

A topological separation condition for fractal attractors

Tim Bedford, Sergiy V. Borodachov, and Jeffrey S. Geronimo¹

Abstract. We consider finite systems of contractive homeomorphisms of a complete metric space, which satisfy the minimality property. In general this separation condition is weaker than the strong open set condition and is not equivalent to the weak separation property. We prove that this separation condition is equivalent to the strong Markov property (see definition below). We also show that the set of N -tuples of contractive homeomorphisms, having the minimality property, is a G_δ set in the topology of pointwise convergence of every component mapping with an additional requirement that the supremum of contraction coefficients of mappings in the sequence be strictly less than one. We find a class of N -tuples of $d \times d$ invertible contraction matrices, which define systems of affine mappings in \mathbb{R}^d having the minimality property for almost every N -tuple of fixed points with respect to the Nd -dimensional Lebesgue measure.

Mathematics Subject Classification (2010). 28A80; 37C70.

Keywords. Minimality property, separation condition, Hausdorff dimension, similarity dimension, open set condition, Markov partition property, self-similar sets.

1. Introduction

Let X be a complete metric space and d be the distance in X . Recall that a mapping $w: X \rightarrow X$ is called a *contracting mapping* (or a *contraction*) if

$$\sigma = \sigma(w) = \sup_{x \neq y \in X} \frac{d(w(x), w(y))}{d(x, y)} < 1.$$

The number $\sigma(w)$ will be referred to as the contraction coefficient of the mapping w .

Let $N \in \mathbb{N}$, $w_1, \dots, w_N: X \rightarrow X$ be contracting homeomorphisms of X onto itself and $A = A(w_1, \dots, w_N) \subset X$ be the unique non-empty compact set such that

$$A = \bigcup_{i=1}^N w_i(A).$$

¹Tim Bedford was partially supported by an NSF grant and a Simons grant.

The set A is known as the invariant set or the attractor of the system $\{w_1, \dots, w_N\}$ and this way to define the attractor first appears in the paper by Hutchinson [6]. Denote by $\mathcal{M}(X)$ the space of all contracting homeomorphisms $w: X \rightarrow X$ of the space X onto itself. Sets defined as above have become generically to be called fractals and those whose parts do not overlap to much seem to be the most amenable to investigate. It has been an area of much study to make precise how much overlap is allowed between each $w_i(A)$. Moran [11] and Hutchinson [6] gave a criterion called the open set condition which guaranteed that there is not to much overlap. A set of contractions $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ satisfies the *open set condition* (OSC), if there is a non-empty open set $\mathcal{O} \subset X$ such that

1. $w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset, i \neq j$;
2. $w_i(\mathcal{O}) \subset \mathcal{O}, i = 1, \dots, N$.

A system of contractions (w_1, \dots, w_N) satisfies the *strong open set condition* (SOSC) if it satisfies the OSC with $\mathcal{O} \cap A \neq \emptyset$.

A mapping $w: X \rightarrow X$ is called a *contracting similitude* if there is a number $r \in (0, 1)$ such that

$$d(w(x), w(y)) = rd(x, y), \quad x, y \in X,$$

and r is sometimes called the similarity coefficient. The attractor of a finite system of contracting similitudes in X is known as a *self-similar set*. If $X = \mathbb{R}^d, d \in \mathbb{N}$, and $w_1, \dots, w_N: \mathbb{R}^d \rightarrow \mathbb{R}^d$ are contracting similitudes Bandt and Graf [1] studied the set

$$\mathcal{E} = \{w_j^{-1}w_i: \mathbf{i}, \mathbf{j} \in \mathcal{F} \cup \{\emptyset\}, \mathbf{i} \neq \mathbf{j}\}.$$

where

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \Sigma^n.$$

with $\Sigma = \{1, \dots, N\}$ and showed that the Hausdorff measure of A is strictly positive if and only if the identity mapping I is not in $\bar{\mathcal{E}}$. Schief [12], using this approach showed that the SOSC and the OSC are equivalent. He accomplished this by showing that if the Hausdorff measure of A is positive then the SOSC holds thus solving an important open problem in the area. For extensions see [15]. The positivity of the Hausdorff measure of A implies that the Hausdorff dimension of A is the same as its similarity dimension (see equation (13)). In general however, the OSC does not imply the SOSC (cf. e.g. [12]). For every vector $\mathbf{i} = \{i_1, \dots, i_n\} \in \Sigma^n$, let

$$w_{\mathbf{i}} = w_{i_1, \dots, i_n} = w_{i_1} \dots w_{i_n} = w_{i_1} \circ \dots \circ w_{i_n}.$$

Let Σ^∞ be the set of all infinite sequences (i_1, i_2, \dots) , where $i_j \in \Sigma, j = 1, 2, \dots$. A sequence $(i_1, i_2, \dots) \in \Sigma^\infty$ is called an *address* of a point $x \in A$, if

$$x \in \bigcap_{n=1}^{\infty} w_{i_1, \dots, i_n}(A).$$

This is equivalent to the fact that for some point $a \in X$,

$$x = \lim_{n \rightarrow \infty} w_{i_1, \dots, i_n}(a).$$

It is not difficult to see that every point $x \in A$ has at least one address and every sequence from Σ^∞ is an address of some point from A . If the set

$$\mathcal{T} = \bigcup_{i \neq j} w_i(A) \cap w_j(A)$$

is non-empty, there are points in A , which have more than one address. An important consequence of the OSC for contracting similitudes on \mathbb{R}^d is that it limits the number of addresses a point in A may have (see [6]). A weaker separation condition than the OCS condition,

$$I \notin \overline{(\mathcal{E} \setminus \{I\})}, \tag{1}$$

was used by Lau and Ngai ([8] and [9]) to study the multifractal spectrum of certain attractors.

Definition 1.1. For every $n \in \mathbb{N}$, denote by \mathcal{V}_n the set of all ordered N -tuples $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ such that for every $\mathbf{i} \in \Sigma^n$, there holds

$$w_{\mathbf{i}}(A) \not\subseteq \bigcup_{j \in \Sigma^n, j \neq \mathbf{i}} w_j(A).$$

A separation property based upon the above definition was introduced by Kigami [7], Section 1.3.

Definition 1.2. A system $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ is said to be *minimal* if

$$(w_1, \dots, w_N) \in \bigcap_{n=1}^{\infty} \mathcal{V}_n.$$

Theorem 1.3.8 in [7] gives different equivalent restatements of the minimality property. A collection $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ satisfies the Markov partition property (MPP) if there exists a subset $V \subset A$ open relative to A such that

1. $\bar{V} = A$;
2. $w_i(V) \cap w_j(V) = \emptyset, i \neq j$.

Definition 1.3. A system of mappings $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ satisfies the *strong Markov property* (SMP) if for every $n \in \mathbb{N}$, there is an open set $\mathcal{O}_n \subset X$ such that

1. $\overline{\mathcal{O}_n} \cap A = A$;
2. $w_i(\mathcal{O}_n) \cap w_j(\mathcal{O}_n) = \emptyset$, for every $\mathbf{i} \neq \mathbf{j} \in \Sigma^n$.

It is not difficult to see that SMP implies MPP if we let $V = \mathcal{O}_1 \cap A$, and that SOSC implies the SMP if we set $\mathcal{O}_n = \mathcal{O}$ for every $n \in \mathbb{N}$. The SMP does not in general imply the SOSC; hence, MPP is also a weaker property than SOSC (see Remark 4.11 for a more detailed discussion). Furthermore, in \mathbb{R}^d , condition (1) combined with the SMP is equivalent to SOSC (provided that the attractor is in general position), see Remark 4.6 and Propositions 4.7 and 4.8 for more details. Condition (1) is also known to be equivalent to the weak separation property introduced in [8], see [14].

One of the results proved in this paper is the equivalence of minimality and SMP. One of our main objectives is to investigate the set \mathcal{T} above to see how much overlap is allowed under the SMP condition. We also show in the case of contracting similitudes on \mathbb{R}^d that if the Hausdorff dimension and the similarity dimension of A are equal then A satisfies the SMP.

An interesting question is how generic are any of the above separation conditions in $\mathcal{M}(X)$. One of the results we present below is to show that the SMP condition is a countable intersection of open sets i.e a G_δ set. We also show that when $X = \mathbb{R}^d$ and all the w_i 's are similitudes the SMP is generic in the sense of Lebesgue measure. This result should be contrasted with that of Falconer [4] where he considered attractors associated with affine maps and obtained a formula for the Hausdorff dimension that was generic in the sense of Lebesgue measure.

We establish the following results.

Theorem 1.4. *Let X be a complete metric space. The system (w_1, \dots, w_N) of contracting homeomorphisms of X onto X satisfies the SMP if and only if it is minimal.*

Definition 1.5. We call a sequence $\{u_m\}_{m \in \mathbb{N}}$ of mappings from $\mathcal{M}(X)$ strongly pointwise convergent to a mapping $w \in \mathcal{M}(X)$ and write

$$u_m \xrightarrow{\text{s.p.}} w, \quad m \rightarrow \infty,$$

if

1. $\lim_{m \rightarrow \infty} u_m(x) = w(x)$ for every $x \in X$;
2. $\sup_{m \in \mathbb{N}} \sigma(u_m) < 1$.

If $\{u_m\}_{m \in \mathbb{N}} \subset \mathcal{M}(X)$ is a sequence of similitudes and $w \in \mathcal{M}(X)$ is a similitude, then strong pointwise convergence is equivalent to the ‘‘usual’’ pointwise convergence.

We introduce a topology \mathcal{B}_N on the space $(\mathcal{M}(X))^N$ by defining a subset $C \subset (\mathcal{M}(X))^N$ to be closed if for every sequence $\{(w_1^m, \dots, w_N^m)\}_{m \in \mathbb{N}} \subset C$, such that $\{w_i^m\} \xrightarrow{s.p.} w_i \in \mathcal{M}(X), i = 1, \dots, N$, we have $(w_1, \dots, w_N) \in C$. We agree here that \emptyset is closed. It is not difficult to see, for example, that the space $(\mathcal{M}(X))^N$ with the topology \mathcal{B}_N is a Hausdorff topological space.

Theorem 1.6. *Let $N \in \mathbb{N}$ and X be a complete metric space. The set of systems of mappings $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$, which satisfy the SMP is a G_δ set in the topology \mathcal{B}_N .*

For a $d \times k$ matrix B , let

$$\|B\| = \max_{\mathbf{x} \in \mathbb{R}^k \setminus \{0\}} \frac{|B\mathbf{x}|}{|\mathbf{x}|} \tag{2}$$

be its norm. We say that B is a *contraction matrix* if $\|B\| < 1$.

Let $X = \mathbb{R}^d$ and B_1, \dots, B_N be invertible $d \times d$ contraction matrices. Denote by $E_d(B_1, \dots, B_N)$ the set of all ordered N -tuples $(\alpha_1, \dots, \alpha_N)$ of points from \mathbb{R}^d such that the system of mappings $w_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \alpha_i) + \alpha_i, \quad i = 1, \dots, N,$$

satisfies the SMP. We will sometimes consider the set $E_d(B_1, \dots, B_N)$ as a subset of \mathbb{R}^{dN} .

Corollary 1.7. *For any collection B_1, \dots, B_N of invertible $d \times d$ contraction matrices, the set $E_d(B_1, \dots, B_N)$ is a G_δ subset of \mathbb{R}^{dN} (in the topology induced by the Euclidean distance).*

The rest of the paper is structured as follows. In Section 2 we prove Theorem 1.4 by showing that SMP holds if and only if (w_1, \dots, w_N) is minimal, i.e. $(w_1, \dots, w_N) \in \bigcap_{n=1}^\infty \mathcal{V}_n$. Next, in Section 3, the genericity of the SMP is taken up and it is shown that the set of all systems of mappings that satisfy SMP is a G_δ set in a suitable topology, thus establishing Theorem 1.6 and Corollary 1.7. In Section 4 we establish certain necessary or sufficient conditions for the SMP. Finally in Section 5 genericity results for the SMP in the case of self-affine sets in \mathbb{R}^d are established. For instance, in Section 5 we prove the following results.

