of length ε, the drift cannot move $f(t)$ by more than $N\varepsilon$ in any given direction. Therefore, $G_{A \cap [(k-1)\varepsilon, k\varepsilon]}(g)$ may be covered with N^d as many cubes of $\mathcal{C}_\varepsilon(k)$ as are required to cover $G_{A \cap [(k-1)\varepsilon, k\varepsilon]}(f)$. It follows that the covering number of $G_A(g)$ with elements of $\mathcal{C}_\varepsilon$ is at most N^d times that of $G_A(f)$. Therefore,

$$\underline{\dim}_M G_A(g) \le \underline{\dim}_M G_A(f) \quad and \quad \overline{\dim}_M G_A(g) \le \overline{\dim}_M G_A(f).$$

Since $f(t) = g(t) - \mu t$, the same argument shows that

$$\underline{\dim}_M G_A(f) \le \underline{\dim}_M G_A(g) \quad and \quad \overline{\dim}_M G_A(f) \le \overline{\dim}_M G_A(g),$$

and completes the proof. $\qquad\square$

The stability of Minkowski dimension under finite unions yields the following corollary.

Corollary 2.7. *Let* $f : [0, 1] \to \mathbb{R}^d$ *be a bounded measurable function, and*

$$h \colon [0, 1] \longrightarrow \mathbb{R}^d$$

be piecewise affine. Put $g = f + h$. *Then, for every subset A of* $[0, 1]$, *we have*

$$\underline{\dim}_M G_A(f) = \underline{\dim}_M G_A(g) \quad and \quad \overline{\dim}_M G_A(f) = \overline{\dim}_M G_A(g).$$

Proof of Theorem 1.3. We only prove the result for the lower Minkowski dimension. The proof for the upper Minkowski dimension is identical.

Consider Lévy's construction of Brownian motion as

$$B = \lim_{n \to \infty} Y_n = \lim_{n \to \infty} \sum_{k=1}^n X_k,$$

where $(X_k, k \in \mathbb{N})$ is an independent sequence of continuous piecewise affine random paths on $[0, 1]$, and the convergence is uniform on $[0, 1]$.

For $n \in \mathbb{N}$, put $Z_n = B - Y_{n-1}$. Since $B + f = Z_n + f + Y_{n-1}$ and Y_{n-1} is piecewise affine, Corollary 2.7 implies that $\underline{\dim}_M G_A(B + f) = \underline{\dim}_M G_A(Z_n + f)$. In particular, for any $a > 0$,

$$\{\underline{\dim}_M G_A(B + f) \leq a\} = \{\underline{\dim}_M G_A(Z_n + f) \leq a\} \in \sigma(X_k, k \geq n).$$

Since this is true for every n, it follows that

$$\Lambda_a = \{\underline{\dim}_M G_A(B + f) \leq a\} \in \mathcal{T} = \bigcap_n \sigma(X_k, k \geq n).$$

Therefore by Kolmogorov's 0–1 law $\mathbb{P}(\Lambda_a) \in \{0, 1\}$. It follows that the Minkowski dimension of $G_A(B + f)$ is almost surely constant. $\square$

Remark 2.8. An alternative proof of Theorem 1.2 can be obtained by combining the above proof with Howroyd's projection theorem [7], Theorem 14.

3. Dimension of the image of $B + f$

In this section we prove Theorems 1.4 and 1.5. We first recall Theorem 1.1 from Peres and Sousi [12], since it is going to be used to prove that the Minkowski dimension of the image and the graph of $B + f$ is larger than that of B.

Theorem 3.1 ([12]). *Let $(B(s))_{s \geq 0}$ be a standard Brownian motion in $d \geq 1$ dimensions and let $(D_s)_{s \geq 0}$ be open sets in $\mathbb{R}^d$. For each s, let $r_s > 0$ be such that $\mathrm{vol}(\mathcal{B}(0, r_s)) = \mathrm{vol}(D_s)$. Then for all t we have that*

$$\mathbb{E}\left[\mathrm{vol}\left(\bigcup_{s \leq t}(B(s) + D_s)\right)\right] \geq \mathbb{E}\left[\mathrm{vol}\left(\bigcup_{s \leq t}\mathcal{B}(B(s), r_s)\right)\right].$$

Definition 3.2. Let $G \subseteq \mathbb{R}^d$. We call a collection of balls $(\mathcal{B}(x_i, \varepsilon))_i$ an ε-packing of G if $x_i \in G$ for all i and the balls are pairwise disjoint.

Given an ε-packing of $f(A)$ by P balls with centers $(f(t_i))$ we want to construct an ε-packing of $(B + f)(A)$. The balls of radius ε centered at $(B + f)(t_i)$ might not all be disjoint; the following lemma controls the number of collisions.

Lemma 3.3. *Let (B_t) be a standard Brownian motion in d dimensions and let $f : [0, 1] \to \mathbb{R}^d$ be a bounded measurable function and A a subset of $[0, 1]$. Then there exists a positive constant c such that for all $\varepsilon > 0$, if $(\mathcal{B}(f(t_i), \varepsilon))_{i \leq P_\varepsilon}$ is an ε-packing of $f(A)$, then*

$$\max_{i \leq P_\varepsilon} \mathbb{E}[N_i] \leq c \log(1/\varepsilon)^{d+1},$$

where $N_i = \#\{j : |(B + f)(t_i) - (B + f)(t_j)| < 2\varepsilon\}$, for $i \in \{1, \ldots, P_\varepsilon\}$.

