

Projections of random covering sets

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Abstract. We show that, almost surely, the Hausdorff dimension s_0 of a random covering set is preserved under all orthogonal projections to linear subspaces with dimension $k > s_0$. The result holds for random covering sets with a generating sequence of ball-like sets, and is obtained by investigating orthogonal projections of a class of random Cantor sets.

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1. Introduction

We begin by giving a definition for random covering sets. Given a sequence of independent random variables (ξ_n) , uniformly distributed on the torus \mathbb{T}^d , and a sequence of subsets of the torus, (g_n) , random covering set E is the set of infinitely often covered points,

$$E = \limsup_{n \rightarrow \infty} (\xi_n + g_n) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} (\xi_n + g_n).$$

Here we interpret $\xi_n + g_n \subset \mathbb{T}^d$. For further details and background on random covering sets, see e.g. the survey [8]. Here we only mention a few key observations.

It is an immediate consequence of Borel–Cantelli lemma and Fubini’s theorem that

$$\mathcal{L}(E) = \begin{cases} 0, & \text{when } \sum_{n=1}^{\infty} \mathcal{L}(g_n) < \infty, \\ 1, & \text{when } \sum_{n=1}^{\infty} \mathcal{L}(g_n) = \infty, \end{cases}$$

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where \mathcal{L} is the Lebesgue measure. In the earlier research on the random covering sets, the main emphasis has been on the full-measure case in the circle \mathbb{T}^1 . Of particular interest have been variants of a problem posed by Dvoretzky in 1956 asking what kind of conditions on g_n guarantee that

$$\mathbb{P}(E = \mathbb{T}^1) = 1.$$

This particular problem was fully solved by Shepp in 1972 [13], but many related problems especially in higher dimensions are still open.

Recently, also the zero measure case has drawn a lot of attention among the researchers. The Hausdorff dimension of the random covering sets in \mathbb{T}^1 when g_n are intervals of length $1/n^\alpha$ was first calculated by Fan and Wu [6]. Durand [2] studied the Hausdorff measure and large intersection properties and reproved the dimension result in \mathbb{T}^1 . For certain box-like random covering sets in \mathbb{T}^d , the dimension was obtained by Järvenpää, Järvenpää, Koivusalo, Li and Suomala [7]. Recently, Persson [10] generalized the result using the large intersection method.

In this article we study the dimensions of orthogonal projections of the random covering sets (after embedding the torus \mathbb{T}^d into \mathbb{R}^d in a natural way). Our motivation stems from the classical results of projections of general sets by Marstrand, Kauffman, Mattila and others. In particular, if the Hausdorff dimension of a Borel set $A \subset \mathbb{R}^d$, $\dim_H A = s$, then it is well known that for almost all k -dimensional subspaces $V \subset \mathbb{R}^d$ the orthogonal projection of A into V , $\pi_V A$ is of Hausdorff dimension $\min\{s, k\}$. Below, we refer to this fact as the *projection theorem*. For background and references, see e.g. [9]. One should bear in mind that since this is an almost all type statement, for most concrete sets there are plenty of exceptional directions for which the dimension is less than the expected value. Moreover, if one fixes a direction, the projection theorem cannot be used to obtain information about the dimension of the projection in this direction.

For suitable random families of sets, the situation is different. A model example is the fractal percolation for which Falconer [4] and Falconer and Grimmett [5] showed that for the principal directions the dimension of the projection of the limit set is almost surely equal to the expected value. It is harder to show, and this was done only recently by Rams and Simon (for $d = 2$) [11] and [12], that the same remains true for all directions simultaneously. That is, if s is the almost sure dimension of the fractal percolation limit set $F \subset \mathbb{R}^2$, then almost surely, $\dim_H \pi_\ell(F) = \min\{s, 1\}$ for all lines $\ell \subset \mathbb{R}^2$.

Our approach is motivated by Shmerkin and Suomala [14] where a variant of the fractal percolation model is used in order to study the dimension of non-tube null sets. Some ideas have also been adapted from the forthcoming work [15],

where Shmerkin and Suomala prove various results for the intersections and projections of families of martingale measures (for which fractal percolation is a central example).

In our main result, Theorem 5.1, we establish that for the random covering sets, there are almost surely no exceptional directions for the projection theorem. This, on the other hand, is a corollary of a similar result for a general class of random Cantor sets, Theorem 2.1. The main difference between our model and the (variations of) fractal percolation in [11], [12], and [14], [15] is the fact that we do not impose any bounds on how fast the diameter of the basic sets (or construction cubes) goes to zero. This is essential for the application to the random covering sets and also the main reason why we cannot derive the result directly from [15].

The paper is organized as follows. The random Cantor sets are introduced in Section 2 together with the required notation and definitions. The main result for these Cantor sets is also presented as Theorem 2.1. In Section 3 we present some geometric lemmata, which have been adapted from [14] to the present setting. Section 4 contains the main probabilistic argument, and concludes the proof of Theorem 2.1. The dimension result for the projections of the random covering sets is then derived as a corollary in Section 5, see Theorem 5.1. Throughout Sections 2–5 we only consider the case $d = 2$. Higher dimensional generalizations are discussed at the end in Section 6.

2. Random Cantor sets and their projections

In this section, we define the random Cantor type sets that we consider and state the main results for the dimension of their orthogonal projections.

Given $k \in \mathbb{N}$, we call the closed squares $[\frac{i}{k}, \frac{i+1}{k}] \times [\frac{j}{k}, \frac{j+1}{k}]$, $0 \leq i, j \leq k-1$, k -adic. In a similar way, we can consider the k -adic grid of any $Q = [x, x + \lambda] \times [y, y + \lambda] \subset [0, 1]^2$ consisting of the k^2 closed subsquares of Q of side-length λ/k .

Given $M \in \mathbb{N}$ and $0 < s < 1$, we consider the following random model. Decompose the unit square into M -adic subsquares and randomly choose $N \leq M^2$ of these subsquares Q_1, \dots, Q_N , all choices being independent and uniformly distributed (i.e. $\mathbb{P}(Q_i = Q) = M^{-2}$ for each Q in the grid and each $1 \leq i \leq N$). We stress that the chosen subsquares need not be disjoint, that is, the same square is allowed to be chosen multiple times.

