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Multifractal analysis of Birkhoff averages for typical infinitely generated self-affine sets

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Abstract. We develop a thermodynamic formalism for quasi-multiplicative potentials on a countable symbolic space and apply these results to the dimension theory of infinitely generated self-affine sets. The first application is a generalisation of Falconer's dimension formula to include typical infinitely generated self-affine sets and show the existence of an ergodic invariant measure of full dimension whenever the pressure function has a root. Considering the multifractal analysis of Birkhoff averages of general potentials Φ taking values in $\mathbb{R}^{\mathbb{N}}$, we give a formula for the Hausdorff dimension of $J_\Phi(\alpha)$, the α-level set of the Birkhoff average, on a typical infinitely generated self-affine set. We also show that for bounded potentials Φ, the Hausdorff dimension of $J_\Phi(\alpha)$ is given by the maximum of the critical value for the pressure and the supremum of Lyapunov dimensions of invariant measures μ for which $\int \Phi \, d\mu = \alpha$. Our multifractal results are new in both the finitely generated and the infinitely generated setting.

Contents

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1. Introduction

Let F be the repeller of a piecewise smooth map

$$g: X \longrightarrow X.$$

Given a continuous function

$$\phi: F \longrightarrow \mathbb{R}^N$$

and $\alpha \in \mathbb{R}^N$, we are interested in the set of points in the repeller for which the Birkhoff average is equal to α,

$$J_\phi(\alpha) = \left\{x \in F: \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(g^i(x)) = \alpha\right\}. \tag{1.1}$$

The central question in the multifractal analysis of Birkhoff averages is to determine the Hausdorff dimension of the level sets $J_\varphi(\alpha)$. For conformal expanding maps on compact repellers the Hausdorff dimension is given by a well known conditional variational principle; see e.g. Pesin and Weiss [33], Fan, Feng, and Wu [9], Barreira and Saussol [3], Feng, Lau, and Wu [17] and Olsen [31] and [32].

Situations in which either the map f is non-conformal or the repeller F is non-compact are far less well understood. The vast majority of the work on non-conformal systems has focused on maps which are obtained as skew products of conformal systems; see e.g. Barral and Mensi [2], Barral and Feng [1] and Reeve [34] and [35]. Jordan and Simon [26] have given a conditional variational principle for typical members of parameterisable families of self-affine iterated function systems with a simultaneously diagonalisable linear part.

Recently there has also been a great deal of work dealing with cases in which the repeller F is a non-compact limit set of a countable collection of contractions; see e.g. [8], [10], [11], [12], [21], [23], [29], [30], and [35]. All but one of these results concern situations in which the map f is conformal. The only exception being [35] which deals with a family of skew products including the direct product of the Gauss map and the doubling map.

There are two facts concerning the space of invariant measures for a continuous map of a compact metric space which make the dimension theory of compact systems a great deal easier to handle. The first is that if the space itself is compact, then the space of invariant probability measures is also compact. Thus given a sequence of invariant measures one can always extract a convergent subsequence. The second fact is that for compact systems entropy is an upper-semicontinuous function on the space of invariant measures, so given a sequence of invariant measures one may extract a weak star limit point with entropy equal to the upper limit of the entropies of the measures in the sequence. Since our setting will be non-compact the main challenge comes from the lack of these two facts.

In this article, the underlying set is an infinitely generated self-affine set. Besides that this set is not compact, we have the added complication coming from the fact that the natural potential associated to the set is usually not multiplicative. As the first main result, stated in Theorem A, we establish a variational principle for quasi-multiplicative potentials. The main task is to show the existence of a Gibbs measure. In Theorem B, by applying the previously developed thermodynamical formalism, we deliver a dimension formula for typical infinitely generated self-affine sets. In Theorems C and D, we further develop the dimension theory of such sets by giving a formula for the Hausdorff dimension of the α-level set corresponding to (1.1). In order to prove these results, we use and generalise some of the ideas presented by Gelfert and Rams [19] and Fan, Jordan, Liao, and Rams [10]. We also introduce the concept of $\mathcal{M}$-trees to be able to better deal with the difficulties arising from quasi-multiplicativity. Theorem D generalises [10], Theorem 1.2, to the self-affine setting.

The article is organised as follows. In §2, we exhibit and motivate the results, and in §3–6, we provide the reader with all the necessary details.

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2. Preliminaries and statement of results

2.1. Thermodynamic formalism for sub-multiplicative potentials. The classical thermodynamical formalism is an effective tool in the dimension theory of conformal dynamical systems. However, in the non-conformal setting this is no longer the appropriate tool. In §3, we develop a suitable formalism to study dimension theory of infinitely generated self-affine sets.

Define

$$\Sigma = \mathbb{N}^{\mathbb{N}}$$

to be the set of all infinite words constructed from natural numbers. Let

$$\Sigma_n = \mathbb{N}^n, \quad n \in \mathbb{N}$$

and

$$\Sigma_* = \bigcup_{n\in\mathbb{N}} \Sigma_n$$

be the collection of all finite words. If $\omega \in \Sigma_*$ and $\tau \in \Sigma_* \cup \Sigma$, then $\omega\tau$ denotes the concatenation of ω and τ. Furthermore, if $\omega \in \Sigma_* \cup \Sigma$ and $n \in \mathbb{N}$, then $\omega|_n$ is the unique word in Σ_n for which there is $\tau \in \Sigma$ so that $\omega|_n\tau = \omega$. If $\omega, \tau \in \Sigma_* \cup \Sigma$, then by $\omega \wedge \tau$ we mean the common beginning of ω and τ. Given $n \in \mathbb{N}$ and $\omega \in \Sigma_n$ we set

$$|\omega| = n$$

and define the cylinder set given by ω to be

$$[\omega] = \{\omega\tau : \tau \in \Sigma\}.$$

We denote the left shift operator by σ and let $\mathcal{M}_\sigma(\Sigma)$ be the set of all σ-invariant Borel probability measures on Σ.

We equip Σ with the product discrete topology and call it a *shift space*. We could have also defined the shift space by using a finite alphabet, i.e. by setting

$$\Sigma = I^{\mathbb{N}}$$

for some finite set $I \subset \mathbb{N}$. In this case, we say that the shift space is *finitely generated*. Observe that the shift space is compact if and only if it is finitely generated. Moreover, the cylinder sets are open and closed and they generate the Borel σ-algebra.

We shall consider maps

$$\varphi : \Sigma_* \longrightarrow (0, \infty)$$

which we refer to as *potentials*. We remark that in the literature, usually the function $\log \varphi$ is termed potential. Since in this article we are more concerned with φ rather than its logarithm we chose to deviate from the usual convention. We say that a potential φ is *sub-multiplicative* if

$$\varphi(\omega\tau) \le \varphi(\omega)\varphi(\tau)$$

for all $\omega, \tau \in \Sigma_*$. A sub-multiplicative potential φ is said to be *quasi-multiplicative* if there exist a constant $c \ge 1$ and a finite subset $\Gamma \subset \Sigma_*$ such that for any given pair $\omega, \tau \in \Sigma_*$ there exists $\kappa \in \Gamma$ with

$$\varphi(\omega)\varphi(\tau) \le c\varphi(\omega\kappa\tau). \tag{2.1}$$

We also define

$$K = \max\{|\omega| : \omega \in \Gamma\} + 1.$$

A sub-multiplicative potential φ is said to be *almost-multiplicative* if there exists a constant $c \ge 1$ such that

$$\varphi(\omega)\varphi(\tau) \le c\varphi(\omega\tau)$$

for all $\omega, \tau \in \Sigma_*$. If the constant c above equals 1, then φ is *multiplicative*. In Proposition 2.3 and Remarks 2.4, we exhibit various conditions introducing different types of potentials.

If φ is a sub-multiplicative potential, then we define the *pressure* $P(\varphi)$ by setting

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log Z_n(\varphi),$$

where $Z_n(\varphi) = \sum_{\omega\in\Sigma_n} \varphi(\omega)$ for all $n \in \mathbb{N}$. Note that by the sub-multiplicativity, the pressure is well-defined, although it may not be finite. It is immediate that $P(\varphi) = \infty$ if and only if $Z_n(\varphi) = \infty$ for all $n \in \mathbb{N}$. Thus, if the shift space is finitely generated, then $P(\varphi) < \infty$. Observe that even if the shift space is finitely generated, the pressure can be negative infinity. Let

$$\psi\colon \Sigma_* \longrightarrow (0,\infty)$$

be a sub-multiplicative potential so that $P(\psi) < \infty$ and $Z_{n+m}(\psi) \geq cZ_n(\psi)Z_m(\psi)$ for some constant $c > 0$. If the shift space is finitely generated, then the potential $\psi \equiv 1$ satisfies these assumptions. Now defining

$$\varphi\colon \Sigma_* \longrightarrow (0,\infty)$$

by setting

$$\varphi(\omega) = (cZ_n(\psi)n!)^{-1}\psi(\omega), \quad \omega \in \Sigma_*,$$

it is easy to see that φ is sub-multiplicative with $P(\varphi) = -\lim_{n\to\infty} \frac{1}{n}\log n! = -\infty$.

We let $\mathcal{M}_\sigma(\Sigma)$ denote the set of all σ-invariant Borel probability measures on Σ. Given $\mu \in \mathcal{M}_\sigma(\Sigma)$ along with a sub-multiplicative potential φ, we define the *measure-theoretical pressure* $P_\mu(\varphi)$ by setting

$$P_\mu(\varphi) = \inf_{n\in\mathbb{N}} \frac{1}{n} \sum_{\omega\in\Sigma_n} \mu([\omega]) \log \frac{\varphi(\omega)}{\mu([\omega])}. \tag{2.2}$$

We adopt the usual convention according to which $0\log(x/0) = 0\log 0 = 0$ for all $x > 0$.

Lemma 2.1. *If φ is a sub-multiplicative potential and $\mu \in \mathcal{M}_\sigma(\Sigma)$, then*

$$P_\mu(\varphi) = \lim_{n\to\infty} \frac{1}{n} \sum_{\omega\in\Sigma_n} \mu([\omega]) \log \frac{\varphi(\omega)}{\mu([\omega])}.$$

Proof. The proof follows from the standard theory of sub-additive sequences by the sub-multiplicativity of φ, the concavity of the function $H(x) = -x\log x$, and the invariance of μ. □

Furthermore, we define the *Lyapunov exponent* $\Lambda_\mu(\varphi)$ for φ and the *entropy* h_μ of μ by setting

$$\begin{aligned}\Lambda_\mu(\varphi) &= \lim_{n\to\infty} \frac{1}{n} \sum_{\omega\in\Sigma_n} \mu([\omega]) \log \varphi(\omega) \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n} \sum_{\omega\in\Sigma_n} \mu([\omega]) \log \varphi(\omega)\end{aligned} \tag{2.3a}$$

and

$$
\begin{aligned}
h_\mu &= \lim_{n\to\infty} \frac{1}{n} \sum_{\omega\in\Sigma_n} -\mu([\omega]) \log \mu([\omega]) \\
&= \inf_{n\in\mathbb{N}} \frac{1}{n} \sum_{\omega\in\Sigma_n} -\mu([\omega]) \log \mu([\omega]) \geq 0,
\end{aligned} \tag{2.3b}
$$

respectively. Observe that if the potential φ is bounded, then $\Lambda_\mu(\varphi) \leq 0$. Similar to the proof of Lemma 2.1, we see that the Lyapunov exponent and the entropy are well-defined by the sub-multiplicativity of φ and the invariance of μ.

Lemma 2.2. *If φ is a sub-multiplicative potential, then*

$$P(\varphi) \geq P_\mu(\varphi)$$

for all $\mu \in \mathcal{M}_\sigma(\Sigma)$. Furthermore, if $h_\mu < \infty$ or $\Lambda_\mu(\varphi)$ is finite, then

$$P_\mu(\varphi) = h_\mu + \Lambda_\mu(\varphi).$$

Proof. To show the first claim, we may assume that $P_\mu(\varphi) > -\infty$ and $P(\varphi) < \infty$. Thus $\sum_{\omega\in\Sigma_n} \mu([\omega]) \log \varphi(\omega)/\mu([\omega]) > -\infty$ for all $n \in \mathbb{N}$ and there is $n_0 \in \mathbb{N}$ so that $Z_n(\varphi) < \infty$ for all $n \geq n_0$. For each $n \geq n_0$ and $C_n \subset \Sigma_n$ we use the concavity of the function $H(x) = -x \log x$ to obtain

$$
\begin{aligned}
&\sum_{\omega\in C_n} \mu([\omega]) \Big(\log \frac{\varphi(\omega)}{\mu([\omega])} - \log \sum_{\kappa\in C_n} \varphi(\kappa) \Big) \\
&\quad = \sum_{\omega\in C_n} \beta(\omega) H(\mu([\omega])/\beta(\omega)) \\
&\quad \leq H\Big(\sum_{\omega\in C_n} \beta(\omega)\mu([\omega])/\beta(\omega) \Big) \in \Big[0, \frac{1}{e}\Big],
\end{aligned} \tag{2.4}
$$

where $\beta(\omega) = \varphi(\omega)/\sum_{\kappa\in C_n} \varphi(\kappa)$. Dividing by n before letting $n \to \infty$ proves the first claim.

To show the second claim, we first assume that $\Lambda_\mu(\varphi)$ is finite. Notice first that if $h_\mu < \infty$, then also $P_\mu(\varphi) = h_\mu + \Lambda_\mu(\varphi)$ is finite. On the other hand, if $P_\mu(\varphi) < \infty$, then there is $n_0 \in \mathbb{N}$ so that

$$-\infty < \Lambda_\mu(\varphi) \leq \frac{1}{n} \sum_{\omega\in\Sigma_n} \mu([\omega]) \log \varphi(\omega)$$

and

$$\frac{1}{n} \sum_{\omega\in\Sigma_n} \mu([\omega]) \log \frac{\varphi(\omega)}{\mu([\omega])} \leq P_\mu(\varphi) + 1 < \infty$$

for all $n \geq n_0$. Thus

$$\frac{1}{n} \sum_{\omega \in \Sigma_n} -\mu([\omega]) \log \mu([\omega]) \leq P_\mu(\varphi) - \Lambda_\mu(\varphi) + 1$$

for all $n \geq n_0$ and $h_\mu < \infty$. Therefore, if $h_\mu = \infty$, then $P_\mu(\varphi) = \infty$ and the desired equality holds.

Finally, we notice that the proof of the second claim in the case $h_\mu < \infty$ is similar. □

Our first main result is the following variational principle. The proof of the result can be found in the end of §3.2.

Theorem A. *If φ is a quasi-multiplicative potential, then*

$$P(\varphi) = \sup\{P_\mu(\varphi) \colon \mu \in \mathcal{M}_\sigma(\Sigma)\}.$$

Moreover, if $P(\varphi) < \infty$, then there exists a unique invariant measure μ for which

$$P(\varphi) = P_\mu(\varphi).$$

If the shift space is finitely generated, then we always have $P_\mu(\varphi) = h_\mu + \Lambda_\mu(\varphi)$. Moreover, the variational principle holds for all sub-multiplicative potentials; see Käenmäki [27], Theorem 2.6, and Cao, Feng, and Huang [5], Theorem 1.1. Quasi-multiplicativity has been a crucial property in the study of Lyapunov exponents for products of matrices; see e.g. Feng and Lau [16], Feng [13], and Feng and Käenmäki [15]. It has also been used in connection with finitely generated self-affine sets; see Feng [14] and Falconer and Sloan [6].

In the infinitely generated case, Iommi and Yayama [22], Theorem 3.1, have recently verified the variational principle for almost-multiplicative potentials. Although the main steps of the proof in our setting are the same as in [22], §4, we have the added complication coming from the quasi-multiplicativity. It should also be noted that the dynamical system considered by Iommi and Yayama is more general than ours.

2.2. Infinitely generated self-affine sets. A classical result of Hutchinson [20] states that for every finite collection of strictly contractive mappings

$$f_1, \ldots, f_N \colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

there is a unique non-empty compact set $F \subset \mathbb{R}^d$ for which

$$F = \bigcup_{i=1}^{N} f_i(F). \tag{2.5}$$

If the mappings above are affine, then the set F is called *self-affine*. We are interested in countable collections of uniformly strictly contractive affine mappings $\{f_i\}_{i\in\mathbb{N}}$. A natural generalisation of the condition (2.5) is to require that a non-empty set F satisfies

$$F = \bigcup_{i\in\mathbb{N}} f_i(F). \tag{2.6}$$

We shall define the infinitely generated self-affine set as a canonical projection of the shift space. This set then satisfies the above condition.

Let $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ be such that $\sup_{i\in\mathbb{N}} \|T_i\| < 1$. Define

$$\mathbf{A} = ([0,1]^d)^{\mathbb{N}}$$

and note that by the Kolmogorov extension theorem $\mathbf{A}$ supports a natural probability measure

$$\mathscr{L}_{\mathbf{A}} = (\mathscr{L}^d|_{[0,1]^d})^{\mathbb{N}}.$$

If

$$\mathbf{a} = (a_i)_{i\in\mathbb{N}} \in \mathbf{A},$$

then we have a collection of uniformly strictly contractive affine mappings $\{f_i = T_i + a_i\}_{i\in\mathbb{N}}$. For each choice of $\mathbf{a} \in \mathbf{A}$ we associate a *projection*

$$\pi_{\mathbf{a}} : \Sigma \longrightarrow \mathbb{R}^d$$

defined by

$$\pi_{\mathbf{a}}(\omega) = \sum_{j=1}^{\infty} T_{\omega|_{j-1}} a_{\omega_j} = \lim_{n\to\infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(0).$$

Here $T_\omega = T_{\omega_1} \cdots T_{\omega_n}$ for all $\omega = \omega_1 \cdots \omega_n \in \Sigma_n$ and $n \in \mathbb{N}$. The set

$$F_{\mathbf{a}} = \pi_{\mathbf{a}}(\Sigma)$$

is termed *self-affine*. Since $f_i(\pi_{\mathbf{a}}(\omega)) = \pi_{\mathbf{a}}(i\omega)$ for all $\omega \in \Sigma$ and $i \in \mathbb{N}$ we see that $F_{\mathbf{a}}$ satisfies (2.6). Furthermore, if the shift space is finitely generated, then $F_{\mathbf{a}}$ is the unique compact set satisfying (2.5).

The dimension theory of finitely generated self-affine sets for a typical choice of $\mathbf{a}$ was first investigated by Falconer [7]. A central tool in Falconer's analysis was the singular value function. Given a matrix $T \in \mathrm{GL}_d(\mathbb{R})$ we let

$$\gamma_1(T) \geq \cdots \geq \gamma_d(T) > 0$$

denote the singular values of T, in non-increasing order of magnitude. The singular values are the square roots of the eigenvalues of T^*T or, equivalently, the principal semiaxis of the ellipsis $T(B(0,1))$. For example, we have $\gamma_1(T) = \|T\|$ and $\gamma_d(T) = \|T^{-1}\|^{-1}$. For the sake of geometric intuition, we notice that in $\mathbb{R}^2$, we need approximately $\gamma_1(T)/\gamma_2(T)$ many balls of radius $\gamma_2(T)$ to cover $T(B(0,1))$.

Thus the s-dimensional Hausdorff content of any subset of $T(B(0,1))$ can be estimated above by a constant times $\gamma_1(T)\gamma_2(T)^{s-1}$, which is a certain singular value function. To cover the self-affine set, Falconer used ellipses arising naturally from the construction; recall (2.5). His crucial observation was that the behaviour of the singular value function, perhaps rather surprisingly, gave also the lower bound for the Hausdorff dimension of a finitely generated self-affine set for almost all $\mathbf{a}$.

We shall now give the precise definition for the *singular value function* φ^s. If $0 \le s = m + \delta \le d$ with $m \in \mathbb{Z}$ and $0 < \delta \le 1$, then we set

$$\varphi^s(T) = \gamma_1(T)\cdots\gamma_m(T)\gamma_{m+1}(T)^\delta$$

for all $T \in \mathrm{GL}_d(\mathbb{R})$. When $s \ge d$, we set

$$\varphi^s(T) = |\det(T)|^{s/d}$$

for completeness. Given $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ with $\sup_{i\in\mathbb{N}} \|T_i\| < 1$ the singular value function introduces a potential by slightly abusing notation and setting

$$\varphi^s(\omega) = \varphi^s(T_\omega) \tag{2.7}$$

for all $\omega \in \Sigma_*$. Note that φ^s is bounded above by 1 for all $0 \le s \le d$. Singular values γ_i introduce potentials in a similar way. For example, if $s \ge 0$, then γ_1^s is the sub-multiplicative potential $\omega \mapsto \|T_\omega\|^s$.

Falconer [7], Lemma 2.1, showed that the singular value function is φ^s is sub-multiplicative. It follows that the corresponding sub-multiplicative pressure $P(\varphi^s)$ is well-defined. Observe that $\exp(nP(\varphi^s))$ describes the asymptotic behaviour of $Z_n(\varphi^s) = \sum_{\omega\in\Sigma_n} \varphi^s(\omega)$, which can be used to estimate the s-dimensional Hausdorff measure from above. Following the proof of [28], Lemma 2.1, we see that the function $s \mapsto P(\varphi^s)$ is strictly decreasing and thus finite on an interval I of $[0,\infty)$. Furthermore, it is convex on connected components of $I \setminus \{1,\dots,d\}$. Note that also the functions $s \mapsto P_\mu(\varphi^s)$ and $s \mapsto \Lambda_\mu(\varphi^s)$ are strictly decreasing and continuous for all $\mu \in \mathcal{M}_\sigma(\Sigma)$.

Falconer [7], Theorem 5.3, proved that given finitely many affine contractions with contraction ratios at most $\frac{1}{3}$ the Hausdorff dimension of the corresponding self-affine set $F_{\mathbf{a}}$ is given by the unique zero of $s \mapsto P(\varphi^s)$ for almost every translation vector $\mathbf{a}$. Later Solomyak extended Falconer's proof to self-affine sets with the contraction ratios up to $\frac{1}{2}$; see [36], Proposition 3.1(i). See Käenmäki [27], Theorem 4.5, and Jordan, Pollicott, and Simon [25], Theorem 1.7, for corresponding results for measures.

In our second main theorem, we generalise Falconer's dimension result to infinitely generated self-affine sets. The proof of the result can be found in §4.

Theorem B. *If* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *satisfies* $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$ *and the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$*, then*

$$\begin{aligned}\dim_{\mathrm{H}}(F_{\mathbf{a}}) &= \min\{d, \inf\{s\colon P(\varphi^s) \le 0\}\} \\ &= \sup\{\dim_{\mathrm{H}}(\pi_{\mathbf{a}}(I^{\mathbb{N}}))\colon I \subset \mathbb{N} \text{ is finite}\}\end{aligned}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$.

In Theorem B, we assume that the singular value function is quasi-multiplicative. We shall next analyse the generality of this assumption.

Proposition 2.3. *The singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$ *if* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *satisfies one of the following conditions.*

(1) *Suppose that* $d = 2$ *and for every line* $\ell \in \mathbb{R}^2$ *there is* $i \in \mathbb{N}$ *with* $T_i(\ell) \ne \ell$.

(2) *Suppose that* $d = 2$ *and the matrices* T_i *have strictly positive entries so that the ratio of the smallest and largest entry of* T_i *is uniformly bounded away from zero for all* $i \in \mathbb{N}$.

(3) *Suppose that* $d \in \mathbb{N}$ *and* $T_i = \mathrm{diag}(t_1^i, \ldots, t_d^i)$*, where* $1 > |t_1^i| > \cdots > |t_d^i| > 0$ *for all* $i \in \mathbb{N}$.

Proof. Assuming (1), [13], Proposition 2.8, shows that the potential γ_1 is quasi-multiplicative. Observe that the proof of [13], Proposition 2.8, applies verbatim in the infinite case. Similarly, assuming (2), [22], Lemma 7.1, shows that γ_1 is almost-multiplicative. The claim in both of these cases follows now by recalling that the determinant is the product of singular values. Finally, assuming (3), the multiplicativity of the singular value function is immediate. □

Remarks 2.4. (1) The assumption (1) in Proposition 2.3 is equivalent to the property that the matrices do not have a common eigenvector. Thus, if the 2×2 matrices cannot simultaneously be presented (in some coordinate system) as upper triangular matrices, then the singular value function φ^s is quasi-multiplicative for all $0 \le s \le d$.

(2) The set of $(T_i)_{i\in\mathbb{N}} \in GL_2(\mathbb{R})^{\mathbb{N}}$ satisfying the assumption (1) in Proposition 2.3 is open and dense set under the product topology. Indeed, the set of pairs $(T_1, T_2) \in GL_2(\mathbb{R})^2$ for which there is no common eigenvector is easily seen to be an open and dense set of full Lebesgue measure.

(3) Falconer and Sloan [6] have introduced a certain condition under which the singular value function is quasi-multiplicative also in higher dimensions; see [6], Corollary 2.3. Observe that a modification of [13], Proposition 2.8, can also be applied here. This condition can be considered as a higher dimensional version of the

assumption (1) in Proposition 2.3 tailored for the singular value function. Falconer and Sloan show in [6], Corollary 1.3, that the condition is open and dense provided that it is non-empty. It seems plausible that the non-emptyness can be proven by choosing the number of matrices large enough so that they all “point in a different direction”.