Proof. Let $\varepsilon > 0$ and $(\mathcal{B}(f(t_i), \varepsilon))_{i \leq P_\varepsilon}$ an ε-packing of $f(A)$. We fix $i \in \{1, \ldots, P_\varepsilon\}$. For every $k \in \mathbb{N}$ we define the sets

$$S(k) = \{j : |f(t_i) - f(t_j)| \in [2^k \varepsilon, 2^{k+1} \varepsilon)\},$$

$$S_1(k) = \left\{j \in S(k): |t_i - t_j| \geq \left(\frac{2^k \varepsilon}{\log(1/\varepsilon)}\right)^2\right\},$$

$$S_2(k) = S(k) \setminus S_1(k).$$

Since f is bounded, it follows that $S(k) = \emptyset$ whenever $k \geq c_1 \log(1/\varepsilon)$, for a positive constant c_1. Furthermore, if $j \in S(k)$, then since $(f(t_j))_j$ is an ε-packing of $f(A)$, the balls $\{\mathcal{B}(f(t_j), \varepsilon)\}_j$ are disjoint and for all such j the ball $\mathcal{B}(f(t_j), \varepsilon)$ is contained in $\mathcal{B}(f(t_i), (2^{k+1} + 1)\varepsilon)$. Therefore

$$|S(k)| \leq \frac{\text{vol}(\mathcal{B}(0, (2^{k+1} + 1)\varepsilon))}{\text{vol}(\mathcal{B}(0, \varepsilon))} \leq (2^{k+1} + 1)^d. \tag{3.1}$$

If we write

$$p(j) = \mathbb{P}(|(B + f)(t_i) - (B + f)(t_j)| < 2\varepsilon),$$

then by the definition of N_i we have

$$\mathbb{E}[N_i] = \sum_{j=1}^{P_\varepsilon} p(j) = \sum_{k=1}^{c_1 \log(1/\varepsilon)} \sum_{j \in S_1(k)} p(j) + \sum_{k=1}^{c_1 \log(1/\varepsilon)} \sum_{j \in S_2(k)} p(j). \tag{3.2}$$

If $j \in S_1(k)$, then for a positive constant c_2 we have

$$p(j) = \frac{1}{(2\pi |t_i - t_j|)^{d/2}} \int_{\mathcal{B}(f(t_i) - f(t_j), 2\varepsilon)} \exp\left\{-\frac{|x|^2}{2|t_i - t_j|}\right\} dx$$

$$\leq \frac{\log(1/\varepsilon)^d}{2^{dk} \varepsilon^d (2\pi)^{d/2}} \text{vol}(\mathcal{B}(0, 2\varepsilon)) \tag{3.3}$$

$$= c_2 \frac{\log(1/\varepsilon)^d}{2^{dk}}.$$

If $j \in S_2(k)$, then for a positive constant c_3 we have by the Gaussian tail estimate if $k \geq 2$

$$p(j) \leq \mathbb{P}(|B(t_i) - B(t_j)| > |f(t_i) - f(t_j)| - 2\varepsilon)$$

$$\leq \mathbb{P}(|B(t_i) - B(t_j)| > (2^k - 2)\varepsilon) \tag{3.4}$$

$$\leq 2 \exp(-c_3 (\log(1/\varepsilon))^2).$$

Plugging the estimates (3.3) and (3.4) in (3.2) and using (3.1) concludes the proof of the lemma. $\square$

Proof of Theorem 1.4. From Proposition 2.1 we infer that a.s.

$$\overline{\dim}_M(B + f)(A) = \limsup_{\varepsilon \to 0} \frac{\log \mathbb{E}[V_{B+f}(A, \varepsilon)]}{\log \frac{1}{\varepsilon}}, \tag{3.5}$$

$$\underline{\dim}_M(B + f)(A) = \liminf_{\varepsilon \to 0} \frac{\log \mathbb{E}[V_{B+f}(A, \varepsilon)]}{\log \frac{1}{\varepsilon}}. \tag{3.6}$$

Let $D_s = \mathcal{B}(f(s), \varepsilon)$ if $s \in A$ and $D_s = \emptyset$ if $s \notin A$. Then applying Theorem 3.1 we get

$$\mathbb{E}[V_{B+f}(A, \varepsilon)] \geq \mathbb{E}[V_B(A, \varepsilon)]. \tag{3.7}$$

From (3.5), (3.6) and (3.7) we deduce the a.s. inequalities

$$\overline{\dim}_M(B + f)(A) \geq \overline{\dim}_M B(A) \quad \text{and} \quad \underline{\dim}_M(B + f)(A) \geq \underline{\dim}_M B(A).$$

It now remains to show that almost surely

$$\overline{\dim}_M(B + f)(A) \geq \overline{\dim}_M f(A) \quad \text{and} \quad \underline{\dim}_M(B + f)(A) \geq \underline{\dim}_M f(A). \tag{3.8}$$

First we note that in the definition of upper and lower Minkowski dimension it suffices to take ε which is tending to 0 along powers of 2.

We fix k and consider a 2^{-k}-packing of $f(A)$ with $P_k(f)$ balls. Let the centers of the balls be $f(t_i)$ with $t_i \in A$ for $i \in \{1, \ldots, P_k(f)\}$.

For every $i \in \{1, \ldots, P_k(f)\}$ define

$$N_i = \#\{j \neq i : |(B + f)(t_i) - (B + f)(t_j)| < 2^{1-k}\}$$

and

$$G = \sum_{i=1}^{P_k(f)} \mathbb{1}(N_i < k^2 \mathbb{E}[N_i]).$$

We call a point t_i *good* if $N_i < k^2 \mathbb{E}[N_i]$ and *bad* otherwise. Thus G counts the number of good points. By applying Markov's inequality twice we get

$$\mathbb{P}\left(G \leq \frac{P_k(f)}{2}\right) = \mathbb{P}\left(\sum_{i=1}^{P_k(f)} \mathbb{1}(N_i \geq k^2 \mathbb{E}[N_i]) \geq \frac{P_k(f)}{2}\right) \leq \frac{2}{k^2}. \tag{3.9}$$