Let $(M_n) \subset \mathbb{N}$ and $(N_n) \subset \mathbb{N}$ be sequences of integers with $N_n \leq M_n^2$ ($M_n \geq 2$) for all n and

$$s = \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n \log N_i}{\sum_{i=1}^n \log M_i} < 1. \quad (1)$$

Let

$$r_n = \left(\prod_{k=1}^n M_k \right)^{-1}, \quad (2)$$

and

$$P_n = \prod_{k=1}^n N_k. \quad (3)$$

We consider a random construction obtained by iterating the above construction with $M = M_n$ and $N = N_n \leq M_n^2$. We first choose randomly $\mathcal{F}_1 = (Q_{1,1}, \dots, Q_{1,N_1})$ among the squares in the M_1 -adic grid of the unit square as in above. Suppose that for $n \in \mathbb{N}$, $\mathcal{F}_n = (Q_{n,1}, \dots, Q_{n,P_n})$ are chosen r_n^{-1} -adic squares. Independently inside each of these $Q_{n,j}$, we consider the M_{n+1} -adic grid and perform the above construction with $M = M_{n+1}$ to obtain N_{n+1} chosen subsquares of each $Q_{n,j}$. Let \mathcal{F}_{n+1} consist of all the chosen subsquares inside the $Q_{n,j} \in \mathcal{F}_n$. Again, it should be pointed out that the elements of \mathcal{F}_{n+1} do not have to be disjoint. For this reason we use the notation $(Q_{n,1}, \dots)$ rather than $\{Q_{n,1}, \dots\}$. For convenience, we however stick to notations like $Q \in \mathcal{F}_n$ instead of the more rigorous $Q = Q_{n,i}, i \in \{1, \dots, P_n\}$.

Denote by ω the elements in the probability space Ω induced by the construction described above. Let $F = F(\omega)$ be the random limit set

$$F = \bigcap_{n=1}^{\infty} \bigcup_{Q \in \mathcal{F}_n} Q.$$

For each line ℓ in the plane, denote by π_ℓ the orthogonal projection to ℓ and by $\dim_H(F)$ the Hausdorff dimension of a set F . Our main result for the random Cantor sets is the following.

Theorem 2.1. *Almost surely for all lines ℓ , $\dim_H \pi_\ell(F) = s$.*

Remark 2.2. It is an immediate consequence of the definitions that we have the dimension upper bound $\dim_H(F) \leq s$ for all F . It is more or less standard to show that almost surely, also $\dim_H(F) \geq s$ (and these estimates remain true also for $1 \leq s \leq 2$). We do not give the proof here, since in the case $s < 1$ that is relevant for us, Theorem 2.1 yields a much stronger result.

When the sequence (M_n) is bounded, Theorem 2.1 could be obtained by adapting the argument of [14]. It would also follow directly from the general results for martingale measures in [15]. Whence the main content in Theorem 2.1 is that we do not impose any bounds on the growth of the numbers M_n . We also remark that if M_n is bounded, then the packing dimension of F almost surely equals $s' = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n \log N_i}{\sum_{i=1}^n \log M_i}$, whereas in the unbounded case the packing dimension can be anything in between s' and 2 depending on the growth speed of M_n .

Let

$$\mu_n = \sum_{Q \in \mathcal{F}_n} c_n \mathcal{L}|_Q, \quad \text{where } c_n = \prod_{i=1}^n M_i^2 N_i^{-1} = r_n^{-2} P_n^{-1}, \quad (4)$$

and $\mathcal{L}|_Q$ is the Lebesgue measure on Q . Then μ_n is a probability measure for each n and since the measure of any r_n^{-1} -adic square remains unchanged after n steps, it follows that the sequence (μ_n) converges in weak* topology to a random probability measure μ on $[0, 1]^2$.

In the following, tubular neighborhoods of lines in the unit cube are called strips. More precisely, a strip S of width $w(S) = \delta > 0$, defined by a line ℓ , is the set

$$S = \{x \in [0, 1]^2 \mid \text{dist}(x, \ell) < \delta/2\}$$

where dist is the Euclidean distance.

$$|\ell \cap \mathcal{F}_n| := \sum_{Q \in \mathcal{F}_n} \mathcal{H}^1(\ell \cap Q),$$

where \mathcal{H}^1 is the 1-dimensional Hausdorff measure. Since the same square can occur many times in \mathcal{F}_n , this is in general different than the length of $\ell \cap \bigcup_{Q \in \mathcal{F}_n} Q$. For a strip S , denote

$$Z(S, n) = \#\{Q \in \mathcal{F}_n \mid Q \cap S \neq \emptyset\},$$

where $\#J$ denotes the cardinality of a set J . Denote the indicator function of an event A by χ_A and the complement of A by A^c . Finally, denote by $f_*\mu$ the image measure of μ under a mapping f .

Theorem 2.1 is easily deduced from the following estimate for the projections of the limit measure μ , which is proved in Sections 3–4.

Theorem 2.3. *Let $t < s$. There is a constant $C_0 < +\infty$ such that there is a positive probability for the event*

$$(\pi_\ell)_*\mu(B(x, r)) \leq C_0 r^t \text{ for all lines } \ell, \text{ for all } x \in \ell \text{ and for all } r > 0.$$

Proof of Theorem 2.1 assuming Theorem 2.3. Obviously

$$\dim_H \pi_\ell(E) \leq \dim_H(E) \leq s$$

for all lines, so we concentrate on the lower bound.

For all $t < s$, by Theorem 2.3 the estimate

$$(\pi_\ell)_* \mu(B(x, r)) \leq C_0 r^t,$$

holds with positive probability for all ℓ, x and r , and thus with positive probability a lower bound $\dim_H \pi_\ell(F) \geq t$ holds for all lines ℓ (See e.g. [3, 4.2]). Since the event “ $\dim_H \pi_\ell(F) \geq t$ for all lines” is a tail event, the Kolmogorov zero-one law implies that this lower bound holds almost surely. Approaching s along a sequence gives, almost surely for all lines ℓ , the lower bound $\dim_H \pi_\ell(F) \geq s$. \square

3. Geometric lemmata

In this section, we present some simple geometric observations. The following lemma is an adaptation of [14, Lemma 3.3] to our setting. We do not repeat the proof here. We call a line ℓ admissible, if it does not contain a side of any r_n^{-1} -adic square, $n \in \mathbb{N}$. Observe that there are only countably many non-admissible lines.