Theorem 5.2. *Let B_1, \dots, B_N be invertible $d \times d$ contraction matrices such that $\sum_{i=1}^N \|B_i\| < 1$. Then the set $E_d(B_1, \dots, B_N)$ is a G_δ -subset of \mathbb{R}^{dN} of full Lebesgue measure.*

Theorem 5.3. *Let $B_i = \sigma_i U_i$, where $\sigma_i \in (0, 1)$, U_i is a 2×2 rotation matrix, $i = 1, \dots, N$, and $\sum_{i=1}^N \sigma_i^2 < 1$. Then the set $E_2(B_1, \dots, B_N)$ is either empty or is a G_δ -subset of \mathbb{R}^{2N} of full Lebesgue measure.*

When $d = 1$ Theorem 5.2 follows from a result of Falconer [3].

2. Proof of Theorem 1.4

We will start the proof with the following statement.

Lemma 2.1. *Let X be a complete metric space and $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$. If $(w_1, \dots, w_N) \in \bigcap_{n=1}^{\infty} \mathcal{V}_n$, then there is an open set $\mathcal{O} \subset X$ such that $\overline{\mathcal{O} \cap A} = A$ and $w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset, i \neq j$. In particular, the system (w_1, \dots, w_N) will satisfy the MPP.*

Proof. In order to prove Lemma 2.1 denote

$$K_i(A) = w_i(A) \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^N w_j(A), \quad i = 1, \dots, N.$$

Let also

$$Z_i = w_i^{-1}(K_i(A)) \quad \text{and} \quad V = \bigcap_{i=1}^N Z_i.$$

For example, if $w_1(x) = x/2$ and $w_2(x) = x/2 + 1/2$, then $A = [0, 1], Z_1 = [0, 1), Z_2 = (0, 1]$, and hence, $V = (0, 1)$.

It is not difficult to see that $Z_i \subset A, i = 1, \dots, N$. We show that $\bar{Z}_i = A, i = 1, \dots, N$. Let $x \in A$ and let $U \subset X$ be any open set containing x . Denote by $B(a, \rho)$ the open ball in X centered at point a of radius $\rho > 0$. Since $w_i(U)$ is also open, there is $\epsilon > 0$ such that $B(w_i(x), \epsilon) \subset w_i(U)$. Let $r_i = \sigma(w_i) \in (0, 1)$ be the contraction coefficient of $w_i, i = 1, \dots, N$, and define

$$r_{\max} = \max_{i=1, \dots, N} r_i.$$

Choose a number $m \in \mathbb{N}$ so that $(r_{\max})^m \cdot \text{diam} A < \epsilon$. There exist indices $i_1, \dots, i_m \in \Sigma$ such that $x \in w_{i_1, \dots, i_m}(A)$. Then $w_i(x) \in w_{i, i_1, \dots, i_m}(A)$ and

$$\text{diam } w_{i, i_1, \dots, i_m}(A) \leq r_i \cdot r_{i_1} \cdot \dots \cdot r_{i_m} \cdot \text{diam} A \leq (r_{\max})^{m+1} \cdot \text{diam} A < \epsilon.$$

Hence,

$$w_{i, i_1, \dots, i_m}(A) \subset B(w_i(x), \epsilon) \subset w_i(U). \tag{3}$$

Since $(w_1, \dots, w_N) \in \mathcal{V}_{m+1}$, we have

$$w_{i, i_1, \dots, i_m}(A) \not\subset \bigcup_{\substack{j_1, \dots, j_{m+1} \in \Sigma \\ j_1 \neq i}}^N w_{j_1, \dots, j_{m+1}}(A) = \bigcup_{\substack{j=1 \\ j \neq i}}^N w_j(A).$$

Hence, there is $z \in A$ such that $w_{i, i_1, \dots, i_m}(z)$ does not belong to $\bigcup_{j: j \neq i} w_j(A)$. Let $t = w_{i_1, \dots, i_m}(z)$. Since $w_i(t)$ does not belong to any $w_j(A)$ with $j \neq i$, we must

have $w_i(t) \in w_i(A)$, that is, $w_i(t) \in K_i(A)$. Hence, $t \in Z_i$. On the other hand, since $w_i(t) \in w_{i,i_1,\dots,i_m}(A)$, in view of (3), we have $w_i(t) \in w_i(U)$, that is $t \in U$, which implies that $\bar{Z}_i = A, i = 1, \dots, N$.

We next show that $\bar{V} = A$. Indeed, since each Z_i is open relative to A , there are open sets $W_i \subset X$ such that $Z_i = W_i \cap A, i = 1, \dots, N$. Let y be any element in A and U be any open neighborhood of y . Since $\bar{Z}_1 = A$, there is $z_1 \in Z_1 \cap U = A \cap W_1 \cap U$. Since $\bar{Z}_2 = A$, there is $z_2 \in Z_2$ in the open neighborhood $W_1 \cap U$ of the point $z_1 \in A$, that is $z_2 \in A \cap U \cap W_1 \cap W_2$. Then by induction, there will be an element $z_N \in A \cap U \cap W_1 \cap \dots \cap W_N = V \cap U$, and the required relation follows.

Note that for every $i \neq j$, there holds

$$\begin{aligned} w_i(V) \cap w_j(V) &\subset w_i(Z_i) \cap w_j(Z_j) = K_i(A) \cap K_j(A) \\ &\subset (w_i(A) \setminus w_j(A)) \cap w_j(A) = \emptyset. \end{aligned}$$

Taking also into account the fact that V is relatively open with respect to A as an intersection of a finite collection of subsets of A , which are open relative to A , we conclude that the system (w_1, \dots, w_N) possesses the MPP.

For every $x \in V$, denote

$$\rho(x) = \min_{i=1,\dots,N} \text{dist}\left(w_i(x), \bigcup_{\substack{j=1 \\ j \neq i}}^N w_j(A)\right).$$

In view of the relations

$$w_i(V) \subset w_i(Z_i) = K_i(A), \quad i = 1, \dots, N,$$

the point $w_i(x), x \in V$, does not belong to the closed set $\bigcup_{j: j \neq i} w_j(A)$. Hence, $\rho(x) > 0, x \in V$, and the set

$$\mathcal{O} = \bigcup_{x \in V} B(x, \rho(x)/2)$$

is open. Since $\bar{V} = A$ and $V \subset \mathcal{O} \cap A \subset A$, we have $\overline{\mathcal{O} \cap A} = A$. To show that $w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset, i \neq j$, assume to the contrary that there exist indices $i \neq j$ such that $w_i(\mathcal{O}) \cap w_j(\mathcal{O})$ contains some element y . Then $y = w_i(p) = w_j(q)$ for some $p, q \in \mathcal{O}$. There are points $c, b \in V$ such that $d(c, p) < \rho(c)/2$ and $d(b, q) < \rho(b)/2$. Note that

$$d(y, w_i(c)) = d(w_i(p), w_i(c)) \leq r_i \cdot d(p, c) < r_i \cdot \rho(c)/2 \tag{4}$$

and

$$d(y, w_j(b)) = d(w_j(q), w_j(b)) \leq r_j \cdot d(q, b) < r_j \cdot \rho(b)/2. \tag{5}$$

There also hold the following relations

$$\rho(c) \leq \text{dist}\left(w_i(c), \bigcup_{\substack{k=1 \\ k \neq i}}^N w_k(A)\right) \leq \text{dist}(w_i(c), w_j(A)) \leq d(w_i(c), w_j(b)) \quad (6)$$

and

$$\rho(b) \leq \text{dist}\left(w_j(b), \bigcup_{\substack{k=1 \\ k \neq j}}^N w_k(A)\right) \leq \text{dist}(w_j(b), w_i(A)) \leq d(w_j(b), w_i(c)). \quad (7)$$

Then, in view of relations (4)–(7), we obtain

$$\begin{aligned} \rho(c) + \rho(b) &\leq 2d(w_i(c), w_j(b)) \\ &\leq 2(d(w_i(c), y) + d(y, w_j(b))) \\ &< r_i \cdot \rho(c) + r_j \cdot \rho(b) \\ &< \rho(c) + \rho(b), \end{aligned}$$

which is impossible. Hence, $w_i(\mathcal{O})$ and $w_j(\mathcal{O})$ are disjoint, which completes the proof of Lemma 2.1. \square

To prove sufficiency in Theorem 1.4, assume that

$$(w_1, \dots, w_N) \in \bigcap_{n=1}^{\infty} \mathcal{V}_n \subset (\mathcal{M}(X))^N.$$

Then for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$, we have $(w_1, \dots, w_N) \in \mathcal{V}_{nm} \subset (\mathcal{M}(X))^N$, which implies that the system $\{w_i\}_{i \in \Sigma^m}$ belongs to the set $\mathcal{V}_n \subset (\mathcal{M}(X))^{N^m}$. Hence, $\{w_i\}_{i \in \Sigma^m} \in \bigcap_{n=1}^{\infty} \mathcal{V}_n \subset (\mathcal{M}(X))^{N^m}$. By Lemma 2.1, there is an open set $\mathcal{O}_m \subset X$ such that $\overline{\mathcal{O}_m} \cap A = A$ and $w_i(\mathcal{O}_m) \cap w_j(\mathcal{O}_m) = \emptyset$ for every $\mathbf{i} \neq \mathbf{j} \in \Sigma^m$, $m \in \mathbb{N}$. Hence, the system (w_1, \dots, w_N) satisfies the SMP.

The proof of the necessity in Theorem 1.4 is preceded by the following proposition.

Lemma 2.2. *Let mappings $w_1, \dots, w_N \in \mathcal{M}(X)$ be such that there is a non-empty open set $\mathcal{O} \subset X$ with the property*

$$w_i(\mathcal{O}) \cap w_j(\mathcal{O}) = \emptyset, \quad i \neq j.$$

Then for every $i = 1, \dots, N$,

$$w_i(\mathcal{O}) \cap \bigcup_{j: j \neq i} w_j(\overline{\mathcal{O}}) = \emptyset.$$

Proof. Assume the contrary. Then for some $j_0 \neq i$, there is $x \in w_i(\mathcal{O}) \cap w_{j_0}(\overline{\mathcal{O}})$. Let $z \in \overline{\mathcal{O}}$ be such that $x = w_{j_0}(z)$. There is a sequence $\{z_m\}_{m \in \mathbb{N}} \subset \mathcal{O}$ such that $z = \lim_{m \rightarrow \infty} z_m$ and hence $x = \lim_{m \rightarrow \infty} w_{j_0}(z_m)$. Since $w_i(\mathcal{O})$ is an open neighborhood of x , we have $w_{j_0}(z_m) \in w_i(\mathcal{O})$ for every m sufficiently large, and hence $w_i(\mathcal{O}) \cap w_{j_0}(\mathcal{O}) \neq \emptyset$, which contradicts the assumptions. Lemma 2.2 is proved. \square

Completion of the proof of Theorem 1.4. Assume that system $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ satisfies the SMP. Let $k \in \mathbb{N}$ be arbitrary. Then there is an open set $\mathcal{O}_k \subset X$ such that $\overline{\mathcal{O}_k} \cap A = A$ and $w_i(\mathcal{O}_k) \cap w_j(\mathcal{O}_k) = \emptyset$ for every $\mathbf{i} \neq \mathbf{j} \in \Sigma^k$. We first show that for every $\mathbf{i} \in \Sigma^k$,

$$w_i(A) = \overline{w_i(\mathcal{O}_k) \cap A}. \tag{8}$$

Taking into account Lemma 2.2 and the fact that $A = \overline{\mathcal{O}_k} \cap A \subset \overline{\mathcal{O}_k}$, we obtain

$$\begin{aligned} w_i(\mathcal{O}_k) \cap A &= (w_i(\mathcal{O}_k) \cap w_i(A)) \cup \left(w_i(\mathcal{O}_k) \cap \bigcup_{\mathbf{j} \in \Sigma^k, \mathbf{j} \neq \mathbf{i}} w_j(A) \right) \\ &\subset w_i(\mathcal{O}_k \cap A) \cup \left(w_i(\mathcal{O}_k) \cap \bigcup_{\mathbf{j} \in \Sigma^k, \mathbf{j} \neq \mathbf{i}} w_j(\overline{\mathcal{O}_k}) \right) \\ &= w_i(\mathcal{O}_k \cap A). \end{aligned}$$

Then

$$\overline{w_i(\mathcal{O}_k) \cap A} \subset \overline{w_i(\mathcal{O}_k \cap A)} = \overline{w_i(\overline{\mathcal{O}_k} \cap A)} = w_i(A).$$

On the other hand,

$$w_i(A) = \overline{w_i(\mathcal{O}_k \cap A)} = \overline{w_i(\mathcal{O}_k \cap A)} = \overline{w_i(\mathcal{O}_k) \cap w_i(A)} \subset \overline{w_i(\mathcal{O}_k) \cap A},$$

and (8) follows.