We now want to get a 2^{-k}-packing of $(B + f)(A)$ from the packing of $f(A)$. We do this by recursively picking good points t_i and removing the N_i balls

$$\mathcal{B}((B + f)(t_j), 2^{-k})$$

that intersect $\mathcal{B}((B + f)(t_i), 2^{-k})$. This leaves us with a 2^{-k}-packing of $(B + f)(A)$ with Ψ_k balls, which on the event $\{G \geq P_k(f)/2\}$ satisfies

$$\Psi_k \geq \frac{P_k(f)}{2(1 + k^2 \max_i \mathbb{E}[N_i])} \geq \frac{P_k(f)}{2(1 + ck^2 \log(2^k)^{d+1})},$$

where the last inequality follows from Lemma 3.3. From (3.9) we now deduce that

$$\mathbb{P}\left(\Psi_k < \frac{P_k(f)}{2(1 + ck^2 \log(2^k)^{d+1})}\right) \le \frac{2}{k^2},$$

and hence by Borel–Cantelli we conclude that a.s. eventually in k

$$\Psi_k \ge \frac{P_k(f)}{2(1 + ck^2 \log(2^k)^{d+1})}.$$

Taking log of both side, dividing by $\log(2^k)$ and taking lim sup and lim inf as $k \to \infty$ finishes the proof. $\qquad \square$

Remark 3.4. We note that $\dim_M(B + f)(A)$ can be much larger than the maximum between $\dim_M B(A)$ and $\dim_M f(A)$. We recall an example given in [11], Example 5.4. Let $d = 3$ and let $f(t) = (f_1(t), 0, 0)$, where f_1 is a fractional Brownian motion independent of B of Hurst index α. Then $\dim_M f[0, 1] = 1$ a.s. For α small we have that almost surely $\dim_H(B + f)[0, 1] = 3 - 2\alpha$, which is a special case of [3], Theorem 1, which was later corrected by Xiao [16]. Since $\dim_M(B + f)[0, 1] \ge \dim_H(B + f)[0, 1]$, we get $\dim_M(B + f)(A) \ge 3 - 2\alpha$.

We will now prove Corollary 1.5. We start with a standard result about Hölder continuous functions and we include its proof here for the sake of completeness.

Claim 3.5. *Let*

$$g \colon \mathbb{R}_+ \longrightarrow \mathbb{R}$$

be a γ-Hölder continuous function and let

$$A_\beta = \{n^{-\beta} : n \in \mathbb{N}\} \cup \{0\}$$

for $\beta \in (0, \infty)$. Then

$$\overline{\dim}_M g(A_\beta) \le \frac{1}{1 + \gamma\beta}.$$

Proof. Without loss of generality we can assume that $g(0) = 0$. Let L be the Hölder constant of g. For $k \in \mathbb{N}$ and $n \ge k$, we have

$$|g(n^{-\beta})| \le Ln^{-\gamma\beta} \le Lk^{-\gamma\beta}.$$

Fix $\varepsilon > 0$. The set $\{g(n^{-\beta}) : n > k\} \cup \{0\}$ may be covered with $\lceil 2Lk^{-\gamma\beta}/\varepsilon \rceil$ closed balls of diameter ε. The set $\{g(n^{-\beta}) : n \le k\}$ may be covered with k such closed balls. Therefore, the covering number satisfies

$$N(\varepsilon) \le \left\lceil \frac{2Lk^{-\gamma\beta}}{\varepsilon} \right\rceil + k.$$

Taking k of the order $\varepsilon^{-1/(\gamma\beta+1)}$ shows that

$$N(\varepsilon) \leq c\varepsilon^{-1/(1+\gamma\beta)},$$

and hence the result follows immediately.   □

Proof of Corollary 1.5. By Theorem 1.4 it suffices to prove the inequalities for $f = 0$. The case $d \geq 2$ follows from [14], Lemma 2.3(a). The case $d = 1$ can then be inferred by projecting a planar Brownian motion on a line in a random direction and applying [6], Theorem 3. However, we give a self-contained proof below.

We set $\alpha = \overline{\dim}_M A$ and $\beta = \underline{\dim}_M A$.

Again we take ε tending to 0 along powers of 2. Let $\delta_k = 2^{-2k/(\alpha+1)}$ and consider a δ_k-packing of A with $P_{\delta_k}(A)$ balls with centers $(t_i)_{i \leq P_{\delta_k}(A)}$. We call a point t_i *good* if

$$N_i = \#\{j \neq i : |B(t_i) - B(t_j)| < 2^{1-k}\} < k^2 \mathbb{E}[N_i],$$

and *bad* otherwise. Let $G = G_k$ denote the number of good points. Then by Markov's inequality as in (3.9) we get that

$$\mathbb{P}\Big(G \leq \frac{P_{\delta_k}(A)}{2}\Big) \leq \frac{2}{k^2}.$$

We now want to get a 2^{-k}-packing of $B(A)$ from the δ_k-packing of A. We do this by recursively picking good points t_i and removing the N_i balls $\mathcal{B}(B(t_j), 2^{-k})$ that intersect $\mathcal{B}(B(t_i), 2^{-k})$. This yields a 2^{-k}-packing of $B(A)$ with Ψ_k balls, which on the event $\{G \geq P_{\delta_k}(A)/2\}$ satisfies

$$\Psi_k \geq \frac{P_{\delta_k}(A)}{2(1 + k^2 \max_i \mathbb{E}[N_i])}. \tag{3.10}$$

Since the points (t_i) are a δ_k-packing of the set A, it follows that $|t_{i+\ell} - t_i| \geq \delta_k \ell$, and hence we can bound

$$\mathbb{E}[N_i] = \sum_{\substack{j=1 \\ j \neq i}}^{P_{\delta_k}(A)} \mathbb{P}\Big(|(B + f)(t_i) - (B + f)(t_j)| < 2^{1-k}\Big)$$

$$\leq \sum_{\substack{\ell = -i+1 \\ \ell \neq 0}}^{P_{\delta_k}(A)} \frac{c2^{-k}}{\sqrt{|\ell|\delta_k}}$$

$$\leq c' \sqrt{P_{\delta_k}(A)}.$$

Substituting this in (3.10) we get

$$\mathbb{P}\Big(\Psi_k < \frac{P_{\delta_k}(A)}{2(1 + c'k^2 \sqrt{P_{\delta_k}(A)})}\Big) \leq \frac{2}{k^2},$$

and hence by Borel-Cantelli again we get that a.s. eventually in k

$$\Psi_k \geq \frac{P_{\delta_k}(A)}{2(1 + c'k^2 \sqrt{P_{\delta_k}(A)})}. \tag{3.11}$$

Taking log of both sides, dividing by $\log(2^k)$ and taking lim sup and lim inf as $k \to \infty$ concludes the proof of the first part of the corollary.