(4) None of the assumptions (1)–(3) in Proposition 2.3 are contained in each other. Indeed, two strictly positive 2×2 matrices having a common eigenvector show that strict positivity does not imply the assumption (1). Furthermore, if

$$T_1 = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & -1 \\ 10 & 11 \end{pmatrix},$$

then (T_1, T_2) satisfies the assumption (1), but it is easy to see that there is no coordinate system in which the matrices are simultaneously strictly positive. The rest of the cases are trivial to check.

2.3. Multifractal analysis of Birkhoff averages. If

$$\phi\colon \Sigma \longrightarrow \mathbb{R}$$

is a continuous function and μ is an ergodic probability measure, then, by the Birkhoff ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(\sigma^j(\omega)) = \int \phi \, \mathrm{d}\mu$$

for μ-almost all $\omega \in \Sigma$. The Birkhoff ergodic theorem does not give any information about the exceptional set where this equality does not hold. Since any two ergodic measures are mutually singular the exceptional set contains other ergodic measures and thus, the exceptional set cannot be considered negligible. Multifractal analysis of Birkhoff averages aims at understanding the structure of this exceptional set.

We shall consider Birkhoff averages of functions

$$\Phi\colon \Sigma \longrightarrow \mathbb{R}^{\mathbb{N}}.$$

The vector space $\mathbb{R}^{\mathbb{N}}$ is endowed with the product topology, so a sequence $(\alpha(n))_{n\in\mathbb{N}}$ with $\alpha(n) = (\alpha_i(n))_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ converges to $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ if $\lim_{n\to\infty} \alpha_i(n) = \alpha_i$ for each $i \in \mathbb{N}$. Given a function

$$\phi\colon \Sigma \longrightarrow \mathbb{R}$$

we define the *variation* $\mathrm{var}_n\, \phi$ for each $n \in \mathbb{N}$ by

$$\mathrm{var}_n\, \phi = \sup\{|\phi(\omega) - \phi(\tau)| : [\omega|_n] = [\tau|_n]\}.$$

The function ϕ is said to have *summable variations* if

$$\sum_{n=1}^{\infty} \mathrm{var}_n\, \phi < \infty.$$

We take a sequence $\Phi = (\phi_i)_{i\in\mathbb{N}}$ of functions $\phi_i : \Sigma \to \mathbb{R}$, each with summable variations, which we think of as a function from Σ to $\mathbb{R}^{\mathbb{N}}$. In this case, we just say that

$$\Phi\colon \Sigma \longrightarrow \mathbb{R}^{\mathbb{N}}, \quad \Phi(\omega) = (\phi_1(\omega), \phi_2(\omega), \dots),$$

has *summable variations*. Moreover, if each ϕ_i is bounded, then we say that Φ is *bounded*. We define the *Birkhoff sum* of Φ for each $n \in \mathbb{N}$ by

$$S_n\Phi = \sum_{j=0}^{n-1} \Phi \circ \sigma^j$$

and the *Birkhoff average* of Φ to be the limit of

$$A_n\Phi = n^{-1} S_n\Phi$$

as $n \to \infty$. We define $S_n\phi$ and $A_n\phi$ similarly when $\phi\colon \Sigma \to \mathbb{R}^k$ for some $k \in \mathbb{N}$. We let the *symbolic level set* to be

$$E_\Phi(\alpha) = \{\omega \in \Sigma\colon \lim_{n\to\infty} A_n\Phi(\omega) = \alpha\}$$

for all $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \overline{\mathbb{R}}^{\mathbb{N}}$, where

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

Suppose we have a self-affine set $F_{\mathbf{a}}$, that is, $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, $\mathbf{a} = (a_i)_{i\in\mathbb{N}} \in \mathbf{A}$, and $F_{\mathbf{a}} = \pi_{\mathbf{a}}(\Sigma)$ is the projection of the shift space. In Theorems C and D, we consider the projections of symbolic level sets,

$$J_\Phi^{\mathbf{a}}(\alpha) = \pi_{\mathbf{a}}(E_\Phi(\alpha)) \subset F_{\mathbf{a}},$$

for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$. Let us denote by f_i the affine maps

$$x \longmapsto T_i x + a_i.$$

If the sequence $\mathbf{a}$ is such that there is a compact set $X \subset \mathbb{R}^d$ with $f_i(X) \subset X$ for all $i \in \mathbb{N}$ and $f_i(X) \cap f_j(X) = \emptyset$ for $i \neq j$, then the projection $\pi_{\mathbf{a}}$ gives a conjugacy between the left shift $\sigma\colon \Sigma \to \Sigma$ and the well-defined map

$$g\colon \bigcup_{i\in\mathbb{N}} f_i(X) \longrightarrow X$$

for which

$$g(x) = f_i^{-1}(x) = T_i^{-1}x - a_i, \quad x \in f_i(X).$$

Thus our considerations with $J_\Phi^{\mathbf{a}}(\alpha)$ make sense in the context of (1.1) when the self-affine set satisfies a separation condition.

Next we state our main results concerning multifractal formalism in this paper. For each $k \in \mathbb{N}$ we let $\mathcal{M}_{\sigma^k}(\Sigma)$ denote the set of all σ^k-invariant Borel probability measures and define $\mathcal{M}^*_{\sigma^k}(\Sigma)$ to be the collection of all measures $\mu \in \mathcal{M}_{\sigma^k}(\Sigma)$ which are compactly supported. If $k \in \mathbb{N}$ and $\mu \in \mathcal{M}^*_{\sigma^k}(\Sigma)$, then we let $D_k(\mu)$ to be the unique $s \geq 0$ satisfying

$$\sum_{\omega \in \Sigma_k} \mu([\omega]) \log \frac{\varphi^s(\omega)}{\mu([\omega])} = 0.$$

The potential φ^s here is the singular value function defined in (2.7). We also set

$$D(\mu) = \inf\{s \colon P_\mu(\varphi^s) \leq 0, \Lambda_\mu(\varphi^s) > -\infty\}$$

for all $\mu \in \mathcal{M}_\sigma(\Sigma)$ and call it a *Lyapunov dimension* of μ. Given $\alpha \in \overline{\mathbb{R}}$ we define an indexed family of neighbourhoods by

$$B_n(\alpha) = \begin{cases} (-\infty, -n), & \text{if } \alpha = -\infty, \\ \left(\alpha - \dfrac{1}{n}, \alpha + \dfrac{1}{n}\right), & \text{if } \alpha \in \mathbb{R}, \\ (n, \infty), & \text{if } \alpha = \infty. \end{cases}$$

We have two main results concerning multifractal analysis of Birkhoff averages. In the first one, we consider general potentials and in the second one, we restrict our analysis to bounded potentials.

Theorem C. *Suppose that* $(T_i)_{i \in \mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i \in \mathbb{N}} \|T_i\| < \frac{1}{2}$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \leq s \leq d$,

$$\Phi \colon \Sigma \longrightarrow \mathbb{R}^{\mathbb{N}}$$

has summable variations, and $\alpha \in \overline{\mathbb{R}}^{\mathbb{N}}$. *Then*

$$\begin{aligned} &\dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi}(\alpha)) \\ &\quad = \min\Big\{d, \lim_{n \to \infty} \lim_{k \to \infty} \sup\Big\{D_k(\mu) \colon \mu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \text{ so that} \\ &\qquad\qquad \int A_k \phi_i \,\mathrm{d}\mu \in B_n(\alpha_i) \text{ for all } i \in \{1, \ldots, n\}\Big\}\Big\} \end{aligned}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$.

The proof of Theorem C is given in §5. Theorem D, our second result on multifractal formalism, generalises the theorem of Fan, Jordan, Liao, and Rams (see [10], Theorem 1.2) to the self-affine setting.

Define

$$s_\infty = \inf\{s\colon P(\varphi^s) < \infty\}$$

and

$$\mathcal{P}(\Phi) = \left\{ \int \Phi \, \mathrm{d}\mu \colon \mu \in \mathcal{M}_\sigma(\Sigma) \right\},$$

where

$$\int \Phi \, \mathrm{d}\mu = \left(\int \phi_1 \, \mathrm{d}\mu, \int \phi_2 \, \mathrm{d}\mu, \dots \right).$$

Let $\overline{\mathcal{P}(\Phi)}$ be the closure of $\mathcal{P}(\Phi)$ with respect to the pointwise topology.

Theorem D. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$, $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ *is bounded with summable variations, and* $\alpha \in \overline{\mathcal{P}(\Phi)}$. *Then*

$$\begin{aligned} &\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) \\ &\quad = \min\left\{ d, \max\left\{ s_\infty, \sup\left\{ D(\mu)\colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi \, \mathrm{d}\mu = \alpha \right\}\right\}\right\} \end{aligned}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$. *Furthermore, if* $\alpha \notin \overline{\mathcal{P}(\Phi)}$, *then*

$$J_\Phi^{\mathbf{a}}(\alpha) = \emptyset$$

for all $\mathbf{a} \in \mathbf{A}$.

The proof of Theorem D is presented in §6. It strongly relies on the boundedness assumption even in the case where there are only finitely many potentials and we are looking at the interior points of the spectrum; see §6.2. To show the upper bound in this case, the main tool is thermodynamical formalism developed in §3: the boundedness assumption guarantees the existence of a Gibbs measure for a suitable potential which then allows us to differentiate the corresponding pressure. The boundedness assumption is also used in obtaining the lower bound; see §6.6.

3. Thermodynamic formalism for quasi-multiplicative potentials

In this section, our main goal is to show that if φ is a quasi-multiplicative potential with finite pressure, then φ has a Gibbs measure. We remark that this is not the case for all sub-multiplicative potentials; see [28], Example 6.4, for a counter-example in a finitely generated self-affine set. After showing the existence of a Gibbs measure, the rest of the section, including the proof of the variational principle, follows by applying standard arguments.

3.1. Existence of Gibbs measures. Suppose we have a sub-multiplicative potential φ along with a subset $I \subset \mathbb{N}$. We define the *pressure* $P(\varphi, I)$ by

$$P(\varphi, I) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, I) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log Z_n(\varphi, I),$$

where $Z_n(\varphi, I) = \sum_{\omega\in I^n} \varphi(\omega)$ for all $n \in \mathbb{N}$. Thus $Z_n(\varphi, \mathbb{N}) = Z_n(\varphi)$ and $P(\varphi, \mathbb{N}) = P(\varphi)$. Observe that $Z_n(\varphi, J) \le Z_n(\varphi, I)$ and hence also $P(\varphi, J) \le P(\varphi, I)$ for all $J \subset I \subset \mathbb{N}$. If $C \ge 1$, then an invariant probability measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is said to be a *C-Gibbs measure for the potential φ on I* if it is supported on $I^{\mathbb{N}}$, the pressure $P(\varphi, I)$ is finite, and

$$C^{-1} \le \frac{\mu([\omega])}{\varphi(\omega)\exp(-nP(\varphi, I))} \le C$$

for all $\omega \in I^n$ and $n \in \mathbb{N}$. An invariant measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is said to be a *Gibbs measure for the potential φ on I* if there exists some $C \ge 1$ such that μ is a C-Gibbs measure for the potential φ on I. Finally, $\mu \in \mathcal{M}_\sigma(\Sigma)$ is said to be a *Gibbs measure for the potential φ* if μ is a Gibbs measure for the potential φ on $\mathbb{N}$.

For a given quasi-multiplicative potential, throughout the section, we let $\Gamma \subset \Sigma_*$, $K \in \mathbb{N}$, and $c \ge 1$ be as in the definition of the quasi-multiplicative potential; see (2.1).

Lemma 3.1. *If φ is a quasi-multiplicative potential and $I \subset \mathbb{N}$ is so that $\Gamma \subset \bigcup_{k=1}^{K} I^k$, then*

$$e^{nP(\varphi,I)} \le Z_n(\varphi, I) \le cK \max\{1, e^{KP(\varphi)}\} e^{nP(\varphi,I)}$$

for all $n \in \mathbb{N}$. In particular, $P(\varphi, I) > -\infty$.

Proof. Since the left-hand side inequality follows immediately from the definition of the pressure, it suffices to show the right-hand side inequality. Fix $n, m \in \mathbb{N}$ and $\omega_i \in I^n$ for all $i \in \{1, \ldots, m\}$. By the quasi-multiplicativity, there are $\kappa_1, \ldots, \kappa_{m-1}$ so that

$$\varphi(\omega_1)\cdots\varphi(\omega_m) \le c^{m-1}\varphi(\omega_1\kappa_1\omega_2\kappa_2\cdots\omega_{m-1}\kappa_{m-1}\omega_m).$$

To simplify notation, for each pair $\omega, \tau \in \Sigma_*$, we fix a choice of κ satisfying (2.1). Denoting

$$\xi_{n,m}(\omega_1\cdots\omega_m) = \omega_1\kappa_1\omega_2\kappa_2\cdots\omega_{m-1}\kappa_{m-1}\omega_m$$

for all $\omega = \omega_1\cdots\omega_m \in (I^n)^m$ defines a mapping

$$\xi_{n,m} : (I^n)^m \longrightarrow \bigcup_{\ell=1}^{K(m-1)} I^{nm+\ell}$$

which is at most K^{m-1} to one. Hence

$$\begin{aligned} Z_n(\varphi, I)^m &= \Big(\sum_{\omega \in I^n} \varphi(\omega) \Big)^m \\ &= \sum_{\omega \in (I^n)^m} \prod_{i=1}^{m} \varphi(\omega_i) \\ &\le c^{m-1} \sum_{\omega \in (I^n)^m} \varphi(\xi_{n,m}(\omega)) \\ &\le (cK)^{m-1} \sum_{\ell=1}^{K(m-1)} \sum_{\omega \in I^{nm+\ell}} \varphi(\omega) \\ &= (cK)^{m-1} \sum_{\ell=1}^{K(m-1)} Z_{nm+\ell}(\varphi, I). \end{aligned}$$

Consequently, for each $m \in \mathbb{N}$ there is $\ell_m \in \mathbb{N}$ with $nm \le \ell_m \le (n+K)m$ satisfying

$$Z_n(\varphi, I)^m \le m(cK)^m Z_{\ell_m}(\varphi, I).$$

Hence

$$\begin{aligned} Z_n(\varphi, I) &\le m^{1/m} cK (Z_{\ell_m}(\varphi, I)^{1/\ell_m})^{\ell_m/m} \\ &\le \begin{cases} m^{1/m} cK (Z_{\ell_m}(\varphi, I))^{n+K}, & \text{if } P(\varphi, I) > 0, \\ m^{1/m} cK (Z_{\ell_m}(\varphi, I))^{n}, & \text{if } P(\varphi, I) \le 0, \end{cases} \end{aligned}$$

for all m large enough. Thus, by letting $m \to \infty$, we get

$$Z_n(\varphi, I) \le \begin{cases} cKe^{(n+K)P(\varphi,I)}, & \text{if } P(\varphi, I) > 0, \\ cKe^{nP(\varphi,I)}, & \text{if } P(\varphi, I) \le 0. \end{cases}$$

The proof follows since $P(\varphi, I) \le P(\varphi)$. □

The following proposition is a finite approximation property for the pressure. It is crucial in our analysis since it makes it possible to construct a Gibbs measure on an infinitely generated shift space via its finitely generated sub-spaces.

Proposition 3.2. *If $(I_\ell)_{\ell \in \mathbb{N}}$ is a sequence of non-empty finite sets $I_\ell \subset \mathbb{N}$ with $I_\ell \subset I_{\ell+1}$ for all $\ell \in \mathbb{N}$ so that $\mathbb{N} = \bigcup_{\ell \in \mathbb{N}} I_\ell$, then*

$$P(\varphi) = \lim_{\ell \to \infty} P(\varphi, I_\ell)$$

for all quasi-multiplicative potentials φ. In particular,

$$P(\varphi) = \sup\{P(\varphi, I) \colon I \subset \mathbb{N} \text{ is finite}\}.$$

Proof. Recall that $P(\varphi, I_\ell) \le P(\varphi, I_{\ell+1}) \le P(\varphi)$ for all $\ell \in \mathbb{N}$. Fix $\varrho < P(\varphi)$, $n \in \mathbb{N}$, and let

$$P = \lim_{\ell \to \infty} P(\varphi, I_\ell).$$

Since $\varrho < \frac{1}{n} \log Z_n(\varphi)$, we may choose $\ell \in \mathbb{N}$ so that

$$\Gamma \subset \bigcup_{k=1}^{K} I_\ell^k \quad \text{and} \quad \varrho < \frac{1}{n} \log Z_n(\varphi, I_\ell).$$

By Lemma 3.1, we have

$$Z_n(\varphi, I_\ell) \le cK \max\{1, e^{KP(\varphi)}\} e^{nP}$$

and thus

$$\varrho < \frac{1}{n}(\log cK + K|P(\varphi)|) + P.$$

The proof is finished by letting $n \to \infty$. □

Lemma 3.3. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$, then there exists a constant $C \ge 1$ such that for each $I \subset \mathbb{N}$ with $\Gamma \subset \bigcup_{k=1}^{K} I^k$ we have*

$$C^{-1} e^{(n+m)P(\varphi,I)} \varphi(\omega) \le \sum_{\kappa \in I^n} \sum_{\tau \in I^m} \varphi(\kappa\omega\tau) \le C e^{(n+m)P(\varphi,I)} \varphi(\omega)$$

for all $m, n \in \mathbb{N}$ and $\omega \in \bigcup_{n \in \mathbb{N}} I^n$.

Proof. The right-hand side inequality follows immediately since

$$\begin{aligned}
\sum_{\kappa \in I^n} \sum_{\tau \in I^m} \varphi(\kappa\omega\tau) &\le \sum_{\kappa \in I^n} \sum_{\tau \in I^m} \varphi(\kappa)\varphi(\omega)\varphi(\tau) \\
&= \varphi(\omega) Z_n(\varphi, I) Z_m(\varphi, I) \\
&\le \big(cK \max\{1, e^{KP(\varphi)}\}\big)^2 e^{(n+m)P(\varphi,I)} \varphi(\omega)
\end{aligned}$$

by Lemma 3.1.

To show the left-hand side inequality, we first notice that the quasi-multiplicativity implies

$$\varphi(\omega)\varphi(\kappa) \le c \sum_{k=1}^{K} \sum_{\alpha \in I^k} \varphi(\omega\alpha\kappa) \tag{3.1}$$

for all $\omega, \kappa \in \Sigma_*$. Applying Lemma 3.1, along with (3.1), we obtain

$$\begin{aligned}
e^{(n+m)P(\varphi,I)}\varphi(\omega) &\le Z_n(\varphi, I) \sum_{\kappa \in I^m} \varphi(\omega)\varphi(\kappa) \\
&\le c Z_n(\varphi, I) \sum_{\tau \in I^m} \sum_{k=1}^{K} \sum_{\alpha \in I^k} \varphi(\omega\alpha\tau) \\
&= c Z_n(\varphi, I) \sum_{k=1}^{K} \sum_{\alpha \in I^k} \sum_{\tau \in I^m} \varphi(\omega\tau\alpha) \\
&\le c Z_n(\varphi, I) \sum_{k=1}^{K} Z_k(\varphi, I) \sum_{\tau \in I^m} \varphi(\omega\tau) \\
&\le c^2 \sum_{k=1}^{K} Z_k(\varphi, I) \sum_{k=1}^{K} \sum_{\alpha \in I^k} \sum_{\kappa \in I^n} \sum_{\tau \in I^m} \varphi(\kappa\alpha\omega\tau) \\
&\le c^2 \Big(\sum_{k=1}^{K} Z_k(\varphi, I) \Big)^2 \sum_{\kappa \in I^n} \sum_{\tau \in I^m} \varphi(\kappa\omega\tau).
\end{aligned}$$

The proof is now finished since

$$\sum_{k=1}^{K} Z_k(\varphi, I) \le cK \max\{1, e^{KP(\varphi)}\} \sum_{k=1}^{K} e^{kP(\varphi)} < \infty$$

by Lemma 3.1. □

We are now ready to show that every finite sub-space carries a Gibbs measure. Observe that, to be able to extend the result into infinitely generated shift space, it is crucial to find a uniform constant.

Proposition 3.4. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$, then there is $C \ge 1$ so that φ has a C-Gibbs measure for φ on I for all finite subsets $I \subset \mathbb{N}$ with $\Gamma \subset \bigcup_{k=1}^{K} I^k$.*

Proof. Let $I \subset \mathbb{N}$ be a finite subset with $\Gamma \subset \bigcup_{k=1}^{K} I^k$. Given a finite word $\omega \in \bigcup_{n\in\mathbb{N}} I^n$ we choose $\tilde{\omega} \in [\omega] \cap I^{\mathbb{N}}$ and let δ_ω denote the point mass concentrated at $\tilde{\omega}$. For each $n \in \mathbb{N}$ we define a probability measure ν_n on Σ by

$$\nu_n = Z_{3n}(\varphi, I)^{-1} \sum_{\omega\in I^{3n}} \varphi(\omega)\delta_\omega.$$

Note that ν_n is supported on $I^{\mathbb{N}}$. If $m, \ell \in \{1, \dots, n\}$ and $\omega \in I^m$, then

$$\begin{aligned}\nu_n \circ \sigma^{-\ell}([\omega]) &= \sum_{\kappa\in I^\ell} \sum_{\tau\in I^{3n-\ell-m}} \nu_n([\kappa\omega\tau]) \\ &= Z_{3n}(\varphi, I)^{-1} \sum_{\kappa\in I^\ell} \sum_{\tau\in I^{3n-\ell-m}} \varphi(\kappa\omega\tau).\end{aligned}$$

According to Lemmas 3.1 and 3.3 there exists a constant $C \geq 1$ so that

$$C^{-1}e^{-mP(\varphi,I)}\varphi([\omega]) \leq \nu_n \circ \sigma^{-\ell}([\omega]) \leq Ce^{-mP(\varphi,I)}\varphi(\omega) \tag{3.2}$$

for all finite subsets $I \subset \mathbb{N}$ with $\Gamma \subset \bigcup_{k=1}^{K} I^k$. Observe that the above estimate remains true if we replace $\nu_n \circ \sigma^{-\ell}$ by the probability measure

$$\mu_n = \frac{1}{n}\sum_{\ell=1}^{n} \nu_n \circ \sigma^{-\ell}. \tag{3.3}$$

Since I is finite and each μ_n is supported on the compact set $I^{\mathbb{N}}$, there is a convergent subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ converging to some limit μ in the weak* topology. It follows from (3.3) that μ is a σ-invariant probability measure. Moreover, by (3.2), μ is a C-Gibbs measure for φ on I. □

Theorem 3.5. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$, then φ has a Gibbs measure μ. Moreover, there is $C \geq 1$ so that for each $\ell \in \mathbb{N}$ there are a finite set $I_\ell \subset \mathbb{N}$ and a C-Gibbs measure μ_ℓ for φ on I_ℓ such that $P(\varphi, I_\ell) \to P(\varphi)$ and $\mu_\ell \to \mu$ in the weak* topology.*

Proof. Let $(I_\ell)_{\ell\in\mathbb{N}}$ be a sequence of non-empty finite sets $I_\ell \subset \mathbb{N}$ with $I_\ell \subset I_{\ell+1}$ and $\Gamma \subset \bigcup_{k=1}^{K} I_\ell^k$ for all $\ell \in \mathbb{N}$ such that $\mathbb{N} = \bigcup_{\ell\in\mathbb{N}} I_\ell$. Recalling Proposition 3.2, we have $\lim_{\ell\to\infty} P(\varphi, I_\ell) = P(\varphi)$. By Proposition 3.4, there exist a constant $C \geq 1$ and for each $\ell \in \mathbb{N}$ a σ-invariant probability measure $\mu_\ell \in \mathcal{M}_\sigma(\Sigma)$ so that

$$C^{-1} \leq \frac{\mu_\ell([\omega])}{\varphi(\omega)\exp(-nP(\varphi, I_\ell))} \leq C \tag{3.4}$$

for all $\omega \in I_\ell^n$ and $\ell \in \mathbb{N}$. It suffices to show that the sequence $(\mu_\ell)_{\ell\in\mathbb{N}}$ is tight, that is, for each $\varepsilon > 0$ there exists a compact set $K \subset \Sigma$ for which $\mu_\ell(K) > 1 - \varepsilon$ for all $\ell \in \mathbb{N}$. Then the sequence $(\mu_\ell)_{\ell\in\mathbb{N}}$ has a converging subsequence and it follows from (3.4) that the limit measure of that subsequence is a Gibbs measure for φ.

