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On McMullen-like mappings

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Abstract. We introduce a generalization of particular dynamical behavior for rational maps. In 1988, C. McMullen showed that the Julia set of $f_\lambda(z) = z^n + \lambda/z^d$ for $|\lambda| \neq 0$ small enough is a Cantor set of circles if and only if $1/n + 1/d < 1$ holds. Several other specific singular perturbations of polynomials have been studied in recent years, all have parameter values where a Cantor set of circles is present in the associated Julia set. We unify these examples by defining a McMullen-like mapping as a rational map f associated to a hyperbolic postcritically finite polynomial P and a pole data $\mathcal{D}$ where we encode the location of every pole of f and the local degree at each pole. As for the McMullen family f_λ, we characterize a McMullen-like mapping using an arithmetic condition depending only on $(P, \mathcal{D})$. We show how to check the definition in practice providing new explicit examples of McMullen-like mappings for which a complete topological description of their Julia sets is made.

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1. Introduction

The *Fatou set* of a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, denoted by $\mathcal{F}(f)$, is defined to be the set of points at which the family of iterates of f is a normal family in the sense of P. Montel. The complement of the Fatou set in the Riemann sphere $\widehat{\mathbb{C}}$, is the *Julia set*, denoted by $\mathcal{J}(f)$. The Julia set is known to be the closure of the set of repelling periodic points of f, and it is the set where f has sensitive dependence on initial conditions. Equivalently, the Julia set is the smallest closed set containing at least three points which is completely invariant under f. For a deep and helpful introduction on iteration of rational maps see [1, 4, 17, 22]. One of the main goals in complex dynamics is to study the topological properties of the Julia set and the dynamics of f restricted to $\mathcal{J}(f)$.

In [15], C. McMullen showed the first example of Julia set which is a *Cantor set of circles*, namely homeomorphic to the product of the Cantor set and a simple closed curve. The rational map that exhibits this phenomenon, hereafter *McMullen family*, is $f_\lambda(z) = z^n + \lambda/z^d$ for some values of $n, d \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. The McMullen family, which can be viewed as a singular perturbation of the polynomial $z \mapsto z^n$ when we add a pole of order d at the origin, has been the focus of attention for several reasons. On the one hand, the parameter space has complex dimension one, since the "free" critical points of f_λ behave in a symmetric way, and on the other hand, it exhibits classical Julia sets including Cantor sets, Sierpiński curves, and Cantor sets of circles (see [8]). The McMullen family has been studied extensively by R. Devaney et al. [8, 9, 2], N. Steinmetz [23] and Qiu W., P. Roesch, Wang X. and Yin Y. [20, 19] among others. We refer to [6] for a survey of the main results about singular perturbations of complex polynomials and references therein.

For the polynomial $z \mapsto z^n$, with $n \geqslant 2$, infinity and the origin are superattracting fixed points and the Julia set is the unit circle. When we add the perturbation λ/z^d with $\lambda \neq 0$, several aspects of the dynamics remain the same, but others change dramatically. For instance the point at infinity is still a superattracting fixed point and there is an immediate attracting basin of ∞ that we call V_∞. However, there is a neighborhood of the pole located at the origin that is now mapped d-to-1 onto a neighborhood of ∞. When this neighborhood of 0 is disjoint from V_∞, we call it the *trap door* and denote it by T_0. Every point that escapes to infinity and does not lie in V_∞ has to do so by passing through T_0. Since the degree of f_λ changes from n to $n+d$, some additional critical points are created. The set of critical points includes ∞ and 0 whose orbits are completely determined, so there are $n+d$ additional "free" critical points. The orbits of these points are of fundamental importance in characterizing the Julia set of f_λ.

McMullen showed that if the arithmetic condition $1/n + 1/d < 1$ holds and if the free critical values lie in the trap door T_0 (this second condition being satisfied as soon as $|\lambda| \neq 0$ is small enough) then the Julia set of f_λ is a Cantor set of circles (see [15, 8, 7]). Under these assumptions we notice that the $n + d$ free critical points belong to a doubly connected Fatou component A separating the trap door T_0 from the immediate attracting basin of infinity V_∞, and such that f_λ maps the annulus A over T_0 with degree $n + d$.

In recent years, several works have appeared in the literature dealing with singular perturbations of polynomials of the form $P_c(z) = z^n + c$ where c is chosen to be the center of a hyperbolic component of the corresponding Multibrot set and adding a perturbation with one or several poles (see for instance [2, 11]). Figure 1 displays the Julia set of three singular perturbations of polynomials. The first one corresponds to a member of the McMullen family, concretely the rational map $z \mapsto z^3 - 10^{-2}/z^3$. The second one is the Julia set of the rational map $z \mapsto z^3 + i - 10^{-7}/z^3$, that corresponds to a singular perturbation of the cubic polynomial $z \mapsto z^3 + i$, that exhibits a super-attracting cycle $0 \mapsto i \mapsto 0$, when we add a pole at the origin. Finally, the third one is the Julia set of the rational map $z \mapsto z^2 - 1 + 10^{-22}/(z^7(z + 1)^5)$ that corresponds to a perturbation of the quadratic polynomial $z^2 - 1$, with super-attracting cycle $0 \mapsto -1 \mapsto 0$, when we add two poles, one at $z = 0$ and another one at $z = -1$. In this figure we also show the dynamical plane of the corresponding polynomial and we mark the Fatou component with a number where we add a pole with corresponding local degree. We remark that all these examples present Cantor sets of circles in their Julia set.

The main goal of this paper is to present a unified approach to this kind of dynamical systems. We firstly define what we call a *McMullen-like mapping*. Here we give an idea and we refer to the next section for a precise definition. Before the definition we need two ingredients: the first one is a hyperbolic postcritically finite polynomial (hereafter HPcFP) and the second one is a pole data that encodes the information about the poles (locations and local degrees).

A McMullen-like mapping of type $(P, \mathcal{D})$, formed by a HPcFP P and a pole data $\mathcal{D}$, is a rational map f verifying the following conditions. The first condition is that ∞ is a super-attracting fixed point of f, whose immediate attracting basin is denoted by V_∞, and that ∂V_∞ is a homeomorphic copy of the Julia set of the polynomial P. Outside the Fatou components that appears in the pole data $\mathcal{D}$, the second condition requires that the dynamics of f is basically that of the polynomial P. The third condition is about the trap doors which are simply connected domains mapped by f onto V_∞ according to the pole data $\mathcal{D}$. Moreover, around every trap door we require the existence of an annulus (containing some critical

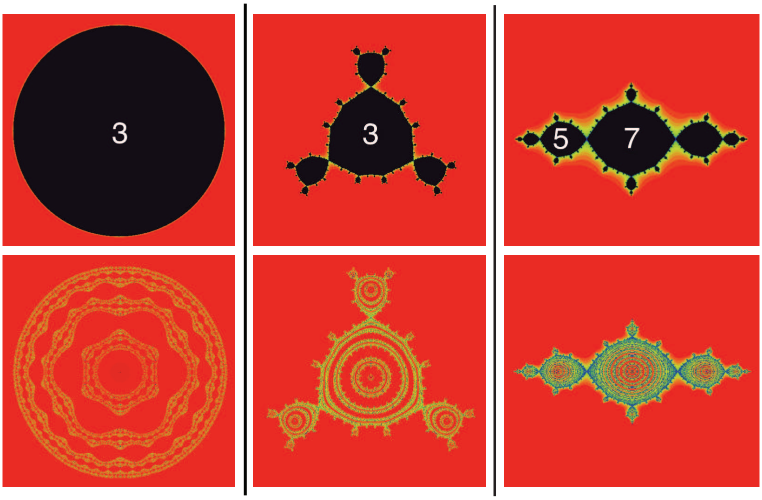

Figure 1. Three examples of McMullen-like mappings. On the left, we show the dynamical plane of the polynomial $z \mapsto z^3$ (up) and the rational map $z \mapsto z^3 - 10^{-2}/z^3$ (down). In the middle, the polynomial $z \mapsto z^3 + i$ and the rational map $z \mapsto z^3 + i - 10^{-7}/z^3$. And finally, on the right, the polynomial $z \mapsto z^2 - 1$ and the rational map $z \mapsto z^2 - 1 + 10^{-22}/(z^7(z+1)^5)$. In each case, the bounded Fatou component where we put a pole has been marked with the local degree.

points) separating the corresponding trap door from V_∞ and mapped by f onto a simply connected domain. Finally, the last condition on the critical points of f ensures that the forward orbit under f of every annulus described above eventually reaches a trap door.

The main result of this paper Theorem A, about the existence of McMullen-like mappings. More precisely, given a pair $(P, \mathcal{D})$, formed by a HPcFP P and a pole data $\mathcal{D}$, we can characterize the existence of a McMullen-like mapping of type $(P, \mathcal{D})$ under an arithmetic condition. We also describe the Julia set of any McMullen-like mapping.

We organize the rest of the paper in the following way. In Section 2 we give the precise definition of a pole data (Definition 2.1) and of a McMullen-like mapping (Definition 2.2). Then we describe the Julia set of any McMullen-like mapping (Theorem 2.5) and we show some examples from the literature regarding this kind of rational maps (Subsection 2.3). In Section 3 we state the main result of this work, namely Theorem A which characterizes the existence of a McMullen-like mapping of type $(P, \mathcal{D})$ using an arithmetic condition $(\star)$ depending on the polynomial P and the pole data $\mathcal{D}$. The rest of Section 3 is devoted to proving Theorem A. We first use the theory developed by W. Thurston about obstructions (see [10, 14]) to prove the necessity of the arithmetic condition ($\star$) (Subsection 3.2). Then using quasiconformal surgery (see [21, 3]) we are able to construct a McMullen-like mapping of given type $(P, \mathcal{D})$ assuming only the arithmetic condition ($\star$) (Subsection 3.3), proving thus the sufficiency. Finally, in Section 4 we give some new examples of McMullen-like mappings. In particular, we give an explicit expression of a McMullen-like mapping of minimal degree 4.

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2. Definition and properties of McMullen-like mappings

2.1. Definition of McMullen-like mappings. Let P be a hyperbolic postcritically finite polynomial (HPcFP for short), namely a polynomial map $P: \mathbb{C} \to \mathbb{C}$ of degree $n = \deg(P) \geqslant 2$ such that every critical point is eventually mapped under iteration to a super-attracting periodic cycle. The Julia set $\mathcal{J}(P)$ is connected and the Fatou set $\mathcal{F}(P)$ contains countably many connected components which are simply connected. Moreover the unbounded Fatou component, denoted by U_∞, is a completely invariant super-attracting basin with $\partial U_\infty = \mathcal{J}(P)$, and every bounded Fatou component U is eventually mapped to a periodic cycle of immediate super-attracting basins (see [1, 4, 17, 22]).