For the second part, let $\alpha > 0$ and set $\beta = 1/\alpha - 1$. It is easy to check that the set A_β from Claim 3.5 satisfies $\dim_M A_\beta = (1 + \beta)^{-1} = \alpha$. Now pick $\gamma < 1/2$. Since almost all Brownian paths are γ-Hölder continuous, Claim 3.5 guarantees that

$$\overline{\dim}_M B(A_\beta) \leq \frac{1}{1 + \gamma\beta} \longrightarrow \frac{1}{1 + \beta/2} = \frac{2\alpha}{\alpha + 1} \quad \text{as } \gamma \to 1/2. \qquad \square$$

4. Dimension of the graph

In this section we give the proofs of the theorems stated in the Introduction concerning the Minkowski dimension of the graph of $B + f$.

We start by proving a lemma analogous to Lemma 3.3 in the case of the graph.

Lemma 4.1. *Let* (B_t) *be a standard Brownian motion in d dimensions and let* $f : [0, 1] \to \mathbb{R}^d$ *be a bounded measurable function, and A a subset of $[0, 1]$. Then there exists a positive constant c such that for all $\varepsilon > 0$, if $(\mathcal{B}((t_i, f(t_i)), 2\varepsilon))_{i \leq P_\varepsilon}$ is a 4ε-packing of $G_A(f)$, then*

$$\max_{i \leq P_\varepsilon} \mathbb{E}[N_i] \leq c \log(1/\varepsilon)^{d+1},$$

where

$$N_i = \#\{j : |(t_i, (B + f)(t_i)) - (t_j, (B + f)(t_j))| < 2\varepsilon\}, \quad for \ i \in \{1, \ldots, P_\varepsilon\}.$$

Proof. Let $\varepsilon > 0$ and $(\mathcal{B}((t_i, f(t_i)), 4\varepsilon))_{i \leq P_\varepsilon}$ a 4ε-packing of $G_A(f)$. We fix $i \in \{1, \ldots, P_\varepsilon\}$. For every $k \in \mathbb{N}$ we define the sets

$$S(k) = \{j : |t_i - t_j| < 2\varepsilon \quad \text{and} \quad |f(t_i) - f(t_j)| \in [2^k \varepsilon, 2^{k+1}\varepsilon)\},$$

$$S_1(k) = \left\{j \in S(k) : |t_i - t_j| \geq \left(\frac{2^k \varepsilon}{\log(1/\varepsilon)}\right)^2\right\},$$

$$S_2(k) = S(k) \setminus S_1(k).$$

Again since f is bounded, $S(k) = \emptyset$ when $k \geq c_1 \log(1/\varepsilon)$. Furthermore, if $j \in S(k)$, then since $(t_j, f(t_j))_j$ is a 4ε-packing of $G_A(f)$, the balls $\{\mathcal{B}(f(t_j), \varepsilon)\}_j$ are

disjoint and for all such j the ball $\mathcal{B}(f(t_j), \varepsilon)$ is contained in $\mathcal{B}(f(t_i), (2^{k+1} + 1)\varepsilon)$. Hence

$$|S(k)| \leq \frac{\text{vol}(\mathcal{B}(0, (2^{k+1} + 1)\varepsilon))}{\text{vol}(\mathcal{B}(0, \varepsilon))} \leq (2^{k+1} + 1)^d. \tag{4.1}$$

We now set

$$\begin{aligned} q(j) &= \mathbb{P}(|(t_i, (B + f)(t_i)) - (t_j, (B + f)(t_j))| < 2\varepsilon) \\ &\leq \mathbb{P}(|(B + f)(t_i) - (B + f)(t_j)| < 2\varepsilon). \end{aligned}$$

Proceeding as for the estimate for $p(j)$ in Lemma 3.3 gives that if $j \in S_1(k)$, then for a positive constant c_2

$$q(j) \leq c_2 2^{-dk} \log(1/\varepsilon)^d.$$

Similarly to the proof of Lemma 3.3, if $j \in S_2(k)$ for $k \geq 2$, we get for a positive constant c_3

$$q(j) \leq 2 \exp(-c_3 (\log(1/\varepsilon))^2).$$

Plugging these two estimates above in the expression for $\mathbb{E}[N_i]$ and using (4.1) we deduce

$$\mathbb{E}[N_i] \leq c \log(1/\varepsilon)^{d+1},$$

where c is a positive constant and this concludes the proof of the lemma. $\qquad \square$

Proof of Theorem 1.7. Let $\varepsilon > 0$ and

$$\mathcal{C}_{\varepsilon, d+1} = \{[(\ell_1 - 1)\varepsilon, \ell_1 \varepsilon] \times \ldots \times [(\ell_{d+1} - 1)\varepsilon, \ell_{d+1}\varepsilon] : \ell_1, \ldots, \ell_{d+1} \in \mathbb{Z}\}.$$