Assume that (w_1, \dots, w_N) does not belong to $\cap_{n=1}^\infty \mathcal{V}_n$. Then there is $n \in \mathbb{N}$ and $\mathbf{i}_n \in \Sigma^n$ such that

$$w_{\mathbf{i}_n}(A) \subset \bigcup_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}_n} w_j(A).$$

Then, taking into account (8) we obtain

$$\begin{aligned} w_{\mathbf{i}_n}(\mathcal{O}_n) \cap A &\subset \overline{w_{\mathbf{i}_n}(\mathcal{O}_n) \cap A} = w_{\mathbf{i}_n}(A) \\ &\subset \bigcup_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}_n} w_j(A) \\ &= \bigcup_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}_n} \overline{w_j(\mathcal{O}_n) \cap A} \\ &\subset \bigcup_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}_n} w_j(\overline{\mathcal{O}_n}). \end{aligned}$$

Since $\overline{w_{i_n}(\mathcal{O}_n) \cap A} = w_{i_n}(A) \neq \emptyset$, there is a point $x \in w_{i_n}(\mathcal{O}_n) \cap A \subset w_{i_n}(\mathcal{O}_n)$. Then $x \in \bigcup_{j \in \Sigma^n, j \neq i_n} w_j(\overline{\mathcal{O}_n})$. Hence,

$$w_{i_n}(\mathcal{O}_n) \cap \bigcup_{j \in \Sigma^n, j \neq i_n} w_j(\overline{\mathcal{O}_n}) \neq \emptyset,$$

which contradicts Lemma 2.2. Theorem 1.4 is proved. \square

3. Proofs of Theorem 1.6 and of Corollary 1.7.

The proof of some statements in this section is standard, but we include it for the convenience of the reader.

Lemma 3.1. *If a sequence $\{u_m\}_{m \in \mathbb{N}} \subset \mathcal{M}(X)$ converges strongly pointwise to a mapping $w \in \mathcal{M}(X)$, then the sequence of fixed points of mappings u_m converges to the fixed point of w .*

Proof. Let $x_m \in X$ be the fixed point of the mapping u_m , $m \in \mathbb{N}$, and $x \in X$ be the fixed point of w . Denote also

$$\sigma = \sup_{m \in \mathbb{N}} \sigma(u_m).$$

Then

$$\begin{aligned} d(x_m, x) &\leq d(x_m, u_m(x)) + d(u_m(x), x) \\ &= d(u_m(x_m), u_m(x)) + d(u_m(x), w(x)) \\ &\leq \sigma d(x_m, x) + d(u_m(x), w(x)). \end{aligned}$$

Hence,

$$d(x_m, x) \leq \frac{1}{1 - \sigma} d(u_m(x), w(x)),$$

and we have

$$\lim_{m \rightarrow \infty} d(x_m, x) = 0.$$

Lemma 3.1 is proved. \square

Lemma 3.2. *Let A be the attractor of a system of mappings $w_1, \dots, w_N \in \mathcal{M}(X)$ with contraction coefficients not exceeding a given number $\sigma \in (0, 1)$. Let also $B[a, r]$ be a closed ball containing the fixed point of every mapping w_1, \dots, w_N . Then $A \subset B[a, R]$, where $R = \frac{1 + \sigma}{1 - \sigma} r$.*

Proof. Assume the contrary. Denote by y_1, \dots, y_N the fixed points of mappings w_1, \dots, w_N respectively. Let z be a point in A furthest from a . Then we must have $d(z, a) > R$. Let $1 \leq i \leq N$ be such index that $z = w_i(z_1)$ for some $z_1 \in A$. Then

$$\begin{aligned} d(z_1, a) &\geq d(z_1, y_i) - d(y_i, a) \\ &\geq \frac{1}{\sigma} d(w_i(z_1), w_i(y_i)) - r \\ &= \frac{1}{\sigma} d(z, y_i) - r \\ &\geq \frac{1}{\sigma} d(z, a) - \frac{1}{\sigma} d(y_i, a) - r \\ &\geq \frac{1}{\sigma} d(z, a) - \frac{r}{\sigma} - r. \end{aligned}$$

Hence,

$$\frac{d(z_1, a)}{d(z, a)} \geq \frac{1}{\sigma} - \frac{(1 + \sigma)r}{\sigma d(z, a)} > \frac{1}{\sigma} - \frac{(1 + \sigma)r}{\sigma R} = 1,$$

which contradicts the fact that z is a point in A furthest from a . \square

Lemma 3.3. *Let $\{w_1^m\}_{m \in \mathbb{N}}, \dots, \{w_n^m\}_{m \in \mathbb{N}}$ be sequences of mappings from $\mathcal{M}(X)$ such that $w_i^m \xrightarrow{s.p.} w_i \in \mathcal{M}(X)$, $i = 1, \dots, n$. Then $w_1^m \circ \dots \circ w_n^m \xrightarrow{s.p.} w_1 \circ \dots \circ w_n$, $m \rightarrow \infty$.*

Proof. We will use induction. For $n = 1$, the assertion of the lemma is trivial. Assume that the assertion is true for a given value of $n \geq 1$ and show that it holds for any $n + 1$ sequences satisfying the assumptions of the lemma. For every $x \in X$, we will have

$$\begin{aligned} &d(w_1^m w_2^m \dots w_{n+1}^m(x), w_1 w_2 \dots w_{n+1}(x)) \\ &\leq d(w_1^m(w_2^m \dots w_{n+1}^m(x)), w_1^m(w_2 \dots w_{n+1}(x))) \\ &\quad + d(w_1^m(w_2 \dots w_{n+1}(x)), w_1(w_2 \dots w_{n+1}(x))) \\ &\leq d(w_2^m \dots w_{n+1}^m(x), w_2 \dots w_{n+1}(x)) \\ &\quad + d(w_1^m(w_2 \dots w_{n+1}(x)), w_1(w_2 \dots w_{n+1}(x))). \end{aligned}$$

By the assumption of the induction, both distances in the last line vanish as $m \rightarrow \infty$ and we have

$$\lim_{m \rightarrow \infty} w_1^m w_2^m \dots w_{n+1}^m(x) = w_1 w_2 \dots w_{n+1}(x), \quad x \in X.$$

Since

$$\sigma = \max_{i=1, \dots, n+1} \sup_{m \in \mathbb{N}} \sigma(w_i^m) < 1,$$

we have

$$\sigma(w_1^m w_2^m \dots w_{n+1}^m) \leq \sigma^{n+1} < 1, \quad m \in \mathbb{N},$$

which implies strong pointwise convergence. Lemma 3.3 is proved. \square

Given a system $W = (w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ and an address $\mathbf{i} \in \Sigma^\infty$, let $\Pi_{\mathbf{i}}(W)$ be the point in the attractor of W with address \mathbf{i} .

Lemma 3.4. *Let $W_m = (w_1^m, \dots, w_N^m)$, $m \in \mathbb{N}$, be a sequence from $(\mathcal{M}(X))^N$ such that for every $i = 1, \dots, N$, the sequence $\{w_i^m\}_{m \in \mathbb{N}}$ converges strongly pointwise to some mapping $w_i \in \mathcal{M}(X)$. Then for every address $\mathbf{i} \in \Sigma^\infty$,*

$$\lim_{m \rightarrow \infty} \Pi_{\mathbf{i}}(W_m) = \Pi_{\mathbf{i}}(W),$$

where $W = (w_1, \dots, w_N)$.

Proof. Given an arbitrary address $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma^\infty$, denote by $x_{i_1 \dots i_n}$ the fixed point of the mapping w_{i_1, \dots, i_n} . Let also

$$\delta = \max_{i=1, \dots, N} \sup_{m \in \mathbb{N}} \sigma(w_i^m).$$

Let $B(a, r)$ be a ball containing the attractor A of the system W and $R = \frac{1+\delta}{1-\delta}r$. Choose an arbitrary $\epsilon > 0$ and let $n \in \mathbb{N}$ be large enough so that

$$d(\Pi_{\mathbf{i}}(W), x_{i_1 \dots i_n}) < \epsilon \quad \text{and} \quad R\delta^n < \epsilon. \quad (9)$$

Denote by $x_{\alpha_1 \dots \alpha_n}^m$ the fixed point of the mapping $w_{\alpha_1}^m \circ \dots \circ w_{\alpha_n}^m$, $\alpha_1, \dots, \alpha_n \in \Sigma$. By Lemma 3.3, we have $w_{\alpha_1}^m \circ \dots \circ w_{\alpha_n}^m \xrightarrow{\text{s.p.}} w_{\alpha_1, \dots, \alpha_n}$, $m \rightarrow \infty$. Then by Lemma 3.1, we have $\lim_{m \rightarrow \infty} x_{\alpha_1 \dots \alpha_n}^m = x_{\alpha_1 \dots \alpha_n}$ for every $\alpha_1, \dots, \alpha_n \in \Sigma$. Since $x_{\alpha_1 \dots \alpha_n} \in A \subset B(a, r)$, there is a number $m_n \in \mathbb{N}$ such that for every $m > m_n$ and $\alpha_1, \dots, \alpha_n \in \Sigma$, we have $x_{\alpha_1 \dots \alpha_n}^m \in B(a, r)$. For every $m > m_n$, we obtain

$$\begin{aligned} d(\Pi_{\mathbf{i}}(W), \Pi_{\mathbf{i}}(W_m)) &\leq d(\Pi_{\mathbf{i}}(W), x_{i_1 \dots i_n}) \\ &\quad + d(w_{i_1, \dots, i_n}(x_{i_1 \dots i_n}), w_{i_1}^m \dots w_{i_n}^m(x_{i_1 \dots i_n})) \\ &\quad + d(w_{i_1}^m \dots w_{i_n}^m(x_{i_1 \dots i_n}), \Pi_{\mathbf{i}}(W_m)) \\ &\leq \epsilon + d(w_{i_1, \dots, i_n}(x_{i_1 \dots i_n}), w_{i_1}^m \dots w_{i_n}^m(x_{i_1 \dots i_n})) \\ &\quad + d(w_{i_1}^m \dots w_{i_n}^m(x_{i_1 \dots i_n}), w_{i_1}^m \dots w_{i_n}^m(z_{\mathbf{i}, m})), \end{aligned}$$

where $z_{\mathbf{i}, m}$ is some point in the attractor A_m of the system W_m . Taking into account Lemma 3.3, we will have

$$d(\Pi_{\mathbf{i}}(W), \Pi_{\mathbf{i}}(W_m)) \leq \epsilon + o(1) + \delta^n d(x_{i_1 \dots i_n}, z_{\mathbf{i}, m}).$$