We arbitrary choose a labeling of the finitely many periodic bounded Fatou components, of the following form

$$\{U_{i,j} \mid 1 \leqslant i \leqslant N \text{ and } j \in \mathbb{Z}/p_i\mathbb{Z}\}$$

where $N \geqslant 1$ is the number of bounded super-attracting periodic cycles, $p_i \geqslant 1$ is the period of the i-th cycle, and so that $P(U_{i,j}) = U_{i,j+1}$ for every $1 \leqslant i \leqslant N$ and $j \in \mathbb{Z}/p_i\mathbb{Z}$. Moreover for every periodic bounded Fatou component $U_{i,j}$, we denote by $n_{i,j}$ the degree of the restriction $P|_{U_{i,j}}$, which coincides with the degree of the restriction $P|_{\partial U_{i,j}}$. Notice that the Riemann–Hurwitz formula gives $n - 1 = \sum_{i,j}(n_{i,j} - 1)$.

Definition 2.1 (pole data). A pole data $\mathcal{D}$ associated to a HPcFP P is a nonempty collection of periodic bounded Fatou components of P, each provided with a positive integer. More precisely, $\mathcal{D}$ is the data of a nonempty subset of $\{U_{i,j} \mid 1 \leqslant i \leqslant N \text{ and } j \in \mathbb{Z}/p_i\mathbb{Z}\}$ and a function from this subset to $\mathbb{N} \setminus \{0\}$ denoted by $U_{i,j} \mapsto d_{i,j}$.

With abuse of notation, we write $U_{i,j} \in \mathcal{D}$ if and only if $U_{i,j}$ is picked in the pole data $\mathcal{D}$, and conversely $U \notin \mathcal{D}$ for every bounded Fatou component U which is not a $U_{i,j} \in \mathcal{D}$.

The degree of the pole data $\mathcal{D}$ is defined to be

$$d = \deg(\mathcal{D}) := \sum_{U_{i,j} \in \mathcal{D}} d_{i,j} \geqslant 1.$$

We remark that if there is a simply connected domain $V_\infty \subset \widehat{\mathbb{C}}$ and an orientation preserving homeomorphism $\varphi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that $\varphi(\partial U_\infty) = \partial V_\infty$ then we may define for every bounded Fatou component U of P a simply connected domain $V(U) \subset \widehat{\mathbb{C}}$ as the unique connected component of $\widehat{\mathbb{C}} \setminus \varphi(\partial U)$ which does not intersect V_∞ (it is well defined because ∂U is a simple closed curve since P is hyperbolic, see for instance Lemma 19.3 in [17]). In order to lighten the notation, we write $V_{i,j} = V(U_{i,j})$ for every $1 \leqslant i \leqslant N$ and $j \in \mathbb{Z}/p_i\mathbb{Z}$.

Definition 2.2 (McMullen-like mapping). Let $\mathcal{D}$ be a pole data associated to a HPcFP P. A McMullen-like mapping of type $(P, \mathcal{D})$ is a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which satisfies the following conditions.

(i) There exist a simply connected domain $V_\infty \subset \widehat{\mathbb{C}}$ and an orientation preserving homeomorphism $\varphi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f(V_\infty) = V_\infty$, $\varphi(\partial U_\infty) = \partial V_\infty$, and $f|_{\partial V_\infty} \circ \varphi = \varphi \circ P|_{\partial U_\infty}$.

(ii) For every $U \notin \mathcal{D}$, $f(V(U)) = V(P(U))$.

(iii) For every $U_{i,j} \in \mathcal{D}$, there exist a simply connected domain $T_{i,j} \subset V_{i,j}$, called a trap door, and a doubly connected domain $A_{i,j} \subset V_{i,j} \setminus \overline{T_{i,j}}$ which separates $T_{i,j}$ from V_∞ such that

- $f(T_{i,j}) = V_\infty$ and $f|_{T_{i,j}}$ has degree $d_{i,j}$;
- $f(A_{i,j})$ is a simply connected domain contained in $V(P(U_{i,j})) = V_{i,j+1}$ and $f|_{A_{i,j}}$ is a proper map (namely $f(\partial A_{i,j}) = \partial f(A_{i,j})$);
- f has no critical points in $\overline{V_{i,j}} \setminus (A_{i,j} \cup T_{i,j})$.

(iv) For every critical point c of f, if c is eventually mapped under iteration of f into $V_{i,j}$ for some $U_{i,j} \in \mathcal{D}$, or equivalently if

$$t_c := \min\{k \geqslant 1 \mid \exists U_{i,j} \in \mathcal{D},\ f^k(c) \in V_{i,j}\} < +\infty,$$

then c is mapped into the corresponding trap door, namely $f^{t_c}(c) \in T_{i,j}$.

According to points (i) and (ii) above, $f|_{V_\infty}\colon V_\infty \to V_\infty$ is a holomorphic branched covering of degree $n = \deg(P)$ with $n-1$ critical points counted with multiplicity, and for every $U \notin \mathcal{D}$, $f|_{V(U)}\colon V(U) \to V(P(U))$ is a holomorphic branched covering of same degree as $P|_U\colon U \to P(U)$ with the same number of critical points counted with multiplicity. The following lemma extends point (iii) above by describing how f acts on each $V_{i,j}$.

Lemma 2.3. *For every $U_{i,j} \in \mathcal{D}$, denote by $\overline{A^{\text{out}}_{i,j}}$ the closed annulus between $\partial V_{i,j}$ and $\partial A_{i,j}$, and by $\overline{A^{\text{in}}_{i,j}}$ the closed annulus between $\partial A_{i,j}$ and $\partial T_{i,j}$. Then the action of f on $\overline{V_{i,j}} = \overline{A^{\text{out}}_{i,j}} \cup A_{i,j} \cup \overline{A^{\text{in}}_{i,j}} \cup T_{i,j}$ is as follows:*

- *$\overline{f(A^{\text{out}}_{i,j})}$ is the closed annulus in $\overline{V_{i,j+1}}$ between $\partial V_{i,j+1}$ and $\partial f(A_{i,j})$, and $f|_{A^{\text{out}}_{i,j}}$ is a holomorphic covering of degree $n_{i,j}$;*
- *$f(A_{i,j})$ is a simply connected domain in $V_{i,j+1}$, and $f|_{A_{i,j}}$ is a holomorphic branched covering of degree $n_{i,j}+d_{i,j}$ with $n_{i,j}+d_{i,j}$ critical points counted with multiplicity;*
- *$\overline{f(A^{\text{in}}_{i,j})}$ is the closed annulus between $\partial f(A_{i,j})$ and ∂V_∞, and $f|_{A^{\text{in}}_{i,j}}$ is a holomorphic covering of degree $d_{i,j}$;*
- *$f(T_{i,j}) = V_\infty$, and $f|_{T_{i,j}}$ is a holomorphic branched covering of degree $d_{i,j}$ with $d_{i,j}-1$ critical points counted with multiplicity.*

In particular $f(V_{i,j}) = \widehat{\mathbb{C}}$ (and hence $f|_{V_{i,j}}$ is not a proper map).

Proof. Since f has no critical points in $\overline{V_{i,j}} \setminus (T_{i,j} \cup A_{i,j})$, it follows that $f|_{A_{i,j}}$ has degree

$$\deg(f|_{A_{i,j}}) = \deg(f|_{\partial V_{i,j}}) + \deg(f|_{T_{i,j}}) = n_{i,j} + d_{i,j}$$

because $\deg(f|_{\partial V_{i,j}}) = \deg(P|_{\partial U_{i,j}}) = n_{i,j}$ from point (i) in Definition 2.2. The remaining easily follows by using the Riemann–Hurwitz formula. □

Notice that each of the critical points $c \in A_{i,j}$ satisfies $f(c) \in V_{i,j+1}$. It follows from Definition 2.2 that $t_c \leqslant p_i < +\infty$ and t_c only depends on $A_{i,j}$, namely on $U_{i,j} \in \mathcal{D}$. The following lemma shows that $A_{i,j}$ may be chosen in Definition 2.2 in order that the whole image $f^{t_c}(A_{i,j})$ is actually the trap door containing $f^{t_c}(c)$.

Lemma 2.4. *For every $U_{i,j} \in \mathcal{D}$, we may assume without loss of generality that $f^{t_{i,j}}(A_{i,j}) = T_{i,j+t_{i,j}}$ where*

$$t_{i,j} := \min\{k \geqslant 1 \mid U_{i,j+k} \in \mathcal{D}\}.$$

In particular, $t_c = t_{i,j}$ for every critical point $c \in A_{i,j}$.

This result is similar to the first proposition of Section 3 in [8] for the McMullen mapping $f_\lambda(z) = z^n + \lambda/z^d$, except we can not use the symmetry of the map here.

Proof. Remark that $f^{t_{i,j}-1}(V_{i,j+1}) = V_{i,j+t_{i,j}}$ from the definition of $t_{i,j}$ and point (ii) in Definition 2.2, and the critical values of $f^{t_{i,j}-1}|_{V_{i,j+1}}$ lie in $T_{i,j+t_{i,j}}$ from point (iv). Therefore the preimage of $T_{i,j+t_{i,j}}$ under $f^{t_{i,j}-1}|_{V_{i,j+1}}$ is a simply connected domain in $V_{i,j+1}$.

Let

$$A'_{i,j} = \left\{z \in V_{i,j} \;\middle|\; f(z) \in (f^{t_{i,j}-1}|_{V_{i,j+1}})^{-1}(T_{i,j+t_{i,j}})\right\}.$$

It follows that $f(A'_{i,j})$ is a simply connected domain in $V_{i,j+1}$, $f|_{A'_{i,j}}$ is a proper map (since $f(V_{i,j}) = \widehat{\mathbb{C}}$ from Lemma 2.3), and $f^{t_{i,j}}(A'_{i,j}) = T_{i,j+t_{i,j}}$. Notice that $A'_{i,j} \subset V_{i,j} \setminus \overline{T_{i,j}}$ because $f(T_{i,j}) = V_\infty$ is outside $V_{i,j+1}$. Furthermore, every critical point $c \in A_{i,j}$ satisfies $f^{t_{i,j}}(c) \in T_{i,j+t_{i,j}}$ from the definition of $t_{i,j}$ and point (iv), and hence is in $A'_{i,j}$. Therefore f has no critical points in $\overline{V_{i,j}} \setminus (A'_{i,j} \cup T_{i,j})$.