By [4], Definition 3.1, the Minkowski dimension of $G_A(B + f)$ is determined by counting the boxes in $\mathcal{C}_{\varepsilon, d+1}$ that intersect $G_A(B + f)$. We take ε tending to 0 along powers of 2, i.e. take $\varepsilon = 2^{-n}$. Let the minimal number of boxes in the covering be $N(\varepsilon) = N_n$. Setting $I_k = [(k - 1)2^{-n}, k2^{-n}]$ and

$$A_{k,n} = \#\{\text{boxes in } \mathcal{C}_{\varepsilon, d} \text{ intersecting } (B + f)(I_k \cap A)\}$$

we have $N_n = \sum_{k=1}^{2^n} A_{k,n}$. Just as in the proof of Claim 2.2 we get that there exist positive constants c_1 and c_2 such that

$$c_1 2^{nd} \sum_{k=1}^{2^n} V_{B+f}(I_k \cap A, 2^{-n}) \leq N_n \leq c_2 2^{nd} \sum_{k=1}^{2^n} V_{B+f}(I_k \cap A, 2^{-n})$$

and

$$c_1 2^{(d+1)n} V_{G(B+f)}(A, 2^{-n}) \leq N_n \leq c_2 2^{(d+1)n} V_{G(B+f)}(A, 2^{-n}),$$

where $G(B + f)$ stands for the process $(t, B(t) + f(t))_t$.

Since the process $(s, B(s))$ is càdlàg and has independent and stationary increments and $(s, f(s))$ is càdlàg, Lemma 2.3 together with the above inequalities give that a.s.

$$\overline{\dim}_M G_A(B + f) = \limsup_{n \to \infty} \frac{\log \Big(\sum_{k=1}^{2^n} \mathbb{E}[V_{B+f}(I_k \cap A, 2^{-n})] \Big)}{\log(2^n)} + d.$$

Fix $k \in \{1, \ldots, 2^n\}$. For $s \in I_k \cap A$ define $D_s = \mathcal{B}(f(s), 2^{-n})$ and for $s \notin I_k \cap A$ let $D_s = \emptyset$. Then Theorem 3.1 gives

$$\mathbb{E}[V_{B+f}(I_k \cap A, 2^{-n})] \geq \mathbb{E}[V_B(I_k \cap A, 2^{-n})],$$

and hence it follows that a.s.

$$\overline{\dim}_M G_A(B + f) \geq \overline{\dim}_M G_A(B).$$

The inequality for the lower Minkowski dimension of $G_A(B + f)$ follows in exactly the same way.

It now remains to show that a.s.

$$\overline{\dim}_M G_A(B + f) \geq \overline{\dim}_M G_A(f)$$

and similarly for lower Minkowski. The proof of that follows in the same way as the proof of Theorem 1.4. We point out the differences. We call a point $(t_i, f(t_i))$ good if $N_i < k^2 \mathbb{E}[N_i]$, where N_i is as defined in the statement of Lemma 4.1 for $\varepsilon = 2^{-k}$. Then we proceed in exactly the same way as in Theorem 1.4 and on the event that the number of good points is at least $P_k(f)/2$ we get a 2^{-k}-packing of $G_A(f)$ with at least

$$\frac{P_k(f)}{2(1 + k^2 \max_i \mathbb{E}[N_i])}$$

balls of radius 2^{-k}. Using Lemma 4.1 and the Borel-Cantelli lemma as in Theorem 1.4 concludes the proof. □

The rest of this section is devoted to the proof of Proposition 1.8.

Proposition 4.2. *Let f and $g \colon [0, 1] \to \mathbb{R}$ be two continuous functions. Assume that $\dim_M G_{[0,1]}(f)$ exists. Then,*

$$\underline{\dim}_M G_{[0,1]}(f + g) \leq \max\{\dim_M G_{[0,1]}(f), \underline{\dim}_M G_{[0,1]}(g)\},$$

$$\overline{\dim}_M G_{[0,1]}(f + g) \leq \max\{\dim_M G_{[0,1]}(f), \overline{\dim}_M G_{[0,1]}(g)\}.$$

Furthermore, in both cases, when the dimensions on the right hand side are different, we even have equality.

Proof. We shall only prove the inequality for the lower Minkowski dimension. The
other case is proved similarly.

Set

$$\alpha = \dim_M G_{[0,1]}(f) \quad \text{and} \quad \beta = \underline{\dim}_M G_{[0,1]}(g)$$

and consider the collection of squares

$$\mathcal{C}_\varepsilon = \{[(k-1)\varepsilon, k\varepsilon] \times [(\ell-1)\varepsilon, \ell\varepsilon] : k, \ell \in \mathbb{Z}\}.$$

Let $h: [0,1] \to \mathbb{R}$ be a continuous function. A covering of $G_A(h)$ is given by taking
all the elements of $\mathcal{C}_\varepsilon$ that intersect $G_{[0,1]}(h)$; and that many are needed. Let $S_\varepsilon(h)$
be the number of these squares. Take $\varepsilon = 2^{-n}$ and set $I_k = [(k-1)\varepsilon, k\varepsilon]$ and
$\Omega_{k,n}(h) = \lceil 2^n (\max_{s \in I_k} h(s) - \min_{s \in I_k} h(s)) \rceil$. Then it is easy to see that

$$\sum_{k=1}^{2^n} \Omega_{k,n}(h) \leq S_{2^{-n}}(h) \leq 2 \sum_{k=1}^{2^n} \Omega_{k,n}. \tag{4.2}$$

It is straightforward to check that

$$\Omega_{k,n}(g) - \Omega_{k,n}(f) \leq \Omega_{k,n}(f+g) \leq \Omega_{k,n}(f) + \Omega_{k,n}(g). \tag{4.3}$$