Lemma 3.1. *There is a collection of admissible lines, \mathcal{A}_n , such that for any admissible line ℓ there is $\tilde{\ell} \in \mathcal{A}_n$ with*

$$|\ell \cap \mathcal{F}_n| \leq |\tilde{\ell} \cap \mathcal{F}_n| + r_n$$

and \mathcal{A}_n has at most $C r_n^{-4}$ elements, where $C < +\infty$ is a constant.

Lemma 3.2. *Let $2 \leq M \in \mathbb{N}$. There is a finite family of strips \mathcal{D} with at most $16M^3$ elements so that for any $M^{-1} \leq \delta \leq 1$ and any strip S of width δ , there is $\tilde{S} \in \mathcal{D}$ such that $w(\tilde{S}) \leq 5w(S)$ and $S \subset \tilde{S}$.*

Proof. Denote by \mathcal{B} the collection of all lines connecting any two disjoint points in the collection

$$\begin{aligned} & \{(0, \frac{k}{M}) \in \mathbb{R}^2 \mid k, l = 1, \dots, M\} \cup \{(1, \frac{k}{M}) \in \mathbb{R}^2 \mid k, l = 1, \dots, M\} \\ & \cup \{(\frac{k}{M}, 0) \in \mathbb{R}^2 \mid k, l = 1, \dots, M\} \cup \{(\frac{k}{M}, 1) \in \mathbb{R}^2 \mid k, l = 1, \dots, M\}. \end{aligned}$$

Denote by \mathcal{D}^1 the collection of all strips of width $5M^{-1}$ defined by the at most $16M^2$ lines in the collection \mathcal{B} . This has the desired property for all strips S of width $M^{-1} \leq w(S) \leq 2M^{-1}$.

Repeat this argument for all $i = 2, \dots, M$ to obtain collections \mathcal{D}^i , that satisfy the claim for strips S of width $iM^{-1} \leq w(S) \leq (i + 1)M^{-1}$. Then choose $\mathcal{D} = \mathcal{D}^1 \cup \dots \cup \mathcal{D}^M$. \square

Lemma 3.3. *Let $0 \leq V \in \mathbb{R}$. If $|\ell \cap \mathcal{F}_n| \leq V$ for all admissible lines ℓ , then for a strip S with width $0 < w(S) \leq r_n$,*

$$Z(S, n) \leq 2(1 + 2\sqrt{2})r_n^{-1}V.$$

Proof. Let ℓ be the line defining S , ℓ' a line perpendicular to ℓ with $\ell \cap \ell' = \{x\}$ for $x \in \mathbb{R}^2$ and ℓ_y lines parallel to ℓ with $\ell' \cap \ell_y = \{y\}$. Denote by r the number $w(S) + 2\sqrt{2}r_n$ and by $B(x, r)$ the ball of center x and radius r on the line ℓ' . Then Fubini's theorem, (4) and the assumption $|\ell' \cap \mathcal{F}_n| \leq V$ imply

$$\mu_n(\pi_{\ell'}^{-1}(B(x, r))) = c_n \int_{B(x, r)} |\ell_y \cap \mathcal{F}_n| dy \leq 2Vrc_n.$$

If $Q \cap S \neq \emptyset$, then $Q \subset \pi_{\ell'}^{-1}(B(x, r))$, and from (4), we have

$$Z(S, n)P_n^{-1} \leq \mu_n(\pi_{\ell'}^{-1}(B(x, r))) \leq 2Vrc_n,$$

as required. \square

4. Main proofs

Throughout this section, we fix a number $t < s$ and let $0 < 2\epsilon < s - t$. By slight abuse of notation, denote by $\mathbb{E}(\cdot | \mathcal{F}_n)$ the expectation conditional on construction squares up to level n being chosen, and by $\mathbb{P}(\cdot | \mathcal{F}_n)$ the corresponding conditional probability.

Notation 4.1. Let ℓ be a line, S a strip and $n \in \mathbb{N}$. Let $G(\ell, n)$ be the event $|\ell \cap \mathcal{F}_n| \leq RP_n r_n^{t+1} + r_n$, where $R > 0$ is a large constant. Denote by $A(S, n)$ the event $Z(S, n) \leq 500 \cdot 5^t RP_n w(S)^t$. Let G_n be the event that $G(\ell, n)$ occurs for all admissible lines ℓ , and A_n the event that $A(S, n)$ occurs for all strips S with $\frac{1}{5}r_n < w(S) \leq \frac{1}{5}r_{n-1}$. Finally, let Γ_n be the event $A_n \cap G_n$.

In order to prove Theorem 2.3 we wish to verify that

$$\mathbb{P}(\bigcap_{n=1}^{\infty} \Gamma_n) > 0, \tag{5}$$

since then for all $\omega \in \bigcap_{n=1}^{\infty} \Gamma_n$ the bound $(\pi_{\ell})_*\mu(B(x, r)) \leq Cr^t$ holds, as will be shown at the end of the section. To that end, we start by estimating the random

variables $Z(S, n + 1)$ conditional on \mathcal{F}_n and bound the probability of the events A_{n+1}^c . Using similar ideas, we then provide estimates for the probability of G_{n+1}^c conditional on \mathcal{F}_n .

Lemma 4.2. *For any strip S with $r_{n+1} < w(S) \leq r_n$, we have*

$$\begin{aligned} \mathbb{P}(Z(S, n + 1) > 500R P_{n+1} w(S)^t \mid \mathcal{F}_n) \\ \leq \exp(-500R w(S)^t P_{n+1} + 20w(S)r_n^{-1} N_{n+1} Z(S, n)). \end{aligned}$$

Proof. Denote $w(S)$ by δ and $Z(S, n)$ by K ; let $\{Q_1, Q_2, \dots, Q_K\}$ be the r_n^{-1} -adic squares from \mathcal{F}_n that hit S , and for $i = 1, \dots, K$ let $\{Q_{i,1}, Q_{i,2} \dots Q_{i,m_i}\}$ be the r_{n+1}^{-1} -adic squares inside Q_i that hit S . Conditional on \mathcal{F}_n these notions are deterministic. For each $1 \leq i \leq K$, let $(U_1^i, \dots, U_{N_{n+1}}^i) = \{U \in \mathcal{F}_{n+1} \mid U \subset Q_i\}$. Further, let $X_{i,j}, 1 \leq j \leq N_{n+1}$ be the random variables defined as

$$X_{i,j} = \begin{cases} 1, & \text{when } U_j^i \cap S \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Then, conditional on \mathcal{F}_n , $X_{i,j}$ are independent random variables and take value 1 with probability $m_i M_{n+1}^{-2}$, where m_i is the number of r_{n+1}^{-1} -adic subsquares of Q_i that touch S . Further,

$$Z(S, n + 1) = \sum_{i=1}^K \sum_{j=1}^{N_{n+1}} X_{i,j}.$$

It is easy to see that $m_i M_{n+1}^{-2} \leq 5\delta \sqrt{2} r_n^{-1}$, so that $\mathbb{P}(X_{i,j} = 1 \mid \mathcal{F}_n) \leq 5\delta \sqrt{2} r_n^{-1}$.