Consequently, it is enough to show that $A'_{i,j}$ is a doubly connected domain which separates $T_{i,j}$ from V_∞. At first remark that $f|_{A'_{i,j}}$ is actually a holomorphic branched covering of degree $\deg(f|_{A'_{i,j}}) = n_{i,j} + d_{i,j}$ with $\nu(f|_{A'_{i,j}}) = n_{i,j} + d_{i,j}$ critical points counted with multiplicity from Lemma 2.3 since every critical point and every cocritical point of $f|_{A_{i,j}}$ is in $A'_{i,j}$ by definition. Denote by $\{C_\ell \mid 1 \leqslant \ell \leqslant L\}$ the collection of $L \geqslant 1$ connected components of $A'_{i,j}$, and by $m_\ell \geqslant 1$ the number of connected components in every ∂C_ℓ. The Riemann–Hurwitz formula gives

$$2 - m_\ell = (2-1)\deg(f|_{C_\ell}) - \nu(f|_{C_\ell}), \quad 1 \leqslant \ell \leqslant L,$$

that leads by summing to

$$2L - \sum_{\ell=1}^{L} m_\ell = \sum_{\ell=1}^{L} \deg(f|_{C_\ell}) - \sum_{\ell=1}^{L} \nu(f|_{C_\ell}) = \deg(f|_{A'_{i,j}}) - \nu(f|_{A'_{i,j}}) = 0. \quad (1)$$

Furthermore, every simply connected components of $V_{i,j} \setminus \overline{A'_{i,j}}$ must contain some points which are mapped into $V_\infty \subset \widehat{\mathbb{C}} \setminus \overline{f(A'_{i,j})}$, and hence must contain $T_{i,j}$ from Lemma 2.3. In particular there is no $m_\ell \geqslant 3$, and (1) implies that every

C_ℓ is a doubly connected domain which separates $T_{i,j}$ from V_∞. Assume the $(C_\ell)_{1\leqslant \ell\leqslant L}$ are ordered labelled from $\partial T_{i,j}$ to $\partial V_{i,j}$, then $\deg(f|_{C_1}) \geqslant d_{i,j}+1$ and $\deg(f|_{C_L}) \geqslant 1+n_{i,j}$ lead to a contradiction as soon as $L\geqslant 2$. □

As a consequence, the orbit of $A_{i,j}$ is as follows:

$$\begin{aligned}
A_{i,j} &\subset V_{i,j}, && \text{with } U_{i,j}\in\mathcal{D},\\
f^k(A_{i,j}) &\subset V_{i,j+k}, && \text{for all } 1\leqslant k< t_{i,j},\ \text{with } U_{i,j+k}\notin\mathcal{D},\\
f^{t_{i,j}}(A_{i,j}) &= T_{i,j+t_{i,j}} \subset V_{i,j+t_{i,j}}, && \text{with } U_{i,j+t_{i,j}}\in\mathcal{D},\\
f^k(A_{i,j}) &= V_\infty, && \text{for all } k>t_{i,j}.
\end{aligned}$$

2.2. Julia sets of McMullen-like mappings. The following result describes the Julia set of any McMullen-like mapping.

Theorem 2.5. *If f is a McMullen-like mapping of type $(P,\mathcal{D})$ for some pole data $\mathcal{D}$ associated to a* HPcFP *P, then f is a hyperbolic rational map of degree* $\deg(f)=\deg(P)+\deg(\mathcal{D})$ *with disconnected Julia set, every Fatou component is either simply or doubly connected, and $\mathcal{J}(f)$ contains*

- *countably many preimages of ∂V_∞ which is a fixed Julia component quasisymetrically equivalent to $\partial U_\infty=\mathcal{J}(P)$;*
- *countably many Cantor sets of circles such that every pair of simple closed curves in each Cantor set of circles belong to different Julia components*;
- *and, if P is not affine conjugate to $z\mapsto z^n$, uncountably many point Julia components which accumulate everywhere on $\mathcal{J}(f)$.*

There is actually much more structure in the Julia set $\mathcal{J}(f)$. Indeed, countably many simple closed curves are eventually mapped under iteration onto a proper subset of ∂V_∞. Therefore every Julia component which contains such a simple closed curve is actually quasisymetrically equivalent to a finite covering of $\partial U_\infty = \mathcal{J}(P)$, and hence comes with infinitely many "decorations" attached to the simple closed curve, provided P is not affine conjugate to $z\mapsto z^n$ (this structure has already been noticed in [7, 11]). We will see in the proof below that except for the preimages of ∂V_∞, all others Julia components are either points or simple closed curves (in particular, every buried Julia component is either a point or a simple closed curve, compare with [12]), that provides a complete topological description of all Julia components.

Proof. We compute the degree of the rational map f. The Riemann–Hurwitz formula gives $2\deg(f) - 2 = \nu(f)$ where $\nu(f)$ denotes the number of critical points of f counted with multiplicity. From the definition of a McMullen-like mapping (Definition 2.2), we have

$$\nu(f) = \nu(f|_{V_\infty}) + \nu'(f) + \sum_{U_{i,j} \in \mathcal{D}} \nu(f|_{A_{i,j}}) + \sum_{U_{i,j} \in \mathcal{D}} \nu(f|_{T_{i,j}})$$

where

- $\nu(f|_{V_\infty}) = n - 1$ is the number of critical points of $f|_{V_\infty}$ counted with multiplicity;
- $\nu'(f)$ is the number of critical points of f counted with multiplicity which are neither in V_∞ nor in $\bigcup_{U_{i,j} \in \mathcal{D}} V_{i,j}$, namely

$$\begin{aligned} \nu'(f) &= \sum_{U \notin \mathcal{D}} (\deg(f|_{V(U)}) - 1) \\ &= \sum_{U \notin \mathcal{D}} (\deg(P|_U) - 1) = (n-1) - \sum_{U_{i,j} \in \mathcal{D}} (n_{i,j} - 1); \end{aligned}$$

- $\nu(f|_{A_{i,j}}) = n_{i,j} + d_{i,j}$ is the number of critical points of $f|_{A_{i,j}}$ counted with multiplicity;
- $\nu(f|_{T_{i,j}}) = d_{i,j} - 1$ is the number of critical points of $f|_{T_{i,j}}$ counted with multiplicity.

Putting everything together leads to

$$\begin{aligned} \deg(f) &= \frac{1}{2}(\nu(f) + 2) \\ &= \frac{1}{2}\Big(2n + \sum_{U_{i,j} \in \mathcal{D}} 2d_{i,j}\Big) \\ &= n + d \\ &= \deg(P) + \deg(\mathcal{D}). \end{aligned}$$

In order to prove that f is a hyperbolic map we study the orbit of every critical point. Let c be a critical point of f. From Definition 2.2 and Lemma 2.4, one of the following holds:

- either $c \in V_\infty$ (and hence $t_c = +\infty$), where V_∞ is a fixed simply connected domain, that corresponds to a fixed immediate attracting or super-attracting basin;
- or $c \in T_{i,j}$ for some $U_{i,j} \in \mathcal{D}$ (and hence $t_c = +\infty$), where $T_{i,j}$ is a simply connected preimage by f of V_∞;
- or $c \in A_{i,j}$ for some $U_{i,j} \in \mathcal{D}$ (and hence $t_c = t_{i,j}$), where $A_{i,j}$ is a doubly connected preimage by $f^{t_{i,j}+1}$ of V_∞;
- or $c \in V(U)$ for some $U \notin \mathcal{D}$ and $t_c < +\infty$, then c lies in a simply connected preimage by f^{t_c+1} of V_∞;
- or $c \in V(U)$ for some $U \notin \mathcal{D}$ and $t_c = +\infty$, then $V(U)$ is a simply connected domain which is eventually mapped onto a periodic cycle of simply connected domains of the form $V(U_{i,j})$ with $U_{i,j} \notin \mathcal{D}$, that corresponds to a periodic cycle of immediate attracting or super-attracting basins.

Therefore the rational map f is hyperbolic, and every Fatou component is either simply or doubly connected.

Now, we focus on the Julia set. $\mathcal{J}(f)$ is disconnected since there is at least one doubly connected Fatou component. From point (i) in Definition 2.2, ∂V_∞ is a fixed Julia component homeomorphic to ∂U_∞. Using the surgery procedure described in [15] (Sections 5 and 6), we can extend the topological conjugation $\varphi|_{\partial U_\infty} \colon \partial U_\infty \to \partial V_\infty$ to a quasiconformal conjugation on a neighborhood of ∂U_∞, and hence ∂V_∞ is quasisymetrically equivalent to $\partial U_\infty = \mathcal{J}(P)$.

Fix $U_{i,j} \in \mathcal{D}$ and consider the orbit of the closed annulus $\overline{A} = \overline{A^{\text{out}}_{i,j}} \cup A_{i,j} \cup \overline{A^{\text{in}}_{i,j}}$ where $\overline{A^{\text{out}}_{i,j}}$ and $\overline{A^{\text{in}}_{i,j}}$ are defined as in Lemma 2.3. It turns out that $\overline{f^{p_i}(A)}$ covers $V_{i,j}$. The preimage of $\overline{A}$ by f^{p_i} contains two disjoint closed annuli both nested in $\overline{A}$ (more precisely, they are nested in $\overline{A^{\text{out}}_{i,j}}$ and $\overline{A^{\text{in}}_{i,j}}$ respectively) which do not contain critical points of f^{p_i}. It is then straightforward to show (see [15] or [8]) that the non-escaping set $\overline{\bigcap_{k\geqslant 1}(f^{p_i})^{-k}(A)} \subset \mathcal{J}(f)$ is homeomorphic to a Cantor set of circles, namely to $\Sigma_2 \times \partial\mathbb{D}$ where $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ is the set of all one-sided sequences on two symbols. Moreover, the topological dynamics is conjugate to the following skew product

$$(\varepsilon_0\varepsilon_1\varepsilon_2\ldots, z) \in \Sigma_2 \times \partial\mathbb{D} \longmapsto \begin{cases} (\varepsilon_1\varepsilon_2\varepsilon_3\ldots\;, z^{n_{i,j}}) & \text{if } \varepsilon_0 = 1 \\ (\sigma(\varepsilon_1\varepsilon_2\varepsilon_3\ldots), z^{-d_{i,j}}) & \text{if } \varepsilon_0 = 0 \end{cases}$$

where

$$\sigma(\varepsilon_1\varepsilon_2\varepsilon_3\ldots) = (1-\varepsilon_1)(1-\varepsilon_2)(1-\varepsilon_3)\ldots.$$

Notice that every copy of $\partial\mathbb{D}$ which is eventually mapped under iteration by the 2-to-1 map σ to the fixed copy coded by $11111\ldots$, corresponds to a simple closed curve in $\mathcal{J}(f)$ which is eventually mapped under iteration by f^{p_i} to $\partial V_{i,j} \subset \partial V_\infty$. Therefore, every Julia component which contains such a simple closed curve is actually a preimage of ∂V_∞. The others copies correspond to buried simple closed curves, and are either preperiodic or wandering. In the first case (for instance the fixed copy coded by $01010\ldots$), the surgery procedure described in [15] (Sections 5 and 6) shows that the simple closed curve is the whole Julia component. The same holds in the second case according to the main result in [18].

