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Measure theoretic trigonometric functions

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Abstract. We study the eigenvalues and eigenfunctions of the Laplacian $\Delta_\mu = \frac{d}{d\mu}\frac{d}{dx}$ for a Borel probability measure μ on the interval [0, 1] by a technique that follows the treatment of the classical eigenvalue equation $f'' = -\lambda f$ with homogeneous Neumann or Dirichlet boundary conditions. For this purpose we introduce generalized trigonometric functions that depend on the measure μ. In particular, we consider the special case where μ is a self-similar measure like e.g. the Cantor measure. We develop certain trigonometric identities that generalize the addition theorems for the sine and cosine functions. In certain cases we get information about the growth of the suprema of normalized eigenfunctions. For several special examples of μ we compute eigenvalues of Δ_μ and L_∞- and L_2-norms of eigenfunctions numerically by applying the formulas we developed.

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Contents

1. Introduction

Assume that μ is a Borel probability measure on the interval $[0, 1]$. We consider the Laplacian Δ_μ on $[0, 1]$ for the measure μ and study the eigenvalue problem

$$\Delta_\mu f = -\lambda f$$

with either homogeneous Dirichlet boundary conditions $f(a) = f(b) = 0$, or homogeneous Neumann boundary conditions $f'(a) = f'(b) = 0$.

The definition of Δ_μ involves the derivative with respect to the measure μ. If a function $g\colon [0, 1] \to \mathbb{R}$ allows the representation

$$g(x) = g(a) + \int_{[a,x]} \frac{dg}{d\mu}\, d\mu \tag{1}$$

for all $x \in [0, 1]$, then $\frac{dg}{d\mu}$ is unique in $L_2(\mu)$ and is called the *μ-derivative* of g. In Freiberg [8] an analytic calculus of the concept of μ derivatives is developed.

The operator Δ_μ is then given by

$$\Delta_\mu f = \frac{d}{d\mu} f'$$

for all $f \in L_2(\mu)$ for which f' and the μ-derivative of f' exist.

It is well known that if μ is a non-atomic Borel measure, Δ_μ has a pure point spectrum consisting only of eigenvalues with multiplicity one, that accumulate at infinity, see Freiberg [8], Lemma 5.1 and Corollary 6.9, or Bird, Ngai and Teplyaev [4], Theorem 2.5. Moreover, we have a pure point spectrum not only in the non-atomic case, see Vladimirov and Sheipak [30].

This operator and the resulting eigenvalue problem has been studied in numerous papers, for example in Feller [7], McKean and Ray [23], Kac and Krein [17], Fujita [14], Naimark and Solomyak [24], Freiberg and Zähle [13], Bird, Ngai and Teplyaev [4], Freiberg [8, 10, 11, 9], Freiberg and Löbus [12], Hu, Lau and Ngai [15], Chen and Ngai [5], and Arzt and Freiberg [2].

In this paper we give a new technique of determining the eigenvalues and eigenfunctions of Δ_μ that involves a generalization of the sine and cosine functions.

In this we follow the classical case, where μ is the Lebesgue measure. There, the Dirichlet eigenvalue problem reads $f'' = -\lambda f$, $f(0) = f(1) = 0$. Then, for every non-negative λ, $f(x) = \sin(\sqrt{\lambda} x)$ satisfies the equation as well as the boundary condition on the left-hand side. On the right-hand side, the boundary condition is only met if $\sqrt{\lambda}$ is a zero point of the sine function, which are, indeed, very well known.

If we impose Neumann boundary conditions $f'(0) = f'(1) = 0$, we take $f(x) = \cos(\sqrt{\lambda}x)$, because this complies automatically with the left-hand side condition. The right-hand side condition again is satisfied if $\sqrt{\lambda}$ is a zero point of the sine function, which leads to the same eigenvalues as in the Dirichlet case (supplemented by zero). But here sine appears as the derivative of cosine, which will make a difference when we take more general measures.

Now let μ be an arbitrary Borel probability measure on $[0, 1]$. We construct functions $\mathrm{s}_{\lambda,\mu}(\cdot,\cdot)$ and $\mathrm{c}_{\lambda,\mu}(\cdot,\cdot)$ as a replacement for sin and cos by generalizing the series

$$\sin(zx) = \sum_{n=0}^{\infty}(-1)^n \frac{(zx)^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos(zx) = \sum_{n=0}^{\infty}(-1)^n \frac{(zx)^{2n}}{(2n)!}.$$

There we replace $x^n/n!$ by appropriate functions $p_n(x)$ or $q_n(x)$, depending on whether we impose Neumann or Dirichlet boundary conditions. These functions fulfill the eigenvalue equation and meet the left-hand side Dirichlet and Neumann boundary condition, respectively.

Putting

$$p_n := p_n(1) \quad \text{and} \quad q_n := q_n(1),$$

we define

$$\sin_\mu^D(z) := \sum_{n=0}^{\infty}(-1)^n q_{2n+1} z^{2n+1} \quad \text{and} \quad \sin_\mu^N(z) := \sum_{n=0}^{\infty}(-1)^n p_{2n+1} z^{2n+1}.$$

For $\mathrm{s}_{\lambda,\mu}(z,\cdot)$ and $\mathrm{c}_{\lambda,\mu}(z,\cdot)$ to also match the right-hand side conditions, z has to be chosen as a zero point of $\sin_\mu^D$ in the Dirichlet case and $\sin_\mu^N$ in the Neumann case. All this is described in Section 3.

In Section 4 we show how to compute the norms in $L_2(\mu)$ of the eigenfunctions by using the sequences p_n and q_n.

The functions $\mathrm{c}_{\lambda,\mu}(z,\cdot)$ and $\mathrm{s}_{\lambda,\mu}(z,\cdot)$ satisfy an identity that generalizes the classical trigonometric identity. This is established in Section 5.

In Section 6 we consider symmetric measures and get some symmetry results.

The main results are established in Section 7. We outline these briefly here. Since the functions $p_n(x)$ and $q_n(x)$ are determined in a process of iterative integration alternately with respect to μ and the Lebesgue measure, the coefficients p_n and q_n are difficult to compute in general. But if μ is a self-similar measure with respect to the mappings $S_1(x) = r_1 x$ and $S_2(x) = r_2(x-1)+1$ as well as the weight factors m_1 and m_2, we develop a recursion formula for p_n and q_n.

To illustrate the structure of this recursion formula, we consider again the classical Lebesgue case. There we have $p_n = q_n = \frac{1}{n!}$ which leads to $\sin_\mu^N(z) = \sin_\mu^D(z) = \sin(z)$. The sequence $p_n = \frac{1}{n!}$ can be viewed as the solution of the problem

$$2^n p_n = \sum_{i=0}^{n} p_i \, p_{n-i}, \quad p_0 = p_1 = 1, \tag{2}$$

which is derived from the equation $2^n = \sum_{i=0}^{n} \binom{n}{i}$. Our recursion formula for self-similar μ looks a little more involved, as it distinguishes between the two different kinds of boundary conditions. Additionally it is different for even and odd values of n, and it involves the parameters r_1, r_2, m_1, m_2 of the measure. However, it has the same basic structure as (2).

Moreover, we establish functional equations involving $\sin_\mu^N$ and $\sin_\mu^D$ that can be viewed as generalizations of the classical addition theorems.

In Section 8 we consider the especially interesting case where $r_1 m_1 = r_2 m_2$. Then the Neumann eigenvalues fulfill a renormalization formula $\lambda_{2n} = R\lambda_n$, where $1/R = r_1 m_1$. This property has been established in a special case by Volkmer [31] and in our setting by Freiberg [11]. This formula allows us to investigate the growth of subsequences

$$(\|\tilde{f}_{k2^n}\|_\infty)_{n\in\mathbb{N}}, \quad \text{for odd } k,$$

where $\tilde{f}_n$ denotes an eigenfunction to the nth Neumann eigenvalue that is normalized to one in $L_2(\mu)$.

We show in Section 9 that, if we assume $r_1 + r_2 = 1$ in addition to $r_1 m_1 = r_2 m_2$, the Dirichlet and Neumann eigenvalues coincide.

Finally, by using the formulas we developed in the course of our investigations, we compute approximations of eigenvalues for certain examples in Section 10.

Several remarks about possible further investigations are made in Section 11.

2. Derivatives and the Laplacian with respect to a measure

As in Freiberg [8, 10], we define a derivative of a function with respect to a measure.

Definition 2.1. Let μ be a non-atomic Borel probability measure on $[0,1]$ and let $f : [0,1] \to \mathbb{R}$. A function $h \in L_2([0,1],\mu)$ is called the *μ-derivative of f*, if

$$f(x) = f(a) + \int_a^x h \, d\mu \quad \text{for all } x \in [0,1].$$

As can be easily seen, the μ-derivative in Definition 2.1 is unique in $L_2(\mu)$. We denote the μ-derivative of a function f by $\frac{df}{d\mu}$. The λ-derivative $\frac{df}{d\lambda}$ we denote by f', where λ denotes the Lebesgue measure on $[0, 1]$.

We define $H^1([0,1],\mu) = H^1(\mu)$ to be the space of all $L_2(\mu)$-functions whose μ-derivative exists. According to our definition, if it exists, the μ-derivative is always in $L_2(\mu)$, and thus it is clear that, for every non-atomic measure μ, all functions in $H^1(\mu)$ are continuous. In case $\mu = \lambda$ is the Lebesgue measure, the definition of $H^1(\lambda)$ is equivalent to the usual definition of the Sobolev space $H^1 = W_2^1$.

The following useful lemma is an analogue to integration by parts and can be found in Freiberg [8], Proposition 3.1.

Lemma 2.2. *For $c, d \in [0, 1]$ with $c < d$ and functions $f \in H^1(\mu)$ and $g \in H^1(\lambda)$ we have*

$$\int_c^d \frac{df}{d\mu}(t)\, g(t)\, d\mu(t) = f\, g\Big|_c^d - \int_c^d f(t)\, g'(t)\, dt.$$

Let ν be another non-atomic Borel probability measure on $[0, 1]$. Then the space $H^2(\nu,\mu)$ is defined to be the collection of all functions in $H^1(\nu)$ whose ν-derivative belongs to $H^1(\mu)$. Now we define the operator Δ_μ for all $f \in H^2(\lambda,\mu)$ as

$$\Delta_\mu f := \frac{d}{d\mu} f'.$$

Remark 2.3. In Freiberg [8], Corollary 6.4, is shown that $H^2(\lambda,\mu)$ is dense in $L_2(\mu)$. Furthermore, it is well known (see e.g. [8], Corollary 3.2) that Δ_μ is a negative symmetric operator on $L_2(\mu)$.

3. Generalized trigonometric functions

Let μ be an atomless Borel probability measure on $[0, 1]$. We construct sequences of functions $p_n(x)$ and $q_n(x)$ depending on μ.

Definition 3.1. For $x \in [0, 1]$ we set $p_0(x) = q_0(x) = 1$ and, for $n \in \mathbb{N}$,

$$p_n(x) := \begin{cases} \int_0^x p_{n-1}(t)\, d\mu(t), & \text{if } n \text{ is odd,} \\ \int_0^x p_{n-1}(t)\, dt, & \text{if } n \text{ is even,} \end{cases}$$

and

$$q_n(x) := \begin{cases} \int_0^x q_{n-1}(t)\,dt, & \text{if } n \text{ is odd,} \\ \int_0^x q_{n-1}(t)\,d\mu(t), & \text{if } n \text{ is even.} \end{cases}$$

Then, for $n \in \mathbb{N}_0$, we have by definition $p_{2n}, q_{2n+1} \in H^1(\lambda)$, $p_{2n+1}, q_{2n} \in H^1(\mu)$ and

$$\frac{d}{d\mu} p_{2n+1} = p_{2n}, \quad q'_{2n+1} = q_{2n}, \quad p'_{2n} = p_{2n-1}, \quad \frac{d}{d\mu} q_{2n} = q_{2n-1}.$$

Remark 3.2. (i) If we take μ to be the Lebesgue measure, then $p_n(x) = q_n(x) = \frac{x^n}{n!}$. In the following, we will transfer classical concepts and techniques to a general measure μ by replacing $\frac{x^n}{n!}$ by $p_n(x)$ or $q_n(x)$. In this sense, we can look at $p_n(x)$ or $q_n(x)$ as a kind of generalized monomials.

(ii) It is easy to see that for $n \in \mathbb{N}$ and $x \in [0, 1]$, $q_{n+1}(x) \leq p_n(x)$ and $p_{n+1}(x) \leq q_n(x)$.

To prove convergence of the series defined below, we will need the following lemma.

Lemma 3.3. *For all* $x \in [0, 1]$, $z \in \mathbb{R}$ *and* $n \in \mathbb{N}_0$,

$$p_{2n+1}(x) \leq \frac{1}{n!} q_2(x)^n,$$

$$q_{2n+1}(x) \leq \frac{1}{n!} p_2(x)^n,$$

and

$$p_{2n}(x) \leq \frac{1}{n!} p_2(x)^n,$$

$$q_{2n}(x) \leq \frac{1}{n!} q_2(x)^n.$$

Proof. The estimates for $q_{2n+1}(x)$ and $q_{2n}(x)$ are proved in Lemma 2.3 in Freiberg and Löbus [12], with complete induction. The proof of the other estimates works analogously. □

Definition 3.4. Using $p_n(x)$ and $q_n(x)$ we now define for $x \in [0,1]$ and $z \in \mathbb{R}$:

$$\mathrm{s}_{\mu,\lambda}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n+1} p_{2n+1}(x),$$

$$\mathrm{s}_{\lambda,\mu}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n+1} q_{2n+1}(x),$$

and

$$\mathrm{c}_{\lambda,\mu}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n} p_{2n}(x),$$

$$\mathrm{c}_{\mu,\lambda}(z,x) := \sum_{n=0}^{\infty} (-1)^n z^{2n} q_{2n}(x).$$

Note that for every $z \in \mathbb{R}$,

$$\mathrm{c}_{\lambda,\mu}(z,\cdot),\ \mathrm{s}_{\lambda,\mu}(z,\cdot) \in H^2(\lambda,\mu) \quad \text{and} \quad \mathrm{s}_{\mu,\lambda}(z,\cdot),\ \mathrm{c}_{\mu,\lambda}(z,\cdot) \in H^2(\mu,\lambda).$$

Remark 3.5. (i) If μ is the Lebesgue measure, then

$$\mathrm{s}_{\mu,\lambda}(z,x) = \mathrm{s}_{\lambda,\mu}(z,x) = \sin(zx)$$

and

$$\mathrm{c}_{\lambda,\mu}(z,x) = \mathrm{c}_{\mu,\lambda}(z,x) = \cos(zx).$$

(ii) Functions corresponding to $\mathrm{s}_{\lambda,\mu}(z,\cdot)$ and $\mathrm{c}_{\mu,\lambda}(z,\cdot)$ have also been constructed in Freiberg and Löbus [12], where they are used to determine the number of zeros of Dirichlet eigenfunctions.

Lemma 3.6. *For every* $z \in \mathbb{R}$ *the series in Definition* 3.4 *converge uniformly absolutely on* $[0,1]$ *and the following differentiation rules hold*:

$$\frac{d}{d\mu}\mathrm{s}_{\mu,\lambda}(z,\cdot) = z\,\mathrm{c}_{\lambda,\mu}(z,\cdot), \qquad \mathrm{s}'_{\lambda,\mu}(z,\cdot) = z\,\mathrm{c}_{\mu,\lambda}(z,\cdot),$$

$$\mathrm{c}'_{\lambda,\mu}(z,\cdot) = -z\,\mathrm{s}_{\mu,\lambda}(z,\cdot), \quad \frac{d}{d\mu}\mathrm{c}_{\mu,\lambda}(z,\cdot) = -z\,\mathrm{s}_{\lambda,\mu}(z,\cdot).$$

Proof. Let $z \in \mathbb{R}$. Since $q_2(x) \le 1$ for $x \in [0,1]$, we get by Lemma 3.3 for $N \in \mathbb{N}$

$$\sup_{x\in[0,1]} \sum_{n=N}^{\infty} |z|^{2n+1} p_{2n+1}(x) \le \sup_{x\in[0,1]} \sum_{n=N}^{\infty} \frac{|z|^{2n+1} q_2(x)^n}{n!} \le \sum_{n=N}^{\infty} \frac{|z|^{2n+1}}{n!}.$$

Hence, for every $z \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} |z|^{2n+1} p_{2n+1}(x)$ converges uniformly in x. The proof for the other series works analogously with the estimates in Lemma 3.3. Thus, we can differentiate term by term and get the above rules. □

Now we show the relation between $c_{\lambda,\mu}(z,\cdot)$ and $s_{\lambda,\mu}(z,\cdot)$ to the eigenvalue problem for Δ_μ. Consider the Neumann problem

$$\frac{d}{d\mu} f' = -\lambda f,\ f'(0) = f'(1) = 0$$

and the Dirichlet problem

$$\frac{d}{d\mu} f' = -\lambda f,\ f(0) = f(1) = 0.$$

It is well known that the Neumann eigenvalues can be sorted according to size such that

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{N,2} < \cdots,$$

where $\lambda_{N,0} = 0$ and $\lim_{m\to\infty} \lambda_{N,m} = \infty$. The same holds for the Dirichlet eigenvalues, we denote them such that

$$\lambda_{D,1} < \lambda_{D,2} < \lambda_{D,3} < \cdots$$

where $\lambda_{D,1} > 0$ and $\lim_{n\to\infty} \lambda_{D,n} = \infty$.