For every $i = 1, \dots, N$, the fixed point x_i^m of w_i^m is also the fixed point of the n -th power of w_i^m , and as it was noted above, $x_i^m \in B(a, r)$, $m > m_n$. By Lemma 3.2, we

have $z_{i,m} \in A_m \subset B[a, R]$. Since $x_{i_1 \dots i_n} \in A \subset B(a, r) \subset B[a, R]$, in view of (9), we obtain

$$d(\Pi_i(W), \Pi_i(W_m)) \leq \epsilon + o(1) + 2R\delta^n \leq 3\epsilon + o(1).$$

Hence,

$$\limsup_{m \rightarrow \infty} d(\Pi_i(W), \Pi_i(W_m)) \leq 3\epsilon.$$

In view of arbitrariness of ϵ , we have

$$\lim_{m \rightarrow \infty} d(\Pi_i(W), \Pi_i(W_m)) = 0,$$

and the assertion of Lemma 3.4 follows. □

Let

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \Sigma^n.$$

Lemma 3.5. *Let $N, n \in \mathbb{N}$. If a sequence $\{(w_1^m, \dots, w_N^m)\}_{m \in \mathbb{N}} \subset (\mathcal{M}(X))^N \setminus \mathcal{V}_n$ converges strongly pointwise in every component to a system $W = (w_1, \dots, w_N) \in (\mathcal{M}(X))^N$, then we have $W \in (\mathcal{M}(X))^N \setminus \mathcal{V}_n$.*

From Lemma 3.5 we obtain the following statement, which in view of Theorem 1.4, implies the assertion of Theorem 1.6.

Corollary 3.6. *For every positive integers n and N , the set \mathcal{V}_n is open in the topology \mathcal{B}_N , and hence, $\bigcap_{n=1}^\infty \mathcal{V}_n$ is a G_δ set.*

Proof of Lemma 3.5. Let $W_m = (w_1^m, \dots, w_N^m) \in (\mathcal{M}(X))^N \setminus \mathcal{V}_n$ be a sequence, where every component is convergent strongly pointwise to the corresponding component of the system $W = (w_1, \dots, w_N) \in (\mathcal{M}(X))^N$. Denote $w_k^m = w_{k_1}^m \circ \dots \circ w_{k_p}^m$, $\mathbf{k} = (k_1, \dots, k_p) \in \mathcal{F}$. For every $m \in \mathbb{N}$, there is a vector $\mathbf{i}_m \in \Sigma^n$ such that

$$w_{\mathbf{i}_m}^m(A_m) \subset \bigcup_{\mathbf{k} \in \Sigma^n, \mathbf{k} \neq \mathbf{i}_m} w_{\mathbf{k}}^m(A_m),$$

where A_m is the attractor of the system W_m . There is an index $\mathbf{i} \in \Sigma^n$ and an infinite subsequence $\mathcal{N} \subset \mathbb{N}$ such that

$$w_{\mathbf{i}}^m(A_m) \subset \bigcup_{\mathbf{k} \in \Sigma^n, \mathbf{k} \neq \mathbf{i}} w_{\mathbf{k}}^m(A_m), \quad m \in \mathcal{N}. \tag{10}$$

Let A be the attractor of the system W and $x \in w_{\mathbf{i}}(A)$ be an arbitrary point. Then $x = \Pi_{\mathbf{i}\beta}(W)$ for some $\beta \in \Sigma^\infty$. In view of (10), for every $m \in \mathcal{N}$, there holds

$$\Pi_{\mathbf{i}\beta}(W_m) \in w_{\mathbf{i}}^m(A_m) \cap w_{\mathbf{j}_m}^m(A_m)$$

for some $\mathbf{j}_m \in \Sigma^n$ distinct from \mathbf{i} . There are index $\mathbf{j} \in \Sigma^n$, $\mathbf{j} \neq \mathbf{i}$, and infinite subsequence $\mathcal{N}' \subset \mathcal{N}$ such that

$$\Pi_{\mathbf{i}\beta}(W_m) \in w_{\mathbf{i}}^m(A_m) \cap w_{\mathbf{j}}^m(A_m), \quad m \in \mathcal{N}'.$$

Hence, there is a sequence $\gamma_m = (\gamma_1^m, \gamma_2^m, \dots) \in \Sigma^\infty$ such that

$$\Pi_{\mathbf{i}\beta}(W_m) = \Pi_{\mathbf{j}\gamma_m}(W_m), \quad m \in \mathcal{N}'. \tag{11}$$

One can find an infinite subsequence $\mathcal{N}_1 \subset \mathcal{N}'$ and an index $\gamma_1 \in \Sigma$ such that $\gamma_1^m = \gamma_1$, $m \in \mathcal{N}_1$. One can find an infinite subsequence $\mathcal{N}_2 \subset \mathcal{N}_1$ and an index $\gamma_2 \in \Sigma$ such that $\gamma_1^m = \gamma_1$ and $\gamma_2^m = \gamma_2$, $m \in \mathcal{N}_2$. Continuing this process indefinitely, we obtain an address $\gamma = (\gamma_1, \gamma_2, \dots) \in \Sigma^\infty$ and a sequence of embedded infinite sets $\mathcal{N}_1 \supset \mathcal{N}_2 \supset \dots \supset \mathcal{N}_k \supset \dots$ such that $\gamma_k^m = \gamma_k$, $m \in \mathcal{N}_k$, $k \in \mathbb{N}$.

Let as above, $B(a, r)$ be an open ball containing A . Since A contains the fixed points of mappings w_1, \dots, w_N , by Lemma 3.1, there is $m_0 \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $m > m_0$, the fixed points of mappings w_1^m, \dots, w_N^m will be in $B(a, r)$. Then, by Lemma 3.2, $A_m \subset B[a, R]$, where $R = \frac{1+\delta}{1-\delta}r$ and $\delta = \max_{i=1, \dots, N} \sup_{m \in \mathbb{N}} \sigma(w_i^m) \in (0, 1)$. For every $k \in \mathbb{N}$ and $m \in \mathcal{N}_k$, $m > m_0$, there are points b and c in A_m such that

$$\begin{aligned} d(\Pi_{\mathbf{j}\gamma_m}(W_m), \Pi_{\mathbf{j}\gamma}(W_m)) &= d(w_{\mathbf{j}}^m w_{\gamma_1^m, \dots, \gamma_k^m}^m(b), w_{\mathbf{j}}^m w_{\gamma_1, \dots, \gamma_k}^m(c)) \\ &= d(w_{\mathbf{j}}^m w_{\gamma_1, \dots, \gamma_k}^m(b), w_{\mathbf{j}}^m w_{\gamma_1, \dots, \gamma_k}^m(c)) \\ &\leq \sigma(w_{\mathbf{j}}^m) \cdot \sigma(w_{\gamma_1}^m) \cdot \dots \cdot \sigma(w_{\gamma_k}^m) d(b, c) \\ &\leq \delta^{n+k} \text{diam} A_m \leq 2R\delta^{n+k}. \end{aligned}$$

By Lemma 3.4 and relation (11), for every $m \in \mathcal{N}_k$, $m > m_0$, we obtain

$$\begin{aligned} d(x, \Pi_{\mathbf{j}\gamma}(W)) &\leq d(\Pi_{\mathbf{i}\beta}(W), \Pi_{\mathbf{i}\beta}(W_m)) \\ &\quad + d(\Pi_{\mathbf{j}\gamma_m}(W_m), \Pi_{\mathbf{j}\gamma}(W_m)) \\ &\quad + d(\Pi_{\mathbf{j}\gamma}(W_m), \Pi_{\mathbf{j}\gamma}(W)) \\ &\leq 2R\delta^{n+k} + o(1). \end{aligned}$$

Hence, letting $m \rightarrow \infty$ along the sequence \mathcal{N}_k , we will have

$$d(x, \Pi_{\mathbf{j}\gamma}(W)) \leq 2R\delta^{n+k}, \quad k \in \mathbb{N}.$$

Letting now $k \rightarrow \infty$ we get that $d(x, \Pi_{\mathbf{j}\gamma}(W)) = 0$, which implies that

$$x = \Pi_{\mathbf{j}\gamma}(W) \in w_{\mathbf{j}}(A) \subset \bigcup_{\mathbf{k} \in \Sigma^n, \mathbf{k} \neq \mathbf{i}} w_{\mathbf{k}}(A),$$

where vector \mathbf{j} was chosen to be distinct from \mathbf{i} . Since $x \in w_i(A)$ was chosen arbitrarily, we obtain that

$$w_i(A) \subset \bigcup_{\mathbf{k} \in \Sigma^n, \mathbf{k} \neq \mathbf{i}} w_{\mathbf{k}}(A),$$

and hence, $W \in (\mathcal{M}(X))^N \setminus \mathcal{V}_n$. Lemma 3.5 is proved, which completes the proof of Theorem 1.6. \square

Proof of Corollary 1.7. Let U_n , $n \in \mathbb{N}$, be the set of ordered N -tuples $(\alpha_1, \dots, \alpha_N) \in (\mathbb{R}^d)^N$ such that the system of mappings

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \alpha_i) + \alpha_i, \quad i = 1, \dots, N, \tag{12}$$

belongs to \mathcal{V}_n . By Theorem 1.4, we have

$$E_d(B_1, \dots, B_N) = \bigcap_{n=1}^{\infty} U_n.$$

It remains to show that for every $n \in \mathbb{N}$, the set U_n is open. Assume the contrary and let $\alpha = (\alpha_1, \dots, \alpha_N) \in U_n$ be not an interior point of U_n . Then there is a sequence $\{\beta_m\}_{m=1}^{\infty} \subset (\mathbb{R}^d)^N \setminus U_n$ such that $\alpha = \lim_{m \rightarrow \infty} \beta_m$. Let $\beta_m = (\beta_1^m, \dots, \beta_N^m)$, $m \in \mathbb{N}$, where $\beta_i^m \in \mathbb{R}^d$, $i = 1, \dots, N$. Then for every $m \in \mathbb{N}$, the system of contracting mappings

$$w_i^m(\mathbf{x}) = B_i(\mathbf{x} - \beta_i^m) + \beta_i^m, \quad i = 1, \dots, N,$$

does not belong to \mathcal{V}_n . Since for every $i = 1, \dots, N$ and $\mathbf{x} \in \mathbb{R}^d$,

$$\lim_{m \rightarrow \infty} w_i^m(\mathbf{x}) = w_i(\mathbf{x}),$$

where w_i is defined as in (12), and

$$\max_{i=1, \dots, N} \|B_i\| < 1,$$

we have a strong pointwise convergence of the sequence $\{w_i^m\}_{m=1}^{\infty}$ to w_i , $i = 1, \dots, N$. By Lemma 3.5, we have that (w_1, \dots, w_N) does not belong to \mathcal{V}_n , i.e. $\alpha \notin U_n$. This contradiction shows that U_n is an open set for every n and the assertion of Corollary 1.7 follows. \square

4. Remarks about conditions related the SMP

In this section we mention several simple statements, which give necessary or sufficient conditions for the SMP.