Now if P is not affine conjugate to $z \mapsto z^n$, every preimage of ∂V_∞ comes with infinitely many "decorations". In particular, $V_{i,j}$ contains some disjoint closed disks which are preimages by f^{p_i} of the whole closed disks $\overline{V_{i,j}}$. Repeating this reasoning gives uncountably many sequences of nested closed disks. It is then straightforward to show (see [7, 11]) that the intersection of such a nested sequence is actually a point connected component of $\mathcal{J}(f)$. The grand orbit of such a point consists of point Julia components as well, and it accumulates everywhere on $\mathcal{J}(f)$. □

Theorem 2.6. *Two McMullen-like mappings are topologically conjugate on their Julia sets* (*by an orientation preserving homeomorphism of* $\widehat{\mathbb{C}}$) *if and only if they have same type* $(P, \mathcal{D})$ *up to conjugation by an affine map.*

Proof. Let f_1 and f_2 be two McMullen-like mappings, of type $(P_1, \mathcal{D}_1)$ and $(P_2, \mathcal{D}_2)$ respectively, which are topologically conjugate on their Julia sets. Since they both have only one Fatou component which contains the image of every trap door, it follows from point (i) in Definition 2.2 that P_1 and P_2 are topologically conjugate on their Julia sets, and hence are affine conjugate as HPcFP. It is straightforward to see that the combinatorial description of all Julia components in the proof of Theorem 2.5 concludes the proof. □

2.3. Known examples of McMullen-like mappings. In this subsection we show some known examples of McMullen-like mappings from the literature. For each example we focus in the pair $(P, \mathcal{D})$ where P is a HPcFP and $\mathcal{D}$ is the pole data in the definition of a McMullen-like mapping (see Definition 2.1 and Definition 2.2).

The first example is the McMullen family given by $f_\lambda(z) = z^n + \lambda/z^d$. This rational map has ∞ as a super-attracting fixed point and the only finite preimage of ∞ is the origin. Thus if V_∞ is the immediate super-attracting basin of ∞ and when V_∞ does not contain the origin, there exists a unique trap door T_0, a neighborhood of the origin, that is mapped d-to-1 onto V_∞. We also recall that f_λ has $n + d$

"free" critical points given by $c_\lambda = (\lambda d/n)^{1/(n+d)}$ and the corresponding free critical values are $v_\lambda = f_\lambda(c_\lambda)$. In Figure 1 we show the dynamical plane of the McMullen-like mapping $z \mapsto z^3 - 10^{-2}/z^3$.

In this case the polynomial $z \mapsto z^n$ is a HPcFP, this is a singular case since the Fatou set $\mathcal{F}(P)$ only contains two simply connected components: the immediate super-attracting basin of ∞ and the immediate super-attracting basin of 0. Thus there is a unique bounded Fatou component, denoted by U_0, which coincides with the unit disk. The pole data $\mathcal{D}$ is formed by U_0 and a positive integer $d = d_0 \geqslant 1$. From [15, 8, 7] we conclude the following result.

Proposition 2.7. *Let $f_\lambda(z) = z^n + \lambda/z^d$ be such that the free critical values v_λ belong to the trap door T_0 (or equivalently $|\lambda| \neq 0$ is mall enough) and the arithmetic condition $1/n + 1/d < 1$ is satisfied, then f_λ is a McMullen-like mapping.*

The next two examples of McMullen-like mappings are related to the polynomial $P_c(z) = z^n + c$ where $c \in \mathbb{C}$ is a center of a hyperbolic component of the corresponding Multibrot set. The choice of the parameter c ensures that P_c is HPcFP since the orbit of the critical point located at 0 is a periodic orbit. We denote by p the period of 0 and by $U_0, U_1, \ldots, U_{p-1}$ the Fatou components containing $0, P_c(0), \ldots, P_c^{p-1}(0)$, respectively.

The first McMullen-like mapping related to P_c was introduced in [2] where the authors studied the family of rational maps $g_\lambda(z) = z^n + c + \lambda/z^n$. In this case the pole data is formed by the Fatou component U_0 and the positive integer n, making thus the first generalization of the McMullen family. In Figure 1 we show the dynamical plane of the McMullen-like mapping $z \mapsto z^3 + i - 10^{-7}/z^3$. In [2] the authors proved the following result.

Proposition 2.8. *Let $g_\lambda(z) = z^n + c + \lambda/z^n$ be such that $|\lambda| \neq 0$ is small enough and the arithmetic condition $n > 2$ is satisfied, then g_λ is a McMullen-like mapping.*

The second McMullen-like mapping related to $P_c(z) = z^2 + c$ for $n = 2$ was introduced in [11] where the authors studied the family $h_\lambda(z) = z^2 + c + \lambda/\prod_{j=0}^{p-1}(z - c_j)^{d_j}$ with $c_j := P_c^j(0)$ for every $0 \leqslant j \leqslant p-1$. In this case h_λ has a pole of order d_j in every point c_j of the super-attracting orbit $c_0 = 0 \mapsto c_1 = P_c(0) \mapsto \ldots \mapsto c_{p-1} = P_c^{p-1}(0)$. The pole data $\mathcal{D}$ is formed by the Fatou components $\{U_0, U_1, \ldots, U_{p-1}\}$, and the corresponding positive integers $d_0, d_1, \ldots, d_{p-1}$. In Figure 1 we show the dynamical plane of the McMullen-like mapping $z \mapsto z^2 - 1 + 10^{-22}/(z^7(z+1)^5)$. In [11], the following is proved.

Proposition 2.9. *Let* $h_\lambda(z) = z^2 + c + \lambda / \prod_{j=0}^{p-1}(z - c_j)^{d_j}$ *be such that* $|\lambda| \neq 0$ *is small enough and the arithmetic conditions* $2d_1 > d_0 + 2$ *and* $d_{j+1} > d_j + 1$ *for every* $1 \leqslant j \leqslant p-1$ *(with* $d_p = d_0$*) are satisfied, then* h_λ *is a McMullen-like mapping.*

We notice that in these three examples, some arithmetic conditions are required to ensure that the corresponding rational maps are McMullen-like mappings. In the next section we state the main result of this paper that characterize a McMullen-like mapping of type $(P, \mathcal{D})$ using an arithmetic condition.

3. Existence of McMullen-like mappings

3.1. Theorem A: the arithmetic condition

Theorem A. *Let* $\mathcal{D}$ *be a pole data associated to a* HPcFP P. *Then there exists a McMullen-like mapping of type* $(P, \mathcal{D})$ *if and only if the following arithmetic condition holds:*

$$\max_{1\leqslant i\leqslant N} \left\{ \prod_{\substack{j\in\mathbb{Z}/p_i\mathbb{Z} \\ U_{i,j}\notin\mathcal{D}}} \frac{1}{n_{i,j}} \times \prod_{\substack{j\in\mathbb{Z}/p_i\mathbb{Z} \\ U_{i,j}\in\mathcal{D}}} \left(\frac{1}{n_{i,j}} + \frac{1}{d_{i,j}}\right) \right\} < 1 \qquad (\star)$$

Let us give some remarks about this arithmetic condition.

Remark 1. It is enough to check the arithmetic condition ($\star$) for every $1 \leqslant i \leqslant N$ such that the set of indices $J_i = \{j \in \mathbb{Z}/p_i\mathbb{Z} \mid U_{i,j} \in \mathcal{D}\}$ is not empty. Indeed, for every $1 \leqslant i \leqslant N$ there is at least one degree $n_{i,j} \geqslant 2$, and hence $\prod_{j\in\mathbb{Z}/p_i\mathbb{Z}} 1/n_{i,j} < 1$. Moreover, notice that if $n_{i,j} \geqslant 2$ and $d_{i,j} \geqslant 3$ for every $U_{i,j} \in \mathcal{D}$, then the arithmetic condition ($\star$) holds. Equivalently speaking, it is sufficient that every periodic bounded Fatou component picked in the pole data contains a critical point, and every degree in the pole data is larger than 3.

Remark 2. If P is affine conjugate to $z \mapsto z^n$, namely if $N = 1$ and $p_1 = 1$, then the arithmetic condition ($\star$) reduces to the well known arithmetic condition $1/n + 1/d < 1$ (see [15] and Proposition 2.7). The same holds if P is affine conjugate to $z \mapsto z^n + c$ where c is a center of a hyperbolic component of the corresponding Multibrot set, and if $\mathcal{D}$ only consists of the bounded Fatou component containing the unique critical point (see Propositions 2.8 and 4.2). The arithmetic conditions introduced in [11] (see Proposition 2.9) implies the arithmetic condition ($\star$), but the converse is not true (see Proposition 4.3).

Remark 3. It is straightforward to show that the degree of a McMullen-like mapping of type $(P, \mathcal{D})$, that is $\deg(P) + \deg(\mathcal{D})$, has a lower bound according to condition $(\star)$. This lower bound is reached for a polynomial P of degree $n = 3$ with two simple critical points in a same super-attracting cycle of period 2, and a pole data which only consists of one of the two bounded Fatou components containing a critical point with $d = 1$ (see Proposition 4.1). Indeed, in that case $(\star)$ reduces to $1/2 \times (1/2 + 1/1) = 3/4 < 1$. In particular, there is no McMullen-like mappings of degree less than 4.

Remark 4. Finally, and according to Theorem 2.6, the set of all types $(P, \mathcal{D})$, where P is a monic centered HPcFP and $\mathcal{D}$ is an associated pole data which satisfies condition $(\star)$, is in 1-to-1 correspondence with the set of all topological conjugation classes of Julia sets of McMullen-like mappings.

3.2. Proof of Theorem A: necessity of the arithmetic condition. The main ingredient to prove the necessity of the arithmetic condition $(\star)$ is the theory of Thurston obstructions for rational maps (see [10, 14]).

Assume there exists a McMullen-like mapping f of a given type $(P, \mathcal{D})$. Denote by $\mathcal{P}_f$ the closure of its postcritical set. Fix $1 \leqslant i \leqslant N$ such that the set of indices $J_i = \{j \in \mathbb{Z}/p_i\mathbb{Z} \mid U_{i,j} \in \mathcal{D}\}$ is not empty. Remark that every $U_{i,j}$ may be uniquely written as $U_{i,j'+k}$ for some $U_{i,j'} \in \mathcal{D}$ and some $1 \leqslant k \leqslant t_{i,j'}$.

For every $j \in \mathbb{Z}/p_i\mathbb{Z}$, consider an arbitrary simple closed curve $\Gamma_{i,j}$ in $V_{i,j}$ such that

- $\Gamma_{i,j}$ separates $\partial V_{i,j}$ from $\overline{f^k(A_{i,j'})} \subset V_{i,j}$ if $U_{i,j} = U_{i,j'+k} \notin \mathcal{D}$ with $1 \leqslant k < t_{i,j'}$;
- $\Gamma_{i,j}$ separates $\partial V_{i,j}$ from $\overline{f^{t_{i,j'}}(A_{i,j'})} = \overline{T_{i,j}}$ (by Lemma 2.4) if $U_{i,j} = U_{i,j'+t_{i,j'}} \in \mathcal{D}$.

From Lemma 2.3, every $f^k(A_{i,j'})$ contains at least $n_{i,j'} + d_{i,j'} \geqslant 2$ postcritical points. Thus, it is straightforward to show that $\Gamma_i = \{\Gamma_{i,j} \mid j \in \mathbb{Z}/p_i\mathbb{Z}\}$ is a multicurve, namely a finite collection of disjoint, non-homotopic, and non-peripheral simple closed curves in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ (recall that a simple closed curve in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ is said to be non-peripheral if each connected component of its complement contains at least two points in $\mathcal{P}_f$).