Proposition 3.7. (i) *The Neumann eigenvalues $\lambda_{N,m}$, $m \in \mathbb{N}_0$, are the squares of the non-negative zeros of the function $\sin_\mu^N$ given by*

$$\sin_\mu^N(z) := s_{\mu,\lambda}(z,1) = \sum_{n=0}^{\infty} (-1)^n p_{2n+1} z^{2n+1}, \quad \textit{for } z \in \mathbb{R},$$

where we write p_n instead of $p_n(1)$ for simplicity. The corresponding eigenfunctions $f_{N,m}$ are given by

$$f_{N,m}(x) := c_{\lambda,\mu}(\sqrt{\lambda_{N,m}}, x) = \sum_{n=0}^{\infty} (-1)^n \lambda_{N,m}^n p_{2n}(x), \quad x \in [0,1].$$

(ii) *The Dirichlet eigenvalues* $\lambda_{D,m}$, $m \in \mathbb{N}$, *are the squares of the positive zeros of the function* $\sin_\mu^D$ *given by*

$$\sin_\mu^D(z) := \mathrm{s}_{\lambda,\mu}(z,1) = \sum_{n=0}^{\infty}(-1)^n q_{2n+1} z^{2n+1}, \quad \textit{for } z \in \mathbb{R}$$

where, as above, q_n *stands for* $q_n(1)$. *The corresponding eigenfunctions* $f_{D,m}$ *are given by*

$$f_{D,m}(x) = \mathrm{s}_{\lambda,\mu}(\sqrt{\lambda_{D,m}}, x) = \sqrt{\lambda_{D,m}} \sum_{n=0}^{\infty}(-1)^n \lambda_{D,m}^n \, q_{2n+1}(x), \quad x \in [0,1].$$

Proof. Using the differentiation rules from Lemma 3.6 it is easy to see that $\mathrm{c}_{\lambda,\mu}(z,\cdot)$ satisfies the eigenvalue equation if $\lambda = z^2$, while it also fulfills the left boundary condition $\mathrm{c}'_{\lambda,\mu}(z,0) = -z\,\mathrm{s}_{\mu,\lambda}(z,0) = 0$. Here, the dash refers to the second argument of $\mathrm{c}_{\lambda,\mu}$. In order that $\mathrm{c}_{\lambda,\mu}(z,\cdot)$ satisfies the right boundary condition, too, z has to be zero itself or it must be chosen such that $\mathrm{s}_{\mu,\lambda}(z,1) = 0$. It is known (see Freiberg [8] p. 40) that the solution of the above problem is unique up to a multiplicative constant. So z is a zero point of $\sin_\mu^N$ if and only if z^2 is a Neumann eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$.

Thus, for $m \in \mathbb{N}_0$, $f_{N,m} = \mathrm{c}_{\lambda,\mu}(\sqrt{\lambda_{N,m}}, x)$ is an eigenfunction to the mth Neumann eigenvalue $\lambda_{N,m}$.

To show the second part of the proposition, note that $\mathrm{s}_{\lambda,\mu}(z,\cdot)$ satisfies the equation if $\lambda = z^2$ and also the left boundary condition $\mathrm{s}_{\lambda,\mu}(z,0) = 0$. The right boundary condition gives $\mathrm{s}_{\lambda,\mu}(z,1) = 0$. So z^2 is a Dirichlet eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$ if and only if z is a zero point of $\sin_\mu^D$ and $z \neq 0$.

Thus, for $m \in \mathbb{N}$, the function $f_{D,m} = \mathrm{s}_{\lambda,\mu}(\sqrt{\lambda_{D,m}}, x)$ is an eigenfunction to the mth Dirichlet eigenvalue $\lambda_{D,m}$. □

So if we only know the sequences $(p_n(1))_n$ and $(q_n(1))_n$, we can determine the Neumann and Dirichlet eigenvalues by means of the functions $\sin_\mu^N$ and $\sin_\mu^D$.

Remark 3.8. (i) As was pointed out to me only recently by V. Kravchenko, a construction analogous to that in Definitions 3.1 and 3.4 has also been done in [18] for Sturm–Liouville equations of the form $(pu')' + qu = z^2 u$. There, the corresponding spectral problem is also transformed to the problem of finding zeros of a power series as in Proposition 3.7. See also Kravchenko and Porter [19].

(ii) An eigenfunction is only unique up to a multiplicative constant. Throughout the chapter we will use the notations $f_{N,m}$ and $f_{D,m}$ for the eigenfunctions as constructed above. One would also get these by imposing the additional conditions $f_{N,m}(0) = 1$ and $f'_{D,m}(0) = \sqrt{\lambda_{D,m}}$.

Analogously to $\sin_\mu^N$ and $\sin_\mu^D$ we define for $z \in \mathbb{R}$

$$\cos_\mu^N(z) := \mathrm{c}_{\lambda,\mu}(z,1) = \sum_{n=0}^{\infty}(-1)^n p_{2n} z^{2n}$$

and

$$\cos_\mu^D(z) := \mathrm{c}_{\mu,\lambda}(z,1) = \sum_{n=0}^{\infty}(-1)^n q_{2n} z^{2n}$$

These functions are linked with the eigenvalue problems with mixed boundary conditions

$$f'(0) = 0, \quad f(1) = 0, \tag{ND}$$

and

$$f(0) = 0, \quad f'(1) = 0. \tag{DN}$$

We treat these problems as the problems in the above Proposition 3.7. If $\lambda > 0$ is chosen such that $\cos_\mu^N(\sqrt{\lambda}) = 0$, the solutions to (ND) are multiples of $\mathrm{c}_{\lambda,\mu}(\sqrt{\lambda},\cdot)$, because

$$\mathrm{c}'_{\lambda,\mu}(\sqrt{\lambda},0) = -\sqrt{\lambda}\,\mathrm{s}_{\mu,\lambda}(\sqrt{\lambda},0) = 0$$

and

$$\mathrm{c}_{\lambda,\mu}(\sqrt{\lambda},1) = \cos_\mu^N(\sqrt{\lambda}).$$

Similarly, if $\lambda > 0$ satisfies $\cos_\mu^D\left(\sqrt{\lambda}\right) = 0$, the solutions to (DN) are multiples of $\mathrm{s}_{\lambda,\mu}(\sqrt{\lambda},\cdot)$, because

$$\mathrm{s}_{\lambda,\mu}(\sqrt{\lambda},0) = 0$$

and

$$\mathrm{s}'_{\lambda,\mu}(\sqrt{\lambda},1) = \sqrt{\lambda}\,\mathrm{c}_{\mu,\lambda}(\sqrt{\lambda},1) = \sqrt{\lambda}\cos_\mu^D(\sqrt{\lambda}),$$

where the derivative refers to the second argument of $\mathrm{s}_{\lambda,\mu}$. Therefore, the (ND) eigenvalues are the squares of the zeros of $\cos_\mu^N$ and the (DN) eigenvalues are the squares of the zeros of $\cos_\mu^D$.

4. Calculation of L_2-norms

It turns out that by knowing the sequences $(p_n)_n$ and $(q_n)_n$ we can not only determine the Neumann and Dirichlet eigenvalues, but also the $L_2(\mu)$-norms of the eigenfunctions $f_{N,m}$ and $f_{D,m}$. We will need the following lemma to achieve that.

Lemma 4.1. *For $k, n \in \mathbb{N}_0$ with $k \leq n$ and for all $x \in [0,1]$ we have*

$$\int_0^x p_{2k}(t)\, p_{2n-2k}(t)\, d\mu(t) = \sum_{j=0}^{2k} (-1)^j\, p_j(x)\, p_{2n+1-j}(x) \tag{3}$$

and

$$\int_0^x q_{2k+1}(t)\, q_{2n+1-2k}(t)\, d\mu(t) = \sum_{j=0}^{2k+1} (-1)^{j+1} q_j(x)\, q_{2n+3-j}(x). \tag{4}$$

Proof. We prove (3) by induction on k. If $k = 0$ and $n \geq 0$, we have

$$\int_0^x p_0(t)\, p_{2n}(t)\, d\mu(t) = p_{2n+1}(x)$$

and so the assertion holds. Now, take $k \in \mathbb{N}_0$ and assume that the assertion holds for k and all $n \geq k$. Then, for all $n \geq k+1$,

$$\begin{aligned}
&\int_0^x p_{2k+2}(t)\, p_{2n-2k-2}(t)\, d\mu(t) \\
&\quad = p_{2k+2}(x)\, p_{2n-2k-1}(x) - \int_0^x p_{2k+1}(t)\, p_{2n-2k-1}(t)\, dt \\
&\quad = p_{2k+2}(x)\, p_{2n-2k-1}(x) - p_{2k+1}(x)\, p_{2n-2k}(x) \\
&\qquad + \int_0^x p_{2k}(t)\, p_{2n-2k}(t)\, d\mu(t),
\end{aligned}$$

by Lemma 2.2. Thus, by the induction hypothesis, we have for all $n \geq k+1$

$$\int_0^x p_{2k+2}(t)\, p_{2n-2k-2}(t)\, d\mu(t) = \sum_{j=0}^{2k+2} (-1)^j\, p_j(x)\, p_{2n+1-j}(x),$$

which proves (3). The proof of (4) works the same way. ☐

Proposition 4.2. *Let $z \in \mathbb{R}$ and $p_j := p_j(1)$ and $q_j := q_j(1)$. Then*

$$\|\mathrm{c}_{\lambda,\mu}(z,\cdot)\|^2_{L_2(\mu)} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \sum_{k=0}^{n} (n+1-2k)\, p_{2k}\, p_{2n+1-2k}, \tag{5}$$

and

$$\|\mathrm{s}_{\lambda,\mu}(z,\cdot)\|^2_{L_2(\mu)} = \sum_{n=0}^{\infty} (-1)^n z^{2n+2} \sum_{k=0}^{n+1} (n+1-2k)\, q_{2k+1}\, q_{2n+2-2k}. \tag{6}$$

Proof. First we prove (5). It holds for all $x \in [0,1]$ and $z \in \mathbb{R}$ that

$$c_{\lambda,\mu}(z,x)^2 = \sum_{n=0}^{\infty}(-1)^n z^{2n} \sum_{k=0}^{n} p_{2k}(x)\, p_{2n-2k}(x).$$

Consequently, applying (3),

$$\begin{aligned}\int_0^1 c_{\lambda,\mu}(z,t)^2\, d\mu(t) &= \sum_{n=0}^{\infty}(-1)^n z^{2n} \sum_{k=0}^{n} \int_0^1 p_{2k}(t)\, p_{2n-2k}(t)\, d\mu(t)\\ &= \sum_{n=0}^{\infty}(-1)^n z^{2n} \sum_{k=0}^{n}\sum_{j=0}^{2k}(-1)^j p_j\, p_{2n+1-j}.\end{aligned}$$

Straightforward computation yields

$$\sum_{k=0}^{n}\sum_{j=0}^{2k}(-1)^j p_j\, p_{2n+1-j} = \sum_{k=0}^{n}(n+1-2k)\, p_{2k}\, p_{2n+1-2k},$$

which proves (5). The proof of (6) works analogously. □

We put $z = \sqrt{\lambda_{N,m}}$ and $z = \sqrt{\lambda_{D,m}}$ to get the following corollary.

Corollary 4.3. *The $L_2(\mu)$-norm of the Neumann eigenfunction $f_{N,m}$ is given by*

$$\| f_{N,m}\|_{L_2(\mu)}^2 = \sum_{n=0}^{\infty}(-1)^n \lambda_{N,m}^n \sum_{k=0}^{n}(n+1-2k)\, p_{2k}\, p_{2n+1-2k}$$

and of the Dirichlet eigenfunction $f_{D,m}$ by

$$\| f_{D,m}\|_{L_2(\mu)}^2 = \sum_{n=0}^{\infty}(-1)^n \lambda_{D,m}^{n+1} \sum_{k=0}^{n+1}(n+1-2k)\, q_{2k+1}\, q_{2n+2-2k}.$$

5. A trigonometric identity

As in the previous section, we consider an atomless Borel probability measure μ on $[0,1]$. We prove a formula that links the functions $c_{\lambda,\mu}, c_{\mu,\lambda}, s_{\mu,\lambda}$, and $s_{\lambda,\mu}$ generalizing the trigonometric identity $\sin^2 + \cos^2 = 1$. For this we need the following lemma.

Lemma 5.1. *For $k, n \in \mathbb{N}$ with $k \le n$ and for all $x \in [0,1]$ we have*

$$\int_0^x q_{2k-1}(t)\, p_{2n-2k}(t)\, d\mu(t) = \sum_{j=0}^{2k-1}(-1)^{j+1} q_j(x)\, p_{2n-j}(x).$$

Proof. We prove this by induction on k. For $k = 1$ and $n \geq 1$, we get by Lemma 2.2

$$\int_0^x q_1(t)\, p_{2n-2}(t)\, d\mu(t) = q_1(x)\, p_{2n-1}(x) - \int_0^x p_{2n-1}(t)\, dt$$
$$= q_1(x)\, p_{2n-1}(x) - p_{2n}(x),$$

and so the assertion holds. Now, take $k \in \mathbb{N}$, and assume that the assertion holds for k and all $n \geq k$. Then, again by using Lemma 2.2, we get

$$\begin{aligned}\int_0^x & q_{2k+1}(t)\, p_{2n-2k-2}(t)\, d\mu(t) \\ &= q_{2k+1}(x)\, p_{2n-2k-1}(x) - \int_0^x q_{2k}(t)\, p_{2n-2k-1}(t)\, dt \\ &= q_{2k+1}(x)\, p_{2n-2k-1}(x) - q_{2k}(x)\, p_{2n-2k}(x) \\ &\quad + \int_0^x q_{2k-1}(t)\, p_{2n-2k}(t)\, d\mu(t).\end{aligned}$$

Thus, by the induction hypothesis, for all $n \geq k + 1$,

$$\int_0^x q_{2k+1}(t)\, p_{2n-2k-2}(t)\, d\mu(t) = \sum_{j=0}^{2k+1} (-1)^{j+1} q_j(x)\, p_{2n-j}(x). \qquad \square$$

Corollary 5.2. *If we set $n = k$ in Lemma 5.1, we get the formula*

$$\sum_{j=0}^{2n} (-1)^j\, q_j(x)\, p_{2n-j}(x) = 0,$$

which holds for all $n \in \mathbb{N}$ and $x \in [0, 1]$.

With the above corollary we can prove the following theorem.

Theorem 5.3. *For all $x \in [0, 1]$ and $z \in \mathbb{R}$,*

$$\mathrm{c}_{\mu,\lambda}(z, x)\ \mathrm{c}_{\lambda,\mu}(z, x) + \mathrm{s}_{\lambda,\mu}(z, x)\ \mathrm{s}_{\mu,\lambda}(z, x) = 1.$$

Proof. Take $x \in [0,1]$ and $z \in \mathbb{R}$. Then, by Corollary 5.2,

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n z^{2n} \sum_{k=0}^{n} q_{2k}(x)\, p_{2n-2k}(x) \\
&\qquad + \sum_{n=0}^{\infty}(-1)^n z^{2n+2} \sum_{k=0}^{n} q_{2k+1}(x)\, p_{2n+1-2k}(x) \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n z^{2n}\Big[\sum_{k=0}^{n} q_{2k}(x)\, p_{2n-2k}(x) - \sum_{k=0}^{n-1} q_{2k+1}(x)\, p_{2n-(2k+1)}(x)\Big] \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n z^{2n} \sum_{k=0}^{2n}(-1)^k\, q_k(x)\, p_{2n-k}(x) = 1. \qquad \square
\end{aligned}$$

6. Symmetric measures

In this section we consider symmetric measures μ on $[0,1]$, meaning that, additionally to being an atomless Borel probability measure, μ shall satisfy for all $x \in [0,1]$

$$\mu([0,x]) = \mu([1-x,1]).$$

Proposition 6.1. *Let μ be symmetric and let $x \in [0,1]$. Then, for $n \in \mathbb{N}_0$,*

$$p_{2n+1}(x) = \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k+1}(x) - p_{2n+1}(1-x), \quad (7)$$

and

$$p_{2n}(x) = \sum_{k=0}^{n-1} p_{2k+1}\, q_{2n-2k-1}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k}(x) + p_{2n}(1-x). \quad (8)$$

Proof. For $p_1(x)$ the formula reduces to $p_1(x) = p_1 - p_1(1-x)$. This holds since

$$p_1(x) = \mu([0,x]) = \mu([1-x,1]) = \int_0^1 d\mu - \int_0^{1-x} d\mu = p_1 - p_1(1-x).$$

With $d\mu(t) = d\mu(1-t)$ the rest of the induction proof is straightforward. $\square$

Corollary 6.2. *Let μ be symmetric. Then, for $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n} p_{2k}\, p_{2n-2k+1} = \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k}. \tag{9}$$

Proof. This follows from Proposition 6.1 by putting $x = 1$ in (7). □

Remark 6.3. In the special case where μ is the Lebesgue measure, the above formula reduces to $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

Corollary 6.4. *Let μ be symmetric. Then the following statements hold*:

(i) $p_{2n} = q_{2n}$ *for all $n \in \mathbb{N}$;*

(ii) $\cos_\mu^N(z) = \cos_\mu^D(z)$ *for all $z \in \mathbb{R}$;*

(iii) $\cos_\mu^N(z)^2 + \sin_\mu^N(z)\,\sin_\mu^D(z) = 1$ *for all $z \in \mathbb{R}$;*

(iv) *we have the recursion formula*

$$p_{2n} = \frac{1}{2}\sum_{k=1}^{2n-1}(-1)^{k+1}\, p_k\, q_{2n-k}. \tag{10}$$

Proof. We prove (i) by induction. By putting $n = 1$ in (9), we find that

$$p_3 + p_2 p_1 = p_1 q_2 + p_3,$$

which implies $p_2 = q_2$. Assume that $p_{2k} = q_{2k}$ for all k smaller than some $n \in \mathbb{N}$, $n \geq 2$. We reverse the order of the summands in the second sum of (9) to get

$$\sum_{k=0}^{n-1} p_{2k}\, p_{2n-2k+1} + p_{2n}\, p_1 = \sum_{k=0}^{n-1} p_{2n-2k+1}\, q_{2k} + p_1\, q_{2n}.$$

From the induction hypothesis follows that $p_{2n} = q_{2n}$. Then, (ii) follows immediately and by Proposition 5.3 also (iii). Clearly, (iv) follows from (i) and Corollary 5.2. □

Proposition 6.5. *Let μ be symmetric. Then, for all $z \in \mathbb{R}$ and $x \in [0,1]$,*

$$\mathrm{c}_{\lambda,\mu}(z, 1-x) = \cos_\mu^N(z)\,\mathrm{c}_{\lambda,\mu}(z,x) + \sin_\mu^N(z)\,\mathrm{s}_{\lambda,\mu}(z,x).$$

Proof. Rearranging (8) gives

$$p_{2n}(1-x) = \sum_{k=0}^{n} p_{2k}\, p_{2n-2k}(x) - \sum_{k=0}^{n-1} p_{2k+1}\, q_{2n-2k-1}(x).$$

Multiplying with $(-1)^n z^{2n}$ and summing from $n = 0$ to infinity gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}(iz)^{2k} p_{2k}\cdot(iz)^{2n-2k} p_{2n-2k}(x)\\
&\qquad - \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(iz)^{2k+1} p_{2k+1}\cdot(iz)^{2n-2k-1} q_{2n-2k-1}(x)\\
&= \sum_{n=0}^{\infty}(-1)^n z^{2n} p_{2n}\cdot\sum_{k=0}^{\infty}(-1)^k z^{2k} p_{2k}(x)\\
&\qquad + \sum_{n=1}^{\infty}(-1)^n z^{2n+1} p_{2n+1}\cdot\sum_{k=0}^{\infty}(-1)^k z^{2k+1} q_{2k+1}(x)\\
&= \cos_\mu^N(z)\,\mathrm{c}_{\lambda,\mu}(z,x) + \sin_\mu^N(z)\,\mathrm{s}_{\lambda,\mu}(z,x).
\end{aligned}$$

□

Corollary 6.6. *Let μ be symmetric. Then the Neumann eigenfunctions $f_{N,m}$ are either symmetric or antisymmetric, that is, either*

$$f_{N,m}(x) = f_{N,m}(1-x)$$

or

$$f_{N,m}(x) = -f_{N,m}(1-x)$$

for all $x \in [0,1]$.