Fix $\varepsilon > 0$ and notice that

$$\sum_{i\in\mathbb{N}} \varphi(i) = Z_1(\varphi) \le Ce^{P(\varphi)} < \infty$$

by Lemma 3.1. Thus, for each $k \in \mathbb{N}$ there is a finite subset $I_k \subset \mathbb{N}$ so that

$$\sum_{i\in\mathbb{N}\setminus I_k} \varphi(i) < \varepsilon 2^{-k} C^{-1} e^{P(\varphi,I_1)} \le \varepsilon 2^{-k} C^{-1} e^{P(\varphi,I_\ell)}$$

for all $\ell \in \mathbb{N}$. We define

$$K = \{\omega \in \Sigma \colon \omega_k \in I_k \text{ for all } k \in \mathbb{N}\}.$$

It follows from (3.4) that

$$\begin{aligned}
\mu_\ell(K) &= \mu_\ell\Big(\Sigma \setminus \bigcup_{k\in\mathbb{N}} \{\omega \in \Sigma \colon \omega_k \notin I_k\}\Big) \\
&= 1 - \sum_{k\in\mathbb{N}} \mu_\ell(\{\omega \in \Sigma \colon \omega_k \notin I_k\}) \\
&= 1 - \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{N}\setminus I_k} \mu_\ell(\sigma^{-k}([i])) \\
&= 1 - \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{N}\setminus I_k} \mu_\ell([i]) \\
&\ge 1 - \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{N}\setminus I_k} Ce^{-P(\varphi,I_\ell)} \varphi(i) \\
&> 1 - \sum_{k\in\mathbb{N}} 2^{-k}\varepsilon = 1 - \varepsilon
\end{aligned}$$

for all $\ell \in \mathbb{N}$. □

3.2. Variational principle. We shall study the properties of the Gibbs measure found in Theorem 3.5. At the end of this section, we prove Theorem A.

Lemma 3.6. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$ and μ is a C-Gibbs measure for φ, then*

$$\sum_{p,q=1}^{K} \mu([\omega] \cap \sigma^{-(n+p+q)}([\tau])) \geq c^{-2} C^{-1} e^{-2K|P(\varphi)|} \mu([\omega])\mu([\tau]).$$

for all $\omega, \tau \in \Sigma$ and $n \geq |\omega|$.

Proof. By the Gibbs property, we have

$$\begin{aligned}\mu([\omega] \cap \sigma^{-(n+p+q)}([\tau])) &= \sum_{\kappa \in \Sigma_{n+p+q-|\omega|}} \mu([\omega\kappa\tau]) \\ &\geq C_0^{-1} e^{-(n+p+q+|\tau|)P(\varphi)} \sum_{\kappa \in \Sigma_{n+p+q-|\omega|}} \varphi(\omega\kappa\tau)\end{aligned}$$

and, by (3.1),

$$\begin{aligned}\sum_{p,q=1}^{K} \sum_{\kappa \in \Sigma_{n+p+q-|\omega|}} \varphi(\omega\kappa\tau) &= \sum_{\kappa \in \Sigma_{n-|\omega|}} \sum_{q=1}^{K} \sum_{\beta \in \Sigma_q} \sum_{p=1}^{K} \sum_{\alpha \in \Sigma_p} \varphi(\omega\alpha\kappa\beta\tau) \\ &\geq c^{-2} \varphi(\omega)\varphi(\tau) \sum_{\kappa \in \Sigma_{n-|\omega|}} \varphi(\kappa)\end{aligned}$$

for all $n \geq |\omega|$. Therefore

$$\begin{aligned}&\sum_{p,q=1}^{K} \mu([\omega] \cap \sigma^{-(n+p+q)}([\tau])) \\ &\quad \geq c^{-2} C_0^{-1} e^{-2K|P(\varphi)|} e^{-(n+|\tau|)P(\varphi)} \varphi(\omega)\varphi(\tau) \sum_{\kappa \in \Sigma_{n-|\omega|}} \varphi(\kappa)\end{aligned}$$

from which the claim follows again by the Gibbs property. □

With the above lemma, the proof of the ergodicity of the Gibbs measure now follows as e.g. in [16], Theorem 3.2.

Theorem 3.7. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$ and μ is a Gibbs measure for φ, then μ is ergodic. In particular, μ is the only Gibbs measure for φ.*

Proof. Suppose $A, B \subset \Sigma$ are Borel sets with $0 < \mu(A), \mu(B) < 1$. Let $\varepsilon > 0$. Since the semi-algebra of cylinder sets generates the Borel σ-algebra we may choose finite disjoint unions of cylinder sets A_ε and B_ε so that $\mu(A \triangle A_\varepsilon) < \varepsilon$ and $\mu(B \triangle B_\varepsilon) < \varepsilon$. According to Lemma 3.6 there is $n_0 \in \mathbb{N}$ depending only on ε (and the set A) so that for some constant $C > 0$ we have

$$\sum_{p,q=1}^{K} \mu(A_\varepsilon \cap \sigma^{-(n+p+q)}(B_\varepsilon)) \geq C\mu(A_\varepsilon)\mu(B_\varepsilon)$$

for all $n \geq n_0$. In particular, for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ so that

$$\mu(A_\varepsilon \cap \sigma^{-n}(B_\varepsilon)) \geq CK^{-2}\mu(A_\varepsilon)\mu(B_\varepsilon) \geq CK^{-2}(\mu(A) - \varepsilon)(\mu(B) - \varepsilon).$$

Since $(A \cap \sigma^{-1}(B)) \triangle (A_\varepsilon \cap \sigma^{-1}(B_\varepsilon)) \subset (\sigma^{-1}(B) \triangle \sigma^{-1}(B_\varepsilon)) \cup (A \triangle A_\varepsilon)$ the invariance of μ implies

$$\mu\bigl(A \cap \sigma^{-n}(B)\bigr) \geq \mu(A_\varepsilon \cap \sigma^{-n}(B_\varepsilon)) - 2\varepsilon.$$

Thus, by taking $A = \Sigma \setminus B$ and $\varepsilon > 0$ small enough, we see that there exists $n \in \mathbb{N}$ with $\mu(\Sigma \setminus B \cap \sigma^{-n}(B)) > 0$. Therefore any invariant B must satisfy $\mu(B) = 0$ or $\mu(B) = 1$.

To show the claimed uniqueness, assume that ν is another Gibbs measure for φ. The proof above shows that also ν is ergodic. It follows from the Gibbs properties of both μ and ν that they are equivalent. This is a contradiction since two different ergodic measures are mutually singular. □

The following two lemmas examine the relations between the Gibbs measure, measure-theoretical pressure, and pressure.

Lemma 3.8. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$ and μ is the Gibbs measure for φ on a set $I \subset \mathbb{N}$, then $P(\varphi, I) = P_\mu(\varphi)$.*

Proof. By the definition of a Gibbs measure, we get

$$\begin{aligned} P_\mu(\varphi) &= \lim_{n\to\infty} \frac{1}{n} \sum_{\omega \in I^n} \mu([\omega]) \log \frac{\varphi(\omega)}{\mu([\omega])} \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{\omega \in I^n} \mu([\omega]) \log e^{nP(\varphi,I)} = P(\varphi, I) \end{aligned}$$

as desired. □

In the proof of the following lemma, we follow the ideas of [4] and [28], Theorem 3.6.

Lemma 3.9. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$ and μ is the Gibbs measure for φ, then any measure $\nu \in \mathcal{M}_\sigma(\Sigma)$ with $P(\varphi) \leq P_\nu(\varphi)$ is absolutely continuous with respect to μ.*

Proof. Let μ be a C-Gibbs measure. Assume to the contrary that there exist a measure $\nu \in \mathcal{M}_\sigma(\Sigma)$ with $P(\varphi) \le P_\nu(\varphi)$ and a Borel set $B \subset \Sigma$ so that $\mu(B) = 0$ and $\nu(B) > 0$. Since the semi-algebra of cylinder sets generates the Borel σ-algebra we may choose a sequence of sets $(B_n)_{n\in\mathbb{N}}$ such that each B_n is a union of cylinders of length n with $(\mu + \nu)(B_n \triangle B) \to 0$ as $n \to \infty$. Let

$$B_n' = \{\omega \in \Sigma_n : [\omega] \subset B_n\}.$$

Hence, by (2.2) and (2.4), we have

$$\begin{aligned}0 &\le \sum_{\omega\in B_n'} \nu([\omega]) \log \frac{\varphi(\omega)}{\nu([\omega])} + \sum_{\omega\in\Sigma\setminus B_n'} \nu([\omega]) \log \frac{\varphi(\omega)}{\nu([\omega])} - nP(\varphi) \\ &\le \nu(B_n) \log \sum_{\omega\in B_n'} \varphi(\omega) + \nu(\Sigma\setminus B_n) \log \sum_{\omega\in\Sigma\setminus B_n'} \varphi(\omega) - nP(\varphi) + \frac{2}{e} \qquad (3.5) \\ &\le \nu(B_n) \log \mu(B_n) + \nu(\Sigma\setminus B_n) \log \mu(\Sigma\setminus B_n) + \log C + \frac{2}{e}\end{aligned}$$

for all n large enough. Since $\nu(B_n) \to \nu(B)$ and $\mu(B_n) \to 0$ the right-hand side of (3.5) tends to $-\infty$ as $n \to \infty$. This contradiction finishes the proof. □

We are now ready to prove Theorem A.

Proof of Theorem A. Let us first assume that $P(\varphi) = \infty$. Let $(I_\ell)_{\ell\in\mathbb{N}}$ be a sequence of non-empty finite sets with $I_\ell \subset \mathbb{N}$ and $\Gamma \subset \bigcup_{k=1}^K I_\ell^k$ for all $\ell \in \mathbb{N}$ such that $\mathbb{N} = \bigcup_{\ell\in\mathbb{N}} I_\ell$. Recalling Proposition 3.4, let μ_ℓ be a Gibbs measure for φ on I_ℓ for all $\ell \in \mathbb{N}$. Now

$$\begin{aligned}P(\varphi) &= \sup\{P(\varphi, I_\ell) : \ell \in \mathbb{N}\} \\ &= \sup\{P_{\mu_\ell}(\varphi) : \ell \in \mathbb{N}\} \\ &\le \sup\{P_\mu(\varphi) : \mu \in \mathcal{M}_\sigma(\Sigma)\}\end{aligned}$$

by Proposition 3.2 and Lemma 3.8.

If $P(\varphi) < \infty$, then it suffices to prove that a Gibbs measure μ is the only invariant measure for which $P(\varphi) = P_\mu(\varphi)$. Theorem 3.7 shows that μ is ergodic and Lemma 3.8 shows that it satisfies $P(\varphi) = P_\mu(\varphi)$. If $\nu \in \mathcal{M}_\sigma(\Sigma)$ is an invariant measure satisfying $P(\varphi) = P_\nu(\varphi)$, then ν is absolutely continuous with respect to μ by Lemma 3.9. Thus the Radon-Nikodym derivative exists and, by the ergodicity of μ, is equal to 1 for μ-almost everywhere; see [37], Theorem 6.10. Hence $\mu = \nu$ and the proof is finished. □

3.3. Differentiation of pressure. We shall study differentiability of the pressure related to a specific potential. The result we obtain is crucial in proving the upper bound in Theorem D.

Given a pair of potentials

$$\varphi_1, \varphi_2 \colon \Sigma_* \longrightarrow [0, \infty)$$

we let $\varphi_1 \cdot \varphi_2$ denote the potential defined by

$$\omega \longmapsto \varphi_1(\omega)\varphi_2(\omega), \quad \omega \in \Sigma_*.$$

Given a function $\phi \colon \Sigma \to \mathbb{R}$ we define an associated potential

$$e_\phi \colon \Sigma_* \longrightarrow [0, \infty)$$

by setting

$$e_\phi(\omega) = \exp(\sup\{S_n\phi(\tau) \colon \tau \in [\omega]\})$$

for all $\omega \in \Sigma_n$ and $n \in \mathbb{N}$. Recall that

$$S_n\phi(\tau) = \sum_{j=0}^{n-1} \phi(\sigma^j(\tau))$$

for all $\tau \in \Sigma$.

Lemma 3.10. *If φ is a quasi-multiplicative potential and $\phi \colon \Sigma \to \mathbb{R}$ has summable variations, then the potential $\varphi \cdot e_\phi$ is quasi-multiplicative.*

Proof. If

$$c = \exp\Big(\sum_{n=1}^{\infty} \mathrm{var}_n(\phi)\Big),$$

then

$$c^{-1} e_\phi(\omega) e_\phi(\kappa) \leq e_\phi(\omega\kappa) \leq e_\phi(\omega) e_\phi(\kappa)$$

for all $\omega, \kappa \in \Sigma_*$. The claim follows from the quasi-multiplicativity of φ. □

Lemma 3.11. *If φ is a sub-multiplicative potential with $P(\varphi) < \infty$ and $\phi \colon \Sigma \to \mathbb{R}$ is bounded with summable variations, then the function*

$$q \longmapsto P(\varphi \cdot e_{q\phi})$$

is convex.

Proof. If $q, p \in \mathbb{R}$ and $0 \le \lambda \le 1$, then

$$\varphi(\omega)e_{(\lambda q+(1-\lambda)p)\phi}(\omega) \le (\varphi(\omega)e_{q\phi}(\omega))^{\lambda}(\varphi(\omega)e_{p\phi}(\omega))^{1-\lambda}$$

for all $\omega \in \Sigma_*$. Thus, by Hölder's inequality, we have

$$\sum_{\omega\in\Sigma_n} \varphi(\omega)e_{(\lambda q+(1-\lambda)p)\phi}(\omega) \le \Big(\sum_{\omega\in\Sigma_n} \varphi(\omega)e_{q\phi}(\omega)\Big)^{\lambda}\Big(\sum_{\omega\in\Sigma_n} \varphi(\omega)e_{p\phi}(\omega)\Big)^{1-\lambda}.$$

Taking logarithms, dividing by n, and letting $n \to \infty$ gives the claim. □

Lemma 3.12. *If φ is a quasi-multiplicative potential with $P(\varphi) < \infty$, μ is the Gibbs measure for φ, and $\phi\colon \Sigma \to \mathbb{R}$ is bounded with summable variations, then the function*

$$q \longmapsto P(\varphi \cdot e_{q\phi})$$

is differentiable at zero with derivative

$$\frac{\partial P(\varphi \cdot e_{q\phi})}{\partial q}\Big|_{q=0} = \int \phi \, \mathrm{d}\mu.$$

Proof. To prove the claim, we use some of the ideas used in the proof of [28], Theorem 4.4. It suffices to show that the right derivative exists at zero and equals to $\int \phi \, \mathrm{d}\mu$ since applying this result with $-\phi$ in place of ϕ gives

$$\lim_{q\uparrow 0} \frac{1}{q}(P(\varphi \cdot e_{q\phi}) - P(\varphi)) = -\lim_{q\downarrow 0} \frac{1}{q}(P(\varphi \cdot e_{q(-\phi)}) - P(\varphi)) = \int \phi \, \mathrm{d}\mu.$$

Throughout the proof of the lemma, to simplify notation, we write $P(q)$ in place of $P(\varphi \cdot e_{q\phi})$. By Lemma 3.11, the function $q \mapsto P(q)$ is convex and hence there is a well-defined right derivative at zero. We shall denote it by $P'_+(0)$.

To prove that $P'_+(0) \le \int \phi \, \mathrm{d}\mu$, take $\beta > \int \phi \, \mathrm{d}\mu$. Define

$$C'_n = \{\omega \in \Sigma_n \colon S_n\phi(\tau) > n\beta \text{ for some } \tau \in [\omega]\}$$

for all $n \in \mathbb{N}$ and let

$$C_n = \bigcup_{\omega\in C'_n} [\omega].$$

Since μ is a Gibbs measure for φ there is $C \ge 1$ so that

$$\varphi(\omega) \le Ce^{nP(0)}\mu([\omega]) \tag{3.6}$$

for all $\omega \in \Sigma_n$ and $n \in \mathbb{N}$. By Theorem 3.7, μ is ergodic and thus, we may apply Birkhoff's ergodic theorem, Egorov's theorem, and the fact that ϕ has summable variations, to obtain $\lim_{n\to\infty} \mu(C_n) = 0$.

Fix $\gamma > 0$. Since $P(q)$ is convex we have $P(\gamma/n) \geq P(0) + \gamma P'_+(0)/n$. Using the sub-multiplicativity of $\varphi \cdot e_{\gamma\phi/n}$ and (3.6), we have

$$\begin{aligned} e^{nP(0)+\gamma P'_+(0)} &\leq e^{nP(\gamma/n)} \\ &\leq \sum_{\omega\in\Sigma_n} \varphi(\omega)\exp(\gamma\|S_n(\phi)|_{[\omega]}\|/n) \\ &\leq \sum_{\omega\in\Sigma_n\setminus C'_n} \varphi(\omega)e^{\gamma\beta} + \sum_{\omega\in C'_n} \varphi(\omega)e^{\gamma\|\phi\|} \\ &\leq Ce^{\gamma\beta}e^{nP(0)}(1-\mu(C_n)) + Ce^{\gamma\|\phi\|}e^{nP(0)}\mu(C_n). \end{aligned}$$

Dividing by $e^{nP(0)}$, letting $n \to \infty$, and then $\gamma \to \infty$ gives $P'_+(0) \leq \beta$ as desired.

To show that $P'_+(0) \geq \int \phi \, \mathrm{d}\mu$, we use Lemma 2.2 for the sub-multiplicative potential $\varphi \cdot e_{q\phi}$ and Lemma 3.8 for the quasi-multiplicative potential φ to obtain

$$P(q) \geq P_\mu(\varphi \cdot e_{q\phi}) \geq P_\mu(\varphi) + q\int \phi \, \mathrm{d}\mu = P(0) + q\int \phi \, \mathrm{d}\mu$$

for all $q \geq 0$. The proof follows. □

4. Dimension of infinitely generated self-affine sets

In this section, we prove Theorem B, that is, we show that the dimension of a typical infinitely generated self-affine set is a supremum of dimensions of its finitely generated subsets. We also examine when the projection of the Gibbs measure is a measure of maximal dimension. The reader is prompted to recall notation from §2.2.

Proof of Theorem B. Define

$$s_0 = \inf\{s\colon P(\varphi^s) \leq 0\}$$

and let $(I_\ell)_{\ell\in\mathbb{N}}$ be a sequence of non-empty finite sets $I_\ell \subset \mathbb{N}$ with

$$I_\ell \subset I_{\ell+1} \quad \text{and} \quad \Gamma \subset \bigcup_{k=1}^{K} I_\ell^k, \quad \ell \in \mathbb{N},$$

such that

$$\mathbb{N} = \bigcup_{\ell\in\mathbb{N}} I_\ell.$$

Fix $\ell \in \mathbb{N}$ and let $0 < s_\ell \leq s_0$ be such that

$$P(\varphi^{s_\ell}, I_\ell) = 0.$$

To show that $s_0 \leq \sup_{\ell \in \mathbb{N}} s_\ell$, take $s < s_0$. Since $P(\varphi^s) > 0$ and $P(\varphi^s, I_\ell) \to P(\varphi^s)$ by Proposition 3.2, we may choose $\ell_0 \in \mathbb{N}$ so that $P(\varphi^s, I_{\ell_0}) > 0$. Therefore $s_{\ell_0} > s$, and, consequently, $s_0 = \sup_{\ell \in \mathbb{N}} s_\ell$.

Recall that $\dim_H(\pi_{\mathbf{a}}(I_\ell^{\mathbb{N}})) = \min\{d, s_\ell\}$ for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$ by [7], Theorem 5.3. Therefore, since $\bigcup_{\ell \in \mathbb{N}} \pi_{\mathbf{a}}(I_\ell^{\mathbb{N}}) \subset F_{\mathbf{a}}$, we have $\min\{d, s_0\} \leq \dim_H(F_{\mathbf{a}})$. To show that $\dim_H(F_{\mathbf{a}}) \leq s_0$, take $s < \dim_H(F_{\mathbf{a}})$. Choose $m \in \mathbb{Z}$ and $0 < \delta \leq 1$ so that $s = m + \delta$ and let Δ be a closed ball such that $f_i(\Delta) \subset \Delta$ for all $i \in \mathbb{N}$. It follows from the definition of singular values that for each $\omega \in \Sigma_*$ we may cover $f_\omega(\Delta)$ with at most a constant times

$$\frac{\gamma_1(\omega)}{\gamma_{m+1}(\omega)} \frac{\gamma_2(\omega)}{\gamma_{m+1}(\omega)} \cdots \frac{\gamma_m(\omega)}{\gamma_{m+1}(\omega)}$$

balls of radius $\gamma_{m+1}(\omega)$. This is just a straightforward generalisation of the geometric intuition explained in §2.2. Thus there exists $c \geq 1$ so that

$$\mathcal{H}^s_{2^{-k}}(F_{\mathbf{a}}) \leq \sum_{\omega \in \Sigma_k} \mathcal{H}^s_{2^{-k}}(f_\omega(\Delta)) \leq c \sum_{\omega \in \Sigma_k} \varphi^s(\omega)$$

for all $k \in \mathbb{N}$. It follows that $\sum_{\omega \in \Sigma_k} \varphi^s(\omega) \geq 1$ for all $k \in \mathbb{N}$ large enough. Thus $P(\varphi^s) \geq 0$ and $s \geq s_0$ which finishes the proof. □

Considering the projection $\pi_{\mathbf{a}}$, we denote the pushforward measure of $\mu \in \mathcal{M}_\sigma(\Sigma)$ by $\pi_{\mathbf{a}}\mu$.

Theorem 4.1. *Suppose that $(T_i)_{i \in \mathbb{N}} \in GL_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i \in \mathbb{N}} \|T_i\| < \frac{1}{2}$, the singular value function φ^s is quasi-multiplicative for all $0 \leq s \leq d$, there exists $0 \leq s_0 \leq d$ so that $P(\varphi^{s_0}) = 0$, μ is the Gibbs measure for φ^{s_0} so that $\Lambda_\mu(\varphi^{s_0}) > -\infty$, and $F'_{\mathbf{a}} \subset F_{\mathbf{a}}$ with $\pi_{\mathbf{a}}\mu(F'_{\mathbf{a}}) > 0$. Then*

$$\dim_H(F'_{\mathbf{a}}) = \dim_H(F_{\mathbf{a}})$$

for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$.

Proof. Let $s < t < s_0$ and recall that by Lemma 3.8, Theorem A, and Lemma 2.2, the measure μ is ergodic and satisfies $h_\mu + \Lambda_\mu(\varphi^{s_0}) = 0$. Hence, by the Shannon–McMillan–Breiman Theorem and Kingman's sub-additive ergodic theorem, we have

$$\lim_{n \to \infty} \frac{\log \mu([\omega|_n])}{\log \varphi^t(\omega|_n)} > 1$$

for μ-almost all $\omega \in \Sigma$. Applying Egorov's theorem, we find for each $\varepsilon > 0$ a compact set $C \subset \Sigma$ and $n_0 \in \mathbb{N}$ so that

$$\mu(C) > 1 - \varepsilon \quad \text{and} \quad \mu([\omega|_n]) \leq \varphi^t(\omega|_n), \omega \in C, n \geq n_0.$$

Now, according to [36], Proposition 3.1(i), we have

$$\begin{aligned}
\int_{\mathbf{A}} \int_C \int_\Sigma \frac{\mathrm{d}\mu(\omega)\,\mathrm{d}\mu(\tau)\,\mathrm{d}\mathbf{a}}{|\pi_{\mathbf{a}}(\omega) - \pi_{\mathbf{a}}(\tau)|^s} &\leq c' \int_C \int_\Sigma \varphi^s(\omega \wedge \tau)^{-1}\,\mathrm{d}\mu(\omega)\,\mathrm{d}\mu(\tau) \\
&= c' \sum_{n=0}^\infty \sum_{\omega \in \Sigma_n} \varphi^s(\omega)^{-1} \mu([\omega]) \mu(C \cap [\mathtt{i}]) \\
&\leq c \sum_{n=0}^\infty \sum_{\omega \in \Sigma_n} \varphi^s(\omega)^{-1} \varphi^t(\omega) \mu([\omega]) \\
&\leq c \sum_{n=0}^\infty 2^{-(t-s)n} < \infty
\end{aligned}$$

for some constants $c, c' > 0$. Observe that [36], Proposition 3.1(i), is a refinement of [7], Lemma 3.1, and it generalises immediately to the infinite case. It follows that

$$\liminf_{r \downarrow 0} \frac{\log \pi_{\mathbf{a}}\mu(B(\pi_{\mathbf{a}}(\tau), r))}{\log r} = \sup\left\{t \geq 0\colon \int_\Sigma \frac{\mathrm{d}\mu(\omega)}{|\pi_{\mathbf{a}}(\omega) - \pi_{\mathbf{a}}(\tau)|^t} < \infty\right\} \geq s$$

for μ-almost all $\tau \in C$ and for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$. Thus $\dim_{\mathrm{H}}(F'_{\mathbf{a}}) \geq s$ for all $F'_{\mathbf{a}} \subset F_{\mathbf{a}}$ with $\pi_{\mathbf{a}}\mu(F'_{\mathbf{a}}) > 0$. The proof is finished by recalling Theorem B. □

To finish this section, we provide the reader with a sufficient condition to guarantee the finiteness of the Lyapunov exponent in Theorem 4.1. Recall that $s_\infty = \inf\{s\colon P(\varphi^s) < \infty\}$.

Lemma 4.2. *If $(T_i)_{i \in \mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i \in \mathbb{N}} \|T_i\| < 1$, the singular value function φ^s is quasi-multiplicative for all $0 \leq s \leq d$, $s_0 > s_\infty$, and μ is the Gibbs measure for φ^{s_0}, then $\Lambda_\mu(\varphi^{s_0}) > -\infty$.*

Proof. Observe that since $P(\varphi^{s_0}) < \infty$ the Gibbs measure μ for φ^{s_0} exists by Theorems 3.5 and 3.7. To prove the claim, let $m \in \mathbb{Z}$ be so that

$$m < s_0 \leq m + 1.$$

By the Gibbs property there is a constant $C \geq 1$ so that

$$\mu([\omega]) \leq C \varphi^{s_0}(\omega) e^{-nP(\varphi^{s_0})}$$

for all $\omega \in \Sigma_n$ and $n \in \mathbb{N}$. Thus

$$\log \frac{\varphi^{s_0}(\omega)}{\mu([\omega])} \geq nP(\varphi^{s_0}) - \log C \tag{4.1}$$

for all $\omega \in \Sigma_n$ and $n \in \mathbb{N}$.