For every $\Gamma_{i,j} \in \Gamma_i$, consider the connected components of $f^{-1}(\Gamma_{i,j})$ which are homotopic in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ to some simple closed curves in Γ_i. According to Definition 2.2 and Lemma 2.3, any such connected component lies in $V_{i,j-1}$, and hence is homotopic in $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ to $\Gamma_{i,j-1}$. More precisely, there are

- only one connected component if $U_{i,j-1} \notin \mathcal{D}$, which is mapped onto $\Gamma_{i,j}$ with degree $n_{i,j-1}$ (from point (ii) in Definition 2.2);

- two connected components if $U_{i,j-1} \in \mathcal{D}$, one in $A^{\text{out}}_{i,j-1}$ and one in $A^{\text{in}}_{i,j-1}$, which are mapped onto $\Gamma_{i,j}$ with degrees $n_{i,j-1}$ and $d_{i,j-1}$ respectively (from Lemma 2.3).

It follows that the transition matrix associated to the multicurve Γ_i may be written as

$$\begin{pmatrix} 0 & m_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & m_{2,3} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & m_{p_i-1,p_i} \\ m_{p_i,1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where

$$m_{j-1,j} = \begin{cases} \dfrac{1}{n_{i,j-1}} & \text{if } U_{i,j-1} \notin \mathcal{D}, \\[2mm] \dfrac{1}{n_{i,j-1}} + \dfrac{1}{d_{i,j-1}} & \text{if } U_{i,j-1} \in \mathcal{D}. \end{cases}$$

It is straightforward to show that the associated leading eigenvalue is

$$\begin{aligned} \lambda(\Gamma_i) &= \Big[\prod_{j \in \mathbb{Z}/p_i\mathbb{Z}} m_{j-1,j} \Big]^{\frac{1}{p_i}} \\ &= \Big[\prod_{\substack{j \in \mathbb{Z}/p_i\mathbb{Z} \\ U_{i,j} \notin \mathcal{D}}} \frac{1}{n_{i,j}} \times \prod_{\substack{j \in \mathbb{Z}/p_i\mathbb{Z} \\ U_{i,j} \in \mathcal{D}}} \Big(\frac{1}{n_{i,j}} + \frac{1}{d_{i,j}} \Big) \Big]^{\frac{1}{p_i}}. \end{aligned}$$

Now applying Theorem B.3 and Theorem B.4 from [14] to the hyperbolic rational map f, we get $\lambda(\Gamma_i) < 1$ for every $1 \leqslant i \leqslant N$ such that $J_i \neq \emptyset$, that implies the arithmetic condition ($\star$).

3.3. Proof of Theorem A: construction of McMullen-like mappings. In this section, we construct a McMullen-like mapping of an arbitrary given type $(P, \mathcal{D})$ which satisfies the arithmetic condition ($\star$), that will show the sufficiency of ($\star$) and conclude the proof of Theorem A. The method is to start from the polynomial P and to add a pole of degree $d_{i,j}$ on every $U_{i,j} \in \mathcal{D}$ by quasiconformal surgery (we refer readers to [3] for a comprehensive treatment on this powerful method). At first, we need the two following technical lemmas that will allow us to divide the Riemann sphere $\widehat{\mathbb{C}}$ into several pieces on which a quasiregular map will be defined.

Lemma 3.1. *If the arithmetic condition* ($\star$) *holds then there exist some positive real numbers* $(\alpha_{i,j})_{U_{i,j}\in\mathcal{D}}$ *and* $(\beta_{i,j})_{U_{i,j}\in\mathcal{D}}$ *such that, for all* $U_{i,j}\in\mathcal{D}$,

$$\alpha_{i,j} < \beta_{i,j}$$

and

$$\Big(\frac{n_{i,j+t_{i,j}}}{d_{i,j+t_{i,j}}}\alpha_{i,j+t_{i,j}} + \beta_{i,j+t_{i,j}}\Big) < \Big(\prod_{k=0}^{t_{i,j}-1} n_{i,j+k}\Big)\alpha_{i,j}.$$

Proof. Fix $1 \leqslant i \leqslant N$ such that the set of indices $J_i = \{j \in \mathbb{Z}/p_i\mathbb{Z} \mid U_{i,j} \in \mathcal{D}\}$ is not empty, and let M be the positive constant

$$M = \Big[\prod_{\substack{j\in\mathbb{Z}/p_i\mathbb{Z}\\ U_{i,j}\notin\mathcal{D}}} \frac{1}{n_{i,j}} \times \prod_{\substack{j\in\mathbb{Z}/p_i\mathbb{Z}\\ U_{i,j}\in\mathcal{D}}} \Big(\frac{1}{n_{i,j}} + \frac{1}{d_{i,j}}\Big)\Big]^{-1/2|J_i|}$$

$$= \prod_{j\in J_i}\Big[\prod_{k=1}^{t_{i,j}-1}\frac{1}{n_{i,j+k}} \times \Big(\frac{1}{n_{i,j}} + \frac{1}{d_{i,j}}\Big)\Big]^{-1/2|J_i|}.$$

Remark that the arithmetic condition ($\star$) precisely implies that $M > 1$.

Now pick one $j_0 \in J_i$, let α_{i,j_0} be any positive real number, and recursively define $\alpha_{i,j}$ for every $j \in J_i$ by

$$\alpha_{i,j+t_{i,j}} = \Big(M^{-2}\frac{n_{i,j}}{n_{i,j+t_{i,j}}}\Big[\prod_{k=1}^{t_{i,j}-1}\frac{1}{n_{i,j+k}} \times \Big(\frac{1}{n_{i,j+t_{i,j}}} + \frac{1}{d_{i,j+t_{i,j}}}\Big)\Big]^{-1}\Big)\alpha_{i,j}. \quad (2)$$

The $(\alpha_{i,j})_{j\in J_i}$ are well defined because the product over $j \in J_i$ of the terms in the biggest brackets is 1 by definition of M. Define $\beta_{i,j}$ for every $j \in J_i$ so that

$$Mn_{i,j}\Big(\frac{1}{n_{i,j}} + \frac{1}{d_{i,j}}\Big)\alpha_{i,j} = \frac{n_{i,j}}{d_{i,j}}\alpha_{i,j} + \beta_{i,j}$$

or equivalently

$$\beta_{i,j} = M\alpha_{i,j} + (M-1)\frac{n_{i,j}}{d_{i,j}}\alpha_{i,j}.$$

Remark that $\beta_{i,j} > \alpha_{i,j}$ since $M > 1$. Moreover, (2) may be rewritten as follows

$$Mn_{i,j+t_{i,j}}\Big(\frac{1}{n_{i,j+t_{i,j}}} + \frac{1}{d_{i,j+t_{i,j}}}\Big)\alpha_{i,j+t_{i,j}} = M^{-1}\Big(\prod_{k=0}^{t_{i,j}-1} n_{i,j+k}\Big)\alpha_{i,j}.$$

Using the definition of $\beta_{i,j+t_{i,j}}$ above concludes the proof since $M^{-1} < 1$. $\square$

For every periodic bounded Fatou component $U_{i,j}$, Böttcher's Theorem provides a Riemann mapping $\phi_{i,j}\colon \mathbb{D} \to U_{i,j}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathbb{D} & \xrightarrow{z \mapsto z^{n_{i,j}}} & \mathbb{D} & \xrightarrow{z \mapsto z^{n_{i,j+1}}} & \mathbb{D} & \longrightarrow & \cdots \\
\phi_{i,j}\downarrow & & \phi_{i,j+1}\downarrow & & \phi_{i,j+2}\downarrow & & \\
U_{i,j} & \xrightarrow[P]{} & U_{i,j+1} & \xrightarrow[P]{} & U_{i,j+2} & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \longrightarrow & \mathbb{D} & \xrightarrow{z \mapsto z^{n_{i,j+p_i-1}}} & \mathbb{D} \\
& & \phi_{i,j+p_i-1}\downarrow & & \phi_{i,j+p_i} = \phi_{i,j}\downarrow \\
\cdots & \longrightarrow & U_{i,j+p_i-1} & \xrightarrow[P]{} & U_{i,j+p_i} = U_{i,j}.
\end{array}$$

An equipotential γ in some $U_{i,j}$ is the image by $\phi_{i,j}$ of an Euclidean circle in $\mathbb{D}$ centered at 0. The radius of this circle is called the level of γ, and is denoted by $L_{i,j}(\gamma) \in]0,1[$ in order that $\gamma = \{z \in U_{i,j} \mid |\phi_{i,j}^{-1}(z)| = L_{i,j}(\gamma)\}$.

Similarly, Böttcher's Theorem provides a Riemann mapping $\phi_\infty\colon \widehat{\mathbb{C}}\setminus\mathbb{D} \to U_\infty$ which conjugates $P|_{U_\infty}$ with $z \mapsto z^n$. The equipotentials in U_∞ are defined as well (with level > 1).

Recall that any pair of disjoint continua γ, γ' in $\widehat{\mathbb{C}}$ uniquely defines a doubly connected domain denoted by $A(\gamma,\gamma')$. If γ, γ' contain at least two points each, $A(\gamma,\gamma')$ is biholomorphic to the round annulus $\{z \in \mathbb{C} \mid r < |z| < 1\}$ where $r \in]0,1[$ does not depend on the choice of biholomorphism. The modulus of $A(\gamma,\gamma')$ is defined to be $\operatorname{mod}(\gamma,\gamma') = (2\pi)^{-1}\log(1/r)$. In particular if γ, γ' are two equipotentials in some $U_{i,j}$ of levels $L_{i,j}(\gamma) > L_{i,j}(\gamma')$ then

$$\operatorname{mod}\big(\gamma,\gamma'\big) = \frac{1}{2\pi}\log\Big(\frac{L_{i,j}(\gamma)}{L_{i,j}(\gamma')}\Big).$$

Lemma 3.2. *If the arithmetic condition* ($\star$) *holds then there exist an equipotential* Γ_∞ *in* U_∞, *and three equipotentials* $\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty}$ *in every* $U_{i,j} \in \mathcal{D}$ *such that*

(i) $L_{i,j}(\gamma_{i,j}^{\text{out}}) > L_{i,j}(\gamma_{i,j}^{\text{in}}) > L_{i,j}(\gamma_{i,j}^{\infty})$;

(ii) $\operatorname{mod}(\gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty}) = \frac{1}{d_{i,j}}\operatorname{mod}(\Gamma_{i,j+1}, \Gamma_\infty)$ *where* $\Gamma_{i,j+1} = P(\gamma_{i,j}^{\text{out}})$;

(iii) $L_{i,j+t_{i,j}}(\gamma_{i,j+t_{i,j}}^{\infty}) > \big(\prod_{k=0}^{t_{i,j}-1} n_{i,j+k}\big)L_{i,j}(\gamma_{i,j}^{\text{out}})$.