Applying Markov's inequality to the random variable $e^{Z(S,n+1)}$ gives

$$\mathbb{P}(Z(S, n + 1) > 500R\delta^t P_{n+1} \mid \mathcal{F}_n) \leq e^{-500R\delta^t P_{n+1}} \mathbb{E}(e^{Z(S,n+1)} \mid \mathcal{F}_n). \tag{6}$$

Since $X_{i,j}$ are mutually independent so are the $e^{X_{i,j}}$, and

$$\mathbb{E}\left(\prod_{i,j} e^{X_{i,j}} \mid \mathcal{F}_n\right) = \prod_{i,j} \mathbb{E}(e^{X_{i,j}} \mid \mathcal{F}_n) = \prod_{i,j} (e m_i M_{n+1}^{-2} + (1 - m_i M_{n+1}^{-2})).$$

Using the fact that $(1 + x) \leq e^x$ for all x , and $m_i M_{n+1}^{-2} \leq 5\delta \sqrt{2} r_n^{-1}$, we have

$$\mathbb{E}(e^{Z(S,n+1)} \mid \mathcal{F}_n) \leq \exp((e - 1)5\delta \sqrt{2} r_n^{-1} N_{n+1} K).$$

Combining the above calculations with (6) and recalling $\delta = w(S)$, $K = Z(S, n)$ gives

$$\begin{aligned} \mathbb{P}(Z(S, n + 1) > 500R w(S)^t P_{n+1} \mid \mathcal{F}_n) \\ \leq \exp(-500R w(S)^t P_{n+1}) \exp((e - 1)5\sqrt{2} w(S) r_n^{-1} N_{n+1} Z(S, n)) \end{aligned}$$

finishing the proof. □

Notation 4.3. Consider the collection given by Lemma 3.2 for $M = 5r_{n+1}^{-1}$. Denote by \mathcal{S}_n the strips in this collection with width bounded from above by r_n . Note that by the construction in Lemma 3.2 all strips in collection \mathcal{S}_n have width bounded from below by r_{n+1} .

Proposition 4.4. *There is a constant $C_3 > 0$ such that for all large n*

$$\mathbb{E}(\chi_{\cap_{k \leq n} \Gamma_k} \mathbb{P}(A_{n+1}^c \mid \mathcal{F}_n)) \leq \mathbb{P}(\cap_{k=1}^n \Gamma_k) 2000 r_{n+1}^{-3} \exp(-C_3 r_{n+1}^\epsilon).$$

Proof. Condition on $\cap_{k \leq n} \Gamma_k$. Note that for large n , when Γ_n and thus G_n occurs, then by Lemma 3.3 and (4),

$$Z(S, n) \leq 4(1 + 2\sqrt{2}) R P_n r_n^t \leq 20 R P_n r_n^t \tag{7}$$

for all strips S of width $r_{n+1} < w(S) \leq r_n$. Recall that $0 < \epsilon < s - t$ and hence for all large enough n we have $P_{n+1} \geq r_{n+1}^{-t-\epsilon}$. Let n be at least this large. When G_n occurs, for a strip S with $r_{n+1} < w(S) \leq r_n$, we obtain using Lemma 4.2 and (7)

$$\begin{aligned} \mathbb{P}(Z(S, n + 1) > 500 R P_{n+1} w(S)^t \mid \mathcal{F}_n) &\leq \exp(-500 R w(S)^t P_{n+1} + 20 w(S) r_n^{-1} N_{n+1} Z(S, n)) \\ &\leq \exp(-500 R w(S)^t P_{n+1} + 400 w(S) r_n^{-1} N_{n+1} R P_n r_n^t). \end{aligned}$$

Plugging in $P_{n+1} = N_{n+1} P_n$, this is bounded from above by

$$\exp(-w(S)^t P_{n+1} R (500 - 400 w(S)^{1-t} r_n^{t-1})) \leq \exp(-C_3 r_{n+1}^{-\epsilon})$$

for a positive constant C_3 . Fix an ω such that $Z(\tilde{S}, n + 1) \leq 500 R P_{n+1} w(\tilde{S})^t$ holds for all $\tilde{S} \in \mathcal{S}_n$ where \mathcal{S}_n is given in Notation 4.3. Then, for any strip S with $\frac{1}{5} r_{n+1} < w(S) \leq \frac{1}{5} r_n$, by Lemma 3.2, there is a strip $\tilde{S} \in \mathcal{S}_n$ with $S \subset \tilde{S}$ and $w(\tilde{S}) \leq 5w(S)$. Thus

$$Z(S, n + 1) \leq Z(\tilde{S}, n + 1) \leq 500 \cdot 5^t R P_{n+1} w(S)^t,$$

and further, A_{n+1} occurs. Apply Lemma 4.2 and Lemma 3.2 to all strips \tilde{S} in the collection \mathcal{S}_n , to obtain from the above calculation that

$$\mathbb{E}(\chi_{\cap_{k \leq n} \Gamma_k} \mathbb{P}(A_{n+1}^c \mid \mathcal{F}_n)) \leq \mathbb{P}(\cap_{k=1}^n \Gamma_k) 2000 r_{n+1}^{-3} \exp(-C_3 r_{n+1}^{-\epsilon}). \quad \square$$

Our next task is to adjust the argument of Lemma 4.2 in order to get, with high probability, a good upper bound for $|\ell \cap \mathcal{F}_{n+1}|$.