If $\max\{s_\infty, m\} < t < s_0$, then $P(\varphi^t) < \infty$ and $Z_n(\varphi^t) < \infty$ for all $n \in \mathbb{N}$ by Lemma 3.1. As in (2.4), Jensen's inequality gives

$$\sum_{\omega \in \Sigma_n} \mu([\omega]) \log \frac{\varphi^t(\omega)}{\mu([\omega])} \leq \log\Big(\sum_{\omega \in \Sigma_n} \varphi^t(\omega)\Big) = \log Z_n(\varphi^t). \tag{4.2}$$

Since $\varphi^{s_0}(\omega) = \gamma_{m+1}(\omega)^{s_0 - t}\varphi^t(\omega)$ for all $\omega \in \Sigma_*$ we have, by (4.1) and (4.2), that

$$\begin{aligned}\big(nP(\varphi^{s_0}) - \log C\big) &\leq \sum_{\omega \in \Sigma_n} \mu([\omega])\Big(\log \gamma_{m+1}(\omega)^{s_0 - t} + \log \frac{\varphi^t(\omega)}{\mu([\omega])}\Big) \\ &\leq \sum_{\omega \in \Sigma_n} \mu([\omega]) \log \gamma_{m+1}(\omega)^{s_0 - t} + \log Z_n(\varphi^t).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{n}\sum_{\omega \in J^n} \mu([\omega]) \log \varphi^{s_0}(\omega) &\geq \frac{1}{n}\sum_{\omega \in I^n} \mu([\omega]) \log \gamma_{m+1}(\omega)^{m+1} \\ &\geq \frac{(m+1)(nP(\varphi^{s_0}) - \log C - \log Z_n(\varphi^t))}{n(s_0 - t)}.\end{aligned}$$

Letting $n \to \infty$ we have

$$\Lambda_\mu(\varphi^{s_0}) \geq \frac{(m+1)(P(\varphi^{s_0}) - P(\varphi^t))}{s_0 - t} > -\infty. \qquad \square$$

5. Multifractal analysis of Birkhoff averages

The aim of this section is to prove Theorem C. The upper bound is proved in Proposition 5.2 and the lower bound in Theorem 5.4. It is worth mentioning that the upper bound in Theorem C holds for all $\mathbf{a} \in \mathbf{A}$. The proof of the upper bound can be considered to be a refinement of the proof of Theorem B; see also [24], §4. The proof of the lower bound is more complicated and it involves a detailed examination of symbolic tree structures.

5.1. Proof of the upper bound in Theorem C. In this section we shall prove the upper bound in Theorem C. The reader is prompted to recall notation from §2.2 and §2.3. We begin with a lemma relating the dimension of $J_\Phi(\alpha)$ to the singular value function. Define

$$A_\Phi(\alpha, n, k) = \{\omega \in \Sigma_k : A_k\phi_i(\tau) \in B_n(\alpha_i) \text{ for all } \tau \in [\omega] \text{ and } i \in \{1, \dots, n\}\}$$

for all $n, k \in \mathbb{N}$.

Lemma 5.1. *If $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i\in\mathbb{N}} \|T_i\| < 1$, $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ has summable variations, $\alpha \in \overline{\mathbb{R}}^{\mathbb{N}}$, $\mathbf{a} \in \mathbf{A}$, $s < \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha))$, and $n \in \mathbb{N}$, then there is $k_0 \in \mathbb{N}$ such that*

$$\sum_{\omega\in A_\Phi(\alpha,n,k)} \varphi^s(\omega) > 1$$

for all $k \ge k_0$.

Proof. Let $\delta = \sup_{i\in\mathbb{N}} \|T_i\|$ and set

$$D_\Phi(\alpha, n, k) = \{\omega \in \Sigma_k : \text{there is } \tau \in [\omega] \text{ such that } A_k\phi_i(\tau) \in B_{2n}(\alpha_i) \text{ for all } i \in \{1, \dots, n\}\}$$

for all $n, k \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $\lim_{k\to\infty} \mathrm{var}_k(A_k\phi_i) = 0$ we may choose k_1 so that $\mathrm{var}_k\,(A_k\phi_i) < (2n)^{-1}$ for all $i \in \{1, \dots, n\}$ and all $k \ge k_1$. Thus we have $D_\Phi(\alpha, n, k) \subset A_\Phi(\alpha, n, k)$ for all $k \ge k_1$. Since

$$J_\Phi(\alpha) \subset \bigcup_{l\in\mathbb{N}} \bigcap_{k=l}^{\infty} \bigcup_{\omega\in D_\Phi(\alpha,n,k)} \pi_{\mathbf{a}}([\omega])$$

and $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) > s$ there is $l \in \mathbb{N}$ with

$$\dim_{\mathrm{H}}\Big(\bigcap_{k=l}^{\infty} \bigcup_{\omega\in D_\Phi(\alpha,n,k)} \pi_{\mathbf{a}}\,([\omega])\Big) > s.$$

Hence, continuing as in the proof of Theorem B, we find a constant $c \ge 1$ so that

$$\mathcal{H}^s_{\delta^k}\Big(\bigcup_{\omega\in D_\Phi(\alpha,n,k)} \pi_{\mathbf{a}}([\omega])\Big) \le c \sum_{\omega\in D_\Phi(\alpha,n,k)} \varphi^s(\omega)$$

for all $k \in \mathbb{N}$. The claim follows. □

Proposition 5.2. *If $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i\in\mathbb{N}} \|T_i\| < 1$, $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ has summable variations, and $\alpha \in \overline{\mathbb{R}}^{\mathbb{N}}$, then*

$$\dim_{\mathrm{H}}(J_{\Phi}^{\mathbf{a}}(\alpha)) \le \lim_{n\to\infty} \lim_{k\to\infty} \sup \Big\{ D_k(\mu) \colon \mu \in \mathcal{M}_{\sigma^k}^*(\Sigma) \text{ so that } \int A_k\phi_i \,\mathrm{d}\mu \in B_n(\alpha_i) \text{ for all } i \in \{1,\ldots,n\} \Big\}$$

for all $\mathbf{a} \in \mathbf{A}$.

Proof. Fix $\mathbf{a} \in \mathbf{A}$, $s < \dim_{\mathrm{H}}(J_{\Phi}^{\mathbf{a}}(\alpha))$, and $n \in \mathbb{N}$. According to Lemma 5.1, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{\omega\in A_\Phi(\alpha,n,k)} \varphi^s(\omega) > 1$$

for all $k \ge k_0$. Let $k \ge k_0$ and choose a finite subset

$$F_\Phi(\alpha,n,k) \subset A_\Phi(\alpha,n,k)$$

with

$$F(k) = \sum_{\omega\in F_\Phi(\alpha,n,k)} \varphi^s(\omega) \ge 1.$$

Define a compactly supported k-th level Bernoulli measure $\mu \in \mathcal{M}_{\sigma^k}(\Sigma)$ by setting

$$\mu([\omega]) = \begin{cases} \varphi^s(\omega)/F(k), & \text{if } \omega \in F_\Phi(\alpha,n,k), \\ 0, & \text{if } \omega \in \Sigma_k \setminus F_\Phi(\alpha,n,k), \end{cases}$$

for all $\omega \in \Sigma_k$. It follows immediately that

$$\sum_{\omega\in\Sigma_k} \mu([\omega]) \log \frac{\varphi^s(\omega)}{\mu([\omega])} = \log F(k) \ge 0$$

yielding $s \le D_k(\mu)$. Since μ is supported on $\bigcup_{\omega\in F_\Phi(\alpha,n,k)}[\omega]$ and $A_k\phi_i(\tau) \in B_n(\alpha_i)$ for all $\omega \in F_\Phi(\alpha,n,k)$, $\tau \in [\omega]$, and $i \in \{1,\ldots,n\}$ we also have

$$\int A_k\phi_i \,\mathrm{d}\mu \in B_n(\alpha_i)$$

for all $i \in \{1,\ldots,n\}$. These observations imply the proof. □

It remains to show the lower bound in Theorem C.

5.2. Symbolic tree structure in level sets. The following proposition contains the essence of the proof of the lower bound in Theorem C. Its proof uses some of the ideas presented in [19], §5.2, and [10], §3.1.

Proposition 5.3. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$, $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ *has summable variations,* $\alpha \in \overline{\mathbb{R}}^{\mathbb{N}}$, *and*

$$s < \lim_{n\to\infty} \lim_{k\to\infty} \sup \Big\{ D_k(\mu)\colon\ \mu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \text{ so that } \int A_k\phi_i \,\mathrm{d}\mu \in B_n(\alpha_i) \text{ for all } i \in \{1,\ldots,n\} \Big\}.$$

Then there exists a set $S \subset E_\Phi(\alpha)$, *a Borel probability measure* μ *supported on* S, *and a constant* $C \ge 1$, *such that* $\mu([\omega]) \le C\varphi^s(\omega)$ *for all* $\omega \in \Sigma_*$.

In fact, with this proposition, the lower bound in Theorem C follows almost immediately.

Theorem 5.4. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$, $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ *has summable variations, and* $\alpha \in \overline{\mathbb{R}}^{\mathbb{N}}$. *Then*

$$\dim_{\mathrm{H}}(J^{\mathbf{a}}_\Phi(\alpha)) \ge \min\Big\{ d, \lim_{n\to\infty} \lim_{k\to\infty} \sup \Big\{ D_k(\mu)\colon \mu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \text{ so that } \int A_k\phi_i \,\mathrm{d}\mu \in B_n(\alpha_i) \text{ for all } i \in \{1,\ldots,n\} \Big\}\Big\}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$.

Proof. Let $s > 0$ be as in Proposition 5.3. Applying the measure given by Proposition 5.3 in the proof of Theorem 4.1, we get $\dim_{\mathrm{H}}(\pi_{\mathbf{a}}(S)) \ge s$ for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$, where $S \subset E_\Phi(\alpha)$ is as in Proposition 5.3. Thus $\dim_{\mathrm{H}}(J^{\mathbf{a}}_\Phi(\alpha)) \ge s$ for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$. □

In the course of the proof of Proposition 5.3, we shall rely on the concept of $\mathcal{M}$-trees. This approach is inspired by a similar notion discussed by Furstenberg in [18]. We shall now define all the required concepts.

If $\omega, \tau \in \Sigma_* \cup \Sigma$ so that $\omega \wedge \tau = \omega$, then we write

$$\omega \preccurlyeq \tau.$$

This defines a partial order on Σ_*. Let $\mathbb{X} \subset \Sigma_*$ be an antichain with respect to $\preccurlyeq$. This means that $\omega \not\preccurlyeq \tau$ for all $\omega, \tau \in \mathbb{X}$. If there is a function

$$\mathcal{M}_{\mathbb{X}} \colon \mathbb{X} \longrightarrow [0,1]$$

so that

$$\sum_{\omega \in \mathbb{X}} \mathcal{M}_{\mathbb{X}}(\omega) = 1,$$

then the ordered pair $(\mathbb{X}, \mathcal{M}_{\mathbb{X}})$ is called an $\mathcal{M}$*-tree*. An $\mathcal{M}$-tree $(\mathbb{X}, \mathcal{M}_{\mathbb{X}})$ is said to be *finite* if $\mathbb{X}$ is a finite set. If $(\mathbb{X}, \mathcal{M}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{M}_{\mathbb{Y}})$ are $\mathcal{M}$-trees so that

$$\mathcal{M}_{\mathbb{X}}(\omega) = \sum_{\tau \in \{\kappa \in \mathbb{Y} \colon \omega \preccurlyeq \kappa\}} \mathcal{M}_{\mathbb{Y}}(\tau)$$

for all $\omega \in \mathbb{X}$, then we write

$$(\mathbb{X}, \mathcal{M}_{\mathbb{X}}) \preccurlyeq (\mathbb{Y}, \mathcal{M}_{\mathbb{Y}}).$$

This defines a partial order on the collection of all $\mathcal{M}$-trees.

Next we shall define a limit for certain $\mathcal{M}$-tree sequences. If $((\mathbb{X}_n, \mathcal{M}_{\mathbb{X}_n}))_{n\in\mathbb{N}}$ is a chain of finite $\mathcal{M}$-trees so that $\lim_{n\to\infty} \min\{|\omega| \colon \omega \in \mathbb{X}_n\} = \infty$, then the limit of that sequence is defined to be

$$\lim_{n\to\infty} (\mathbb{X}_n, \mathcal{M}_{\mathbb{X}_n}) = (\mathbb{X}_\infty, \mathcal{M}_\infty),$$

where

$$\mathbb{X}_\infty = \{\tau \in \Sigma \colon \text{for each } n \in \mathbb{N} \text{ there is } \omega \in \mathbb{X}_n \text{ so that } \omega \preccurlyeq \tau\}$$

and $\mathcal{M}_\infty$ is a Borel probability measure supported on $\mathbb{X}_\infty$ defined as follows. Observe first that since each $\mathbb{X}_n$ is a finite antichain, it is readily checked that the collection

$$\mathcal{A}(\mathbb{X}_\infty) = \{\emptyset, X_\infty\} \cup \Big\{[\omega] \cap \mathbb{X}_\infty \colon \omega \in \bigcup_{n\in\mathbb{N}} \mathbb{X}_n\Big\}$$

is a semi-algebra of subsets of $\mathbb{X}_\infty$. Moreover, since

$$\lim_{n\to\infty} \min\{|\omega| \colon \omega \in \mathbb{X}_n\} = \infty,$$

it is clear that this semi-algebra generates the Borel σ-algebra restricted to $\mathbb{X}_\infty$. We define $\mathcal{M}_\infty$ on $\mathcal{A}(\mathbb{X}_\infty)$ by setting

$$\mathcal{M}_\infty(\emptyset) = 0, \quad \mathcal{M}_\infty(\mathbb{X}_\infty) = 1,$$

and

$$\mathcal{M}_\infty([\omega] \cap \mathbb{X}_\infty) = \mathcal{M}_{\mathbb{X}_n}(\omega).$$

for all $\omega \in \mathbb{X}_n$ and $n \in \mathbb{N}$. It follows from the fact that $(\mathbb{X}_n, \mathcal{M}_{\mathbb{X}_n}) \preccurlyeq (\mathbb{X}_{n+1}, \mathcal{M}_{\mathbb{X}_{n+1}})$ for each $n \in \mathbb{N}$, that this set function is well-defined and countably additive. Thus $\mathcal{M}_\infty$ extends to a measure on Σ. Finally, given a subset $\Omega \subset \Sigma_* \cup \Sigma$ let $\mathbb{D}(\Omega) \subset \mathbb{N}$ be the collection of all digits contained within words from Ω, that is,

$$\mathbb{D}(\Omega) = \{l \in \mathbb{N} \colon \text{there is } \omega \in \Omega \text{ with } \omega_i = l \text{ for some } i\}.$$

Before proving Proposition 5.3, let us sketch the main idea of the proof. We start by choosing a compactly supported k-th level Bernoulli measure ρ_n so that $D_k(\rho_n)$ is close to the right-hand value of the inequality in the formulation of Proposition 5.3 and

$$\int A_k \phi_i \, \mathrm{d}\rho_n \in B_n(\alpha_i)$$

for all $i \in \{1, \ldots, n\}$. Then we use the strong law of large numbers and Egorov's theorem to see that appropriate averages converge to the values $D_k(\rho_n)$ and $\int A_k \phi_i \, \mathrm{d}\rho_n$ uniformly on a set S_n having large ρ_n measure. We continue by constructing an antichain $\mathbb{T}_n$ so that the set S_n is contained in the union of all cylinders obtained from the finite words in $\mathbb{T}_n$. We also construct a function $\mathcal{M}_n$ from the measure ρ_n and require that $((\mathbb{T}_n, \mathcal{M}_n))_{n \in \mathbb{N}}$ is a chain of finite $\mathcal{M}$-trees with

$$\lim_{n \to \infty} \min\{|\omega| \colon \omega \in \mathbb{T}_n\} = \infty.$$

After some organising of symbolic levels (this will be the role of the mapping Ψ_n), we obtain a measure μ supported on a set S as the limit of the chain. By choosing the averages close enough to the limiting values $D_k(\rho_n)$ and $\int A_k \phi_i \, \mathrm{d}\rho_n$, it is possible, albeit a bit technical, to show that μ and S are exactly the objects we claimed to exist.

Major technical obstacles in the proof come from the quasi-multiplicativity and the fact that the shift space is infinitely generated. Quasi-multiplicativity obliges us to organise the symbolic levels and because of the infinitely generated shift space, we have to keep track of the symbols used at each level of the construction. Let us next turn to the rigorous proof of the proposition.

Proof of Proposition 5.3. We begin by noting that, without loss of generality, we may assume that if ϕ is in the sequence Φ, then also $-\phi$ is in Φ. Indeed, if $\Phi = (\phi_i)_{i \in \mathbb{N}}$, $\alpha = (\alpha_i)_{i \in \mathbb{N}}$, and the right-hand side of the inequality in the formulation of Proposition 5.3 is denoted by $D(\Phi, \alpha)$, then we clearly have $D(\Phi, \alpha) = D(\Phi', \alpha')$, where $\Phi' = (\phi'_i)_{i \in \mathbb{N}}$ and $\alpha' = (\alpha'_i)_{i \in \mathbb{N}}$ are defined so that

$$\phi'_{2i} = \phi_i, \quad \phi'_{2i-1} = -\phi_i$$

and

$$\alpha'_{2i} = \alpha_i, \quad \alpha'_{2i-1} = -\alpha_i$$

for all $i \in \mathbb{N}$.

Choose $s < t < D(\Phi, \alpha)$ and define

$$C_i = \sup\{|\phi_i(\omega)| : \omega \in [\tau] \text{ and } \tau \in \mathbb{D}(\Gamma \cup \{1\})\}, \quad i \in \mathbb{N}. \tag{5.1}$$

For each $n \in \mathbb{N}$ we choose

$$k = k(n) \geq 4Kn(\max_{i \leq n} C_i + 1)$$

so that

$$\mathrm{var}_{k(n)} A_{k(n)}\phi_i < \frac{1}{2n}, \quad i \in \{1, \dots, n\},$$

and there exists $\nu_n \in \mathcal{M}^*_{\sigma^k}(\Sigma)$ with $D_k(\nu_n) > t$ and

$$\int A_k\phi_i \, \mathrm{d}\nu_n \in B_{2n}(\alpha_i) \tag{5.2}$$

for all $i \in \{1, \dots, n\}$. Let $\rho_n \in \mathcal{M}^*_{\sigma^k}(\Sigma)$ be the compactly supported $k(n)$-th level Bernoulli measure given by

$$\rho_n([\omega_1 \cdots \omega_{k(n)q}]) = \prod_{j=0}^{q-1} \nu_n([\omega_{jk(n)+1} \cdots \omega_{(j+1)k(n)}]).$$

For each potential ϕ_i we define a k-th level locally constant potential $\underline{A}_k\phi_i$ by

$$\underline{A}_k\phi_i(\omega) = \inf\Big\{\frac{1}{k}\sum_{l=0}^{k-1} \phi_i(\sigma^l(\tau)) : \tau_j = \omega_j \text{ for } j \in \{1, \dots, k\}\Big\}.$$

Note that $A_k\phi_i(\omega) - \mathrm{var}_k A_k\phi_i \leq \underline{A}_k\phi_i(\omega) \leq A_k\phi_i(\omega)$ for all $\omega \in \Sigma$ and $i \in \mathbb{N}$. Since $\mathrm{var}_k A_k(\phi_i) < \frac{1}{2n}$ it follows from (5.2) that

$$\int \underline{A}_k\phi_i \, \mathrm{d}\rho_n \in B_n(\alpha_i), \quad i \in \{1, \dots, n\}.$$

Moreover, it is immediate from $D_k(\nu_n) > t$ that

$$\sum_{\omega \in \Sigma_{k(n)}} \rho_n(\omega) \log \frac{\varphi^t(\omega)}{\rho_n(\omega)} > 0.$$

We let

$$\mathcal{D}(n) = \{\omega \in \Sigma_{k(n)} : \rho_n(\omega) > 0\}.$$

Since $\rho_n \in \mathcal{M}^*_{\sigma^k}(\Sigma)$ the number of words in $\mathcal{D}(n)$ is finite. Hence, for each n there is a finite set of digits $\mathbb{D}^*(n) \subset \mathbb{N}$ defined by

$$\mathbb{D}^*(n) = \mathbb{D}\Big(\bigcup_{l=1}^{n+2} \mathcal{D}(l) \cup \{1\} \cup \Gamma\Big).$$

Since we also have $\mathrm{var}_1\, \phi_i < \infty$ the quantities

$$\mathcal{A}(n) = \sup\big\{|\phi_i(\tau)| \colon \tau \in [\omega] \text{ for some } \omega \in \mathbb{D}^*(n) \text{ and } i \in \{1,\dots,n\}\big\} \tag{5.3a}$$

and

$$\mathcal{B}(n) = \sup\{\rho_n(\omega)^{-1} \colon \omega \in \mathcal{D}(n)\} \tag{5.3b}$$

are both finite. By Kolmogorov's strong law of large numbers, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \log \frac{\varphi^t(\omega_{jk(n)+1} \cdots \omega_{(j+1)k(n)})}{\rho_n([\omega_{jk(n)+1} \cdots \omega_{(j+1)k(n)}])} = \sum_{\tau\in\Sigma_{k(n)}} \rho_n(\tau) \log \frac{\varphi^t(\tau)}{\rho_n([\tau])} > 0$$

for ρ_n-almost all $\omega \in \Sigma$, and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \underline{A}_{k(n)}\phi_i(\sigma^{jk(n)}(\omega)) \in B_n(\alpha_i)$$

and for all $i \in \{1,\dots,n\}$. By Egorov's theorem we find $S_n \subset \mathrm{supp}(\rho_n)$ with $\rho_n(S_n) > 1/2$ so that each of the above convergences are uniform upon S_n. Hence there is $L(n) \in \mathbb{N}$ such that

$$\prod_{j=0}^{N-1} \varphi^t(\omega_{jk(n)+1} \cdots \omega_{(j+1)k(n)}) > \prod_{j=0}^{N-1} \rho_n([\omega_{jk(n)+1} \cdots \omega_{(j+1)k(n)}]) \tag{5.4a}$$

and

$$\frac{1}{N} \sum_{j=0}^{N-1} \underline{A}_{k(n)}\phi_i(\sigma^{jk(n)}(\omega)) \in B_n(\alpha_i), \quad i \in \{1,\dots,n\}. \tag{5.4b}$$

for all $\omega \in S_n$ and all $N \geq L(n)$. We also let

$$M(n) = \max\{\varphi^t(\tau)^{-1} \colon [\tau] \cap \mathrm{supp}(\rho_n) \neq \emptyset\}.$$

For every $\alpha, \beta \in \Sigma_*$, according to the quasi-multiplicativity of φ^t, there exists $\omega \in \Gamma$ such that

$$\varphi^t(\alpha\omega\beta) \geq c(t)\varphi^t(\alpha)\varphi^t(\beta),$$

where $c(t) > 0$ is a constant depending only on t. For each pair α, β we fix a choice of such ω. We let

$$\alpha \star \beta$$

denote the word $\alpha\omega\beta$, so

$$\varphi^t(\alpha \star \beta) \geq c(t)\varphi^t(\alpha)\varphi^t(\beta).$$

Note that for any given $\alpha, \beta \in \Sigma_*$ there are at most $K = \max\{|\omega| : \omega \in \Gamma\}$ finite words $\beta' \in \Sigma_*$ with $\alpha \star \beta' = \alpha \star \beta$ (including β itself). We also write

$$\alpha \star \beta \star \omega = (\alpha \star \beta) \star \omega.$$

Our aim is to construct a sequence of $\mathcal{M}$-trees $((\mathbb{T}_n, \mathcal{M}_n))_{n \in \mathbb{N} \cup \{0\}}$ with

$$(\mathbb{T}_{n-1}, \mathcal{M}_{n-1}) \preccurlyeq (\mathbb{T}_n, \mathcal{M}_n), \quad n \in \mathbb{N},$$

along with functions $(\Psi_n)_{n \in \mathbb{N}}$ of the form

$$\Psi_n : \mathbb{T}_n \longrightarrow \Sigma_*,$$

together with a sequence $(\gamma_n)_{n \in \mathbb{N}}$ with the property that every $\tau \in \mathbb{T}_n$ satisfies

$$\gamma_n - K \leq |\Psi_n(\tau)| \leq \gamma_n.$$

We begin by letting $\mathbb{T}_0 = \{\emptyset\}$, $\mathcal{M}_0(\emptyset) = 1$, $\Psi_0 = \{\emptyset \mapsto \emptyset\}$, and $\gamma_0 = 0$. Suppose we have defined $(\mathcal{M}_{n-1}, \mathbb{T}_{n-1})$,

$$\Psi_{n-1} : \mathbb{T}_{n-1} \longrightarrow \Sigma_*,$$

and γ_{n-1} with the required properties. For each $\omega \in \mathbb{T}_{n-1}$ we let

$$\mathbb{Z}_{n-1}(\omega) = \{\tau \in \mathbb{T}_{n-1} : \Psi_{n-1}(\tau) \preccurlyeq \Psi_{n-1}(\omega) \text{ or } \Psi_{n-1}(\omega) \preccurlyeq \Psi_{n-1}(\tau)\}.$$

We shall construct $(\mathcal{M}_n, \mathbb{T}_n)$, $\Psi_n : \mathbb{T}_n \to \Sigma_*$, and γ_n as follows. First take $q_n \in \mathbb{N}$ so that

$$\begin{aligned} q_n > {} & \frac{4M(n)^{L(n)} M(n+1)^{L(n+1)}}{\min\{\varphi^t(\Psi_{n-1}(\omega)) : \omega \in \mathbb{T}_{n-1}\}} + \max\{\#\Psi_{n-1}^{-1}(\omega) : \omega \in \mathbb{T}_{n-1}\} \\ & + nL(n+1)(\mathcal{A}(n)+1)(\mathcal{B}(n)+1)(\gamma_{n-1} + 4K \\ & \qquad + k(n+1) + k(n+2) + 1) \\ & + \#\mathcal{D}(n)\#\mathcal{D}(n+1) \max\{\#\mathbb{Z}_{n-1}(\omega) : \omega \in \mathbb{T}_{n-1}\} + q_{n-1} \\ & + \#\mathcal{D}(n+1)\#\big(\{1\} \cup \Gamma \cup \mathbb{D}(\mathcal{D}(n)) \cup \mathbb{D}(\mathcal{D}(n+1))\big)^{5K + k(n+1) + k(n) + 2}, \end{aligned} \tag{5.5}$$

where $\mathcal{A}(n)$ and $\mathcal{B}(n)$ are as in (5.3).