Note that $\Gamma_{i,j+1}$ is actually an equipotential in $U_{i,j+1}$ of level

$$L_{i,j+1}(\Gamma_{i,j+1}) = n_{i,j}L_{i,j}(\gamma_{i,j}^{\text{out}}).$$

Proof. Fix Γ_∞ to be any arbitrary equipotential in U_∞. Let r be a real number in $]0,1[$. In each $U_{i,j} \in \mathcal{D}$, define $\gamma_{i,j}^{\text{out}}$ and $\gamma_{i,j}^{\text{in}}$ to be the equipotentials of levels $L_{i,j}(\gamma_{i,j}^{\text{out}}) = r^{\alpha_{i,j}}$ and $L_{i,j}(\gamma_{i,j}^{\text{in}}) = r^{\beta_{i,j}}$ respectively, where $\alpha_{i,j}$ and $\beta_{i,j}$ come from Lemma 3.1. Notice that $L_{i,j}(\gamma_{i,j}^{\text{out}}) > L_{i,j}(\gamma_{i,j}^{\text{in}})$ since $\alpha_{i,j} < \beta_{i,j}$. Define $\gamma_{i,j}^{\infty}$ in every $U_{i,j} \in \mathcal{D}$ so that point (ii) holds, or equivalently $L_{i,j}(\gamma_{i,j}^{\infty}) = r^{\delta_{i,j}}$ with

$$\delta_{i,j} = \beta_{i,j} + \frac{\frac{1}{d_{i,j}}\operatorname{mod}(\Gamma_{i,j+1}, \Gamma_\infty)}{\frac{1}{2\pi}\ln(\frac{1}{r})}.$$

In particular, $L_{i,j}(\gamma_{i,j}^{\text{in}}) > L_{i,j}(\gamma_{i,j}^{\infty})$ since $\beta_{i,j} < \delta_{i,j}$ that completes point (i). For the last point, using the reversed Grötzch inequality (see Appendix B in [5]),

$$\begin{aligned}\operatorname{mod}(\Gamma_{i,j+t_{i,j}+1}, \Gamma_\infty) &\leqslant C + \frac{1}{2\pi}\ln\left(\frac{1}{L_{i,j+t_{i,j}+1}(\Gamma_{i,j+t_{i,j}+1})}\right) \\ &= C + \frac{n_{i,j+t_{i,j}}}{2\pi}\alpha_{i,j+t_{i,j}}\ln\left(\frac{1}{r}\right),\end{aligned}$$

where $C > 0$ does not depend on r. Putting this in the expression of $\delta_{i,j+t_{i,j}}$ leads to

$$\begin{aligned}\delta_{i,j+t_{i,j}} &\leqslant \left(\frac{n_{i,j+t_{i,j}}}{d_{i,j+t_{i,j}}}\alpha_{i,j+t_{i,j}} + \beta_{i,j+t_{i,j}}\right) + \frac{2\pi C}{d_{i,j+t_{i,j}}\ln\left(\frac{1}{r}\right)} \\ &< \left(\prod_{k=0}^{t_{i,j}-1} n_{i,j+k}\right)\alpha_{i,j},\end{aligned}$$

provided $r > 0$ is small enough according to Lemma 3.1. Point (iii) follows. □

Now we are going to piecewisely define a quasiregular map F on $\widehat{\mathbb{C}}$ according to a partition induced by the equipotentials coming from Lemma 3.2. Let D_∞ be the unbounded connected component of $\widehat{\mathbb{C}} \setminus \Gamma_\infty$, and W_∞ be the unbounded connected component of $\widehat{\mathbb{C}} \setminus \bigcup_{U_{i,j}\in\mathcal{D}} \gamma_{i,j}^{\text{out}}$. Denote by $D(\gamma)$ the bounded connected component of $\widehat{\mathbb{C}} \setminus \gamma$ for every simple closed curve γ in $\mathbb{C}$. Consider the partition of $\widehat{\mathbb{C}}$

$$\widehat{\mathbb{C}} = W_\infty \bigcup_{U_{i,j}\in\mathcal{D}} (\overline{A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})} \cup A(\gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty}) \cup \overline{D(\gamma_{i,j}^{\infty})}).$$

On W_∞. Define $F|_{W_\infty}$ to be the polynomial P in order that F continuously extend to every $\gamma_{i,j}^{\text{out}}$ with

$$F(\gamma_{i,j}^{\text{out}}) = P(\gamma_{i,j}^{\text{out}}) = \Gamma_{i,j+1}$$

and $\deg(F|_{\gamma_{i,j}^{\text{out}}}) = n_{i,j}$.

On every $A(\gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty})$. Define $F|_{A(\gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty})}$ in order that

$$F(A(\gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty})) = A(\Gamma_{i,j+1}, \Gamma_\infty)$$

and $F|_{A(\gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty})}$ is a holomorphic covering of degree $d_{i,j}$ which continuously extends on the boundary with $F(\gamma_{i,j}^{\text{in}}) = \Gamma_{i,j+1}$ and $F(\gamma_{i,j}^{\infty}) = \Gamma_\infty$. Point (ii) in Lemma 3.2 ensures that such a holomorphic covering exists. Note that

$$\deg(F|_{\gamma_{i,j}^{\text{in}}}) = \deg(F|_{\gamma_{i,j}^{\infty}}) = d_{i,j}.$$

On every $\overline{A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})}$. Continuously extend F so that

$$F(A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})) = D(\Gamma_{i,j+1})$$

and $F|_{A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})}$ is a quasiregular branched covering. This extension must have degree $\deg(F|_{\gamma_{i,j}^{\text{out}}}) + \deg(F|_{\gamma_{i,j}^{\text{in}}}) = n_{i,j} + d_{i,j}$, and $n_{i,j} + d_{i,j}$ critical points counted with multiplicity by the Riemann–Hurwitz formula (compare with the action of a McMullen-like mapping on every $A_{i,j}$ in Lemma 2.3). The existence of such an annulus-disk map is discussed in [3] (the section about annulus-disk surgery by K. Pilgrim and Tan L.) and [12].

On every $\overline{D(\gamma_{i,j}^{\infty})}$. Continuously extend F so that

$$F(D(\gamma_{i,j}^{\infty})) = D_\infty$$

and $F|_{D(\gamma_{i,j}^{\infty})}$ is a quasiregular branched covering. This extension must have degree $\deg(F|_{\gamma_{i,j}^{\infty}}) = d_{i,j}$, and $d_{i,j} - 1$ critical points counted with multiplicity by the Riemann–Hurwitz formula (compare with the action of a McMullen-like mapping on every $T_{i,j}$ in Lemma 2.3). To construct such a quasiregular branched covering, we may start from the map

$$\phi_\infty \circ \left(z \longmapsto L_\infty(\Gamma_\infty)\left(\frac{L_{i,j}(\gamma_{i,j}^{\infty})}{z}\right)^{d_{i,j}}\right) \circ \phi_{i,j}^{-1}$$

which is a holomorphic branched covering of degree $d_{i,j}$ from $D(\gamma_{i,j}^{\infty})$ onto D_∞, and then quasiconformally modify it in a neighborhood of $\gamma_{i,j}^{\infty}$ in order that the continuous extension on $\gamma_{i,j}^{\infty}$ agrees with the definition of F on $A(\gamma_{i,j}^{\text{in}}, \gamma_{i,j}^{\infty})$.

Remark that F is holomorphic outside $\bigcup_{U_{i,j}\in\mathcal{D}}(\overline{A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})} \cup \overline{D(\gamma_{i,j}^{\infty})})$. The following lemma shows that this set does not intersect its forward orbit after finitely many iterations.

Lemma 3.3. *For every $U_{i,j} \in \mathcal{D}$, the following hold:*

$$F^{t_{i,j}}(\overline{A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})}) \subset D(\gamma_{i,j+t_{i,j}}^{\infty}),$$

$$F(\overline{D(\gamma_{i,j}^{\infty})}) = \overline{D_\infty},$$

and

$$F(\overline{D_\infty}) \subset D_\infty \subset W_\infty.$$

Proof. By definition of F on $\overline{A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})}$ and W_∞,

$$\begin{aligned} F^{t_{i,j}}(\overline{A(\gamma_{i,j}^{\text{out}}, \gamma_{i,j}^{\text{in}})}) &= F^{t_{i,j}-1}(\overline{D(\Gamma_{i,j+1})}) \\ &= P^{t_{i,j}-1}(\overline{D(\Gamma_{i,j+1})}) \\ &= \overline{D(\Gamma_{i,j+t_{i,j}})}, \end{aligned}$$

where $\Gamma_{i,j+t_{i,j}} = P^{t_{i,j}-1}(\Gamma_{i,j+1}) = P^{t_{i,j}}(\gamma_{i,j}^{\text{out}})$ is an equipotential in $U_{i,j+t_{i,j}}$ of level

$$L_{i,j+t_{i,j}}(\Gamma_{i,j+t_{i,j}}) = \Big(\prod_{k=0}^{t_{i,j}-1} n_{i,j+k}\Big) L_{i,j}(\gamma_{i,j}^{\text{out}}).$$

Point (iii) in Lemma 3.2 gives $L_{i,j+t_{i,j}}(\gamma_{i,j+t_{i,j}}^{\infty}) > L_{i,j+t_{i,j}}(\Gamma_{i,j+t_{i,j}})$ that implies the first inclusion. The equality follows from definition of F on $\overline{D(\gamma_{i,j}^{\infty})}$. Finally, $D_\infty \subset U_\infty \subset W_\infty$ by definition of W_∞, and hence $F|_{\overline{D_\infty}} = P|_{\overline{D_\infty}}$ is conjugate to the action of $z \mapsto z^n$ on $\widehat{\mathbb{C}} \setminus L_\infty(\Gamma_\infty)\mathbb{D}$ that concludes the proof. $\square$

As a consequence, the iterated pullback by F of the standard complex structure provides a F-invariant Beltrami form on $\widehat{\mathbb{C}}$ with uniformly bounded dilatation. Then, after integrating, there exists a quasiconformal map $\Phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f = \Phi \circ F \circ \Phi^{-1}$ is holomorphic on $\widehat{\mathbb{C}}$, namely a rational map. We refer readers to [21, 3] for more details about this result, which is known as the Shishikura principle for quasiconformal surgery.

Finally, it is straightforward to see that f is actually a McMullen-like mapping of type $(P, \mathcal{D})$ by construction.

4. Further examples of McMullen-like mappings

In this section we present several new specific examples of McMullen-like mappings showing how to check in practice the conditions in Definition 2.2.

In the first example we give an explicit expression of a McMullen-like mapping of minimal degree 4 (see Remark 3 below Theorem A). We notice that the first example of a rational map of degree less than 5 with buried Julia components was recently founded in [12] for an explicit family of cubic rational maps. We prove the following result.