Proposition 4.1. *Let X be a complete metric space and (w_1, \dots, w_N) be a collection of contracting homeomorphisms of X onto X . If (w_1, \dots, w_N) satisfies the SOSC, then (w_1, \dots, w_N) satisfies the SMP.*

Proof. Let $\mathcal{O} \subset X$ be the open set from the definition of the SOSC. Show that $\overline{A \cap \mathcal{O}} = A$. Indeed, if $x \in A$ and $\epsilon > 0$ are arbitrary, for some $m \in \mathbb{N}$ sufficiently large and $\mathbf{i} \in \Sigma^m$ we have $x \in w_{\mathbf{i}}(A) \subset B(x, \epsilon)$. Since $w_{\mathbf{i}}(\mathcal{O} \cap A) \neq \emptyset$, $w_{\mathbf{i}}(\mathcal{O} \cap A) \subset B(x, \epsilon)$, and

$$w_{\mathbf{i}}(\mathcal{O} \cap A) = w_{\mathbf{i}}(\mathcal{O}) \cap w_{\mathbf{i}}(A) \subset \mathcal{O} \cap A,$$

we have $(\mathcal{O} \cap A) \cap B(x, \epsilon) \neq \emptyset$. Hence, $A \subset \overline{\mathcal{O} \cap A}$. Since the opposite inclusion is trivial, we have $\overline{\mathcal{O} \cap A} = A$.

If for every $n \in \mathbb{N}$, we let $\mathcal{O}_n = \mathcal{O}$, then conditions 1 and 2 in the definition of the SMP will hold and, by Theorem 1.4, we have $(w_1, \dots, w_N) \in \bigcap_{n=1}^{\infty} \mathcal{V}_n$. Proposition 4.1 is proved. \square

The converse of Proposition 4.1 is not true (see Remark 4.11 below).

Recall that

$$\mathcal{T} = \bigcup_{i \neq j} w_i(A) \cap w_j(A)$$

and denote

$$\mathcal{D} = \mathcal{D}(w_1, \dots, w_N) = A \setminus \bigcup_{i \in \mathcal{F}} w_i^{-1}(\mathcal{T}).$$

Theorem 1.3.8 in [7], in particular, implies the following statement.

Proposition 4.2. *Let X be a complete metric space and w_1, \dots, w_N be contracting homeomorphisms of the space X onto X . The system (w_1, \dots, w_N) is minimal if and only if $\mathcal{D}(w_1, \dots, w_N) \neq \emptyset$.*

Hence, in view of Proposition 4.1, the condition $\mathcal{D} = \emptyset$ implies that the SOSC does not hold. We also remark that the following statement holds.

Proposition 4.3. *Let X be a complete metric space and w_1, \dots, w_N be contracting homeomorphisms of the space X onto X . Then*

1. $w_i(\mathcal{D}) \subset \mathcal{D}$, $i = 1, \dots, N$;
2. $w_i(\mathcal{D}) \cap w_j(\mathcal{D}) = \emptyset$, $i \neq j$.

If in addition, this system satisfies the SMP, then

3. $\overline{\mathcal{D}} = A$.

Proof. If $\mathcal{D} = \emptyset$, conditions 1 and 2 hold trivially. Assume that $\mathcal{D} \neq \emptyset$ (which is equivalent to the SMP). To prove the first statement assume the contrary, i.e. for some $1 \leq k \leq N$, there is $y \in w_k(\mathcal{D}) \setminus \mathcal{D}$. Then there is a point $x \in \mathcal{D}$ such that $y = w_k(x)$. On the other hand, since y is not in \mathcal{D} , there is a vector $\mathbf{p} = (p_1, \dots, p_s) \in \mathcal{F}$ such that $w_{\mathbf{p}}(y) \in \mathcal{T}$. Hence, $w_{p_1, \dots, p_s, k}(x) \in \mathcal{T}$, which contradicts the fact that $x \in \mathcal{D}$.

To prove the second statement, assume again the contrary, i.e. for some indexes $1 \leq i \neq j \leq N$, there is a point $x \in w_i(\mathcal{D}) \cap w_j(\mathcal{D})$. Then $x = w_i(t)$, $t \in \mathcal{D}$. Since

$$w_i(t) \in w_i(\mathcal{D}) \cap w_j(\mathcal{D}) \subset w_i(A) \cap w_j(A) \subset \mathcal{T},$$

we have a contradiction with the fact that $t \in \mathcal{D}$.

To show the third statement, choose any point $z \in A$ and a ball $B(z, \epsilon)$, $\epsilon > 0$. Denote $r_{\max} = \max_{i=1, \dots, N} \sigma(w_i)$. Let $m \in \mathbb{N}$ be such number that $r_{\max}^m \cdot \text{diam} A < \epsilon$ and $\mathbf{i} = (i_1, \dots, i_m) \in \Sigma^m$ be such that $z \in w_{\mathbf{i}}(A)$. Let point $q \in A$ be such that $z = w_{\mathbf{i}}(q)$ and x be some point in \mathcal{D} . Then, by the first statement, $w_{\mathbf{i}}(x) \in \mathcal{D}$. Since

$$\begin{aligned} d(z, w_{\mathbf{i}}(x)) &= d(w_{\mathbf{i}}(q), w_{\mathbf{i}}(x)) \\ &\leq \sigma(w_{i_1}) \cdot \dots \cdot \sigma(w_{i_m}) \cdot d(q, x) \\ &\leq r_{\max}^m \cdot \text{diam} A < \epsilon, \end{aligned}$$

we have $\mathcal{D} \cap B(z, \epsilon) \neq \emptyset$ for every $z \in A$ and $\epsilon > 0$. Taking into account that $\mathcal{D} \subset A$, we have $\mathcal{D} = A$. Proposition 4.3 is proved. \square

The following statement shows the relation between the cardinality of the overlaps of sets $w_i(A)$ and the SMP.

Proposition 4.4. *Let $w_1, \dots, w_N \in \mathcal{M}(X)$ be such that the corresponding attractor A is uncountable and every set $w_i(A) \cap w_j(A)$, $i \neq j$, is at most countable. Then the system (w_1, \dots, w_N) satisfies the SMP.*

Proof. By assumption, the set \mathcal{T} is at most countable. Then the set $\cup_{\mathbf{i} \in \mathcal{F}} w_{\mathbf{i}}^{-1}(\mathcal{T})$ is also at most countable. Since A is uncountable, we have $D(w_1, \dots, w_N) \neq \emptyset$. By Proposition 4.2 and Theorem 1.4, the system (w_1, \dots, w_N) has the SMP. Proposition 4.4 is proved. \square

The following statement is obtained if one combines Corollaries 1.4.8 and 1.4.9 from [7]

Proposition 4.5. *Let X be a complete metric space and $w_1, \dots, w_N: X \rightarrow X$ be contracting homeomorphisms of X onto X . Assume that every point in the attractor A of this system has a finite number of addresses. Then the system (w_1, \dots, w_N) satisfies the SMP.*

We say that two vectors $\mathbf{i}, \mathbf{j} \in \mathcal{F}$ are *incomparable* if neither \mathbf{i} is an initial word of \mathbf{j} nor \mathbf{j} is an initial word of \mathbf{i} . Denote

$$\mathcal{E} = \{\overline{w_{\mathbf{j}}^{-1} w_{\mathbf{i}}}: \mathbf{i}, \mathbf{j} \in \mathcal{F} \cup \{\emptyset\}, \mathbf{i}, \mathbf{j} \text{ incomparable}\}.$$

Denote by I the identity mapping from X to X .

Remark 4.6. In the case when $X = \mathbb{R}^d$ and w_i 's are contractive similitudes, the results of papers by Hutchinson [6], Bandt and Graf [1], and Schief [12] imply the equivalence of the following three conditions: SOSC, OSC, and the condition that $I \notin \mathcal{E}$ in the topology of pointwise convergence of similitudes. The weak separation property (WSP) introduced by Lau and Ngai in [8] was shown to be equivalent to the condition that $I \notin \overline{\mathcal{E} \setminus \{I\}}$ for finite systems of contractive similitudes whose attractor is not contained in a hyperplane (cf. the work by Zerner [14]).

Moreover, the following statement holds.

Proposition 4.7 (see [14], Proposition 1). *Assume that attractor A of a system $W = (w_1, \dots, w_N)$ of contractive similitudes in \mathbb{R}^d is not contained in any hyperplane. Then the OSC holds if and only if the WSP holds and $w_{\mathbf{i}} \neq w_{\mathbf{j}}$ for every $\mathbf{i} \neq \mathbf{j}$ from \mathcal{F} .*

This proposition implies the following statement.

Proposition 4.8. *Let $W = (w_1, \dots, w_N)$ be a system of contractive similitudes in \mathbb{R}^d , whose attractor is not contained in any hyperplane. Then system W satisfies the SOSC if and only if it satisfies the WSP and the SMP.*

Proof. In view of Propositions 4.1 and 4.7, the SOSC implies the SMP and the WSP. If the system W satisfies the WSP and the SMP, assume to the contrary that for some $\mathbf{i} = (i_1, \dots, i_m) \neq \mathbf{j} = (j_1, \dots, j_n)$ from \mathcal{F} , we had $w_{\mathbf{i}} = w_{\mathbf{j}}$ ($m \leq n$). If \mathbf{i} is a prefix of \mathbf{j} , then $m < n$ and the contraction w_{j_{m+1}, \dots, j_n} must be the identity mapping which is a contradiction. If \mathbf{i} is not a prefix of \mathbf{j} , then $(i_1, \dots, i_m) \neq (j_1, \dots, j_m)$ and

$$w_{i_1, \dots, i_m}(A) = w_{j_1, \dots, j_n}(A) \subset w_{j_1, \dots, j_m}(A),$$

which contradicts the fact that W has the SMP and shows that our assumption is false. Then by Proposition 4.7, the system W possesses the SOSC. Proposition 4.8 is proved. \square

From the results cited above we obtain that that SOSC does not allow $I \in \mathcal{E}$. The WSP allows I to be in \mathcal{E} as an isolated point.

Proposition 4.9. *Let X be a complete metric space and let the system $(w_1, \dots, w_N) \in (\mathcal{M}(X))^N$ satisfy the SMP. Then $I \notin \mathcal{E}$. The converse is not true.*

Proof. The proof of the fact that $I \notin \mathcal{E}$ is analogous to the proof of the Proposition 4.8 when \mathbf{i} is not a prefix of \mathbf{j} . The following counterexample shows that the converse is not true. Let $w_1(x) = x/2$, $w_2(x) = (x + 1)/2$, and $w_3(x) = (x + a)/2$, where a is an irrational number from $(0, 1)$. It is not difficult to see that interval $[0, 1]$ is the attractor of the system (w_1, w_2, w_3) . Since $w_3([0, 1]) \subset w_1([0, 1]) \cup w_2([0, 1])$, the system (w_1, w_2, w_3) does not satisfy the SMP. If we assumed that $I \in \mathcal{E}$, there would be incomparable indexes $\mathbf{i} = (i_1, \dots, i_n), \mathbf{j} = (j_1, \dots, j_m) \in \mathcal{F}$ such that $w_{\mathbf{i}} = w_{\mathbf{j}}$. Hence,

$$w_{\mathbf{i}}(x) = \frac{1}{2^n}x + \sum_{\substack{k=1 \\ i_k=2}}^n \frac{1}{2^k} + \sum_{\substack{k=1 \\ i_k=3}}^n \frac{1}{2^k}a = w_{\mathbf{j}}(x) = \frac{1}{2^m}x + \sum_{\substack{k=1 \\ j_k=2}}^m \frac{1}{2^k} + \sum_{\substack{k=1 \\ j_k=3}}^m \frac{1}{2^k}a.$$

Then $n = m$ and

$$a \left(\sum_{k: j_k=3} \frac{1}{2^k} - \sum_{k: i_k=3} \frac{1}{2^k} \right) = \sum_{k: i_k=2} \frac{1}{2^k} - \sum_{k: j_k=2} \frac{1}{2^k}.$$