Let (ε_n) be a subsequence of (2^{-n}) along which $\log S_{\varepsilon_n}(g) / \log(1/\varepsilon_n) \to \beta$ as
$n \to \infty$. Fix $\delta \in (0, \infty)$. Then $S_{\varepsilon_n}(g) \leq \varepsilon_n^{-\beta-\delta}$ for all n large enough. Since
$\dim_M G_{[0,1]}(f)$ exists, it follows that for all n large enough $S_{\varepsilon_n}(f) \leq \varepsilon_n^{-\alpha-\delta}$. Thus
for all n sufficiently large we obtain

$$S_{\varepsilon_n}(f+g) \leq 2S_{\varepsilon_n}(f) + 2S_{\varepsilon_n}(g) \leq 4\varepsilon_n^{-\max\{\alpha,\beta\}-\delta}.$$

Taking logarithms of both sides, dividing by $\log(1/\varepsilon_n)$ and taking the limit as $n \to \infty$
gives that for all $\delta > 0$

$$\underline{\dim}_M G_{[0,1]}(f+g) \leq \max\{\alpha, \beta\} + \delta,$$

and hence letting $\delta \to 0$ gives $\underline{\dim}_M G_{[0,1]}(f+g) \leq \max\{\alpha, \beta\}$.

It only remains to show the final statement of the proposition. Suppose that $\alpha < \beta$.
We will show that

$$\underline{\dim}_M G_{[0,1]}(f+g) \geq \beta.$$

The other cases are treated similarly.

Take $\delta > 0$ small enough so that $\beta > \alpha + 2\delta$. Using (4.2) and the left hand side
inequality of (4.3) we deduce that for all n sufficiently large

$$S_n(f+g) \geq \frac{2^{n(\beta-\delta)}}{2} - 2^{n(\alpha+\delta)},$$

and hence it easily follows that in this case $\underline{\dim}_M G_{[0,1]}(f+g) \geq \beta$, which together
with the inequality previously shown completes the proof. □

Proof of Proposition 1.8. The proposition follows directly from Proposition 4.2 and
Theorem 1.7. □

5. Example

Example 5.1. Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}$ be a function with period 1 defined in $[0, 1]$ via $\varphi(x) = \max\{x, 1 - x\}$. For $n \in \mathbb{N}$ we define $\psi_n \colon [0, 1] \to \mathbb{R}_+$ via

$$\psi_n(x) = n^{-3/4} \lfloor \sqrt{n}\varphi(nx) \rfloor$$

adjusted to be càdlàg. Let $n_k = 2^{6^k}$ and define $f = \sum_{k=1}^{\infty} \psi_{n_k}$. Since f is the uniform limit of càdlàg functions, it is also càdlàg. We will show that

$$\overline{\dim}_M G_{[0,1]}(f) = \frac{5}{3} \quad \text{and} \quad \overline{\dim}_M G_{[0,1]}(B + f) \geq \frac{7}{4} \text{ a.s.}$$

The idea motivating this construction is that the graph of f can be covered efficiently due to the large jumps; $B + f$ interpolates many of these jumps and hence has a larger Minkowski dimension. Moreover, the graph of B can be covered efficiently due to cancellation of the upward and downward movement; adding f to B eliminates much of this cancellation, so that the graph of $B + f$ has Minkowski dimension greater than the graph of f.

Claim 5.2. $\overline{\dim}_M G_{[0,1]}(f) = \frac{5}{3}$.

Proof. Let $\varepsilon = n_\ell^{-3/4}$ and suppose that we want to cover the graph of f with boxes of side length ε. We now argue that the number of boxes needed is up to constants the number of boxes needed to cover the graph of ψ_{n_ℓ}.

Indeed, since for all x

$$\sum_{k=\ell+1}^{\infty} \psi_{n_k}(x) \leq \frac{1}{2} \sum_{k=\ell+1}^{\infty} (2^{6^k})^{-1/4} \leq c_1 (2^{6^\ell})^{-3/2}, \tag{5.1}$$

where c_1 is a positive constant, the number of ε-boxes needed to cover the graph of $\sum_{k \geq \ell} \psi_{n_k}$ is up to constants the same as the number of ε-boxes needed to cover the graph of ψ_{n_ℓ}.

Note that $n_k^{3/2}$ divides n_{k+1} for all k. Also for all $k < \ell$ the function ψ_{n_k} is constant on each standard subinterval of length $n_{\ell-1}^{-3/2}$. Thus on a subinterval of length $n_{\ell-1}^{-3/2}$ adding the functions ψ_{n_k} for $k < \ell$ to the function $\sum_{k=\ell}^{\infty} \psi_{n_k}$ only shifts the graph of $\sum_{k=\ell}^{\infty} \psi_{n_k}$ by a constant on each standard interval of length $n_{\ell-1}^{-3/2}$. Since we want to cover the graph of f with boxes of side length $n_\ell^{-3/4}$ and we can fit an integer number of those in the subinterval of length $n_{\ell-1}^{-3/2}$, it follows that the number of ε-boxes needed is the same as the number of ε-boxes needed to cover the graph of $\sum_{k=\ell}^{\infty} \psi_{n_k}$.

So it only remains to calculate the number of ε-boxes needed to cover the graph of ψ_{n_ℓ}. By the construction of the function ψ_{n_ℓ} it is easy to see that the number of ε-boxes needed to cover the graph of ψ_{n_ℓ} is of order $\dfrac{1}{n_\ell^{-3/4}} \dfrac{n_\ell^{-1/4}}{n_\ell^{-3/4}} = n_\ell^{5/4}$.
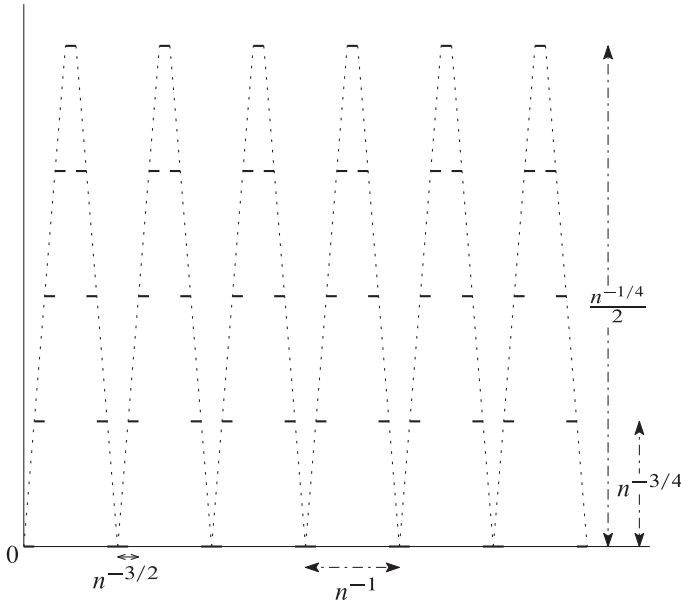
Figure 1. Graph of $\sum_{k \geq \ell} \psi_{n_k}$.