Proof. Let z^2 be a Neumann eigenvalue. Then, by Proposition 3.7, $\sin_\mu^N(z) = 0$ and hence, by Corollary 6.4 (iii), $|\cos_\mu^N(z)| = 1$. Thus, by Proposition 6.5, we get $\mathrm{c}_{\lambda,\mu}(z, 1-x) = \pm\,\mathrm{c}_{\lambda,\mu}(z,x)$. Since $\mathrm{c}_{\lambda,\mu}(z,\cdot) = f_{N,m}$ for $z^2 = \lambda_m$ the corollary is proved. □

Remark 6.7. All statements in this section have a counterpart in the Dirichlet case.

7. Self-similar measures

In this section we impose that the measure μ has a self-similar structure. For definitions of the concept of iterated function systems and self-similar measures, see Hutchinson [16]. For reasons of simplicity, we take an IFS consisting only of two mappings, but it should not raise considerable problems to generalize this to an arbitrary number.

Let r_1, r_2, m_1 and m_2 be positive numbers satisfying

$$r_1 + r_2 \leq 1 \quad \text{and} \quad m_1 + m_2 = 1.$$

Let $\mathcal{S} = (S_1, S_2)$ be the IFS given by

$$S_1(x) = r_1 x \quad \text{and} \quad S_2(x) = r_2 x + 1 - r_2, \quad x \in [0,1].$$

By K we denote the invariant set of $\mathcal{S}$ and by μ its invariant measure with vector of weights (m_1, m_2). In this case we are able to prove several properties of the functions $p_n(x)$ and $q_n(x)$ that resemble corresponding ones of $\frac{x^n}{n!}$. These we will employ to examine the Neumann and Dirichlet eigenfunctions and eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$. In particular, we will develop a recursion law for $p_n(1)$ and $q_n(1)$.

In the following proposition we present a formula that can be viewed as an analogue of the binomial theorem, adapted to the self-similar measure μ. It relates values on the left part of K, contained in $[0, r_1]$, to values on the right part, contained in $[1 - r_2, 1]$.

Proposition 7.1. *For $x \in [0,1]$ and $n \in \mathbb{N}_0$,*

$$\begin{aligned} &p_{2n+1}(1 - r_2 + r_2 x) \\ &\quad = [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1) \Big(\frac{r_2}{r_1}\Big)^{n-i-1} \Big(\frac{m_2}{m_1}\Big)^{n-i} p_{2n-2i-1}(r_1 x) \\ &\qquad + \sum_{i=0}^{n} p_{2i+1}(r_1) \Big(\frac{r_2 m_2}{r_1 m_1}\Big)^{n-i} q_{2n-2i}(r_1 x) \\ &\qquad + \sum_{i=0}^{n} p_{2i}(r_1) \Big(\frac{r_2}{r_1}\Big)^{n-i} \Big(\frac{m_2}{m_1}\Big)^{n-i+1} p_{2n-2i+1}(r_1 x) \end{aligned} \tag{11}$$

where a sum from 0 *to* -1 *is regarded as zero, and, for* $n \in \mathbb{N}$,

$$
\begin{aligned}
&p_{2n}(1 - r_2 + r_2 x) \\
&\quad = [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1) \Big(\frac{r_2 m_2}{r_1 m_1}\Big)^{n-i-1} p_{2n-2i-2}(r_1 x) \\
&\qquad + \sum_{i=0}^{n} p_{2i}(r_1) \Big(\frac{r_2 m_2}{r_1 m_1}\Big)^{n-i} p_{2n-2i}(r_1 x) \\
&\qquad + \sum_{i=0}^{n-1} p_{2i+1}(r_1) \Big(\frac{r_2}{r_1}\Big)^{n-i} \Big(\frac{m_2}{m_1}\Big)^{n-i-1} q_{2n-2i-1}(r_1 x).
\end{aligned}
\tag{12}
$$

Remark 7.2. If $r_1 = m_1$ and $r_2 = m_2$ and $r_1 + r_2 = 1$ (and hence, μ is the Lebesgue measure), the above formulas reduce to

$$
(r_1 + r_2 x)^n = \sum_{i=0}^{n} \binom{n}{i} r_1^i \, (r_2 x)^{n-i}, \quad n \in \mathbb{N}.
$$

Proof. We prove the proposition by induction. We have $\mu = m_1(S_1\mu)$ on $[0, r_1]$ and $\mu = m_2(S_2\mu)$ on $[1 - r_2, 1]$ and therefore,

$$
\begin{aligned}
p_1(1 - r_2 + r_2 x) &= \int_0^{1-r_2+r_2 x} d\mu \\
&= \int_0^{r_1} d\mu + \int_{1-r_2}^{1-r_2+r_2 x} d\mu \\
&= p_1(r_1) + m_2 \int_{1-r_2}^{1-r_2+r_2 x} d(S_2\mu) \\
&= p_1(r_1) + m_2 \int_0^{x} d\mu \\
&= p_1(r_1) + m_2 \int_0^{r_1 x} d(S_1\mu) \\
&= p_1(r_1) + \frac{m_2}{m_1} \int_0^{r_1 x} d\mu \\
&= p_1(r_1) + \frac{m_2}{m_1} p_1(r_1 x),
\end{aligned}
$$

which proves the assertion for p_1.

Assume that the formula for p_{2n+1} holds for some $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
&p_{2n+2}(1-r_2+r_2x) \\
&\quad = \int_0^{r_1} p_{2n+1}(t)\,dt + \int_{r_1}^{1-r_2} p_{2n+1}(t)\,dt + \int_{1-r_2}^{1-r_2+r_2x} p_{2n+1}(t)\,dt \\
&\quad = p_{2n+2}(r_1) + [1-(r_1+r_2)]p_{2n+1}(r_1) + r_2 \int_0^x p_{2n+1}(1-r_2+r_2t)\,dt.
\end{aligned}$$

Applying the induction hypothesis and basic sum transformations, we receive the formula for p_{2n+2}, for a more detailed calculation, see [1].

Furthermore, suppose that the assertion is true for p_{2n} for some $n \in \mathbb{N}$. Then, transforming μ as in the proof of the initial step and applying the induction hypothesis we get the formula for p_{2n+1}. □

Analogous formulas hold for the functions q_n.

Proposition 7.3. *For $x \in [0,1]$ and $n \in \mathbb{N}_0$,*

$$\begin{aligned}
q_{2n+1}(1-r_2+r_2x) = [1-(r_1+r_2)] &\sum_{i=0}^{n} q_{2i}(r_1)\Big(\frac{r_2m_2}{r_1m_1}\Big)^{n-i} p_{2n-2i}(r_1x) \\
&+ \sum_{i=0}^{n} q_{2i+1}(r_1)\Big(\frac{r_2m_2}{r_1m_1}\Big)^{n-i} p_{2n-2i}(r_1x) \\
&+ \sum_{i=0}^{n} q_{2i}(r_1)\Big(\frac{r_2}{r_1}\Big)^{n-i+1}\Big(\frac{m_2}{m_1}\Big)^{n-i} q_{2n-2i+1}(r_1x),
\end{aligned} \tag{13}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned}
&q_{2n}(1-r_2+r_2x) \\
&\quad = [1-(r_1+r_2)]\sum_{i=0}^{n-1} q_{2i}(r_1)\Big(\frac{r_2}{r_1}\Big)^{n-i-1}\Big(\frac{m_2}{m_1}\Big)^{n-i} p_{2n-2i-1}(r_1x) \\
&\qquad + \sum_{i=0}^{n} q_{2i}(r_1)\Big(\frac{r_2m_2}{r_1m_1}\Big)^{n-i} q_{2n-2i}(r_1x) \\
&\qquad + \sum_{i=0}^{n-1} q_{2i+1}(r_1)\Big(\frac{r_2}{r_1}\Big)^{n-i-1}\Big(\frac{m_2}{m_1}\Big)^{n-i} p_{2n-2i-1}(r_1x).
\end{aligned} \tag{14}$$

Proof. The proof works by induction analogously to that of Proposition 7.1. □

We translate the formulas about the functions $p_n(x)$ and $q_n(x)$ into formulas about $\mathrm{c}_{\lambda,\mu}(z,x)$ and $\mathrm{s}_{\lambda,\mu}(z,x)$. In the Lebesgue case, these are the usual addition theorems for $\cos(r_1 z + r_2 xz)$ and $\sin(r_1 z + r_2 xz)$.

Corollary 7.4. *Let* $z \in \mathbb{R}$ *and* $x \in [0,1]$. *With the abbreviation*

$$\bar{z} := \sqrt{\frac{r_2 m_2}{r_1 m_1}}\, z$$

we get

$$\begin{aligned} \mathrm{c}_{\lambda,\mu}(z, 1 - r_2 + r_2 x) = {} & \mathrm{c}_{\lambda,\mu}(z, r_1)\, \mathrm{c}_{\lambda,\mu}(\bar{z}, r_1 x) \\ & - \sqrt{\tfrac{r_2 m_1}{r_1 m_2}}\, \mathrm{s}_{\mu,\lambda}(z, r_1)\, \mathrm{s}_{\lambda,\mu}(\bar{z}, r_1 x) \\ & - [1 - (r_1 + r_2)] z\, \mathrm{s}_{\mu,\lambda}(z, r_1)\, \mathrm{c}_{\lambda,\mu}(\bar{z}, r_1 x) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mathrm{s}_{\lambda,\mu}(z, 1 - r_2 + r_2 x) = {} & \mathrm{s}_{\lambda,\mu}(z, r_1)\, \mathrm{c}_{\lambda,\mu}(\bar{z}, r_1 x) \\ & + \sqrt{\tfrac{r_2 m_1}{r_1 m_2}}\, \mathrm{c}_{\mu,\lambda}(z, r_1)\, \mathrm{s}_{\lambda,\mu}(\bar{z}, r_1 x) \\ & + [1 - (r_1 + r_2)] z\, \mathrm{c}_{\mu,\lambda}(z, r_1)\, \mathrm{c}_{\lambda,\mu}(\bar{z}, r_1 x). \end{aligned} \tag{16}$$

Proof. We prove (16). We multiply (13) by

$$(-1)^n z^{2n+1} = \frac{1}{i} (iz)^{2n+1},$$

sum from $n = 0$ to infinity and get

$$\begin{aligned} & \mathrm{s}_{\lambda,\mu}(z, 1 - r_2 + r_2 x) \\ & = \frac{1}{i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k+1} q_{2k+1}(r_1) \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2n-2k} p_{2n-2k}(r_1 x) \\ & \quad + \sqrt{\tfrac{r_2 m_1}{r_1 m_2}} \frac{1}{i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k} q_{2k}(r_1) \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2n-2k+1} q_{2n-2k+1}(r_1 x) \\ & \quad + [1 - (r_1 + r_2)] z \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k} q_{2k}(r_1) \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2n-2k} p_{2n-2k}(r_1 x). \end{aligned}$$

This we transform to

$$\frac{1}{i}\Big(\sum_{n=0}^{\infty}(iz)^{2n+1}q_{2n+1}(r_1)\Big)\Big(\sum_{k=0}^{\infty}\Big(i\sqrt{\frac{r_2m_2}{r_1m_1}}z\Big)^{2k}p_{2k}(r_1x)\Big)$$
$$+\sqrt{\frac{r_2m_1}{r_1m_2}}\frac{1}{i}\Big(\sum_{n=0}^{\infty}(iz)^{2n}q_{2n}(r_1)\Big)\Big(\sum_{k=0}^{\infty}\Big(i\sqrt{\frac{r_2m_2}{r_1m_1}}z\Big)^{2k+1}q_{2k+1}(r_1x)\Big)$$
$$+[1-(r_1+r_2)]z\Big(\sum_{n=0}^{\infty}(iz)^{2n}q_{2n}(r_1)\Big)\Big(\sum_{k=0}^{\infty}\Big(i\sqrt{\frac{r_2m_2}{r_1m_1}}z\Big)^{2k}p_{2k}(r_1x)\Big)$$
$$=\mathrm{s}_{\lambda,\mu}(z,r_1)\,\mathrm{c}_{\lambda,\mu}(\bar{z},r_1x)+\sqrt{\frac{r_2m_1}{r_1m_2}}\,\mathrm{c}_{\mu,\lambda}(z,r_1)\,\mathrm{s}_{\lambda,\mu}(\bar{z},r_1x)$$
$$+[1-(r_1+r_2)]z\,\mathrm{c}_{\mu,\lambda}(z,r_1)\,\mathrm{c}_{\lambda,\mu}(\bar{z},r_1x).$$

By multiplying (12) with $(-1)^n z^{2n}$ and summing up, (15) is proved in the same way. □

The following scaling properties are a replacement of the property

$$\Big(\frac{1}{2}x\Big)^n=\frac{1}{2^n}x^n$$

for p_n and q_n.

Proposition 7.5. *For $x\in[0,1]$ and $n\in\mathbb{N}_0$ we have*

$$p_{2n+1}(r_1x)=r_1^n m_1^{n+1}\,p_{2n+1}(x),$$
$$q_{2n+1}(r_1x)=r_1^{n+1}m_1^n\,q_{2n+1}(x),$$

and, for $n\in\mathbb{N}$,

$$p_{2n}(r_1x)=(r_1m_1)^n\,p_{2n}(x),$$
$$q_{2n}(r_1x)=(r_1m_1)^n\,q_{2n}(x).$$

Proof. We prove the asserted property for p_n by induction on $n\in\mathbb{N}$. Since μ satisfies $\mu(B)=m_1(S_1\mu)(B)$ for all Borel sets $B\subseteq[0,r_1]$, we have

$$p_1(r_1x)=\int_0^{r_1x}d\mu$$
$$=m_1\int_0^{r_1x}d(S_1\mu)$$
$$=m_1\int_0^{x}d\mu=m_1p_1(x).$$

Suppose the assertion is true for p_{2n+1} for some $n \in \mathbb{N}_0$. Then

$$\begin{aligned} p_{2n+2}(r_1 x) &= \int_0^{r_1 x} p_{2n+1}(t)\,dt \\ &= r_1 \int_0^x p_{2n+1}(r_1 t)\,dt \\ &= (r_1 m_1)^{n+1} p_{2n+2}(x). \end{aligned}$$

Assuming that the formula holds for p_{2n} for some $n \in \mathbb{N}$, we get

$$\begin{aligned} p_{2n+1}(r_1 x) &= \int_0^{r_1 x} p_{2n}(t)\,d\mu(t) \\ &= m_1 \int_0^x p_{2n}(r_1 t)\,d\mu(t) \\ &= r_1^n m_1^{n+1} p_{2n+1}(x). \end{aligned}$$

The formula for q_n is proved analogously. □

Next, we deduce formulas corresponding to those in Proposition 7.5 that relate values of $\mathrm{c}_{\lambda,\mu}(z,\cdot)$ and $\mathrm{s}_{\lambda,\mu}(z,\cdot)$ at $S_1(x)$ to values of $\mathrm{c}_{\lambda,\mu}(\sqrt{r_1 m_1} z,\cdot)$ and $\mathrm{s}_{\lambda,\mu}(\sqrt{r_1 m_1} z,\cdot)$ at x.