Let

$$\mathcal{F}_n = \{\tau \in \Sigma_{k(n)q_n} : [\tau] \cap S_n \neq \emptyset\}$$

and

$$\eta(n) = \sum_{\tau \in \mathcal{F}_n} \rho_n([\tau]) \geq \rho_n(S_n) > 1/2.$$

Define

$$\mathcal{F}_n^l = \{\tau \in \Sigma_{k(n)l} : \text{there is } \beta \in \Sigma_{k(n)(q_n - l)} \text{ with } \tau\beta \in \mathcal{F}_n\}, \quad l \in \{1, \ldots, q_n\}.$$

In the process of constructing $(\mathbb{T}_n, \mathcal{M}_n)$, Ψ_n, and γ_n, we shall construct a sequence of intermediary $\mathcal{M}$-trees $((\mathbb{T}_n^l, \mathcal{M}_n^l))_{l=0}^{q_n}$ so that

$$(\mathbb{T}_n^l, \mathcal{M}_n^l) \preccurlyeq (\mathbb{T}_n^{l+1}, \mathcal{M}_n^{l+1})$$

and

$$(\mathbb{T}_{n-1}, \mathcal{M}_{n-1}) \preccurlyeq (\mathbb{T}_n^l, \mathcal{M}_n^l) \preccurlyeq (\mathbb{T}_n, \mathcal{M}_n) \tag{5.6}$$

for all $l \in \{1, \ldots, q_n\}$. In addition, we construct intermediary maps

$$\Psi_n^l : \mathbb{T}_n^l \longrightarrow \Sigma_*$$

and $(\gamma_n^l)_{l=0}^{q_n}$ so that

$$\gamma_n^l - K \leq |\Psi_n^l(\tau)| \leq \gamma_n^l, \quad \tau \in \mathbb{T}_n^l$$

and

$$\Psi_n^l(\omega^l) \preccurlyeq \Psi_n^{l+1}(\omega^{l+1}), \quad \omega^l \in \mathbb{T}_n^l,\ \omega^{l+1} \in \mathbb{T}_n^{l+1},\ \omega^l \preccurlyeq \omega^{l+1}.$$

First take $(\mathbb{T}_n^0, \mathcal{M}_n^0) = (\mathbb{T}_{n-1}, \mathcal{M}_{n-1})$, $\Psi_n^0 = \Psi_{n-1}$, and $\gamma_n^0 = \gamma_{n-1}$. Clearly $(\mathbb{T}_n^0, \mathcal{M}_n^0)$, Ψ_n^0, and γ_n^0 satisfy the required properties. For each $l \in \{1, \ldots, q_n\}$ we let

$$\mathbb{T}_n^l = \{\kappa\tau : \kappa \in \mathbb{T}_{n-1} \text{ and } \tau \in \mathcal{F}_n^l\}.$$

For each $\omega \in \mathbb{T}_n^l$ we take the (unique) pair $\kappa \in \mathbb{T}_{n-1}$ and $\tau \in \mathcal{F}_n^l$ with $\omega = \kappa\tau$ and let

$$\mathcal{M}_n^l(\omega) = \frac{\mathcal{M}_{n-1}(\kappa)}{\eta(n)} \sum_{\beta \in \{\alpha : \tau\alpha \in \mathcal{F}_n\}} \rho_n([\tau\beta]).$$

It is clear that $(\mathbb{T}_n^l, \mathcal{M}_n^l) \preccurlyeq (\mathbb{T}_n^{l+1}, \mathcal{M}_n^{l+1})$ and if we let

$$(\mathbb{T}_n, \mathcal{M}_n) = (\mathbb{T}_n^{q_n}, \mathcal{M}_n^{q_n}),$$

we have shown (5.6). We shall construct the functions $(\Psi_n^l)_{l=1}^{q_n}$ and numbers $(\gamma_n^l)_{l=1}^{q_n}$ recursively.

Suppose $l \in \{1, \dots, q_n\}$ and we have constructed Ψ_n^{l-1} and γ_n^{l-1} satisfying the required properties. Define

$$\gamma_n^l = \gamma_n^{l-1} + 2K + k(n)$$

and let

$$\omega = \kappa\tau \in \mathbb{T}_n^l$$

so that $\kappa \in \mathbb{T}_{n-1}$ and $\tau \in \mathcal{F}_n^l$. Choose $\tau' \in \mathcal{F}_n^{l-1}$ so that $\tau' \preccurlyeq \tau$ and set

$$\omega' = \kappa\tau' \in \mathbb{T}_n^{l-1}.$$

Thus there exists $\tau_l \in \Sigma_{k(n)}$ so that $\omega = \omega'\tau_l$. The function Ψ_n^l is defined by setting

$$\Psi_n^l(\omega) = \Psi_n^{l-1}(\omega') \star \tau_l \star 1 \cdots 1,$$

where the length of $1 \cdots 1$ is $\gamma_n^l - K - |\Psi_n^{l-1}(\omega') \star \tau_l|$. Since

$$\gamma_n^{l-1} - K \leq |\Psi_n^{l-1}(\omega')| \leq \gamma_n^{l-1},$$

we have

$$\gamma_n^{l-1} - K + k(n) \leq |\Psi_n^l(\omega') \star \tau_l| \leq \gamma_n^{l-1} + k(n) + K.$$

Thus $\Psi_n^l(\omega)$ is well-defined, the length of $1 \cdots 1$ is at most $2K$, and $\gamma_n^l - K \leq |\Psi_n^l(\omega)| \leq \gamma_n^l$. Moreover, if we let

$$c_0 = c(t)^2 \varphi^t(1 \cdots 1),$$

where the length of $1 \cdots 1$ is $2K$, then a simple induction gives

$$\varphi^t(\Psi_{n-1}(\kappa)) \prod_{j=1}^{l} \varphi^t(\tau_j) \leq c_0^{-l} \varphi^t(\Psi_n^l(\kappa\tau)), \tag{5.7}$$

where each τ_j has length $k(n)$. We emphasise that c_0 is independent of n and l. Recalling that $\mathbb{T}_n = \mathbb{T}_n^{q_n}$, we set

$$\Psi_n = \Psi_n^{q_n}.$$

To finish the construction of $\mathcal{M}$-trees $((\mathbb{T}_n, \mathcal{M}_n))_{n \in \mathbb{N} \cup \{0\}}$, functions $(\Psi_n)_{n \in \mathbb{N}}$, and the sequence $(\gamma_n)_{n \in \mathbb{N}}$, we shall show that

$$\max\{\#\mathbb{Z}_n^l(\omega) \colon \omega \in \mathbb{T}_n^l\} \leq q_{n-1}(2K)^{l+q_{n-1}} \tag{5.8}$$

for all $l \in \{1, \dots, q_n\}$, and $n \in \mathbb{N}$, where

$$\mathbb{Z}_n^l(\omega) = \{\omega' \in \mathbb{T}_n^l \colon \Psi_n^l(\omega) \preccurlyeq \Psi_n^l(\omega') \text{ or } \Psi_n^l(\omega') \preccurlyeq \Psi_n^l(\omega)\}$$

for all $\omega \in \mathbb{T}_n^l$, $l \in \{1, \dots, q_n\}$, and $n \in \mathbb{N}$.

Fix $\omega \in \mathbb{T}_n^l$ and choose $\kappa \in \mathbb{T}_{n-1}$ and $\tau \in \mathcal{F}_n^l$ so that $\omega = \kappa\tau$. Write

$$\tau = \tau_1 \cdots \tau_l, \quad \tau_j \in \Sigma_{k(n)}.$$

Now suppose $\omega' \in \mathbb{Z}_n^l(\omega)$ and similarly take $\kappa' \in \mathbb{T}_{n-1}$ and $\tau' = \tau_1' \cdots \tau_l' \in \mathcal{F}_n^l$ so that $\omega' = \kappa'\tau'$ and $\tau_j' \in \Sigma_{k(n)}$. It is clear that either $\Psi_{n-1}(\kappa) \preccurlyeq \Psi_{n-1}(\kappa')$ or $\Psi_{n-1}(\kappa') \preccurlyeq \Psi_{n-1}(\kappa)$. Thus we have $\kappa' \in \mathbb{Z}_{n-1}(\kappa)$. Now since either $\Psi_n^l(\omega) \preccurlyeq \Psi_n^l(\omega')$ or $\Psi_n^l(\omega') \preccurlyeq \Psi_n^l(\omega)$ and $|\Psi_n^{l-1}(\omega'')| \le \gamma_n^{l-1} < \gamma_n^l - K \le |\Psi_n^l(\omega)|$, we have $\Psi_n^{l-1}(\omega'') \preccurlyeq \Psi_n^l(\omega)$, where $\omega'' = \kappa'\tau_1' \cdots \tau_{l-1}' \in \mathbb{T}_n^{l-1}$. Thus, for $j \le l-1$ there is a subword of $\Psi_n^l(\omega)$ which starts between positions $\gamma_n^{j-1} - K$ and $\gamma_n^{j-1} + K$ (or between positions $\gamma_{n-1} - K$ and $\gamma_{n-1} + K$ if $j = 1$). Also, $\tau_l' \in \mathcal{D}(n)$. This shows that

$$\#\mathbb{Z}_n^l(\omega) \le \#\mathbb{Z}_{n-1}(\kappa)(2K)^{l-1}\#\mathcal{D}(n).$$

Hence

$$\begin{aligned}\max\{\#\mathbb{Z}_n^l(\omega) \colon \omega \in \mathbb{T}_n^l\} &\le \max\{\#\mathbb{Z}_{n-1}(\kappa) \colon \kappa \in \mathbb{T}_{n-1}\}(2K)^{l-1}\#\mathcal{D}(n)\\ &= \max\{\#\mathbb{Z}_{n-1}^{q_{n-1}}(\kappa) \colon \omega \in \mathbb{T}_{n-1}^{q_{n-1}}\}(2K)^{l-1}\#\mathcal{D}(n).\end{aligned}$$

Iterating this inequality and applying the definition of q_{n-1} we obtain

$$\begin{aligned}&\max\{\#\mathbb{Z}_n^l(\omega) \colon \omega \in \mathbb{T}_n^l\}\\ &\quad\le \max\{\#\mathbb{Z}_{n-2}(\kappa) \colon \kappa \in \mathbb{T}_{n-2}\}(2K)^{q_{n-1}+l-2}\#\mathcal{D}(n-1)\#\mathcal{D}(n)\\ &\quad\le q_{n-1}(2K)^{q_{n-1}+l}.\end{aligned}$$

Let

$$(T, \nu) = \lim_{n\to\infty}(\mathbb{T}_n, \mathcal{M}_n).$$

Then T consists of all $\omega \in \Sigma$ such that there is a sequence $(\omega_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{T}_n$ such that $\omega_n \preccurlyeq \omega_{n+1} \preccurlyeq \omega$ for all $n \in \mathbb{N}$. It follows from the construction of $(\Psi_n)_{n\in\mathbb{N}}$ that $\Psi_n(\omega_n) \preccurlyeq \Psi_{n+1}(\omega_{n+1})$ and $|\Psi_n(\omega_n)| < |\Psi_{n+1}(\omega_{n+1})|$. Thus, there is a unique infinite word $\Psi(\omega) \in \Sigma$ with $\Psi_n(\omega_n) \preccurlyeq \Psi(\omega)$ for all $n \in \mathbb{N}$. This defines a map

$$\Psi \colon T \longrightarrow \Sigma.$$

We let

$$S = \Psi(T)$$

and

$$\mu = \nu \circ \Psi^{-1}$$

and let

$$S^* = \{\omega \in \Sigma_* \colon [\omega] \cap S \neq \emptyset\}.$$

In Lemmas 5.5–5.10 we shall verify that S and μ defined above have the required properties, that is, $S \subset E_\Phi(\alpha)$ and $\mu([\omega]) \le C\varphi^s(\omega)$ for all $\omega \in \Sigma_*$. The proof of Proposition 5.3 thus follows. □

The following six lemmas should be considered to be part of the proof of Proposition 5.3. The statements were formulated as lemmas in order to help further structure the proof. In the first two lemmas, we estimate the measure ν which was obtained as the limit of the chain of finite $\mathcal{M}$-trees.

Lemma 5.5. *In the setting of the proof of Proposition 5.3, let $n \in \mathbb{N}$, $l \in \{1, \ldots, q_n\}$, and $\omega = \kappa\tau \in \mathbb{T}_n^l$ so that $\kappa \in \mathbb{T}_{n-1}$ and $\tau \in \mathcal{F}_n^l$. Then*

$$\nu([\omega]) \leq 2\nu([\kappa])M(n)^{L(n)}C_0^l\Big(\frac{\varphi^t\left(\Psi_n^l(\omega)\right)}{\varphi^t(\Psi_{n-1}(\kappa))}\Big).$$

Proof. Since $\omega \in \mathbb{T}_n^l$ and $\kappa \in \mathbb{T}_{n-1}$ we have $\nu([\kappa]) = \mathcal{M}_{n-1}(\kappa)$ and

$$\begin{aligned}\nu([\omega]) &= \mathcal{M}_n^l(\omega)\\ &= \frac{\mathcal{M}_{n-1}(\kappa)}{\eta(n)} \sum_{\beta \in \{\alpha :\, \tau\alpha \in \mathcal{F}_n\}} \rho_n([\tau\beta])\\ &= \frac{\nu([\kappa])}{\eta(n)} \sum_{\beta \in \{\alpha :\, \tau\alpha \in \mathcal{F}_n\}} \rho_n([\tau])\rho_n([\beta])\\ &\leq 2\nu([\kappa])\rho_n([\tau]).\end{aligned}$$

So it suffices to show that

$$\rho_n([\tau]) = \prod_{j=1}^{l} \rho_n([\tau_j]) \leq M(n)^{L(n)}C_0^l\Big(\frac{\varphi^t\left(\Psi_n^l(\kappa\tau)\right)}{\varphi^t(\Psi_{n-1}(\kappa))}\Big).$$

Thus, by (5.7), it suffices to show that

$$\prod_{j=1}^{l} \rho_n([\tau_j]) \leq M(n)^{L(n)} \prod_{j=1}^{l} \varphi^t([\tau_j]). \tag{5.9}$$

Now either $l \geq L(n)$, in which case it follows from $\tau \in \mathcal{F}_n^l$ and (5.4) that

$$\prod_{j=1}^{l} \rho_n([\tau_j]) \leq \prod_{j=1}^{l} \varphi^t([\tau_j]),$$

or $l < L(n)$, in which case we have

$$\prod_{j=1}^{l} \varphi^t([\tau_j])^{-1} \leq M(n)^l \leq M(n)^{L(n)}.$$

This shows (5.9) and thus completes the proof of the lemma. □

Lemma 5.6. *In the setting of the proof of Proposition 5.3, let $n \in \mathbb{N}$, $l \in \{1, \dots, q_n\}$, and $\omega \in \mathbb{T}_n^l$. Then*

$$\nu([\omega]) \le q_{n-1} C_0^{q_{n-1}+l} \varphi^t(\Psi_n^l(\omega)).$$

Proof. Since $\omega \in \mathbb{T}_n^l$ we may take $\kappa \in \mathbb{T}_{n-1}$ and $\tau \in \mathcal{F}_n^l$ so that $\omega = \kappa\tau$, so by Lemma 5.5, we have

$$\nu([\omega]) \le 2\nu([\kappa]) M(n)^{L(n)} C_0^l \Big(\frac{\varphi^t(\Psi_n^l(\omega))}{\varphi^t(\Psi_{n-1}(\kappa))} \Big).$$

Moreover, since $\kappa \in \mathbb{T}_{n-1} = \mathbb{T}_{n-2}^{q_{n-1}}$ there exists $\kappa_- \in \mathbb{T}_{n-2}$ and $\tau_- \in \mathcal{F}_{n-1}^{q_{n-1}}$ so that $\kappa = \kappa_- \tau_-$. Applying Lemma 5.5 once more, we obtain

$$\nu([\kappa]) \le 2\nu([\kappa_-]) M(n-1)^{L(n-1)} C_0^{q_{n-1}} \Big(\frac{\varphi^t\big(\Psi_{n-1}^{q_{n-1}}(\kappa)\big)}{\varphi^t(\Psi_{n-2}(\kappa_-))} \Big).$$

Combining these two estimates we get

$$\nu([\omega]) \le \frac{4M(n-1)^{L(n-1)} M(n)^{L(n)}}{\varphi^t(\Psi_{n-2}(\kappa_-))} C_0^{q_{n-1}+l} \varphi^t(\Psi_n^l(\omega)).$$

Noting that the definition of q_{n-1} implies

$$q_{n-1} \ge \frac{4M(n-1)^{L(n-1)} M(n)^{L(n)}}{\min\{\varphi^t(\Psi_{n-2}(\kappa')) \colon \kappa' \in \mathbb{T}_{n-2}\}}$$

completes the proof. □

In the following three lemmas, we turn to estimate the measure μ which was defined as the pushforward of ν.

Lemma 5.7. *In the setting of the proof of Proposition 5.3, let $n \in \mathbb{N}$, $l \in \{1, \dots, q_n\}$, and $\omega \in \mathbb{T}_n^l$. Then,*

$$\mu([\Psi_n^l(\omega)]) \le q_{n-1}^2 \#\mathcal{D}(n) (2KC_0)^{q_{n-1}+l} \varphi^t(\Psi_n^l(\omega)).$$

Proof. Note that $\mu([\Psi_n^l(\omega)]) = \nu \circ \Psi^{-1}([\Psi_n^l(\omega)])$. Moreover, $\Psi^{-1}([\Psi_n^l(\omega)]) \subset \bigcup [\eta]$, where the union is taken over all $\eta \in \mathbb{T}_n^{l+1}$ satisfying $\Psi_n^l(\omega) \preccurlyeq \Psi_n^{l+1}(\eta)$. This follows from the fact that every $\eta \in \mathbb{T}_n^{l+1}$ maps to a string $\Psi_n^{l+1}(\eta)$ of length $|\Psi_n^{l+1}(\eta)| \ge \gamma_n^{l+1} - K \ge \gamma_n^l \ge |\Psi_n^l(\omega)|$. Since

$$\nu([\eta]) \le q_{n-1} C_0^{q_{n-1}+l+1} \varphi^t(\Psi_n^{l+1}(\eta)) \le q_{n-1} C_0^{q_{n-1}+l+1} \varphi^t(\Psi_n^l(\omega))$$

for all $\eta \in \mathbb{T}_n^{l+1}$ satisfying $\Psi_n^l(\omega) \preccurlyeq \Psi_n^{l+1}(\eta)$, it suffices to show that

$$\#\{\eta \in \mathbb{T}_n^{l+1} \colon \Psi_n^l(\omega) \preccurlyeq \Psi_n^{l+1}(\eta)\} \le \#\mathcal{D}(n) q_{n-1} (2K)^{q_{n-1}+l}.$$

Now if $\eta \in \mathbb{T}_n^{l+1}$ satisfies $\Psi_n^l(\omega) \preccurlyeq \Psi_n^{l+1}(\eta)$, then there exists some $\eta_- \in \mathbb{T}_n^l$ with $\eta_- \preccurlyeq \eta$ and $\Psi_n^l(\omega) \preccurlyeq \Psi_n^l(\eta_-)$ or $\Psi_n^l(\eta_-) \preccurlyeq \Psi_n^l(\omega)$. By Lemma 5.8, there are at most $q_{n-1}(2K)^{q_{n-1}+l}$ such η_-. Moreover, each such η_- is continued by at most $\#\mathcal{D}(n)$ strings in $\mathbb{T}_n^{l+1}$. This finishes the proof. □

Lemma 5.8. *In the setting of the proof of Proposition 5.3, let $\tau \in \Sigma_*$. If $n = n(\tau)$ is minimal so that $|\tau| \leq \gamma_n^l - K$ for some $l \in \{1, \dots, q_n\}$ and let $l = l(\tau)$ be the least such l. Then $|\tau| > \gamma_n^l/2$ and*

$$\mu([\tau]) \leq q_{n-1}^3 (2KC_0)^{q_{n-1}+l} \varphi^t(\tau).$$

Proof. Since $|\tau| \leq \gamma_n^l - K$ every $\omega \in \mathbb{T}_n^l$ satisfies $|\tau| \leq |\Psi_n^l(\omega)|$. Hence $\mu([\tau]) \leq \sum \mu([\Psi_n^l(\omega)])$, where the sum is taken over all $\omega \in \mathbb{T}_n^l$ with $\tau \preccurlyeq \Psi_n^l(\omega)$. By Lemma 5.7, for each $\omega \in \mathbb{T}_n^l$ with $\tau \preccurlyeq \Psi_n^l(\omega)$, we have

$$\begin{aligned} \mu([\Psi_n^l(\omega)]) &\leq q_{n-1}^2 \#\mathcal{D}(n)(2KC_0)^{q_{n-1}+l} \varphi^t(\Psi_n^l(\omega)) \\ &\leq q_{n-1}^2 \#\mathcal{D}(n)(2KC_0)^{q_{n-1}+l} \varphi^t(\tau). \end{aligned}$$

As such, we must estimate the number of $\omega \in \mathbb{T}_n^l$ with $\tau \preccurlyeq \Psi_n^l(\omega)$. Either $l > 1$, in which case $|\tau| > \gamma_n^{l-1} - K$, or $l = 1$, in which case $|\tau| > \gamma_{n-1}^{q_{n-1}-1} - K$. In either case $|\tau| > \gamma_n^l - (5K + k(n) + k(n-1) + 2) \geq \gamma_n^l/2$. Now each $\omega \in \mathbb{T}_n^l$ with $\tau \preccurlyeq \Psi_n^l(\omega)$ is of length no more than γ_n^l and each of the final $|\Psi_n^l(\omega)| - |\tau|$ digits is chosen from, $\{1\} \cup \Gamma \cup \mathbb{D}(\mathcal{D}(n-1)) \cup \mathbb{D}(\mathcal{D}(n))$. Thus, there are at most

$$\# \left(\{1\} \cup \Gamma \cup \mathbb{D}(\mathcal{D}(n-1)) \cup \mathbb{D}(\mathcal{D}(n))\right)^{5K+k(n)+k(n-1)+2}$$

words $\omega \in \mathbb{T}_n^l$ with $\tau \preccurlyeq \Psi_n^l(\omega)$.

By the choice of q_{n-1}, we have

$$q_{n-1} > \#\mathcal{D}(n) \# \left(\{1\} \cup \Gamma \cup \mathbb{D}(\mathcal{D}(n-1)) \cup \mathbb{D}(\mathcal{D}(n))\right)^{5K+k(n)+k(n-1)+2}$$

and the claim follows. □

We are now ready to show that μ satisfies the desired property.

Lemma 5.9. *In the setting of the proof of Proposition 5.3, there exists a constant $C \geq 1$ with $\mu([\tau]) \leq C\varphi^s(\tau)$ for all $\tau \in \Sigma_*$.*

Proof. Clearly we may assume $\tau \in S^*$ since otherwise $\mu([\tau]) = 0$. Since each $\mathbb{T}_n$ consists of finitely many elements all of length at least γ_n, the set $S^* \cap \Sigma_m$ is finite for all $m \in \mathbb{N}$. As such, it suffices to show that there is $N \in \mathbb{N}$ so that $\mu([\tau]) \leq \varphi^s(\tau)$ for all $\tau \in S^*$ with $|\tau| > N$. Choose N so that $(2KC_0)^{2/(N-1)} < (3/2)^{(t-s)}$ and $N^3(3/4)^{N(t-s)} < 1$ and let $M = \max\{N, \gamma_N\}$.