Lemma 4.5. *For all $0 < \lambda < r_{n+1}^{-1}\sqrt{2}$ and all admissible lines ℓ , we have the bound*

$$\begin{aligned} \mathbb{P}(|\ell \cap \mathcal{F}_{n+1}| > P_{n+1}Rr_{n+1}^{t+1} \mid \mathcal{F}_n) \\ \leq \exp(-\lambda P_{n+1}Rr_{n+1}^{t+1}) \exp\left(\frac{2\lambda N_{n+1}|\ell \cap \mathcal{F}_n|}{M_{n+1}^2(2 - r_{n+1}\lambda\sqrt{2})}\right). \end{aligned}$$

Proof. Condition on \mathcal{F}_n . Let Q_1, Q_2, \dots, Q_K ($K = K(\ell, \mathcal{F}_n)$) be the squares in \mathcal{F}_n hitting ℓ and $L_i = \mathcal{H}^1(\ell \cap Q_i)$. For each Q_i , let $Q_{i,1}, Q_{i,2}, \dots, Q_{i,m_i}$ denote all the r_{n+1}^{-1} -adic subsquares of Q_i touching ℓ and put $L_{i,j} = \mathcal{H}^1(\ell \cap Q_{i,j})$. We have

$$|\ell \cap \mathcal{F}_n| = \sum_{i=1}^K L_i \quad \text{and} \quad L_i = \sum_{j=1}^{m_i} L_{i,j}.$$

For each $1 \leq i \leq K$, let $(U_1^i, \dots, U_{N_{n+1}}^i) = \{U \in \mathcal{F}_{n+1} \mid U \subset Q_i\}$. Then let $X_{i,k}, 1 \leq k \leq N_{n+1}$ be random variables with

$$X_{i,k} = \mathcal{H}^1(\ell \cap U_k^i).$$

Note that, conditional on \mathcal{F}_n , $X_{i,k}$ are independent. We may write

$$|\ell \cap \mathcal{F}_{n+1}| = \sum_{i=1}^K \sum_{k=1}^{N_{n+1}} X_{i,k}.$$

Let $0 < \lambda < r_{n+1}^{-1}\sqrt{2}$. We apply Markov's inequality for the random variable $e^{\lambda|\ell \cap \mathcal{F}_{n+1}|}$ to obtain an estimate

$$\mathbb{P}(|\ell \cap \mathcal{F}_{n+1}| > P_{n+1}Rr_{n+1}^{t+1} \mid \mathcal{F}_n) \leq e^{-\lambda P_{n+1}Rr_{n+1}^{t+1}} \mathbb{E}(e^{\lambda|\ell \cap \mathcal{F}_{n+1}|} \mid \mathcal{F}_n). \tag{8}$$

Now we estimate $\mathbb{E}(e^{\lambda|\ell \cap \mathcal{F}_{n+1}|} \mid \mathcal{F}_n)$. Firstly, notice that

$$\mathbb{E}(e^{\lambda X_{i,k}} \mid \mathcal{F}_n) = 1 - \frac{m_i}{M_{n+1}^2} + \frac{1}{M_{n+1}^2} \sum_{j=1}^{m_i} e^{\lambda L_{i,j}}.$$

For all $|x| < 2$, we use the fact $e^x \leq 1 + 2x/(2 - x)$ and $\lambda L_{i,j} < 2$, to obtain

$$e^{\lambda L_{i,j}} \leq 1 + \lambda L_{i,j} \frac{2}{2 - \lambda r_{n+1}\sqrt{2}},$$

and further

$$\begin{aligned} \mathbb{E}(e^{\lambda X_{i,k}} \mid \mathcal{F}_n) &\leq 1 + \lambda L_i \frac{1}{M_{n+1}^2} \cdot \frac{2}{2 - \lambda r_{n+1}\sqrt{2}} \\ &\leq \exp\left(\frac{\lambda L_i}{M_{n+1}^2} \cdot \frac{2}{2 - \lambda r_{n+1}\sqrt{2}}\right). \end{aligned} \tag{9}$$

Since $X_{i,k}$ are independent, (9) yields

$$\begin{aligned} \mathbb{E}(e^{|\ell \cap \mathcal{F}_{n+1}|} \mid \mathcal{F}_n) &= \prod_{i,k} \mathbb{E}(e^{\lambda X_{i,k}} \mid \mathcal{F}_n) \\ &\leq \exp\left(\frac{\lambda N_{n+1} |\ell \cap \mathcal{F}_n|}{M_{n+1}^2} \cdot \frac{2}{2 - \lambda r_{n+1} \sqrt{2}}\right). \end{aligned}$$

Combining this with (8) finishes the proof. □

Next we estimate the probability of the event G_{n+1}^c .

Proposition 4.6. *There is a constant $C_2 > 0$ independent of ϵ , and $N = N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N$,*

$$\mathbb{E}(\chi_{\cap_{k \leq n} \Gamma_k} \mathbb{P}(G_{n+1}^c \mid \mathcal{F}_n)) \leq \mathbb{P}(\cap_{k=1}^n \Gamma_k) C r_{n+1}^{-4} \exp(-C_2 r_{n+1}^{-\epsilon})$$

where C is from Lemma 3.1.

Proof. To begin with, recall that when Γ_n occurs, so does G_n and then for all admissible lines ℓ

$$|\ell \cap \mathcal{F}_n| \leq P_n R r_n^{t+1} + r_n. \tag{10}$$

Further, when G_n occurs, by Lemma 4.5 and (10), the probability $\mathbb{P}(|\ell \cap \mathcal{F}_{n+1}| > P_{n+1} R r_{n+1}^{t+1} \mid \mathcal{F}_n)$ is bounded from above by

$$\begin{aligned} &\exp(-\lambda P_{n+1} R r_{n+1}^{t+1}) \exp\left(\frac{2\lambda N_{n+1} |\ell \cap \mathcal{F}_n|}{M_{n+1}^2 (2 - r_{n+1} \lambda \sqrt{2})}\right) \\ &\leq \exp(-\lambda P_{n+1} R r_{n+1}^{t+1}) \exp\left(\frac{2\lambda N_{n+1} r_n P_n R r_n^t (1 + P_n^{-1} R^{-1} r_n^{-t})}{M_{n+1}^2 (2 - r_{n+1} \lambda \sqrt{2})}\right) \\ &\leq \exp\left(-\lambda P_{n+1} R r_{n+1}^{t+1} \left(1 - \frac{2M_{n+1}^{t-1} (1 + P_n^{-1} R^{-1} r_n^{-t})}{(2 - r_{n+1} \lambda \sqrt{2})}\right)\right), \end{aligned}$$

for any admissible ℓ . Choose $\lambda = r_{n+1}^{-1+\epsilon}$. Recall that for large n , by (1) and the choice of ϵ we have $P_n \geq r_n^{-t-2\epsilon}$. Then the term $1 - \frac{2M_{n+1}^{t-1} (1 + P_n^{-1} R^{-1} r_n^{-t})}{(2 - r_{n+1} \lambda \sqrt{2})}$ is bounded from below by a constant $C_2 > 0$ for large values of n .