Proposition 7.6. *For all* $x \in [0,1]$ *and* $z \in \mathbb{R}$ *we have*

$$\mathrm{c}_{\lambda,\mu}(z, S_1(x)) = \mathrm{c}_{\lambda,\mu}(\sqrt{r_1 m_1} z, x) \tag{17}$$

and

$$\mathrm{s}_{\lambda,\mu}(z, S_1(x)) = \sqrt{\frac{r_1}{m_1}}\,\mathrm{s}_{\lambda,\mu}(\sqrt{r_1 m_1} z, x). \tag{18}$$

Furthermore, we have

$$\mathrm{s}_{\mu,\lambda}(z, S_1(x)) = \sqrt{\frac{m_1}{r_1}}\,\mathrm{s}_{\mu,\lambda}(\sqrt{r_1 m_1} z, x)$$

and

$$\mathrm{c}_{\mu,\lambda}(z, S_1(x)) = \mathrm{c}_{\lambda,\mu}(\sqrt{r_1 m_1} z, x).$$

Proof. With Proposition 7.5 we get

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n z^{2n} p_{2n}(r_1 x) &= \sum_{n=0}^{\infty}(-1)^n(\sqrt{r_1 m_1}z)^{2n} p_{2n}(x)\\ &= \mathrm{c}_{\lambda,\mu}(\sqrt{r_1 m_1}z, x)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n z^{2n+1} q_{2n+1}(r_1 x) &= \sqrt{\frac{r_1}{m_1}}\sum_{n=0}^{\infty}(-1)^n(\sqrt{r_1 m_1}z)^{2n+1} q_{2n+1}(x)\\ &= \sqrt{\frac{r_1}{m_1}}\,\mathrm{s}_{\lambda,\mu}(\sqrt{r_1 m_1}z, x).\end{aligned}$$

The other two equations are obtained by deriving. □

The counterparts of (17) and (18) are the following formulas for $\mathrm{c}_{\lambda,\mu}(z, S_2(x))$ and $\mathrm{s}_{\lambda,\mu}(z, S_2(x))$.

Proposition 7.7. *For all $x \in [0,1]$ and $z \in \mathbb{R}$ we have*

$$\begin{aligned}\mathrm{c}_{\lambda,\mu}(z, S_2(x)) = -[1-(r_1+r_2)]\sqrt{\frac{m_1}{r_1}}z\sin_\mu^N(\sqrt{r_1 m_1}z)\,\mathrm{c}_{\lambda,\mu}(\sqrt{r_2 m_2}z, x)\\ + \cos_\mu^N(\sqrt{r_1 m_1}z)\,\mathrm{c}_{\lambda,\mu}(\sqrt{r_2 m_2}z, x)\\ - \sqrt{\frac{r_2 m_1}{r_1 m_2}}\sin_\mu^N(\sqrt{r_1 m_1}z)\,\mathrm{s}_{\lambda,\mu}(\sqrt{r_2 m_2}z, x)\end{aligned} \tag{19}$$

and

$$\begin{aligned}\mathrm{s}_{\lambda,\mu}(z, S_2(x)) = [1-(r_1+r_2)]z\cos_\mu^D(\sqrt{r_1 m_1}z)\,\mathrm{c}_{\lambda,\mu}(\sqrt{r_2 m_2}z, x)\\ + \sqrt{\frac{r_1}{m_1}}\sin_\mu^D(\sqrt{r_1 m_1}z)\,\mathrm{c}_{\lambda,\mu}(\sqrt{r_2 m_2}z, x)\\ + \sqrt{\frac{r_2}{m_2}}\cos_\mu^D(\sqrt{r_1 m_1}z)\,\mathrm{s}_{\lambda,\mu}(\sqrt{r_2 m_2}z, x).\end{aligned} \tag{20}$$

Furthermore, we have

$$\begin{aligned}\mathrm{s}_{\mu,\lambda}(z, S_2(x)) = -[1-(r_1+r_2)]\sqrt{\frac{m_1 m_2}{r_1 r_2}}\,z\sin_\mu^N(\sqrt{r_1 m_1}z)\,\mathrm{s}_{\mu,\lambda}(\sqrt{r_2 m_2}z, x)\\ \sqrt{\frac{m_1}{r_1}}\sin_\mu^N(\sqrt{r_1 m_1}z)\,\mathrm{c}_{\mu,\lambda}(\sqrt{r_2 m_2}z, x)\\ + \sqrt{\frac{m_2}{r_2}}\cos_\mu^N(\sqrt{r_1 m_1}z)\,\mathrm{s}_{\mu,\lambda}(\sqrt{r_2 m_2}z, x)\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{c}_{\mu,\lambda}(z, S_2(x)) = -[1-(r_1+r_2)]\, z \sqrt{\frac{m_2}{r_2}} \cos^D_\mu(\sqrt{r_1 m_1} z)\, \mathrm{s}_{\mu,\lambda}(\sqrt{r_2 m_2} z, x) \\
\cos^D_\mu(\sqrt{r_1 m_1} z)\, \mathrm{c}_{\mu,\lambda}(\sqrt{r_2 m_2} z, x) \\
- \sqrt{\frac{r_1 m_2}{r_2 m_1}} \sin^D_\mu(\sqrt{r_1 m_1} z)\, \mathrm{s}_{\mu,\lambda}(\sqrt{r_2 m_2} z, x).
\end{aligned}
$$

Proof. By (15) and Proposition 7.6 we get

$$
\begin{aligned}
&\mathrm{c}_{\lambda,\mu}(z, 1 - r_2 + r_2 x) \\
&\quad = \mathrm{c}_{\lambda,\mu}(z, r_1)\, \mathrm{c}_{\lambda,\mu}\left(\sqrt{\frac{r_2 m_2}{r_1 m_1}} z, r_1 x\right) \\
&\qquad - \sqrt{\frac{r_2 m_1}{r_1 m_2}}\, \mathrm{s}_{\mu,\lambda}(z, r_1)\, \mathrm{s}_{\lambda,\mu}\left(\sqrt{\frac{r_2 m_2}{r_1 m_1}} z, r_1 x\right) \\
&\qquad - [1-(r_1+r_2)] z\, \mathrm{s}_{\mu,\lambda}(z, r_1)\, \mathrm{c}_{\lambda,\mu}\left(\sqrt{\frac{r_2 m_2}{r_1 m_1}} z, r_1 x\right) \\
&\quad = \cos^N_\mu(\sqrt{r_1 m_1} z)\, \mathrm{c}_{\lambda,\mu}(\sqrt{r_2 m_2} z, x) \\
&\qquad - \sqrt{\frac{r_2 m_1}{r_1 m_2}} \sin^N_\mu(\sqrt{r_1 m_1} z)\, \mathrm{s}_{\lambda,\mu}(\sqrt{r_2 m_2} z, x) \\
&\qquad - [1-(r_1+r_2)] \sqrt{\frac{m_1}{r_1}} z \sin^N_\mu(\sqrt{r_1 m_1} z)\, \mathrm{c}_{\lambda,\mu}(\sqrt{r_2 m_2} z, x).
\end{aligned}
$$

Analogously, (20) is proved using (16).

The other two equations are obtained by deriving. □

If the functions $\cos^N_\mu$, $\sin^N_\mu$ and $\sin^D_\mu$ are assumed to be known, then equations (17) and (19) allow to compute basically all relevant values of the function $\mathrm{c}_{\lambda,\mu}(z, \cdot)$. If, namely, x is a point in the invariant set K, then there is a sequence $(x_n)_n$ that converges to x and takes only values of the form

$$S_{w_1} \circ S_{w_2} \circ \cdots \circ S_{w_n}(0)$$

or

$$S_{w_1} \circ S_{w_2} \circ \cdots \circ S_{w_n}(1),$$

where $n \in \mathbb{N}$ and $w_1, \ldots w_n \in \{1, 2\}$. For each of these values, (17) and (19) can be applied n times to get a formula containing only values of $\cos^N_\mu$, $\sin^N_\mu$ and $\sin^D_\mu$.

For example

$$
\begin{aligned}
&\mathrm{c}_{\lambda,\mu}(z, S_2(S_1(1))) \\
&\quad = \cos_\mu^N(\sqrt{r_1 m_1} z) \cos_\mu^N(\sqrt{r_2 m_2 r_1 m_1} z) \\
&\qquad - \sqrt{\frac{r_2 m_1}{r_1 m_2}} \sin_\mu^N(\sqrt{r_1 m_1} z) \sin_\mu^D(\sqrt{r_2 m_2 r_1 m_1} z) \\
&\qquad - [1 - (r_1 + r_2)] \sqrt{\frac{m_1}{r_1}} z \sin_\mu^N(\sqrt{r_1 m_1} z) \cos_\mu^N(\sqrt{r_2 m_2 r_1 m_1} z).
\end{aligned}
$$

The same holds for $\mathrm{s}_{\lambda,\mu}$ and formulas (18) and (20). This procedure we will use to compute approximate values of the maxima and to give plots of eigenfunctions in Section 10.

Therefore we are interested in the functions $\sin_\mu^D$, $\sin_\mu^N$, $\cos_\mu^N$, and $\cos_\mu^D$. These have power series representations with coefficients $p_n = p_n(1)$ and $q_n = q_n(1)$. For these numerical sequences we prove a recursion formula in the following.

Proposition 7.8. *For $n \in \mathbb{N}_0$,*

$$
\begin{aligned}
p_{2n+1} &= \sum_{i=0}^{n} r_1^i m_1^{i+1} (r_2 m_2)^{n-i} \, p_{2i+1} \, q_{2n-2i} \\
&\quad + \sum_{i=0}^{n} (r_1 m_1)^i r_2^{n-i} m_2^{n-i+1} \, p_{2i} \, p_{2n-2i+1} \\
&\quad + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i-1} m_2^{n-i} \, p_{2i+1} \, p_{2n-2i-1},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
p_{2n} &= \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} \, p_{2i} \, p_{2n-2i} \\
&\quad + \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i} m_2^{n-i-1} \, p_{2i+1} \, q_{2n-2i-1} \\
&\quad + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i-1} \, p_{2i+1} \, p_{2n-2i-2},
\end{aligned}
\tag{22}
$$

$$q_{2n+1} = \sum_{i=0}^{n} r_1^{i+1} m_1^i (r_2 m_2)^{n-i} q_{2i+1}\, p_{2n-2i} + \sum_{i=0}^{n} (r_1 m_1)^i r_2^{n-i+1} m_2^{n-i} q_{2i}\, q_{2n-2i+1} + [1-(r_1+r_2)] \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i}\, p_{2n-2i}, \tag{23}$$

and

$$q_{2n} = \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i}\, q_{2n-2i} + \sum_{i=0}^{n-1} r_1^{i+1} m_1^i r_2^{n-i-1} m_2^{n-i} q_{2i+1}\, p_{2n-2i-1} + [1-(r_1+r_2)] \sum_{i=0}^{n-1} (r_1 m_1)^i r_2^{n-i-1} m_2^{n-i} q_{2i}\, p_{2n-2i-1}. \tag{24}$$

Remark 7.9. If we take $r_1 = m_1$ and $r_2 = m_2$ (and thus $r_1 + r_2 = 1$ and μ is the Lebesgue measure), the above formulas reduce to $\sum_{i=0}^{n} \binom{n}{i} r_1^i r_2^{n-i} = 1$.

Proof. We put $x = 1$ in Propositions 7.1, 7.3, and 7.5. Then we eliminate all terms of the form $p_n(r_1)$ and $q_n(r_1)$ to obtain formulas that contain only the members of the sequences $(p_n)_n$ and $(q_n)_n$ (as well as r_1, r_2, m_1 and m_2). □

To get the desired recursion formulas, we solve the above formulas for the highest order terms.

Corollary 7.10. *For* $n \in \mathbb{N}$,

$$p_{2n+1} = \frac{1}{1 - r_1^n m_1^{n+1} - r_2^n m_2^{n+1}} \Bigg(\sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i} p_{2i+1}\, q_{2n-2i} + \sum_{i=1}^{n} (r_1 m_1)^i r_2^{n-i} m_2^{n-i+1} p_{2i}\, p_{2n-2i+1} + [1-(r_1+r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i-1} m_2^{n-i} p_{2i+1}\, p_{2n-2i-1} \Bigg), \tag{25}$$

$$p_{2n} = \frac{1}{1-(r_1m_1)^n-(r_2m_2)^n}\Big(\sum_{i=1}^{n-1}(r_1m_1)^i(r_2m_2)^{n-i}\,p_{2i}\,p_{2n-2i}$$
$$+\sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i} m_2^{n-i-1}\,p_{2i+1}\,q_{2n-2i-1} \tag{26}$$
$$+[1-(r_1+r_2)]\sum_{i=0}^{n-1} r_1^i m_1^{i+1}(r_2m_2)^{n-i-1}\,p_{2i+1}\,p_{2n-2i-2}\Big),$$

$$q_{2n+1} = \frac{1}{1-r_1^{n+1}m_1^n-r_2^{n+1}m_2^n}\Big(\sum_{i=0}^{n-1} r_1^{i+1}m_1^i(r_2m_2)^{n-i}q_{2i+1}\,p_{2n-2i}$$
$$+\sum_{i=1}^{n}(r_1m_1)^i r_2^{n-i+1}m_2^{n-i}q_{2i}\,q_{2n-2i+1} \tag{27}$$
$$+[1-(r_1+r_2)]\sum_{i=0}^{n}(r_1m_1)^i(r_2m_2)^{n-i}q_{2i}\,p_{2n-2i}\Big),$$

and

$$q_{2n} = \frac{1}{1-(r_1m_1)^n-(r_2m_2)^n}\Big(\sum_{i=1}^{n-1}(r_1m_1)^i(r_2m_2)^{n-i}q_{2i}\,q_{2n-2i}$$
$$+\sum_{i=0}^{n-1} r_1^{i+1}m_1^i r_2^{n-i-1}m_2^{n-i}q_{2i+1}\,p_{2n-2i-1} \tag{28}$$
$$+[1-(r_1+r_2)]\sum_{i=0}^{n-1}(r_1m_1)^i r_2^{n-i-1}m_2^{n-i}q_{2i}\,p_{2n-2i-1}\Big).$$

Example 7.11. We take $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Then, K is the middle third Cantor set and μ is the normalized $\frac{\log 2}{\log 3}$-dimensional Hausdorff measure restricted to K. We calculate the first members of the sequences $(p_n)_n$ and $(q_n)_n$ using formulas (25) and (27) for p_{2n+1} and q_{2n+1}, which simplify to

$$p_{2n+1} = \frac{1}{2\cdot 6^n-2}\Big(\sum_{i=1}^{2n} p_i\,p_{2n+1-i}+\sum_{i=0}^{n-1}p_{2i+1}\,p_{2n-2i-1}\Big),$$

$$q_{2n+1} = \frac{1}{3\cdot 6^n-2}\Big(\sum_{i=1}^{2n} q_i\,q_{2n+1-i}+\sum_{i=0}^{n}q_{2i}\,q_{2n-2i}\Big).$$

Since μ is symmetric, we can use for p_{2n} and q_{2n} the simpler formula (10)

$$p_{2n} = q_{2n} = \frac{1}{2} \sum_{i=1}^{2n-1} (-1)^{i+1} p_i \, q_{2n-i}$$

from Corollary 6.4. Then,

$$\begin{aligned} p_1 &= 1, & q_1 &= 1, & p_2 &= \frac{1}{2}, \\ p_3 &= \frac{1}{5}, & q_3 &= \frac{1}{8}, & p_4 &= \frac{3}{80}, \\ p_5 &= \frac{27}{2\,800}, & q_5 &= \frac{21}{4\,240}, & p_6 &= \frac{311}{296\,800}, \\ p_7 &= \frac{6\,383}{31\,906\,000}, & q_7 &= \frac{33\,253}{383\,465\,600}, & p_8 &= \frac{4\,716\,349}{329\,780\,416\,000}, \end{aligned}$$

and therefore

$$\begin{aligned} \sin_\mu^N(z) &= z - \frac{6}{5}\frac{z^3}{3!} + \frac{81}{70}\frac{z^5}{5!} - \frac{57\,447}{56\,975}\frac{z^7}{7!} + \cdots, \\ \sin_\mu^D(z) &= z - \frac{3}{4}\frac{z^3}{3!} + \frac{63}{106}\frac{z^5}{5!} - \frac{299\,277}{684\,760}\frac{z^7}{7!} + \cdots 2 \end{aligned}$$

and

$$\cos_\mu^N(z) = \cos_\mu^D(z) = 1 - \frac{z^2}{2!} + \frac{9}{10}\frac{z^4}{4!} - \frac{2\,799}{3\,710}\frac{z^6}{6!} + \frac{42\,447\,141}{73\,611\,700}\frac{z^8}{8!} - \cdots.$$

Plots of $\sin_\mu^N$ and $\sin_\mu^D$ as well as further examples can be found in Section 10.

The functions $\sin_\mu^N$, $\sin_\mu^D$, $\cos_\mu^N$ and $\cos_\mu^D$ can be characterized by the following system of functional equations.