Given $\tau \in S^*$ with $|\tau| > N$ we let $n = n(\tau)$ to be minimal so that $|\tau| \le \gamma_n^l - K$ for some $l \in \{1, \dots, q_n\}$ and take $l = l(\tau)$ to be the least such l. Then, by Lemma 5.8, we have $|\tau| > \gamma_n^l/2$ and

$$\mu([\tau]) \le q_{n-1}^3 (2KC_0)^{q_{n-1}+l} \varphi^t(\tau)\,.$$

Hence $2|\tau| \ge \gamma_n^l \ge lk(n) + q_{n-1}k(n+1) \ge (l + q_{n-1})(n-1)$ and

$$\mu([\tau]) \le |\tau|^3 (2KC_0)^{|\tau|/(n-1)} \varphi^t(\tau)\,.$$

Since $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$ we have $\varphi^t(\tau) \le 2^{-|\tau|(t-s)}\varphi^s(\tau)$ and so

$$\mu([\tau]) \le |\tau|^3 \big((2KC_0)^{2/(n-1)} 2^{-(t-s)}\big)^{|\tau|} \varphi^s(\tau)\,.$$

Since $\gamma_n \ge \gamma_l^n \ge |\tau| > M \ge \gamma_N$ we have $n > N$, so

$$(2KC_0)^{2/(n-1)}(1/2)^{(t-s)} < (3/4)^{(t-s)}.$$

Moreover, since $|\tau| > N$ we have $|\tau|^3(3/4)^{|\tau|(t-s)} < 1$ and

$$\begin{aligned}\mu([\tau]) &\le |\tau|^3((2KC_0)^{2/(n-1)}(1/2)^{(t-s)})^{|\tau|}\varphi^s(\tau)\\ &\le |\tau|^3(3/4)^{|\tau|(t-s)}\varphi^s(\tau) \le \varphi^s(\tau)\end{aligned}$$

finishing the proof. □

The following lemma completes the proof of Proposition 5.3, and hence of Theorem 5.4, which gives the lower bound in Theorem C. The lower bound together with Proposition 5.2 completes the proof of Theorem C.

Lemma 5.10. *In the setting of the proof of Proposition 5.3, $S \subset E_\Phi(\alpha)$.*

Proof. Recall that we previously made the assumption that if ϕ_i is in the sequence Φ, then also $-\phi_i$ is in Φ. As such it suffices to fix ϕ_i and show that for each $\tau \in S$ we have

$$\liminf_{m\to\infty} \frac{1}{m}\sum_{j=0}^{m-1} \phi_i(\sigma^j(\tau)) \ge \alpha_i\,. \tag{5.10}$$

Given $m \ge \gamma_i$ we choose $n = n(m)$ to be maximal so that $\gamma_n \le m$ and choose $l = l(m) \le q_{n+1}$ to be maximal so that $\gamma_{n+1}^l \le m$. Since $\tau \in S = \Psi(T)$ there is $\omega \in \mathbb{T}_{n+1}^l$ with $\Psi_{n+1}^l(\omega) \preccurlyeq \tau$. It follows from the construction of $\mathbb{T}_{n+1}^l$ that $\omega = \kappa\omega^1\omega^2$, where $\kappa \in \mathbb{T}_{n-1}$, $\omega^1 \in \mathbb{T}_n^{q_n}$ and $\omega^2 \in \mathbb{T}_n^l$. We deal with these three segments separately.

Since l is maximal we have $m < \gamma_{n+1}^{l+1}$, or $m < \gamma_{n+2}^{1}$ if $l = q_{n+1}$. This implies that

$$m - |\Psi_{n+1}^{l}(\omega)| \leq 2K + k(n+1) + k(n+2) + 1.$$

It also follows that $\tau|_m$, the initial segment of τ of length m, consists entirely of digits from

$$\mathbb{D}^*(n) = \mathbb{D}\Big(\bigcup_{l=1}^{n+2} \mathcal{D}(l) \cup \{1\} \cup \Gamma\Big).$$

Hence, for all $j \in \{1, \ldots, m\}$ we have

$$-\phi_i(\sigma^j(\omega)) \leq \mathcal{A}(n), \tag{5.11}$$

where $\mathcal{A}(n)$ is as in (5.3). Since $m \geq \gamma_n \geq q_n > n\mathcal{A}(n)(\gamma_{n-1} + 2K + k(n+1) + k(n+2) + 1)$, we thus get

$$\begin{aligned} &\sum_{j=0}^{|\Psi_{n-1}(\kappa)|-1} \phi_i(\sigma^j(\tau)) + \sum_{j=|\Psi_{n+1}^{l}(\omega)|}^{m-1} \phi_i(\sigma^j(\tau)) \\ &\quad \geq -\gamma_{n-1}\mathcal{A}(n) - (2K + k(n+1) + k(n+2) + 1)\mathcal{A}(n) \\ &\quad \geq -\frac{2m}{n}. \end{aligned} \tag{5.12}$$

Observe that $q_n > L(n)$ and $\omega^1 \in \mathbb{T}_n^{q_n}$ imply

$$\frac{1}{q_n} \sum_{j=0}^{q_n-1} \underline{A}_{k(n)}\phi_i(\sigma^{jk(n)}(\tilde{\omega}^1)) \in B_n(\alpha_i)$$

for all $\tilde{\omega}^1 \in [\omega^1]$. Here we have used the fact that $\underline{A}_{k(n)}\phi_i$ is constant on cylinders of length $k(n)$. Write ω^1 in the form

$$\omega^1 = \omega_1^1 \cdots \omega_{q_n}^1, \quad \omega_\nu^1 \in \mathcal{D}(n) \subset \Sigma_{k(n)}.$$

It follows from the construction of Ψ_n, along with the fact that $\Psi_n(\kappa\omega^1) \preccurlyeq \tau$, that some set $\widetilde{\mathbb{A}} \subset [\gamma_{n-1}, \gamma_n] \cap \mathbb{N}$ of cardinality q_n has the property that for each $j \in \widetilde{\mathbb{A}}$ there is $\nu \in \{1, \ldots, q_n\}$ such that $\sigma^j\tau \in [\omega_\nu^1]$. Let

$$\mathbb{A} = \widetilde{\mathbb{A}} + \{0, 1, \ldots, k(n) - 1\}.$$

We may choose $\widetilde{\mathbb{A}}$ so that $\tau_\nu \in \{1\} \cup \mathbb{D}(\Gamma)$ for all integers $\nu \in [\gamma_{n-1}+1, \gamma_n] \setminus (\mathbb{A}+1)$.

Since $\sigma^j(\tau) \in [\omega_\nu^1]$, for each $j \in \widetilde{\mathbb{A}}$, we have

$$\frac{1}{q_n k(n)} \sum_{j \in \mathbb{A}} \phi_i(\sigma^j(\tau)) \geq \frac{1}{q_n} \sum_{j \in \mathbb{A}}^{q_n - 1} A_{k(n)} \phi_i(\sigma^j(\tau))$$

$$\geq \frac{1}{q_n} \sum_{\nu=0}^{q_n-1} \underline{A}_{k(n)} \phi_i(\sigma^{\nu k(n)}(\tilde{\omega}^1)) \in B_n(\alpha_i).$$

By the construction of Ψ_n, the cardinality of $[\gamma_{n-1}, \gamma_n - 1] \setminus \mathbb{A}$ is at most $4Kq_n$, $k(n) \geq 4Kn$, and $m \geq \gamma_n \geq k(n)q_n \geq 4Knq_n$. Thus

$$\#\mathbb{A} \geq \gamma_n - \gamma_{n-1} - 4Kq_n \geq \gamma_n - \frac{2m}{n}$$

and

$$\sum_{j \in \mathbb{A}} \phi_i(\sigma^j(\tau)) > \begin{cases} \gamma_n\Big(\alpha_i - \dfrac{1}{n}\Big), & \text{if } \alpha_i \leq \frac{1}{n}, \\ \Big(\gamma_n - \dfrac{2m}{n}\Big)\Big(\alpha_i - \dfrac{1}{n}\Big), & \text{if } \alpha_i > \frac{1}{n}, \\ \Big(\gamma_n - \dfrac{2m}{n}\Big)n, & \text{if } \alpha_i = \infty. \end{cases}$$

As noted, for each $j \in [\gamma_{n-1}, \gamma_n - 1] \setminus \mathbb{A}$ there is $\eta \in \{1\} \cup \mathbb{D}(\Gamma)$ so that $\sigma^j \tau \in [\eta]$. Thus, for all such j, we have $\phi_i(\sigma^j(\tau)) \geq -A_i$, where A_i is as in (5.1). Moreover, since $k(n) \geq 4KnA_i$ for $i \in \{1, \ldots, n\}$ and $m \geq \gamma_n \geq k(n)q_n$, we get

$$\sum_{j \in [\gamma_{n-1}, \gamma_n - 1] \setminus \mathbb{A}} \phi_i(\sigma^j(\tau)) \geq -4KA_i q_n \geq -\frac{m}{n}.$$

Putting these inequalities together we have

$$\sum_{j=\gamma_{n-1}}^{\gamma_n - 1} \phi_i(\sigma^j(\tau)) > \begin{cases} \gamma_n\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i \leq \frac{1}{n}, \\ \Big(\gamma_n - \dfrac{2m}{n}\Big)\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i > \frac{1}{n}, \\ \Big(\gamma_n - \dfrac{2m}{n}\Big)n - \dfrac{m}{n}, & \text{if } \alpha_i = \infty. \end{cases} \tag{5.13}$$

For the sum $\sum_{j=\gamma_n}^{|\Psi_{n+1}^l(\omega)|-1} \phi_i(\sigma^j(\tau))$, there are two cases. Either $l \geq L(n+1)$, in which case there exists $\tilde{\omega}^2 \in [\omega^2]$ with

$$\frac{1}{l} \sum_{j=0}^{l-1} \underline{A}_{k(n+1)} \phi_i(\sigma^{jk(n+1)} \tilde{\omega}^2) \in B_n(\alpha_i), \tag{5.14}$$

so we may proceed as in the previous case to deduce

$$\sum_{j=\gamma_n}^{|\Psi^l_{n+1}(\omega)|-1} \phi_i(\sigma^j(\tau)) > \begin{cases} \Big(|\Psi^l_{n+1}(\omega)| - \gamma_n\Big)\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i \leq \frac{1}{n}, \\ \Big(|\Psi^l_{n+1}(\omega)| - \gamma_n - \dfrac{2m}{n}\Big)\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i > \frac{1}{n}, \\ \Big(|\Psi^l_{n+1}(\omega)| - \gamma_n - \dfrac{2m}{n}\Big)n - \dfrac{m}{n}, & \text{if } \alpha_i = \infty. \end{cases}$$

Recall that $m - |\Psi^l_{n+1}(\omega)| \leq 2K + k(n+1) + k(n+2) + 1 \leq \gamma_n/n \leq m/n$, so we may combine the above inequalities to obtain

$$\sum_{j=\gamma_n}^{|\Psi^l_{n+1}(\omega)|-1} \phi_i(\sigma^j(\tau)) > \begin{cases} \Big(m - \gamma_n\Big)\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i \leq \frac{1}{n}, \\ \Big(m - \gamma_n - \dfrac{3m}{n}\Big)\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i > \frac{1}{n}, \\ \Big(m - \gamma_n - \dfrac{3m}{n}\Big)n - \dfrac{m}{n}, & \text{if } \alpha_i = \infty. \end{cases} \tag{5.15}$$

Combining (5.15) with (5.12) and (5.13) we conclude that whenever $l \geq L(n+1)$, we have

$$\frac{1}{m}\sum_{j=0}^{m-1} \phi_i(\sigma^j(\tau)) > \begin{cases} \Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{3}{n}, & \text{if } \alpha_i \leq \frac{1}{n}, \\ \Big(1 - \dfrac{3}{n}\Big)\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{3}{n}, & \text{if } \alpha_i > \frac{1}{n}, \\ \Big(1 - \frac{3}{n}\Big)n - \frac{3}{n}, & \text{if } \alpha_i = \infty. \end{cases} \tag{5.16}$$

If $l \leq L(n+1)$, then we apply (5.11) once more to obtain

$$\sum_{j=\gamma_n}^{|\Psi^l_{n+1}(\omega)|-1} \phi_i(\sigma^j(\tau)) \geq -k(n+1)L(n+1)\mathcal{A}(n) \geq -\frac{\gamma_n}{n} \geq -\frac{m}{n}. \tag{5.17}$$

Notice that if $l \leq L(n+1)$, then $|\Psi^l_{n+1}(\omega)| - \gamma_n \leq (4K + k(n+1))L(n+1) \leq m/n$. We also have $m - |\Psi^l_{n+1}(\omega)| \leq m/n$, so $m - \gamma_n \leq 2m/n$ which combined with (5.13) gives

$$\sum_{j=\gamma_{n-1}}^{\gamma_n - 1} \phi_i(\sigma^j(\tau)) > \begin{cases} m\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i \leq \frac{1}{n}, \\ \Big(m - \dfrac{4m}{n}\Big)\Big(\alpha_i - \dfrac{1}{n}\Big) - \dfrac{m}{n}, & \text{if } \alpha_i > \frac{1}{n}, \\ \Big(m - \dfrac{4m}{n}\Big)n - \dfrac{m}{n}, & \text{if } \alpha_i = \infty. \end{cases} \tag{5.18}$$

Combining (5.18) with (5.17) and (5.12) gives

$$\frac{1}{m}\sum_{j=0}^{m-1}\phi_i(\sigma^j(\tau)) > \begin{cases} \left(\alpha_i - \frac{1}{n}\right) - \frac{3}{n}, & \text{if } \alpha_i \le \frac{1}{n}, \\ \left(1-\frac{4}{n}\right)\left(\alpha_i - \frac{1}{n}\right) - \frac{3}{n}, & \text{if } \alpha_i > \frac{1}{n}, \\ \left(1-\frac{4}{n}\right)n - \frac{3}{n}, & \text{if } \alpha_i = \infty. \end{cases} \tag{5.19}$$

Since either (5.16) or (5.19) holds for all $m \ge \gamma_i$ and $n(m) \to \infty$ as $m \to \infty$ we have shown (5.10) and thus finished the proof. □

6. Conditional variational principle for bounded potentials

In this section we shall prove Theorem D. The progression from Theorem C to Theorem D relies upon the thermodynamic formalism developed in §3. The main challenge is to prove the upper bound which is given in §6.2–6.5.

We begin by proving some elementary lemmas in §6.1. The upper bound for the interior points of the spectrum for finitely many potentials is given in §6.2. In §6.3, we prove an upper-semicontinuity lemma which is a crucial technical tool in forthcoming sections. In §6.4, we prove a lemma which allows us to extend the upper bound to the boundary of the spectrum. The proof of the upper bound, for all points of the spectrum and for a countable infinity of potentials, is given in §6.5. The lower bound in Theorem D follows reasonably straightforwardly from Theorem C and it is proved in §6.6.

6.1. Space of integrals with respect to invariant measures. We restrict our attention to potentials $\Phi\colon \Sigma \to \mathbb{R}^N$ taking values in some finite dimensional vector space. We begin by recalling an elementary lemma concerning convex sets of $\mathbb{R}^N$. If $\kappa \in \{-1, 1\}^N$, then we define the open *κ-orthant* $O(\kappa)$ to be the set

$$O(\kappa) = \{(x_i)_{i=1}^N : \kappa_i \cdot x_i > 0 \text{ for each } i\}.$$

Lemma 6.1. *If C is a convex set, then $a \in \mathbb{R}^N$ lies in the interior of C if and only if $C \cap (a + O(\kappa)) \neq \emptyset$ for all $\kappa \in \{-1, 1\}^N$.*

Suppose $\Phi = (\phi_i)_{i=1}^N$ is bounded with summable variations. If $\nu \in \mathcal{M}(\Sigma)$, then we write

$$\int \Phi \,\mathrm{d}\nu = \left(\int \phi_1 \,\mathrm{d}\nu, \ldots, \int \phi_N \,\mathrm{d}\nu\right).$$

The space of integrals with respect to invariant measures is

$$\mathcal{P}(\Phi) = \left\{\int \Phi \,\mathrm{d}\nu : \nu \in \mathcal{M}_\sigma(\Sigma)\right\} \subset \mathbb{R}^N.$$

Lemma 6.2. *The set $\mathcal{P}(\Phi)$ is bounded and convex. Moreover, either $\mathcal{P}(\Phi)$ is contained within some $(N-1)$-dimensional hyperplane or $\mathcal{P}(\Phi) \subset \overline{\operatorname{int}(\mathcal{P}(\Phi))}$.*

Proof. The first statement follows immediately from the fact that the mapping

$$\nu \longmapsto \int \Phi \, \mathrm{d}\nu$$

defined on $\mathcal{M}_\sigma(\Sigma)$ is bounded and affine and $\mathcal{M}_\sigma(\Sigma)$ is convex. The second statement follows from elementary properties of convex sets in Euclidean spaces. □

If $I \subset \mathbb{N}$ is finite, then we define $\mathcal{P}(\Phi, I) \subset \mathcal{P}(\Phi)$ by

$$\mathcal{P}(\Phi, I) = \{\Phi(\nu) \colon \nu \in \mathcal{M}_\sigma(\Sigma) \text{ and } \nu(I^{\mathbb{N}}) = 1\}$$

and

$$\mathcal{P}_e(\Phi, I) = \{\Phi(\nu) \colon \nu \in \mathcal{M}_\sigma(\Sigma) \text{ is ergodic and } \nu(I^{\mathbb{N}}) = 1\}.$$

Lemma 6.3. *It holds that*

$$\mathcal{P}(\Phi) \subset \overline{\bigcup_I \mathcal{P}_e(\Phi, I)}$$

and

$$\operatorname{int}(\mathcal{P}(\Phi)) \subset \bigcup_I \operatorname{int}(\mathcal{P}(\Phi, I)),$$

where the unions are taken over all finite subsets $I \subset \mathbb{N}$.

Proof. Let $\nu \in \mathcal{M}_\sigma(\Sigma)$, $\alpha = \int \Phi \, \mathrm{d}\nu \in \mathcal{P}(\Phi)$ and $\varepsilon > 0$. Since each ϕ_i has summable variations we may choose $n \in \mathbb{N}$ with $\operatorname{var}_n(A_n\phi_i) < \varepsilon$. For each $\omega \in \mathbb{N}^n$ we let $\tilde{\omega} \in \Sigma$ denote the unique periodic point with $\sigma^{qn}(\tilde{\omega}) \in [\omega]$ for all $q \in \mathbb{N} \cup \{0\}$.

Note that since ν is σ-invariant,

$$\int A_n\phi_i \, \mathrm{d}\nu = \int \phi_i \, \mathrm{d}\nu = \alpha_i, \quad i \in \{1, \ldots, N\}.$$

Hence, as $\operatorname{var}_n(A_n\phi_i) < \varepsilon$ for each i, we have

$$\sum_{\omega \in \Sigma_n} \nu([\omega]) \inf_\tau \{A_n\phi_i(\tau)\} > \alpha_i - \varepsilon \tag{6.1a}$$

and

$$\sum_{\omega \in \Sigma_n} \nu([\omega]) \sup_\tau \{A_n\phi_i(\tau)\} < \alpha_i + \varepsilon. \tag{6.1b}$$

Given a finite set $I \subset \mathbb{N}$ we let

$$c(\nu, I) = \sum_{\omega \in I^n} \nu([\omega]).$$

Note that I may be chosen so that $c(\nu, I)$ is arbitrarily close to one. Hence, (6.1) implies that there exists a finite set $I \subset \mathbb{N}$ such that

$$c(\nu, I)^{-1} \sum_{\omega \in I^n} \nu([\omega]) \inf_{\tau} \{A_n \phi_i(\tau)\} > \alpha_i - \varepsilon \tag{6.2a}$$

and

$$c(\nu, I)^{-1} \sum_{\omega \in I^n} \nu([\omega]) \sup_{\tau} \{A_n \phi_i(\tau)\} < \alpha_i + \varepsilon. \tag{6.2b}$$

for all $i \in \{1, \ldots, N\}$. Let μ' be the unique nth level Bernoulli measure which satisfies

$$\mu'([\omega]) = c(\nu, I)^{-1} \nu([\omega]), \quad \omega \in I^n.$$

By (6.2), we have

$$\int A_n \phi_i \, \mathrm{d}\mu' \in (\alpha_i - \varepsilon, \alpha_i + \varepsilon).$$

Now let

$$\mu = \frac{1}{n} \sum_{j=0}^{n-1} \mu' \circ \sigma^{-j}.$$

Since μ' is σ^n-invariant and ergodic with respect to σ^n, the measure μ is σ-invariant and ergodic with respect to σ. It is also clear that μ is supported on $I^{\mathbb{N}}$. Moreover, since

$$\int \phi_i \, \mathrm{d}\mu = \frac{1}{n} \sum_{j=0}^{n-1} \int \phi_i \, \mathrm{d}\mu' \circ \sigma^{-j} = \int A_n \phi_i \, \mathrm{d}\mu' \in (\alpha_i - \varepsilon, \alpha_i + \varepsilon)$$

for all $i \in \{1, \ldots, N\}$, we have shown the first claim.

To prove the second claim, we apply Lemma 6.1. Indeed, if $\alpha \in \mathrm{int}(\mathcal{P}(\Phi))$, then

$$\mathcal{P}(\Phi) \cap (\alpha + O(\kappa)) \neq \emptyset, \quad \kappa \in \{-1, 1\}^N.$$

Since each set $\alpha + O(\kappa)$ is open it follows from the first claim that for each $\kappa \in \{-1, 1\}^N$ there is a finite set $I(\kappa) \subset \mathbb{N}$ with

$$\mathcal{P}(\Phi, I(\kappa)) \cap (\alpha + O(\kappa)) \neq \emptyset.$$

Letting

$$I = \bigcup_{\kappa \in \{-1,1\}^N} I(\kappa),$$

we obtain a finite set with $\mathcal{P}(\Phi, I) \cap (\alpha + O(\kappa)) \neq \emptyset$ for all $\kappa \in \{-1, 1\}^N$. Moreover, since $\mathcal{P}(\Phi, I)$ is convex it follows from Lemma 6.1 that $\alpha \in \mathrm{int}\,(\mathcal{P}(\Phi, I))$. This completes the proof. $\square$

6.2. Upper bound for interior points of the spectrum. In this section we give the proof of the upper bound in Theorem D for interior points of the spectrum in the special case where there are only finitely many potentials. The proof uses a variational type argument which is a standard tool in multifractal analysis.

Proposition 6.4. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < 1$*, the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$*,* $\Phi\colon \Sigma \to \mathbb{R}^N$ *is bounded with summable variations, and* $\alpha \in \mathrm{int}(\mathcal{P}(\Phi))$*. Then*

$$\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) \le \min\left\{d, \max\left\{s_\infty, \sup\left\{D(\mu)\colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi\, \mathrm{d}\mu = \alpha\right\}\right\}\right\}$$

for all $\mathbf{a} \in \mathbf{A}$.

The proof of Proposition 6.4 requires two lemmas. Lemma 6.5 uses Lemma 5.1 to relate the dimension of $J_\Phi(\alpha)$ to the pressure. Then Lemma 6.7 proves the upper bound by showing the existence of an appropriate maximising measure.

Lemma 6.5. *If* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < 1$*,* $\mathbf{a} \in \mathbf{A}$*, and* $\Phi\colon \Sigma \to \mathbb{R}^N$ *is bounded with summable variations,* $\alpha \in \mathbb{R}^N$*, and* $s < \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha))$*, then*

$$P(\varphi^s \cdot e_{\langle q, \Phi-\alpha\rangle}) \ge 0$$

for all $q \in \mathbb{R}^N$.

Proof. Fix $s < \dim_{\mathrm{H}}(J_\Phi(\alpha))$ and $q \in \mathbb{R}^N$. By Lemma 5.1, for each $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that

$$\sum_{\omega\in A_\Phi(\alpha,n,k)} \varphi^s(\omega) > 1$$

for all $k \ge k(n)$. Thus

$$\sum_{\omega\in\Sigma_k} \varphi^s(\omega) \sup_{\tau\in[\omega]} \exp(S_n\langle q, \Phi-\alpha\rangle(\tau))$$

$$= \sum_{\omega\in\Sigma_k} \varphi^s(\omega) \sup_{\tau\in[\omega]} \exp\Big(\sum_{i=1}^N q_i(S_n\phi_i - n\alpha_i)\Big)$$

$$\ge \sum_{\omega\in A_\Phi(\alpha,n,k)} \varphi^s(\omega) \sup_{\tau\in[\omega]} \exp\Big(\sum_{i=1}^N q_i(S_n\phi_i - n\alpha_i)\Big)$$

$$\geq \sum_{\omega \in A_\Phi(\alpha,n,k)} \varphi^s(\omega) \cdot e^{-N\bar{q}k/n}$$

$$> e^{-N\bar{q}k/n},$$

where $\bar{q} = \max_{i \in \{1,\dots,N\}} q_i$. Hence

$$P(\varphi^s \cdot e_{\langle q, \Phi-\alpha \rangle}) = \lim_{k\to\infty} \frac{1}{k} \log \Big(\sum_{\omega\in\Sigma_k} \varphi^s(\omega) \sup_{\tau\in[\omega]} \exp(S_n \langle q, \Phi - \alpha\rangle (\tau)) \Big)$$

$$\geq -\frac{N\bar{q}}{n}.$$

Letting $n \to \infty$ completes the proof of the lemma. □

Lemma 6.6. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < 1$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \leq s \leq d$, $\Phi\colon \Sigma \to \mathbb{R}^N$ *is bounded with summable variations,* $\alpha \in \mathbb{R}^N$, $q \in \mathbb{R}^N$, *and* $s > s_\infty$. *Then the potential* $\varphi^s \cdot e_{\langle q,\Phi-\alpha\rangle}$ *is quasi-multiplicative and* $P(\varphi^s \cdot e_{\langle q,\Phi-\alpha\rangle}) < \infty$. *Moreover, if* μ *is the Gibbs measure for* $\varphi^s \cdot e_{\langle q,\Phi-\alpha\rangle}$, *then* $\Lambda_\mu(\varphi^s) > -\infty$.