Thus, for all large n and for each admissible ℓ , recalling $P_{n+1} \geq r_{n+1}^{-t-2\epsilon}$, we arrive at the estimate

$$\mathbb{P}(|\ell \cap \mathcal{F}_{n+1}| > P_{n+1} R r_{n+1}^{t+1} \mid \mathcal{F}_n) \leq \exp(-\lambda r_{n+1}^{1-2\epsilon} C_2) = \exp(-C_2 r_{n+1}^{-\epsilon}).$$

Fix an ω such that the estimate $|\tilde{\ell} \cap \mathcal{F}_{n+1}| \leq P_{n+1} R r_{n+1}^{t+1}$ holds for all lines $\tilde{\ell} \in \mathcal{A}_{n+1}$, where \mathcal{A}_{n+1} is given by Lemma 3.1. Then by Lemma 3.1 for any admissible line ℓ we find a line $\tilde{\ell} \in \mathcal{A}_{n+1}$ with

$$|\ell \cap \mathcal{F}_{n+1}| \leq |\tilde{\ell} \cap \mathcal{F}_{n+1}| + r_{n+1} \leq P_{n+1} R r_{n+1}^{t+1} + r_{n+1},$$

that is, $G(\ell, n+1)$ occurs. Thus by above calculations

$$\begin{aligned} \mathbb{E}(\chi_{\cap_{k \geq n} \Gamma_k} \mathbb{P}(G_{n+1}^c \mid \mathcal{F}_n)) \\ \leq \mathbb{E}(\chi_{\cap_{k \geq n} \Gamma_k} \mathbb{P}(|\ell \cap \mathcal{F}_{n+1}| \geq P_{n+1} R r_{n+1}^t \text{ for some } \tilde{\ell} \in \mathcal{A}_{n+1} \mid \mathcal{F}_n)) \\ \leq \mathbb{P}(\cap_{k=1}^n \Gamma_k) C r_{n+1}^{-4} \exp(-C_2 r_{n+1}^{-\epsilon}), \end{aligned}$$

as required. \square

Theorem 4.7. *We have $\mathbb{P}(\cap_{k=1}^{\infty} \Gamma_k) > 0$.*

Proof. Choose N so large that $1 - 2000 r_n^{-3} \exp(-C_3 r_n^{-\epsilon}) - C r_n^{-4} \exp(-C_2 r_n^{-\epsilon})$ are positive and the claims of Propositions 4.4 and 4.6 hold for $n \geq N$. Choosing the constant R in Notation 4.1 large enough we can make sure that $\mathbb{P}(\cap_{i=1}^N \Gamma_i) > 0$. Observe that the choice of the constants C, C_2, C_3 can be made independent of $R \geq 1$. Then, by Propositions 4.4 and 4.6, for all $m \geq N+1$,

$$\begin{aligned} \mathbb{P}(\Gamma_m \mid \cap_{k=1}^{m-1} \Gamma_k) \\ &= \frac{\mathbb{P}(\Gamma_m \cap \cap_{k=1}^{m-1} \Gamma_k)}{\mathbb{P}(\cap_{k=1}^{m-1} \Gamma_k)} \\ &= \frac{1}{\mathbb{P}(\cap_{k=1}^{m-1} \Gamma_k)} \mathbb{E}(\chi_{\cap_{k \leq m-1} \Gamma_k} \mathbb{P}(\Gamma_m \mid \mathcal{F}_{m-1})) \\ &\geq \frac{1}{\mathbb{P}(\cap_{k=1}^{m-1} \Gamma_k)} \mathbb{E}(\chi_{\cap_{k \leq m-1} \Gamma_k} (1 - \mathbb{P}(A_m^c \mid \mathcal{F}_{m-1}) - \mathbb{P}(G_m^c \mid \mathcal{F}_{m-1}))) \\ &\geq 1 - 2000 r_m^{-3} \exp(-C_3 r_m^{-\epsilon}) - C r_m^{-4} \exp(-C_2 r_m^{-\epsilon}). \end{aligned}$$

Iterating the observation

$$\begin{aligned} \mathbb{P}(\cap_{k=1}^n \Gamma_k) &= \mathbb{P}(\cap_{k=1}^N \Gamma_k) \prod_{m=N+1}^n \mathbb{P}(\Gamma_m \mid \cap_{i=1}^{m-1} \Gamma_i) \\ &\geq \mathbb{P}(\cap_{k=1}^N \Gamma_k) \prod_{m=N+1}^n (1 - 2000 r_m^{-3} \exp(-C_3 r_m^{-\epsilon}) \\ &\quad - C r_m^{-4} \exp(-C_2 r_m^{-\epsilon})) \end{aligned}$$

and noticing that the series

$$\sum_{m=1}^{\infty} 2000r_m^{-3} \exp(-C_3r_m^{-\epsilon}) + Cr_m^{-4} \exp(-C_2r_m^{-\epsilon})$$

converges, finishes the proof. \square

We finish the proof of Theorem 2.1 by giving a proof to Theorem 2.3.

Proof of Lemma 2.3. Let $\omega \in \bigcap_{k=1}^{\infty} \Gamma_k$. Fix a line ℓ , a point $x \in \ell$ and $r > 0$. Let n be number with $\frac{1}{5}r_{n+1} < 2r \leq \frac{1}{5}r_n$ and ℓ' the line perpendicular to ℓ with $\ell \cap \ell' = \{x\}$, and

$$S = \{y \in \mathbb{R}^2 \mid d(y, \ell') < r\}.$$

Then $S = \pi_{\ell}^{-1}(B(x, r))$. Since $\omega \in \Gamma_{n+1}$, the event A_{n+1} occurs and, in particular, $A(S, n + 1)$ occurs as well. Thus, taking (4) into account,

$$\mu_{n+1}(S) \leq 500 \cdot 5^t RP_{n+1}w(S)^t c_{n+1}r_{n+1}^2 \leq 500 \cdot 5^t R w(S)^t.$$