Theorem 7.12. *For $z \in \mathbb{R}$ we have*

$$\begin{aligned} \sin_\mu^N(z) = {} & \sqrt{\frac{m_1}{r_1}} \sin_\mu^N(\sqrt{r_1 m_1} z) \cos_\mu^D(\sqrt{r_2 m_2} z) \\ & + \sqrt{\frac{m_2}{r_2}} \cos_\mu^N(\sqrt{r_1 m_1} z) \sin_\mu^N(\sqrt{r_2 m_2} z) \qquad (29) \\ & - [1 - (r_1 + r_2)] \sqrt{\frac{m_1 m_2}{r_1 r_2}} \, z \sin_\mu^N(\sqrt{r_1 m_1} z) \sin_\mu^N(\sqrt{r_2 m_2} z), \end{aligned}$$

$$\begin{aligned}\sin^D_\mu(z) = {} & \sqrt{\frac{r_1}{m_1}}\sin^D_\mu(\sqrt{r_1m_1}z)\cos^N_\mu(\sqrt{r_2m_2}z)\\ & + \sqrt{\frac{r_2}{m_2}}\cos^D_\mu(\sqrt{r_1m_1}z)\sin^D_\mu(\sqrt{r_2m_2}z)\\ & + [1-(r_1+r_2)]\,z\cos^D_\mu(\sqrt{r_1m_1}z)\cos^N_\mu(\sqrt{r_2m_2}z),\end{aligned} \tag{30}$$

$$\begin{aligned}\cos^N_\mu(z) = {} & \cos^N_\mu(\sqrt{r_1m_1}z)\cos^N_\mu(\sqrt{r_2m_2}z)\\ & - \sqrt{\frac{r_2m_1}{r_1m_2}}\sin^N_\mu(\sqrt{r_1m_1}z)\sin^D_\mu(\sqrt{r_2m_2}z)\\ & - [1-(r_1+r_2)]\sqrt{\frac{m_1}{r_1}}z\sin^N_\mu(\sqrt{r_1m_1}z)\cos^N_\mu(\sqrt{r_2m_2}z),\end{aligned} \tag{31}$$

and

$$\begin{aligned}\cos^D_\mu(z) = {} & \cos^D_\mu(\sqrt{r_1m_1}z)\cos^D_\mu(\sqrt{r_2m_2}z)\\ & - \sqrt{\frac{r_1m_2}{r_2m_1}}\sin^D_\mu(\sqrt{r_1m_1}z)\sin^N_\mu(\sqrt{r_2m_2}z)\\ & - [1-(r_1+r_2)]\sqrt{\frac{m_2}{r_2}}z\cos^D_\mu(\sqrt{r_1m_1}z)\sin^N_\mu(\sqrt{r_2m_2}z).\end{aligned} \tag{32}$$

Furthermore, the functions $\sin^N_\mu$, $\sin^D_\mu$, $\cos^N_\mu$ *and* $\cos^D_\mu$ *are the only analytic functions that solve the above system of functional equations and satisfy the conditions that* $\sin^N_\mu$ *and* $\sin^D_\mu$ *are odd,* $\cos^N_\mu$ *and* $\cos^D_\mu$ *are even, and*

$$\lim_{z\to 0}\frac{\sin^N_\mu(z)}{z} = \lim_{z\to 0}\frac{\sin^D_\mu(z)}{z} = 1$$

and

$$\cos^N_\mu(0) = \cos^D_\mu(0) = 1.$$

Remark 7.13. If we would know all the values of all four functions on a given interval, say, $[0,a]$, then, using the formulas above, we could calculate all values of all four functions on $[0,(\max_i\sqrt{r_im_i})^{-1}a]$. Then, iteratively, we get the values on $[0,(\max_i\sqrt{r_im_i})^{-2}a]$ and so on. So, the functions are determined on $[0,\infty)$ by their values on an arbitrary small interval $[0,a]$.

Furthermore, the theorem describes a kind of "self-similarity" of our four functions.

Proof. To show that $\sin_\mu^N$, $\sin_\mu^D$, $\cos_\mu^N$ and $\cos_\mu^D$ satisfy the equations, put $x = 1$ in Proposition 7.7.

Suppose that f_1, f_2, g_1 and g_2 are real analytic functions that satisfy the above equations, and that f_1, f_2 are odd, g_1, g_2 are even, $\lim_{z\to 0} \frac{f_1(z)}{z} = \lim_{z\to 0} \frac{f_2(z)}{z} = 1$, and $g_1(0) = g_2(0) = 1$. Then, power series representations exist, that is, there are real sequences (a_n), (b_n), (c_n) and (d_n) such that, for all $z \in \mathbb{R}$,

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^{2n+1}, \quad f_2(z) = \sum_{n=0}^{\infty} b_n z^{2n+1},$$

$$g_1(z) = \sum_{n=0}^{\infty} c_n z^{2n}, \qquad g_2(z) = \sum_{n=0}^{\infty} d_n z^{2n},$$

where $a_0 = b_0 = c_0 = d_0 = 1$. Since these functions satisfy (29), we get for all $z \in \mathbb{R}$

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n z^{2n+1} \\
&= \sqrt{\frac{m_1}{r_1}} \sum_{n=0}^{\infty} z^{2n+1} \sum_{k=0}^{n} a_k \sqrt{r_1 m_1}^{2k+1} d_{n-k} \sqrt{r_2 m_2}^{2n-2k} \\
&\quad + \sqrt{\frac{m_2}{r_2}} \sum_{n=0}^{\infty} z^{2n+1} \sum_{k=0}^{n} c_k \sqrt{r_1 m_1}^{2k} a_{n-k} \sqrt{r_2 m_2}^{2n+1-2k} \\
&\quad - [1 - (r_1 + r_2)] \sqrt{\frac{m_1 m_2}{r_1 r_2}} \sum_{n=0}^{\infty} z^{2n+3} \sum_{k=0}^{n} a_k \sqrt{r_1 m_1}^{2k+1} a_{n-k} \sqrt{r_2 m_2}^{2n+1-2k}.
\end{aligned}$$

If we derive this equation $2j + 1$ times and put $z = 0$, we receive formula (21) for a_j. Analogously, one can show that b_j satisfies (23), c_j satisfies (22) and d_j satisfies (24). Together with the initial condition $a_0 = b_0 = c_0 = d_0 = 1$ it follows that $a_j = p_{2j+1}$, $b_j = q_{2j+1}$, $c_j = p_{2j}$ and $d_j = q_{2j}$ for all $j \in \mathbb{N}$. Thus, $f_1 = \sin_\mu^N$, $f_2 = \sin_\mu^D$, $g_1 = \cos_\mu^N$ and $g_2 = \cos_\mu^D$. □

Example 7.14. (i) If we take $r_1 = m_1$ and $r_2 = m_2$ and $r_1 + r_2 = 1$, then K is the unit interval and μ the Lebesgue measure. The functions $\sin_\mu^N$, $\sin_\mu^D$, $\cos_\mu^N$ and $\cos_\mu^D$ equal the usual sine and cosine functions, and the formulas in Theorem 7.12 simplify to

$$\sin(z) = \sin(r_1 z + r_2 z) = \sin(r_1 z)\cos(r_2 z) + \cos(r_1 z)\sin(r_2 z),$$

$$\cos(z) = \cos(r_1 z + r_2 z) = \cos(r_1 z)\cos(r_2 z) - \sin(r_1 z)\sin(r_2 z).$$

(ii) Let $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Then μ is the Cantor measure and the formulas in Theorem 7.12 can be rewritten as

$$\sin_\mu^N(\sqrt{6}z) = \frac{\sqrt{6}}{2}\sin_\mu^N(z)(2\cos_\mu^N(z) - z\sin_\mu^N(z)),$$

$$\sin_\mu^D(\sqrt{6}z) = \frac{\sqrt{6}}{3}\cos_\mu^N(z)(2\sin_\mu^D(z) + z\cos_\mu^N(z)),$$

$$\cos_\mu^N(\sqrt{6}z) = \cos_\mu^N(z)^2 - \sin_\mu^N(z)\sin_\mu^D(z) - z\cos_\mu^N(z)\sin_\mu^N(z).$$

Since K is symmetric, $\cos_\mu^N = \cos_\mu^D$.

Observe that Theorem 7.12 in combination with the recursive rules in Corollary 7.10 supply a technique for investigation of further properties of the eigenvalues. On a given interval $[0, a]$ we can approximate the functions $\sin_\mu^N$, $\sin_\mu^D$, $\cos_\mu^N$ and $\cos_\mu^D$ arbitrarily exact by polynomials consisting of sufficiently many members of the corresponding power series. Then, by Theorem 7.12, we can extend all four functions successively to larger intervals.

8. Self-similar measures with $r_1 m_1 = r_2 m_2$

In this section we suppose μ is a self-similar measure as in the last section but with parameters additionally satisfying $r_1 m_1 = r_2 m_2$. This case is particularly interesting because there we have the following property.

Theorem 8.1. *Let $r_1 m_1 = r_2 m_2$. If λ is the mth Neumann eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$, then $\frac{1}{r_1 m_1}\lambda$ is the $2m$th Neumann eigenvalue, that is, for all $m \in \mathbb{N}$,*

$$r_1 m_1 \lambda_{N,2m} = \lambda_{N,m}.$$

This Theorem has been proved with the method of Prüfer angles by Volkmer [31] for the case $r_1 = r_2 = \frac{1}{3}$, $m_1 = m_2 = \frac{1}{2}$ and by Freiberg [11] in a more general setting. It delivers the foundation for the statements in this section. An analogous property for Dirichlet eigenvalues does not seem to hold. However, in the symmetric case there is a similar relation between Dirichlet eigenvalues and eigenvalues of the problems (DN) or (ND) posed in Section 3.6. Remember, (DN) has boundary conditions $f(0) = f'(1) = 0$ and (ND) has $f'(0) = f(1) = 0$.

Proposition 8.2. *Let μ be symmetric, that is $r := r_1 = r_2$ and $m_1 = m_2 = \frac{1}{2}$ and let λ be an eigenvalue of (DN) or (ND). Then $\frac{2}{r}\lambda$ is a Dirichlet eigenvalue and if f is a $\frac{2}{r}\lambda$-Dirichlet eigenfunction, then $f \circ S_1$ is a λ-(DN) eigenfunction, and $f \circ S_2$ is a λ-(ND) eigenfunction.*

Proof. In Corollary 6.4 we showed that since μ is symmetric, we have $\cos_\mu^N = \cos_\mu^D$. Then we can factorize (30) and get

$$\sin_\mu^D(\sqrt{\tfrac{2}{r}}z) = \cos_\mu^N(z) \cdot [2\sqrt{2r}\sin_\mu^D(z) + (1-2r)\, z \cos_\mu^N(z)].$$

Since λ is an eigenvalue of the (DN) and the (ND) problem, $\cos_\mu^N(\sqrt{\lambda}) = 0$. Then, $\sin_\mu^D\left(\sqrt{\tfrac{2}{r}\lambda}\right) = 0$ and thus, $\tfrac{2}{r}\lambda$ is a Dirichlet eigenvalue. From Propositions 7.6 and 7.7 we get for $x \in [0,1]$ that

$$\mathrm{s}_{\lambda,\mu}\left(\sqrt{\frac{2}{r}\lambda}, S_1(x)\right) = \sqrt{2r}\,\mathrm{s}_{\lambda,\mu}(\sqrt{\lambda}, x)$$

and

$$\mathrm{s}_{\lambda,\mu}\left(\sqrt{\frac{2}{r}\lambda}, S_2(x)\right) = \sqrt{2r}\sin_\mu^D(\sqrt{\lambda})\,\mathrm{c}_{\lambda,\mu}(\sqrt{\lambda}, x),$$

which proves the proposition. □

In the following we treat only the Neumann eigenvalue problem for a (not necessarily symmetric) measure μ using Theorem 8.1. With the formula

$$\cos_\mu^D(z)\,\cos_\mu^N(z) + \sin_\mu^D(z)\,\sin_\mu^N(z) = 1, \tag{33}$$

which follows from Theorem 5.3 by setting $x = 1$, we rearrange the functional equations from Theorem 7.12. With the abbreviation

$$h(z) := r_1\cos_\mu^N(z) + r_2\cos_\mu^D(z) - [1-(r_1+r_2)]z\sin_\mu^N(z) \tag{34}$$

we can write

$$\sin_\mu^N(z) = \frac{\sqrt{r_1 m_1}}{r_1 r_2}\sin_\mu^N(\sqrt{r_1 m_1}z)\,h(\sqrt{r_1 m_1}z), \tag{35}$$

$$\cos_\mu^N(z) = -\frac{r_2}{r_1} + \frac{1}{r_1}\cos_\mu^N(\sqrt{r_1 m_1}z)\,h(\sqrt{r_1 m_1}z), \tag{36}$$

$$\sin_\mu^D(z) = [1-(r_1+r_2)]z + \frac{1}{\sqrt{r_1 m_1}}\sin_\mu^D(\sqrt{r_1 m_1}z)\,h(\sqrt{r_1 m_1}z), \tag{37}$$

$$\cos_\mu^D(z) = -\frac{r_1}{r_2} + \frac{1}{r_2}\cos_\mu^D(\sqrt{r_1 m_1}z)\,h(\sqrt{r_1 m_1}z). \tag{38}$$

Employing the above formulas we can calculate the values of $\cos_\mu^N$, $\cos_\mu^D$ and $\sin_\mu^D$ at the zero points of $\sin_\mu^N$.

Lemma 8.3. *Let $m \in \mathbb{N}$ and let $v(m)$ be the multiplicity of the prime factor 2 in m. Let $z_m := \sqrt{\lambda_{N,m}}$ be the square root of the mth Neumann eigenvalue, that is, the mth zero point of $\sin_\mu^N$. Then*

$$\cos_\mu^N(z_m) = \Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)}}, \tag{39}$$

$$\cos_\mu^D(z_m) = \Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)}}, \tag{40}$$

$$\sin_\mu^D(z_m) = a_{v(m)} \cdot z_m \tag{41}$$

where $(a_k)_k$ is determined by $a_0 = 1-(r_1+r_2)$ and, for $k \in \mathbb{N}$,

$$a_k = 1-(r_1+r_2) + a_{k-1}\Big(r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{k-1}} + r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{k-1}}\Big).$$

Proof. Suppose m is odd. Then $\sin_\mu^N(z_m) = 0$ and $\sin_\mu^N(\sqrt{r_1 m_1} z_m) \neq 0$. To see this, suppose $\sin_\mu^N(\sqrt{r_1 m_1} z_m) = 0$. Then $r_1 m_1 z_m^2$ would be a Neumann eigenvalue, say $r_1 m_1 z_m^2 = \lambda_{N,l}$ for some $l \in \mathbb{N}$, and because of Theorem 8.1, z_m^2 would be the eigenvalue $\lambda_{N,2l}$. Thus, $m = 2l$, which is a contradiction. Hence, it follows by (35) that $h(\sqrt{r_1 m_1} z_m) = 0$. Then, by (36), $\cos_\mu^N(z_m) = -\frac{r_2}{r_1}$.

By (31) follows that, for all $z \in \mathbb{R}$, if $\sin_\mu^N\big(\sqrt{r_1 m_1} z\big) = 0$, then $\cos_\mu^N(z) = \cos_\mu^N(\sqrt{r_1 m_1} z)^2$. Thus, if $m = 2l$ for some odd l, then $\sqrt{r_1 m_1} z_m = z_l$ and hence, $\cos_\mu^N(z_m) = \cos_\mu^N(z_l)^2 = (-\frac{r_2}{r_1})^2$. Iteratively, we get that, if $m = 2^k l$ for some odd l then $\cos_\mu^N(z_m) = \big(-\frac{r_2}{r_1}\big)^{2^k}$, which proves (39). Since $\sin_\mu^N(z_m) = 0$ for all $m \in \mathbb{N}$ we get by (33) that

$$\cos_\mu^D(z_m) = \cos_\mu^N(z_m)^{-1}$$

which implies (40).

Now we show (41). At first, suppose $v(m) = 0$, that is, m is odd. Then, as above, $h(\sqrt{r_1 m_1} z_m) = 0$ and thus, by (37), $\sin_\mu^D(z_m) = [1-(r_1+r_2)]z_m$. Observe that we have for all m

$$h(z_m) = r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)}} + r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)}}. \tag{42}$$

Suppose $v(m) \geq 1$. Then

$$\sqrt{r_1 m_1} z_m = z_{\frac{m}{2}}$$

and thus,

$$\begin{aligned}\frac{\sin_\mu^D(z_m)}{z_m} &= 1-(r_1+r_2)+\frac{\sin_\mu^D(\sqrt{r_1m_1}z_m)}{\sqrt{r_1m_1}z_m}h(\sqrt{r_1m_1}z_m)\\ &= 1-(r_1+r_2)+\frac{\sin_\mu^D(z_{\frac{m}{2}})}{z_{\frac{m}{2}}}h(z_{\frac{m}{2}})\\ &= 1-(r_1+r_2)+\frac{\sin_\mu^D(z_{\frac{m}{2}})}{z_{\frac{m}{2}}}\Big(r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)-1}}+r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)-1}}\Big).\end{aligned}$$

Hence, $\frac{\sin_\mu^D(z_m)}{z_m}$ depends only on $v(m)$ and so, with

$$a_{v(m)} = \frac{\sin_\mu^D(z_m)}{z_m},$$

we get

$$a_{v(m)} = 1-(r_1+r_2)+a_{v(m)-1}\Big(r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)-1}}+r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)-1}}\Big),$$

which proves the assertion. □

We use the above computed values of $\cos_\mu^N(z_m)$ and Propositions 7.6 and 7.7 to get a relation between the mth and the $2m$th Neumann eigenfunction.

Proposition 8.4. *Let $m \in \mathbb{N}$ and $v(m)$ be the 2-multiplicity of m. We denote the mth Neumann eigenfunction by $f_m := \mathrm{c}_{\lambda,\mu}(z_m,\cdot)$. Then, for all $x \in [0,1]$,*

$$f_{2m}\big(S_1(x)\big) = f_m(x) \tag{43}$$

and

$$f_{2m}\big(S_2(x)\big) = \Big(-\frac{m_1}{m_2}\Big)^{2^{v(m)}} f_m(x). \tag{44}$$

Proof. Because of Theorem 8.1 we have $\lambda_m = r_1m_1\lambda_{2m}$ and thus,

$$\sin_\mu^N\big(\sqrt{r_1m_1}z_{2m}\big) = 0.$$

Since $f_m = \mathrm{c}_{\lambda,\mu}(z_m,\cdot)$, Propositions 7.6 and 7.7 give $f_{2m}\big(S_1(x)\big) = f_m(x)$ and $f_{2m}\big(S_2(x)\big) = \cos_\mu^N(z_m)\, f_m(x)$ for $x \in [0,1]$. Noting that $\frac{r_2}{r_1} = \frac{m_1}{m_2}$, (44) follows with (39). □

The above proposition can be employed to work out the relationship between the suprema and the $L_2(\mu)$ norms of f_m and f_{2m}.