Proof. Observe that the quasi-multiplicativity follows immediately from Lemma 3.10. Since Φ is bounded we have $B = \sup\{|\langle q, \Phi(\omega) - \alpha\rangle| : \omega \in \Sigma\} < \infty$. This together with $P(\varphi^s) < \infty$ gives $P(\varphi^s \cdot e_{\langle q,\Phi-\alpha\rangle}) = P < \infty$. Thus, by Theorems 3.5 and 3.7, the Gibbs measure μ for $\varphi^s \cdot e_{\langle q,\Phi-\alpha\rangle}$ exists.

To prove the last claim, let $m \in \mathbb{Z}$ be so that $m < s \leq m+1$. By the Gibbs property of μ there is a constant $C \geq 1$ so that

$$\mu([\omega]) \leq C\varphi^s(\omega) e_{\langle q,\Phi-\alpha\rangle}(\omega) e^{-nP} \leq C\varphi^s(\omega) e^{n(B-P)}$$

for all $\omega \in \Sigma_n$ and $n \in \mathbb{N}$. Now, following the proof of Lemma 4.2, we get $\Lambda_\mu(\varphi^s) > -\infty$. □

Lemma 6.7. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < 1$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \leq s \leq d$, $\Phi\colon \Sigma \to \mathbb{R}^N$ *is bounded with summable variations,* $\alpha \in \mathrm{int}(\mathcal{P}(\Phi)) \subset \mathbb{R}^N$, *and* $s > s_\infty$ *satisfies*

$$\inf_{q\in\mathbb{R}^N} P(\varphi^s \cdot e_{\langle q,\Phi-\alpha\rangle}) \geq 0.$$

Then there exists an ergodic invariant measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ *with* $\int \Phi \,\mathrm{d}\mu = \alpha$ *and* $D(\mu) \geq s$.

Proof. We shall consider the function

$$F: \mathbb{R}^N \longrightarrow \mathbb{R}$$

defined by

$$F(q) = P(\varphi^s \cdot e_{\langle q, \Phi - \alpha \rangle}).$$

Since $\alpha \in \operatorname{int}(\mathcal{P}(\Phi))$ we may apply the latter claim of Lemma 6.3 to obtain a finite subset $I \subset \mathbb{N}$ with $\alpha \in \operatorname{int}(\mathcal{P}(\Phi, I))$. Since I is finite and all of the matrices T_i are non-singular we have

$$\varphi^s(\omega) \geq c^n, \quad \omega \in \Sigma_n,$$

where

$$c = \min\{\gamma_d(T_i) : i \in I\} > 0.$$

Furthermore, $\Lambda_\mu(\varphi^s) \geq \log c > -\infty$ for all $\mu \in \mathcal{M}_\sigma(\Sigma)$ with $\mu(I^{\mathbb{N}}) = 1$. Since $\alpha \in \operatorname{int}(\mathcal{P}(\Phi, I))$ there exists $\varepsilon > 0$ with $\overline{B_\varepsilon(\alpha)} \subset \mathcal{P}(\Phi, I)$. Hence for each $q \in \mathbb{R}^N \setminus \{0\}$ we have $\alpha + \varepsilon q/\|q\| \in \mathcal{P}(\Phi, I)$ and there exists a measure $\nu_q \in \mathcal{M}_\sigma(\Sigma)$ with $\Lambda_{\nu_q}(\varphi^s) \geq \log c$ satisfying

$$\int \Phi \, d\nu_q = \alpha + \varepsilon q/\|q\|.$$

By Lemma 2.2 and the boundedness of Φ, we have

$$\begin{aligned} F(q) &\geq h_{\nu_q} + \left\langle q, \int \Phi \, d\nu_q - \alpha \right\rangle + \Lambda_{\nu_q}(\varphi^s) \\ &\geq \langle q, \varepsilon q/\|q\| \rangle + \log c \\ &= \varepsilon \|q\| + \log c. \end{aligned}$$

Hence $F(q) \to \infty$, as $\|q\| \to \infty$. It follows from Lemma 3.11 that F attains a global minimum on a bounded set. Let $q(\alpha)$ denote a point at which this global minimum is attained.

By Lemma 6.6, the Gibbs measure μ_q for $\varphi^s \cdot e_{\langle q, \Phi - \alpha \rangle}$ satisfies $\Lambda_{\mu_q}(\varphi^s) > -\infty$. Thus, applying Lemmas 3.8 and 2.2, we obtain

$$F(q) = h_{\mu_q} + \left\langle q, \int \Phi \, d\mu_q - \alpha \right\rangle + \Lambda_{\mu_q}(\varphi^s).$$

Moreover, by Lemma 3.12 for each $i \in \{1, \ldots, N\}$ we have

$$\begin{aligned} \frac{\partial F(q)}{\partial q_i}\Big|_{q=q_*} &= \lim_{q_i \to q_i^*} \frac{P(\varphi^s \cdot e_{\langle q_*, \Phi - \alpha \rangle} \cdot e_{(q_i - q_i^*)(\phi_i - \alpha_i)}) - P(\varphi^s \cdot e_{\langle q_*, \Phi - \alpha \rangle})}{q_i - q_i^*} \\ &= \frac{\partial P((\varphi^s \cdot e_{\langle q_*, \Phi - \alpha \rangle}) \cdot e_{q_i(\phi_i - \alpha_i)})}{\partial q_i}\Big|_{q_i = 0} \\ &= \int \phi_i \, d\mu_{q_*} - \alpha_i. \end{aligned}$$

Since F attains a minimum at $q(\alpha)$ it follows that

$$\int \phi_i \,\mathrm{d}\mu_{q(\alpha)} = \alpha_i, \quad i \in \{1, \ldots, N\}.$$

Thus, denoting

$$\mu = \mu_{q(\alpha)},$$

we have

$$\int \Phi \,\mathrm{d}\mu = \alpha,$$

$$\begin{aligned} h_\mu + \Lambda_\mu(\varphi^s) &= h_\mu + \left\langle q(\alpha), \int \Phi \,\mathrm{d}\mu - \alpha \right\rangle + \Lambda_\mu(\varphi^s) \\ &= P(\varphi^s \cdot e_{\langle q, \Phi - \alpha \rangle}) \\ &\geq 0, \end{aligned} \tag{6.3}$$

and

$$D(\mu) \geq s. \qquad \square$$

Now we are ready to show the upper bound in Theorem D for interior points of the spectrum in the case where there are only finitely many potentials.

Proof of Proposition 6.4. Take $\alpha \in \operatorname{int}(\mathcal{P}(\Phi))$. Either $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) \leq s_\infty$, in which case the upper bound holds, or $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) > s_\infty$. If $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) > s_\infty$, then we may choose $s_\infty < s < \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha))$. By Lemmas 6.5 and 6.7 there exists $\mu \in \mathcal{M}_\sigma(\Sigma)$ with $D(\mu) \geq s$. This completes the proof of the proposition. $\square$

6.3. Quasi upper-semicontinuity lemma. Recall that because of the non-compactness of the shift space, the space of invariant probability measures is non-compact and the entropy is not upper semi-continuous. Nonetheless we do have the following proposition. It generalises the statement in [10], Lemma 6.5.

Proposition 6.8. *Suppose that $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i\in\mathbb{N}} \|T_i\| < 1$, and $(\mu_n)_{n\in\mathbb{N}}$ is a sequence with*

$$\mu_n \in \mathcal{M}_\sigma(\Sigma), \quad n \in \mathbb{N}$$

and

$$\limsup_{n\to\infty} D(\mu_n) > s_\infty.$$

Then there exists a sub-sequence $(\mu_{n_j})_{j\in\mathbb{N}}$ and a measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ which is a weak limit point of $(\mu_{n_j})_{j\in\mathbb{N}}$ and satisfies*

$$D(\mu) \geq \limsup_{n\to\infty} D(\mu_n).$$

Before proving the proposition, we first prove a few elementary lemmas. Let $\mathbb{P}$ be the set of all infinite probability vectors, that is,

$$\mathbb{P} = \Big\{(q_i)_{i\in\mathbb{N}} \in [0,1]^{\mathbb{N}} \colon \sum_{i=1}^{\infty} q_i = 1\Big\},$$

and equip it with the usual product topology. If $a = (a_i)_{i\in\mathbb{N}}$ is a sequence of numbers in $(0,1)$ and $C > 0$, then we set

$$\mathbb{P}(a, C) = \Big\{(q_i)_{i\in\mathbb{N}} \in \mathbb{P} \colon \sum_{i\in\mathbb{N}} q_i \log a_i \geq -C\Big\}.$$

Lemma 6.9. *If $a = (a_i)_{i\in\mathbb{N}}$ is a sequence of numbers in $(0,1)$ and $C > 0$, then $\mathbb{P}(a, C)$ is closed.*

Proof. Let $p = (p_i)_{i\in\mathbb{N}}$ be an accumulation point of $\mathbb{P}(a, C)$ and let $(p(n))_{n\in\mathbb{N}}$ be a sequence so that

$$p(n) = (p_i(n))_{i\in\mathbb{N}} \in \mathbb{P}(a, C), \quad n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} p(n) = p.$$

Suppose for a *reductio ad absurdum* that

$$\sum_{i=1}^{\infty} p_i \log a_i < -C.$$

Then there exists $k \in \mathbb{N}$ so that

$$\sum_{i=1}^{k} p_i \log a_i < -C.$$

Choosing now $n_0 \in \mathbb{N}$ so that

$$\sum_{i=1}^{k} p_i(n) \log a_i < -C, \quad n \geq n_0,$$

we have arrived at a contradiction since

$$\sum_{i=1}^{\infty} p_i(n) \log a_i \leq \sum_{i=1}^{k} p_i(n) \log a_i < -C, \quad n \geq n_0. \qquad \square$$

Lemma 6.10. *If $a = (a_i)_{i\in\mathbb{N}}$ is a non-increasing sequence of numbers in $(0,1)$ with $\sum_{i=1}^{\infty} a_i < \infty$ and $C > 0$, the function*

$$F\colon \mathbb{P}(a, C) \longrightarrow \overline{\mathbb{R}},$$

defined by

$$F(q) = \sum_{i=1}^{\infty} q_i \log(a_i/q_i),$$

is upper semi-continuous.

Proof. Let

$$\mathbb{P}_k = \{(q_i)_{i\in\mathbb{N}} \in \mathbb{P} : q_i = 0 \text{ for all } i \in \{k+1, k+2, \ldots\}\}, \quad k \in \mathbb{N},$$

and define a map

$$\xi_k \colon \mathbb{P} \longrightarrow \mathbb{P}_k$$

by setting

$$\xi_k(q)_i = \begin{cases} q_i, & \text{if } i \in \{1, \ldots, k-1\}, \\ \sum_{j=k}^{\infty} q_j, & \text{if } i = k, \\ 0, & \text{if } i \in \{k+1, k+2, \ldots\}, \end{cases}$$

for all $q = (q_i)_{i\in\mathbb{N}} \in \mathbb{P}$. Take a sequence of vectors $(p(n))_{n\in\mathbb{N}}$ with each $p(n) = (p_i(n))_{i\in\mathbb{N}} \in \mathbb{P}(a, C)$ along with $p = (p_i)_{i\in\mathbb{N}} \in \mathbb{P}$ such that for each $i \in \mathbb{N}$ we have $\lim_{n\to\infty} p_i(n) = p_i$. In particular, we have $\sum_{i=1}^{\infty} p_i(n) \log a_i \geq -C$. Our goal is to show that $\limsup_{n\to\infty} F(p(n)) \leq F(p)$.

Since $\lim_{n\to\infty} p_i(n) = p_i$ for all $i \in \{1, \ldots, k-1\}$ and $\sum_{i=k}^{\infty} q_i = 1 - \sum_{i=1}^{k-1} q_i$ for all $q \in \mathbb{P}$ we have $\lim_{n\to\infty} \xi_k(p(n)) = \xi_k(p)$ with respect to the supremum metric. Hence $\lim_{n\to\infty} F(\xi_k(p(n))) = F(\xi_k(p))$. Similarly, as in (2.4), we see that

$$\sum_{i=k}^{\infty} q_i \Big(\log \frac{a_i}{q_i} - \log \sum_{j=k}^{\infty} a_j \Big) \leq -\sum_{i=k}^{\infty} q_i \log \sum_{j=k}^{\infty} q_j,$$

and, consequently,

$$\begin{aligned} F(q) &\leq \sum_{i=1}^{k-1} q_i \log \frac{a_i}{q_i} + \sum_{i=k}^{\infty} q_i \log \frac{\sum_{j=k}^{\infty} a_j}{\sum_{j=k}^{\infty} q_j} \\ &= F(\xi_k(q)) - \sum_{i=k}^{\infty} q_i \log a_k + \sum_{i=k}^{\infty} q_i \log \sum_{j=k}^{\infty} a_j \end{aligned}$$

for all $q \in \mathbb{P}(a, C)$ and $k \in \mathbb{N}$. Choosing $k_0 \in \mathbb{N}$ so that

$$\sum_{j=k}^{\infty} a_j < 1, \quad k \geq k_0,$$

this implies

$$\begin{aligned} F(\xi_k(p)) &= \lim_{n\to\infty} F(\xi_k(p(n))) \\ &\geq \limsup_{n\to\infty} \Big(F(p(n)) + \sum_{i=k}^{\infty} p_i(n) \log a_k \Big) \\ &\geq \limsup_{n\to\infty} F(p(n)) + \liminf_{n\to\infty} \sum_{i=k}^{\infty} p_i(n) \log a_k \\ &\geq \limsup_{n\to\infty} F(p(n)) + \sum_{i=k}^{\infty} p_i \log a_k \end{aligned}$$

for all $k \geq k_0$ by Fatou's lemma. Note that since the sequence $(a_i)_{i\in\mathbb{N}}$ is non-increasing we have

$$\sum_{i=k}^{\infty} p_i \log a_k \geq \sum_{i=k}^{\infty} p_i \log a_i, \quad k \in \mathbb{N}.$$

Moreover, let $\varepsilon > 0$ and, by recalling Lemma 6.9, choose $k_\varepsilon \in \mathbb{N}$ so that

$$\sum_{i=k}^{\infty} p_i \log a_i > -\varepsilon \quad \text{and} \quad \sum_{i=k}^{\infty} p_i \log \sum_{j=k}^{\infty} p_j > -\varepsilon, \quad k \geq k_\varepsilon.$$

Since

$$\begin{aligned} F(p) - F(\xi_k(p)) &= \sum_{i=k}^{\infty} p_i \log \frac{a_i}{p_i} - \sum_{i=k}^{\infty} p_i \log \frac{a_k}{\sum_{j=k}^{\infty} p_j} \\ &\geq \sum_{i=k}^{\infty} p_i \log a_i + \sum_{i=k}^{\infty} p_i \log \sum_{j=k}^{\infty} p_j \\ &> -2\varepsilon \end{aligned}$$

for all $k \geq k_\varepsilon$ we have

$$\begin{aligned}\limsup_{n\to\infty} F(p(n)) &\leq \limsup_{k\to\infty}\Big(F(\xi_k(p)) - \sum_{i=k}^{\infty} p_i \log a_k\Big)\\ &\leq \limsup_{k\to\infty} F(\xi_k(p)) + \varepsilon\\ &\leq F(p) + 3\varepsilon.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ finishes the proof. □

Now we are ready to prove the announced quasi upper-semicontinuity property.

Proof of Proposition 6.8. We begin by showing that $(\mu_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Let

$$\delta = \limsup_{n\to\infty} D(\mu_n)$$

and choose $m < \delta \leq m+1$. If $\max\{s_\infty, m\} < t_0 < t_1 < \delta$, then there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ with $D(\mu_{n_j}) > t_1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} D(\mu_{n_j}) = \delta$. It follows that

$$\Lambda_{\mu_{n_j}}(\varphi^{t_0}) \geq \Lambda_{\mu_{n_j}}(\varphi^{t_1}) > -\infty$$

and that

$$0 \leq P_{\mu_{n_j}}(\varphi^{t_1}) \leq \frac{1}{k}\sum_{\omega\in\Sigma_k} \mu_{n_j}([\omega]) \log \frac{\varphi^{t_1}(\omega)}{\mu_{n_j}([\omega])}, \quad k, j \in \mathbb{N}.$$

Furthermore, recalling (2.4) and Lemma 3.1, we have

$$\sum_{\omega\in\Sigma_k} \mu_{n_j}([\omega]) \log \frac{\varphi^{t_0}(\omega)}{\mu_{n_j}([\omega])} \leq \log Z_k(\varphi^{t_0}) \leq k \log Z_1(\varphi^{t_0}) < \infty, \quad k, j \in \mathbb{N}.$$

Since $\varphi^{t_1}(\omega) = \varphi^{t_0}(\omega)\gamma_{m+1}(\omega)^{t_1-t_0}$ we get

$$\sum_{\omega\in\Sigma_k} \mu_{n_j}([\omega]) \log \gamma_{m+1}(\omega) \geq -\frac{k \log Z_1(\varphi^{t_0})}{t_1 - t_0}, \quad k, j \in \mathbb{N}. \tag{6.4}$$

Note that for every $\varepsilon > 0$ there is $M \in \mathbb{N}$ so that

$$\sum_{i=M}^{\infty} \gamma_{m+1}(T_i)^{m+1} \leq \sum_{i=M}^{\infty} \varphi^{t_0}(T_i) < \varepsilon.$$

Thus for each $\varepsilon > 0$ there are only finitely many i's so that

$$\log \gamma_{m+1}(T_i) \geq (m+1)^{-1} \log \varepsilon.$$

Therefore, (6.4) implies that the sequence $(\mu_{n_j})_{j\in\mathbb{N}}$ is tight and thus has a converging subsequence. We keep denoting the subsequence by $(\mu_{n_j})_{j\in\mathbb{N}}$ and let $\mu \in \mathcal{M}_\sigma(\Sigma)$ be its weak* limit.

Let $\max\{s_\infty, m\} < s < \delta$. Since $\varphi^s(\omega) \geq \gamma_{m+1}(\omega)^{m+1}$ for all $\omega \in \Sigma_k$ it follows from (6.4) that

$$\sum_{\omega\in\Sigma_k} \mu_{n_j}([\omega]) \log \varphi^s(\omega) \geq -\frac{k(m+1)\log Z_1(\varphi^{t_0})}{t_1 - t_0}$$

for all $k, j \in \mathbb{N}$. According to Lemma 6.9, the same estimate holds when the measure μ_{n_j} is replaced by μ. Thus $\Lambda_\mu(\varphi^s) \geq -(m+1)(t_1-t_0)^{-1} \log Z_1(\varphi^{t_0}) > -\infty$. Furthermore, since $s < \delta$ there is $j_0 \in \mathbb{N}$ so that $D(\mu_{n_j}) > s$ for all $j \geq j_0$. Therefore, Lemma 6.10 implies

$$\sum_{\omega\in\Sigma_k} \mu([\omega]) \log \frac{\varphi^s(\omega)}{\mu([\omega])} \geq \limsup_{j\to\infty} \sum_{\omega\in\Sigma_k} \mu_{n_j}([\omega]) \log \frac{\varphi^s(\omega)}{\mu_{n_j}([\omega])} \geq 0$$

and $D(\mu) \geq s$. The proof is finished since $\max\{s_\infty, m\} < s < \limsup_{n\to\infty} D(\mu_n)$ was arbitrary. □

6.4. Finitely many potentials lemma. In this section we prove a technical lemma which, together with Proposition 6.8, allows us to prove the upper bound in Theorem D for boundary points of the spectrum.

Lemma 6.11. *Suppose that $(T_i)_{i\in\mathbb{N}} \in GL_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, the singular value function φ^s is quasi-multiplicative for all $0 \leq s \leq d$, $\Phi\colon \Sigma \to \mathbb{R}^N$ is bounded with summable variations, $\mathcal{P}(\Phi)$ is not contained within any $(N-1)$-dimensional hyperplane, and $\alpha \in \overline{\mathcal{P}(\Phi)}$. Then for each $\varepsilon > 0$ there is $\gamma \in \mathrm{int}(\mathcal{P}(\Phi))$ with $|\alpha - \gamma| < \varepsilon$ and*

$$\dim_H(J_\Phi^{\mathbf{a}}(\gamma)) \geq \dim_H(J_\Phi^{\mathbf{a}}(\alpha)) - \varepsilon$$

for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$.

Proof. Fix $\varepsilon > 0$ and let $\dim_H(J_\Phi^{\mathbf{a}}(\alpha)) - \varepsilon < s < t < \dim_H(J_\Phi^{\mathbf{a}}(\alpha))$. By Lemma 6.2 and the first part of Lemma 6.3, we may choose $\beta \in \mathrm{int}(\mathcal{P}(\Phi)) \cap \mathcal{P}_e(\Phi, I)$, with respect to some finite subset $I \subset \mathbb{N}$, satisfying $|\beta - \alpha| < 1/n$. Since $\beta \in \mathcal{P}_e(\Phi, I)$ there is an ergodic invariant measure $\nu \in \mathcal{M}_\sigma(\Sigma)$ with $\nu(I^{\mathbb{N}}) = 1$ and $\int \phi_i \, d\nu = \beta_i$ for all $i \in \{1, \ldots, N\}$. Since $\nu(I^{\mathbb{N}}) = 1$ we also have $\Lambda_\nu(\varphi^s) > -\infty$. By the sub-additive ergodic theorem there exist both $\tau \in \Sigma$, and a constant $\theta(\tau) > 0$, as well as $L(\tau) \in \mathbb{N}$ such that

$$|A_l\phi_i(\tau) - \beta_i| < l^{-1} \quad \text{and} \quad \varphi^s(\tau|_l) \geq \theta(\tau)^l, \quad l \geq L(\tau),\ i \in \{1, \dots, N\}.$$

Choose $0 < \rho < \min\{1, \varepsilon/|\beta - \alpha|\}$ so that

$$2^{(1-\rho)(t-s)}\theta(\tau)^\rho > 1$$

and let

$$\gamma = \rho\beta + (1-\rho)\alpha.$$

Since $\beta \in \operatorname{int}(\mathcal{P}(\Phi))$ and $\alpha \in \overline{\mathcal{P}(\Phi)}$ we have $\gamma \in \operatorname{int}(\mathcal{P}(\Phi))$ by the elementary properties of convex sets in $\mathbb{R}^N$. Moreover, since $\rho < \varepsilon/|\beta-\alpha|$ we have $|\alpha-\gamma| < \varepsilon$. We shall now show that $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\gamma)) \geq \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) - \varepsilon$.