Since a is irrational, we must have

$$\sum_{k: j_k=3} \frac{1}{2^k} = \sum_{k: i_k=3} \frac{1}{2^k}.$$

Hence, $\{k: i_k = 3\} = \{k: j_k = 3\}$. But then

$$\sum_{k: i_k=2} \frac{1}{2^k} = \sum_{k: j_k=2} \frac{1}{2^k}.$$

Hence, $\{k: i_k = 2\} = \{k: j_k = 2\}$. This implies that $\{k: i_k = 1\} = \{k: j_k = 1\}$ and $\mathbf{i} = \mathbf{j}$, which also contradicts the incomparability of \mathbf{i} and \mathbf{j} . This contradiction shows that for the system (w_1, w_2, w_3) we have $I \notin \mathcal{E}$ but SMP does not hold. \square

Let $W = (w_1, \dots, w_N)$, where $w_1, \dots, w_N: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $d \in \mathbb{N}$, are similitudes with similarity coefficients $r_1, \dots, r_N \in (0, 1)$ respectively. Denote by $\alpha = \alpha(W)$ the unique positive number such that

$$r_1^\alpha + \dots + r_N^\alpha = 1. \tag{13}$$

This number is known as the similarity dimension of the attractor A associated with the system W . Denote by $\dim A$ the Hausdorff dimension of the set A and by \mathcal{H}_λ , $\lambda > 0$, the λ -dimensional Hausdorff measure in \mathbb{R}^d . The standard covering argument shows that

$$\dim A(W) \leq \alpha(W). \tag{14}$$

Proposition 4.10. *Let $W = (w_1, \dots, w_N)$ be a system of contracting similitudes in \mathbb{R}^d , $d \in \mathbb{N}$, and $\dim A(W) = \alpha(W)$. Then A satisfies the SMP.*

Remark 4.11. The results by Hutchinson [6], Theorem 1, Section 5.3], combined with the results by Schief [12], Theorem 2.1, imply that for the attractor $A(W)$ of a finite system W of contracting similitudes in \mathbb{R}^d , we have $\mathcal{H}_{\alpha(W)}(A(W)) > 0$ if and only if W satisfies the OSC. Proposition 4.10 implies that any finite system of contracting similitudes W such that $\dim A(W) = \alpha(W)$ and $\mathcal{H}_{\alpha(W)}(A(W)) = 0$ (examples of such systems were given by Mattila (see [10]) and Solomyak [13]), will have the SMP. But such system will not satisfy the OSC, which shows non-equivalence of these two properties. Since SMP implies MPP as asserted by Lemma 2.1, we conclude that MPP is also weaker than OSC.

Proof of Proposition 4.10. Assume the contrary. Then in view of Theorem 1.4, there is $n \in \mathbb{N}$ such that (w_1, \dots, w_N) does not belong to \mathcal{V}_n . Then there is a vector $\mathbf{i} \in \Sigma^n$ such that

$$w_{\mathbf{i}}(A) \subset \bigcup_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}} w_{\mathbf{j}}(A).$$

Hence,

$$A = \bigcup_{\mathbf{j} \in \Sigma^n} w_{\mathbf{j}}(A) = \bigcup_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}} w_{\mathbf{j}}(A)$$

and A will be also the attractor for the system of mappings $S = \{w_{\mathbf{j}}\}_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}}$. In this case the similarity dimension of A associated with system S satisfies

$$\sum_{\mathbf{j} \in \Sigma^n, \mathbf{j} \neq \mathbf{i}} r_{\mathbf{j}}^{\alpha(S)} = 1,$$

where $r_{\mathbf{j}}$ is the contraction coefficient of the mapping $w_{\mathbf{j}}$, $\mathbf{j} \in \Sigma^n$. Since

$$\sum_{\mathbf{j} \in \Sigma^n} r_{\mathbf{j}}^{\alpha(W)} = 1,$$

we have $\alpha(S) < \alpha(W)$. Then, by (14), we obtain $\dim A \leq \alpha(S) < \alpha(W)$, which contradicts the assumptions of the proposition. Proposition 4.10 is proved. \square

5. Genericity of the SMP on certain classes of self-affine sets

Let B_1, \dots, B_N be invertible $d \times d$ contraction matrices, $d \in \mathbb{N}$. Recall that $E_d(B_1, \dots, B_N)$ is the set of ordered point collections $(\alpha_1, \dots, \alpha_N) \in (\mathbb{R}^d)^N$ such that the system of mappings

$$u_i(\mathbf{x}) = B_i(\mathbf{x} - \alpha_i) + \alpha_i, \quad i = 1, \dots, N,$$

has the SMP. We will consider the set $E_d(B_1, \dots, B_N)$ as a subset of \mathbb{R}^{dN} .

Remark 5.1. When $B_1 = \lambda_1 U_1, \dots, B_N = \lambda_N U_N$, where matrices U_1, \dots, U_N are orthogonal and

- 1) $0 < \lambda_i < \frac{1}{2}, i = 1, \dots, N;$
- 2) $\sum_{i=1}^N \lambda_i^d < 1,$

the set $E_d(B_1, \dots, B_N)$ is a subset of \mathbb{R}^{dN} of full measure. This follows from results of Falconer [4], Theorem 5.3, Solomyak [13], Proposition 3.1, and Proposition 4.10. (Recent results of Falconer and Miao [5] also imply upper estimate for the Hausdorff dimension of the complement of $E_d(B_1, \dots, B_N)$.) In this paper we can show that $E_d(B_1, \dots, B_N)$ has full measure when assumption 1) is replaced with certain other assumptions.

Theorem 5.2. *Let B_1, \dots, B_N be invertible $d \times d$ contraction matrices such that $\sum_{i=1}^N \|B_i\| < 1$. Then the set $E_d(B_1, \dots, B_N)$ is a G_δ -subset of \mathbb{R}^{dN} of full Lebesgue measure.*

When $d = 1$ the result of Theorem 5.2 immediately follows from the result of Falconer [3] (also cited in [10], Theorem 9.13) and Corollary 1.7.

Theorem 5.3. *Let $B_i = \sigma_i U_i$, where $\sigma_i \in (0, 1)$, U_i is a 2×2 rotation matrix, $i = 1, \dots, N$, and $\sum_{i=1}^N \sigma_i^2 < 1$. Then the set $E_2(B_1, \dots, B_N)$ is either empty or is a G_δ -subset of \mathbb{R}^{2N} of full Lebesgue measure.*

Remark 5.4. The set $E_2(B_1, \dots, B_N)$ can be empty under assumptions of Theorem 5.3 as the following example shows. Let $\sigma_1, \sigma_2 > 0$ be such that $\sigma_1 + \sigma_2 > 1$ and $\sigma_1^2 + \sigma_2^2 < 1$, and $B_1 = \sigma_1 I_2, B_2 = \sigma_2 I_2$ (here and below I_d denotes the $d \times d$ identity matrix). For any ordered pair (α_1, α_2) of points in \mathbb{R}^2 , the attractor A of the system of mappings

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \alpha_i) + \alpha_i = \sigma_i \mathbf{x} + (1 - \sigma_i)\alpha_i, \quad i = 1, 2,$$

is the closed segment with endpoints α_1 and α_2 . The set $w_1(A) \cap w_2(A)$ is a segment of positive length. For $n \in \mathbb{N}$ sufficiently large and some index $\mathbf{i} \in \Sigma^n$, there holds $w_{\mathbf{i}}(A) \subset w_1(A) \cap w_2(A)$. If \mathbf{i} starts with 1, we have

$$w_{\mathbf{i}}(A) \subset w_2(A) = \bigcup_{\mathbf{j} \in \Sigma^{n-1}} w_2 w_{\mathbf{j}}(A) \subset \bigcup_{\substack{\mathbf{j} \in \Sigma^n \\ \mathbf{j} \neq \mathbf{i}}} w_{\mathbf{j}}(A).$$

If \mathbf{i} starts with 2 we use analogous argument. Thus, the system (w_1, w_2) does not possess the SMP for any collection of fixed points (α_1, α_2) and hence, $E_2(B_1, B_2) = \emptyset$. This example also shows that in the case $\sum_{i=1}^N \|B_i\| > 1$ the set $E_d(B_1, \dots, B_N)$ in Theorem 5.2 can even be empty.

The proof of Theorems 5.2 and 5.3 will follow from the statement presented below. For an ordered collection of points $\beta = (\beta_1, \dots, \beta_N) \in (\mathbb{R}^d)^N$, denote by $\Pi_{\mathbf{k}}(\beta)$ the element with the address $\mathbf{k} \in \Sigma^\infty$ in the attractor of the system of mappings

$$u_i(\mathbf{x}) = B_i(\mathbf{x} - \beta_i) + \beta_i, \quad i = 1, \dots, N.$$

Proposition 5.5. *Let $1 \leq k \leq d$ be an integer and B_1, \dots, B_N be invertible $d \times d$ contraction matrices such that $\sum_{i=1}^N \|B_i\|^k < 1$. Assume that there is an ordered collection $\gamma_1 = (\gamma_1^1, \dots, \gamma_N^1) \in (\mathbb{R}^d)^N$ such that the system $W = (w_1, \dots, w_N)$, where*

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \gamma_i^1) + \gamma_i^1, \quad i = 1, \dots, N,$$

has the SMP. In the case $k \geq 2$ assume also that there are collections $\gamma_j = (\gamma_1^j, \dots, \gamma_N^j) \in (\mathbb{R}^d)^N$, $j = 2, \dots, k$, such that for every pair of addresses $\mathbf{i} \neq \mathbf{j} \in \Sigma^\infty$ such that $\Pi_{\mathbf{i}}(\gamma_1) \neq \Pi_{\mathbf{j}}(\gamma_1)$, the system of vectors

$$\{\Pi_{\mathbf{i}}(\gamma_i) - \Pi_{\mathbf{j}}(\gamma_j) : i = 1, \dots, k\}$$

is linearly independent.

Then the set $E_d(B_1, \dots, B_N)$ is a G_δ -subset of \mathbb{R}^{dN} of full Lebesgue measure.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_N) \in (\mathbb{R}^d)^N$ be arbitrary. For every $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$, denote by $W_{\mathbf{t}} = (w_1^{\mathbf{t}}, \dots, w_N^{\mathbf{t}})$ the system of mappings

$$\begin{aligned} w_i^{\mathbf{t}}(\mathbf{x}) &= B_i(\mathbf{x} - \alpha_i - t_1\gamma_i^1 - \dots - t_k\gamma_i^k) \\ &\quad + \alpha_i + t_1\gamma_i^1 + \dots + t_k\gamma_i^k, \quad i = 1, \dots, N. \end{aligned}$$

Let $A_{\mathbf{t}} = A(W_{\mathbf{t}})$ be the attractor of the system $W_{\mathbf{t}}$ and $A = A(W)$ be the attractor of the system W . Denote

$$P(\alpha) = \{\mathbf{t} \in \mathbb{R}^k : W_{\mathbf{t}} \text{ has no SMP}\}$$

and for an index $\mathbf{i} = (i_1, \dots, i_n) \in \Sigma^n$, let $w_{\mathbf{i}}^{\mathbf{t}} = w_{i_1}^{\mathbf{t}} \circ \dots \circ w_{i_n}^{\mathbf{t}}$. We next estimate the Hausdorff dimension of the set $P(\alpha)$. This set can be represented as

$$P(\alpha) = \bigcup_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \Sigma^n} \left\{ \mathbf{t} \in \mathbb{R}^k : w_{\mathbf{i}}^{\mathbf{t}}(A_{\mathbf{t}}) \subset \bigcup_{\substack{\mathbf{j} \in \Sigma^n, \\ \mathbf{j} \neq \mathbf{i}}} w_{\mathbf{j}}^{\mathbf{t}}(A_{\mathbf{t}}) \right\}.$$