Proposition 8.5. *Let $m \in \mathbb{N}$ and $v(m)$ the 2-multiplicity of m. Then*

$$\|f_{2m}\|^2_{L_2(\mu)} = \Big(m_1 + m_2\Big(\frac{m_1}{m_2}\Big)^{2^{v(m)+1}}\Big)\|f_m\|^2_{L_2(\mu)} \tag{45}$$

and

$$\|f_{2m}\|_\infty = \max\Big\{1, \Big(\frac{m_1}{m_2}\Big)^{2^{v(m)}}\Big\}\|f_m\|_\infty. \tag{46}$$

Proof. At first we prove (45). For $m \in \mathbb{N}$ we have

$$\begin{aligned}\|f_{2m}\|^2_{L_2(\mu)} &= \int_{S_1(0)}^{S_1(1)} f_{2m}(t)^2\,d\mu(t) + \int_{S_2(0)}^{S_2(1)} f_{2m}(t)^2\,d\mu(t)\\ &= m_1\int_{S_1(0)}^{S_1(1)} f_{2m}(t)^2\,d(S_1\mu)(t) + m_2\int_{S_2(0)}^{S_2(1)} f_{2m}(t)^2\,d(S_2\mu)(t)\\ &= m_1\int_0^1 f_{2m}\big(S_1(t)\big)^2\,d\mu(t) + m_2\int_0^1 f_{2m}\big(S_2(t)\big)^2\,d\mu(t).\end{aligned}$$

By (43) and (44) we get

$$\begin{aligned}\|f_{2m}\|^2_{L_2(\mu)} &= m_1\int_0^1 f_m(t)^2\,d\mu(t) + m_2\Big(-\frac{m_1}{m_2}\Big)^{2^{v(m)+1}}\int_0^1 f_m(t)^2\,d\mu(t)\\ &= \Big[m_1 + m_2\Big(\frac{m_1}{m_2}\Big)^{2^{v(m)+1}}\Big]\|f_m\|^2_{L_2(\mu)}.\end{aligned}$$

Now we show (46). With (43) and (44) we have

$$\sup_{x\in[S_1(0),S_1(1)]}|f_{2m}(x)| = \sup_{x\in[0,1]}\big|f_{2m}\big(S_1(x)\big)\big| = \sup_{x\in[0,1]}|f_m(x)| = \|f_m\|_\infty$$

and

$$\sup_{x\in[S_2(0),S_2(1)]}|f_{2m}(x)| = \sup_{x\in[0,1]}\big|f_{2m}\big(S_2(x)\big)\big| = \Big(\frac{m_1}{m_2}\Big)^{2^{v(m)}}\|f_m\|_\infty.$$

Therefore, since f_{2m} is linear on $[S_1(1), S_2(0)]$ and continuous,

$$\sup_{x\in[0,1]}|f_{2m}(x)| = \max\Big\{1, \Big(\frac{m_1}{m_2}\Big)^{2^{v(m)}}\Big\}\|f_m\|_\infty. \qquad \square$$

Now we consider the normalized Neumann eigenfunctions. For $m \in \mathbb{N}_0$ we set $\tilde{f}_m := \|f_m\|^{-1}_{L_2(\mu)} f_m$. We are interested in the asymptotic behaviour of the sequence $(\|\tilde{f}_m\|_\infty)_m$. With Proposition 8.5 we get some information about certain subsequences stated in the following theorem.

Theorem 8.6. *Let μ be a self-similar measure with $r_1 m_1 = r_2 m_2$. Then, for all $m \in \mathbb{N}_0$,*

$$\|\tilde{f}_{2m}\|_\infty = \frac{\max\left\{1, \left(\frac{m_1}{m_2}\right)^{2^{v(m)}}\right\}}{\sqrt{m_1 + m_2\left(\frac{m_1}{m_2}\right)^{2^{v(m)+1}}}} \|\tilde{f}_m\|_\infty. \tag{47}$$

Suppose $m_1 \leq m_2$ and let l be an odd number. Then, for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^k l}\|_\infty = m_1^{-\frac{k}{2}} \prod_{j=1}^{k} \left(1 + \left(\frac{m_1}{m_2}\right)^{2^j - 1}\right)^{-\frac{1}{2}} \|\tilde{f}_l\|_\infty. \tag{48}$$

Proof. (47) follows directly from (45) and (46). Suppose $m_1 \leq m_2$ and $l \in \mathbb{N}$ is odd. Then iterative application of (47) gives (48). □

Corollary 8.7. *Let $l \in \mathbb{N}$ be odd. Then the following statements hold.*

(i) *If $m_1 = m_2$, then for all $k \in \mathbb{N}$, $\|\tilde{f}_{2^k l}\|_\infty = \|\tilde{f}_l\|_\infty$.*

(ii) *If $m_1 < m_2$, then*

$$C := \left(m_1\left(1 + \frac{m_1}{m_2}\right)\right)^{-1/2} > 1,$$

and we have for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^k l}\|_\infty \geq C^k \|\tilde{f}_l\|_\infty.$$

Additionally, for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^k l}\|_\infty \leq m_1^{-\frac{k}{2}} \left(\frac{m_2}{m_1}\right)^{\frac{k}{2}(2^k - 1)} \|\tilde{f}_l\|_\infty.$$

Proof. (i) follows directly from (48) by putting $m_1 = m_2 = \frac{1}{2}$.

If $m_1 < m_2$, then, for all $j \in \mathbb{N}$, $1 + \left(\frac{m_1}{m_2}\right)^{2^j - 1} \leq 1 + \frac{m_1}{m_2}$. Then,

$$\|\tilde{f}_{2^k l}\|_\infty \geq m_1^{-\frac{k}{2}} \left(1 + \frac{m_1}{m_2}\right)^{-\frac{k}{2}} \|\tilde{f}_l\|_\infty,$$

and since $m_1 < m_2$ implies $m_1 < \frac{1}{2}$, we have $m_1\left(1 + \frac{m_1}{m_2}\right) < 1$. For the upper estimate, we write

$$\prod_{j=1}^{k} \left(1 + \left(\frac{m_1}{m_2}\right)^{2^j - 1}\right) \geq \left(1 + \left(\frac{m_1}{m_2}\right)^{2^k - 1}\right)^k \geq \left(\frac{m_1}{m_2}\right)^{k(2^k - 1)},$$

which proves (ii). □

9. Self-similar measures with $r_1 m_1 = r_2 m_2$ and $r_1 + r_2 = 1$

As in the previous section we have the condition $r_1 m_1 = r_2 m_2$. We treat the special case where $r_1 + r_2 = 1$ from which follows that $r_1 = m_2$ and $r_2 = m_1$. Such measures have been investigated e.g. by Sabot [26] and [27].

Theorem 9.1. *Let μ be a self-similar measure where $r_1 = m_2$ and $r_2 = m_1$ (and therefore $r_1 + r_2 = 1$). Then the positive eigenvalues of $-\frac{d}{d\mu}\frac{d}{x}$ with Neumann boundary conditions coincide with those with Dirichlet boundary conditions.*

Proof. Since the eigenvalues are the squares of the zeros of $\sin_\mu^N$ and $\sin_\mu^D$, respectively, it is sufficient to show that $\sin_\mu^N = \sin_\mu^D$. To do that we show that for all $n \in \mathbb{N}_0$ $p_{2n+1} = q_{2n+1}$. We do this by complete induction using the recursion formulas from Corollary 7.10. By Definition 3.1 we have

$$p_1 = \int_0^1 d\mu = 1 \quad \text{and} \quad q_1 = \int_0^1 dt = 1.$$

Now, let $n \in \mathbb{N}$ and suppose that, for $i = 0, \ldots, n-1$, $p_{2i+1} = q_{2i+1}$. By (25) and rearrangement of the order of the terms in the sums we get

$$\begin{aligned} p_{2n+1} &= \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \Bigg(\sum_{i=0}^{n-1} m_2^i r_2^{i+1} (r_1 m_1)^{n-i} p_{2i+1}\, q_{2n-2i} \\ &\qquad + \sum_{i=1}^{n} (r_2 m_2)^i m_1^{n-i} r_1^{n-i+1} p_{2i}\, p_{2n-2i+1} \Bigg) \\ &= \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \Bigg(\sum_{i=1}^{n} m_2^{n-i} r_2^{n+1-i} (r_1 m_1)^i p_{2n+1-2i}\, q_{2i} \\ &\qquad + \sum_{i=0}^{n-1} (r_2 m_2)^{n-i} m_1^i r_1^{i+1} p_{2n-2i}\, p_{2i+1} \Bigg). \end{aligned}$$

Then, by the induction hypothesis and (27),

$$\begin{aligned} p_{2n+1} &= \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \Bigg(\sum_{i=1}^{n} m_2^{n-i} r_2^{n+1-i} (r_1 m_1)^i q_{2n+1-2i}\, q_{2i} \\ &\qquad + \sum_{i=0}^{n-1} (r_2 m_2)^{n-i} m_1^i r_1^{i+1} p_{2n-2i}\, q_{2i+1} \Bigg) \\ &= q_{2n+1}. \end{aligned}$$

□

With the above theorem we can reformulate Theorem 5.3 to get a property of the Wronskian of $f_{N,m}$ and $f_{D,m}$.

Corollary 9.2. *Let μ be as above, let λ_m be the mth eigenvalue, let*

$$f_{N,m} = \mathrm{c}_{\lambda,\mu}(\sqrt{\lambda_m},\cdot) \quad \textit{and} \quad f_{D,m} = \mathrm{s}_{\lambda,\mu}(\sqrt{\lambda_m},\cdot)$$

be the corresponding Neumann and Dirichlet eigenfunctions constructed in Section 3.6. Then, for all $x \in [0,1]$,

$$f_{N,m}(x)\, f'_{D,m}(x) - f_{D,m}(x)\, f'_{N,m}(x) = \sqrt{\lambda_m}.$$

Proof. We put $z = \sqrt{\lambda_m}$ in Theorem 5.3 and observe that

$$f'_{N,m} = \mathrm{c}'_{\lambda,\mu}(\sqrt{\lambda_m},\cdot) = -\sqrt{\lambda_m}\,\mathrm{s}_{\mu,\lambda}(\sqrt{\lambda_m},\cdot)$$

and

$$f'_{D,m} = \mathrm{s}'_{\lambda,\mu}(\sqrt{\lambda_m},\cdot) = \sqrt{\lambda_m}\,\mathrm{c}_{\mu,\lambda}(\sqrt{\lambda_m},\cdot). \qquad \square$$

Since eigenfunctions can be multiplied with any non-zero number, the above equation states basically that the Wronskian is constant. A similar property of a different Wronskian has been established in Freiberg [8], p. 41.

10. Figures and numbers

In this section we give some explicit results and figures calculated by using formulas we developed in the preceding sections for several examples of self-similar measures. For the calculations we used Sagemath cloud [29]. The program code that we used can be found in the appendix of the longer version of this article published on arXiv [1]. Also in this longer version more examples are recorded.

Example 10.1. Table 1 collects the first few values of the sequences $(p_n)_n$ and $(q_n)_n$ for the classical Cantor set with evenly distributed measure, that is, for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. We computed these values with the recursion formulas in Corollary 7.10 that we implemented for that purpose in Sagemath.

Table 1. The first members of (p_n) and (q_n) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

n	p_{2n+1}
1	$\frac{1}{5}$
2	$\frac{27}{2800}$
3	$\frac{6383}{31906000}$
4	$\frac{928\,046\,087}{427\,065\,638\,720\,000}$
5	$\frac{18\,312\,146\,532\,699}{1\,290\,321\,173\,531\,252\,800\,000}$
n	q_{2n+1}
1	$\frac{1}{8}$
2	$\frac{21}{4240}$
3	$\frac{33253}{383465600}$
4	$\frac{76118969}{91537621184000}$
5	$\frac{20165083798890939}{41033972469990228915200000}$
n	p_{2n}, q_{2n}
1	$\frac{1}{2}$
2	$\frac{3}{80}$
3	$\frac{311}{296800}$
4	$\frac{4716349}{329780416000}$
5	$\frac{186511983201}{1659577072065920000}$

Figure 1 shows a plot of the functions $\sin_\mu^N$ and $\sin_\mu^D$ for $x \in (0, 50)$, where the first 100 terms of the series are taken into account. The zero points of these functions squared give the Dirichlet and Neumann eigenvalues, respectively. Observe that the pictures suggest that the eigenvalues are in the order

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{D,1} < \lambda_{D,2} < \lambda_{N,2} < \lambda_{N,3} < \lambda_{D,3} < \lambda_{D,4} < \cdots .$$

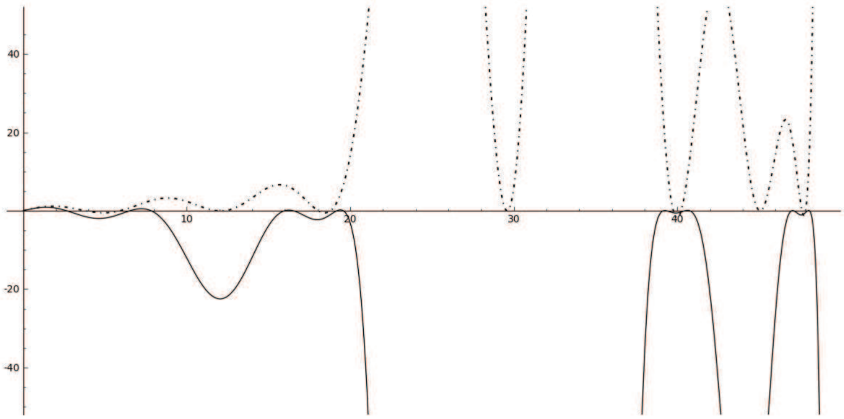


Figure 1. $\sin_\mu^N$ (solid) and $\sin_\mu^D$ (dash-dot) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

Table 2 contains the first 16 positive Neumann eigenvalues correct to 5 decimal places (rounded down).

Table 2. Neumann eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\lambda_{N,m}$	m	$\lambda_{N,m}$
1	7.09743	9	1548.05582
2	42.58458	10	1637.90142
3	61.34420	11	1662.62743
4	255.50751	12	2208.39134
5	272.98357	13	2220.76944
6	368.06522	14	2301.31729
7	383.55288	15	2312.58212
8	1533.04511	16	9198.27070

These values have been calculated as zero points of the polynomial

$$\sum_{n=0}^{a} (-1)^n \, p_{2n+1} \, z^n \, \left(\approx \frac{\sin_\mu^N(\sqrt{z})}{\sqrt{z}} \right)$$

where a is sufficiently large. Note that by Lemma 3.3 we have

$$p_{2n+1} \leq \frac{1}{n!} q_2(1)^n = \frac{1}{n! \cdot 2^n},$$

from which a more detailed error estimate can be obtained. More decimals and more eigenvalues can be found in [1].

Observe that, as stated in Theorem 8.1, we have that $\lambda_{N,2m} = 6 \cdot \lambda_{N,m}$ for all m.

In Table 3 we give approximate values of the sup norms of the normalized eigenfunctions

$$\|\tilde{f}_{N,m}\|_\infty = \frac{\|f_{N,m}\|_\infty}{\|f_{N,m}\|_{L_2(\mu)}}.$$

For that, the L_2 norms have been calculated with the formula in Corollary 4.3 where we put in the values for λ from Table 2. The number of summands had to be chosen higher with bigger eigenvalues, so that the limit value could be approximated with sufficient accuracy. For the supremum norms we calculated

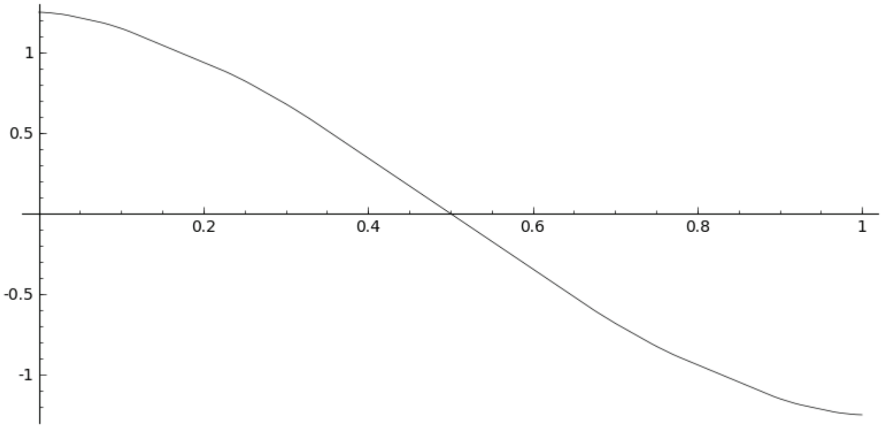
Table 3. Norms of Neumann eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

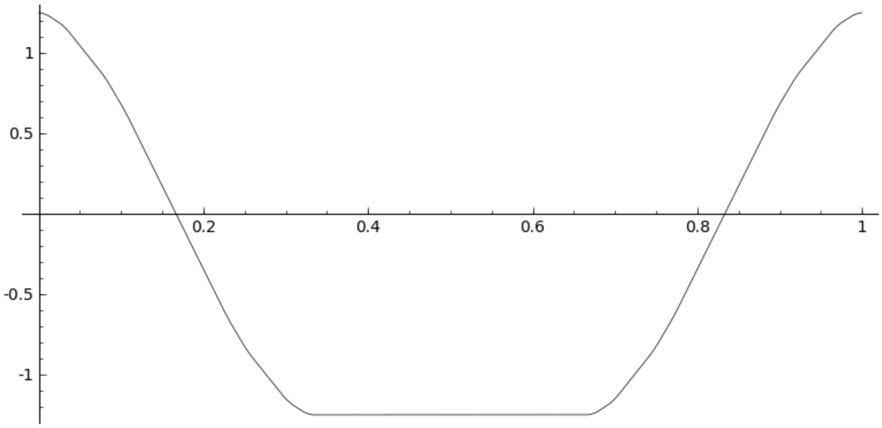
m	$\|\tilde{f}_{N,m}\|_\infty$	m	$\|\tilde{f}_{N,m}\|_\infty$
1	1.248	9	1.467
2	1.248	10	1.405
3	1.306	11	1.512
4	1.248	12	1.306
5	1.405	13	1.474
6	1.306	14	1.401
7	1.401	15	1.508
8	1.248	16	1.248

$f_{N,m}(S_w(0))$ and $f_{N,m}(S_w(1))$ for all words $w \in \{1,2\}^n$ for a certain iteration level n and determined the biggest of these values. We varied n between 5 and 8 to get the values. These calculations were made with the formulas in Proposition 7.7. For that, the eigenvalue λ_m and values of the functions $\sin_\mu^N$, $\sin_\mu^D$ and $\cos_\mu^N$ were needed.