For $k \geq n + 1$, we have $\mu_k(Q) = \mu_{n+1}(Q)$ for all $Q \in \mathcal{F}_{n+1}$ and thus

$$\mu_k(\pi_{\ell}^{-1}(B(x, r))) \leq 500 \cdot 5^t 2^t R r^t,$$

as well. Letting $k \rightarrow \infty$, the claim follows. \square

5. Application to random covering sets

In this section Theorem 2.1 is applied to the problem of calculating dimensions of projections of random covering sets. The random covering set is defined as a subset of torus, but then projected as a subset of plane. We first recall the definition of random covering sets with generating sequence of ball-like sets. A sequence (g_n) is a sequence of ball-like subsets of the torus \mathbb{T}^2 if there exists a sequence of balls $B(x_n, \delta_n) \subset g_n$ such that $\limsup_{n \rightarrow \infty} \frac{\rho_n}{\delta_n} < +\infty$, where ρ_n is the diameter of g_n and decreases to zero. More generally, we can replace this by the slightly weaker condition

$$\lim_{n \rightarrow \infty} \frac{\log \rho_n}{\log \delta_n} = 1. \tag{11}$$

Let ξ_n be a sequence of independent random variables uniformly distributed on \mathbb{T}^2 , and denote the induced probability space by (Ω, \mathbb{P}) . The random covering set is defined as

$$E := \limsup_{n \rightarrow \infty} (\xi_n + g_n).$$

As shown in [7, Proposition 4.7], the mass transference principle [1, Theorem 2], easily implies that almost surely,

$$\dim_H(E) = \inf\{s \geq 0 : \sum_{n=1}^{\infty} \rho_n^s < \infty\} = \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \rho_n} := s_0,$$

provided $s_0 \leq 2$. In the above formula, ρ_n can be replaced by δ_n due to (11).

The following theorem is the main result of the paper.

Theorem 5.1. *Let $s_0 < 1$. Then almost surely for every line ℓ , we have*

$$\dim_H \pi_\ell(E) = s_0.$$

Proof. Since the upper bound $\dim_H \pi_\ell(E) \leq s_0$ is clear, it suffices to show the almost sure lower bound $\dim_H \pi_\ell(E) \geq s_0$.

For simplicity, we assume that the balls $B(x_n, \delta_n)$ are centered at the origin. This will not cause loss of generality since they are to be translated by uniformly distributed random variables. Denote by $B_n = \xi_n + B(0, \delta_n) = B(\xi_n, \delta_n) \subset \mathbb{T}^2$ the randomly translated balls. Then $\limsup_{n \rightarrow \infty} B_n \subset E$.

Our aim is to construct, with positive probability, a random Cantor-like set $F \subset E$ in order to apply Theorem 2.1.

To that end, choose a sequence $\{n_k\}$ of natural numbers such that

$$\lim_{k \rightarrow \infty} \frac{\log n_k}{-\log \delta_{n_k}} = \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \delta_n} = s_0, \tag{12}$$

$$n_k \geq 256 k^2 n_{k-1}^2 \delta_{n_{k-1}}^{-2}, \tag{13}$$

$$\lim_{k \rightarrow \infty} \frac{\log \delta_{n_{k-1}}}{\log \delta_{n_k}} = 0. \tag{14}$$

We now construct the Cantor-like subset by induction. Our construction is a simplification of the one in [7], but we repeat the argument for the convenience of the reader. Let $N_1 = \tilde{N}_1 = \lfloor \frac{1}{2} n_1 \rfloor$ and define $I(1, \mathbb{T}^2) = \{1, \dots, N_1\}$. Decompose \mathbb{T}^2 (which we identify with the unit square) into M_1^2 disjoint subsquares, where $M_1 = \lceil 2\delta_{n_1}^{-1} \rceil$, and denote by \mathcal{D}_1 the collection of closed M_1 -adic subsquares of \mathbb{T}^2 . Here and hereafter, for all $x \in \mathbb{R}$, the notation $\lfloor x \rfloor$ is used for the largest integer smaller than x and $\lceil x \rceil$ for the smallest integer larger than x . For each $i \in I(1, \mathbb{T}^2)$, pick $Q_i \in \mathcal{D}_1$ satisfying

$$\xi_i \in Q_i \subset B_i, \tag{15}$$

Note that, since δ_i are in decreasing order and the sidelength of the cubes satisfies $M_1^{-1} \leq \frac{\delta_{n_1}}{2}$, such a choice is possible. In case there are more than one cubes satisfying (15), that is, ξ_i is on the boundary of a cube, just choose any of them (Since the boundaries of the cubes are of zero Lebesgue measure, the event that any of the ξ_i lies on the boundary of a cube has zero probability anyway). Denote the collection of such Q_i 's by

$$\mathcal{C}_1 = \{Q_i \in \mathcal{D}_1 : i \in I(1, \mathbb{T}^2)\}.$$

For completeness (see the definition of Ω_k and q_k for $k > 1$ below), we define $\Omega_1 = \Omega$ and $q_1 = \mathbb{P}(\Omega_1) = 1$.

We continue the random construction inductively as follows. Assume that numbers M_1, \dots, M_{k-1} and $\tilde{N}_1, \dots, \tilde{N}_{k-1}$ have been defined and suppose that for all $i = 1, \dots, k - 1$ we are given a collection \mathcal{C}_i of \tilde{N}_i squares in \mathcal{D}_i , where \mathcal{D}_i is the family of $(\prod_{j=1}^i M_j)$ -adic subsquares of \mathbb{T}^2 .

Denote

$$\mathcal{C}_{k-1} = \{\tilde{Q}_1, \dots, \tilde{Q}_{\tilde{N}_{k-1}}\}.$$

Let $m_k = \lfloor \frac{n_k - n_{k-1}}{\tilde{N}_{k-1}} \rfloor$. For any $1 \leq l \leq \tilde{N}_{k-1}$, consider the random family of indices consisting of those $n_{k-1} + (l - 1)m_k + 1 \leq j \leq n_{k-1} + lm_k$, for which $\xi_j \in \tilde{Q}_l$. If there are at least $\lfloor \frac{1}{2}m_k \mathcal{L}(\tilde{Q}_l) \rfloor := N_k$ such indices j , let $I(k, \tilde{Q}_l)$ denote the first N_k of them. Let Ω_k be the event that $I(k, \tilde{Q}_l)$ is well defined for all $\tilde{Q}_l \in \mathcal{C}_{k-1}$.