Since $t < \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\gamma))$, it follows from Lemma 5.1 that for all $l \in \mathbb{N}$ there exists $q(l) \in \mathbb{N}$ such that

$$\sum_{\kappa \in A_\Phi(\alpha,l,q)} \varphi^t(\kappa) > 1.$$

for all $q \geq q(l)$. Since $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$ and $s < t$ it follows that

$$\sum_{\kappa \in A_\Phi(\alpha,l,q)} \varphi^s(\kappa) > 2^{l(t-s)}.$$

for all $q \geq q(l)$. For every $\alpha, \beta \in \Sigma_*$, according to the quasi-multiplicativity of φ^s, there exists $\omega \in \Gamma$ such that

$$\varphi^s(\alpha\omega\beta) \geq c\varphi^s(\alpha)\varphi^s(\beta),$$

where $c > 0$ is a constant depending only on s. As in §5.2, we write $\alpha \star \beta$ for $\alpha\omega\beta$. Note that for any given $\alpha, \beta \in \Sigma_*$ there are at most $K = \max\{|\omega| \colon \omega \in \Gamma\}$ finite words $\beta' \in \Sigma_*$ with $\alpha \star \beta' = \alpha \star \beta$

Our choice of ρ implies that for each $l \in \mathbb{N}$ there exists

$$A(l) > \max\{(1-\rho)^{-1}q(l), \rho^{-1}L(\tau), l\}$$

such that

$$c\theta(\tau)\big(2^{(1-\rho)(t-s)}\theta(\tau)^\rho\big)^k > 1$$

for all $k \geq A(l)$. It follows that

$$\begin{aligned}
\sum_{\kappa \in A_\Phi(\alpha,l,\lceil k(1-\rho)\rceil)} \varphi^s(\kappa \star (\tau|_{\lceil k\rho\rceil})) &\geq c \sum_{\kappa \in A_\Phi(\alpha,l,\lceil k(1-\rho)\rceil)} \varphi^s(\kappa)\varphi^s(\tau|_{\lceil k\rho\rceil}) \\
&> c2^{\lceil k(1-\rho)\rceil(t-s)}\theta(\tau)^{\lceil \rho k\rceil} \\
&> c\theta(\tau)(2^{(1-\rho)(t-s)}\theta(\tau)^\rho)^k \\
&> 1
\end{aligned} \tag{6.5}$$

for all for all $k \geq A(l)$ and $l \in \mathbb{N}$. We shall temporarily fix $l \geq d$ and $k \geq A(l)$, and for each $\kappa \in A_\Phi(\alpha, l, \lceil k(1-\rho)\rceil)$ we let $r(\kappa)$ be $|\kappa \star (\tau|_{\lceil k\rho\rceil})| - \lceil k\rho\rceil$. Since $[\hat{\kappa}] \subset [\kappa]$ for $\hat{\kappa} = (\kappa \star (\tau|_{\lceil k\rho\rceil}))|_k$ we have

$$\lceil k(1-\rho)\rceil(\alpha_i - l^{-1}) < S_{\lceil k(1-\rho)\rceil}\phi_i(\omega) < \lceil k(1-\rho)\rceil(\alpha_i + l^{-1})$$

for each $i \in \{1, \dots, N\}$ by the definition of $A_\Phi(\alpha, l, \lceil k(1-\rho)\rceil)$. We also have

$$\lceil k\rho\rceil(\beta_i - l^{-1}) - (K+1)\|\phi_i\| < S_{k-r}\phi_i(\sigma^r(\omega)) < \lceil k\rho\rceil(\beta_i + l^{-1}) + (K+1)\|\phi_i\|.$$

Since $r - \lceil k(1-\rho)\rceil \leq K$ it follows that

$$\begin{aligned}
&\lceil k(1-\rho)\rceil(\alpha_i - l^{-1}) + \lceil k\rho\rceil(\beta_i - l^{-1}) - (2K+1)\|\phi_i\| \\
&\quad < S_k\phi_i(\omega) \\
&\quad < \lceil k\rho\rceil(\beta_i + l^{-1}) + (2K+1)\|\phi_i\| + \lceil k(1-\rho)\rceil(\alpha_i + l^{-1}).
\end{aligned}$$

Furthermore, since $|\alpha_i|, |\beta| < \|\phi_i\|$, $\gamma_i = (1-\rho)\alpha_i + \rho\beta_i$, and $k \geq l$ we have

$$\gamma_i - l^{-1}\left((2K+3)\|\phi_i\| + 1\right) < A_k\phi_i(\omega) < \gamma_i + l^{-1}\left((2K+3)\|\phi_i\| + 1\right).$$

Hence, if $Q = ((2K+3)\max_{i \in \{1,\dots,N\}} \|\phi_i\| + 1)^{-1}$, then we have $\hat{\kappa} \in A_\Phi(\gamma, n, k)$ for all $n \leq \lfloor Ql \rfloor$. Since $\hat{\kappa}$ is an initial substring of $\kappa \star (\tau|_{\lceil k\rho\rceil})$ for any given $\kappa \in A_\Phi(\alpha, l, \lceil k(1-\rho)\rceil)$ it follows from (6.5) that

$$\sum_{\hat{\kappa} \in A_\Phi(\gamma,n,k)} \varphi^s(\hat{\kappa}) \geq \sum_{\kappa \in A_\Phi(\alpha,l,\lfloor k(1-\rho)\rfloor)} \varphi^s(\kappa \star (\tau|_{\lceil k\rho\rceil})) > 1$$

for all $n \leq \lfloor Ql \rfloor$ and $k \geq A(l)$. For each n we choose $l(k) \in \mathbb{N}$ so that $n \leq \lfloor Ql(n) \rfloor$ and let $B(n) = A(l(n))$. It follows that

$$\sum_{\hat{\kappa} \in A_\Phi(\gamma,n,k)} \varphi^s(\hat{\kappa}) > 1$$

for all $n \in \mathbb{N}$ and for all $k \geq B(n)$. As in the proof of Proposition 5.2, we get

$$\begin{aligned}
&\dim_H(J_\Phi^{\mathbf{a}}(\alpha)) - \varepsilon \\
&\quad < s \\
&\quad \leq \lim_{n\to\infty} \lim_{k\to\infty} \sup \Big\{ D_k(\mu) \colon \mu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \text{ so that} \\
&\qquad\qquad \int A_k\phi_i \,\mathrm{d}\mu \in B_n(\gamma_i) \text{ for all } i \in \{1, \dots, n\} \Big\}
\end{aligned}$$

for all $\mathbf{a} \in \mathbf{A}$. Theorem C finishes the proof. □

6.5. Proof of the upper bound in Theorem D. For potentials $\Phi = (\phi_i)_{i\in\mathbb{N}}$ taking values in $\mathbb{R}^{\mathbb{N}}$ we similarly write

$$\int \Phi \,\mathrm{d}\nu = \left(\int \phi_1 \,\mathrm{d}\nu, \int \phi_2 \,\mathrm{d}\nu, \dots \right), \quad \nu \in \mathcal{M}_\sigma(\Sigma),$$

and

$$\mathcal{P}(\Phi) = \left\{ \int \Phi \,\mathrm{d}\nu \colon \nu \in \mathcal{M}_\sigma(\Sigma) \right\} \subset \mathbb{R}^{\mathbb{N}}.$$

The closure of $\mathcal{P}(\Phi)$ with respect to the product topology is denoted by $\overline{\mathcal{P}(\Phi)}$.

The following proposition proves the upper bound in Theorem D. In Lemma 6.15, we show that if $\alpha \notin \overline{\mathcal{P}(\Phi)}$, then $J_\Phi^{\mathbf{a}}(\alpha) = \emptyset$ for all $\mathbf{a} \in \mathbf{A}$.

Proposition 6.12. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$, $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ *is bounded with summable variations, and* $\alpha \in \overline{\mathcal{P}(\Phi)}$. *Then*

$$\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) \le \min\left\{ d, \max\left\{ s_\infty, \sup\left\{ D(\mu) \colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi \,\mathrm{d}\mu = \alpha \right\}\right\}\right\}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$.

Dealing first with the special case in which Φ takes values in $\mathbb{R}^N$, we extend the upper bound for the interior points of the spectrum found in Proposition 6.4 to the closure of the spectrum. This is done in the following two propositions. The proof of Proposition 6.12 is given after this.

Proposition 6.13. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$, $\Phi\colon \Sigma \to \mathbb{R}^N$ *is bounded with summable variations, and* $\alpha \in \overline{\mathrm{int}(\mathcal{P}(\Phi))}$. *Then*

$$\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) \le \min\left\{ d, \max\left\{ s_\infty, \sup\left\{ D(\mu) \colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi \,\mathrm{d}\mu = \alpha \right\}\right\}\right\}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$.

Proof. If $\mathrm{int}(\mathcal{P}(\Phi)) = \emptyset$, then there is nothing to prove, so we may assume that $\mathrm{int}(\mathcal{P}(\Phi)) \ne \emptyset$. Note that in this case, $\mathcal{P}(\Phi)$ cannot be contained within any $(N-1)$-dimensional hyperplane. In addition, if $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) \le s_\infty$, then the conclusion of the proposition holds, so we may as well assume that $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) > s_\infty$.

Fix $\alpha \in \overline{\operatorname{int}(\mathcal{P}(\Phi))}$. By Lemma 6.11, for each $n \in \mathbb{N}$ we may choose $\gamma_n \in \operatorname{int}(\mathcal{P}(\Phi))$ with $|\alpha - \gamma_n| < 1/n$ and $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\gamma_n)) \geq \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) - 1/n$ for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$. Since $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) > s_\infty$ we have $\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\gamma_n)) - 1/n > s_\infty$ for all sufficiently large n. By Proposition 6.4 it follows that for all such n there is a measure $\mu_n \in \mathcal{M}_\sigma(\Sigma)$ so that

$$\int \Phi \,\mathrm{d}\mu_n = \gamma_n$$

and

$$D(\mu_n) > \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\gamma_n)) - 1/n > s_\infty.$$

Now by Proposition 6.8 this implies that the sequence $(\mu_n)_{n\in\mathbb{N}}$ has a weak* limit $\mu \in \mathcal{M}_\sigma(\Sigma)$ with $D(\mu) \geq \limsup_{n\to\infty} D(\mu_n)$. That is, $D(\mu) = \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha))$ for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a}$. Moreover, since $\lim_{n\to\infty} \gamma_n = \alpha$ we have $\int \Phi \,\mathrm{d}\mu = \alpha$. □

Proposition 6.14. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, *the singular value function* φ^s *is quasi-multiplicative for all* $0 \leq s \leq d$, $\Phi\colon \Sigma \to \mathbb{R}^N$ *is bounded with summable variations, and* $\alpha \in \overline{\mathcal{P}(\Phi)}$. *Then*

$$\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)) \leq \min\left\{d, \max\left\{s_\infty, \sup\left\{D(\mu)\colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi \,\mathrm{d}\mu = \alpha\right\}\right\}\right\}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$.

Proof. We let $(\phi_i)_{i=1}^N$ denote the collection of real-valued maps with

$$\Phi(\omega) = (\phi_i(\omega))_{i=1}^N, \quad \omega \in \Sigma.$$

We begin by taking the smallest possible integer $M \leq N$ so that there is an M-dimensional affine subspace of $\mathbb{R}^N$ which contains $\mathcal{P}(\Phi)$. Then there exist $(j_l)_{l=1}^M$ with $j_l \in \{1, \ldots, N\}$ and for each $i \in \{1, \ldots, N\}$ a tuple of reals $(\gamma_{il})_{l=0}^M$ such that

$$\int \phi_i \,\mathrm{d}\mu = \gamma_{i0} + \sum_{l=1}^M \gamma_{il} \int \phi_{j_l} \,\mathrm{d}\mu, \quad \mu \in \mathcal{M}_\sigma(\Sigma),\ i \in \{1, \ldots, N\}.$$

Now define

$$\Phi'\colon \Sigma \longrightarrow \mathbb{R}^M$$

by setting

$$\Phi'(\omega) = (\phi_{j_l}(\omega))_{l=1}^M, \quad \omega \in \Sigma.$$

Given $\alpha = (\alpha_i)_{i=1}^N \in \overline{\mathcal{P}(\Phi)}$ we let

$$\alpha' = (\alpha_{j_l})_{l=1}^M.$$

It follows that $J^{\mathbf{a}}_{\Phi}(\alpha) \subset J^{\mathbf{a}}_{\Phi'}(\alpha')$. Moreover, by our choice of M, $\mathcal{P}(\Phi') \subset \mathbb{R}^M$ cannot be contained within any proper $(M-1)$-dimensional affine space. Thus, by Lemma 6.2 we have $\overline{\mathcal{P}(\Phi')} = \overline{\mathrm{int}(\mathcal{P}(\Phi'))}$. Moreover, since $\alpha \in \overline{\mathcal{P}(\Phi)}$ we have $\alpha' \in \overline{\mathcal{P}(\Phi')}$. Consequently, by Proposition 6.13, we have

$$\begin{aligned}
&\dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi}(\alpha)) \\
&\quad \le \dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi'}(\alpha')) \\
&\quad \le \min\left\{d, \max\left\{s_\infty, \sup\left\{D(\mu)\colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi' \,\mathrm{d}\mu = \alpha'\right\}\right\}\right\}
\end{aligned}$$

for $\mathcal{L}_{\mathbf{A}}$-almost all $\mathbf{a} \in \mathbf{A}$. Now given any $\mu \in \mathcal{M}_\sigma(\Sigma)$ with $\int \Phi' \,\mathrm{d}\mu = \alpha'$ we have $\int \phi_{j_l} = \alpha_{j_l}$ for $l \in \{1, \ldots, M\}$. Thus

$$\int \phi_i \,\mathrm{d}\mu = \gamma_{i0} + \sum_{l=1}^{M} \gamma_{il} \int \phi_{j_l} \,\mathrm{d}\mu = \gamma_{i0} + \sum_{l=1}^{M} \gamma_{il}\alpha_{j_l} = \alpha_i$$

for all $i \in \{1, \ldots, N\}$ and, consequently, $\int \Phi \,\mathrm{d}\mu = \alpha$. The proof is finished. □

We are now ready to prove the upper bound in Theorem D.

Proof of Proposition 6.12. Take a bounded potential $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ with summable variations and fix $\alpha \in \overline{\mathcal{P}(\Phi)}$. We shall apply Proposition 6.8 in a similar way to the proof of Proposition 6.13. Again, if $\dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi}(\alpha)) \le s_\infty$ then the conclusion of the proposition holds trivially, so we may as well assume that $\dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi}(\alpha)) > s_\infty$.

We take $\phi_i : \Sigma \to \mathbb{R}$ and $\alpha_i \in \mathbb{R}$ so that $\Phi = (\phi_i)_{i\in\mathbb{N}}$ and $\alpha = (\alpha_i)_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$ we define $\Phi_n = (\phi_i)_{i=1}^n$ and $\alpha_n = (\alpha_i)_{i=1}^n$. Then for each $n \in \mathbb{N}$ we have $J^{\mathbf{a}}_{\Phi}(\alpha) \subset J^{\mathbf{a}}_{\Phi_n}(\alpha_n)$. Thus, by applying Proposition 6.14 we have

$$\begin{aligned}
&\dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi}(\alpha)) \\
&\quad \le \dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi_n}(\alpha_n)) \\
&\quad \le \min\left\{d, \max\left\{s_\infty, \sup\left\{D(\mu)\colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi_n \,\mathrm{d}\mu = \alpha_n\right\}\right\}\right\}
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi}(\alpha)) > s_\infty$ we see that for each $n \in \mathbb{N}$ we may choose $\mu_n \in \mathcal{M}_\sigma(\Sigma)$ so that

$$D(\mu_n) > \max\{\dim_{\mathrm{H}}(J^{\mathbf{a}}_{\Phi}(\alpha)) - 1/n, s_\infty\}$$

and

$$\int \Phi_n \,\mathrm{d}\mu_n = \alpha_n.$$

By applying Proposition 6.8 we see that the sequence $(\mu_n)_{n\in\mathbb{N}}$ has a limit $\mu \in \mathcal{M}_\sigma(\Sigma)$ with

$$D(\mu) \geq \limsup_{n\to\infty} D(\mu_n) \geq \dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha)).$$

Moreover, since $\int \Phi_n \,\mathrm{d}\mu = \alpha_n$ for each $n \in \mathbb{N}$ we have $\int \Phi \,\mathrm{d}\mu = \alpha$. The proof is finished. □

We finish this section by showing that $E_\Phi(\alpha) = \emptyset$ outside of the closure of the spectrum.

Lemma 6.15. *If $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ is bounded with summable variations and $\alpha \in \mathbb{R}^{\mathbb{N}}$ satisfies $E_\Phi(\alpha) \neq \emptyset$, then $\alpha \in \overline{\mathcal{P}(\Phi)}$.*

Proof. It suffices to show that for each $q \in \mathbb{N}$ there exists a measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ such that

$$\int \phi_i \,\mathrm{d}\mu \in (\alpha_i - 1/q, \alpha_i + 1/q)$$

for all $i \in \{1, \ldots, q\}$. Now since each ϕ_i is uniformly continuous there exists $N \in \mathbb{N}$ for which $\mathrm{var}_n(A_n\phi_i) < (2q)^{-1}$ for all $i \in \{1, \ldots, q\}$ and $n \geq N_0$. Moreover, since $E_\Phi(\alpha) \neq \emptyset$ we may take $\omega \in E_\Phi(\alpha)$. In particular, there exists $N_1 \in \mathbb{N}$ such that for all $i \in \{1, \ldots, q\}$ and $n \geq N_1$ we have $A_n\phi_i(\omega) \in (\alpha_i - (2q)^{-1}, \alpha_i + (2q)^{-1})$. From these two facts it follows that if we take $N = \max\{N_0, N_1\}$ and let $\tau \in \Sigma$ denote the σ^N fixed point with $\sigma^{lN}(\tau) \in [\omega|_N]$ for all $l \in \mathbb{N} \cup \{0\}$, then

$$A_N\phi_i(\tau) \in (\alpha_i - 1/q, \alpha_i + 1/q)$$

for all $i \in \{1, \ldots, q\}$. Thus, if $\mu = N^{-1}\sum_{i=0}^{N-1} \delta_{\sigma^i(\tau)}$, then

$$\int \phi_i \,\mathrm{d}\mu \in (\alpha_i - 1/q, \alpha_i + 1/q)$$

for all $i \in \{1, \ldots, q\}$. Moreover, since τ is a fixed point for σ^N we conclude that μ is σ-invariant. □

6.6. Proof of the lower bound in Theorem D. In this section, we shall prove the lower bound in Theorem D. Together with Proposition 6.12 it finishes the proof of Theorem D.

Proposition 6.16. *Suppose that $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ is such that $\sup_{i\in\mathbb{N}} \|T_i\| < \frac{1}{2}$, the singular value function φ^s is quasi-multiplicative for all $0 \leq s \leq d$, $\Phi\colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ is bounded with summable variations, and $\alpha \in \overline{\mathcal{P}(\Phi)}$. Then*

$$\dim_{\mathrm{H}}(J_\Phi^{\mathbf{a}}(\alpha))$$
$$\geq \min\left\{d, \max\left\{s_\infty, \sup\left\{D(\mu)\colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi \,\mathrm{d}\mu = \alpha\right\}\right\}\right\}$$

for $\mathcal{L}_{\mathbf{A}}$*-almost all* $\mathbf{a} \in \mathbf{A}$.

By Theorem 5.4, to prove Proposition 6.16, it suffices to show that

$$\lim_{n\to\infty} \lim_{k\to\infty} \sup \Big\{ D_k(\mu) \colon \mu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \text{ so that } \int A_k \phi_i \,\mathrm{d}\mu \in B_n(\alpha_i) \text{ for all } i \in \{1,\dots,n\} \Big\} \geq \max \Big\{ s_\infty, \sup \Big\{ D(\mu) \colon \mu \in \mathcal{M}_\sigma(\Sigma) \text{ so that } \int \Phi \,\mathrm{d}\mu = \alpha \Big\} \Big\}.$$

This inequality is shown in the following two lemmas.

Lemma 6.17. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < 1$*, the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$*,* $\Phi \colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ *is bounded with summable variations, and* $\alpha \in \mathbb{R}^{\mathbb{N}}$*. Then*

$$D(\mu) \le \lim_{n\to\infty} \lim_{k\to\infty} \sup \Big\{ D_k(\mu) \colon \mu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \textit{ so that } \int A_k \phi_i \,\mathrm{d}\mu \in B_n(\alpha_i) \textit{ for all } i \in \{1,\dots,n\} \Big\}$$

for all $\mu \in \mathcal{M}_\sigma(\Sigma)$ *with* $\int \Phi \,\mathrm{d}\mu = \alpha$.

Proof. Fix $\mu \in \mathcal{M}_\sigma(\Sigma)$ with $\int \Phi \,\mathrm{d}\mu = \alpha$ and let $0 \le s < D(\mu)$. It follows that $P_\mu(\varphi^s) \ge 0$. Thus

$$\sum_{\omega\in\Sigma_k} \mu([\omega]) \log \frac{\varphi^s(\omega)}{\mu([\omega])} \ge 0$$

and $D_k(\mu) \ge s$ for all $k \in \mathbb{N}$. Moreover, since μ is σ-invariant we have

$$\int A_k \Phi \,\mathrm{d}\mu = \int \Phi \,\mathrm{d}\mu = \alpha \in B_n(\alpha)$$

for all $k, n \in \mathbb{N}$. Hence

$$s < \lim_{k\to\infty} \sup \Big\{ D_k(\nu) \colon \nu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \text{ so that } \int A_k \phi_i \,\mathrm{d}\nu \in B_n(\alpha_i) \text{ for all } i \in \{1,\dots,n\} \Big\}.$$

Letting $n \to \infty$ and $s \uparrow D(\mu)$ completes the proof of the lemma. □

Lemma 6.18. *Suppose that* $(T_i)_{i\in\mathbb{N}} \in \mathrm{GL}_d(\mathbb{R})^{\mathbb{N}}$ *is such that* $\sup_{i\in\mathbb{N}} \|T_i\| < 1$*, the singular value function* φ^s *is quasi-multiplicative for all* $0 \le s \le d$*,* $\Phi \colon \Sigma \to \mathbb{R}^{\mathbb{N}}$ *is bounded with summable variations. Then*

$$s_\infty \leq \lim_{n\to\infty} \lim_{k\to\infty} \sup \Big\{ D_k(\mu) \colon \mu \in \mathcal{M}^*_{\sigma^k}(\Sigma) \text{ so that } \int A_k\phi_i \, d\mu \in B_n(\alpha_i) \text{ for all } i \in \{1,\ldots,n\} \Big\}$$

for all $\alpha \in \overline{\mathcal{P}(\Phi)}$.

Proof. It suffices to show that for any $s < s_\infty$ and $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that for all $n \geq k(n)$ there exists $\mu \in \mathcal{M}^*_{\sigma^k}(\Sigma)$ with $D_k(\mu) \geq s$ and $\int A_k\phi_i \, d\mu \in B_n(\alpha_i)$ for $i \in \{1,\ldots,n\}$.

First take $k_0(n)$ so that $\mathrm{var}_k(A_k\phi_i) < (4n)^{-1}$ for all $i \in \{1,\ldots,n\}$. Let $\Phi^n \colon \Sigma \to \mathbb{R}^n$ denote the potential $(\phi_i)_{i=1}^n$. Since $\alpha \in \overline{\mathcal{P}(\Phi)}$ we have $(\alpha_i)_{i=1}^n \in \overline{\mathcal{P}(\Phi^n)} \subset \overline{\bigcup_I \mathcal{P}_e(\Phi^n, I)}$ by Lemma 6.3. Thus there exists an ergodic invariant measure ν with $\int \phi_i \, d\nu \in B_{4n}(\alpha_i)$ for all $i \in \{1,\ldots,n\}$. Since ν is ergodic we obtain $\tau \in \Sigma$ and $k(n) \geq k_0(n)$ such that for all $k \geq k(n)$ we have $A_k\phi_i(\tau) \in B_{4n}(\alpha_i)$. Since $k(n) \geq k_0(n)$ we have $A_k\phi_i(\kappa) \in B_{2n}(\alpha_i)$ for all $k \geq k(n)$ and $\kappa \in [\tau|_k]$. Now choose $\rho \in (0,1)$ sufficiently large that

$$\rho\Big(\alpha_i - \frac{1}{2n}\Big) - (1-\rho)\|\phi_i\| > \alpha_i - \frac{1}{n}$$

and

$$\rho\Big(\alpha_i + \frac{1}{2n}\Big) + (1-\rho)\|\phi_i\| < \alpha_i + \frac{1}{n}$$

for all $i \in \{1,\ldots,n\}$. It follows that for any $k \geq k(n)$, any measure $\tilde{\mu}$ such that $\tilde{\mu}([\tau|_k]) = \rho$ will satisfy

$$\int A_k\phi_i \, d\tilde{\mu} \in B_n(\alpha_i), \quad i \in \{1,\ldots,n\}.$$

Since $s < s_\infty$ we have

$$\sum_{\omega\in\Sigma_k} \varphi^s(\omega) = \infty, \quad k \in \mathbb{N}.$$

As such, for each $k \geq k(n)$ we choose a finite subset $C(k) \subset \Sigma_k \setminus \{\tau|_k\}$ with

$$\sum_{\omega\in C(k)} \varphi^s(\omega) > (\varphi^s(\tau|_k))^{-\rho/(1-\rho)}.$$

Let μ denote the unique k-th level Bernoulli measure satisfying

$$\mu(\omega) = \begin{cases} (1-\rho)\varphi^s(\omega)/\sum_{\kappa\in C(k)} \varphi^s(\kappa), & \text{if } \omega \in C(k), \\ \rho, & \text{if } \omega = \tau|_k, \\ 0, & \text{if } \omega \notin C(k) \cup \{\tau|_k\}. \end{cases}$$

Since $\mu([\tau|_k]) = \rho$ we have

$$\int A_k \phi_i \, \mathrm{d}\mu \in B_n(\alpha_i), \quad i \in \{1, \dots, n\}.$$

Moreover,

$$\begin{aligned}
&\sum_{\omega \in \Sigma_k} \mu([\omega]) \log \frac{\varphi^s(\omega)}{\mu([\omega])} \\
&= \rho \log \frac{\varphi^s(\tau|_k)}{\rho} + \sum_{\omega \in C(k)} \frac{(1-\rho)\varphi^s(\omega)}{\sum_{\kappa \in C(k)} \varphi^s(\kappa)} \log \frac{\sum_{\kappa \in C(k)} \varphi^s(\kappa)}{(1-\rho)} \\
&\geq \rho \log \varphi^s(\tau|_k) + (1-\rho) \log \Big(\sum_{\kappa \in C(k)} \varphi^s(\kappa) \Big) \\
&> 0.
\end{aligned}$$

Hence $D_k(\mu) > s$. This completes the proof of the lemma. □

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