Observe that, as stated in Equation (47), the values for even m are the same as for $\frac{m}{2}$, respectively.

Figure 2 shows plots of $f_{1,N}$ to $f_{3,N}$ and $f_{1,D}$ to $f_{3,D}$. These were done by iterative use of the formulas in Propositions 7.6 and 7.7 as for the calculation of the sup-norms.


1
0.5
-0.5
-1
0.2
0.4
0.6
0.8
1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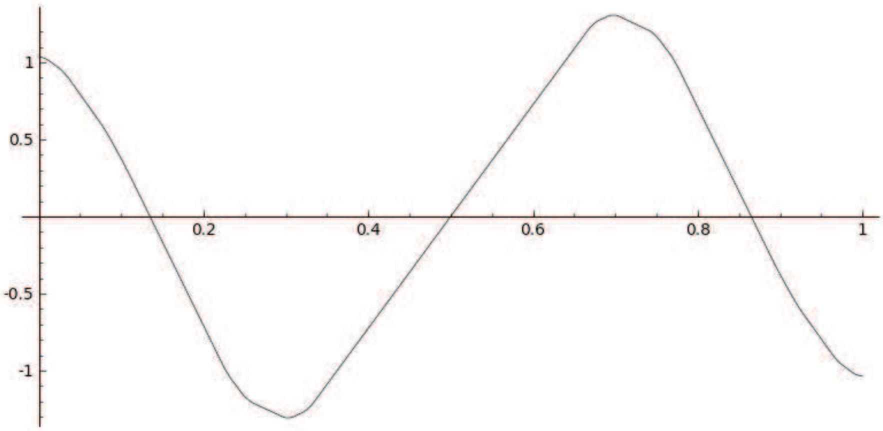
1
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-0.5
-1
0.2
0.4
0.6
0.8
1

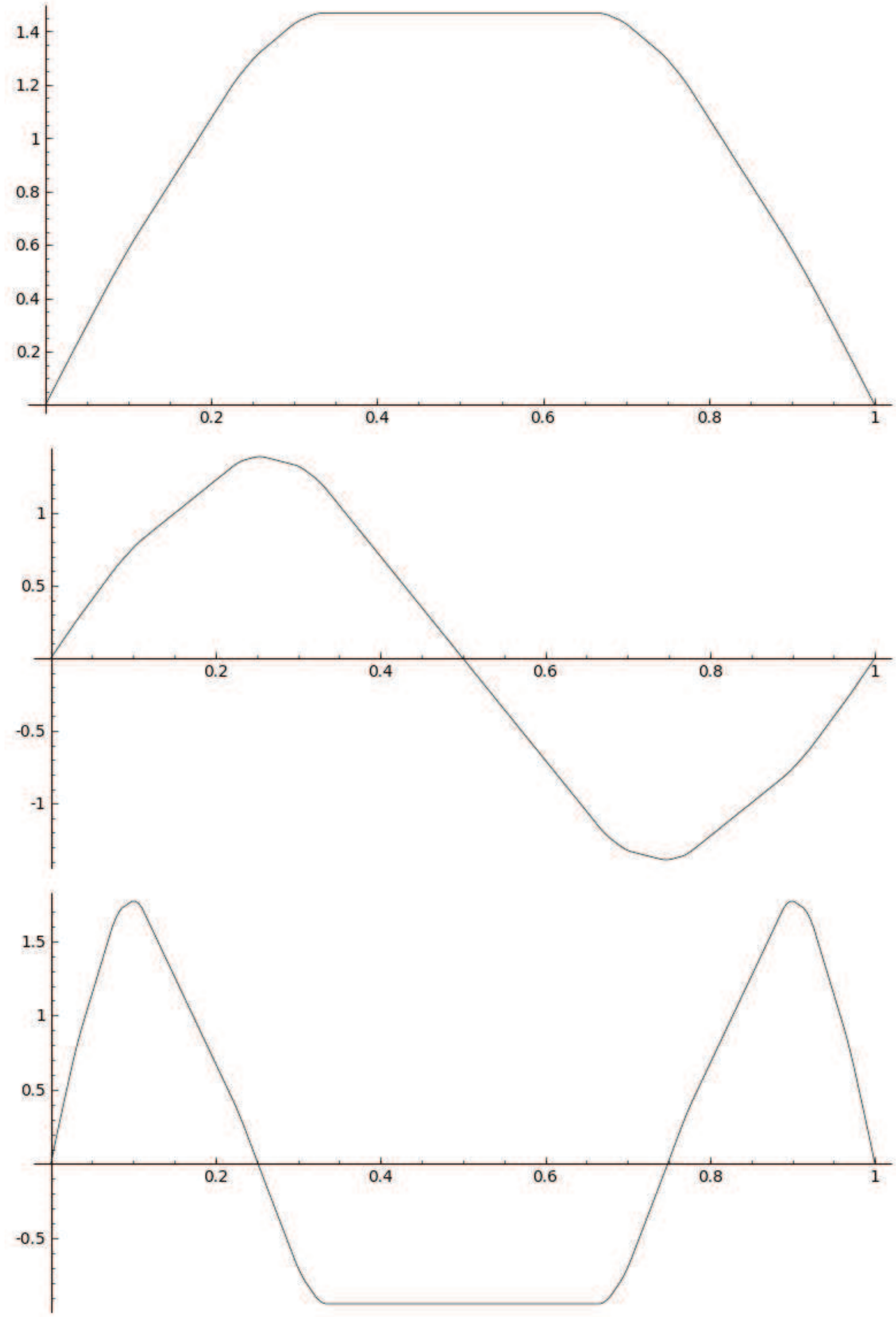

Figure 2. The first three Neumann (top) and Dirichlet (bottom) eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

In Table 4 we state the first 16 eigenvalues with Dirichlet boundary conditions exact to 5 decimals. Estimates of the Dirichlet eigenvalues have also been obtained by Vladimirov and Sheipak in [30] and by Etienne [6] with completely different methods.

Table 4. Dirichlet eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\lambda_{D,m}$	m	$\lambda_{D,m}$
1	14.43524	9	1581.17702
2	35.26023	10	1619.40072
3	140.78105	11	2029.61356
4	151.29061	12	2033.85281
5	326.05732	13	2268.79163
6	353.41692	14	2289.60406
7	876.27445	15	5258.33939
8	876.50531	16	5258.33940

As in the Neumann case, we calculated norms of Dirichlet eigenfunctions, see Table 5.

m	$\|\tilde{f}_{D,m}\|_\infty$	m	$\|\tilde{f}_{D,m}\|_\infty$
1	1.469	9	1.799
2	1.387	10	1.767
3	1.770	11	2.032
4	1.734	12	2.233
5	1.461	13	1.857
6	1.654	14	1.809
7	2.469	15	3.369
8	2.468	16	3.491

Table 5. Norms of Dirichlet eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

Example 10.2. For the next example, we take the asymmetric self-similar measure with $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = 3^{-d_H}$ and $m_2 = 4^{-d_H}$ where d_H is the Hausdorff dimension of the invariant set. That is, d_H is the solution of the equation

$$\frac{1}{3^{d_H}} + \frac{1}{4^{d_H}} = 1.$$

For the calculations we used 0.56049 88652 for d_H. Variation of this value led to no change in the first 5 digits of the eigenvalues. Plots of $\sin_\mu^N$ and $\sin_\mu^D$ are shown in Figure 3 and the first eigenvalues exact to 5 decimal places are displayed in Table 6. Note that here $m_1 r_1 \neq m_2 r_2$. There seem to be no fixed order of Neumann and Dirichlet eigenvalues as in Example 10.1 and there are no clear pairings of the values.

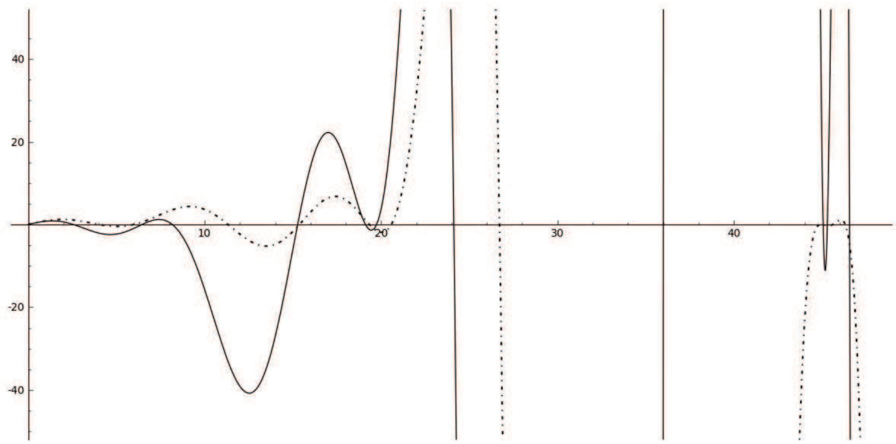

Figure 3. $\sin_\mu^N$ (solid) and $\sin_\mu^D$ (dash-dot) for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = 3^{-d_H}$ and $m_2 = 4^{-d_H}$

Table 6. Neumann and Dirichlet eigenvalues for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = 3^{-d_H}$ and $m_2 = 4^{-d_H}$.

m	$\lambda_{N,m}$	$\lambda_{D,m}$
1	6.56703	16.10784
2	41.63279	35.90760
3	66.82276	128.33044
4	233.35501	236.46367
5	365.58421	373.70192
6	389.94561	423.63815
7	582.13820	713.78698
8	1295.88893	2013.16488

Example 10.3. Figure 4 shows plots of $\sin_\mu^N$ and $\sin_\mu^D$ for $r_1 = \frac{1}{3}$, $r_2 = \frac{1}{4}$ and $m_1 = \frac{3}{7}$, $m_2 = \frac{4}{7}$. The invariant set is geometrically the same as in Example 10.2, but m_1 and m_2 are chosen such that $r_1 m_1 = r_2 m_2 = \frac{1}{7}$ and thus,

$$\lambda_{N,2m} = 7 \cdot \lambda_{N,m}.$$

The first eigenvalues are given in Table 7. Comparing with Example 10.1, we observe that the Neumann eigenvalues behave qualitatively similar, but the Dirichlet eigenvalues do not appear in such close pairs. However, it seems to hold again, that two Neumann and two Dirichlet eigenvalues appear in turns.

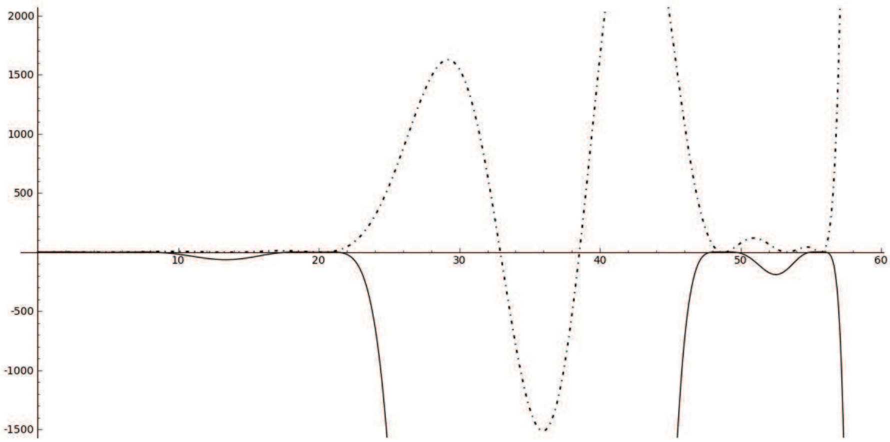

Figure 4. $\sin_\mu^N$ (solid) and $\sin_\mu^D$ (dash-dot) for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{3}{7}$ and $m_2 = \frac{4}{7}$.

Table 7. Neumann and Dirichlet eigenvalues for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{3}{7}$ and $m_2 = \frac{4}{7}$.

m	$\lambda_{N,m}$	$\lambda_{D,m}$
1	6.75228	16.45251
2	47.26598	36.90424
3	62.06687	154.57752
4	330.86192	212.37652
5	345.19467	395.52681
6	434.46810	417.53270
7	446.40799	1083.25327
8	2316.03349	1485.47011

11. Remarks and outlook

In this section we state several remarks and thoughts that could be subject of future studies.

Conjecture 1. *Due to the examination of several examples* (*see e.g. Examples* 10.1 *and* 10.3) *we conjecture that in case of a self-similar measure* μ *with* $r_1 m_1 = r_2 m_2$ *the Neumann and Dirichlet eigenvalues satisfy*

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{D,1} < \lambda_{D,2} < \lambda_{N,2} < \lambda_{N,3} < \lambda_{D,3} < \lambda_{D,4} < \cdots .$$

Remark 2. It would be very interesting to find out, if there was a relation between our sequences $(p_n)_n$ and $(q_n)_n$ to any known number sequences as e.g. Bernoulli or Euler numbers. Indeed, the definition of $p_n(x)$ or $q_n(x)$ (Definition 3.1) is reminiscent of the recursive definition of the Euler polynomials $E_n(x)$ by $E_0(x) := 1$ and $E_n(x) := \int_c^x n E_{n-1}(t)\, dt$, where $c = \frac{1}{2}$ if n is odd and $c = 0$ for even n. Then the nth Euler number is $E_n = 2^n E_n(1/2)$.

Furthermore, (10) has a similar structure as the recursion rule

$$\alpha_n = \frac{1}{2n} \sum_{j=0}^{n-1} \alpha_j\, \alpha_{n-1-j}$$

with $\alpha_0 = \alpha_1 = 1$, where $\alpha_n = \frac{1}{n!}|E_{2n}|$.

Remark 3. One could investigate the functional equations in Theorem 7.12 further. In the simple case where $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$, for instance, we can transform them (after eliminating terms containing $\sin_\mu^D$ by using $\cos_\mu^N(z)^2 + \sin_\mu^N(z)\,\sin_\mu^D(z) = 1$) with the abbreviations $u(z) = z\,\sin_\mu^N(z)$ and $v(z) = 2\,\cos_\mu^N(z)$ to

$$\begin{aligned} u(\sqrt{6}z) &= 3\,u(z)\,v(z) - 3\,u(z)^2 \\ v(\sqrt{6}z) &= v(z)^2 - v(z)\,u(z) - 2. \end{aligned}$$

From this one can derive recursion formulas for the sequence $(p_n)_n$ that contain only members of p_n and not, as in Corollary 7.10, both p_n and q_n. Furthermore, it could be possible to somehow solve these functional equations to get a more direct representation of $\sin_\mu^N$ and $\cos_\mu^N$.

Remark 4. We defined our functions $\sin_\mu^N$, $\sin_\mu^D$, $\cos_\mu^N$ and $\cos_\mu^D$ only for real arguments. However, one can just allow the argument to be complex. Then these power series can be treated with methods of complex analysis.

Remark 5. It is also interesting to consider the eigenvalue problem $\frac{d}{d\mu}\frac{d}{d\nu}f = -\lambda f$ with appropriate boundary conditions, where both μ and ν are non-atomic finite Borel measures. This can be done by modifying the above considerations by replacing λ with ν. The case where both derivatives are with respect to the same measure, that is, $\mu = \nu$ is much simpler. There we get $\mathrm{c}_{\mu,\mu}(z,x) = \cos\big(z\, p_1(x)\big)$ and $\mathrm{s}_{\mu,\mu}(z,x) = \sin\big(z\, p_1(x)\big)$. The eigenvalues are $\lambda_k = k^2\pi^2$, $k \in \mathbb{N}$, as in the classical Lebesgue measure case. This is treated in Arzt and Freiberg [3]. See also Freiberg and Zähle [13].

Remark 6. Our recursion law for p_n and q_n works only for self-similar measures with $r_1 + r_2 \leq 1$. It would be interesting to develop similar formulas for measures with overlaps, i.e. with $r_1 + r_2 > 1$. Such measures are treated for example in Ngai [25] and Chen and Ngai [5], which contains, in particular, numerical solutions of the eigenvalue problem by the finite elements method.

Remark 7. In this work, we examined the eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ by following the basic lines of the treatment of the classical second derivative operator on the interval. In this classical case all eigenvalues are multiples of π^2 and have therefore direct representations in many forms, e.g. by using the series expansion of arctan. Maybe one can find a series representation of eigenvalues of the generalized operator, too, by using such functions as $\sin_\mu^N$, $\sin_\mu^D$, $\cos_\mu^N$ and $\cos_\mu^D$.