Proposition 4.1. *Let* $q_\lambda(z) = 2z^3 - 3z^2 + 1 + \lambda/z$ *such that* $|\lambda| \neq 0$ *is small enough, then* q_λ *is a McMullen-like mapping of minimal degree 4.*

Proof. We first consider the basic dynamics of the cubic polynomial $Q(z) = 2z^3 - 3z^2 + 1$. It is a hyperbolic postcritically finite polynomial. The critical points of Q are ∞, 0, and 1 since $Q'(z) = 6z(z-1)$. There are two super-attracting cycles, the first one is $\infty \mapsto \infty$ and the second one is $0 \mapsto 1 \mapsto 0$. We denote by U_∞, U_0, and U_1 the Fatou components of $\mathcal{F}(Q)$ containing ∞, 0, and 1, respectively. The dynamics of Q on these Fatou components is $Q : U_\infty \to U_\infty$ with degree 3, $Q : U_0 \to U_1$ with degree 2, and $Q : U_1 \to U_0$ also with degree 2. In Figure 2 we show the dynamical planes of Q and q_λ.

We show that q_λ is a McMullen-like mapping of type $(Q, \mathcal{D})$ checking in turn the conditions in Definition 2.2, where $Q(z) = 2z^3 - 3z^2 + 1$ and the pole data $\mathcal{D}$ is formed by the Fatou component U_0 (containing the origin) associated to the integer $d = d_0 = 1$. We prove the first two conditions using a holomorphic motion, for λ small, of U_∞ parametrized by λ obtaining the immediate super-attractive basin of ∞ of q_λ as a result of this movement. Applying the Λ−Lemma, established by R. Mañé, P. Sad and D. Sullivan in [13], we extend this holomorphic motion to the boundary of U_∞ which coincides with $\mathcal{J}(Q)$. This establish that the boundary of the immediate super-attracting basin of ∞ of q_λ is a holomorphic motion of $\mathcal{J}(Q)$ and this new holomorphic motion is precisely the conjugacy between $q_0 = Q$ acting on $\mathcal{J}(Q)$ and q_λ acting on the boundary of the immediate super-attracting basin of ∞ of q_λ. Finally, we prove the last two conditions in Definition 2.2 studying the behavior of the critical points of q_λ.

First of all we can compute the critical points and the critical values of q_λ. Obviously $z = \infty$ is a super-attracting fixed point of q_λ, near infinity the rational map q_λ is conformally conjugate to $z \mapsto z^3$. Moreover, since the degree of q_λ is 4 we have that q_λ has six critical points counted with multiplicity: ∞ with multiplicity 2 and the four other critical points are solutions of $q'_\lambda(z) = 0$. Easy computations show that these four "free" critical points are the solutions of

$$6z^3(z-1) = \lambda.$$

For small values of $|\lambda|$, the above equation has three solutions near 0, denoted by $c_0(\lambda)$, and one solution near 1, denoted by $c_1(\lambda)$. We have

$$c_0(\lambda) \simeq \sqrt[3]{\frac{-\lambda}{6}}$$

and

$$c_1(\lambda) \simeq 1 + \frac{\lambda}{6}.$$

The corresponding free critical values

$$v_0(\lambda) = q_\lambda(c_0(\lambda))$$

and

$$v_1(\lambda) = q_\lambda(c_1(\lambda)),$$

are given by

$$v_0(\lambda) \simeq 1 - \frac{9}{\sqrt[3]{36}}(-\lambda)^{2/3} - \frac{1}{3}\lambda$$

and

$$\begin{aligned} v_1(\lambda) &\simeq \frac{\lambda}{1+\frac{\lambda}{6}} + \frac{1}{12}\lambda^2 + \frac{1}{108}\lambda^3 \\ &= \lambda - \frac{1}{12}\lambda^2 + \mathcal{O}(\lambda^3). \end{aligned}$$

Therefore $v_0(\lambda) \to 1$ and $v_1(\lambda) \to 0$ as $\lambda \to 0$.

We denote by $V_\infty(\lambda)$ the immediate super-attracting basin of $z = \infty$ of q_λ. As it is well known, there is a Böttcher coordinate ϕ_λ defined in a neighborhood of ∞ in $V_\infty(\lambda)$ that conjugates q_λ to $z \mapsto z^3$ in a neighborhood of ∞. If none of the free critical points lie in $V_\infty(\lambda)$, then it is well known that we may extend ϕ_λ so that it takes the entire immediate super-attracting basin univalently onto $\mathbb{C} \setminus \overline{\mathbb{D}}$. If λ is sufficiently small then none of the free critical points lie in $V_\infty(\lambda)$, since they are close to 0 and 1.

Thus, we define $\delta > 0$ such that the Böttcher map ϕ_λ extends to the whole immediate super-attracting basin $V_\infty(\lambda)$ for all $|\lambda| < \delta$. We denote by $\mathbb{D}_\delta$ the round disk $\{\lambda \in \mathbb{C} \mid |\lambda| < \delta\}$. We consider the map

$$\begin{aligned} H : V_\infty(0) \times \mathbb{D}_\delta &\longrightarrow \widehat{\mathbb{C}}, \\ (z, \lambda) &\longmapsto \phi_\lambda^{-1} \circ \phi_0(z). \end{aligned}$$

Next we verify that H is a holomorphic motion. By construction, we have that $H(z,0) = \phi_0^{-1} \circ \phi_0(z) = z$. If we fix the parameter λ we can see that the map $z \mapsto H(z,\lambda)$ is injective. This is immediate since the Böttcher map ϕ_λ is conformal. Finally, if we fix a point $z \in V_\infty(0)$ we can see that $\lambda \mapsto H(z,\lambda)$ is a holomorphic map. Indeed this map is a composition of holomorphic maps, since the Böttcher map depends analytically on the parameter λ.

Applying the Λ-Lemma to H, we obtain a new holomorphic motion

$$\bar{H} : \overline{V_\infty(0)} \times \mathbb{D}_\delta \longrightarrow \widehat{\mathbb{C}}.$$

Hence, it follows that the boundary of $V_\infty(\lambda)$ is the continuous image under

$$\bar{H}(\cdot,\lambda) : z \longmapsto \bar{H}(z,\lambda)$$

of the Julia set of Q and the following diagram commutes

$$\begin{array}{ccc} \mathcal{J}(Q) & \xrightarrow{\;Q\;} & \mathcal{J}(Q) \\ {\scriptstyle \bar{H}(\cdot,\lambda)}\downarrow & & \downarrow{\scriptstyle \bar{H}(\cdot,\lambda)} \\ \partial V_\infty(\lambda) & \xrightarrow[q_\lambda]{} & \partial V_\infty(\lambda) \end{array}$$

proving thus conditions (i) and (ii) in Definition 2.2.

Hereafter we consider $|\lambda| < \delta$. We observe that $z = 0$ is the unique finite preimage of $z = \infty$, so in this case q_λ has a unique trap door $T_0(\lambda)$ containing the origin and such that $q_\lambda : T_0(\lambda) \to V_\infty(\lambda)$ has degree 1. We can compute now the preimage of $T_0(\lambda)$. We claim that the preimage of $T_0(\lambda)$ is formed by three simply connected domains $W_1(\lambda)$, $W_{-1/2}(\lambda)$, and $W_0(\lambda)$. To see the claim we observe that the polynomial $Q(z) = 2z^3 - 3z^2 + 1 = 2(z-1)^2(z+1/2)$ has three roots counted with multiplicity: a double root at $z = 1$, and a simple root at $z = -1/2$. By continuity, a neighborhood of $z = -1/2$, denoted by $W_{-1/2}(\lambda)$, is still mapped under q_λ to $T_0(\lambda)$ with degree 1 and a neighborhood of $z = 1$, denoted by $W_1(\lambda)$ is still mapped under q_λ to $T_0(\lambda)$ with degree 2. Notice that $W_1(\lambda)$ contains the free critical point $c_1(\lambda)$. Near the origin there exists another preimage of the trap door, denoted by $W_0(\lambda)$ since q_λ sends the trap door $T_0(\lambda)$ onto $V_\infty(\lambda)$.

We can show now that the three remaining critical points, denoted by $c_0(\lambda)$, belong to a doubly connected domain $A_0(\lambda)$ separating $T_0(\lambda)$ from $V_\infty(\lambda)$. We can compute the preimage of $W_1(\lambda)$ by q_λ. First we observe that $Q(3/2) = 1$ since $Q(z) = z^2(2z-3)+1$, so again by continuity there exists a simply connected domain near $z = 3/2$ that is mapped under q_λ onto $W_1(\lambda)$. As we show before the corresponding free critical values $v_0(\lambda) = q_\lambda(c_0(\lambda))$ are close to 1, hence in $W_1(\lambda)$ for λ sufficiently small. Thus the three critical points $c_0(\lambda)$ belong to a preimage of $W_1(\lambda)$ and applying the Riemann–Hurwitz formula we obtain that the three critical points belong to a doubly connected domain $A_0(\lambda)$, obtaining thus that $q_\lambda\colon A_0(\lambda) \to W_1(\lambda)$ with degree 3. Since $W_1(\lambda)$ is a preimage of the trap door we have that $q_\lambda^2(A_0(\lambda)) = q_\lambda(W_1(\lambda)) = T_0(\lambda)$ proving condition (iii) in Definition 2.2. Since q_λ has no other finite critical points than $c_0(\lambda)$ and $c_1(\lambda)$, condition (iv) follows that proves that q_λ is a McMullen-like mapping for λ sufficiently small.

Finally, we can check the arithmetic condition ($\star$) for the McMullen-like mapping q_λ. We have that $U_0 \in \mathcal{D}$, $Q\colon U_0 \to U_1$ with degree 2, $U_1 \notin \mathcal{D}$, $Q\colon U_1 \to U_0$ with degree 2, and $d = d_0 = 1$, so the arithmetic condition writes as

$$\frac{1}{2}\Big(\frac{1}{2} + \frac{1}{1}\Big) = \frac{3}{4} < 1. \qquad \square$$

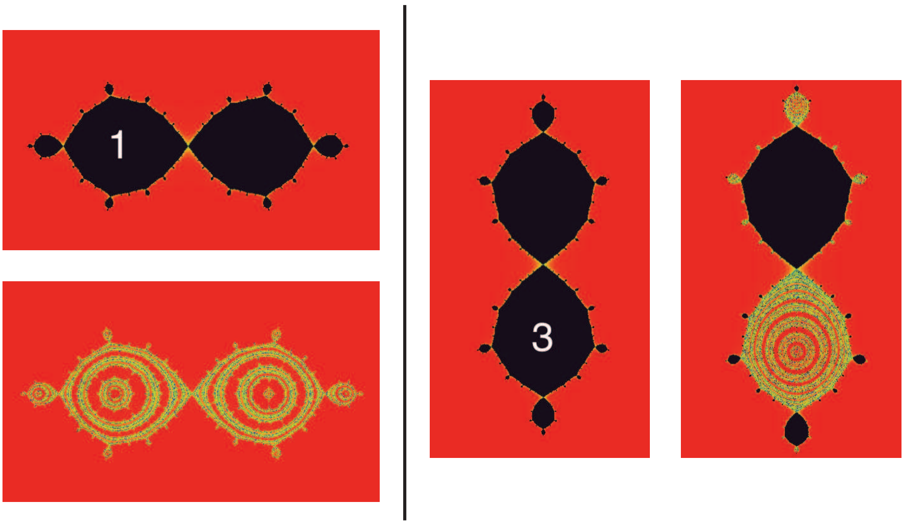

Figure 2. Two examples of McMullen-like mappings. On the left, we show the dynamical plane of the polynomial $Q(z) = 2z^3 - 3z^2 + 1$ (up) and the rational map $z \mapsto 2z^3 - 3z^2 + 1 + 10^{-5}/z$ (down). On the right, the Milnor cubic polynomial $R_{i\sqrt{2}}(z) = z^3 - 3i\sqrt{2}z^2/2$ (left) and the rational map $z \mapsto z^3 - 3i\sqrt{2}z^2/2 + 10^{-9}/z^3$ (right). In both cases, the bounded Fatou component where we put a pole has been marked with the local degree.