Therefore the number of boxes of side length $n_\ell^{-3/4}$ needed to cover the graph of f is of order $n_\ell^{5/4}$. From that it follows that

$$\overline{\dim}_M G_{[0,1]}(f) \geq \frac{5}{3}.$$

It remains to show $\overline{\dim}_M G_{[0,1]}(f) \leq \frac{5}{3}$.

Take ε converging to 0 along powers of 2. Let $\varepsilon = 2^{-r}$ and k be such that

$$n_k^{-3/2} \leq \varepsilon < n_{k-1}^{-3/2}.$$

We consider two separate cases. To simplify notation we write $n = n_k$.

- If $\varepsilon < n^{-3/4}$, then we need order $\varepsilon^{-1}\sqrt{n}$ boxes to cover the graph of f. This follows from (5.1), the fact that $\varepsilon > n^{-3/2}$ and that ε divides $n_\ell^{-3/2}$ for all $\ell < k$.

- If $\varepsilon > n^{-3/4}$, then the number of boxes needed to cover the graph of ψ_n is of order $n^{-1/4}/\varepsilon^2$. Since the contributions of the other functions in the sum do not matter as discussed above, this is indeed the covering number for the graph of f.

From the two cases above it follows that $\overline{\dim}_M G_{[0,1]}(f) \leq \frac{5}{3}$ and this concludes the proof. $\square$
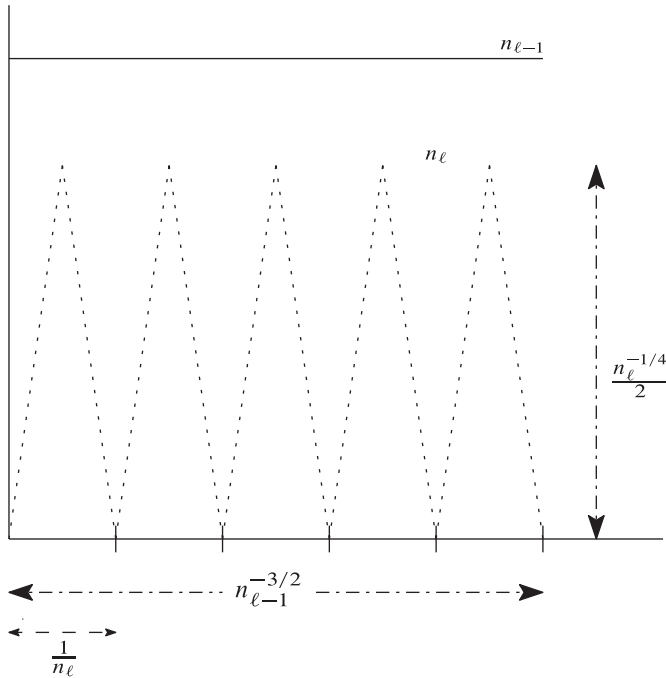
Figure 2. Two scales.

Proof. Suppose that we want to cover the graph of $B + f$ with boxes of side length n_ℓ^{-1}. Then arguing as above it is enough to find the number of n_ℓ^{-1}-boxes needed to cover the graph of $B + \psi_{n_\ell}$.

First we subdivide the interval $[0, 1]$ into subintervals of length

$$n^{-3/2} + (\log n)^2 n^{-3/2}.$$

Thus the number of such subintervals we obtain is of order $n^{3/2}/(\log n)^2$. For each such subinterval $I_{j,n}$ we write $s_{j,n} = \inf I_{j,n}$ and we define the events

$$A_n = \left\{ \text{for all } j, \, B_{s_{j,n}} - \inf_{t \in I_{j,n}} B_t \le \frac{n^{-3/4}(\log n)^2}{2} \right\}.$$

Using the Gaussian tail estimate gives

$$\mathbb{P}(A_n^c) \le c_1 \frac{n^{3/2}}{(\log n)^2} e^{-c_2 (\log n)^2},$$

which is summable. Hence by Borel-Cantelli we get that almost surely for all n sufficiently large we get that in none of the intervals $I_{j,n}$ Brownian motion goes down by more than $n^{-3/4}(\log n)^2/2$.

We now look at the first part of these subintervals $\widetilde{I}_{j,n}$ of length $n^{-3/2}$ and we define the event

$$\widetilde{A}_{j,n} = \left\{ n^{-3/4} \leq \sup_{t \in \widetilde{I}_{j,n}} B_t - B_{\tilde{s}_{j,n}} \leq 2n^{-3/4} \right\},$$

where $\tilde{s}_{j,n} = \inf \widetilde{I}_{j,n}$. Then there exists a constant $c \in (0, 1)$ so that for all j and n

$$\mathbb{P}(\widetilde{A}_{j,n}) \geq c.$$

The events $(\widetilde{A}_{j,n})_j$ are independent by the independence of the increments of Brownian motion. Using the Chernoff bound for Bernoulli random variables we obtain for a positive constant $c_3 < 1$

$$\mathbb{P}\left(\sum_{j=1}^{n^{3/2}} \mathbb{1}(\widetilde{A}_{j,n}) \geq \frac{cn^{3/2}}{4} \right) \geq 1 - c_3^{n^{3/2}}.$$

Thus applying Borel-Cantelli again we deduce that almost surely for all n large enough at least $cn^{3/2}/4$ of the events $\widetilde{A}_{j,n}$ will happen.