Remark 8. In Corollary 8.7 we stated upper and lower estimates for subsequences $(\|\tilde{f}_{2^k l}\|_\infty)_k$, l odd, of the suprema of the normed eigenfunctions. We have no information about the growth of the sequence $\big(\|\tilde{f}_{2k+1}\|_\infty\big)_k$, though.

Such estimates could be used to prove estimates of the heat kernel

$$K(t,x,y) = \sum_{m=1}^{\infty} e^{-\lambda_m t}\, \tilde{f}_m(x)\, \tilde{f}_m(y)$$

for the corresponding quasi-diffusion process. This process has been investigated for example in Löbus [22] and Küchler [20, 21].

Remark 9. We used the functions $p_n(x)$ and $q_n(x)$, $x \in [0,1]$, defined in Definition 3.1 to replace monomials $\frac{1}{n!}x^n$ in the classical case. One could use these functions to build a kind of generalized polynomials that are adjusted to the measure μ. For instance, we take the sequence

$$\tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = q_1(x), \quad \tilde{P}_2(x) = p_2(x), \quad \tilde{P}_3(x) = q_3(x), \quad \ldots$$

and orthogonalize it in $L_2(\mu)$ by using the Gram-Schmidt process. We take odd numbered $q_n(x)$ and even numbered $p_n(x)$, because they are the building blocks for the eigenfunctions $\mathrm{s}_{\lambda,\mu}(z,\cdot)$ and $\mathrm{c}_{\lambda,\mu}(z,\cdot)$. We get

$$P_0(x) = 1,$$

$$P_1(x) = q_1(x) - q_2,$$

$$P_2(x) = p_2(x) - \frac{p_4 - p_2\,p_3}{q_3 - p_2\,q_2}\,q_1(x) + \frac{q_2\,p_4 - q_3\,p_3}{q_3 - p_2\,q_2}.$$

In this fashion one can calculate a sequence of $L_2(\mu)$-orthogonal "polynomials".

As an example we take the Lebesgue measure for μ and put

$$p_n(x) = q_n(x) = \frac{1}{n!}x^n.$$

Then

$$P_0(x) = 1,$$

$$P_1(x) = x - \frac{1}{2},$$

$$P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$$

which are the first Legendre polynomials on [0, 1] (not normed).

If μ is the standard Cantor measure, then $p_2 = q_2 = \frac{1}{2}$, $p_3 = \frac{1}{5}$ and $q_3 = \frac{1}{8}$ and we get

$$P_0(x) = 1,$$

$$P_1(x) = q_1(x) - \frac{1}{2},$$

$$P_2(x) = p_2(x) - \frac{1}{2}q_1(x) + \frac{1}{20}.$$

Maybe one can use these functions for further analytical studies.

Remark 10 (fourier series). It is well known that the normed eigenfunctions $(\tilde{f}_{N,k})_{k=0}^{\infty}$ and $(\tilde{f}_{D,k})_{k=1}^{\infty}$ form orthonormal bases in $L_2(\mu)$ (see [8]).

We set

$$n_{N,k} := \left\| \mathrm{c}_{\lambda,\mu}\left(\sqrt{\lambda_{N,k}},\cdot\right)_{L_2(\mu)} \right\|$$

and

$$n_{D,k} := \left\| \mathrm{s}_{\lambda,\mu}\left(\sqrt{\lambda_{D,k}},\cdot\right)_{L_2(\mu)} \right\|$$

so that

$$\tilde{f}_{N,k} = \frac{1}{n_{N,k}} \, \mathrm{c}_{\lambda,\mu}\left(\sqrt{\lambda_{N,k}}, \cdot\right)$$

and

$$\tilde{f}_{D,k} = \frac{1}{n_{D,k}} \, \mathrm{s}_{\lambda,\mu}\left(\sqrt{\lambda_{D,k}}, \cdot\right).$$

We decompose some functions $f \in L_2(\mu)$ into series of eigenfunctions (Fourier series), ignoring questions about convergence for the moment. Assume that for $x \in [0, 1]$

$$f(x) = \sum_{k=0}^{\infty} a_k \tilde{f}_{N,k}(x)$$

with

$$a_k = \int_0^1 f(t) \, \tilde{f}_{N,k}(t) \, d\mu(t).$$

For reasons of simplicity, we take μ to be a symmetric measure. Then $\cos_\mu^N = \cos_\mu^D$ and we have $\cos_\mu^N(z)^2 + \sin_\mu^N(z) \sin_\mu^D(z) = 1$. From that follows that $\cos_\mu^N(\sqrt{\lambda_{N,k}})^2 = 1$ and it is heuristically clear that $\cos_\mu^N(\sqrt{\lambda_{N,k}}) = (-1)^k$. Employing this fact and Lemma 2.2, the computations can be made explicitly, following the lines of the classical (Euclidean) case.

As a first example, take $f(x) = x$. Then, for $k \in \mathbb{N}$,

$$\begin{aligned}
a_k &= \frac{1}{n_{N,k}} \int_0^1 t \cdot \mathrm{c}_{\lambda,\mu}(\sqrt{\lambda_k}, t) \, d\mu(t) \\
&= \frac{1}{n_{N,k}} \Big[\frac{1}{\sqrt{\lambda_{N,k}}} t \;\, \mathrm{s}_{\mu,\lambda}\big(\sqrt{\lambda_{N,k}}, t\big)\Big|_0^1 - \frac{1}{\sqrt{\lambda_{N,k}}} \int_0^1 \mathrm{s}_{\mu,\lambda}\big(\sqrt{\lambda_{N,k}}, t\big) \, dt \Big] \\
&= \frac{1}{n_{N,k}\lambda_{N,k}} \, \mathrm{c}_{\lambda,\mu}\left(\sqrt{\lambda_{N,k}}, t\right)\Big|_0^1 \\
&= \frac{1}{n_{N,k}\lambda_{N,k}} \big(\cos_\mu^N\left(\sqrt{\lambda_{N,k}}\right) - 1 \big).
\end{aligned}$$

Thus, $a_k = 0$ for even $k \geq 1$ and $a_k = -2(n_{N,k}\lambda_{N,k})^{-1}$ for odd k. Furthermore, we have

$$a_0 = \int_0^1 t \, d\mu(t) = q_2(1) = q_2.$$

Therefore, we have the decomposition into Neumann eigenfunctions

$$x = q_2 - 2\sum_{k=0}^{\infty} \frac{1}{n_{N,2k+1}\lambda_{N,2k+1}}\, \tilde{f}_{N,2k+1}(x).$$

Note that the required norms $n_{N,k}$ can be computed with Corollary 4.3.

We apply Parseval's identity to this series. This gives

$$\int_0^1 t^2\, d\mu(t) = q_2^2 + \sum_{k=0}^{\infty} \frac{4}{c_{N,2k+1}^2\, \lambda_{N,2k+1}^2},$$

and with

$$\int_0^1 t^2\, d\mu(t) = t\, q_2(t)\big|_0^1 - \int_0^1 q_2(t)\, dt = q_2 - q_3$$

and

$$1 - q_2 = p_2$$

we get

$$\sum_{k=0}^{\infty} \frac{1}{c_{N,2k+1}^2\, \lambda_{N,2k+1}^2} = \frac{1}{4}(p_2\, q_2 - q_3).$$

If we choose the Lebesgue measure for μ (then $p_2 = q_2 = \frac{1}{2}$, $q_3 = \frac{1}{6}$ and $c_{N,2k+1}^2 = \frac{1}{2}$), the above equation becomes the well known identity

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.$$

In the same fashion we compute the decomposition of some more examples (μ symmetric):

$$x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{n_{D,k}\sqrt{\lambda_{D,k}}} \tilde{f}_{D,k}(x)$$

$$1 = \sum_{k=0}^{\infty} \frac{2}{c_{D,2k+1}\sqrt{\lambda_{D,2k+1}}}\, \tilde{f}_{D,2k+1}(x)$$

$$f_{D,2n+1}(x) = \frac{2}{\sqrt{\lambda_{D,2n+1}}} - 2\sqrt{\lambda_{D,2n+1}} \sum_{k=1}^{\infty} \frac{1}{(\lambda_{N,2k} - \lambda_{D,2n+1})c_{N,2k}}\, \tilde{f}_{N,2k}(x),$$

for every $n \in \mathbb{N}_0$.

Applying Parseval's identity to these decompositions leads, as above, to

$$\sum_{k=1}^{\infty} \frac{1}{n_{D,k}^2 \lambda_{D,k}} = q_2 - q_3$$

$$\sum_{k=0}^{\infty} \frac{1}{c_{D,2k+1}^2 \lambda_{D,2k+1}} = \frac{1}{4}$$

$$\sum_{k=1}^{\infty} \frac{1}{\left(\lambda_{N,2k} - \lambda_{D,2n+1}\right)^2 c_{N,2k}^2} = \frac{c_{D,2n+1}^2}{4\lambda_{D,2n+1}} - \frac{1}{\lambda_{D,2n+1}^2}.$$

If we again take the Lebesgue measure for μ, we receive some well known identities.

Remark 11. The definition of the operator $-\frac{d}{d\mu}\frac{d}{dx}$ can be extended to subsets of $\mathbb{R}^d$, $d \in \mathbb{N}$, see, for example, Solomyak and Verbitsky [28], Naimark and Solomyak [24] and Hu, Lau and Ngai [15]. This case, however, is substantially more difficult and the techniques presented here can probably not be readily extended to it.

Remark 12. Analogously to our measure trigonometric functions we can define measure theoretic exponential functions. For $x \in [0, 1]$ and $z \in \mathbb{C}$ put

$$\mathrm{e}_{\lambda,\mu}(z, x) := \sum_{n=0}^{\infty} z^{2n}\, p_{2n}(x) + \sum_{n=0}^{\infty} z^{2n+1}\, q_{2n+1}(x)$$

and

$$\mathrm{e}_{\mu,\lambda}(z, x) := \sum_{n=0}^{\infty} z^{2n}\, q_{2n}(x) + \sum_{n=0}^{\infty} z^{2n+1}\, p_{2n+1}(x).$$

Then, $\mathrm{e}_{\lambda,\mu}(z, \cdot) \in H^2(\lambda, \mu)$ and $\mathrm{e}_{\mu,\lambda}(z, \cdot) \in H^2(\mu, \lambda)$ for every $z \in \mathbb{C}$. Furthermore, for all $t \in \mathbb{R}$ and $x \in [0, 1]$, we have Euler's formula

$$\mathrm{e}_{\lambda,\mu}(it, x) = \mathrm{c}_{\lambda,\mu}(t, x) + i\, \mathrm{s}_{\lambda,\mu}(t, x)$$

and

$$\mathrm{e}_{\mu,\lambda}(it, x) = \mathrm{c}_{\mu,\lambda}(t, x) + i\, \mathrm{s}_{\mu,\lambda}(t, x).$$

References

[1] P. Arzt, Measure theoretic trigonometric functions. Longer version of this article. Preprint 2014. arXiv:1405.4693 [math.SP]

[2] P. Arzt and U. Freiberg, Spectral exponents of gap diffusions on random homogeneous Cantor-sets. In preparation.

[3] P. Arzt and U. Freiberg, The spectrum of Laplacians on intervals equipped with singular measures. In preparation.

[4] E. J. Bird, S.-M. Ngai, and A. Teplyaev, Fractal Laplacians on the unit interval. *Ann. Sci. Math. Québec* **27** (2003), no. 2, 135–168. MR 2103098 Zbl 1102.34066

[5] J. Chen and S.-M. Ngai, Eigenvalues and eigenfunctions of one-dimensional fractal Laplacians defined by iterated function systems with overlaps. *J. Math. Anal. Appl.* **364** (2010), no. 1, 222–241. MR 2576066 Zbl 1261.35095

[6] R. J. Etienne, *On the Asymptotic Distribution of the Dirichlet eigenvalues of fractal chains.* Ph.D. Thesis. Universität Siegen, Siegen, 2014.

[7] W. Feller, Generalized second order differential operators and their lateral conditions. *Illinois J. Math.* **1** (1957), 459–504. MR 0092046 Zbl 0077.29102

[8] U. Freiberg, Analytical properties of measure geometric Krein–Feller-operators on the real line. *Math. Nachr.* **260** (2003), 34–47. MR 2017701 Zbl 1055.28003

[9] U. Freiberg, Dirichlet forms on fractal subsets of the real line. *Real Anal. Exchange* **30** (2004/05), no. 2, 589–603. MR 2177421 Zbl 1107.28005

[10] U. Freiberg, Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets. *Forum Math.* **17** (2005), no. 1, 87–104. MR 2110540 Zbl 1135.28302

[11] U. Freiberg, Prüfer angle methods in spectral analysis of Krein–Feller-operators. In M. Fukushima and I. Shigekawa (eds.), *Proceedings of RIMS Workshop on Stochastic Analysis and Applications.* Held in Kyoto, September 11–15, 2006. RIMS Kôkyûroku Bessatsu, B6. Research Institute for Mathematical Sciences (RIMS), Kyoto, 2008, 74–81. MR 2407555 MR 2404085 (collection) Zbl 1147.28002 Zbl 1141.60301 (collection)

[12] U. Freiberg and J.-U. Löbus, Zeros of eigenfunctions of a class of generalized second order differential operators on the Cantor set. *Math. Nachr.* **265** (2004), 3–14. MR 2033064 Zbl 1042.35040

[13] U. Freiberg and M. Zähle, Harmonic calculus on fractals – a measure geometric approach. I. *Potential Anal.* **16** (2002), no. 3, 265–277. MR 1885763 Zbl 1055.28002

[14] T. Fujita, A fractional dimension, self similarity and a generalized diffusion operator. In K. Itô and N. Ikeda (eds.), *Probabilistic methods in mathematical physics.* Proceedings of the Taniguchi International Symposium held in Katata, June 20–26, 1985, and at Kyoto University, Kyoto, June 27–29, 1985. Academic Press, Boston, MA, 1987, 83–90. MR 0933819 MR 0933814 (collection) Zbl 0652.60084 Zbl 0633.00021 (collection)

[15] J. Hu, K.-S. Lau, and S.-M. Ngai, Laplace operators related to self-similar measures on $\mathbb{R}^d$. *J. Funct. Anal.* **239** (2006), no. 2, 542–565. MR 2261337 Zbl 1109.47038

[16] J. E. Hutchinson, Fractals and self similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747. MR 0625600 Zbl 0598.28011

[17] I. S. Kac and M. G. Krein, On the spectral functions of the string. *Amer. Math. Soc. Transl.* (2) **103** (1974), 19–102. Zbl 0291.34017

[18] V. V. Kravchenko, A representation for solutions of the Sturm–Liouville equation. *Complex Var. Elliptic Equ.* **53** (2008), no. 8, 775–789. MR 2436253 Zbl 1183.30052

[19] V. V. Kravchenko and R. M. Porter, Spectral parameter power series for Sturm–Liouville problems. *Math. Methods Appl. Sci.* **33** (2010), no. 4, 459–468. MR 2641623 Zbl 1202.34060

[20] U. Küchler, Some asymptotic properties of the transition densities of one-dimensional quasidiffusions. *Publ. Res. Inst. Math. Sci.* **16** (1980), no. 1, 245–268. MR 0574035 Zbl 0443.60077

[21] U. Küchler, On sojourn times, excursions and spectral measures connected with quasidiffusions. *J. Math. Kyoto Univ.* **26** (1986), no. 3, 403–421. MR 0857226 Zbl 0625.60093

[22] J.-U. Löbus, Constructions and generators of one-dimensional quasidiffusions with applications to selfaffine diffusions and Brownian motion on the Cantor set. *Stochastics Stochastics Rep.* **42** (1993), no. 2, 93–114. MR 1275814 Zbl 0805.60071

[23] H. P. McKean and D. B. Ray, Spectral distribution of a differential operator. *Duke Math. J.* **29** (1962), 281–292. MR 0146444 Zbl 0114.04902

[24] K. Naimark and M. Solomyak, The eigenvalue behaviour for the boundary value problems related to self-similar measures on $\mathbb{R}^d$. *Math. Res. Lett.* **2** (1995), no. 3, 279–298. MR 1338787 Zbl 0836.60016

[25] S.-M. Ngai, Spectral asymptotics of Laplacians associated with one-dimensional iterated function systems with overlaps. *Canad. J. Math.* **63** (2011), no. 3, 648–688. MR 2828537 Zbl 1232.28012

[26] C. Sabot, Density of states of diffusions on self-similar sets and holomorphic dynamics in $\mathbb{P}^k$: the example of the interval $[0,1]$. *C. R. Acad. Sci. Paris Sér.* I *Math.* **327** (1998), no. 4, 359–364. MR 1649959 Zbl 0922.60040

[27] C. Sabot, Integrated density of states of self-similar Sturm–Liouville operators and holomorphic dynamics in higher dimension. *Ann. Inst. H. Poincaré Probab. Statist.* **37** (2001), no. 3, 275–311. MR 1831985 Zbl 1038.37036

[28] M. Solomyak and E. Verbitsky, On a spectral problem related to self-similar measures. *Bull. London Math. Soc.* **27** (1995), no. 3, 242–248. MR 1328700 Zbl 0823.34071

[29] W. A. Stein et al., Sage mathematics software (version 5.12). The Sage development team, 2013. http://cloud.sagemath.com

[30] A. A. Vladimirov and I. A. Sheipak, Self-similar functions in $l_2[0, 1]$ and the Sturm–Liouville problem with a singular indefinite weight. *Mat. Sb.* **197** (2006), no. 11, 13–30. In Russian. English transl., *Sb. Math.* **197** (2006), no. 11-12, 1569–1586. MR 2437086 Zbl 1177.34039

[31] H. Volkmer, Eigenvalues associated with Borel sets. *Real Anal. Exchange* **31** (2005/06), no. 1, 111–124. MR 2218192 Zbl 1108.34022

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