In the second example we consider the family of rational maps given by

$$g_\lambda(z) = z^n + c + \lambda/z^d$$

where c is chosen so that $P_c(z) = z^n + c$ is a HPcFP (equivalently c is chosen to be the center of a hyperbolic component of the corresponding Multibrot set). The case $n = d$ was considered previously in [2] (see also Proposition 2.8). When $n \neq d$ the only difference is the arithmetic condition ($\star$) that in this case writes as $1/n + 1/d < 1$. Using the same ideas as in the proof of Proposition 4.1, we have the following result.

Proposition 4.2. *Let $g_\lambda(z) = z^n + c + \lambda/z^d$ be such that $|\lambda| \neq 0$ is small enough and the arithmetic condition $1/n + 1/d < 1$ is satisfied, then g_λ is a McMullen-like mapping.*

In particular, Theorem 2.5 directly provides a description of the Julia set $\mathcal{J}(g_\lambda)$ similar to that one already obtained in [2] for the case $n = d$.

In the third example we also consider the polynomial $P_c(z) = z^n + c$, where c is such that the critical point located at 0 is periodic of period p. We denote by $U_0, U_1, \ldots, U_{p-1}$ the Fatou components containing 0, $P_c(0), \ldots, P_c^{p-1}(0)$, respectively. As in [11] we can consider McMullen-like mappings of type $(P_c, \mathcal{D})$ where the pole data $\mathcal{D}$ is formed by the Fatou components $\{U_0, U_1, \ldots, U_{p-1}\}$ and the corresponding positive integers $d_0, d_1, \ldots, d_{p-1}$. According to Theorem A, the arithmetic condition ($\star$) writes as

$$\left(\frac{1}{n} + \frac{1}{d_0}\right)\left(1 + \frac{1}{d_1}\right) \ldots \left(1 + \frac{1}{d_{p-1}}\right) < 1,$$

since U_0 is the only bounded Fatou component containing a critical point where $P_c \colon U_0 \to U_1$ is n-to-1 while P_c acts conformally on all the other bounded Fatou components. In [11] (see also Proposition 2.9), which deals with the specific case $n = 2$, the arithmetic conditions to conclude that the rational map $h_\lambda(z) = z^2 + c + \lambda / \prod_{j=0}^{p-1}(z - c_j)^{d_j}$ is a McMullen-like mapping are $2d_1 > d_0 + 2$ and $d_{j+1} > d_j + 1$ for every $1 \leqslant j \leqslant p-1$ (with $d_p = d_0$). We claim that these conditions imply the arithmetic condition ($\star$). To see the claim, we observe that

$$\begin{aligned}
\frac{1}{2} + \frac{1}{d_0} &= \frac{d_0 + 2}{2d_0} < \frac{2d_1}{2d_0} = \frac{d_1}{d_0}, \\
1 + \frac{1}{d_1} &= \frac{d_1 + 1}{d_1} < \frac{d_2}{d_1}, \\
&\vdots \\
1 + \frac{1}{d_{p-1}} &= \frac{d_{p-1} + 1}{d_{p-1}} < \frac{d_0}{d_{p-1}}.
\end{aligned}$$

So, the arithmetic condition (⋆)

$$\Big(\frac{1}{2}+\frac{1}{d_0}\Big)\Big(1+\frac{1}{d_1}\Big)\dots\Big(1+\frac{1}{d_{p-1}}\Big) < \frac{d_1}{d_0}\times\frac{d_2}{d_1}\times\dots\times\frac{d_0}{d_{p-1}} = 1,$$

is satisfied. However the conserve is not true. Take for example $p = 2$ and $d_0 = d_1 = 4$. For these values the arithmetic condition (⋆) is $(1/2 + 1/4)(1 + 1/4) = 15/16 < 1$, whereas the arithmetic conditions in Proposition 2.9 are $2d_1 > d_0 + 2$ and $d_0 > d_1 + 1$ which are not satisfied.

However, it turns out that the explicit rational map

$$h_\lambda(z) = z^2 + c + \frac{\lambda}{\prod_{j=0}^{p-1}(z-c_j)^{d_j}}$$

is not a McMullen-like mapping for every $|\lambda| \neq 0$ small enough if the arithmetic conditions in Proposition 2.9 are not satisfied. Indeed we can prove conditions (i) and (ii) in Definition 2.2 using a holomorphic motion (see the proof of Proposition 4.1), but studying the behavior of the "free" critical points shows that the free critical values do not belong to the trap doors as soon as the arithmetic conditions $2d_1 > d_0 + 2$ and $d_{j+1} > d_j + 1$ for every $1 \leqslant j \leqslant p-1$ (with $d_p = d_0$) are not satisfied (see Lemma 2.1 and Lemma 4.1 in [11]), and thus condition (iii) in Definition 2.2 does not hold.

The following result is a direct consequence of Theorem A, and a straightforward generalization of the ideas in [11] and in the proof of Proposition 4.1.

Proposition 4.3. *Let*

$$h_\lambda(z) = z^n + c + \frac{\lambda}{\prod_{j=0}^{p-1}(z-c_j)^{d_j}}$$

be such that $|\lambda| \neq 0$ is small enough and the arithmetic conditions $nd_1 > d_0 + n$ and $d_{j+1} > d_j + 1$ for every $1 \leqslant j \leqslant p-1$ (with $d_p = d_0$) are satisfied, then h_λ is a McMullen-like mapping of type $(P_c, \mathcal{D})$. Furthermore, there exists a McMullen-like mapping of type $(P_c, \mathcal{D})$ if and only if the strictly weaker arithmetic condition

$$\Big(\frac{1}{n}+\frac{1}{d_0}\Big)\Big(1+\frac{1}{d_1}\Big)\dots\Big(1+\frac{1}{d_{p-1}}\Big) < 1$$

is satisfied.

If the arithmetic condition ($\star$) is satisfied but not the arithmetic conditions $nd_1 > d_0 + n$ and $d_{j+1} > d_j + 1$ for every $1 \leqslant j \leqslant p-1$ (with $d_p = d_0$), we guess there should be McMullen-like mappings of type $(P_c, \mathcal{D})$ of the form $\widehat{h_\lambda}(z) = z^n + c + \Lambda(z)/\prod_{j=0}^{p-1}(z-c_j)^{d_j}$ where $\Lambda(z)$ is a non constant polynomial with $\deg(\Lambda) < \deg(\mathcal{D}) := \sum_{j=0}^{p-1} d_j$. But we have not been able to produce an explicit example because of the large number of parameters (the case $p = 2$ and $d_0 = d_1 = 4$ mentioned above leads to a polynomial $\Lambda(z)$ of degree less than 8, which is the smallest value of $\deg(\mathcal{D})$ so that the stronger arithmetic conditions are not satisfied).

In the next example we introduce a McMullen-like mapping obtained from a HPcFP with more that one bounded super-attracting periodic cycle. We consider the Milnor cubic polynomials (see [16]), that is a particular slice of the space of cubic polynomials fixing the behavior of one of the two critical points. More precisely, we consider the family of polynomials given by $R_a(z) = z^3 - 3az^2/2$. There is a super-attracting fixed point at the origin and the other critical point is located at $z = a$. In particular for $a = i\sqrt{2}$, the polynomial $R_{i\sqrt{2}}$ has two finite super-attracting fixed points: one at 0 and another one at $i\sqrt{2}$. We restrict to this value of a, however the same result is true assuming that the polynomial R_a is HPcFP. We denote by U_0 and $U_{i\sqrt{2}}$ the Fatou components containing 0 and $i\sqrt{2}$, respectively, and we consider the pole data $\mathcal{D}$ formed by the Fatou component U_0 and a positive integer $d = d_0$. Using the same ideas as in the proof of Proposition 4.1, we have the following result.

Proposition 4.4. *Let $r_\lambda(z) = z^3 - 3i\sqrt{2}z^2/2 + \lambda/z^d$ be such that $|\lambda| \neq 0$ is small enough and the arithmetic condition $d > 2$ is satisfied, then r_λ is a McMullen-like mapping.*

Indeed, in this case we have two cycles, so $N = 2$, and according to Theorem A the arithmetic condition ($\star$) writes as

$$\max\left\{\frac{1}{2}, \frac{1}{2} + \frac{1}{d}\right\} < 1,$$

since $U_0 \in \mathcal{D}$, $R_{i\sqrt{2}}\colon U_0 \to U_0$ with degree two, $U_{i\sqrt{2}} \notin \mathcal{D}$, and

$$R_{i\sqrt{2}}\colon U_{i\sqrt{2}} \longrightarrow U_{i\sqrt{2}}$$

with degree two. In Figure 2 we show the dynamical planes of the Milnor cubic polynomial $R_{i\sqrt{2}}$ and the McMullen-like mapping r_λ.

Finally we can obtain two other examples of McMullen-like mappings topologically conjugate on their Julia set to the McMullen family f_λ (see Theorem 2.6). We first fix a small value of $|\lambda| \neq 0$ and we consider the family of rational maps given by $\widetilde{f_\lambda}(z) = z^n + \lambda/(z-a)^d$. Then for every $|a|$ sufficiently small, if the arithmetic condition $1/n + 1/d < 1$ is satisfied then $\widetilde{f_\lambda}$ is a McMullen-like mapping of same type as f_λ. In this case the pole is moved to a and not located at the origin as in the McMullen family. The Julia set of $\widetilde{f_\lambda}$ has already been studied in [9] but Theorem 2.6 directly shows that $\mathcal{J}(\widetilde{f_\lambda})$ is a Cantor set of circles. Similarly, consider the family of rational maps given by $\widehat{f_\lambda}(z) = z^n + \Lambda(z)/z^d$ where $\Lambda(z)$ is any polynomial with $\deg(\Lambda) < d$ and $|\Lambda(0)| \neq 0$ small enough, if the arithmetic condition $1/n + 1/d < 1$ is satisfied then $\widehat{f_\lambda}$ is a McMullen-like mapping of same type as f_λ.

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