We now take n sufficiently large so that A_n holds and at least $cn^{3/2}/4$ of the events $\widetilde{A}_{j,n}$ occur.

We set $n = n_\ell$ and we consider the subintervals of length $n^{-3/2}$ that correspond to the events $\widetilde{A}_{j,n}$ that occur. In each of these subintervals the function ψ_n is constant, and by the definition of the event $\widetilde{A}_{j,n}$, it is easy to see that the number of boxes of side n^{-1} needed to cover the graph of $B + \psi_n$ in this time interval is at least of order $n^{-3/4}/n^{-1} = n^{1/4}$. Next we skip a time interval of length $n^{-3/2}(\log n)^2$. Since the event A_n holds, during this time interval the Brownian motion did not go down by more than $(\log n)^2 n^{-3/4}/2$. At the same time the function f increased by $n^{-3/4}(\log n)^2$. So it follows that we need at least of order $n^{1/4}n^{3/2}/(\log n)^2$ boxes of side length n^{-1} to cover the graph of $B + \psi_n$. Therefore we deduce that a.s.

$$\overline{\dim}_M G_{[0,1]}(B + f) \geq \frac{7}{4}. \qquad \square$$

Remark 5.3. A modification of the example yields a continuous function f and a closed set A in $[0, 1]$ so that

$$\overline{\dim}_M G_A(f) = \frac{5}{3} \quad \text{and} \quad \overline{\dim}_M G_A(B + f) \geq \frac{7}{4}.$$

References

[1] F. Bayart and Y. Heurteaux, On the Hausdorff dimension of graphs of prevalent continuous functions on compact sets. In J. Barral and S. Seuret (eds.), *Further developments in fractals and related fields.* Mathematical foundations and connections. Including papers from the 2nd International Conference on Fractals and Related Fields held on Porquerolles Island, June 2011. Trends in Mathematics. Birkhäuser/Springer, New York, N.Y., 2013, 25–34. MR 3184186 MR 3185036 (collection) Zbl 1268.28003 Zbl 1260.28001 (collection)

[2] J. Bertoin, *Lévy processes.* Cambridge Tracts in Mathematics 121. Cambridge University Press, Cambridge, 1996. MR 1406564 Zbl 0861.60003

[3] J. Cuzick, Some local properties of Gaussian vector fields. *Ann. Probab.* **6** (1978), 984–994. MR 0512415 Zbl 0395.60038

[4] K. J. Falconer, *Fractal geometry. Mathematical foundations and applications.* Second edition. John Wiley & Sons, Hoboken, N.J., 2003. MR 2118797 Zbl 1060.28005

[5] K. J. Falconer and J. M. Fraser. The horizon problem for prevalent surfaces. *Math. Proc. Cambridge Philos. Soc.* **151** (2011), 355–372. MR 2823141 Zbl 1235.28005

[6] K. J. Falconer and J. D. Howroyd, Projection theorems for box and packing dimensions. *Math. Proc. Cambridge Philos. Soc.* **119** (2001), 287–295. MR 1357045 Zbl 0846.28004

[7] J. D. Howroyd, Box and packing dimensions of projections and dimension profiles. *Math. Proc. Cambridge Philos. Soc.* **130** (2001), 135–160. MR 1797735 Zbl 1007.28004

[8] D. Khoshnevisan, R. L. Schilling, and Y. Xiao, Packing dimension profiles and Lévy Processes. *Bull. London Math. Soc.* **44** (2012), 931-943. MR 2975152 Zbl 1266.60071

[9] D. Khoshnevisan and Y. Xiao, Packing-dimension profiles and fractional Brownian motion. *Math. Proc. Cambridge Philos. Soc.* **145** (2008), 205–213. MR 2431650 Zbl 1152.28009

[10] P. Mörters and Y. Peres, *Brownian motion.* With an appendix by Oded Schramm and Wendelin Werner. Cambridge Series in Statistical and Probabilistic Mathematics 30. Cambridge University Press, Cambridge, 2010. MR 2604525 Zbl 1243.60002

[11] Y. Peres and P. Sousi, Brownian motion with variable drift: 0–1 laws, hitting probabilities and Hausdorff dimension. *Math. Proc. Cambridge Philos. Soc.* **153** (2012), 215–234. MR 2981924 Zbl 1260.60168

[12] Y. Peres and P. Sousi, An isoperimetric inequality for the Wiener sausage. *Geom. Funct. Anal.* **22** (2012), 1000–1014. MR 2984124 Zbl 1270.60087

[13] Y. Peres and P. Sousi, Dimension of Fractional Brownian motion with variable drift. Preprint 2013. arXiv:1310.7002 [math.PR]

[14] E. A. Perkins and S. J. Taylor, Uniform measure results for the image of subsets under Brownian motion. *Probab. Theory Related Fields* **76** (1987), 257–289. MR 0912654 Zbl 0613.60071

[15] M. Talagrand and Y. Xiao, Fractional Brownian motion and packing dimension. *J. Theoret. Probab.* **9** (1996), 579–593. MR 1400588 Zbl 0860.60064

[16] Y. Xiao, Dimension results for Gaussian vector fields and index-α stable fields. *Ann. Probab.* **23** (1995), 273–291. MR 1330771 Zbl 0834.60040

[17] Y. Xiao, Packing dimension of the image of fractional Brownian motion. *Statist. Probab. Lett.* **33** (1997), 379–387. MR 1458008 Zbl 0903.60073

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