Conditioning on Ω_k , decompose every $Q \in \mathcal{D}_{k-1}$ to M_k^2 disjoint subsquares, where $M_k = \lceil 2\delta_{n_k}^{-1} (\prod_{j=1}^{k-1} M_j)^{-1} \rceil$, and denote by \mathcal{D}_k the collection of closed $(\prod_{j=1}^k M_j)$ -adic subsquares of \mathbb{T}^2 . For any $i \in I(k, \tilde{Q}_l)$, pick $Q_i \in \mathcal{D}_k$ satisfying

$$\xi_i \in Q_i \subset B_i.$$

Since the sidelength of these squares is $(\prod_{j=1}^k M_j)^{-1} \leq \frac{\delta_{n_k}}{2}$, one can always find such Q_i , and it is unique outside a zero measure set of ξ_i -s. Set

$$\mathcal{C}_k = \{Q_i \in \mathcal{D}_k : Q \in \mathcal{C}_{k-1}, i \in I(k, Q)\}$$

and $\tilde{N}_k := \#\mathcal{C}_k = \tilde{N}_{k-1} N_k$.

For $\omega \in \Omega_\infty = \bigcap_{k \geq 1} \Omega_k$, consider the random Cantor set $F = F(\omega)$ defined as

$$F = \bigcap_{k=1}^{\infty} \bigcup_{Q \in \mathcal{C}_k} Q \subset \limsup_{n \rightarrow \infty} B_n \subset E.$$

We next check that $\mathbb{P}(\Omega_\infty) > 0$. Using [7, Proposition 2.6] and the construction of \mathcal{C}_k , we have the following lower bound for the conditional probabilities $q_k := \mathbb{P}(\Omega_k | \Omega_1, \dots, \Omega_{k-1})$,

$$q_k \geq 1 - \tilde{N}_{k-1} \frac{4(1 - \mathcal{L}(Q))}{m_k \mathcal{L}(Q)} \geq 1 - \frac{4\tilde{N}_{k-1} (\prod_{j=1}^{k-1} M_j)^2}{m_k}$$

where $Q \in \mathcal{C}_{k-1}$. Note that since

$$m_k \geq \frac{n_k - n_{k-1}}{2\tilde{N}_{k-1}} \geq \frac{n_k}{4\tilde{N}_{k-1}} \tag{16}$$

and $\prod_{j=1}^{k-1} M_j \leq 4\delta_{n_{k-1}}^{-1}$, we have

$$q_k \geq 1 - \frac{256\tilde{N}_{k-1}^2 \delta_{n_{k-1}}^{-2}}{n_k} \geq 1 - \frac{256n_{k-1}^2 \delta_{n_{k-1}}^{-2}}{n_k} \geq 1 - \frac{1}{k^2},$$

where the second inequality follows because $\tilde{N}_{k-1} \leq n_{k-1}$ and the third one is due to (13). Therefore, $\mathbb{P}(\Omega_\infty) = \prod_{k=1}^\infty q_k > 0$.

Now F is a random Cantor set as defined in Section 2 with the defining sequences (M_n) and (N_n) . Thus, we can apply Theorem 2.1 to obtain with positive probability for all lines $\ell \subset \mathbb{R}^2$,

$$\dim_H \pi_\ell(E) \geq \dim_H \pi_\ell(F) = \liminf_{k \rightarrow \infty} \frac{\sum_{j=1}^k \log N_j}{\sum_{j=1}^k \log M_j}.$$

Since $\prod_{j=1}^k M_j \leq 4\delta_{n_k}^{-1}$ and using (16)

$$N_k \geq \frac{m_k}{4 \prod_{j=1}^{k-1} M_j^2} \geq \frac{n_k \delta_{n_{k-1}}^2}{16\tilde{N}_{k-1}} \geq \frac{n_k \delta_{n_{k-1}}^3}{16},$$

recalling (14) we have,

$$\frac{\sum_{j=1}^k \log N_j}{\sum_{j=1}^k \log M_j} \geq \lim_{k \rightarrow \infty} \frac{\log n_k}{-\log \delta_{n_k}} = s_0.$$

Finally, since

$$\dim_H \pi_\ell(E) \geq s_0 \text{ for all lines } \ell$$

is a tail event of positive probability, the Kolmogorov zero-one law implies that it must have full probability. □

Remark 5.2. In [7] and [10], dimension formulas are obtained for certain affine type random covering sets. For these classes of random covering sets there is no similar result to Theorem 5.1. Here is a counterexample.

Let (g_n) be a sequence of rectangles in \mathbb{T}^2 with the sidelength of the sides parallel to the x -axis $\alpha_1(g_n) = n^{-\alpha}$ and of those parallel to the y -axis $\alpha_2(g_n) = n^{-\beta}$, where $\beta > \alpha > 1$. Then the Hausdorff dimension $\dim_H(E) = \alpha^{-1}$ almost surely. However, for the projection to the y -axis, π_y , instead $\dim_H(\pi_y(E)) = \beta^{-1}$.

6. Generalizations

For simplicity, in above we have considered the case $d = 2$ only. In a similar way, one can define the random Cantor sets F with defining sequences $(M_n), (N_n)$, $N_n \leq M_n^d$ and almost sure Hausdorff dimension

$$s = \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n \log N_i}{\sum_{i=1}^n \log M_i}$$

also on the higher dimensional torus \mathbb{T}^d for any $d \geq 3$. Embedding \mathbb{T}^d in a natural way into the unit cube of \mathbb{R}^d one can then consider orthogonal projections of F into k -dimensional linear subspaces of \mathbb{R}^d . Let $\mathbb{G}_{d,k}$ denote the family of all k -dimensional linear subspaces of \mathbb{R}^d and for $V \in \mathbb{G}_{d,k}$, let π_V denote the orthogonal projection onto V . The proofs from Sections 2–4 can be easily modified to this setting in order to obtain the following generalization of Theorem 2.1.

Theorem 6.1. *If $s < k \leq d$, then almost surely for every $V \in \mathbb{G}_{d,k}$, we have $\dim_H \pi_V(F) = s$.*

As in case $d = 2$, this result can then be applied for the random covering sets of the torus in order to prove

Theorem 6.2. *Let $E \subset \mathbb{T}^d$ be a ball-like random covering set with almost sure Hausdorff dimension $s_0 < k$. Then almost surely for all $V \in \mathbb{G}_{d,k}$,*

$$\dim_H \pi_V(E) = s_0.$$

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