Denote by $\Pi_{\mathbf{k}}, \mathbf{k} \in \Sigma^\infty$, the element \mathbf{x} in A with address \mathbf{k} . Let also $\Pi_{\mathbf{k}}^{\mathbf{t}}, \mathbf{k} \in \Sigma^\infty$, be the element in $A_{\mathbf{t}}$ with address \mathbf{k} . For every $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma^n$, let $\mathbf{k}(\mathbf{i}) \in \Sigma^\infty$ be such sequence that

$$\Pi_{\mathbf{i}\mathbf{k}(\mathbf{i})} \notin \bigcup_{\substack{\mathbf{j} \in \Sigma^n \\ \mathbf{j} \neq \mathbf{i}}} w_{\mathbf{j}}(A)$$

(such $\mathbf{k}(\mathbf{i})$ exists since W satisfies the SMP). For every pair of indices $\mathbf{i} \neq \mathbf{j}$ from Σ^n , let

$$Q_{\mathbf{i},\mathbf{j}} = \{\mathbf{t} \in \mathbb{R}^k : \Pi_{\mathbf{ik}(\mathbf{i})}^{\mathbf{t}} \in w_{\mathbf{j}}^{\mathbf{t}}(A_{\mathbf{t}})\}.$$

Then

$$P(\boldsymbol{\alpha}) \subset \bigcup_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \Sigma^n} \bigcup_{\substack{\mathbf{j} \in \Sigma^n \\ \mathbf{j} \neq \mathbf{i}}} Q_{\mathbf{i},\mathbf{j}}.$$

We now fix a number $n \in \mathbb{N}$ and indices $\mathbf{i} \neq \mathbf{j}$ from Σ^n . For every $m \in \mathbb{N}$ and $\mathbf{k} \in \Sigma^m$, denote

$$Q_{\mathbf{i},\mathbf{j}}^{\mathbf{k}} = \{\mathbf{t} \in \mathbb{R}^k : \Pi_{\mathbf{ik}(\mathbf{i})}^{\mathbf{t}} \in w_{\mathbf{jk}(\mathbf{j})}^{\mathbf{t}}(A_{\mathbf{t}})\} = \{\mathbf{t} \in \mathbb{R}^k : \Pi_{\mathbf{ik}(\mathbf{i})}^{\mathbf{t}} = \Pi_{\mathbf{jkp}}^{\mathbf{t}} \text{ for some } \mathbf{p} \in \Sigma^m\}.$$

Then

$$Q_{\mathbf{i},\mathbf{j}} = \bigcup_{\mathbf{k} \in \Sigma^m} Q_{\mathbf{i},\mathbf{j}}^{\mathbf{k}}.$$

It is a straightforward argument to verify that for every address $\mathbf{q} = (q_1, q_2, \dots) \in \Sigma^\infty$, we have

$$\begin{aligned} \Pi_{\mathbf{q}}^{\mathbf{t}} &= \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (I_d - B_{q_i}) (\boldsymbol{\alpha}_{q_i} + t_1 \boldsymbol{\gamma}_{q_i}^1 + \dots + t_k \boldsymbol{\gamma}_{q_i}^k) \\ &= \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (I_d - B_{q_i}) \boldsymbol{\alpha}_{q_i} \\ &\quad + t_1 \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (I_d - B_{q_i}) \boldsymbol{\gamma}_{q_i}^1 + \dots + \\ &\quad + t_k \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (I_d - B_{q_i}) \boldsymbol{\gamma}_{q_i}^k \\ &= \Pi_{\mathbf{q}}(\boldsymbol{\alpha}) + t_1 \Pi_{\mathbf{q}}(\boldsymbol{\gamma}_1) + \dots + t_k \Pi_{\mathbf{q}}(\boldsymbol{\gamma}_k). \end{aligned}$$

Then

$$\begin{aligned} Q_{\mathbf{i},\mathbf{j}}^{\mathbf{k}} &= \left\{ \mathbf{t} \in \mathbb{R}^k : \Pi_{\mathbf{ik}(\mathbf{i})}(\boldsymbol{\alpha}) + \sum_{i=1}^k t_i \Pi_{\mathbf{ik}(\mathbf{i})}(\boldsymbol{\gamma}_i) \right. \\ &\quad \left. = \Pi_{\mathbf{jkp}}(\boldsymbol{\alpha}) + \sum_{i=1}^k t_i \Pi_{\mathbf{jkp}}(\boldsymbol{\gamma}_i) \text{ for some } \mathbf{p} \in \Sigma^\infty \right\} \\ &= \left\{ \mathbf{t} \in \mathbb{R}^k : \sum_{i=1}^k t_i (\Pi_{\mathbf{ik}(\mathbf{i})}(\boldsymbol{\gamma}_i) - \Pi_{\mathbf{jkp}}(\boldsymbol{\gamma}_i)) \right. \\ &\quad \left. = \Pi_{\mathbf{jkp}}(\boldsymbol{\alpha}) - \Pi_{\mathbf{ik}(\mathbf{i})}(\boldsymbol{\alpha}) \text{ for some } \mathbf{p} \in \Sigma^\infty \right\}. \end{aligned}$$

Given an address $\mathbf{q} \in \Sigma^\infty$, let

$$B(\mathbf{q}) = [\Pi_{\mathbf{ik}(\mathbf{i})}(\mathcal{Y}_1) - \Pi_{\mathbf{jq}}(\mathcal{Y}_1), \dots, \Pi_{\mathbf{ik}(\mathbf{i})}(\mathcal{Y}_k) - \Pi_{\mathbf{jq}}(\mathcal{Y}_k)]$$

be the $d \times k$ matrix with columns

$$\Pi_{\mathbf{ik}(\mathbf{i})}(\mathcal{Y}_i) - \Pi_{\mathbf{jq}}(\mathcal{Y}_i), \quad i = 1, \dots, k.$$

Let also

$$\mathbf{b}(\mathbf{q}) = \Pi_{\mathbf{jq}}(\boldsymbol{\alpha}) - \Pi_{\mathbf{ik}(\mathbf{i})}(\boldsymbol{\alpha}),$$

$$\sigma = \max_{i=1, \dots, N} \|B_i\|,$$

and

$$a = \max\{\text{diam } A(\boldsymbol{\alpha}), \text{diam } A(\mathcal{Y}_1), \dots, \text{diam } A(\mathcal{Y}_k)\}, \tag{15}$$

where $A(\mathbf{c})$, $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_N) \in (\mathbb{R}^d)^N$, denotes the attractor of the system

$$u_i(\mathbf{x}) = B_i(\mathbf{x} - \mathbf{c}_i) + \mathbf{c}_i, \quad i = 1, \dots, N,$$

and for an index $\mathbf{j} = (j_1, \dots, j_n) \in \Sigma^n$, denote

$$\sigma_{\mathbf{j}} = \|B_{j_1}\| \cdot \dots \cdot \|B_{j_n}\|.$$

Then

$$Q_{\mathbf{i}, \mathbf{j}}^{\mathbf{k}} = \{\mathbf{t} \in \mathbb{R}^k : B(\mathbf{kp}) \cdot \mathbf{t} = \mathbf{b}(\mathbf{kp}) \text{ for some } \mathbf{p} \in \Sigma^\infty\}.$$

We will need the following auxiliary statement.

Lemma 5.6. *Let \mathcal{C} be a set of $d \times k$ matrices of rank $k \leq d$, which has diameter δ with respect to the matrix norm (2), and \mathcal{P} be a set of vectors from \mathbb{R}^d , which has diameter ϵ with respect to the Euclidean distance. Assume that there exists a finite and positive number $M > 0$ such that for every matrix $B \in \mathcal{C}$,*

$$\|(B^T B)^{-1}\| \leq M.$$

Denote also by L and K positive numbers such that $\|B\| \leq L$ for every matrix $B \in \mathcal{C}$, and $|\mathbf{b}| \leq K$ for every vector $\mathbf{b} \in \mathcal{P}$. Let Q be the set of all vectors $\mathbf{t} \in \mathbb{R}^k$, which are solutions to the equation

$$B\mathbf{t} = \mathbf{b}$$

for some matrix $B \in \mathcal{C}$ and vector $\mathbf{b} \in \mathcal{P}$. Then

$$\text{diam } Q \leq \epsilon ML + \delta MK + 2\delta M^2 L^2 K. \tag{16}$$

Proof. Let \mathbf{t}_1 and \mathbf{t}_2 be arbitrary points from Q . There exist matrices $B_1, B_2 \in \mathcal{C}$ and vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{P}$ such that

$$B_i \mathbf{t}_i = \mathbf{b}_i, \quad i = 1, 2. \tag{17}$$

Since matrices B_1 and B_2 have rank k , vector $\mathbf{t}_i = (B_i^T B_i)^{-1} B_i^T \mathbf{b}_i$ is the unique solution for the i -th equation (17), $i = 1, 2$. Then

$$\begin{aligned} |\mathbf{t}_1 - \mathbf{t}_2| &= |(B_1^T B_1)^{-1} B_1^T \mathbf{b}_1 - (B_2^T B_2)^{-1} B_2^T \mathbf{b}_2| \\ &\leq |(B_1^T B_1)^{-1} B_1^T \mathbf{b}_1 - (B_2^T B_2)^{-1} B_1^T \mathbf{b}_1| \\ &\quad + |(B_2^T B_2)^{-1} B_1^T \mathbf{b}_1 - (B_2^T B_2)^{-1} B_2^T \mathbf{b}_1| \\ &\quad + |(B_2^T B_2)^{-1} B_2^T \mathbf{b}_1 - (B_2^T B_2)^{-1} B_2^T \mathbf{b}_2| \\ &\leq \|(B_1^T B_1)^{-1} - (B_2^T B_2)^{-1}\| \cdot \|B_1^T\| \cdot |\mathbf{b}_1| \\ &\quad + \|(B_2^T B_2)^{-1}\| \cdot \|B_1^T - B_2^T\| \cdot |\mathbf{b}_1| \\ &\quad + \|(B_2^T B_2)^{-1}\| \cdot \|B_2^T\| \cdot |\mathbf{b}_1 - \mathbf{b}_2|. \end{aligned}$$

Due to equality $\|B^T\| = \|B\|$ and definition of numbers M, L and K , we have

$$|\mathbf{t}_1 - \mathbf{t}_2| \leq LK \cdot \|(B_1^T B_1)^{-1} - (B_2^T B_2)^{-1}\| + \delta MK + \epsilon ML.$$

Using the estimate

$$\begin{aligned} &\|(B_1^T B_1)^{-1} - (B_2^T B_2)^{-1}\| \\ &= \|(B_2^T B_2)^{-1} B_2^T B_2 (B_1^T B_1)^{-1} - (B_2^T B_2)^{-1} B_1^T B_1 (B_1^T B_1)^{-1}\| \\ &= \|(B_2^T B_2)^{-1} (B_2^T B_2 - B_1^T B_1) (B_1^T B_1)^{-1}\| \\ &\leq M^2 \|B_2^T B_2 - B_1^T B_1\| \\ &\leq M^2 (\|B_2^T B_2 - B_2^T B_1\| + \|B_2^T B_1 - B_1^T B_1\|) \\ &\leq M^2 (\|B_2^T\| \cdot \|B_2 - B_1\| + \|B_2^T - B_1^T\| \cdot \|B_1\|) \\ &\leq 2\delta M^2 L, \end{aligned} \tag{18}$$

for every $\mathbf{t}_1, \mathbf{t}_2 \in Q$, we obtain

$$|\mathbf{t}_1 - \mathbf{t}_2| \leq 2\delta M^2 L^2 K + \delta MK + \epsilon ML,$$

and estimate (16) follows. Lemma 5.6 is proved. \square

Completion of the proof of Proposition 5.5. We apply Lemma 5.6 with $\mathcal{C} = \{B(\mathbf{kp}) : \mathbf{p} \in \Sigma^\infty\}$ and $\mathcal{P} = \{\mathbf{b}(\mathbf{kp}) : \mathbf{p} \in \Sigma^\infty\}$. For a matrix $B = [\mathbf{c}_1, \dots, \mathbf{c}_k]$, denote

$$\|B\|_{2,\infty} = \max_{i=1,\dots,k} |\mathbf{c}_i|.$$

It is not difficult to see that for any $d \times k$ matrix B ,

$$\|B\|_{2,\infty} \leq \|B\| \leq \sqrt{k} \|B\|_{2,\infty}. \tag{19}$$

Let

$$M_{i,j} = \sup_{\mathbf{q} \in \Sigma^\infty} \|(B(\mathbf{q})^T B(\mathbf{q}))^{-1}\|.$$

Denote $\mathcal{Y} = \{B(\mathbf{q}) : \mathbf{q} \in \Sigma^\infty\}$. By assumption, since $\Pi_{\mathbf{jk}(i)} \notin w_j(A)$, the columns of matrix $B(\mathbf{q})$ are linearly independent for every $\mathbf{q} \in \Sigma^\infty$. In view of the fact that $\det B^T B \neq 0$, $B \in \mathcal{Y}$, and continuity of $\det B^T B$ and of the algebraic complement to every element of $B^T B$ (with respect to the matrix norm (2)), we have that $\|(B^T B)^{-1}\|$ is also continuous with respect to matrix norm (2) on the set \mathcal{Y} . Since \mathcal{Y} is compact with respect to the matrix norm (2), we obtain that $M_{i,j}$ is finite. It is not difficult to see that $\text{diam } \mathcal{C} \leq a\sqrt{k}\sigma_j\sigma_k$ and $\text{diam } \mathcal{P} \leq a\sigma_j\sigma_k$, where a is defined by (15). Denote

$$L_{i,j} = \sup_{\mathbf{q} \in \Sigma^\infty} \|B(\mathbf{q})\|,$$

and let

$$K_{i,j} = \sup_{\mathbf{q} \in \Sigma^\infty} |\mathbf{b}(\mathbf{q})|.$$

Then by Lemma 5.6,

$$\text{diam } Q_{i,j}^k \leq \sigma_{jk} a M_{i,j} L_{i,j} + \sigma_{jk} \sqrt{k} a M_{i,j} K_{i,j} + 2\sigma_{jk} \sqrt{k} a M_{i,j}^2 L_{i,j}^2 K_{i,j} =: \sigma_j \sigma_k U_{i,j}.$$

Denote by λ such number that

$$\sum_{i=1}^N \|B_i\|^\lambda = 1.$$

Then

$$\mathcal{H}_\lambda(Q_{i,j}) \leq \limsup_{m \rightarrow \infty} \sum_{\mathbf{k} \in \Sigma^m} (\text{diam } Q_{i,j}^{\mathbf{k}})^\lambda \leq \lim_{m \rightarrow \infty} \sum_{\mathbf{k} \in \Sigma^m} \sigma_j^\lambda \sigma_k^\lambda U_{i,j}^\lambda = \sigma_j^\lambda U_{i,j}^\lambda < \infty.$$

Since $P(\alpha)$ is covered by a countable collection of sets of Hausdorff dimension at most λ , we have $\dim P(\alpha) \leq \lambda < k$. To complete the proof of the proposition denote $V = \mathbb{R}^{dN} \setminus E_d(B_1, \dots, B_N)$ and let $\Gamma = [\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_k]$ be the $dN \times k$ matrix with columns $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_k$, and $l = \dim \text{Ker } \Gamma$. For every vector $\alpha \in (\text{Im } \mathbb{L})^\perp$, we also let $Q(\alpha) = P(\alpha) \cap (\text{Ker } \Gamma)^\perp$. Then we have $P(\alpha) = Q(\alpha) \oplus \text{Ker } \mathbb{L}$. Since $\dim P(\alpha) < k$, the set $P(\alpha)$ has k -dimensional Lebesgue measure zero and hence, the set $Q(\alpha)$ as a subset of the space $(\text{Ker } \Gamma)^\perp$ has $(k - l)$ -dimensional Lebesgue measure zero. Since mapping $f : (\text{Ker } \Gamma)^\perp \rightarrow (\alpha + \text{Im } \mathbb{L})$, $f(\mathbf{t}) = \alpha + \Gamma \mathbf{t}$ is affine, bijective, and $f(Q(\alpha)) = \alpha + T(\alpha)$, where

$$T(\alpha) = \{\mathbf{y} \in \text{Im } \Gamma : \alpha + \mathbf{y} \in V\},$$

we have

$$\mathcal{L}'(T(\alpha)) = 0, \tag{20}$$

where \mathcal{L}' is the $(k - l)$ -dimensional Lebesgue measure in the space $\text{Im}\Gamma$. Corollary 1.7 implies that V is Lebesgue measurable, since it is a complement of a G_δ -set. Moreover,

$$V = \bigcup_{\alpha \in (\text{Im}\Gamma)^\perp} (\alpha + T(\alpha)),$$

where the union is disjoint. Denote by \mathcal{L}'' the $(dN - k + l)$ -dimensional Lebesgue measure in the space $(\text{Im}\Gamma)^\perp$. In view of (20) we obtain

$$\mathcal{L}_{dN}(V) = \int_{(\text{Im}\Gamma)^\perp} \mathcal{L}'(T(\alpha)) d\mathcal{L}''(\alpha) = 0,$$

which shows that $E_d(B_1, \dots, B_N)$ has full measure. The fact that it is a G_δ -set follows from Corollary 1.7. Proposition 5.5 is proved. \square

Proof of Theorem 5.2. Let $\mathbf{u} \in \mathbb{R}^d$ be a unit vector. Since $\sum_{i=1}^N \|B_i\| < 1$, there are numbers $c_1, \dots, c_N \in (-1, 1)$ such that balls $B[c_i \mathbf{u}, \|B_i\|]$, $i = 1, \dots, N$, are pairwise disjoint and are contained in $B[\mathbf{0}, 1]$. Let $\boldsymbol{\gamma}_i^1 = c_i(I_d - B_i)^{-1} \mathbf{u}$, and

$$w_i(\mathbf{x}) = B_i \mathbf{x} + c_i \mathbf{u} = B_i(\mathbf{x} - \boldsymbol{\gamma}_i^1) + \boldsymbol{\gamma}_i^1, \quad i = 1, \dots, N,$$

and $A = A(w_1, \dots, w_N)$ be the attractor of the system (w_1, \dots, w_N) . It is not difficult to see that

$$w_i(B[\mathbf{0}, 1]) \subset B[c_i \mathbf{u}, \|B_i\|] \subset B[\mathbf{0}, 1], \quad i = 1, \dots, N.$$

This implies that $A \subset B[\mathbf{0}, 1]$. Indeed, for every element $\mathbf{x} \in A$, there is a sequence $(i_1, i_2, \dots) \in \Sigma^\infty$ such that $\mathbf{x} = \lim_{n \rightarrow \infty} w_{i_1, \dots, i_n}(\mathbf{0})$. Since $w_{i_1, \dots, i_n}(\mathbf{0}) \in B[\mathbf{0}, 1]$ for every $n \in \mathbb{N}$, we have $\mathbf{x} \in B[\mathbf{0}, 1]$. We also have

$$\begin{aligned} w_i(A) \cap w_j(A) &\subset w_i(B[\mathbf{0}, 1]) \cap w_j(B[\mathbf{0}, 1]) \\ &\subset B[c_i \mathbf{u}, \|B_i\|] \cap B[c_j \mathbf{u}, \|B_j\|] = \emptyset, \quad i \neq j, \end{aligned}$$

which implies $w_i(A) \cap w_j(A) = \emptyset$, $\mathbf{i}, \mathbf{j} \in \Sigma^n$, $\mathbf{i} \neq \mathbf{j}$, $n \in \mathbb{N}$. Hence, system of mappings (w_1, \dots, w_N) has the SMP and we have $\boldsymbol{\gamma}_1 = (\boldsymbol{\gamma}_1^1, \dots, \boldsymbol{\gamma}_N^1) \in E_d(B_1, \dots, B_N)$. Since $k = 1$, the other assumption of Proposition 5.5 does not apply and we obtain that $E_d(B_1, \dots, B_N)$ has full measure and is a G_δ -set. Theorem 5.2 is proved. \square

Proof of Theorem 5.3. Assume that $E_2(B_1, \dots, B_N) \neq \emptyset$ and let $\boldsymbol{\gamma}_1 = (\boldsymbol{\gamma}_1^1, \dots, \boldsymbol{\gamma}_N^1) \in (\mathbb{R}^2)^N$ be such collection of points that the system of mappings

$$w_i(\mathbf{x}) = B_i(\mathbf{x} - \boldsymbol{\gamma}_i^1) + \boldsymbol{\gamma}_i^1, \quad i = 1, \dots, N,$$

satisfies the SMP. Denote

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(V is a rotation matrix) and let $\boldsymbol{\gamma}_2 = (V\boldsymbol{\gamma}_1^1, \dots, V\boldsymbol{\gamma}_N^1)$. Note that for any non-zero vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, we have

$$\det[\mathbf{x}, V\mathbf{x}] = \det \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} = x_1^2 + x_2^2 \neq 0. \quad (21)$$

Since rotation matrices commute, for every address $\mathbf{q} = (q_1, q_2, \dots) \in \Sigma^\infty$, we obtain,

$$\begin{aligned} \Pi_{\mathbf{q}}(\boldsymbol{\gamma}_2) &= \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (I_2 - B_{q_i}) V \boldsymbol{\gamma}_{q_i}^1 \\ &= \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (V - B_{q_i} V) \boldsymbol{\gamma}_{q_i}^1 \\ &= \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (V - V B_{q_i}) \boldsymbol{\gamma}_{q_i}^1 \\ &= V \sum_{i=1}^{\infty} B_{q_1} \cdots B_{q_{i-1}} (I_2 - B_{q_i}) \boldsymbol{\gamma}_{q_i}^1 \\ &= V \Pi_{\mathbf{q}}(\boldsymbol{\gamma}_1). \end{aligned}$$

Then for every pair of addresses $\mathbf{i} \neq \mathbf{j} \in \Sigma^\infty$ such that $\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) \neq \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1)$, in view of (21), we have

$$\begin{aligned} &\det[\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1), \Pi_{\mathbf{i}}(\boldsymbol{\gamma}_2) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_2)] \\ &= \det[\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1), V(\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_1) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_1))] \neq 0. \end{aligned}$$

Then vectors $\Pi_{\mathbf{i}}(\boldsymbol{\gamma}_i) - \Pi_{\mathbf{j}}(\boldsymbol{\gamma}_i)$, $i = 1, 2$, are linearly independent and by Proposition 5.5 we obtain that $E_2(B_1, \dots, B_N)$ is a G_δ -subset of \mathbb{R}^{2N} of full Lebesgue measure. \square

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Received September 27, 2013

Tim Bedford, Business School, University of Strathclyde, Glasgow, G1 1QX, Scotland
E-mail: tim.bedford@strath.ac.uk

Sergiy Borodachov, Department of Mathematics, Towson University, 7800 York Road,
Towson, MD, 21252, USA
E-mail: sborodachov@towson.edu

Jeffrey Geronimo, Department of Mathematics, Georgia Institute of Technology, Atlanta,
GA, 30332-0160, USA
E-mail: geronimo@math.gatech.edu